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The convergence Newton polygon of a p-adic differential equation I: Affinoid domains of the Berkovich affine line

by

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Introduction

The fundamental work of Christol and Mebkhout [CM1]–[CM4], together with [A1], [Me], and [Ke1], achieved a program (firstly initiated by P. Robba and B. Dwork [Dw], [Ro1], [DR1], [Ro2], [CD], ...) concerning differential equations "*coming from rigid cohomology*". These differential equations have maximal radius of convergence at the "*generic point*", and have over-convergent coefficients.

This paper, and its sequel [NP2], deal with a more general program concerning locally free $\mathscr{O}_X$-modules with connection, over a rig-smooth K-analytic Berkovich curve X, with no conditions on the size of the radii of convergence.

In this context there is a lack of results of global nature in the sense of Berkovich. Even the case of an open disk is not well understood. The existing results on curves mainly concern differential modules over a field of power series at a rational point, or over the so-called Robba ring. From the point of view of Berkovich curves this means a germ of segment out of a point of type 1, 2, or 3.

The most basic, but central tool of the theory is the so-called *convergence Newton polygon* of a p-adic differential equation.(1) Roughly speaking the slopes of that polygon at $x\in X$ are the logarithms of the radii of convergence of the Taylor solutions at x, in increasing order, counted with multiplicity (cf. Definition 1 below). The continuity of the convergence Newton polygon, as a function on X, appears in this program as the

(1) We consider p-adic differential equations in the large sense of the word. This also covers the case of any ultrametric complete valued field K. In particular we treat the case of differential modules over the field of formal power series $K(\!(T)\!)$, where K is trivially valued. The formal Newton polygon (in the sense of B. Malgrange and J. P. Ramis) is in fact the derivative of the convergence Newton polygon (cf. Remark 3.12).

fundamental step, and the major tool in the classification of the equations, as illustrated (in the solvable case) by the work of G. Christol and Z. Mebkhout.

Moreover the convergence Newton polygon carries important numerical invariants of the equation, in analogy with the Swan conductor, that are *highly related to the residual wild ramification* in the spirit of [Mat], [Ts], [Cr], [A1], [Mar], [CP].

In the more global setting of Berkovich curves there is an additional geometrical datum furnished by the convergence Newton polygon: a graph $\Gamma\subseteq X$ called *controlling graph* of the differential equation. Roughly speaking Γ is a locally finite graph such that $X\setminus\Gamma$ is a disjoint union of virtual open disks on which the polygon is constant. So, by continuity, the behavior of the polygon as a function on X is determined by its restriction to Γ.

As an example, if $f\colon Y\to X$ is an étale morphism between rig-smooth K-analytic Berkovich curves, the controlling graph Γ of $f_*(\mathscr{O}_Y)$, and more precisely the derivative of the polygon as a function on Γ, is an invariant of the morphism, highly related to its residual wild ramification.

The main goals of this paper, and of its sequel [NP2] are the following:

(1) An unconditional definition of the convergence Newton polygon, based on [Ba], not involving formal models, and resulting completely *within the framework of Berkovich analytic spaces*;

(2) The *continuity* of each slope of the polygon, as a function on X;

(3) The *local finiteness* of the graph Γ.

In this paper, we focus on the case of affinoid domains of the affine line. A great part of the literature about p-adic differential equations is devoted to the affine line. So this case has its own interest, and it is important to treat it explicitly and completely. Point (1) is not really relevant in this setting, but we prove that points (2) and (3) hold, by using techniques from p-adic differential equations.

In [NP2], we extend the results to arbitrary smooth curves, by using techniques from Berkovich geometry to reduce to the case treated in this paper.

We prove that the controlling graph Γ of a differential equation is always a *locally finite graph* without particular assumptions (no solvability, no exponents, no Frobenius map, ...). This implies that the entire convergence Newton polygon is determined, as a function on X, by a *locally finite family of numbers* (finite in the case of this paper).

The continuity of the polygon (which is a consequence of the local finiteness of Γ) is the major ingredient for decomposition theorems of global nature [NP3]. The finiteness of Γ also represents the fundamental point permitting a computation of the de Rham cohomology of the equation. In particular the global finiteness of Γ is the crucial property that gives the finite-dimensionality of the de Rham cohomology of the differential equation [NP4]. These results were unknown even in the elementary case of a non-solvable differential equation over a disk or an annulus.

Essential ingredients are the work of K. S. Kedlaya about subsidiary radii [Ke3], and that of F. Baldassarri and L. Di Vizio about the generic radius of convergence [BD], [Ba], where the finiteness of Γ was originally conjectured. The work of Kedlaya is a determinant refinement of classical ideas (together with the introduction of the crucial notion of super-harmonicity), while Baldassarri's work is a change in perspective which opened up a whole new line of investigation. Namely, in several recent talks, Baldassarri conjectured that the radii should factorize through some unspecified *finite* graph that he baptized *controlling graph*. He also established a link between the graph and the cohomology, and suggested some partial idea of proofs. The conjecture was supported by effective computations obtained by Christol for rank-1 equations [Ch2] (cf. final notes in §7).

We now enter more specifically in the contents of this paper. In the introduction we assume for simplicity that the base field K is algebraically closed. Let X be an affinoid domain of the affine line, and let $x \in X$ be a Berkovich point. In order to define Taylor solutions "at x", we need to consider a field extension Ω/K where x becomes rational. Let $t \in X_\Omega$ be any rational point lifting x. The fiber $\pi^{-1}(x)$ of the projection $\pi_{\Omega/K}: X_\Omega \to X$ has a nice structure: it has a peaked point $\sigma_{\Omega/K}(x)$ (cf. [Be, p. 98]) with the property that $\pi^{-1}_{\Omega/K}(x) \setminus \{\sigma_{\Omega/K}(x)\}$ is a disjoint union of open disks, all having $\sigma_{\Omega/K}(x)$ as a relative boundary in X_Ω.

We call these disks *Dwork generic disks*. Up to further extensions of the ground field K, they are all isomorphic, and independent on X. We call $D(x)$ the one of them containing t.

We now introduce the *maximal disk* $D(x, X)$. This is the largest open disk in X_Ω containing t.

The topological structure of X is the following. The set of points without neighborhoods isomorphic to an open disk form a finite graph such that $X \setminus \Gamma_X$ is a disjoint union of open disks. All these disks are the maximal disks of their points. While the maximal disk of a point in Γ_X is by definition its Dwork generic disk $D(x)$.

