Acta Math., 216 (2016), 127–176
DOI: 10.1007/s11511-016-0137-1

Regularity of Kähler–Ricci flows on Fano manifolds

by

Gang Tian

Peking University
Beijing, China
and
Princeton University
Princeton, NJ, U.S.A.

Zhenlei Zhang

Capital Normal University
Beijing, China

Contents

1. Introduction

The goal of this paper is to study the long-time behavior of Kähler–Ricci flows on Fano manifolds. We will solve a long-standing conjecture in a low-dimensional case.

The first author was supported by NSF grants. The second author was supported by a grant of Beijing MCE 11224010007 and NSFC 13210010022.

Let M be a Fano n-manifold, that is a compact Kähler manifold with positive first Chern class. Consider the normalized Kähler–Ricci flow

$$\frac{\partial g}{\partial t} = g - \mathrm{Ric}(g). \tag{1.1}$$

It was proved in [6] that (1.1) has a global solution $g=g(t)$ in the case that $g(0)=g_0$ represents $2\pi c_1(M)$. The main problem is to understand the limit of $g(t)$ as $t\to\infty$. A desirable result for the limit is given by the following folklore conjecture.(1)

Conjecture 1.1. ([38]) $(M, g(t))$, as $t\to\infty$, converges (at least along a subsequence) to a shrinking Kähler–Ricci soliton with mild singularities.

Here, "mild singularities" may be understood in two ways: (i) as a singular set of codimension at least 4 in the metric topology, or (ii) as a singular set of a normal variety. The first interpretation concerns the differential-geometric part of the problem where the convergence is taken in the Gromov–Hausdorff topology, while in the second interpretation the spaces $(M, g(t))$ converge as algebraic varieties in some projective space. By extending the partial C^0-estimate conjecture [40] to the Kähler–Ricci flow, one can show that these two interpretations are actually equivalent for the limiting space (see Theorem 1.6 and §5 for more details).

Conjecture 1.1 implies another famous conjecture, the Yau–Tian–Donaldson conjecture, in the case of Fano manifolds. The Yau–Tian–Donaldson conjecture states that a Fano manifold M admits a Kähler–Einstein metric if and only if M is K-stable. The necessary part of the conjecture is proved by the first named author in [38]. Recently, the same author gave a proof for the sufficient part (see [42]) by establishing an extension of the Cheeger–Colding–Tian theory for compactness and the partial C^0-estimate for conic Kähler–Einstein metrics. Another proof was given in [13]–[14]. As we will see in the subsequent sections, the crucial step in the resolution of conjecture (1.1), as for proving the Yau–Tian–Donaldson conjecture, is to establish the Cheeger–Gromov convergence of the Kähler–Ricci flow.

Let us recall some facts about the Kähler–Ricci flow. Perelman has shown that the diameter and scalar curvature of $g(t)$ are uniformly bounded (see [34] for the proof). Then, by the non-collapsing result of Perelman [27], there is a positive constant $\varkappa$ depending only on g_0 such that, for all $x\in M$,

$$\mathrm{vol}_{g(t)}(B_{g(t)}(x,r)) \geqslant \varkappa r^{2n} \quad \text{for all } t\geqslant 0 \text{ and } r\leqslant 1. \tag{1.2}$$

Since the volume stays the same along the Kähler–Ricci flow (1.1), it follows from the non-collapsing property (1.2) that for any sequence $t_i\to\infty$, by taking a subsequence if

(1) It has often been referred to as the Hamilton–Tian conjecture in the literature, e.g., in [27]. Also see [38] for a formulation of this conjecture.

necessary, $(M, g(t_i))$ converges to a limiting length-metric space (M_∞, d) in the Gromov–Hausdorff topology, which we write as

$$(M, g(t_i)) \xrightarrow{d_{\mathrm{GH}}} (M_\infty, d). \tag{1.3}$$

The question is the regularity of the limit M_∞. In the case of del Pezzo surfaces, or in higher dimensions with additional assumption of uniformly bounded Ricci curvature or Bakry–Émery–Ricci curvature, the regularity of M_∞ has been established (cf. [33], [15] and [44]). In the case that M admits Kähler–Einstein metrics, Perelman asserted that the Kähler–Ricci flow converges to a smooth Kähler–Einstein metric. He also showed some crucial estimates towards his proof. A complete proof of this was given by Tian–Zhu and also generalized to the case of Kähler–Ricci solitons under the assumption that the metric is invariant under the S^1-action induced by the imaginary part of the holomorphic vector field associated with the Ricci soliton (see [46], [47] and [43]).

The following theorem is the main result of this paper.

THEOREM 1.2. *Let $(M, g(t))$, t_i and (M_∞, d) be given as above. Suppose that, for some uniform constants $p>n$ and $\Lambda<\infty$,*

$$\int_M |\mathrm{Ric}(g(t))|^p \, dv_{g(t)} \leqslant \Lambda. \tag{1.4}$$

Then the limit M_∞ is smooth outside a closed subset $\mathcal{S}$ of (real) codimension $\geqslant 4$ and d is induced by a smooth Kähler–Ricci soliton g_∞ on $M_\infty \setminus \mathcal{S}$. Moreover, $g(t_i)$ converges to g_∞ in the C^∞-topology outside $\mathcal{S}$.(2)

Remark 1.3. In view of the main result in [43], one should be able to prove that under the assumption of Theorem 1.2, the spaces $(M, g(t))$ converge globally to (M_∞, g_∞) in the Cheeger–Gromov topology as t tends to ∞. If M admits a shrinking Kähler–Ricci soliton, then by the uniqueness theorem of Berndtsson [4] and Berman–Boucksom–Essydieux–Guedj–Zeriahi [3], the Kähler–Ricci flow should converge to the Ricci soliton. This will be discussed in a future paper.

The proof of Theorem 1.2 is based on Perelman's pseudolocality theorem [27] of Ricci flow and a regularity theory for manifolds with integral-bounded Ricci curvature. The latter is a generalization of the regularity theory of Cheeger–Colding [9], [10] and Cheeger–Colding–Tian [11] for manifolds with bounded Ricci curvature. We remark that the uniform non-collapsing condition (1.2) also plays a role in our regularity theory, we refer the readers to §2 for a further discussion.

It is crucial to check if the integral condition of Ricci curvature holds along the Kähler–Ricci flow. Indeed, we can prove the following partial result.

(2) Convergence with these properties is also referred to as *convergence in the Cheeger–Gromov topology*; see [36] for instance.

THEOREM 1.4. *Let $(M, g(t))$ be as above. There exists a constant Λ depending on g_0 such that*

$$\int_M |\mathrm{Ric}(g(t))|^4 \, dv_{g(t)} \leqslant \Lambda. \tag{1.5}$$

Therefore, by applying the regularity result in Theorem 1.2, we have the following consequence.

COROLLARY 1.5. *Conjecture 1.1 holds for dimension $n \leqslant 3$.*

Inspired by [19], as well as [42] and [41], as an application of Theorem 1.2 we can also establish the partial C^0-estimate for the Kähler–Ricci flow (see §5 for details). As a direct consequence, we can refine the regularity in Theorem 1.2.

THEOREM 1.6. *Suppose that $(M, g(t_i)) \xrightarrow{d_{\mathrm{GH}}} (M_\infty, g_\infty)$ as in Theorem 1.2. Then M_∞ is a normal projective variety and $\mathcal{S}$ is a subvariety of complex codimension at least 2.*

Remark 1.7. If we consider a Kähler–Ricci flow on a normal Fano orbifold, then the limit M_∞ is also a normal variety. The main ingredients in the proof of our regularity of Kähler–Ricci flows still hold for orbifolds. Using the convexity of the regular set, we can generalize the regularity theory of Cheeger–Colding and Cheeger–Colding–Tian to orbifolds with integral-bounded Ricci curvature. Moreover, Perelman's estimates on Ricci potentials, local volume non-collapsings as well as the pseudolocality theorem are still valid for the orbifold Kähler–Ricci flow.

The partial C^0-estimate of Kähler–Einstein manifolds plays the key role in Tian's program to resolve the Yau–Tian–Donaldson conjecture, see [36], [38], [40], [19] and [42] for examples. An extension of the partial C^0-estimate to shrinking Kähler–Ricci solitons was given in [31]. These works are based on the compactness of Cheeger–Colding–Tian [11] and its generalizations to Kähler–Ricci solitons by [44]. In [36] and [37] the partial C^0-estimate conjecture was proposed for metrics with positive lower bound on the Ricci curvature. This conjecture has been actively studied in recent years.

Then, we show that the Yau–Tian–Donaldson conjecture follows from the Hamilton–Tian conjecture using the Kähler–Ricci flow. As discussed before, the key is the partial C^0-estimate. One can follow the arguments in [40] and [42]. Let M be K-stable as defined in [38]. Suppose $(M, g(t_i))$, as $t_i \to \infty$, converges in the Cheeger–Gromov topology to a shrinking Kähler–Ricci soliton (M_∞, g_∞), possibly with singularities, as in Theorem 1.2. In §6 we are going to show that M_∞ is isomorphic to M and g_∞ is Einstein, that is we have the following result.

THEOREM 1.8. *Suppose that M is* K*-stable. If $(M, g(t_i)) \xrightarrow{d_{\mathrm{GH}}} (M_\infty, g_\infty)$ as in Theorem 1.2, then M_∞ coincides with M and g_∞ is a Kähler–Einstein metric.*

In particular, we have the following consequence.

COROLLARY 1.9. *The Yau–Tian–Donaldson conjecture holds for dimension* $\leqslant 3$.

The authors would like to thank the referees and the technical editors for their comments which were helpful in improving the presentation of the paper.

2. Manifolds with integral-bounded Ricci curvature

In this section, following the lines of Cheeger–Colding [8]–[10], Cheeger–Colding–Tian [11] and Colding–Naber [17], we develop a regularity theory for manifolds with integral-bounded Ricci curvature. Let (M,g) be an m-dimensional Riemannian manifold satisfying

$$\int_M |\mathrm{Ric}_-|^p \, dv \leqslant \Lambda, \tag{2.1}$$

for some constants $\Lambda<\infty$ and $p>\frac{1}{2}m$, where $\mathrm{Ric}_-=\max_{|v|=1}\{0,-\mathrm{Ric}(v,v)\}$. We may assume $\Lambda\geqslant 1$ in generality. For applications to the regularity theory of the Kähler–Ricci flow, we shall focus on the case when the manifold (M,g) is uniformly locally non-collapsing in the sense that

$$\mathrm{vol}(B(x,r)) \geqslant \varkappa r^m \quad \text{for all } x\in M \text{ and } r\leqslant 1, \tag{2.2}$$

where $\varkappa>0$ is a fixed constant. It is remarkable that different phenomena would happen if we replace the condition (2.2) by a non-collapsing condition in a definite scale such as $\mathrm{vol}(B(p,1))\geqslant\varkappa$. Actually, due to an example of Yang [48], for any $p>0$, there exists a Gromov–Hausdorff limit space of m-manifolds with uniformly L^p-bounded Riemannian curvature and $\mathrm{vol}(B(x,1))\geqslant\varkappa$ for any x whose tangent cones may collapse.

The geometry of manifolds with integral-bounded Ricci curvature has been studied extensively by Dai, Petersen, Wei et al., see [30] and references therein. It is also pointed out in [30] that there should exist a Cheeger–Gromov convergence theory for such manifolds. The critical assumption added here is (2.2). The regularity theory without this uniform non-collapsing condition is much more subtle and needs further study.

We start by reviewing some known results for manifolds satisfying (2.1) which are proved in [29] and [30]. Then we prove the segment inequalities under the non-collapsing condition (2.2) in §2.4. These estimates will be sufficient to give a direct generalization of the regularity theory of Cheeger–Colding [8]–[10] and Cheeger–Colding–Tian [11]; cf. [30]. Then we derive some analytical results including the short-time heat kernel estimate and apply these to derive the Hessian estimate to the parabolic approximations of distance functions as in [17]. This makes it possible to give a generalization of Colding–Naber's work on the Hölder continuity of tangent cones [17] on the limit spaces of manifolds satisfying (2.1) and (2.2).

For simplicity we will denote by $C(a_1, a_2, \dots)$ a positive constant which depends on the variables $a_1, a_2, \dots$, but may vary in different situations.

2.1. Preliminary results

For any $x \in M$, let $(t, \theta) \in \mathbb{R}^+ \times S_x^{m-1}$ be the polar coordinate at x where S_x^{m-1} is the unit sphere bundle restricted at x. Write the Riemannian volume form in this coordinate as

$$dv = \mathcal{A}(t, \theta)\, dt \wedge d\theta. \tag{2.3}$$

Let $r(y) = d(x, y)$ denote the distance function to x. Then an immediate computation in the polar coordinate shows that

$$\Delta r = \frac{\partial}{\partial r} \log \mathcal{A}(r, \cdot). \tag{2.4}$$

As in [29], we introduce the error function of the Laplacian comparison of distances

$$\psi(r, \theta) = \left(\Delta r(\exp_x(r\theta)) - \frac{m-1}{r} \right)_+, \tag{2.5}$$

where $a_+ = \max\{a, 0\}$. Notice that ψ depends on the base point x. For any subset $\Gamma \subset S_x$ we define

$$B_\Gamma(x, r) = \{\exp_x(t\theta) : 0 \leqslant t < r \text{ and } \theta \in \Gamma\}.$$

The following estimate, which is proved in [29], is fundamental in the theory of integral-bounded Ricci curvature:

$$\int_{B_\Gamma(x,r)} \psi^{2p}\, dv \leqslant C(m, p) \int_{B_\Gamma(x,r)} |\mathrm{Ric}_-|^p\, dv \quad \text{for all } r > 0 \text{ and } p > \frac{m}{2}, \tag{2.6}$$

where

$$C(m, p) = \left(\frac{(m-1)(2p-1)}{2p-m} \right)^p.$$

Based on this integral estimate, Petersen–Wei proved the following relative volume comparison theorem.

Theorem 2.1. ([29]) *For any $p > \frac{1}{2}m$ there exists $C(m, p)$ such that*

$$\frac{d}{dr} \left(\frac{\mathrm{vol}(B_\Gamma(x, r))}{r^m} \right)^{1/2p} \leqslant C(m, p) \left(\frac{1}{r^m} \int_{B_\Gamma(x,r)} |\mathrm{Ric}_-|^p\, dv \right)^{1/2p} \quad \textit{for all } r > 0. \tag{2.7}$$

Integration gives, for any $r_2 > r_1 > 0$,

$$\left(\frac{\mathrm{vol}(B_\Gamma(x,r_2))}{r_2^m}\right)^{1/2p} - \left(\frac{\mathrm{vol}(B_\Gamma(x,r_1))}{r_1^m}\right)^{1/2p} \leqslant C(m,p)\left(r_2^{2p-m}\int_{B_\Gamma(x,r_2)} |\mathrm{Ric}_-|^p\,dv\right)^{1/2p}. \tag{2.8}$$

Remark 2.2. The quantity $r^{2p-m}\int_{B_\Gamma(x,r)}|\mathrm{Ric}_-|^p\,dv$ in the above inequality (2.8) is scaling invariant. Therefore, under the global integral condition of Ricci curvature (2.1), the volume ratio $\mathrm{vol}(B_\Gamma(x,r))/r^m$ will become almost monotone whenever the radius r in consideration is sufficiently small. In particular this implies the metric cone structure of the tangent cone on non-collapsing limit spaces.

Remark 2.3. Under the additional assumption (2.2), the relative volume comparison (2.8) gives rise to a volume doubling property of concentric metric balls of small radii [30].

COROLLARY 2.4. *Under the assumption* (2.1), *the volume has the upper bound*

$$\mathrm{vol}(B_\Gamma(x,r)) \leqslant |\Gamma| r^m + C(m,p)\Lambda r^{2p} \quad \textit{for all } r>0, \tag{2.9}$$

where $|\Gamma|$ *denotes the measure of* Γ *as a subset of the unit sphere.*

The upper bound of the volume of geodesic balls can be refined as follows.

LEMMA 2.5. *Under the assumption* (2.1), *we have*

$$\mathrm{vol}(\partial B(x,r)) \leqslant C(m,p,\Lambda)\cdot r^{m-1}, \quad \textit{when } r\leqslant 1, \tag{2.10}$$

$$\mathrm{vol}(\partial B(x,r)) \leqslant C(m,p,\Lambda) r^{2p-1}, \quad \textit{when } r>1. \tag{2.11}$$

Proof. When $r\leqslant 1$, this is exactly [18, Lemma 3.2]. We next use iteration to prove (2.11). For simplicity, we only consider the case $r=2^k$ for some positive integer k. The estimate for other radii bigger than 1 can be attained by a finite step iteration starting from a unique radius between $\frac{1}{2}$ and 1.

By (2.4) and (2.5),

$$\frac{\partial}{\partial t}\frac{\mathcal{A}(t,\theta)}{t^{m-1}} \leqslant \psi(t,\theta)\frac{\mathcal{A}(t,\theta)}{t^{m-1}}.$$

Integrating over the direction space S_x^{m-1} gives

$$\frac{d}{dt}\frac{\int_{S_x}\mathcal{A}(t,\theta)\,d\theta}{t^{m-1}} \leqslant \frac{\int_{S_x}\psi(t,\theta)\mathcal{A}(t,\theta)\,d\theta}{t^{m-1}}.$$

Integrating over an interval of radius $[r,2r]$ gives

$$\frac{\int_{S_x}\mathcal{A}(2r,\theta)\,d\theta}{(2r)^{m-1}} - \frac{\int_{S_x}\mathcal{A}(r,\theta)\,d\theta}{r^{m-1}} \leqslant \int_r^{2r}\frac{\int_{S_x}\psi(t,\theta)\mathcal{A}(t,\theta)\,d\theta}{t^{m-1}}\,dt \leqslant \frac{1}{r^{m-1}}\int_{B(x,2r)}\psi\,dv.$$

By the integral version of mean curvature comparison (2.6) and volume comparison (2.9),

$$\int_{B(x,2r)} \psi\, dv \leqslant \left(\int_{B(x,2r)} \psi^{2p}\, dv\right)^{1/2p} \operatorname{vol}(B(x,2r))^{(2p-1)/2p} \leqslant C(m,p,\Lambda)(2r)^{2p-1}.$$

Thus,

$$\frac{\int_{S_x} \mathcal{A}(2r,\theta)\, d\theta}{(2r)^{m-1}} \leqslant \frac{\int_{S_x} \mathcal{A}(r,\theta)\, d\theta}{r^{m-1}} + C(m,p,\Lambda)(2r)^{2p-m}.$$

Put $r_k = 2^k$, $k \geqslant 0$. An iteration then gives

$$\int_{S_x} \mathcal{A}(r_k,\theta)\, d\theta \leqslant C(m,p,\Lambda) r_k^{2p-1},$$

as desired. □

Let $\partial B_\Gamma(x,r) = \{y = \exp_x(r\theta) : \theta \in \Gamma \text{ and } d(x,y) = r\}$. By the proof of [18, Lemma 3.2], we also have the following volume estimate of ∂B_Γ in terms of $|\Gamma|$.

Lemma 2.6. *Under the assumption* (2.1), *we have*

$$\operatorname{vol}(\partial B_\Gamma(x,r)) \leqslant C(m,p,\Lambda)(|\Gamma| r^{m-1} + r^{2p-1}), \quad \textit{when } r \leqslant 1. \tag{2.12}$$

Next we recall a nice cut-off function which is constructed by Petersen–Wei following the idea of Cheeger–Colding [8]. In the rest of this subsection we assume (2.1) and (2.2).