Now fix a coordinate $T: X \to \mathbb{A}^{1,\mathrm{an}}_K$, and call $r(x)$ and $\varrho_{x,X}$ the radii of the generic disk $D(x)$ and of the maximal disk $D(x, X)$, respectively. Now let $\mathscr{F}$ be a locally free $\mathscr{O}_X$-module of finite rank r, endowed with a connection $\nabla: \mathscr{F} \to \mathscr{F} \otimes \Omega^1_X$.

Definition 1. Denote by $\mathcal{R}^{\mathscr{F}}_i(x) \leqslant \varrho_{x,X}$ the radius of the largest open disk centered at t_x, contained in $D(x, X)$, on which $\mathscr{F}$ has at least $r-i+1$ linearly independent solutions. We define the ith radius of convergence of $\mathscr{F}$ at x as

$$\mathcal{R}_i(x,\mathscr{F}) := \frac{\mathcal{R}_i^{\mathscr{F}}(x)}{\varrho_{x,X}} \leqslant 1. \tag{0.1}$$

The convergence Newton polygon of $\mathscr{F}$ is the polygon with slopes

$$\log \mathcal{R}_1(x,\mathscr{F}) \leqslant \ldots \leqslant \log \mathcal{R}_r(x,\mathscr{F}).$$

We let the ith *spectral radius* of $\mathscr{F}$ at x be the number

$$\mathcal{R}_i^{\mathrm{sp}}(x,\mathscr{F}) = \min\left\{\mathcal{R}_i(x,\mathscr{F}), \frac{r(x)}{\varrho_{x,X}}\right\}.$$

We say that the index i is *spectral*, *solvable*, or *over-solvable* at x if

$$\mathcal{R}_i(x,\mathscr{F}) \leqslant \frac{r(x)}{\varrho_{x,X}}, \quad \mathcal{R}_i(x,\mathscr{F}) = \frac{r(x)}{\varrho_{x,X}}, \quad \text{or} \quad \mathcal{R}_i(x,\mathscr{F}) > \frac{r(x)}{\varrho_{x,X}},$$

respectively.

The function $x \mapsto \min\{\mathcal{R}_1^{\mathscr{F}}(x), r(x)\}$ is the ancient definition of spectral radius, that one finds for example in [CD], while $\mathcal{R}_1^{\mathscr{F}}(x)$ is the radius of convergence of [BD]. The normalized definition (0.1) is that of [Ba], and it has the merit of being independent of the coordinate. For $i \geqslant 2$ the definition is due to Kedlaya [Ke3] (following Young [Y]).

Spectral radii are related to the spectral norm of the connection, their nature is hence quite algebraic. They are not continuous (cf. (4.6)). The novelty of [BD] and [Ba] consists in allowing over-solvable radii, and hence working with a more geometric notion.

The continuity results of [BD] and [Ba] are proved using the same ingredients of the original proof of [CD]: they are obtained as a consequence of a certain Dwork–Robba theorem [DR2], that gives a bound on the growth of the coefficients of the Taylor solution matrix. Unfortunately the Dwork–Robba bound is not helpful in the understanding of the ith radii for $i \geqslant 2$, because it does not apply to an individual solution, but only to the entire solution matrix.

Definition 2. (Cf. §2) Let $\mathcal{T}$ be a set, and let $F: X \to \mathcal{T}$ be a function. We define the *controlling graph* (also called *constancy skeleton*) of F as the set $\Gamma(F)$ formed by the points $x \in X$ without neighborhoods in X isomorphic to open disks on which F is constant.

We say that F is a *finite function* if $\Gamma(F)$ is a finite graph (i.e. it is a finite union of intervals).

The set $\Gamma(F)$ is always a graph containing the skeleton Γ_X of X. Moreover $X\setminus\Gamma(F)$ is a disjoint union of open disks. In particular there exists a canonical retraction $X\to\Gamma(F)$, which is continuous as soon as $\Gamma(F)$ is a finite graph. As an example one easily proves that for all $i=1,...,\mathrm{rank}(\mathscr{F})$ one has $\Gamma(\mathcal{R}_i^{\mathrm{sp}}(-,\mathscr{F}))=X$.

The following theorem is our main result.

THEOREM 3. (Cf. Theorem 3.9) *For all $i=1,...,r$ the graph $\Gamma(\mathcal{R}_i(-,\mathscr{F}))$ is finite, and the function $\mathcal{R}_i(-,\mathscr{F})\colon X\to]0,1]$ factorizes through the retraction $X\to\Gamma(\mathcal{R}_i(-,\mathscr{F}))$. As a consequence $\mathcal{R}_i(-,\mathscr{F})$ is a continuous function.*

The statement of Theorem 3.9 is more complex and complete. It assembles the main properties satisfied by the radii. It is structured in analogy with [Ke3, Theorem 11.3.2], where the same properties are stated for spectral radii. Roughly speaking we establish the following properties:

(1) Finiteness of each $\mathcal{R}_i(-,\mathscr{F})$, and of each partial height

$$H_i(-,\mathscr{F}):=\prod_{j=1}^{i}\mathcal{R}_j(-,\mathscr{F});$$

(2) Integrality of the slopes of each $\mathcal{R}_i(-,\mathscr{F})$ along the segments of X;

(3) Concavity locus of each $\mathcal{R}_i(-,\mathscr{F})$;

(4) Super-harmonicity of the partial heights $H_i(-,\mathscr{F})$ outside a (locally) finite subset $\mathscr{C}_i$;

(5) Description of $\mathscr{C}_i$.

We now give some ideas about the proof. Our approach is different in nature from that of [BD] and [Ba]. It basically consists in applying Frobenius push-forward to make the spectral radii small, and then read them on the coefficient of the operator in a cyclic basis by the theorem of Young [Y]. This is a well-known method (at least) since [CD]. The problem consists, in fact, in making this process global in the sense of Berkovich, and in particular in managing solvable or over-solvable radii for which the reduction of the radii by Frobenius push-forward fails.

The proof is based on a criterion (cf. §2.4) providing the finiteness of a real-valued function $F\colon X\to\mathbb{R}_{>0}$ satisfying six technical properties. One of them is the super-harmonicity outside a finite set $\mathscr{C}$, which is the crucial assumption of the criterion.

Now the proof of Theorem 3 consists in verifying the six assumptions of the criterion for $F=H_i(-,\mathscr{F})$. This is done by induction on i. Now, we are able to prove that the potential failure of the super-harmonicity of $H_i(-,\mathscr{F})$ can only happen at the end-points of some $\Gamma(\mathcal{R}_k(-,\mathscr{F}))$, with $k\leqslant i-1$. This ensures, by induction, that the locus $\mathscr{C}_i$ of non-super-harmonicity is a finite set.

The potential failure of the super-harmonicity for $i \geqslant 2$ is actually the major theoretical difference with the case $i=1$ (indeed the first radius $\mathcal{R}_1^{\mathscr{F}}$ is super-harmonic outside the Shilov boundary). This is one of the deeper difficulties of the paper.

The main points permitting us to deal with this are the super-harmonicity in the spectral non-solvable case (cf. Proposition 5.9, generalizing [Ke3, Theorem 11.3.2 (c)]), a description of the nature of the graphs around solvable points (cf. Lemma 6.5), and a concavity property of the radii generalizing the transfer principle for the first radius (cf. Proposition 6.2).

Remark 4. Recently similar results have been announced by Baldassarri and Kedlaya [Ke4] (cf. final notes in §7).

Independently, J. Poineau and A. Thuillier pointed out that, if a rig-smooth K-analytic curve X has no boundary, then the continuity of $\mathcal{R}_1(-, \mathscr{F})$ on X is a consequence of the super-harmonicity. This is now a theorem [PP].

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1. Notation

In this paper all rings are commutative with unit element. $\mathbb{R}$ is the field of real numbers, and $\mathbb{R}_{\geqslant 0} := \{r \in \mathbb{R} : r \geqslant 0\}$. For any field L we denote its algebraic closure by L^{alg}, by $L[T]$ the ring of polynomials with coefficients in L, and by $L(T)$ the fraction field of $L[T]$. If L is valued, $\hat{L}$ will be its completion.

Throughout the paper $(K, |\cdot|)$ will be a complete field of characteristic 0 with respect to an ultrametric absolute value $|\cdot| : K \to \mathbb{R}_{\geqslant 0}$, that is, satisfying $|1|=1$, $|ab|=|a||b|$, and $|a+b| \leqslant \max\{|a|, |b|\}$ for all $a, b \in K$, and $|a|=0$ if and only if $a=0$. We set

$$|K| := \{r \in \mathbb{R}_{\geqslant 0} : r = |t| \text{ for some } t \in K\}.$$

The semi-norm of a matrix will always mean the maximum of the semi-norms of its entries.

Let $E(K)$ be the category of isometric ring morphisms $(K, |\cdot|) \to (\Omega, |\cdot|)$. A morphism $\Omega \to \Omega'$ in $E(K)$ is an isometric ring morphism inducing the identity on K. For all

$\Omega,\Omega'\in E(K)$ there exists $\Omega''\in E(K)$ together with two morphisms $\Omega\subseteq\Omega''$ and $\Omega'\subseteq\Omega''$ of $E(K)$.

We refer to [Be] for the definition of Berkovich spaces. For any point x we denote by $\mathscr{H}(x)$ the residual field of x. By convention an open disk D always has finite radius. Similarly an open annulus $C=\{x\in\mathbb{A}^{1,\mathrm{an}}_K : R_1<|T-c|(x)<R_2\}$ always satisfies $0<R_1\leqslant R_2<\infty$. We recall that if $\Omega\in E(X)$, and if $D\subset\mathbb{A}^{1,\mathrm{an}}_\Omega$ is an open disk of radius R centered at $t\in\Omega$ we have

$$\mathscr{O}(D):=\Big\{\sum_{n\geqslant 0}a_n(T-t)^n : a_n\in\Omega \text{ and, for all } \varrho<R,\ \lim_{n\to\infty}|a_n|\varrho^n=0\Big\}. \qquad (1.1)$$

A *virtual disk* (resp. *annulus*) is a non-empty connected analytic domain of $\mathbb{A}^{1,\mathrm{an}}_K$ which becomes isomorphic to a union of disks (resp. annuli whose orientation is preserved by $\mathrm{Gal}(\widehat{K^{\mathrm{alg}}}/K)$) over $\widehat{K^{\mathrm{alg}}}$ (cf. [Du, 3.6.32 and 3.6.35]).

If K is algebraically closed, an affinoid domain X of the Berkovich affine line $\mathbb{A}^{1,\mathrm{an}}_K$ (cf. [Be, §2.2]) is a disjoint union of connected affinoid domains of the form

$$X=D^+(c_0,R_0)\setminus\bigcup_{i=1}^{n}D^-(c_i,R_i), \qquad (1.2)$$

where $D^+(c_0,R_0)$ (resp. $D^-(c_i,R_i)$) denotes the closed (resp. open) disk centered at c_0 (resp. c_i) with radius R_0 (resp. R_i), $c_0,\dots,c_n\in K$ satisfy $|c_i-c_0|\leqslant R_0$ for all i, and $0<R_1,\dots,R_n\leqslant R_0$. In order to avoid overlaps we implicitly assume that for $0<i<j\leqslant n$ we have $D^-(c_i,R_i)\cap D^-(c_j,R_j)=\varnothing$.

If K is general, an affinoid domain X of $\mathbb{A}^{1,\mathrm{an}}_K$ is the quotient of $X_{\widehat{K^{\mathrm{alg}}}}$ by $\mathrm{Gal}(\widehat{K^{\mathrm{alg}}}/K)$ (cf. [Be, Proposition 1.3.6]). Without loss of generality we will always assume that X is connected. So the holes of $X_{\widehat{K^{\mathrm{alg}}}}$ are permuted by $\mathrm{Gal}(\widehat{K^{\mathrm{alg}}}/K)$,(2) which acts isometrically. Hence we can speak about the holes of X, and of their radii $R_1,\dots,R_n$. And also about the larger virtual disk containing X whose radius is R_0. Such notation is fixed from now on.

We denote by $\mathscr{O}(X)$ the K-algebra of the global sections of X, and by ∂X its Shilov boundary.

1.0.1. For all $c\in K$, and all $\varrho\geqslant 0$, we denote by $x_{c,\varrho}\in\mathbb{A}^{1,\mathrm{an}}_K$ the semi-norm defined by

$$x_{c,\varrho}(f):=\sup_{n\geqslant 0}\left|\frac{f^{(n)}(c)}{n!}\right|_K\varrho^n,\qquad f\in K[T], \qquad (1.3)$$

(2) Including the holes of $X_{\widehat{K^{\mathrm{alg}}}}$ at ∞, i.e. the complements in $\mathbb{P}^{1,\mathrm{an}}_{\widehat{K^{\mathrm{alg}}}}$ of the disk $D^+(c_0,R_0)$.

where $f^{(n)}$ is the n-th derivative of f with respect to a coordinate T of X. This definition actually depends on T.