Lemma 2.7. ([30]) *There exist* $r_0 = r_0(m,p,\varkappa,\Lambda)$ *and* $C = C(m,p,\varkappa,\Lambda)$ *such that, on any* $B(x,r)$, $r \leqslant r_0$, *there exists a cut-off function* $\phi \in C_0^\infty(B(x,r))$ *which satisfies*

$$\phi \geqslant 0, \quad \phi \equiv 1 \textit{ in } B\left(x, \tfrac{1}{2}r\right), \tag{2.13}$$

and

$$\|\nabla\phi\|_{C^0}^2 + \|\Delta\phi\|_{C^0} \leqslant C r^{-2}. \tag{2.14}$$

As in [17] one can extend the construction to a slightly more general case, by using a covering technique based on the volume doubling property. Let E be a closed subset of M. Denote the r-neighborhood of E by

$$U_r(E) = \{x \in M : d(x,E) < r\},$$

and let $A_{r_1,r_2}(E) = U_{r_2} \setminus \overline{U}_{r_1}$ be the open annulus of radii $0 < r_1 < r_2$.

Corollary 2.8. *For any* $R > 0$ *there exists* $C = C(m,p,\varkappa,\Lambda,R)$ *such that the following holds. Let* E *be any closed subset and* $0 < r_1 < 10 r_2 < R$. *There exists a cut-off function* $\phi \in C^\infty(U_R(E))$ *which satisfies*

$$\phi \geqslant 0, \quad \phi \equiv 1 \textit{ in } A_{3r_1, r_2/3}(E), \quad \phi \equiv 0 \textit{ outside } A_{2r_1, r_2/2}(E) \tag{2.15}$$

and

$$\|\nabla\phi\|_{C^0}^2 + \|\Delta\phi\|_{C^0} \leqslant C r_1^{-2} \quad \textit{in } A_{2r_1, 3r_1}(E), \tag{2.16}$$

$$\|\nabla\phi\|_{C^0}^2 + \|\Delta\phi\|_{C^0} \leqslant C r_2^{-2} \quad \textit{in } A_{r_2/3, r_2/2}(E). \tag{2.17}$$

Finally we recall a bound of Sobolev constants which is essential for Nash–Moser iteration on manifolds with integral-bounded Ricci curvature. When the Riemannian manifold is a spatial slice of a Kähler–Ricci flow on a Fano manifold, the Sobolev constant C_s has a global estimate; see [50] and [49]. In the general setting, we have the following result.

LEMMA 2.9. ([48]) *There exist* $r_0=r_0(m,p,\varkappa,\Lambda)$ *and* $C=C(m,p,\varkappa,\Lambda)$ *such that*

$$C_s(B(x,r_0))\leqslant C \quad \text{for all } x\in M. \tag{2.18}$$

By a covering technique, one can directly prove the following corollary.

COROLLARY 2.10. *For any* $R>0$, *there exists* $C=C(m,p,\varkappa,\Lambda,R)$ *such that*

$$C_s(B(x,R))\leqslant C \quad \text{for all } x\in M. \tag{2.19}$$

Here, the local Sobolev constant $C_s(B(x,R))$ is defined to be the minimum value of all constants C_s satisfying

$$\left(\int f^{2m/(m-2)}\,dv\right)^{(m-2)/m}\leqslant C_s\int(|\nabla f|^2+f^2)\,dv \quad \text{for all } f\in C_0^\infty(B(x,R)). \tag{2.20}$$

2.2. A heat kernel estimate

The aim of this subsection is to prove a heat kernel estimate as well as some geometric inequalities for heat equations on manifolds with integral-bounded Ricci curvature.

Let M be a Riemannian manifold satisfying (2.1) and (2.2) for some constants $p>\frac{1}{2}m$, $\Lambda>1$ and $\varkappa>0$. We start with the mean value inequality and gradient estimate for heat equations.

Denote by

$$⨏_A u\,dv=\frac{1}{\operatorname{vol}(A)}\int_A u\,dv$$

the average of u over the set A.

LEMMA 2.11. *There exists* $C=C(m,p,\varkappa,\Lambda)$ *such that the following holds. For any* $0<t_0\leqslant 1$ *and any function* $u=u(x,t)$ *in* $B(x,\sqrt{t_0})\times[0,t_0]$ *satisfying*

$$\frac{\partial}{\partial t}u=\Delta u, \tag{2.21}$$

we have

$$u_+(x,t_0)\leqslant Ct_0^{-1}\int_{t_0/2}^{t_0}⨏_{B(x,\sqrt{t_0})}u_+\,dv\,dt, \tag{2.22}$$

$$|\nabla u|^2(x,t_0)\leqslant Ct_0^{-2}\int_{t_0/2}^{t_0}⨏_{B(x,\sqrt{t_0})}u^2\,dv\,dt. \tag{2.23}$$

Proof. The estimates follow from the iteration argument of Nash–Moser; see [16, pp. 306–316] for details. The proof of the mean value inequality (2.22) is standard. We give a proof of (2.23).

First of all, applying the iteration to the evolution of $|\nabla u|^2$,

$$\frac{\partial}{\partial t}|\nabla u|^2=\Delta|\nabla u|^2-2|\nabla\nabla u|^2-2\,\mathrm{Ric}(\nabla u,\nabla u)\leqslant\Delta|\nabla u|^2+|\mathrm{Ric}_-|\,|\nabla u|^2,$$

where $|\mathrm{Ric}_-|$ is L^p integrable, gives

$$|\nabla u|^2(x,t_0)\leqslant C(m,p,\varkappa,\Lambda)t_0^{-1}\int_{t_0/2}^{t_0}\fint_{B(x,\sqrt{t_0}/2)}|\nabla u|^2\,dv\,dt.$$

Then we estimate $\int_{t_0/2}^{t_0}\fint_{B(x,\sqrt{t_0}/2)}|\nabla u|^2\,dv\,dt$ in terms of the L^2-norm of u to complete the proof. Consider the evolution equation

$$\frac{\partial}{\partial t}u^2=\Delta u^2-2|\nabla u|^2.$$

Let $\phi\in C_0^\infty(B(x,r))$, $r=t_0^2$, be a non-negative cut-off function such that $\phi\equiv1$ on $B\big(x,\tfrac12 r\big)$ and, for some $C=C(m,p,\varkappa,\Lambda)\geqslant2$,

$$|\nabla\phi|^2+|\Delta\phi|\leqslant Cr^{-2}.$$

See Lemma 2.7. Multiplying by the cut-off function and integrating over space-time, we get

$$\begin{aligned}2\int_{t_0/2}^{t_0}\int_M\phi^2|\nabla u|^2\,dv\,dt&=\int_{t_0/2}^{t_0}\int_M\phi^2\Delta u^2\,dv\,dt-\int_M\phi^2u^2(t_0)\,dv+\int_M\phi^2u^2\left(\frac{t_0}{2}\right)dv\\&\leqslant Cr^{-2}\int_{t_0/2}^{t_0}\int_{B(x,r)}u^2\,dv\,dt+\int_M\phi^2u^2\left(\frac{t_0}{2}\right)dv.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}\int_M\phi^2u^2\,dv=2\int_M\phi^2u\Delta u\,dv&=-2\int_M\phi^2|\nabla u|^2\,dv-4\int_M\phi u\nabla\phi\nabla u\,dv\\&\geqslant-2\int_M|\nabla\phi|^2u^2\,dv\geqslant-Cr^{-2}\int_{B(x,r)}u^2\,dv.\end{aligned}$$

We claim that

$$\int_M\phi^2u^2\left(\frac{t_0}{2}\right)dv\leqslant3Cr^{-2}\int_{t_0/2}^{t_0}\int_{B(x,r)}u^2\,dv\,dt,$$

which is sufficient to complete the proof. Actually, if it fails, then for any $t\in\big[\tfrac12t_0,t_0\big]$,

$$\int_M\phi^2u^2(t)\,dv\geqslant2Cr^{-2}\int_{t_0/2}^{t_0}\int_{B(x,r)}u^2\,dv\,dt,$$

consequently

$$\int_{t_0/2}^{t_0}\int_{B(x,r)}\phi^2u^2\,dv\,dt \geqslant \frac{t_0}{2}2Cr^{-2}\int_{t_0/2}^{t_0}\int_{B(x,r)}u^2\,dv\,dt = C\int_{t_0/2}^{t_0}\int_{B(x,r)}u^2\,dv\,dt,$$

which gives a contradiction, since $C\geqslant 2$. □

Corollary 2.12. *With the same assumptions as in the above lemma, if u is harmonic in $B(x,r)$, $r\leqslant 1$, then*

$$|\nabla u|^2(x) \leqslant Cr^{-2} \fint_{B(x,r)} u^2\,dv. \tag{2.24}$$

Theorem 2.13. (Heat kernel upper bound) *Let M be a complete Riemannian manifold of dimension m which satisfies (2.1) and (2.2). Let $H(x,y,t)$ be its heat kernel. There exists a positive constant $C=C(m,p,\varkappa,\Lambda)$ such that*

$$H(x,y,t) \leqslant Ct^{-m/2}e^{-d^2(x,y)/5t} \quad \textit{for all } x,y\in M \textit{ and } 0<t\leqslant 1. \tag{2.25}$$

Proof. There exists a unique heat kernel on a manifold satisfying (2.1) due to a criterion of Grogor′yan [22]. The mean value inequality (2.22) gives the following upper bound of H:

$$H(x,y,t) \leqslant C(m,p,\varkappa,\Lambda)t^{-m/2} \quad \text{for all } x\in M \text{ and } t\leqslant 1.$$

The Gaussian upper bound of $H(x,y,t)$ is concluded from [21]. □

Theorem 2.14. (Heat kernel lower bound) *With the same assumptions as in the above theorem, there exist constants $\tau=\tau(m,p,\varkappa,\Lambda)$ and $C=C(m,p,\varkappa,\Lambda)$ such that*

$$H(x,y,t) \geqslant C^{-1}t^{-m/2}, \tag{2.26}$$

whenever

$$0<t\leqslant\tau \quad \textit{and} \quad d(x,y)\leqslant 10\sqrt{t}. \tag{2.27}$$

Proof. We follow the argument of Cheeger–Yau [12]; see [18] for a closer situation. Let $\overline{H}(x,y,t)=(4\pi t)^{-m/2}e^{-d^2(x,y)/4t}$. By DuHamel's principle for the heat equation,

$$H(x,y,t)-\overline{H}(x,y,t) = \int_0^t\int_M \frac{\partial}{\partial s}\overline{H}(x,z,t-s)H(z,y,s)\,dv(z)\,ds \tag{2.28}$$

$$+\int_0^t\int_M \overline{H}(x,z,t-s)\frac{\partial}{\partial s}H(z,y,s)\,dv(z)\,ds. \tag{2.29}$$

Fix $x\in M$ and let $r(z)=d(x,z)$. An easy calculation shows that

$$\frac{\partial}{\partial s}\overline{H}(x,z,t-s)=-\Delta\overline{H}(x,z,t-s)+\frac{r(z)}{2(t-s)}\left(\frac{n-1}{r(z)}-\Delta r(z)\right)\overline{H}(x,z,t-s).$$

Let

$$\psi(z)=\left(\Delta r(z)-\frac{n-1}{r(z)}\right)_+.$$

As shown in [18], this implies that

$$\begin{aligned}H(x,y,t)&-\overline{H}(x,y,t)\\&\geqslant-\int_0^t\int_M\frac{r(z)}{2(t-s)}\psi(z)\overline{H}(x,z,t-s)H(z,y,s)\,dv(z)\,ds\\&\geqslant-C(m,p,\varkappa,\Lambda)\int_0^t\int_M\psi(z)(t-s)^{-(m+1)/2}s^{-m/2}e^{-r^2(z)/6(t-s)-d^2(y,z)/5s}\,dv(z)\,ds,\end{aligned}$$

where we in the last inequality use that $re^{-r^2/5(t-s)+r^2/6(t-s)}$ has a universal upper bound when $t\leqslant 1$. By (2.6),

$$\int_M\psi^{2p}\,dv\leqslant C(m,p)\int_M|\mathrm{Ric}_-|^p\leqslant C(m,p,\Lambda)\,dv.$$

By the Hölder inequality,

$$\begin{aligned}&\int_M\psi(z)e^{-r^2(z)/6(t-s)-d^2(y,z)/5s}\,dv(z)\\&\qquad\leqslant C(m,p,\Lambda)\left(\int_M e^{-(2p/(2p-1))(r^2(z)/6(t-s)+d^2(y,z)/5s)}\,dv(z)\right)^{(2p-1)/2p}.\end{aligned}$$

For $0<s\leqslant\frac{1}{2}t$,

$$\begin{aligned}&\int_M e^{-(2p/(2p-1))(r^2(z)/6(t-s)+d^2(y,z)/5s)}\,dv(z)\\&\qquad\leqslant\int_M e^{-(2p/(2p-1))(d^2(y,z)/5s)}\,dv(z)\leqslant C(m,p,\Lambda)s^{m/2},\end{aligned}$$

and for $\frac{1}{2}t\leqslant s\leqslant t$, similarly,

$$\int_M e^{-(p/(p-1))(r^2(z)/6(t-s)+d^2(y,z)/5s)}\,dv(z)\leqslant C(m,p,\Lambda)(t-s)^{m/2}.$$

Here, in order to derive the explicit upper bound of the integral, we use the upper bound of volume growth of the geodesic spheres centered at x and y, namely the estimates (2.10) and (2.11) in Corollary 2.5.

Summing up the estimates we obtain

$$\begin{aligned}H(x,y,t)-\overline{H}(x,y,t) \geqslant -C(m,p,\varkappa,\Lambda)\bigg(\int_0^{t/2}(t-s)^{-(m+1)/2}s^{-(m/2)(1/2p)}\,ds\\ +\int_{t/2}^{t}(t-s)^{-(m+1)/2+(m/2)((2p-1)/2p)}s^{-m/2}\,ds\bigg)\\ =-C(m,p,\varkappa,\Lambda)t^{-m/2+(2p-m)/4p}.\end{aligned}$$

This is sufficient to get the required lower bound of $H(x,y,t)$ when t is small and

$$d(x,y)\leqslant 10\sqrt{t}. \qquad \square$$

COROLLARY 2.15. *With the same assumptions as in the above theorem, there exist constants* $\tau=\tau(m,p,\varkappa,\Lambda)$ *and* $C=C(m,p,q,\varkappa,\Lambda)$ *such that the following holds. Let* f *be a non-negative function satisfying*

$$\frac{\partial}{\partial t}f\geqslant \Delta f-\xi, \tag{2.30}$$

where ξ *is a space-time function such that* ξ_+ *is* L^q*-integrable for some* $q>\frac{1}{2}m$ *at any time slice* t. *Then*

$$⨏_{B(x,r)}f(\,\cdot\,,0)\,dv\leqslant C\Big(f(x,r^2)+r^{2-m/q}\sup_{t\in[0,r^2]}\|\xi_+(t)\|_q\Big)\quad \textit{for all } x\in M \textit{ and } r\leqslant\sqrt{\tau}. \tag{2.31}$$

Proof. The idea of the proof follows [17]. A direct calculation shows that

$$\begin{aligned}\frac{d}{dt}\int_M f(y,t)H(x,y,r^2-t)\,dv(y)&=\int_M H(x,y,r^2-t)\bigg(\frac{\partial}{\partial t}-\Delta\bigg)f(y,t)\,dv(y)\\ &\geqslant -\int_M H(x,y,r^2-t)\xi_+(y,t)\,dv(y).\end{aligned}$$

Then, using the upper bound of H,

$$\begin{aligned}&\int_M H(x,y,r^2-t)\xi_+(y,t)\,dv(y)\\ &\quad\leqslant C(r^2-t)^{-m/2}\int_M \xi_+(y,t)e^{-d^2(x,y)/5(r^2-t)}\,dv(y)\\ &\quad\leqslant C(r^2-t)^{-m/2}\|\xi_+(t)\|_q\bigg(\int_M e^{-(q/(q-1))(d^2(x,y)/5(r^2-t))}\,dv(y)\bigg)^{1-1/q}\\ &\quad\leqslant C(r^2-t)^{-m/2q}\|\xi_+(t)\|_q.\end{aligned}$$

Integrating from 0 to r^2 and applying the lower bound of H and upper bound of $\operatorname{vol}(B(x,r))$, we have

$$f(x,r^2) \geqslant \int_M f(y,0)H(x,y,r^2)\,dv(y) - \int_0^{r^2} C(r^2-t)^{-m/2q}\|\xi_+(t)\|_q\,dt$$
$$\geqslant C^{-1} \fint_{B(x,r)} f(y,0)\,dv(y) - Cr^{2(1-m/2q)} \sup_{t\in[0,r^2]} \|\xi_+(t)\|_q.$$

The required estimate now follows directly. □

Corollary 2.16. *With the same assumptions as above, there exist constants* $\tau = \tau(m,p,\varkappa,\Lambda)$ *and* $C = C(m,p,q,\varkappa,\Lambda)$ *such that the following holds. Let* f *be a non-negative function on* M *satisfying*

$$\Delta f \leqslant \xi, \tag{2.32}$$

where $\xi \in L^q$ *for some* $q > \frac{1}{2}m$. *Then*

$$\fint_{B(x,r)} f\,dv \leqslant C(f(x) + r^{2-m/q}\|\xi\|_q) \quad \text{for all } x\in M \text{ and } r \leqslant \sqrt{\tau}. \tag{2.33}$$

The crucial application is when f is the distance function d, in which case we have

$$\Delta d \leqslant \frac{n-1}{d} + \psi,$$

where ψ has a uniform L^{2p}-bound in terms of $\int |\mathrm{Ric}_-|^p\,dv$ by (2.6).

Remark 2.17. There exists an estimate of the same type as in Corollary 2.15 even if $\|\xi(t)\|_q$ is not bounded but satisfies certain growth conditions as t approaches 0, for example $\|\xi(t)\|_q \leqslant Ct^{-1+\varepsilon}$ for some $\varepsilon > 0$. See Lemma 2.23 for an application.

Remark 2.18. Trivial examples show that the order of r, namely $2-m/q$, in the estimates (2.31) and (2.33) is sharp. It follows that the estimates for the parabolic approximations in the next subsection are the best possible.

2.3. Parabolic approximations

Let M be a complete Riemannian manifold of dimension m which satisfies (2.1) and (2.2) for some $\varkappa > 0$, $p > \frac{1}{2}m$ and $\Lambda \geqslant 1$.

We first present some notation we shall use in this subsection. Let $\tau = \tau(m,p,\varkappa,\Lambda)$ denote the constant in Corollary 2.16 and $\delta < \tau$ be a fixed small positive constant. Furthermore, $C = C(m,p,\varkappa,\Lambda,\delta)$ will always be a positive constant depending on the parameters m, p, $\varkappa$, Λ and δ.

Pick two base points $p^{\pm}\in M$ with $d=d(p^+,p^-)\leqslant\frac{1}{20}\sqrt{\tau}$. Define the annulus

$$A_{r,s}=A_{rd,sd}(\{p,q\}),\quad 0<r<s<20.$$

Define functions

$$b^+(x)=d(p^+,x)-d(p^+,p^-),\quad b^-(x)=d(p^-,x) \tag{2.34}$$

and

$$e(x)=d(p^+,x)+d(p^-,x)-d(p^+,p^-) \tag{2.35}$$

on M. The function e is known as the *excess function*. Let ϕ be a non-negative cut-off function as in Corollary 2.8 with $E=\{p^{\pm}\}$ such that

$$\phi\equiv 1 \text{ in } A_{\delta/4,8},\quad \phi\equiv 0 \text{ outside } A_{\delta/16,16}$$

and

$$|\nabla\phi|^2+|\Delta\phi|\leqslant C.$$

Define space-time functions $\mathbf{b}_t^{\pm}$ and $\mathbf{e}_t$ by

$$\mathbf{b}_t^{\pm}(x)=\int_M H(x,y,t)\phi(y)b^{\pm}(y)\,dv(y)$$

and

$$\mathbf{e}_t(x)=\int_M H(x,y,t)\phi(y)e(y)\,dv(y).$$

They are heat solutions with initial data $\phi b^{\pm}$ and ϕe respectively. It is obvious that

$$\mathbf{e}_t\equiv\mathbf{b}_t^{+}+\mathbf{b}_t^{-}.$$

The aim is to investigate the approximating properties of $\mathbf{b}_t^{\pm}$ to the distance functions $b^{\pm}$ on the annulus domain $A_{\delta/4,8}$. The argument goes through as in [17] without essential difficulties.

We start by noticing that

$$\Delta d(p^{\pm},x)\leqslant\frac{n-1}{d(p^{\pm},x)}+\psi^{\pm},$$

where

$$\psi^{\pm}=\left(\Delta d(p^{\pm},x)-\frac{n-1}{d(p^{\pm},x)}\right)_+$$

is the error term of the Laplacian comparison of distance functions. Then Corollary 2.16 immediately implies the following result.

Corollary 2.19. *For any* $0<\varepsilon<\frac{1}{100}\delta$, *we have*

$$\fint_{B(x,\varepsilon d)} e\,dv \leqslant C(e(x)+\varepsilon^{2-m/2p}d) \quad \text{for all } x\in A_{\delta/4,16}. \tag{2.36}$$

In particular, if $e(x)\leqslant\varepsilon^{2-m/2p}d$, *then*

$$e(y)\leqslant C\varepsilon^{1+\alpha}d \quad \text{for all } y\in B\big(x,\tfrac{1}{2}\varepsilon d\big), \tag{2.37}$$

where

$$\alpha=\frac{1}{m+1}\left(1-\frac{m}{2p}\right)>0.$$

A similar argument as in the proof of Corollary 2.15 also gives the following.

Lemma 2.20. *We have*

$$\Delta\mathbf{b}_t^+, \Delta\mathbf{b}_t^-, \Delta\mathbf{e}_t \leqslant C(d^{-1}+t^{-m/4p}) \quad \text{for all } 0<t\leqslant\tau. \tag{2.38}$$

Proof. By direct computation,

$$\Delta(\phi b^+)=\Delta\phi b^+ +2\langle\nabla\phi,\nabla b^+\rangle+\phi\Delta b^+\leqslant C(m,p,\varkappa,\Lambda,\delta)d^{-1}+\psi^+.$$

Thus,

$$\begin{aligned}\Delta\mathbf{b}_t^+(x)&=\int_M \Delta_x H(x,y,t)\phi(y)b^+(y)\,dv(y)=\int_M \Delta_y H(x,y,t)\phi(y)b^+(y)\,dv(y)\\&=\int_M H(x,y,t)\Delta_y(\phi(y)b^+(y))\,dv(y)\leqslant Cd^{-1}+\int_M H(x,y,t)\psi^+(y)\,dv(y).\end{aligned}$$

The last term can be estimated using the upper bound of H when $t\leqslant 1$,

$$\int_M H(x,y,t)\psi^+(y)\,dv(y)\leqslant Ct^{-m/2}\int_M e^{-d^2(x,y)/5t}\psi^+(y)\,dv(y)\leqslant Ct^{-m/4p}\|\psi^+\|_{2p}.$$

The desired upper bound of $\Delta\mathbf{b}_t^+$ then follows from (2.6). The proofs of the other two estimates are similar. □

Lemma 2.21. *For* $t\leqslant\tau$ *we have*

$$\mathbf{e}_t(y)\leqslant C(e(x)+d^{-1}t+t^{1-m/4p}) \qquad \text{for all } y\in B(x,\sqrt{t}), \tag{2.39}$$

$$|\nabla\mathbf{e}_t|(x)\leqslant Ct^{-1/2}(e(x)+d^{-1}t+t^{1-m/4p}). \tag{2.40}$$

Proof. First of all, when t is small,

$$\mathbf{e}_t(x)=e(x)+\int_0^t \Delta\mathbf{e}_s(x)\,ds \leqslant e(x)+C(d^{-1}t+t^{1-m/4p}).$$

Then, by Lemma 2.15,

$$\fint_{B(x,3\sqrt{t})} \mathbf{e}_t\,dv \leqslant C\mathbf{e}_{7t}(x) \leqslant C(e(x)+d^{-1}t+t^{1-m/4p}).$$

The mean value inequality shows that, for all $y\in B(x,\sqrt{t})$,

$$\begin{aligned}\mathbf{e}_t(y) &\leqslant Ct^{-1}\int_{t/2}^t \fint_{B(y,\sqrt{t})} \mathbf{e}_s\,dv\,ds \leqslant Ct^{-1}\int_{t/2}^t \fint_{B(y,2\sqrt{s})} \mathbf{e}_s\,dv\,ds\\ &\leqslant C(e(x)+d^{-1}t+t^{1-m/4p}),\end{aligned}$$

where we also used the volume doubling property. The second estimate is a consequence of the mean value inequality. □

We also have the following lemma, as in [17].

LEMMA 2.22. *For $x\in A_{\delta/2,4}$ and $t\leqslant \frac{1}{100}\delta^2$ the following estimates hold*:

$$|\mathbf{b}_t^{\pm}(x)-b^{\pm}(x)| \leqslant C(e(x)+d^{-1}t+t^{1-m/4p}); \tag{2.41}$$

$$|\nabla\mathbf{b}_t^{\pm}(x)|^2 \leqslant 1+Ct^{1-m/2p}; \tag{2.42}$$

$$\fint_{B(x,\sqrt{t})} \big||\nabla\mathbf{b}_t^{\pm}|^2-1\big|\,dv \leqslant C(e(x)t^{-1/2}+d^{-1}t^{1/2}+t^{1/2-m/4p}); \tag{2.43}$$

$$\int_{t/2}^t \fint_{B(x,\sqrt{t})} |\mathrm{Hess}\,\mathbf{b}_t^{\pm}|^2\,dv\,ds \leqslant C(e(x)t^{-1/2}+d^{-1}t^{1/2}+t^{1/2-m/4p}). \tag{2.44}$$

Proof. We prove the first two estimates; the last two integral estimates can be proved exactly as in [17]. Estimate (2.41) can be derived from the upper bound of $\mathbf{e}_t=\mathbf{b}_t^++\mathbf{b}_t^-$ in (2.39) and the estimate

$$\mathbf{b}_t^{\pm}(x)-b^{\pm}(x) \leqslant C(d^{-1}t+t^{1-m/4p}),$$

which can be proved as in the above lemma. To show the upper bound of $|\nabla\mathbf{b}_t^{\pm}|$, we first apply the gradient estimate (2.23) together with the C^0-bound of $\mathbf{b}_t^{\pm}$ to get

$$|\nabla\mathbf{b}_t^{\pm}(x)| \leqslant Ct^{-1/2}.$$

Then we apply the same trick as in the proof of Corollary 2.15 to the formula

$$\frac{\partial}{\partial t}|\nabla\mathbf{b}_t^{\pm}| \leqslant \Delta|\nabla\mathbf{b}_t^{\pm}|+|\mathrm{Ric}_-|\,|\nabla\mathbf{b}_t^{\pm}|$$

to get

$$|\nabla\mathbf{b}_t^{\pm}(x)| \leqslant 1+C\int_0^t\int_M H(x,y,t-s)s^{-1/2}|\mathrm{Ric}_-|\,dv\,ds \leqslant 1+Ct^{1/2-m/2p}.$$

Finally, repeating this argument we get the desired estimate (2.42). □

Remark 2.23. Notice that the order $\frac{1}{2}-m/4p$ of t in (2.43) and (2.44) is sharp. This will play an important role in the proof of Hölder continuity of tangent cones of non-collapsing limit spaces of manifolds with integral-bounded Ricci curvature.

Remark 2.24. The estimates up to now combined with the segment inequality in the next subsection are sufficient to generalize the regularization theory of Cheeger–Colding [8]–[10], Cheeger–Colding–Tian [11] and Cheeger [7] to manifolds satisfying (2.1) and (2.2). The point is to get appropriate functions $\mathbf{b}_t^{\pm}$ of distance functions $b^{\pm}$ with small L^2-Hessian. We refer to [30] for related discussions and results.

Let σ be any ε-geodesic connecting p^+ and p^- whose length is less than $(1+\varepsilon^2)d$. Obviously $e(x)\leqslant\varepsilon^2 d$ for all $x\in\sigma$. As in [17], we have a better L^2-estimate for $\operatorname{Hess}\mathbf{b}_t^{\pm}$ along ε-geodesics. Since the proof is the same it will be omitted.

THEOREM 2.25. *The following estimates hold for any* $\delta d\leqslant t_0<t_0+\sqrt{t}\leqslant(1-\delta)d$:

$$\int_{t_0}^{t_0+\sqrt{t}}\fint_{B(\sigma(s),\sqrt{t})}\big||\nabla\mathbf{b}_t^{\pm}|^2-1\big|\,dv\,ds\leqslant C(\varepsilon^2 d+d^{-1}t+t^{1-m/4p});\tag{2.45}$$

$$\int_{t/2}^{t}\int_{t_0}^{t_0+\sqrt{t}}\fint_{B(\sigma(s),\sqrt{t})}|\operatorname{Hess}\mathbf{b}_\tau^{\pm}|^2\,dv\,ds\,d\tau\leqslant C(\varepsilon^2 d+d^{-1}t+t^{1-m/4p}).\tag{2.46}$$

Taking $t=d_\varepsilon^2=(\varepsilon d)^2$, we obtain the following result.

COROLLARY 2.26. *There is* $r\in\left[\frac{1}{2},1\right]$ *such that the following estimates hold for any* $\delta d\leqslant t_0<t_0+d_\varepsilon\leqslant(1-\delta)d$:

$$\int_{t_0}^{t_0+rd_\varepsilon}\fint_{B(\sigma(s),rd_\varepsilon)}\big||\nabla\mathbf{b}_{r^2d_\varepsilon^2}^{\pm}|^2-1\big|\,dv\,ds\leqslant C(\varepsilon^2 d+d_\varepsilon^{2-m/2p});\tag{2.47}$$

$$\int_{t_0}^{t_0+rd_\varepsilon}\fint_{B(\sigma(s),rd_\varepsilon)}|\operatorname{Hess}\mathbf{b}_{r^2d_\varepsilon^2}^{\pm}|^2\,dv\,ds\leqslant C(d^{-1}+d_\varepsilon^{-m/2p}).\tag{2.48}$$

The following lemma is also needed.

LEMMA 2.27. *The following holds for any* $\delta d\leqslant t'<t\leqslant(1-\delta)d$ *with* $t-t'\leqslant d_\varepsilon$:

$$\int_{t'}^{t}|\nabla\mathbf{b}_{r^2d_\varepsilon^2}^{\pm}-\nabla b^{\pm}|(\sigma(s))\,ds\leqslant C(\varepsilon d^{1/2}+d_\varepsilon^{1-m/4p})\sqrt{t-t'}.\tag{2.49}$$

Proof. Notice that

$$\int_{t'}^{t}|\nabla\mathbf{b}_{r^2d_\varepsilon^2}^{\pm}-\nabla b^{\pm}|^2(\sigma(s))\,ds=\int_{t'}^{t}(|\nabla\mathbf{b}_{r^2d_\varepsilon^2}^{\pm}|^2-1+2(1-\langle\nabla\mathbf{b}_{r^2d_\varepsilon^2}^{\pm},\nabla b^{\pm}\rangle))\,ds.$$

A direct calculation, as in [17], gives

$$\int_{t'}^{t'+d_\varepsilon}|\nabla\mathbf{b}_{r^2d_\varepsilon^2}^{\pm}-\nabla b^{\pm}|^2(\sigma(s))\,ds\leqslant C(\varepsilon^2 d+d_\varepsilon^{2-m/2p}).$$

Then we use the Cauchy–Schwartz inequality to get the required estimate. □

2.4. Segment inequalities

The segment inequality of Cheeger–Colding [8] plays an important role in their proof of the local almost-rigidity structure [8], [9] of manifolds with Ricci curvature bounded from below. However the proof of the segment inequality depends highly on the pointwise comparison of the mean curvature along a radial geodesic [8] which does not remain valid on manifolds with integral-bounded Ricci curvature. We will prove two modified versions of the segment inequality which are sufficient for our applications. The first one applies to scalar functions; the second one is a substitution for the segment inequality for the Hessian estimate.

For any fixed $x\in M$ let $\mathcal{A}$ and ψ be the volume element and error function in polar coordinates defined by (2.3) and (2.5), respectively.

LEMMA 2.28. *For any* $0<\frac{1}{2}r\leqslant t\leqslant r$ *we have*

$$\mathcal{A}(r,\theta)\leqslant 2^{m-1}\mathcal{A}(t,\theta)+2^{m-1}\int_t^r \psi(\tau,\theta)\mathcal{A}(\tau,\theta)\,d\tau. \tag{2.50}$$

Proof. By (2.6),

$$\frac{\partial}{\partial t}\frac{\mathcal{A}(t,\theta)}{t^{m-1}}\leqslant \psi(t,\theta)\frac{\mathcal{A}(t,\theta)}{t^{m-1}}. \tag{2.51}$$

Integrating from t to r gives

$$\frac{\mathcal{A}(r,\theta)}{r^{n-1}}\leqslant \frac{\mathcal{A}(t,\theta)}{t^{n-1}}+\int_t^r \psi(\tau,\theta)\frac{\mathcal{A}(\tau,\theta)}{\tau^{n-1}}\,d\tau\leqslant \frac{\mathcal{A}(t,\theta)}{t^{m-1}}+\frac{1}{t^{m-1}}\int_t^r \psi(\tau,\theta)\mathcal{A}(\tau,\theta)\,d\tau.$$

The desired estimate now follows immediately. □

For non-negative functions f we define $\mathcal{F}_f\colon M\times M\to\mathbb{R}^+$ by

$$\mathcal{F}_f(x,y)=\inf\Big\{\textstyle\int_\gamma f(\gamma(t))\,dt:\gamma \text{ is a minimal normal geodesic from } x \text{ to } y\Big\}.$$

PROPOSITION 2.29. *Let* M *be a complete Riemannian manifold of dimension* m *which satisfies* (2.1) *for some* $p>\frac{1}{2}m$ *and* $\Lambda\geqslant 1$. *Then, for any* $B=B(z,R)$, $R\leqslant 1$, *the following holds*:

$$\begin{aligned}\int_{B\times B}\mathcal{F}_f(x,y)\,dv(x)\,dv(y)\leqslant{}& 2^{m+1}R\,\mathrm{vol}(B)\int_{B(z,2R)} f\,dv\\ &+C(m,p,\Lambda)R^{m+2-m/2p}\,\mathrm{vol}(B)\|f\|_{C^0(B(z,2R))}.\end{aligned} \tag{2.52}$$

Proof. Denote by $\gamma=\gamma_{x,y}$ a minimal geodesic from x to y. Put

$$\mathcal{F}_1(x,y)=\int_0^{d(x,y)/2} f(\gamma(t))\,dt \quad\text{and}\quad \mathcal{F}_2(x,y)=\int_{d(x,y)/2}^{d(x,y)} f(\gamma(t))\,dt.$$

By symmetry, as in [8], it is sufficient to establish a bound for

$$\int_{B\times B} \mathcal{F}_2(x,y)\,dv(x)\,dv(y).$$

Fix $x\in B$. For $y=\exp_x(r\theta)\in B$, $r=d(x,y)$,

$$\begin{aligned}\mathcal{A}(r,\theta)\int_{r/2}^{r} f(\gamma(t))\,dt &\leqslant 2^{m-1}\int_{r/2}^{r} f(\gamma(t))\mathcal{A}(t,\theta)\,dt\\ &\quad +2^{m-1}\int_{r/2}^{r}\int_{t}^{r} f(\gamma(t))\psi(\tau,\theta)\mathcal{A}(\tau,\theta)\,d\tau\,dt\\ &\leqslant 2^{m-1}\int_0^r f(\gamma(t))\mathcal{A}(t,\theta)\,dt+2^m R\|f\|_{C^0}\int_0^r \psi(\tau,\theta)\mathcal{A}(\tau,\theta)\,d\tau.\end{aligned}$$

Integrating over B gives

$$\int_B \mathcal{F}_2(x,y)\,dv(y)\leqslant 2^m R\int_{B(z,2R)} f\,dv+2^{m+1}R^2\|f\|_{C^0}\int_{B(x,2R)}\psi.$$

By (2.6) and the volume growth estimate (2.9), we get

$$\int_{B(x,2R)}\psi\leqslant C(m,p,\Lambda)R^{m(1-1/2p)}.$$

This is sufficient to complete the proof. □

For any $x,y\in M$ let $\gamma_{x,y}$ be a minimizing normal geodesic connecting x and y.

PROPOSITION 2.30. *Let $f\in C^\infty(B(z,3R))$, $R\leqslant 1$, satisfy $|\nabla f|\leqslant\Lambda'$. For any $\eta>0$ the following estimate holds*:

$$\begin{aligned}&\int_{B(z,R)\times B(z,R)} |\langle\nabla f,\dot\gamma_{x,y}\rangle(x)-\langle\nabla f,\dot\gamma_{x,y}\rangle(y)|\,dv(x)\,dv(y)\\ &\quad\leqslant C(m)\eta^{-1}R^{m+1}\int_{B(z,3R)}|\mathrm{Hess}\,f|\,dv+C(m,p,\Lambda)\Lambda'(R^{2p}+\eta R^m)\,\mathrm{vol}(B).\end{aligned}\tag{2.53}$$

Proof. Let $B=B(z,R)$. Fix $x\in B$ and view points of B in polar coordinates at x. For any $\theta\in S_x$ let $r(\theta)$ be the maximum radius such that $\exp_x(r\theta)\in B$ and $d(x,\exp_x(r\theta))=r$. Obviously $r(\theta)\leqslant 2R$. Let $\gamma_\theta(t)=\exp_x(t\theta)$, $t\leqslant r(\theta)$, be a radial geodesic for $\theta\in S_x$. Then

$$\begin{aligned}&\int_{\{x\}\times B}|\langle\nabla f,\dot\gamma_{x,y}\rangle(x)-\langle\nabla f,\dot\gamma_{x,y}\rangle(y)|\,dv(y)\\ &\quad\leqslant\int_{S_x}\int_0^{r(\theta)}|\langle\nabla f,\dot\gamma_\theta\rangle(0)-\langle\nabla f,\dot\gamma_\theta\rangle(t)|\mathcal{A}(t,\theta)\,dt\,d\theta\\ &\quad\leqslant\int_0^{2R}\int_{S_x}|\langle\nabla f,\dot\gamma_\theta\rangle(0)-\langle\nabla f,\dot\gamma_\theta\rangle(t)|\mathcal{A}(t,\theta)\,d\theta\,dt.\end{aligned}$$

Then we divide the integration into two parts, for each $t\in[0,2r]$:

$$\int_{\{\theta:\mathcal{A}(t,\theta)\leqslant\eta^{-1}t^{m-1}\}}|\langle\nabla f,\dot\gamma_\theta\rangle(0)-\langle\nabla f,\dot\gamma_\theta\rangle(t)|\mathcal{A}(t,\theta)\,d\theta \leqslant \eta^{-1}t^{m-1}\int_{S_x}|\langle\nabla f,\dot\gamma_\theta\rangle(0)-\langle\nabla f,\dot\gamma_\theta\rangle(t)|\,d\theta$$

and

$$\int_{\{\theta:\mathcal{A}(t,\theta)>\eta^{-1}t^{m-1}\}}|\langle\nabla f,\dot\gamma_\theta\rangle(0)-\langle\nabla f,\dot\gamma_\theta\rangle(t)|\mathcal{A}(t,\theta)\,d\theta\leqslant 2\Lambda'\int_{\{\theta:\mathcal{A}(t,\theta)>\eta^{-1}t^{m-1}\}}\mathcal{A}(t,\theta)\,d\theta.$$