For all $\Omega\in E(K)$ we have a map

$$i_\Omega: X(\Omega) \longrightarrow X \tag{1.4}$$

associating to $t\in X(\Omega)$ the image $\pi_{\Omega/K}(t)$ of $t\in X_\Omega$ by the projection $\pi_{\Omega/K}: X_\Omega \to X$, where $X_\Omega := X\widehat{\otimes}_K \Omega$. If $x=i_\Omega(t)$ we say that $t\in X(\Omega)$ is a *Dwork generic point* for x. Each point $x\in X$ admits a canonical Dwork generic point $t_x\in X_{\mathscr{H}(x)}$. Indeed, by the canonical property of the cartesian diagram $X_{\mathscr{H}(x)}/\mathscr{H}(x)\to X/K$, the point $x: \mathscr{M}(\mathscr{H}(x))\to X$ lifts uniquely into a rational point $t_x: \mathscr{M}(\mathscr{H}(x))\to X_{\mathscr{H}(x)}$. However for a given field $\Omega\in E(K)$ we can have several embeddings $\mathscr{H}(x)\to\Omega$, and hence there is no canonical lifting of x in X_Ω.

1.0.2. More generally, for all $x\in\mathbb{A}_K^{1,\mathrm{an}}$ we let $\lambda_x(0):=x$ and for all $\varrho>0$ we set

$$\lambda_x(\varrho)(f) := \sup_{n\geqslant 0} x\left(\frac{f^{(n)}}{n!}\right)\varrho^n, \quad f\in K[T]. \tag{1.5}$$

One sees that $\lambda_x(\varrho)\in X$ if an only if x lies in the maximal virtual disk containing X and $\varrho\in I_x$, where I_x is either equal to $[0,R_0]$ if $x\in X$, or $I_x=[R_i,R_0]$ if x lies in a hole of X with radius R_i.

It follows from the definition that if $t\in X(\Omega)$ is a Dwork generic point for x, then $\lambda_x(\varrho)=\pi_{\Omega/K}(x_{t,\varrho})$. In particular, the path $\varrho\mapsto\lambda_x(\varrho): I_x\to X$ is continuous.

We call *generic radius* of x the number

$$r_K(x) := \max\{\varrho\in[0,R_0] : \lambda_x(\varrho)=\lambda_x(0)\}. \tag{1.6}$$

We write $r(x):=r_K(x)$ if no confusion is possible.

LEMMA 1.1. *Let $x\in\mathbb{A}_K^{1,\mathrm{an}}$, and let $t\in X(\Omega)$ be a Dwork generic point for x. Assume that $K^{\mathrm{alg}}\subset\Omega$. Then $r_K(x)$ equals the distance of t from K^{alg}, i.e.*

$$r_K(x) = \inf_{c\in K^{\mathrm{alg}}} |t-c|_\Omega. \tag{1.7}$$

Proof. Let $d_t := \inf_{c\in K^{\mathrm{alg}}}|t-c|_\Omega$. The norm of a polynomial $f\in K[T]$ is constant on each disk without zeros of f, and thus $|f(y)|=|f(t)|$ for all $y\in D^-(t,d_t)\subset\mathbb{A}_\Omega^{1,\mathrm{an}}$. Hence $\lambda_x(d_t)=\lambda_x(0)$ and $d_t\leqslant r(x)$. To show that $r(x)\leqslant d_t$ observe that the norm of a polynomial f is not constant on a disk containing a zero of f. So $D^-(t,r(x))$ has no K^{alg}-rational points. □

COROLLARY 1.2. *The canonical path λ_x is constant on $[0,r(x)]$, and it induces a homeomorphism between $[r(x),R_0]$ and its image in X.* □

COROLLARY 1.3. *Let $t\in\Omega\in E(K)$ be a Dwork generic point for x. Then for all $\Omega'\in E(\Omega)$ each Ω'-rational point of $D^-(t,r(x))$ is a Dwork generic point for x.* □

1.0.3. The following proposition describes the structure of the fiber $\pi_{\Omega/K}^{-1}(x)$ of a point $x\in X$.

PROPOSITION 1.4. *Assume that the field K is algebraically closed. Let $\Omega\in E(K)$, let $\pi_{\Omega/K}\colon X_\Omega\to X$ be the canonical projection, and let $x\in X$. Then there exists a point $\sigma_{\Omega/K}(x)\in\pi_{\Omega/K}^{-1}(x)$ such that*

$$\pi_{\Omega/K}^{-1}(x)\setminus\{\sigma_{\Omega/K}(x)\} \tag{1.8}$$

is a (possibly empty) disjoint union of open disks, all having $\sigma_{\Omega/K}(x)$ as relative boundary in X_Ω.

Moreover, if Ω/K is algebraically closed and spherically complete, we have that the group $\mathrm{Gal}^{\mathrm{cont}}(\Omega/K)$, of K-linear continuous automorphisms of Ω, fixes $\sigma_{\Omega/K}(x)$ and acts transitively on those disks, and also on the set $i_\Omega^{-1}(x)$ of their Ω-rational points.

Proof. We may assume that Ω is algebraically closed. Let $t\in i_\Omega^{-1}(x)$. By Corollary 1.3 one has $D^-(t,r(x))\subseteq\pi_{\Omega/K}^{-1}(x)$. It is then enough to show that all $t'\in i_\Omega^{-1}(x)$ satisfies $|t'-t|\leqslant r(x)$.

This follows from the fact that $\pi_{\Omega/K}^{-1}(x)$ is the spectrum $\mathscr{M}(\mathscr{H}(x)\widehat{\otimes}_K\Omega)$, so it is contained in all affinoid domains containing x. Hence we can replace X by any K-rational closed disk in $\mathbb{A}_K^{1,\mathrm{an}}$ containing x. Now by Lemma 1.1 we can find a sequence of closed K-rational disks with intersection $\mathrm{D}^+(t,r(x))$. This proves the claim.

The assertion about the Galois action follows from Lemma 1.5 below. □

LEMMA 1.5. *Let $x\in X$. If $\Omega\in E(K)$ is algebraically closed and maximally complete, then $i_\Omega^{-1}(x)$ is either the empty set, or $\mathrm{Gal}^{\mathrm{cont}}(\Omega/K)$ acts transitively on it.*

Namely for all $t,t'\in i_\Omega^{-1}(x)$ there is $\sigma\in\mathrm{Gal}^{\mathrm{cont}}(\Omega/K)$ such that $\sigma(t)=t'$.

Proof. Identify $\mathbb{A}_K^{1,\mathrm{an}}(\Omega)$ with Ω, and $X(\Omega)$ with a subset of Ω. With this identification we have to find an automorphism of Ω sending t into t'. Let $\widehat{K(t)}$ and $\widehat{K(t')}$ be the completions of the sub-fields of Ω generated by t and t'. We consider the K-isomorphism $K(t)\xrightarrow{\sim}K(t')$ sending t into t'. Since $x=\pi_{\Omega/K}(x_{t,0})=\pi_{\Omega/K}(x_{t',0})$, these semi-norms coincide on $K[T]\subset\mathscr{O}(X_\Omega)$. Hence this K-isomorphism is isometric, and $\widehat{K(t)}\cong\mathscr{H}(x)\cong\widehat{K(t')}$. More precisely, there exists a continuous isometric K-linear isomorphism $\sigma\colon\widehat{K(t)}\xrightarrow{\sim}\widehat{K(t')}$ such that $\sigma(t)=t'$. Now σ extends to an isometric automorphism of Ω/K (cf. [DR1, Lemma 8.3], [MR], [Po]; see [NP2] for more details). □

Definition 1.6. (Generic and maximal disks) Let $x\in X$, let $\Omega\in E(\mathscr{H}(x))$, and let $t\in X(\Omega)$ be a Dwork generic point for $x\in X$. We call *generic disk* of x the virtual open disk

$$D(x)\subset X_\Omega \tag{1.9}$$

which is the connected component of $\pi^{-1}_{\Omega/K}(x)\setminus\{\sigma_{\Omega/K}(x)\}$ containing the point t. Its radius is $r(x)$.

We define the *maximal disk*

$$D(x,X)\subset X_\Omega \tag{1.10}$$

of x to be the maximum virtual open disk in X_Ω containing t. It is also the connected component of $X\setminus\Gamma_{X_\Omega}$ containing t. With the notation of (1.12), its radius is $\varrho_{\Gamma_X}(x)$, and it will be denoted by

$$\varrho_{x,X}=\varrho_{\Gamma_X}(x). \tag{1.11}$$

By Lemma 1.5, up to extensions of Ω, all generic and maximal disks are isomorphic. The notation does not depend on t, and the definitions and results of this paper will be independent on its choice.

Lemma 1.7. *One has* $r(x_{t,\varrho})=\max\{\varrho,r(x_t)\}$. *In particular, if* $t\in X(\widehat{K^{\mathrm{alg}}})$, *then* $r(x_{t,\varrho})=\varrho$. □

1.1. Graphs

As a topological space X is a tree, in particular it is uniquely arcwise connected.(3) If $x,y\in X$ we denote by $[x,y]\subset X$ the image of an injective continuous path $[0,1]\to X$ with initial point x and end-point y. In particular the image of $\lambda_x\colon I_x\to X$ is the closed segment $\Lambda(x):=[x,x_{c_0,R_0}]$.(4) We define in an evident way open and semi-open segments, denoted by $]x,y[$, $[x,y[$, and $]x,y]$.

Following [Du] we say that a graph Γ in X is *admissible* if $X\setminus\Gamma$ is a disjoint union of virtual open disks, in particular Γ is closed in X, and also connected (since we assume that X is connected).

An example of an admissible graph is the analytic skeleton $\Gamma_X\subseteq X$ defined as the locus of points without open neighborhoods in X isomorphic to virtual open disks. More explicitly Γ_X is the union of the segments $\Lambda(x)$ for all points x at the boundary of a hole of X (i.e. for all x in the Shilov boundary ∂X). Γ_X is also the set of semi-norms

(3) This means that for all $x,y\in X$ there exists an injective continuous path $[0,1]\to X$ with initial point x and end-point y. Moreover two such paths have the same image in X.

(4) By abuse here and below we identify $x_{c_0,R_0}\in X_{\widehat{K^{\mathrm{alg}}}}$ with its image in X.

on $\mathscr{O}(X)$ that are maximal with respect to the partial order given by $x \leqslant x'$ if and only if $x(f) \leqslant x'(f)$ for all $f \in \mathscr{O}(X)$.

For any subset $A \subseteq X$ we set $\mathrm{Sat}(A) := \bigcup_{x \in A} \Lambda(x)$. This is a tree in X. As an example $\Gamma_X = \mathrm{Sat}(\partial X)$. We say that a subset of X is *saturated* if it coincides with $\mathrm{Sat}(A)$, for some set A.

LEMMA 1.8. *A graph $\Gamma \subset X$ is admissible if and only if the following conditions hold*:
(i) $\Gamma_X \subseteq \Gamma$;
(ii) $\Gamma = \mathrm{Sat}(\Gamma)$;
(iii) Γ *contains its end-points.* □

Definition 1.9. Let Γ be a non-empty saturated subset and let $x \in X$. We set

$$\varrho_\Gamma(x) := \inf\{\varrho \geqslant r(x) : \lambda_x(\varrho) \in \Gamma\} \quad \text{and} \quad \delta_\Gamma(x) := \lambda_x(\varrho_\Gamma(x)). \tag{1.12}$$

The map $\delta_\Gamma : X \to X$ is a retraction onto the graph $\bar{\Gamma}$ obtained from Γ by adding its end-points. In particular, δ_Γ induces the identity on $\bar{\Gamma}$ and $\delta_\Gamma(X) = \bar{\Gamma}$. We call $\delta_\Gamma : X \to \bar{\Gamma}$ the *canonical retraction*.

If Γ is admissible, then each point $x \in X \setminus \Gamma$ lies in a virtual open disk D_x with boundary in Γ, and δ_Γ associates to x that boundary. If Γ is finite, admissible, and endowed with the topology induced by X, then $\delta_\Gamma : X \to \Gamma$ is continuous, and the topology of Γ is also the quotient topology by δ_Γ.

Remark 1.10. If $t_x \in X(\Omega)$ is a Dwork generic point for x, then the radius $\varrho_{x,X}$ of $D(x, X)$ satisfies $\varrho_{x,X} = \varrho_{t,X_\Omega} := \min\{\min_{i=1,\dots,n} |t - c_i|_\Omega, R_0\}$. We notice that if $x \leqslant x'$, then $\varrho_{x,X} = \varrho_{x',X}$. In fact the inequality $x \leqslant x'$ applied to $T - c_i$ and $(T - c_i)^{-1}$ provides $|t_x - c_i| = |t_{x'} - c_i|$.

Remark 1.11. For all $t \in X(\Omega)$ and all $\sigma \geqslant 0$ one has $\varrho_{x_{t,\sigma},X} = \max\{\sigma, \varrho_{t,X}\}$.

1.2. Directions, slopes, directional finiteness, and harmonicity

We define an equivalence relation between the open segments $]x, y[$ with $x \in X$ in their boundary. We say that $]x, y[\sim]x, z[$ if there exists $]x, t[\subseteq]x, z[\cap]x, y[$. An equivalence class b is called a *germ of segment out of* x, or *direction*, or again a *branch*.