By (2.10),

$$|\{\theta:\mathcal{A}(t,\theta)>\eta^{-1}t^{m-1}\}|\leqslant C(m,p,\Lambda)\eta.$$

Then (2.12) gives

$$\int_{\{\theta:\mathcal{A}(t,\theta)>\eta^{-1}t^{m-1}\}}\mathcal{A}(t,\theta)\,d\theta\leqslant C(m,p,\Lambda)(\eta t^{m-1}+t^{2p-1}).$$

Therefore,

$$\begin{aligned}
&\int_{\{x\}\times B}|\langle\nabla f,\dot\gamma_{x,y}\rangle(x)-\langle\nabla f,\dot\gamma_{x,y}\rangle(y)|\,dv(y)\\
&\leqslant\eta^{-1}\int_0^{2R}\int_{S_x}|\langle\nabla f,\dot\gamma_\theta\rangle(0)-\langle\nabla f,\dot\gamma_\theta\rangle(t)|t^{m-1}\,d\theta\,dt+C(m,p,\Lambda)\Lambda'(R^{2p}+\eta R^m)\\
&\leqslant C(m)\eta^{-1}R^{m-1}\int_0^{2R}\int_{S_x}|\langle\nabla f,\dot\gamma_\theta\rangle(0)-\langle\nabla f,\dot\gamma_\theta\rangle(t)|\,d\theta\,dt+C(m,p,\Lambda)\Lambda'(R^{2p}+\eta R^m)\\
&\leqslant C(m)\eta^{-1}R^{m}\int_0^{2R}\int_{S_x}|\mathrm{Hess}\,f|(\gamma_\theta(t))\,d\theta\,dt+C(m,p,\Lambda)\Lambda'(R^{2p}+\eta R^m).
\end{aligned}$$

Integrating over $x\in B$ gives

$$\begin{aligned}
&\int_{B\times B}|\langle\nabla f,\dot\gamma_{x,y}\rangle(x)-\langle\nabla f,\dot\gamma_{x,y}\rangle(y)|\,dv(x)\,dv(y)\\
&\quad\leqslant C(m)\eta^{-1}R^{m}\int_0^{2R}\int_{SB}|\mathrm{Hess}\,f|(\gamma_\theta(t))\,d\theta\,dt+C(m,p,\Lambda)\cdot\Lambda'\cdot(R^{2p}+\eta R^m)\,\mathrm{vol}(B)\\
&\quad\leqslant C(m)\eta^{-1}R^{m+1}\int_{B(z,3R)}|\mathrm{Hess}\,f|\,dv+C(m,p,\Lambda)\cdot\Lambda'\cdot(R^{2p}+\eta R^m)\,\mathrm{vol}(B).
\end{aligned}$$

The last inequality uses the invariance of Liouville measure under geodesic flows. □

2.5. Almost rigidity structures

Let M be a complete Riemannian manifold of dimension m which satisfies (2.1) and (2.2) for some $\varkappa>0$, $p>\frac{1}{2}m$ and $\Lambda\geqslant 1$. The local almost-rigidity properties below can be proved exactly as in [8] and [9].

For any $\varepsilon>0$ small, there exist positive constants δ and r_0 depending on m, p, $\varkappa$, Λ and ε such that the following Theorems 2.31–2.34 hold.

THEOREM 2.31. (Almost splitting, [30]) *Let $p^{\pm}\in M$ with $d=d(p^+,p^-)\leqslant r_0$. If $x\in M$ satisfies $d(p^{\pm},x)\geqslant\frac{1}{5}d$ and*

$$d(p^+,x)+d(p^-,x)-d\leqslant\delta^2 d, \tag{2.54}$$

then there exist a complete length space X and $B((0,x^*),r)\subset\mathbb{R}\times X$ such that

$$d_{\mathrm{GH}}(B(x,\delta d),B((0,x^*),\delta d))\leqslant\varepsilon d. \tag{2.55}$$

THEOREM 2.32. (Volume convergence, [30]) *If $x\in M$ satisfies*

$$d_{\mathrm{GH}}(B(x,r),B_r)\leqslant\delta r, \tag{2.56}$$

for some $r\leqslant r_0$, where B_r denotes a Euclidean ball of radius r, then

$$\mathrm{vol}(B(x,r))\geqslant(1-\varepsilon)\,\mathrm{vol}(B_r). \tag{2.57}$$

THEOREM 2.33. (Almost metric cone) *If $x\in M$ satisfies*

$$\frac{\mathrm{vol}(B(x,2r))}{\mathrm{vol}(B_{2r})}\geqslant(1-\delta)\frac{\mathrm{vol}(B(x,r))}{\mathrm{vol}(B_r)} \tag{2.58}$$

for some $r\leqslant r_0$, then there exists a compact length space X with

$$\mathrm{diam}(X)\leqslant(1+\varepsilon)\pi \tag{2.59}$$

such that, for any metric ball $B(o^*,r)\subset C(X)$ centered at the vertex o^*,

$$d_{\mathrm{GH}}(B(x,r),B(o^*,r))\leqslant\varepsilon r. \tag{2.60}$$

THEOREM 2.34. *If $x\in M$ satisfies*

$$\mathrm{vol}(B(x,2r))\geqslant(1-\delta)\,\mathrm{vol}(B_{2r}) \tag{2.61}$$

for some $r\leqslant r_0$, then

$$d_{\mathrm{GH}}(B(x,r),B_r)\leqslant\varepsilon r. \tag{2.62}$$

2.6. C^{α}-structure in almost Euclidean regions

Let M be a complete Riemannian manifold of dimension m which satisfies (2.2) for some $\varkappa>0$. Instead of (2.1) we assume an L^p-bound of the Ricci curvature, according to

$$\int_M |\mathrm{Ric}|^p\,dv\leqslant\Lambda, \tag{2.63}$$

for some $p>\frac{1}{2}m$ and $\Lambda\geqslant 1$.

Fix $\alpha\in(0,1)$ and $\theta>0$. For $x\in M$, we define the C^α *harmonic radius* at x, denoted by $r_g^{\alpha,\theta}(x)$, to be the maximal radius r such that there exist harmonic coordinates $\mathbf{x}=(x^1,...,x^{2n})\colon B(x,r)\to\mathbb{R}^{2n}$ satisfying

$$e^{-\theta}(\delta_{ij})\leqslant(g_{ij})\leqslant e^{\theta}(\delta_{ij}) \tag{2.64}$$

as matrices, and

$$\sup_{i,j}(\|g_{ij}\|_{C^0}+r^\alpha\|g_{ij}\|_{C^\alpha})\leqslant e^\theta, \tag{2.65}$$

where $g_{ij}=(\mathbf{x}^{-1})^*g(\partial/\partial x^i,\partial/\partial x^j)$ is defined on the domain $\mathbf{x}(B(x,r))$. In harmonic coordinates, the L^p-bound of the Ricci curvature gives the $L^{2,p}$-bound of the metric tensor g_{ij} which in turn implies the C^α-regularity of the metric. Following the arguments in [2] and [28] one can prove the following result.

THEOREM 2.35. *For any $\delta,\theta\in(0,1)$ and $0<\alpha<2-m/p$, there exist $\eta>0$ and $r_0>0$ such that the following statement holds: if $x\in M$ satisfies*

$$\mathrm{vol}(B(x,r))\geqslant(1-\eta)\,\mathrm{vol}(B_r) \tag{2.66}$$

for some $r\leqslant r_0$, then

$$r_g^{\alpha,\theta}(x)\geqslant\delta r. \tag{2.67}$$

COROLLARY 2.36. *With the same assumptions as in the above theorem, if $x\in M$ satisfies* (2.66), *then the isoperimetric constant of $B(x,\delta r)$ has a lower bound*

$$\mathrm{Isop}(B(x,\delta r))\geqslant(1-\theta)\,\mathrm{Isop}(\mathbb{R}^m). \tag{2.68}$$

2.7. Structure of the limit space

Let (M_i,g_i) be a sequence of Riemannian manifolds of dimension m satisfying (2.2) and (2.63) for some $\varkappa,\Lambda>0$ and $p>\frac{1}{2}$ independent of i. Then (2.9) gives us the uniform upper bound of the volume growth. By Gromov's first convergence theorem, there exists a complete length metric space (Y,d) such that

$$(M_i,g_i)\xrightarrow{d_{\mathrm{GH}}}(Y,d) \tag{2.69}$$

along a subsequence in the pointed Gromov–Hausdorff topology.

THEOREM 2.37. *With the same assumptions as above, the following facts hold*:

(i) *For any $r>0$ and $x_i\in M_i$ such that $x_i\to x_\infty\in Y$, we have*

$$\mathrm{vol}(B(x_i,r))\longrightarrow\mathcal{H}^m(B(x_\infty,r)), \tag{2.70}$$

where $\mathcal{H}^m$ *denotes the* m*-dimensional Hausdorff measure.*

(ii) *For any* $x\in Y$ *and any sequence* $\{r_j\}_j$ *with* $\lim_{j\to\infty} r_j=0$, *there is a subsequence of* $(Y, r_j^{-2}d, x)$ *converging to a metric space* $(\mathcal{C}_x, d_x, o)$ *as* $j\to\infty$. *Any such space* $(\mathcal{C}_x, d_x, o)$ *is a metric cone with vertex* o *and splits off lines isometrically.*

(iii) $Y=\mathcal{S}\cup\mathcal{R}$, *where* $\mathcal{S}$ *is a closed set of codimension* $\geqslant 2$ *and* $\mathcal{R}$ *is convex in* Y; $\mathcal{R}$ *consists of points whose tangent cone is* $\mathbb{R}^m$.

(iv) *There is a* $C^{1,\alpha}$*-smooth structure on* $\mathcal{R}$ *and a* C^α*-metric* g_∞, *for all* $\alpha< 2-m/p$, *which induces* d; *moreover*, g_i *converges to* g_∞ *in the* C^α*-topology on* $\mathcal{R}$.

(v) *The singular set* $\mathcal{S}$ *has codimension* $\geqslant 4$ *if each* (M_i, g_i) *is Kähler.*

The proofs of (i)–(iv), except the convexity of $\mathcal{R}$, are standard, following the same line as that of Cheeger–Colding and Cheeger–Colding–Tian; see [7]–[9] and [11]. In the Kähler setting, the convergence of the metric and complex structure takes place in the $C^\alpha\cap L^{2,p}$-topology on $\mathcal{R}$ (cf. [28]), so g_∞ is Kähler with respect to the limit complex structure in the weak sense. However, the $L^{2,p}$-convergence of g_i will be enough to carry out the slice argument as in [11] or [7] to show the codimension-4 property of the singular set $\mathcal{S}$. The convexity of $\mathcal{R}$ is a consequence of the following local Hölder continuity of geodesic balls in the interior of geodesic segments and the local C^α-structure of the regular set; see [17] for details.

THEOREM 2.38. *Let* (M, g) *be a complete Riemannian manifold of dimension* m *which satisfies* (2.2) *and* (2.63) *for some* $\varkappa, \Lambda>0$ *and* $p>\frac{1}{2}m$. *There is* $\alpha=\alpha(p,m)>0$ *such that the following statement holds*: *for any* $\delta>0$ *small, we can find positive constants* C *and* r_0 *depending on* m, p, $\varkappa$, Λ *and* δ *such that, on any normal geodesic* $\gamma\colon[0,l]\to M$ *of length* $l\leqslant 1$,

$$d_{\mathrm{GH}}(B_r(\gamma(s)), B_r(\gamma(t)))\leqslant \frac{C}{\delta l}|s-t|^\alpha r, \tag{2.71}$$

whenever

$$0<r\leqslant r_0\delta l, \quad \delta l\leqslant s\leqslant t\leqslant (1-\delta)l \quad \textit{and} \quad |s-t|\leqslant r.$$

Proof. The proof is exactly the same as that of Theorem 1.1 in [17, §3]. As $\alpha\leqslant 1$ and $l\leqslant 1$, we may assume that $l=1$ by a scaling. The conditions (2.2) and (2.63) remain valid for the same constants $\varkappa, \Lambda>0$ and $p>\frac{1}{2}m$. To apply the segment inequality established in §2.4, we replace the Hessian estimate along a geodesic connecting x and y, namely $\int_{\gamma_{x,y}}|\mathrm{Hess}\,\mathbf{b}^\pm_{r^2}|\,\gamma_{x,y}(s)\,ds$, where $\mathbf{b}^\pm_{r^2}$ is the parabolic approximation of distance function defined in §2.3 with base points $p=\gamma(0)$ and $q=\gamma(l)$, by the integrand in (2.53)

$$F(x,y)=|\langle\nabla\mathbf{b}^\pm_{r^2}, \dot\gamma_{x,y}\rangle(x)-\langle\nabla\mathbf{b}^\pm_{r^2}, \dot\gamma_{x,y}\rangle(y)|.$$

Define $I^r_{t-t'}$, T^r_η and $T^r_\eta(x)$ for $x\in T^r_\eta$, $\eta>0$, as in [17], just by replacing the upper bound of $e_{p,q}$ in T^r_η by $\eta^{-1}r^{2-m/2p}$. The points in $T^r_\eta(x)$ behave very well under the gradient flow associated with the distance function at p. In the simplest case, by the Hessian estimate and (2.49) in §2.3, if $\gamma(t)\in T_\eta^r$, $t\in[\delta,1-\delta]$ and $x\in T_\eta^r(\gamma(t))\cap T_\eta^r$ as considered in [17, p. 1210], the distortion of distance under the geodesic flow can be estimated as

$$\begin{aligned}d(\gamma_{p,x}(t'),\gamma(t'))-d(x,\gamma(t)) &\leqslant C\eta^{-2}(\delta^{-1}r^{-m/4p}\sqrt{t-t'}+\delta r^{-1}(t-t')+r^{2p-m-1}(t-t'))r\\ &\leqslant C\eta^{-2}(\delta^{-1}(t-t')^{1/2-m/4p}+\delta+(t-t')^{2p-m})r\end{aligned}$$

for all $\delta>0$ and $t'\leqslant t\leqslant t'+r$. It follows, by letting $\delta=(t-t')^{1/4-m/8p}$, that

$$d(\gamma_{p,x}(t'),\gamma(t'))-d(x,\gamma(t))\leqslant C\eta^{-2}(t-t')^{1/4-m/8p}r \quad \text{for all } \delta\leqslant t'\leqslant t\leqslant t'+r\leqslant 1-\delta.$$

For the general case where $\gamma(t)$ is not in T_η^r, one can follow [17] to get a precise $\alpha=\alpha(p,m)$ such that (2.71) holds for a certain constant C. □

3. Regularity under the Kähler–Ricci flow

In this section, we prove Theorem 1.2. We will first show that the regular set $\mathcal{R}$ of the limit space is smooth, and then apply the pseudolocality theorem of Perelman [27] to prove the smooth convergence on $\mathcal{R}$.

Let us fix some notions first. Let $(M;J)$ be a compact Kähler manifold of (complex) dimension n. Let g be a Kähler metric and ∇^L be the Levi-Civita connection of g. In local complex coordinates $(z^1,...,z^n)$, we set

$$g_{i\bar{j}}=g\left(\frac{\partial}{\partial z^i},\frac{\partial}{\partial \bar{z}^j}\right),\quad R_{i\bar{j}}=\mathrm{Ric}\left(\frac{\partial}{\partial z^i},\frac{\partial}{\partial \bar{z}^j}\right),\quad \text{etc.}$$

For simplicity, let ∇_i and $\nabla_{\bar{j}}$ denote $\nabla^L_{\partial/\partial z^i}$ and $\nabla^L_{\partial/\partial \bar{z}^j}$, respectively. Define the projections of the Levi-Civita connection onto the $(1,0)$ and $(0,1)$ spaces as

$$\nabla=\nabla_i\otimes dz^i \quad\text{and}\quad \overline{\nabla}=\nabla_{\bar{i}}\otimes d\bar{z}^i.$$

Define the rough Laplacian acting on tensor fields by $\Delta=g^{i\bar{j}}\nabla_i\nabla_{\bar{j}}$.

From now on, $(M;J)$ will be a compact Fano n-manifold and g_0 be a Kähler metric in the anti-canonical class $2\pi c_1(M;J)$. Let $g=g(t)$ be the solution to the volume normalized Kähler–Ricci flow

$$\frac{\partial g}{\partial t}=g-\mathrm{Ric}(g) \tag{3.1}$$

with initial data $g(0)=g_0$. By the $\partial\bar{\partial}$-lemma, there exists a family of real-valued functions $u=u(t)$, called *Ricci potentials* of g, which are determined by

$$g_{i\bar{j}}-R_{i\bar{j}}=\partial_i\partial_{\bar{j}}u \quad\text{and}\quad \frac{1}{\mathrm{V}}\int_M e^{-u(t)}\,dv_{g(t)}=1, \tag{3.2}$$

where $\mathrm{V}=\int_M dv_g$ denotes the volume of the Kähler–Ricci flow. By Perelman's estimate (see [34] for a proof), there exists C depending only on the initial metric g_0 such that

$$\|u(t)\|_{C^0}+\|\nabla u(t)\|_{C^0}+\|\Delta u(t)\|_{C^0}\leqslant C \quad \text{for all } t\geqslant 0. \tag{3.3}$$

By Perelman's non-collapsing theorem for Ricci flows [27], there exist positive constants $\varkappa$ and D depending on g_0 such that

$$\operatorname{vol}(B_{g(t)}(x,r))\geqslant \varkappa r^{2n} \quad \text{for all } x\in M \text{ and } r\leqslant 1, \tag{3.4}$$

and

$$\operatorname{diam}(M,g(t))\leqslant D. \tag{3.5}$$

The following formulas for u can easily be verified under the Kähler–Ricci flow:

$$\frac{\partial}{\partial t}u=\Delta u+u-a; \tag{3.6}$$

$$\frac{\partial}{\partial t}|\nabla u|^2=\Delta|\nabla u|^2-|\nabla\nabla u|^2-|\nabla\overline{\nabla}u|^2+|\nabla u|^2; \tag{3.7}$$

$$\frac{\partial}{\partial t}\Delta u=\Delta\Delta u-|\nabla\overline{\nabla}u|^2+\Delta u; \tag{3.8}$$

$$\frac{\partial}{\partial t}|\nabla\overline{\nabla}u|^2=\Delta|\nabla\overline{\nabla}u|^2-2|\nabla\nabla\overline{\nabla}u|^2+2R_{i\bar{j}k\bar{l}}\nabla_{\bar{i}}\nabla_l u\nabla_j\nabla_{\bar{k}}u; \tag{3.9}$$

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla\Delta u|^2=\Delta|\nabla\Delta u|^2-|\nabla\nabla\Delta u|^2-|\nabla\overline{\nabla}\Delta u|^2+|\nabla\Delta u|^2\\ -\nabla_i|\nabla\overline{\nabla}u|^2\nabla_{\bar{i}}\Delta u-\nabla_i\Delta u\nabla_{\bar{i}}|\nabla\overline{\nabla}u|^2.\end{aligned} \tag{3.10}$$

Here,

$$a(t)=\frac{1}{\mathrm{V}}\int_M u(t)e^{-u(t)}\,dv_{g(t)} \tag{3.11}$$

is the average of $u(t)$. By the Jensen inequality, $a(t)\leqslant 0$. It is known that $a(t)$ increases along the Kähler–Ricci flow [51], so we may assume that

$$\lim_{t\to\infty}a(t)=a_\infty. \tag{3.12}$$

3.1. Long-time behavior of the Ricci potentials

We will show that the Ricci potentials $u(t)$ behaves very well as $t\to\infty$ under the Kähler–Ricci flow, namely the corresponding gradient fields have a holomorphic limit in the L^2 topology. This implies that the limit of the Kähler–Ricci flow should be a Kähler–Ricci soliton (in some weak topology).