We denote by $\Delta_X(x)$, or simply by $\Delta(x)$ if no confusion is possible, the set of all directions out of x, and if Γ is a graph containing x, we denote by $\Delta(x, \Gamma)$ the set of germs of segment out of x that are contained in Γ. If $\Delta(x, \Gamma)$ is a finite set we say that Γ is *directionally finite* at x.

Let $x\in X$ and let $b=]x,y[$ be a germ of segment out of x. We will always provide b with the orientation as outside x. Clearly, if y is close to x, then either

$$]x,y[\subset\Lambda(x)=[x,x_{c_0,R_0}] \quad\text{or}\quad]x,y[\subset\Lambda(y)=[y,x_{c_0,R_0}].$$

Assume that $b\subset\Lambda(x)$, and let $I\subset\mathbb{R}_{>0}$ be the inverse image of $]x,y[$ in $I_x=[0,R_0]$ (cf. §1.0.2). We recall that, for all $x\in X$, the path

$$\lambda_x\colon[0,R_0]\longrightarrow[x,x_{c_0,R_0}]\subset X \tag{1.13}$$

is continuous, it is constant on $[0,r(x)]$ with value x, and it identifies $[r(x),R_0]$ with $[x,x_{c_0,R_0}]$. So $I=]r(x),\varrho[$ for some $\varrho>r(x)$.

With these conventions let $F\colon X\to\mathbb{R}_{>0}$ be such that $\log\circ F\circ\lambda_x\circ\exp\colon\log(I)\to\mathbb{R}$ is an affine function. We say that F is log-affine along $b=]x,y[$ and we set

$$L_xF:=\log\circ F\circ\lambda_x\circ\exp\colon]-\infty,\log R_0]\longrightarrow\mathbb{R}. \tag{1.14}$$

We say that L_xF is the log-function attached to F. We denote its slope by $\partial_bF(x)$, this is the right derivative of $\log\circ F\circ\lambda_x\circ\exp$ at $\log r(x)$. If now $b=]x,y[\subset\Lambda(y)$ we call I the inverse image of $]x,y[$ in I_y, and we denote by $\partial_bF(x)$ the negative of the slope of $\log\circ F\circ\lambda_y\circ\exp\colon\log(I)\to\mathbb{R}$.

Definition 1.12. If $]z,u[\subset\Lambda(x)=[x,x_{c_0,R_0}]$, we say that F is *log-affine* (resp. *log-concave*, *log-decreasing*, etc.) along $]z,u[$, if L_xF is affine (resp. concave, decreasing, etc.) over $(\lambda_x\circ\exp)^{-1}(]z,u[)$.

Notation 1.13. Assume that F is log-affine along all directions $b\in\Delta(x)$ out of $x\in X$, and that $\partial_bF(x)=0$ for all but a finite number of them.

Definition 1.14. The *Laplacian* of F at x is the finite sum

$$dd^cF(x)=\sum_{b\in\Delta(x)}m_b\partial_bF(x), \tag{1.15}$$

where $m_b\in\mathbb{N}$ is the multiplicity of b (i.e. the number of germs of segment in $X_{\widehat{K^{\mathrm{alg}}}}$ lying over b).

If now $x\notin\partial X$, we say that F is *super-harmonic* (resp. *sub-harmonic*, *harmonic*) at x if

$$dd^cF(x)\leqslant 0 \quad (\text{resp. } dd^cF(x)\geqslant 0,\ dd^cF(x)=0). \tag{1.16}$$

We say that F is (globally) *super-harmonic* (resp. *sub-harmonic*, *harmonic*) on X, if it is so at every point $x\notin\partial X$.

LEMMA 1.15. *Let $x \in X \setminus \partial X$, and let $F, G\colon X \to \mathbb{R}$ be two functions on X as in Notation 1.13. Assume that $F|_b \leqslant G|_b$ along each germ of segment b out of x, that $F(x) = G(x)$, and that G is super-harmonic at x. Then F is super-harmonic at x.* □

Remark 1.16. The Laplacian of F at the points of ∂X does not give information since some directions out of x are "*removed*". As an example, functions $f \in \mathscr{O}(X)$ are harmonic, but their Laplacian at the points $x \in \partial X$ of the boundary is not always negative.

Definition 1.14 is less general than the usual definition of super-harmonicity, as for example those in [FJ], [Th] and [BR]. The general definition allows for an infinite number of directions of non-zero slope and the finite sum (1.16) is replaced by an infinite one.

2. Constancy skeleton of a function on X

Let $\mathcal{T}$ be a set and let $F\colon X \to \mathcal{T}$ be an arbitrary function.

Definition 2.1. We define the *controlling graph* (also called *constancy skeleton*)

$$\Gamma(X, F) \subseteq X \tag{2.1}$$

of F as the set of points of X without neighborhoods in X isomorphic to an open virtual disk on which F is constant. We write $\Gamma(F)$ if no confusion is possible.

Definition 2.2. (Constancy radius) For all $x \in X$ let $t_x \in X_{\mathscr{H}(x)}$ be the canonical point of §1.0.1. We define the *constancy radius* $\varrho_F(x) := \varrho_{\Gamma(F)}(x)$ of F at x as the radius of the largest open disk in $X_{\mathscr{H}(x)}$ centered at t_x on which the composite map $F_{\mathscr{H}(x)}\colon X_{\mathscr{H}(x)} \to X \to \mathcal{T}$ is constant.

2.1. Basic properties

Since $D(x) = D^-(t_x, r(x)) \subset \pi^{-1}_{\mathscr{H}(x)/K}(x)$, from the definition one immediately has

$$r(x) \leqslant \varrho_F(x) \leqslant \varrho_{x,X} \leqslant R_0. \tag{2.2}$$

LEMMA 2.3. *The following conditions are equivalent*:

(i) $x \in \Gamma(F)$;

(ii) $r(x) = \varrho_F(x)$;

(iii) *there exists $y \in X$ such that $x = \lambda_y(\varrho_F(y)) \in X$.*

Proof. If $x \in \Gamma(F)$, then $\varrho_F(x) = r(x)$, because if $r(x) < \varrho_F(x)$, the image in X of $D^-(t_x, \varrho_F(x))$ is a virtual disk containing x on which F is constant. Hence (i) ⇒ (ii). Now $x = \lambda_x(r(x))$, so (ii) ⇒ (iii). Assume now (iii). If $D \subseteq X$ is a virtual open disk containing x on which F is constant, then $y \in D$ and $D_{\mathscr{H}(y)} \subseteq D^-(t_y, \varrho_F(y))$. Hence we obtain the contradiction $x \neq \lambda_y(\varrho_F(y))$. So (iii) ⇒ (i). □