PROPOSITION 3.1. *Under the Kähler–Ricci flow*,

$$\int_0^\infty \int_M |\nabla\nabla u|^2 \, dv \, dt < \infty. \tag{3.13}$$

In particular,

$$\int_M |\nabla\nabla u|^2 \, dv \to 0, \quad \textit{as } t \to \infty. \tag{3.14}$$

PROPOSITION 3.2. *Under the Kähler–Ricci flow*,

$$\int_t^{t+1} \int_M |\nabla(\Delta u - |\nabla u|^2 + u)|^2 \, dv \, dt \to 0, \quad \textit{as } t \to \infty, \tag{3.15}$$

$$\int_M (\Delta u - |\nabla u|^2 + u - a)^2 \, dv \to 0, \quad \textit{as } t \to \infty. \tag{3.16}$$

Remark 3.3. Proposition 3.1 gives a hint of the global convergence of a Kähler–Ricci flow. Assuming the boundedness of curvature, this has been proved by Ache [1].

Remark 3.4. Recall that a Riemannian manifold (M,g) is a *shrinking Ricci soliton* if

$$\mathrm{Ric}(g) + \mathrm{Hess}\, f = \lambda g \tag{3.17}$$

for some $f \in C^\infty(M;\mathbb{R})$ and $\lambda > 0$. In the case where M is Fano and $g \in 2\pi c_1(M)$, the manifold is a shrinking Ricci soliton (called a *shrinking Kähler–Ricci soliton*) only if $\lambda = 1$. In this case, f equals the Ricci potential u, so (M,g) is a shrinking Kähler–Ricci soliton if and only if

$$\nabla\nabla u = 0. \tag{3.18}$$

Moreover, applying the Bianchi identity, it can be checked that (M,g) is a shrinking Kähler–Ricci soliton if and only if the Schur-type identity

$$\Delta u - |\nabla u|^2 + u = a \tag{3.19}$$

holds. Actually, since u is real-valued, the condition (3.18) means that $u - a$ is an eigenfunction of the weighted Laplacian $g^{i\bar{j}}(\partial_i \partial_{\bar{j}} - \partial_i u \partial_{\bar{j}})$, associated with the eigenvalue 1, i.e., u satisfies (3.19).

To prove the proposition, we need Perelman's entropy functional (compare with Perelman's original definition in [27]): For any Kähler metric $g \in 2\pi c_1(M)$, let

$$\mathcal{W}(g,f) = \frac{1}{\mathrm{V}} \int_M (s + |\nabla f|^2 + f - n) e^{-f} \, dv \quad \text{for all } f \in C^\infty(M;\mathbb{R}), \tag{3.20}$$

and define

$$\mu(g) = \inf \left\{ \mathcal{W}(g,f) : \int_M e^{-f} \, dv = \mathrm{V} \right\}, \tag{3.21}$$

where s is the scalar curvature of g. It is known that a smooth minimizer of μ, though it may not be unique, always exists [32]. The entropy admits a natural upper bound

$$\mu(g)\leqslant \frac{1}{\mathrm{V}}\int_M ue^{-u}\,dv =: a\leqslant 0.$$

Consider the entropy under the Kähler–Ricci flow $g=g(t)$: for any solution $f=f(t)$ to the backward heat equation

$$\frac{\partial f}{\partial t}=-\Delta f+|\nabla f|^2+\Delta u, \tag{3.22}$$

we have

$$\frac{d}{dt}\mathcal{W}(g,f)=\frac{1}{\mathrm{V}}\int_M(|\nabla\overline{\nabla}(u-f)|^2+|\nabla\nabla f|^2)e^{-f}\,dv. \tag{3.23}$$

This implies Perelman's monotonicity

$$\mu(g_0)\leqslant\mu(g(t))\leqslant 0\quad\text{for all } t\geqslant 0. \tag{3.24}$$

We also need the following lemma to prove the propositions.

Lemma 3.5. *For any $g=g(t)$ and smooth function f we have*

$$\int_M|\nabla\nabla f|^2\,dv\leqslant C(g_0)\int_M|\nabla\overline{\nabla}f|^2\,dv. \tag{3.25}$$

Proof. By adding a constant we may assume that f satisfies $\int_M fe^{-u}\,dv=0$. Then the weighted Poincaré inequality (cf. [20, Theorem 2.4.3]) gives

$$\int_M f^2e^{-u}\,dv\leqslant\int_M|\nabla f|^2e^{-u}\,dv.$$

By Perelman's estimate applied to u (3.3), we have

$$\int_M f^2\,dv\leqslant C(g_0)\int_M|\nabla f|^2\,dv.$$

Thus,

$$\int_M|\nabla f|^2\,dv=-\int_M f\Delta f\,dv\leqslant\frac{1}{2C}\int_M f^2\,dv+2C\int_M(\Delta f)^2\,dv,$$

from which it follows that

$$\int_M|\nabla f|^2\,dv\leqslant C(g_0)\int_M(\Delta f)^2\,dv.$$

Integrating by parts gives

$$\int_M|\nabla\nabla f|^2\,dv=\int_M((\Delta f)^2-R_{i\bar{j}}\nabla_{\bar{i}}f\nabla_j f)\,dv=\int_M((\Delta f)^2-|\nabla f|^2+\nabla_i\nabla_{\bar{j}}u\nabla_{\bar{i}}f\nabla_j f)\,dv.$$

The last term on the right-hand side can be estimated as

$$\begin{aligned}\int_M \nabla_i \nabla_{\bar{j}} u \nabla_{\bar{i}} f \nabla_j f \, dv &= -\int_M \nabla_{\bar{j}} u (\Delta f \nabla_j f + \nabla_{\bar{i}} f \nabla_i \nabla_j f) \, dv \\ &\leqslant \int_M \left((\Delta f)^2 + \frac{1}{2} |\nabla\nabla f|^2 + \|\nabla u\|_{C^0}^2 |\nabla f|^2 \right) dv.\end{aligned}$$

Combining these estimates, we have

$$\int_M |\nabla\nabla f|^2 \, dv \leqslant \int_M (4(\Delta f)^2 + 2C|\nabla f|^2) \, dv \leqslant C \int_M (\Delta f)^2 \, dv \leqslant C \int_M |\nabla\overline{\nabla} f|^2 \, dv,$$

which is the desired estimate. □

We now give the proofs of the propositions.

Proof of Proposition 3.1. For any time $t=k\geqslant 1$, choose a normalized minimizer of $\mu(g(k))$, say f_k, satisfing $\int_M e^{-f_k} \, dv = \mathrm{V}$. Let $f_k(t)$ be the solution to (3.22) on the time interval $[k-1,k]$. Then we have

$$\frac{1}{\mathrm{V}} \int_{k-1}^{k} \int_M (|\nabla\overline{\nabla}(u-f_k)|^2 + |\nabla\nabla f_k|^2) e^{-f_k} \, dv \, dt \leqslant \mu(g(k)) - \mu(g(k-1)).$$

It is proved that $|f_k(t)|\leqslant C(g_0)$ for any $t\in[k-1,k]$ (see [44], [47] and [43]). Thus,

$$\int_{k-1}^{k} \int_M (|\nabla\overline{\nabla}(u-f_k)|^2 + |\nabla\nabla f_k|^2) \, dv \, dt \leqslant C(g_0)(\mu(g(k)) - \mu(g(k-1))).$$

Summing over $k=1,2,\dots$, and using $\mu(g(t))\leqslant 0$ for all t, we conclude that

$$\sum_{k=1}^{\infty} \int_{k-1}^{k} \int_M (|\nabla\overline{\nabla}(u-f_k)|^2 + |\nabla\nabla f_k|^2) \, dv \, dt \leqslant C(g_0). \tag{3.26}$$

Applying Lemma 3.5 to $u-f_k$, we get

$$\int_M |\nabla\nabla(u-f_k)|^2 \, dv \leqslant C(g_0) \int_M |\nabla\overline{\nabla}(u-f_k)|^2 \, dv \quad \text{for all } t\in[k-1,k].$$

Combining this with (3.26) gives

$$\int_0^{\infty} \int_M |\nabla\nabla u|^2 \, dv \leqslant \sum_{k=1}^{\infty} \int_M (2|\nabla\nabla(u-f_k)|^2 + 2|\nabla\nabla f_k|^2) \, dv \leqslant C(g_0).$$

This proves (3.13).

To prove (3.14), it will be sufficient to show that

$$\frac{d}{dt}\int_M |\nabla\nabla u|^2\,dv \leqslant C(g_0) \quad \text{for all } t\geqslant 0. \tag{3.27}$$

An easy calculation shows that

$$\frac{\partial}{\partial t}|\nabla\nabla u|^2 = \Delta|\nabla\nabla u|^2 - |\overline{\nabla}\nabla\nabla u|^2 - |\nabla\nabla\nabla u|^2 - 2R_{i\bar{j}k\bar{l}}\nabla_{\bar{i}}\nabla_{\bar{k}}u\nabla_j\nabla_l u. \tag{3.28}$$

Integrating, and denoting by Rm the Riemannian curvature tensor of the metric, we get

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla\nabla u|^2\,dv &= \int_M (-|\overline{\nabla}\nabla\nabla u|^2 - |\nabla\nabla\nabla u|^2 + \Delta u|\nabla\nabla u|^2 - 2R_{i\bar{j}k\bar{l}}\nabla_{\bar{i}}\nabla_{\bar{k}}u\nabla_j\nabla_l u)\,dv\\ &\leqslant \int_M (\Delta u|\nabla\nabla u|^2 + 2\nabla_{\bar{j}}R_{k\bar{l}}\nabla_{\bar{k}}u\nabla_j\nabla_l u + 2R_{i\bar{j}k\bar{l}}\nabla_{\bar{k}}u\nabla_{\bar{i}}\nabla_j\nabla_l u)\,dv\\ &\leqslant \int_M ((\|\Delta u\|_{C^0} + \|\nabla u\|^2_{C^0})|\nabla\nabla u|^2 + |\nabla\nabla\overline{\nabla}u|^2 + \|\nabla u\|^2_{C^0}|\mathrm{Rm}|^2)\,dv,\end{aligned}$$

The desired estimate (3.27) now follows from Perelman's estimate applied to u (3.3) and the general estimates (4.4) and (4.8) in the next section. □

Proof of Proposition 3.2. First of all, by the Ricci potential equation,

$$\nabla_i(\Delta u - |\nabla u|^2 + u) = \nabla_{\bar{j}}\nabla_i\nabla_j u - \nabla_i\nabla_j u\nabla_{\bar{j}}u.$$

Thus,

$$|\nabla(\Delta u - |\nabla u|^2 + u)|^2 \leqslant 2(|\nabla u|^2|\nabla\nabla u|^2 + |\nabla_{\bar{j}}\nabla_i\nabla_j u|^2).$$

To prove (3.15) we only need to show that

$$\int_t^{t+1}\int_M |\nabla_{\bar{j}}\nabla_i\nabla_j u|^2\,dv\,dt \to 0.$$

Integrating by parts and using the second Bianchi identity,

$$\begin{aligned}\int_M |\nabla_{\bar{j}}\nabla_i\nabla_j u|^2\,dv &= \int_M \nabla_{\bar{j}}\nabla_i\nabla_j u\nabla_k\nabla_{\bar{i}}\nabla_{\bar{k}}u\,dv\\ &= -\int_M \nabla_i\nabla_j u\nabla_{\bar{j}}(\nabla_{\bar{i}}\Delta u + R_{\bar{i}k}\nabla_{\bar{k}}u)\,dv\\ &= -\int_M \nabla_i\nabla_j u(\nabla_{\bar{j}}\nabla_{\bar{i}}\Delta u + \nabla_{\bar{j}}R_{\bar{i}k}\nabla_{\bar{k}}u + R_{\bar{i}k}\nabla_{\bar{j}}\nabla_{\bar{k}}u)\,dv\\ &= -\int_M \nabla_i\nabla_j u\nabla_{\bar{j}}\nabla_{\bar{i}}\Delta u\,dv + \int_M \nabla_i\nabla_j u\nabla_{\bar{j}}\nabla_{\bar{i}}\nabla_k u\nabla_{\bar{k}}u\,dv\\ &\quad -\int_M |\nabla\nabla u|^2\,dv + \int_M \nabla_i\nabla_j u\nabla_{\bar{i}}\nabla_k u\nabla_{\bar{j}}\nabla_{\bar{k}}u\,dv\\ &= -\int_M \nabla_i\nabla_j u\nabla_{\bar{j}}\nabla_{\bar{i}}\Delta u\,dv + \int_M \nabla_i\nabla_j u\nabla_{\bar{j}}\nabla_{\bar{i}}\nabla_k u\nabla_{\bar{k}}u\,dv\\ &\quad -\int_M |\nabla\nabla u|^2\,dv - \int_M \nabla_k u(\nabla_i\nabla_j u\nabla_{\bar{i}}\nabla_{\bar{j}}\nabla_{\bar{k}}u + \nabla_{\bar{i}}\nabla_i\nabla_j u\nabla_{\bar{j}}\nabla_{\bar{k}}u)\,dv.\end{aligned}$$

Then, by Schwarz inequality,

$$\begin{aligned}\int_{t-1}^{t}\int_M |\nabla_{\bar j}\nabla_i\nabla_j u|^2\,dv\,dt \\ \leqslant \biggl(\int_{t-1}^{t}\int_M |\nabla\nabla u|^2\,dv\,dt\biggr)^{1/2}\biggl[\biggl(\int_{t-1}^{t}\int_M |\nabla\nabla\Delta u|^2\,dv\,dt\biggr)^{1/2} \\ +\biggl(\int_{t-1}^{t}\int_M |\nabla u|^2(|\nabla\nabla\nabla u|^2+|\nabla\nabla\overline{\nabla} u|^2+|\overline{\nabla}\nabla\nabla u|^2)\biggr)^{1/2}\,dv\,dt\biggr].\end{aligned}$$

Applying (4.8) and the L^2-bound of $\nabla\nabla\Delta u$ (see Remark 4.6), we get (3.15).

Set $h=\Delta u-|\nabla u|^2+u-a$. Noticing that $\int_M he^{-u}\,dv=0$, by the weighted Poincaré inequality, and using the uniform bound of u, we derive

$$\int_M h^2\,dv \leqslant C(g_0)\int_M |\nabla(\Delta u-|\nabla u|^2+u)|^2\,dv.$$

Thus,

$$\int_t^{t+1}\int_M h^2\,dv\,dt\to 0 \quad \text{as } t\to\infty.$$

To show (3.16), it is sufficient to prove that

$$\frac{d}{dt}\int_M h^2\,dv \leqslant C(g_0)\biggl(1+\int_M h^2\,dv\biggr).$$

Actually, by

$$\frac{\partial}{\partial t}h=\Delta h+h-\frac{d}{dt}a+|\nabla_i\nabla_j u|^2,$$

we have

$$\begin{aligned}\frac{d}{dt}\int_M h^2\,dv &= \int_M 2h\biggl(\Delta h+h-\frac{d}{dt}a+|\nabla_i\nabla_j u|^2+\frac{1}{2}h\Delta u\biggr)\,dv \\ &\leqslant \int_M 2h\biggl(h+|\nabla_i\nabla_j u|^2+\frac{1}{2}h\Delta u\biggr)\,dv \\ &\leqslant (3+\|\Delta u\|_{C^0})\int_M h^2\,dv+\int_M |\nabla\nabla u|^4\,dv.\end{aligned}$$

The required estimate follows from (4.8) in the next section. □

3.2. Regularity of the limit

For any sequence $t_i\to\infty$, we define a family of Kähler–Ricci flows $g_i=g_i(t)$ by

$$(M,g_i(t))=(M,g(t_i+t)),\quad t\geqslant -1. \tag{3.29}$$

Let $u_i(t)$ be the associated Ricci potentials, which satisfy the uniform bound

$$\|u_i(t)\|_{C^0}+\|\nabla u_i(t)\|_{C^0}+\|\Delta u_i(t)\|_{C^0}\leqslant C(g_0)\quad\text{for all } t\geqslant -1. \tag{3.30}$$

Furthermore, by (3.14), for any $t\geqslant -1$,

$$\int_M \left|\nabla\nabla u_i(t)\right|^2 dv_{g_i}\to 0\quad\text{as } i\to\infty. \tag{3.31}$$

By the convergence theorem in §2, passing to a subsequence if necessary, we may assume, at time $t=0$, that

$$(M,g_i(0))\xrightarrow{d_{\mathrm{GH}}}(M_\infty,d). \tag{3.32}$$

We have the decomposition $M_\infty=\mathcal{S}\cup\mathcal{R}$, where $\mathcal{R}$ is a smooth complex manifold with a C^α-complex structure J_∞ and a C^α-metric g_∞ which induces d, while $\mathcal{S}$ is a closed singular set of codimension $\geqslant 4$. Moreover, under Gromov–Hausdorff convergence,

$$(g_i(0),u_i(0))\xrightarrow{C^\alpha\cap L^{2,p}}(g_\infty,u_\infty)\quad\text{on } \mathcal{R}. \tag{3.33}$$

The convergence of $u_i(0)$ follows from the elliptic regularity of $\Delta u_i(0)=n-s(g_i(0))\in L^p$. It is obvious that u_∞ is globally Lipschitz on M_∞ by Perelman's estimate.

Proposition 3.6. *If* (3.33) *holds, then* g_∞ *is smooth and satisfies*

$$\mathrm{Ric}(g_\infty)+\mathrm{Hess}\,u_\infty=g_\infty\quad\text{on } \mathcal{R}. \tag{3.34}$$

Moreover, J_∞ *is smooth on* $\mathcal{R}$ *and* g_∞ *is Kähler with respect to* J_∞*.*

Proof. We first show that g_∞ is smooth and satisfies (3.34). The strategy is to apply a bootstrap argument as in [28]; the difference is the existence of a twisted function term. In local harmonic coordinates $(x^1,...,x^{2n})$, the soliton equation (3.34) is equivalent to

$$g^{\alpha\beta}\frac{\partial^2 g_{\gamma\delta}}{\partial x^\alpha\partial x^\beta}=-\frac{\partial^2 u_\infty}{\partial x^\gamma\partial x^\delta}+Q(g,\partial g)_{\gamma\delta}+T(g^{-1},\partial g,\partial u)_{\gamma\delta}+g_{\gamma\delta}, \tag{3.35}$$

where Q is a quadratical term while T is a trilinear term of the corresponding variables. By Proposition 3.1, (3.35) holds in $L^2(\mathcal{R})$. Since both u_∞ and g_∞ are in $L^{2,p}$, equation (3.35) holds even in $L^p(\mathcal{R})$. On the other hand, by Proposition 3.2 and (3.33), we have

$$g^{\alpha\beta}\frac{\partial^2 u_\infty}{\partial x^\alpha\partial x^\beta}=g^{\alpha\beta}\frac{\partial u_\infty}{\partial x^\alpha}\frac{\partial u_\infty}{\partial x^\beta}-2u_\infty+2a_\infty \tag{3.36}$$

in the L^p-topology. A bootstrap argument appied to the elliptic systems (3.36) and (3.35) shows that g_∞ and u_∞ are actually smooth on $\mathcal{R}$.

Since g_∞ is smooth and $\nabla_{g_\infty}J_\infty=0$, the elliptic regularity shows that J_∞ is also smooth. □

3.3. Smooth convergence on the regular set

In order to prove the smooth convergence of the Kähler–Ricci flow on the regular set, we need the following version of Perelman's pseudolocality theorem: there exist $\varepsilon_P, \delta_P > 0$ and $r_P > 0$, which depend on p and Λ in Theorem 1.2, such that for any space-time point $(x_0, t_0) \in M \times [-1, \infty)$ in any flow g_i constructed in the previous subsection, if

$$\mathrm{vol}_{g_i(t_0)}(B_{g_i(t_0)}(x_0, r)) \geqslant (1-\varepsilon_P)\,\mathrm{vol}(B_r) \tag{3.37}$$

for some $r \leqslant r_P$, where $\mathrm{vol}(B_r)$ denotes the volume of a Euclidean ball of radius r in $\mathbb{R}^{2n}$, then we have the curvature estimate

$$|\mathrm{Rm}_{g_i}(x,t)| \leqslant \frac{1}{t-t_0} \quad \text{for all } x \in B_{g_i(t)}(x_0, \varepsilon_P r), \quad t_0 < t \leqslant t_0 + \varepsilon_P^2 r^2, \tag{3.38}$$

where Rm_{g_i} is the Riemannian curvature tensor of g_i, and the volume estimate

$$\mathrm{vol}_{g_i(t)}(B_{g_i(t)}(x_0, \delta_P\sqrt{t-t_0})) \geqslant (1-\eta)\,\mathrm{vol}(B_{\delta_P\sqrt{t-t_0}}), \quad t_0 < t \leqslant t_0 + \varepsilon_P^2 r^2, \tag{3.39}$$

where $\eta > 0$ is the constant in (2.66). One can assume $\eta \leqslant \varepsilon_P$ in the following application. In other words, in view of Shi's higher derivative estimate to curvature [35], the region around x_0 is almost Euclidean in the C^∞-topology at time $t_0 + \varepsilon_P^2 r^2$.

Notice that Perelman's pseudolocality theorem is originally stated for Ricci flow [27]. In our application, the sequence of Kähler flows g_i comes from a Ricci flow by scaling with a definite control (by Perelman's estimate to scalar curvature (3.3)). The condition (3.37) implies the local C^α-structure at x_0 (see §2.6).

We start to prove our main theorem.

Proof of Theorem 1.2. Recall that we have a family of spaces $(M, g_i(t))$, $-1 \leqslant t < \infty$, which converges at $t=0$ in the Cheeger–Gromov topology to a limit space (M_∞, d). The regular set $\mathcal{R}$ is a smooth complex manifold with a smooth metric g_∞ which induces d; the singular set $\mathcal{S}$ is closed and has codimension $\geqslant 4$. Moreover, the metric $g_i(0)$ converges to g_∞ in C^α-sense on $\mathcal{R}$. The goal is to show that $g_i(0)$ converges smoothly to g_∞.

For any radius $0 < r \leqslant r_P$, integer i and time $t \geqslant -1$, let us define

$$K_{r,i,t} = \{x \in M : \text{estimate (3.37) holds for the ball } B_{g_i(t)}(x, r)\}.$$

Then (3.39) implies that

$$K_{r,i,t} \subset K_{\delta_P\sqrt{s},i,t+s} \quad \text{for all } i \text{ and } 0 < s \leqslant \varepsilon_P^2 r^2. \tag{3.40}$$

First of all, observe that by the volume continuity under the Cheeger–Gromov convergence there exists for any $\varepsilon > 0$, $i \geqslant 1$ and $-1 \leqslant t_0 \leqslant 0$ a radius $0 < r_\varepsilon \leqslant r_P$ such that

$$\mathrm{vol}_{g_i(t_0)}(M \setminus K_{r_\varepsilon,i,t_0}) \leqslant \varepsilon. \tag{3.41}$$

Now, let r_j be a decreasing sequence of radii such that $\lim_{j\to\infty} r_j=0$ and $t_j=-\varepsilon_P r_j$ be a sequence of times. Estimate (3.41) implies that

$$\mathrm{vol}_{g_i(t)}(M\setminus K_{r_j,i,t})\to 0 \tag{3.42}$$

uniformly as $j\to\infty$. By (3.42), after a suitable adjustment of the radii r_j, we may assume that

$$K_{r_j,i,t_j}\subset K_{r_{j+1},i,t_{j+1}}\quad \text{for all } i,j\geqslant 1. \tag{3.43}$$

By the pseudolocality theorem,

$$|\mathrm{Rm}(g_i(t))(x)|\leqslant (t-t_j)^{-1} \tag{3.44}$$

for all (x,t) satisfying

$$d_{g_i(t)}(x,K_{r_j,i,t_j})\leqslant \varepsilon_P r_j,\quad t_j<t\leqslant 0.$$

By Shi's derivative estimate applied to the curvature under Ricci flow [35], there exists a sequence of constants $C_{k,j,i}$ such that

$$|(\nabla^L)^k\,\mathrm{Rm}(g_i(0))|\leqslant C_{k,j,i}\quad \text{on } K_{r_j,i,t_j}, \tag{3.45}$$

where ∇^L denotes the Levi-Civita connection of the corresponding Riemannian metric. Passing to a subsequence of $\{j\}$ if necessary, one can find a subsequence $\{i_j\}$ of $\{i\}$ such that

$$(K_{r_j,i_j,t_j},g_{i_j}(t_j))\xrightarrow{C^\alpha}(\Omega',g_{\Omega'}) \tag{3.46}$$

and

$$(K_{r_j,i_j,t_j},g_{i_j}(0))\xrightarrow{C^\infty}(\Omega,g_\Omega), \tag{3.47}$$

where $(\Omega',g_{\Omega'})$ and (Ω,g_Ω) are smooth Riemannian manifolds (which may not be complete). We may also assume that

$$(M,g_{i_j}(t_j))\xrightarrow{d_{\mathrm{GH}}}(M'_\infty,d'),$$

where (M'_∞,d') is a complete length space as described in §2. Let $M'_\infty=\mathcal{R}'\cup\mathcal{S}'$ be the regular-singular decomposition of M'_∞ and g'_∞ be a Riemannian metric on $\mathcal{R}'$ which induces d'. Then we have the following two claims.

Claim 3.7. *$(\Omega',g_{\Omega'})$ is isometric to $(\mathcal{R}'_\infty,g'_\infty)$.*

Claim 3.8. *$(\Omega',g_{\Omega'})$ is isometric to (Ω,g_Ω).*

The smooth convergence of $g_i(0)$ to g_∞ on $\mathcal{R}$ will follow directly from (3.47), once the two claims are proved.

Proof of Claim 3.7. Obviously $(\Omega', g_{\Omega'})$ can be viewed as a subset of $(\mathcal{R}'_\infty, g'_\infty)$. To show the equality, just note that a point of M'_∞ belongs to $\mathcal{R}'_\infty$ if and only if there is a local C^α-structure around it, and then by the continuity of volume under the Cheeger–Gromov convergence this point is a limit of points in K_{r_j,i_j,t_j}. □

Proof of Claim 3.8. Using the curvature estimate (3.44), by a similar argument as in the proof of [27, Lemma 8.3 (b)], we can show that, for any endpoints $x,y\in K_{r_j,i_j,t_j}$,

$$\frac{d}{dt} d_{g_{i_j}(t)}(x,y) \geqslant -C(n,g_0)(t-t_j)^{-1/2} \quad \text{for all } t_j < t \leqslant t_j+\varepsilon_P^2 r_j^2.$$

Integrating from time t_j to 0 gives

$$d_{g_{i_j}(0)}(x,y) \geqslant d_{g_{i_j}(t_j)}(x,y) - C(n,g_0)\sqrt{-t_j} \quad \text{for all } x,y\in K_{r_j,i_j,t_j}. \tag{3.48}$$

Passing to the limit, we get an expanding map

$$\psi_\infty : \mathcal{R}'_\infty \longrightarrow \Omega.$$

Since the volume

$$\mathrm{vol}_{g_\Omega}(\Omega) \leqslant \mathrm{V} = \mathrm{vol}_{g'_\infty}(\mathcal{R}'),$$

the map ψ_∞ must be an isometry. □

The proof of Therem 1.2 is now complete. □

4. An L^4-bound of the Ricci curvature under the Kähler–Ricci flow

In this section, we prove that the L^4-norm of the Ricci curvature is uniformly bounded along the Kähler–Ricci flow. This is crucial in the application of our regularity theory established in the previous section to the Hamilton–Tian conjecture on low-dimensional manifolds.

Let M be a compact Fano n-manifold and g_0 be a Kähler metric in the class $2\pi c_1(M;J)$. Let $g=g(t)$ be the solution to the volume normalized Kähler–Ricci flow (1.1) with initial data $g(0)=g_0$.

THEOREM 4.1. *There exists a constant* $C=C(g_0)$ *such that*

$$\int_M |\mathrm{Ric}(g(t))|^4 \, dv_{g(t)} \leqslant C \quad \textit{for all } t \geqslant 0. \tag{4.1}$$

Let $u(t)$ be the Ricci potential of $g(t)$, determined by (3.2). Then (4.1) is equivalent to the uniform L^4-bound of $\nabla\overline{\nabla}u$. The proof relies on the heat equation

$$\frac{\partial}{\partial t}u=\Delta u+u-a, \tag{4.2}$$

where $a=a(t)$ is a family of constants defined by (3.11), as well as the uniform L^2-bound of the total Riemannian curvature. An elliptic version of such integral Hessian estimates and its application are discussed by Cheeger in [7]. We remark that it is much more subtle in our parabolic case under the Kähler–Ricci flow. Actually, Perelman's gradient estimate and Laplacian estimate applied to u will be essential to the proof.

Recall Perelman's estimate:

$$\|u(t)\|_{C^0}+\|\nabla u(t)\|_{C^0}+\|\Delta u(t)\|_{C^0}\leqslant C \quad \text{for all } t\geqslant 0, \tag{4.3}$$

where $C=C(g_0)$. To prove the L^4-bound of $\nabla\overline{\nabla}u$, we need several lemmas.

Lemma 4.2. *There exists $C=C(g_0)$ such that*

$$\int_M (|\nabla\nabla u|^2+|\nabla\overline{\nabla}u|^2+|\mathrm{Rm}|^2)\,dv\leqslant C \quad \textit{for all } t\geqslant 0. \tag{4.4}$$

Proof. The L^2-bound of $\nabla\overline{\nabla}u$ follows from the observation

$$\int_M |\nabla\overline{\nabla}u|^2\,dv=\int_M(\Delta u)^2\,dv.$$

The L^2 bound of $\nabla\nabla u$ follows from an integration by parts:

$$\begin{aligned}\int_M|\nabla\nabla u|^2\,dv&=\int_M((\Delta u)^2-R_{i\bar j}\nabla_{\bar i}u\nabla_j u)\,dv\\&=\int_M((\Delta u)^2-|\nabla u|^2+\nabla_i\nabla_{\bar j}u\nabla_{\bar i}u\nabla_j u)\,dv\\&\leqslant\int_M((\Delta u)^2+|\nabla\overline{\nabla}u|^2+|\nabla u|^4)\,dv.\end{aligned}$$

The L^2-bound of the Riemannian curvature tensor follows from the Chern–Weil theory. Denote the ith Chern class by c_i. Let W be the Weyl tensor and define U and Z as

$$U=\frac{s}{2n(2n-1)}g\odot g \quad\text{and}\quad Z=\frac{1}{2n-2}\left(\mathrm{Ric}-\frac{s}{n}g\right)\odot g,$$

where s is the scalar curvature of g and $\odot$ is the Kulkarni–Nomizu product. Then we have the general formula [5, p. 80]

$$\int c_2\wedge c_1^{n-2}=\frac{(n-2)!}{2(2\pi)^n}\int_M((2n-3)(n-1)|U|^2-(2n-3)|Z|^2+|W|^2)\,dv.$$

The L^2-norms of Z and U are uniformly bounded in terms of $\int_M |\nabla\overline{\nabla}u|^2\,dv$. Since the left-hand side of the above formula is a topological invariant, the L^2-norm of the Weyl tensor is uniformly bounded, and this in turn gives the uniform L^2-bound of the total curvature tensor. □

LEMMA 4.3. *There exists a constant* $C=C(g_0)$ *such that*

$$\int_M |\nabla\overline{\nabla}u|^4\,dv \leqslant C\int_M (|\nabla\nabla\overline{\nabla}u|^2+|\overline{\nabla}\nabla\nabla u|^2)\,dv \quad \text{for all } t\geqslant 0, \tag{4.5}$$

and

$$\int_M |\nabla\nabla u|^4\,dv \leqslant C\int_M (|\nabla\nabla\nabla u|^2+|\nabla\nabla\overline{\nabla}u|^2+|\overline{\nabla}\nabla\nabla u|^2)\,dv \quad \text{for all } t\geqslant 0. \tag{4.6}$$

Proof. Recall the Bochner formula:

$$\Delta|\nabla u|^2 = |\nabla\nabla u|^2+|\nabla\overline{\nabla}u|^2+\Delta\nabla_i u\nabla_{\bar{i}}u+\nabla_i u\Delta\nabla_{\bar{i}}u.$$

Multiplying by $|\nabla\overline{\nabla}u|^2$ and integrating over M, we have

$$\begin{aligned}
&\int_M (|\nabla\nabla u|^2+|\nabla\overline{\nabla}u|^2)|\nabla\overline{\nabla}u|^2\,dv\\
&\quad= \int_M \Delta|\nabla u|^2|\nabla\overline{\nabla}u|^2-\Delta\nabla_i u\nabla_{\bar{i}}u|\nabla\overline{\nabla}u|^2-\nabla_i u\Delta\nabla_{\bar{i}}u|\nabla\overline{\nabla}u|^2\,dv\\
&\quad= -\int_M (\nabla_{\bar{i}}\nabla_j u\nabla_{\bar{j}}u+\nabla_{\bar{i}}\nabla_{\bar{j}}u\nabla_j u)(\nabla_i\nabla_j\nabla_{\bar{k}}u\nabla_{\bar{j}}\nabla_k u+\nabla_i\nabla_{\bar{j}}\nabla_k u\nabla_j\nabla_{\bar{k}}u)\,dv\\
&\qquad\quad -\int_M (\Delta\nabla_i u\nabla_{\bar{i}}u|\nabla\overline{\nabla}u|^2+\nabla_i u\Delta\nabla_{\bar{i}}u|\nabla\overline{\nabla}u|^2)\,dv\\
&\quad\leqslant 2\int_M |\nabla u|\,|\nabla\overline{\nabla}u|\,|\nabla\nabla\overline{\nabla}u|(|\nabla\overline{\nabla}u|+|\nabla\nabla u|)\,dv\\
&\qquad\quad +n\int_M |\nabla u|\,|\nabla\overline{\nabla}u|^2(|\nabla\nabla\overline{\nabla}u|+|\overline{\nabla}\nabla\nabla u|)\,dv\\
&\quad\leqslant \frac{1}{2}\int_M |\nabla\overline{\nabla}u|^2(|\nabla\overline{\nabla}u|^2+|\nabla\nabla u|^2)\,dv+8(n^2+1)\int_M |\nabla u|^2(|\nabla\nabla\overline{\nabla}u|^2+|\overline{\nabla}\nabla\nabla u|^2)\,dv.
\end{aligned}$$

This gives the first estimate by Perelman's estimate (4.3). The second estimate can be derived similarly. □

LEMMA 4.4. *There exists a constant* $C=C(g_0)$ *such that*

$$\begin{aligned}
&\int_M (|\overline{\nabla}\nabla\nabla u|^2+|\nabla\nabla\overline{\nabla}u|^2+|\nabla\nabla\nabla u|^2)\,dv\\
&\qquad\qquad \leqslant C\int_M (|\nabla\Delta u|^2+|\mathrm{Rm}|^2+|\nabla\nabla u|^2)\,dv \quad \text{for all } t\geqslant 0.
\end{aligned} \tag{4.7}$$

Proof. We prove the estimate using integration by parts. For example,

$$\begin{aligned}\int_M |\nabla\nabla\overline{\nabla}u|^2\,dv &= \int_M \nabla_i\nabla_j\nabla_{\bar{k}}u\nabla_{\bar{i}}\nabla_{\bar{j}}\nabla_k u\,dv\\ &= \int_M \nabla_i\nabla_j\nabla_{\bar{k}}u(\nabla_k\nabla_{\bar{i}}\nabla_{\bar{j}}u+R_{\bar{i}kl\bar{j}}\nabla_{\bar{l}}u)\,dv\\ &= \int_M (-\nabla_i\nabla_j\Delta u\nabla_{\bar{i}}\nabla_{\bar{j}}u+R_{\bar{i}kl\bar{j}}\nabla_{\bar{l}}u\nabla_i\nabla_j\nabla_{\bar{k}}u))\,dv\\ &= \int_M (\nabla_j\Delta u(\nabla_{\bar{j}}\Delta u+R_{i\bar{j}}\nabla_{\bar{i}}u)+R_{\bar{i}kl\bar{j}}\nabla_{\bar{l}}u\nabla_i\nabla_j\nabla_{\bar{k}}u))\,dv\\ &\leqslant \int_M \biggl(\frac{1}{2}|\nabla\nabla\overline{\nabla}u|^2+2|\nabla\Delta u|^2+n|\nabla u|^2|\mathrm{Rm}|^2\biggr)\,dv.\end{aligned}$$

Then, applying Perelman's estimate (4.3), we get the desired estimate of $\int_M |\nabla\nabla\overline{\nabla}u|^2\,dv$. The estimate of $\int_M |\overline{\nabla}\nabla\nabla u|^2\,dv$ is exactly the same as that of $\int_M |\nabla\nabla\overline{\nabla}u|^2\,dv$. The additional term $\int_M |\nabla\nabla u|^2\,dv$ on the right-hand side of (4.7) comes from the integration by parts in proving the estimate of $\int_M |\nabla\nabla\nabla u|^2\,dv$. □

We are now ready to prove the L^4-bound of $\nabla\overline{\nabla}u$ under the Kähler–Ricci flow.