Lemma 2.4. *Let* $F: X \to \mathcal{T}$ *and* $F': X \to \mathcal{T}'$ *be two functions. We have* $\Gamma(F)=\Gamma(F')$ *if and only if* $\varrho_F(x)=\varrho_{F'}(x)$ *for all* $x\in X$. □

Proposition 2.5. $\Gamma(F)$ *is an admissible graph in* X. *Moreover it satisfies the following properties*:

(i) $x\in\Gamma(F)$ *if and only if* $\varrho_F(x)=r(x)$;

(ii) $\lambda_x(\varrho_F(x))\in\Gamma_X$ *if and only if* $\varrho_F(x)=\varrho_{x,X}$;

(iii) $\varrho_F(x)=\varrho_{\Gamma(F)}(x)$ *for all* $x\in X$ (*cf.* (1.12));

(iv) *for all* $x\in X$, *and all* $\varrho\in[0,R_0]$ *one has* $\varrho_F(\lambda_x(\varrho))=\max\{\varrho,\varrho_F(x)\}$;

(v) *if* F *is constant on a virtual open disk* $D\subset X$, *then* $D\cap\Gamma(F)$ *is empty.*

Proof. We may assume $K=\widehat{K^{\mathrm{alg}}}$. By definition, $\Gamma(F)$ is the complement of a union of disks, and so it is admissible. Point (i) follows from Lemma 2.3, and property (v) is evident.

Now (ii), (iii), and (iv) are straightforward. □

For all $x\in X$ we set $\delta_F(x):=\delta_{\Gamma(F)}(x)=\lambda_x(\varrho_F(x))$.

We say that F is *finite* if $\Gamma(F)$ is a finite graph. In this case $\delta_F: X\to\Gamma(F)$ is a continuous retraction (cf. comment after Definition 1.9).

Remark 2.6. The correspondence $F\mapsto\delta_F$ is idempotent: $\delta_{\delta_F}=\delta_F$. More precisely, if $\Gamma\subseteq X$ is a saturated subset, $\bar\Gamma$ denotes its closure in X, and $F=\delta_\Gamma: X\to\bar\Gamma$ is its retraction, then

$$\delta_{\delta_\Gamma}=\delta_{\Gamma\cup\Gamma_X}=\delta_{\bar\Gamma\cup\Gamma_X}. \tag{2.3}$$

Every admissible graph Γ is the skeleton of its retraction map δ_Γ (i.e. $\Gamma=\Gamma(\delta_\Gamma)$).

Remark 2.7. Let $F_i: X\to\mathcal{T}_i$, $i=1,2$, and let $g:\mathcal{T}_1\times\mathcal{T}_2\to\mathcal{T}_3$ be any functions. Then

$$\Gamma(g\circ(F_1\times F_2))\subseteq\Gamma(F_1)\cup\Gamma(F_2). \tag{2.4}$$

Indeed clearly $\varrho_{F_3}(x)\geqslant\min\{\varrho_{F_1}(x),\varrho_{F_2}(x)\}$, and $\Gamma(F_1)\cup\Gamma(F_2)$ is saturated.

If $\mathcal{T}_i=\mathbb{R}$, this holds in particular for $\max\{F_1,F_2\}$ and $\min\{F_1,F_2\}$.

Remark 2.8. Let $X'\subseteq X$ be a sub-affinoid, and let $F': X'\to\mathcal{T}$ be the restriction of $F: X\to\mathcal{T}$ to X'. In order to avoid confusion we denote by $\Gamma(X,F)\subseteq X$, $\Gamma(X',F')\subset X'$, $\varrho_F(X,-)$, and $\varrho_{F'}(X',-)$ the respective skeletons and constancy radii. If $x'\in X'$ one has $\varrho_{F'}(X',x')=\min\{\varrho_F(X,x'),\varrho_{x',X'}\}$, so

$$\Gamma(X',F')=(\Gamma(X,F)\cap X')\cup\Gamma_{X'}. \tag{2.5}$$

Hence the directional finiteness of F at $x'\in X'$ is equivalent to that of F' at x'. Moreover the finiteness of F on X implies that of F' on X'.

PROPOSITION 2.9. (Scalar extension) *Let $\Omega \in E(K)$ and let as usual $\pi_{\Omega/K}: X_\Omega \to X$ be the canonical projection. Denote by $F_\Omega: X_\Omega \to \mathcal{T}$ the composite map $F \circ \pi_{\Omega/K}$. One has $\Gamma(F) = \pi_{\Omega/K}(\Gamma(F_\Omega))$. Moreover, if K is algebraically closed, then $\pi_{\Omega/K}$ induces a bijection between $\Gamma(F_\Omega)$ and $\Gamma(F)$ with inverse $\sigma_{\Omega/K}$ (cf. Proposition 1.4). In particular F is finite if and only if F_Ω is finite.*

Proof. We have $X = X_{\widehat{K^{\mathrm{alg}}}}/G$, where $G = \mathrm{Gal}(K^{\mathrm{alg}}/K)$. Hence $\Gamma(F) = \Gamma(F_{\widehat{K^{\mathrm{alg}}}})/G$. As a consequence, F is a finite function if and only if $F_{\widehat{K^{\mathrm{alg}}}}$ is. So we may assume that K is algebraically closed. By Proposition 1.4, there is an open disk containing $x \in X$ on which F is constant if and only if there is an open disk containing $\sigma_{\Omega/K}(x) \in X_\Omega$ on which F_Ω is constant. The claim follows. □

Remark 2.10. (i) Let $F = \mathrm{Id}_X: X \to X$ be the identity, then $\Gamma(\mathrm{Id}_X) = \Gamma(r_K) = X$ (cf. definition (1.6)).

(ii) Let $F = 1: X \to \{\mathrm{pt}\}$ be a constant map. Then $\Gamma(1) = \Gamma(\varrho_{-,X}) = \Gamma_X$ is the skeleton of X.

(iii) Let $f_1, ..., f_n \in \mathscr{O}(X)$, let $\alpha_1, ..., \alpha_n > 0$, and define $F(x) := \min_i |f_i(x)|^{\alpha_i}$. Then $\Gamma(F) = \mathrm{Sat}(\{z_1, ..., z_r\}) \cup \Gamma_X$, where $\{z_1, ..., z_r\} \subset X(K^{\mathrm{alg}})$ is the union of all the zeros of $f_1, ..., f_n$.