THEOREM 4.5. *There exists $C=C(g_0)$ such that*

$$\int_M (|\nabla\nabla\overline{\nabla}u|^2+|\overline{\nabla}\nabla\nabla u|^2+|\nabla\nabla\nabla u|^2+|\nabla\nabla u|^4+|\nabla\overline{\nabla}u|^4)\,dv\leqslant C \quad \textit{for all } t\geqslant 0. \tag{4.8}$$

Proof. It is sufficient to show a uniform bound of $\int_M |\nabla\Delta u|^2\,dv$ under the Kähler–Ricci flow. To this purpose, we consider the evolution of $(\Delta u)^2$:

$$\frac{\partial}{\partial t}(\Delta u)^2=\Delta(\Delta u)^2-2|\nabla\Delta u|^2-2\Delta u|\nabla\overline{\nabla}u|^2+2(\Delta u)^2.$$

Integrating this identity gives

$$\begin{aligned}2\int_M |\nabla\Delta u|^2\,dv &= \int_M \biggl(-2\Delta u|\nabla\overline{\nabla}u|^2+2(\Delta u)^2-\frac{\partial}{\partial t}(\Delta u)^2\biggr)\,dv\\ &= \int_M (-2\Delta u|\nabla\overline{\nabla}u|^2+2(\Delta u)^2+(\Delta u)^3)\,dv-\frac{d}{dt}\int_M (\Delta u)^2\,dv.\end{aligned}$$

Together with Perelman's estimate (4.3) and (4.4), this implies

$$\int_t^{t+1}\int_M |\nabla\Delta u|^2\,dv\,ds\leqslant C(g_0) \quad \text{for all } t\geqslant 0. \tag{4.9}$$

Next, we calculate the derivative

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla\Delta u|^2\,dv &= \int_M (-|\nabla\nabla\Delta u|^2-|\nabla\overline{\nabla}\Delta u|^2+|\nabla\Delta u|^2\\ &\qquad -\nabla_i|\nabla\overline{\nabla}u|^2\nabla_{\bar{i}}\Delta u-\nabla_{\bar{i}}|\nabla\overline{\nabla}u|^2\nabla_i\Delta u+\Delta u|\nabla\Delta u|^2)\,dv.\end{aligned}$$

By integration by parts,

$$\begin{aligned}\left|\int_M -\nabla_i|\nabla\overline{\nabla}u|^2\nabla_{\bar i}\Delta u\,dv\right| &= \left|-\int_M (\nabla_i\nabla_j\nabla_{\bar k}u\nabla_{\bar j}\nabla_k u+\nabla_j\nabla_{\bar k}u\nabla_i\nabla_{\bar j}\nabla_k u)\nabla_{\bar i}\Delta u\,dv\right|\\ &\leqslant \left|\int_M \nabla_{\bar j}u(\nabla_i\nabla_j\Delta u\nabla_{\bar i}\Delta u+\nabla_i\nabla_j\nabla_{\bar k}u\nabla_k\nabla_{\bar i}\Delta u)\,dv\right|\\ &\quad+\left|\int_M \nabla_{\bar k}u(\nabla_i\nabla_k\Delta u\nabla_{\bar i}\Delta u+\nabla_i\nabla_{\bar j}\nabla_k u\nabla_j\nabla_{\bar i}\Delta u)\,dv\right|\\ &\leqslant \frac{1}{4}\int_M (|\nabla\nabla\Delta u|^2+|\nabla\overline{\nabla}\Delta u|^2)\,dv\\ &\quad+C(g_0)\int_M (|\nabla\Delta u|^2+|\nabla\nabla\overline{\nabla}u|^2)\,dv,\end{aligned}$$

and, similarly,

$$\begin{aligned}\left|\int_M -\nabla_i\Delta u\nabla_{\bar i}|\nabla\overline{\nabla}u|^2\,dv\right| &\leqslant \frac{1}{4}\int_M (|\nabla\nabla\Delta u|^2+|\nabla\overline{\nabla}\Delta u|^2)\,dv\\ &\quad+C(g_0)\int_M (|\nabla\Delta u|^2+|\nabla\nabla\overline{\nabla}u|^2)\,dv.\end{aligned}$$

Thus, by (4.7) and Perelman's estimate (4.3), we obtain

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla\Delta u|^2\,dv &\leqslant -\frac{1}{2}\int_M (|\nabla\nabla\Delta u|^2+|\nabla\overline{\nabla}\Delta u|^2)\,dv+C(g_0)\left(1+\int_M |\nabla\Delta u|^2\,dv\right)\\ &\leqslant C(g_0)\left(1+\int_M |\nabla\Delta u|^2\,dv\right).\end{aligned}$$

Together with estimate (4.9), this implies the uniform bound of $\int_M |\nabla\Delta u|^2\,dv$. The proof is complete. □

Remark 4.6. From the proof of the above theorem, we also have that, under the Kähler–Ricci flow,

$$\int_t^{t+1}\int_M (|\nabla\nabla\Delta u|^2+|\nabla\overline{\nabla}\Delta u|^2)\,dv\,ds\leqslant C(g_0)\quad\text{for all } t\geqslant 0. \tag{4.10}$$

5. Proof of Theorem 1.6

In this section, we shall prove Theorem 1.6 by generalizing the partial C^0 estimate to Kähler–Ricci flows on Fano manifolds.

As before, let M be a Fano manifold and $g=g(t)$ be a solution to the Kähler–Ricci flow (1.1) in the canonical class $2\pi c_1(M)$ with initial data $g(0)=g_0$. Let $u(t)$ be the Ricci potential of $g(t)$ defined by (3.2). Then we have Perelman's estimate (3.3)

$$\|u(t)\|_{C^0}+\|\nabla u(t)\|_{C^0}+\|\Delta u(t)\|_{C^0}\leqslant C(g_0)\quad\text{for all } t\geqslant 0. \tag{5.1}$$

Let $\tilde{g}(t)=e^{-u(t)/n}g(t)$ and $h(t)$ be the induced metric of $\tilde{g}(t)$ on $K_M^{-\ell}$, the ℓth power of the anti-canonical bundle ($\ell\geqslant 1$). Let D be the Chern connection of $h(t)$. For simplicity, we also use ∇ and $\overline{\nabla}$ to denote $\nabla\otimes D$ and $\overline{\nabla}\otimes D$, respectively, on the $K_M^{-\ell}$-valued tensor fields. The rough Laplacian on tensor fields is $\Delta=g^{i\bar{j}}\nabla_{\partial/\partial z^i}\nabla_{\partial/\partial\bar{z}^j}$.

Under this notation, the curvature form of the Chern connection satisfies

$$\mathrm{Ric}(h(t))=\ell\omega(t), \tag{5.2}$$

where $\omega(t)$ is the Kähler form of $g(t)$.

Set $N_\ell=\dim H^0(M,K_M^{-\ell})-1$, where $\ell\geqslant 1$. At any time t, we choose an orthonormal basis $\{s_{t,l,k}\}_{k=0}^{N_\ell}$ of $H^0(M,K_M^{-\ell})$ relative to the L^2-norm defined by $h(t)$ and the Riemannian volume form, and put

$$\varrho_{t,\ell}(x)=\sum_{k=0}^{N_\ell}|s_{t,\ell,k}|^2_{h(t)}(x)\quad\text{for all } x\in M. \tag{5.3}$$

Inspired by [42], [41] and [19], we have the following extension of the partial C^0-estimates to Kähler–Ricci flows.

THEOREM 5.1. *Assume that* $(M,g(t_i))\xrightarrow{d_{\mathrm{GH}}}(M_\infty,g_\infty)$, *as in Theorem* 1.2. *Then*

$$\inf_{t_i}\inf_{x\in M}\varrho_{t_i,\ell}(x)>0 \tag{5.4}$$

for a sequence $\ell\to\infty$.

In the proof of Theorem 5.1, two ingredients are important: the gradient estimate to pluri-anti-canonical sections and Hörmander's L^2-estimate for the $\bar{\partial}$-operator on (0,1)-forms. When the Ricci curvature is bounded from below, these estimates are standard and well known; cf. [36] and [19]. In our case of the Kähler–Ricci flow, the arguments should be modified because there is no Ricci curvature bound.

At any time t, for any holomorphic section $\sigma\in H^0(M,K_M^{-\ell})$, we have

$$\Delta|\sigma|^2=|\nabla\sigma|^2-n\ell|\sigma|^2 \tag{5.5}$$

and the Bochner formula

$$\Delta|\nabla\sigma|^2=|\nabla\nabla\sigma|^2+|\overline{\nabla}\nabla\sigma|^2-(n+2)\ell|\nabla\sigma|^2+\langle\mathrm{Ric}(\nabla\sigma,\cdot),\nabla\sigma\rangle. \tag{5.6}$$

The above formula can be rewritten as

$$\Delta|\nabla\sigma|^2=|\nabla\nabla\sigma|^2+|\overline{\nabla}\nabla\sigma|^2-((n+2)\ell-1)|\nabla\sigma|^2-\langle\partial\bar{\partial}u(\nabla\sigma,\cdot),\nabla\sigma\rangle. \tag{5.7}$$

Recall that, by [50] or [49], there is a uniform bound of the Sobolev constant along the Kähler–Ricci flow. So we may apply the standard iteration arguments of Nash–Moser to the above equations for σ and $\nabla\sigma$. The extra, and "bad", term $\langle\partial\bar\partial u(\nabla\sigma,\cdot),\nabla\sigma\rangle$ can be canceled in the iteration process by integrating by parts and applying Perelman's gradient estimate to u. Then we may conclude the following L^∞-estimate and gradient estimate for σ.

LEMMA 5.2. *There exists a constant* $C=C(g_0)$ *such that, for any* $\ell\geqslant 1$, $t\geqslant 0$ *and* $\sigma\in H^0(M,K_M^{-\ell})$, *we have*

$$\|\sigma\|_{C^0}+\ell^{-1/2}\|\nabla\sigma\|_{C^0}\leqslant C\ell^{n/2}\biggl(\int_M|\sigma|^2\,dv\biggr)^{1/2}. \tag{5.8}$$

The L^2-estimate for the $\bar\partial$ operator is first established for Kähler–Einstein surfaces in [36]. The following is a similar estimate for the Kähler–Ricci flow.

LEMMA 5.3. *There exists* ℓ_0 *depending on* g_0 *such that, for any* $\ell\geqslant\ell_0$, $t\geqslant 0$ *and* $\sigma\in C^\infty(M,T^{0,1}M\otimes K_M^{-\ell})$ *with* $\bar\partial\sigma=0$, *we can find a solution to* $\bar\partial\vartheta=\sigma$ *which satisfies*

$$\int_M|\vartheta|^2\,dv\leqslant 4\ell^{-1}\int_M|\sigma|^2\,dv. \tag{5.9}$$

Proof. It is sufficient to show that the Hodge Laplacian $\Delta_{\bar\partial}=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial\geqslant\frac14\ell$ as an operator on $C^\infty(M,T^{0,1}M\otimes K_M^{-\ell})$, when ℓ is sufficiently large. Actually, this implies that (i) $H^{0,1}(M,K_M^{-\ell})=0$ and thus $\bar\partial\vartheta=\sigma$ is solvable when $\bar\partial\sigma=0$ and (ii) the first positive eigenvalue of $\Delta_{\bar\partial}$ on $C^\infty(M,K_M^{-\ell})$ is bigger than $\frac14\ell$ so that (5.9) is satisfied for some solution ϑ.

The following Weitzenböch-type formulas hold for any $\sigma\in C^\infty(M,T^{0,1}M\otimes K_M^{-\ell})$:

$$\Delta_{\bar\partial}\sigma=\overline\nabla^*\overline\nabla\sigma+\mathrm{Ric}(\sigma,\cdot)+\ell\sigma, \tag{5.10}$$

$$\Delta_{\bar\partial}\sigma=\nabla^*\nabla\sigma-(n-1)\ell\sigma. \tag{5.11}$$

A combination of these gives

$$\Delta_{\bar\partial}\sigma=\biggl(1-\frac{1}{2n}\biggr)\overline\nabla^*\overline\nabla\sigma+\frac{1}{2n}\nabla^*\nabla\sigma+\biggl(1-\frac{1}{2n}\biggr)\mathrm{Ric}(\sigma,\cdot)+\frac{\ell}{2}\sigma. \tag{5.12}$$

Multiplying by σ and integrating over M, we obtain

$$\begin{aligned}\int_M\langle\Delta_{\bar\partial}\sigma,\sigma\rangle\,dv&=\int_M\biggl(\biggl(1-\frac{1}{2n}\biggr)|\overline\nabla\sigma|^2+\frac{1}{2n}|\nabla\sigma|^2+\frac{\ell}{2}|\sigma|^2\biggr)\,dv\\&\quad+\biggl(1-\frac{1}{2n}\biggr)\int_M(|\sigma|^2-\langle\nabla\overline\nabla u(\sigma,\cdot),\sigma\rangle)\,dv,\end{aligned}$$

where the bad term $\int_M \langle \nabla\overline{\nabla} u(\sigma,\cdot),\sigma\rangle\, dv$ can be estimated as follows:

$$\begin{aligned}\int_M \langle \nabla\overline{\nabla} u(\sigma,\cdot),\sigma\rangle\, dv &= -\int_M \overline{\nabla} u(\langle\nabla\sigma,\sigma\rangle+\langle\sigma,\overline{\nabla}\sigma\rangle)\, dv\\ &\leqslant \frac{1}{2n}\int_M (|\nabla\sigma|^2+|\overline{\nabla}\sigma|^2)\, dv + C\int_M |\sigma|^2\, dv,\end{aligned}$$

where C depends on n and $\|\nabla u\|_{C^0}$. Thus,

$$\int_M \langle \Delta_{\bar\partial}\sigma,\sigma\rangle\, dv \geqslant \left(\frac{\ell}{2}-C\right)\int_M |\sigma|^2\, dv \quad \text{for all } \sigma\in C^\infty(M, T^{0,1}M\otimes K_M^{-\ell}).$$

In particular, $\Delta_{\bar\partial}\geqslant \frac{1}{4}\ell$ when ℓ is large enough. □

The partial C^0-estimate for the Kähler–Ricci flow will follow from a parallel argument to the one for the Kähler–Einstein case in [42], [41] and [19]. We will adopt the notation and follow the arguments in [41].

According to our results in §3, for any $r_j\to 0$, by taking a subsequence if necessary, we have a tangent cone $\mathcal{C}_x$ of (M_∞,ω_∞) at x, where $\mathcal{C}_x$ is the limit $\lim_{j\to\infty}(M_\infty, r_j^{-2}\omega_\infty, x)$ in the Gromov–Hausdorff topology, satisfying the following two conditions:

($\mathbf{TZ}_1$) Each $\mathcal{C}_x$ is regular outside a closed subcone $\mathcal{S}_x$ of complex codimension at least 2. Such a $\mathcal{S}_x$ is the singular set of $\mathcal{C}_x$.

($\mathbf{TZ}_2$) There is a natural Kähler–Ricci-flat metric g_x on $\mathcal{C}_x\setminus\mathcal{S}_x$ which is also a cone metric. Its Kähler form ω_x is equal to $\sqrt{-1}\,\partial\bar\partial\varrho_x^2$ on the regular part of $\mathcal{C}_x$, where ϱ_x denotes the distance function from the vertex of $\mathcal{C}_x$, denoted by x for simplicity.

We will denote by L_x the trivial bundle $\mathcal{C}_x\times\mathbb{C}$ over $\mathcal{C}_x$ equipped with the Hermitian metric $e^{-\varrho_x^2}|\cdot|^2$. The curvature of this Hermitian metric is given by ω_x.

Without loss of generality, we may assume that, for each j, $r_j^{-2}=k_j$ is an integer.

For any $\varepsilon>0$, we put

$$V(x;\varepsilon)=\{y\in\mathcal{C}_x : y\in B_{\varepsilon^{-1}}(0,g_x)\setminus\overline{B_\varepsilon(0,g_x)} \text{ and } d(y,\mathcal{S}_x)>\varepsilon\},$$

where $B_R(o,g_x)$ denotes the geodesic ball of $(\mathcal{C}_x,g_x)$ centered at the vertex o and with radius R.

For any $\varepsilon>0$, whenever j is sufficiently large, there are diffeomorphisms

$$\phi_j\colon V\big(x;\tfrac{1}{4}\varepsilon\big)\longrightarrow M_\infty\setminus\mathcal{S}$$

satisfying

(1) $d(x,\phi_j(V(x;\varepsilon)))<10\varepsilon r_j$ and $\phi_j(V(x;\varepsilon))\subset B_{(1+\varepsilon^{-1})r_j}(x)$, where $B_R(x)$ is the geodesic ball of (M_∞,ω_∞) with radius R and center x;

(2) if g_∞ is the Kähler metric with Kähler form ω_∞ on $M_\infty\setminus\mathcal{S}$, then

$$\lim_{j\to\infty}\|r_j^{-2}\phi_j^* g_\infty - g_x\|_{C^6(V(x;\varepsilon/2))}=0, \tag{5.13}$$

where the norm is defined in terms of the metric g_x.

LEMMA 5.4. *For any δ sufficiently small, there exist a sufficiently large $\ell=k_j$ and an isomorphism ψ from the trivial bundle $\mathcal{C}_x\times\mathbb{C}$ onto $K_{M_\infty}^{-\ell}$ over $V(x;\varepsilon)$ commuting with $\phi=\phi_j$, satisfying*

$$|\psi(1)|_\infty^2=e^{-\varrho_x^2} \quad \textit{and} \quad \|\nabla\psi\|_{C^4(V(x;\varepsilon))}\leqslant\delta, \tag{5.14}$$

where $|\cdot|_\infty^2$ denotes the induced norm on $K_{M_\infty}^{-\ell}$ by $e^{-u_\infty/n}g_\infty$, and ∇ denotes the covariant derivative with respect to the norms $|\cdot|_\infty^2$ and $e^{-\varrho_x^2}|\cdot|^2$.

We refer the reader to [41] for a proof of the lemma. Actually, it is easier in our case since the singularity $\mathcal{S}_x$ is of complex codimension at least 2.

Let $\varepsilon>0$ and $\delta>0$ be sufficiently small and to be determined later. Choose ℓ, ϕ and ψ as in Lemma 5.4. Then there is a section $\tau=\psi(1)$ of $K_{M_\infty}^{-\ell}$ on $\phi(V(x;\varepsilon))$ satisfying

$$|\tau|_\infty^2=e^{-\varrho_x^2}.$$

By Lemma 5.4, for some uniform constant C, we have

$$|\bar\partial\tau|_\infty\leqslant C\delta.$$

Since $\mathcal{S}_x$ has codimension at least 4, we can easily construct a smooth function $\gamma_{\bar\varepsilon}$ on $\mathcal{C}_x$ for each $\bar\varepsilon>0$ with the following properties: $\gamma_{\bar\varepsilon}(y)=1$ if $d(y,\mathcal{S}_x)\geqslant\bar\varepsilon$, $0\leqslant\gamma_{\bar\varepsilon}\leqslant1$, $\gamma_{\bar\varepsilon}(y)=0$ in a neighborhood of $\mathcal{S}_x$ and

$$\int_{B_{\bar\varepsilon^{-1}}(o,g_x)}|\nabla\gamma_{\bar\varepsilon}|^2\omega_x^n\,dv\leqslant\bar\varepsilon.$$

Moreover, we may have $|\nabla\gamma_{\bar\varepsilon}|\leqslant C$ for some constant $C=C(\bar\varepsilon)$.

We define, for any $y\in V(x;\varepsilon)$,

$$\tilde\tau(\phi(y))=\eta(2\delta\varrho_x(y))\gamma_{\bar\varepsilon}(y)\tau(\phi(y)),$$

where η is a cut-off function satisfying

$$\eta(t)=\begin{cases}1, & \text{for } t\leqslant1,\\ 0, & \text{for } t\geqslant2,\end{cases} \quad \text{and} \quad |\eta'(t)|\leqslant1 \ \text{ for all } t\geqslant0.$$

Choose $\bar\varepsilon$ such that $V(x;\varepsilon)$ contains the support of $\gamma_{\bar\varepsilon}$, and $\gamma_{\bar\varepsilon}=1$ on $V(x;\delta_0)$, where $\delta_0>0$ will be determined later.

It is easy to see that $\tilde\tau$ vanishes outside $\phi(V(x;\varepsilon))$, so it extends to a smooth section of $K_{M_\infty}^{-\ell}$ on M_∞. Furthermore, $\tilde\tau$ satisfies

(i) $\tilde\tau=\tau$ on $\phi(V(x;\delta_0))$;

(ii) there is a $\nu=\nu(\delta,\varepsilon)$ such that

$$\int_{M_\infty}|\bar\partial\tilde\tau|_\infty^2\omega_\infty^n\,dv\leqslant\nu r^{2n-2}.$$

Note that we can make ν as small as we want so long as δ, ε and $\bar\varepsilon$ are sufficiently small.

Since $(M, g(t_i))$ converges to (M_∞, g_∞) and the Hermitian metrics $h(t_i)$ on $K_M^{-\ell}$ converge to h_∞ on $M_\infty\backslash\mathcal{S}$ in the C^∞-topology, there are diffeomorphisms

$$\tilde{\phi}_i: M_\infty\backslash\mathcal{S} \longrightarrow M$$

and smooth isomorphisms

$$F_i: K_{M_\infty}^{-\ell} \longrightarrow K_M^{-\ell}$$

over M, satisfying the following properties:

($\mathbf{C}_1$) $\tilde{\phi}_i(M_\infty\backslash N_{1/i}(\mathcal{S}))\subset M$, where $N_\varepsilon(\mathcal{S})$ is the ε-neighborhood of $\mathcal{S}$;

($\mathbf{C}_2$) $\pi_i\circ F_i=\tilde{\phi}_i\circ\pi_\infty$, where π_i and π_∞ are the projections corresponding to the line bundles $K_M^{-\ell}$ and $K_{M_\infty}^{-\ell}$, respectively;

($\mathbf{C}_3$) $\|\tilde{\phi}_i^* g(t_i)-g_\infty\|_{C^2(M_\infty\backslash T_{1/i}(\mathcal{S}))}\to 0$ as $i\to\infty$;

($\mathbf{C}_4$) $\|F_i^* h(t_i)-h_\infty\|_{C^4(M_\infty\backslash T_{1/i}(\mathcal{S}))}\to 0$ as $i\to\infty$.

Put $\tilde{\tau}_i=F_i(\tilde{\tau})$. Then we deduce from the above

(i) $\tilde{\tau}_i=F_i(\tau)$ on $\tilde{\phi}_i(\phi(V(x;\delta_0)))$;

(ii) for i sufficiently large, we have

$$\int_M |\bar{\partial}\tilde{\tau}_i|_i^2\, dV_{g(t_i)} \leqslant 2\nu r^{2n-2},$$

where $|\cdot|_i$ denotes the Hermitian norm corresponding to $h(t_i)$.

By the L^2-estimate in Lemma 5.3, we get a section v_i of $K_M^{-\ell}$ such that

$$\bar{\partial} v_i = \bar{\partial}\tilde{\tau}_i$$

and

$$\int_{M_\infty} |v_i|_i^2\, dV_{g(t_i)} \leqslant \frac{1}{\ell}\int_M |\bar{\partial}\tilde{\tau}_i|_i^2\, dV_{g(t_i)} \leqslant 3\nu r^{2n}.$$

Put $\sigma_i=\tilde{\tau}_i-v_i$, a holomorphic section of $K_M^{-\ell}$. One can show that the C^4-norm of $\bar{\partial} v_i$ on $\tilde{\phi}_i(\phi(V(x;\delta_0)))$ is bounded from above by $c\delta$, for a uniform constant c. By the standard elliptic estimates, we have

$$\sup_{\tilde{\phi}(\phi(V(x;2\delta_0)\cap B_1(o,g_x)))} |v_i|_i^2 \leqslant C(\delta_0 r)^{-2n}\int_{M_i} |v_i|_i^2\, dV_{g(t_i)} \leqslant C\delta_0^{-2n}\nu.$$

Here C denotes a uniform constant. For any given δ_0, if δ and ε are sufficiently small, then we can make ν such that

$$8C\nu \leqslant \delta_0^{2n}.$$

It follows that

$$|\sigma_i|_i \geqslant |F_i(\tau)|_i - |v_i|_i \geqslant \tfrac{1}{2} \quad \text{on } \tilde{\phi}_i(\phi(V(x;\delta_0)\cap B_1(o,g_x))).$$

On the other hand, by applying Lemma 5.2 to σ_i, we get

$$\sup_M |\nabla \sigma_i|_i \leqslant C' \ell^{(n+1)/2} \left(\int_M |\sigma_i|_i^2 \, dV_{g(t_i)} \right)^{1/2} \leqslant C' r^{-1}.$$

Since the distance $d(x, \phi(\delta_0 u))$ is less than $10\delta_0 r$ for some $u \in \partial B_1(o, g_x)$, if i is sufficiently large, we deduce from the above estimates that

$$|\sigma_i|_i(x_i) \geqslant \tfrac{1}{4} - C'\delta_0.$$

Hence, if we choose δ_0 such that $C'\delta_0 < \frac{1}{8}$, then $\varrho_{\omega_i,\ell}(x_i) > \frac{1}{8}$. Theorem 5.1, i.e., the partial C^0-estimate for $g(t_i)$ in the Kähler–Ricci flow, is therefore proved.

Using the same arguments as those in proving [41, Theorem 5.9], we can deduce Theorem 1.6 from Theorem 5.1.

6. A corollary of Conjecture 1.1

In this last section, we will show how to deduce the Yau–Tian–Donaldson conjecture in case of Fano manifolds from Conjecture 1.1. The key is to prove that there is a uniform lower bound for Mabuchi's K-energy along the Kähler–Ricci flow provided the partial C^0-estimate and K-stability of the manifold.

Let $\omega = \omega(t)$ be the Kähler form of the Kähler–Ricci flow $g = g(t)$. For Kähler metrics $\omega_1, \omega_2 \in 2\pi c_1$, denote by $K(\omega_1, \omega_2)$ the relative Mabuchi's K-energy from ω_1 to ω_2 (the function M in [23]).

Theorem 6.1. *Suppose that the partial C^0-estimate (5.4) holds for a sequence of times $t_i \to \infty$. If M is* K*-stable, then the* K*-energy is bounded from below under the Kähler–Ricci flow,*

$$K(\omega(0), \omega(t)) \geqslant -C(g_0). \tag{6.1}$$

Proof. It is well known that $K(\omega(0), \omega(t))$ is non-increasing in t (cf. [46]). So it is sufficient to show a uniform lower bound of $K(\omega(0), \omega_i)$, where $\omega_i = \omega(t_i)$. We will prove this by using a result of S. Paul [24], [25]: if M is K-stable, then the K-energy is bounded from below on the space of Bergman metrics which arise from the Kodaira embedding via bases of $K_M^{-\ell}$.

Fix an integer $\ell > 0$ sufficiently large such that $K_M^{-\ell}$ is very ample and M is K-stable with respect to $K_M^{-\ell}$. Any orthonormal basis $\{s_{t_i,\ell,k}\}_{k=0}^{N_\ell}$ of $H^0(M, K_M^{-\ell})$ at t_i defines an embedding

$$\Phi_i : M \longrightarrow \mathbb{C}P^{N_\ell}.$$

Let ω_{FS} be the Fubini–Study metric on $\mathbb{C}P^{N_\ell}$ and put $\widetilde{\omega}_i=\Phi_i^*\omega_{FS}/\ell$, the Bergman metric associated with Φ_i. For any $i\geqslant 1$, there exists a $\sigma_i\in\mathrm{SL}(N_\ell+1,\mathbb{C})$ such that $\Phi_i=\sigma_i\circ\Phi_1$. By the result of [25], we have

$$K(\widetilde{\omega}_1,\widetilde{\omega}_i)\geqslant -C,$$

where C is a uniform constant. By the cocycle condition of the K-energy,

$$K(\omega(0),\omega_i)+K(\omega_i,\widetilde{\omega}_i)=K(\omega(0),\widetilde{\omega}_i)=K(\omega(0),\widetilde{\omega}_1)+K(\widetilde{\omega}_1,\widetilde{\omega}_i)\geqslant -C.$$

Therefore, to show that $K(\omega(0),\omega_i)$ is bounded from below, we only need to get an upper bound of $K(\omega_i,\widetilde{\omega}_i)$.

Put $\tilde{\varrho}_i=\varrho_{t_i,\ell}/\ell$, where $\varrho_{t_i,\ell}$ is defined by (5.3) with $t=t_i$. Then

$$\omega_i=\widetilde{\omega}_i+\sqrt{-1}\partial\bar{\partial}\,\tilde{\varrho}_i.$$

The K-energy has the following explicit expression [39]:

$$K(\omega_i,\widetilde{\omega}_i)=\int_M \log\frac{\widetilde{\omega}_i^n}{\omega_i^n}\widetilde{\omega}_i^n+\int_M u(t_i)(\widetilde{\omega}_i^n-\omega_i^n)-\sum_{k=0}^{n-1}\frac{n-k}{n+1}\int_M \sqrt{-1}\partial\tilde{\varrho}_i\wedge\bar{\partial}\tilde{\varrho}_i\wedge\omega_i^k\wedge\widetilde{\omega}_i^{n-k-1},$$

where $u(t_i)$ is the Ricci potential at time t_i of the Kähler–Ricci flow. Thus,

$$K(\omega_i,\widetilde{\omega}_i)\leqslant\int_M \log\frac{\widetilde{\omega}_i^n}{\omega_i^n}\widetilde{\omega}_i^n+\int_M u(t_i)(\widetilde{\omega}_i^n-\omega_i^n).$$

By Perelman's estimate, we have $|u(t_i)|\leqslant C(g_0)$. It follows that

$$K(\omega_i,\widetilde{\omega}_i)\leqslant\int_M \log\frac{\widetilde{\omega}_i^n}{\omega_i^n}\widetilde{\omega}_i^n+C.$$

Finally, by using the partial C^0-estimate and applying the gradient estimate in Lemma 5.2 to each $s_{t_i,\ell,k}$, we have

$$\widetilde{\omega}_i\leqslant C(g_0)\omega_i.$$

This gives the desired upper bound of $K(\omega_i,\widetilde{\omega}_i)$, and consequently, a lower bound of $K(\omega(0),\omega_i)$. The proof is now completed. □

Theorem 6.1 implies that the limit M_∞ must be Kähler–Einstein (see [46] for an example). Then its automorphism group must be reductive as a corollary of the uniqueness theorem due to Berndtsson and Berman (see [4]). It follows that if M_∞ is not equal to M, then there is a $\mathbb{C}^*$-action $\{\sigma(s)\}_{s\in\mathbb{C}^*}\subset\mathrm{SL}(N_\ell+1,\mathbb{C})$ such that $\sigma(s)\Phi_1(M)$ converges to the embedding of M_∞ in $\mathbb{C}P^{N_\ell}$. This contradicts the K-stability, since the Futaki invariant of M_∞ vanishes. Hence, there is a Kähler–Einstein metric on $M=M_\infty$.

Remark 6.2. In fact, using a very recent result of Paul [26] and the same arguments as those in the proof of Theorem 6.1, we can prove directly that the K-energy is proper along the Kähler–Ricci flow, so the flow converges to a Kähler–Einstein metric on the same underlying Kähler manifold.

As a final remark, we outline a method of directly producing a non-trivial holomorphic vector field on M_∞ if it is different from M. Suppose that M_∞ is not isomorphic to M. Let $\lambda(t)$ be the smallest eigenvalue of the weighted Laplacian $\Delta_u=\Delta-g^{i\bar{j}}\partial_i u\partial_{\bar{j}}$ at time t, where $u=u(t)$ is the Ricci potential of $g(t)$ defined in (3.2). The weighted Poincaré inequality [20] shows that $\lambda(t)>1$. According to [51, Theorem 1.5], $\lambda(t_i)\to 1$ as $i\to\infty$. If we denote by $\theta_i=\theta(t_i)$ an eigenfunction of $\lambda(t_i)$, satisfying the normalization

$$\int_M |\theta_i|^2 e^{-u(t_i)}\,dv_{g(t_i)}=1,$$

then, by the Nash–Moser iteration, we have the following gradient estimate.

Lemma 6.3. *There exists* $C=C(g_0)$ *such that any eigenfunction* θ, *at any time* t, *satisfying*

$$g^{i\bar{j}}(\partial\partial_{\bar{j}}\theta-\partial_i u\partial_{\bar{j}}\theta)=\lambda\theta \tag{6.2}$$

has the gradient estimate

$$\|\bar\partial\theta\|_{C^0}+\|\partial\theta\|_{C^0}\leqslant C\lambda^{(n+1)/2}\|\theta\|_{L^2}. \tag{6.3}$$

It follows that θ_i converges to a non-trivial eigenfunction θ_∞ with eigenvalue 1 on the limit variety M_∞. By an easy calculation,

$$\int_M |\overline{\nabla}\overline{\nabla}\theta_i|^2 e^{-u(t_i)}\,dv_{g(t_i)}=\lambda(t_i)(\lambda(t_i)-1)\to 0.$$

Together with Perelman's C^0-estimate on u, we see that the gradient field of θ_∞ gives rise to a bounded holomorphic vector field on M_∞.

References

[1] Ache, A. G., On the uniqueness of asymptotic limits of the Ricci flow. Preprint, 2012. arXiv:1211.3387 [math.DG].

[2] Anderson, M. T., Convergence and rigidity of manifolds under Ricci curvature bounds. *Invent. Math.*, 102 (1990), 429–445.

[3] Berman, R. J., Boucksom, S., Essydieux, P., Guedj, V. & Zeriahi, A., Kähler–Einstein metrics and the Kähler–Ricci flow on log Fano varieties. Preprint, 2011. arXiv:1111.7158 [math.CV].

[4] Berndtsson, B., A Brunn–Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry. *Invent. Math.*, 200 (2015), 149–200.
[5] Besse, A. L., *Einstein Manifolds*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 10. Springer, Berlin–Heidelberg, 1987.
[6] Cao, H. D., Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds. *Invent. Math.*, 81 (1985), 359–372.
[7] Cheeger, J., Integral bounds on curvature elliptic estimates and rectifiability of singular sets. *Geom. Funct. Anal.*, 13 (2003), 20–72.
[8] Cheeger, J. & Colding, T. H., Lower bounds on Ricci curvature and the almost rigidity of warped products. *Ann. of Math.*, 144 (1996), 189–237.
[9] — On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.*, 46 (1997), 406–480.
[10] — On the structure of spaces with Ricci curvature bounded below. II. *J. Differential Geom.*, 54 (2000), 13–35.
[11] Cheeger, J., Colding, T. H. & Tian, G., On the singularities of spaces with bounded Ricci curvature. *Geom. Funct. Anal.*, 12 (2002), 873–914.
[12] Cheeger, J. & Yau, S. T., A lower bound for the heat kernel. *Comm. Pure Appl. Math.*, 34 (1981), 465–480.
[13] Chen, X., Donaldson, S. & Sun, S., Kähler–Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities. *J. Amer. Math. Soc.*, 28 (2015), 183–197.
[14] — Kähler–Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof. *J. Amer. Math. Soc.*, 28 (2015), 235–278.
[15] Chen, X. & Wang, B., Space of Ricci flows I. *Comm. Pure Appl. Math.*, 65 (2012), 1399–1457.
[16] Chow, B., Chu, S.-C., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F. & Ni, L., *The Ricci Flow*: *Techniques and Applications*. Part III. *Geometric-Analytic Aspects*. Mathematical Surveys and Monographs, 163. Amer. Math. Soc., Providence, RI, 2010.
[17] Colding, T. H. & Naber, A., Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications. *Ann. of Math.*, 176 (2012), 1173–1229.
[18] Dai, X. & Wei, G., A heat kernel lower bound for integral Ricci curvature. *Michigan Math. J.*, 52 (2004), 61–69.
[19] Donaldson, S. & Sun, S., Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry. *Acta Math.*, 213 (2014), 63–106.
[20] Futaki, A., *Kähler–Einstein Metrics and Integral Invariants*. Lecture Notes in Math., 1314. Springer, Berlin–Heidelberg, 1988.
[21] Grigor′yan, A., Gaussian upper bounds for the heat kernel on arbitrary manifolds. *J. Differential Geom.*, 45 (1997), 33–52.
[22] — Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. *Bull. Amer. Math. Soc.*, 36 (1999), 135–249.
[23] Mabuchi, T., K-energy maps integrating Futaki invariants. *Tohoku Math. J.*, 38 (1986), 575–593.
[24] Paul, S. T., Hyperdiscriminant polytopes, Chow polytopes, and Mabuchi energy asymptotics. *Ann. of Math.*, 175 (2012), 255–296.
[25] — A numerical criterion for K-energy maps of algebraic manifolds. Preprint, 2012. arXiv:1210.0924 [math.DG].
[26] — Stable pairs and coercive estimates for the Mabuchi functional. Preprint, 2013. arXiv:1308.4377 [math.AG].

[27] Perelman, G., The entropy formula for the Ricci flow and its geometric applications. Preprint, 2002. arXiv:math/0211159 [math.DG].

[28] Petersen, P., Convergence theorems in Riemannian geometry, in *Comparison Geometry* (Berkeley, CA, 1993–94), Math. Sci. Res. Inst. Publ., 30, pp. 167–202. Cambridge Univ. Press, Cambridge, 1997.

[29] Petersen, P. & Wei, G., Relative volume comparison with integral curvature bounds. *Geom. Funct. Anal.*, 7 (1997), 1031–1045.

[30] — Analysis and geometry on manifolds with integral Ricci curvature bounds. II. *Trans. Amer. Math. Soc.*, 353 (2001), 457–478.

[31] Phong, D. H., Song, J. & Sturm, J., Degeneration of Kähler–Ricci solitons on Fano manifolds. Preprint, 2012. arXiv:1211.5849 [math.DG].

[32] Rothaus, O. S., Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators. *J. Funct. Anal.*, 42 (1981), 110–120.

[33] Sesum, N., Convergence of a Kähler–Ricci flow. *Math. Res. Lett.*, 12 (2005), 623–632.

[34] Sesum, N. & Tian, G., Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman). *J. Inst. Math. Jussieu*, 7 (2008), 575–587.

[35] Shi, W.-X., Ricci deformation of the metric on complete noncompact Riemannian manifolds. *J. Differential Geom.*, 30 (1989), 303–394.

[36] Tian, G., On Calabi's conjecture for complex surfaces with positive first Chern class. *Invent. Math.*, 101 (1990), 101–172.

[37] — Kähler–Einstein metrics on algebraic manifolds, in *Proceedings of the International Congress of Mathematicians*, Vol. I (Kyoto, 1990), pp. 587–598. Math. Soc. Japan, Tokyo, 1991.

[38] — Kähler–Einstein metrics with positive scalar curvature. *Invent. Math.*, 130 (1997), 1–37.

[39] — *Canonical Metrics in Kähler Geometry*. Lectures in Mathematics ETH Zürich. Birkhäuser, Basel, 2000.

[40] — Existence of Einstein metrics on Fano manifolds, in *Metric and Differential Geometry*, Progr. Math., 297, pp. 119–159. Birkhäuser/Springer, Basel, 2012.

[41] — Partial C^0-estimate for Kähler–Einstein metrics. *Commun. Math. Stat.*, 1 (2013), 105–113.

[42] — K-stability and Kähler–Einstein metrics. *Comm. Pure Appl. Math.*, 68 (2015), 1085–1156.

[43] Tian, G., Zhang, S., Zhang, Z. & Zhu, X., Perelman's entropy and Kähler–Ricci flow on a Fano manifold. *Trans. Amer. Math. Soc.*, 365 (2013), 6669–6695.

[44] Tian, G. & Zhang, Z., Degeneration of Kähler–Ricci solitons. *Int. Math. Res. Not. IMRN*, 5 (2012), 957–985.

[45] — Regularity of the Kähler–Ricci flow. *C. R. Math. Acad. Sci. Paris*, 351 (2013), 635–638.

[46] Tian, G. & Zhu, X., Convergence of Kähler–Ricci flow. *J. Amer. Math. Soc.*, 20 (2007), 675–699.

[47] — Convergence of the Kähler–Ricci flow on Fano manifolds. *J. Reine Angew. Math.*, 678 (2013), 223–245.

[48] Yang, D., Convergence of Riemannian manifolds with integral bounds on curvature. I. *Ann. Sci. École Norm. Sup.*, 25 (1992), 77–105.

[49] Ye, R., The logarithmic Sobolev and Sobolev inequalities along the Ricci flow. *Commun. Math. Stat.*, 3 (2015), 1–36.

[50] Zhang, Q. S., A uniform Sobolev inequality under Ricci flow. *Int. Math. Res. Not. IMRN*, 17 (2007), Art. ID rnm056, 17 pp.

[51] Zhang, Z., Kähler Ricci flow on Fano manifolds with vanished Futaki invariants. *Math. Res. Lett.*, 18 (2011), 969–982.

Gang Tian
School of Mathematical Sciences
and Beijing International Center
for Mathematical Research
Peking University
Beijing 100871
China

and

Department of Mathematics
Princeton University
Princeton, NJ 08544
U.S.A.
tian@math.princeton.edu

Zhenlei Zhang
School of Mathematics
Capital Normal University
Beijing 100048
China
zhleigo@aliyun.com

Received October 22, 2013