(iv) With the above notation, if $F(x) := \max_i |f_i(x)|^{-\alpha_i}$, intended as a function with values in the set $\mathcal{T} := \mathbb{R}_{>0} \cup \{\infty\}$, then one again has $\Gamma(F) = \mathrm{Sat}(\{x_{z_1}, ..., x_{z_r}\}) \cup \Gamma_X$.

(v) Assume now that $F(x) := \max_i |f_i(x)|^{\alpha_i}$ (resp. $F(x) := \min_i |f_i(x)|^{-\alpha_i}$ as a function with values in $\mathcal{T} := \mathbb{R}_{>0} \cup \{\infty\}$). In this case the explicit description of the skeleton $\Gamma(F)$ is more complicate. However, one can easily deduce its finiteness from Remark 2.7.

2.2. Branch continuity and dag-skeleton

We investigate now whether the function F admits a factorization as $F = F|_{\Gamma(F)} \circ \delta_F$:

$$\begin{array}{ccccc} & & X & \xrightarrow{\ F\ } & \mathcal{T} \\ & \nearrow & & \searrow^{\delta_F} & \nearrow_{F|_{\Gamma(F)}} \\ \Gamma(F) & & = & & \Gamma(F). \end{array} \tag{2.6}$$

This is not automatically satisfied. In fact, for a given $x \in X$, the restriction

$$F \circ \lambda_x: [0, R_0] \longrightarrow \mathcal{T}$$

is constant for $\varrho \in [0, \varrho_F(x)[$, but one may have a different value at $\varrho = \varrho_F(x)$.

We say that F is *branch continuous* if for all $x\in X$ one has

$$F(\lambda_x(\varrho_F(x)))=\lim_{\varrho\to\varrho_F(x)^-}F(\lambda_x(\varrho))=F(x). \tag{2.7}$$

A branch continuous map factorizes as $F=F|_{\Gamma(F)}\circ\delta_F$ and is determined by its values on $\Gamma(F)$.

A continuous function with values in a Hausdorff space $\mathcal{T}$ is branch continuous. Conversely a finite and branch continuous function is continuous if and only if its restriction to $\Gamma(F)$ is continuous.

For some purposes this situation may be unsatisfactory since we want to factorize all functions. For this we define the *dag-skeleton* $\Gamma(F)^\dagger$ as the union of $\Gamma(F)$ together with an (unspecified) germ of segment out of all points x of $\Gamma(F)$, for all directions $b\in\Delta(x)$. Clearly, any function F factorizes through its dag-skeleton $\Gamma(F)^\dagger$. This situation will not occur in this paper since all the functions will be branch continuous. This idea can be better expressed in term of Huber spaces [H], indeed germs of segment correspond to Huber points, but this lies outside the scope of this paper.

2.3. Minimal triangulation

Assume that K is algebraically closed. We denote by S_X the union of the Shilov boundary ∂X and of the bifurcation points of Γ_X. Since X is of the form (1.2), we explicitly have

$$S_X:=\{x_{c_i,R_i}:i=0,...,n\}_{i=0,\dots,n}\cup\{x_{c_i,|c_i-c_j|}:i,j=1,...,n,i\neq j\}. \tag{2.8}$$

If K is general, define S_X as the image of $S_{X_{\widehat{K^{\mathrm{alg}}}}}$ by the projection. We notice that all points of S_X are of type 2 or 3, and that $X\setminus S_X$ is a disjoint union of virtual open disks or annuli that are relatively compact in X. This is called a *triangulation* of X in [Du], and it is related to the existence of a formal model of X. More precisely, S_X is the minimum triangulation of X.

2.4. A criterion for the finiteness of a positive real-valued function *F*

Let as usual X be an affinoid domain of the affine line. Let

$$F: X\longrightarrow\mathbb{R}_{>0} \tag{2.9}$$

be a positive function. We know that L_xF is constant at least on $]-\infty,\log r(x)]$ (cf. §1.2).

Let $\Gamma\subseteq X$ be a finite admissible graph. We consider the following conditions:

(C1) For all $x\in X(\widehat{K^{\mathrm{alg}}})$ one has $\varrho_F(x)>0$ (equivalently $\Gamma(F)$ has no points of type 1).

(C2) For all $x\in X$ the function $L_xF\colon]-\infty,\log R_0]\to\mathbb{R}$ is continuous on $]-\infty,\log R_0]$, piecewise affine on it, and with a finite number of breaks all along $]-\infty,\log R_0]$.

(C3) For all $x\in X$ the function L_xF is concave on $]-\infty,\log\varrho_\Gamma(x)[$. This implies in particular that if $]x,y[\cap\Gamma=\varnothing$, then F is log-concave on $]x,y[$ (cf. Definition 1.12).

(C4) The *non-zero* slopes of F cannot be arbitrarily small. Namely there exists a positive constant $\nu_F>0$ such that for all $x\in X$, and all germs of segment b out of x one has

$$\partial_b F(x)\in\,]-\infty,-\nu_F[\cup\{0\}\cup]\nu_F,\infty[\,. \tag{2.10}$$

(C5) $\Gamma(F)$ is directionally finite at all its bifurcation points (cf. §1.2).

(C6) There exists a finite set $\mathscr{C}(F)\subseteq X$ such that if x is a bifurcation point of $\Gamma(F)$ not in $\mathscr{C}(F)\cup\partial X$, then F is super-harmonic at x (cf. Definition 1.14).

Remark 2.11. (1) By (2.2), we have $\varrho_F(x)>0$ for all $x\notin X(\widehat{K^{\mathrm{alg}}})$.

(2) If F satisfies (C1) and (C2) then it is branch continuous (cf. §2.2).

(3) (C1) plus (C3) imply that F is logarithmically non-increasing over each segment $]x,y[$ (oriented towards ∞) such that $]x,y[\cap\Gamma=\varnothing$.

(4) Conditions (C2) and (C5) ensure Notation 1.13, so that $dd^cF(x)$ is defined for all $x\in X$.

PROPOSITION 2.12. (Permanence of (C1)–(C6) by scalar extension) *With the notation of Proposition 2.9, if $i\in\{1,...,6\}$, then $F_\Omega:=F\circ\pi_{\Omega/K}$ satisfies* (Ci) *if and only if F verifies* (Ci).

Proof. The claim holds immediately for (C1)–(C4), since for all $x\in X_\Omega$ and all $\varrho>0$ one has $\pi_{\Omega/K}(\lambda_x(\varrho))=\lambda_{\pi_{\Omega/K}(x)}(\varrho)$, and $\varrho_{F_\Omega}(x)=\varrho_F(\pi_{\Omega/K}(x))$.

For (C5) and (C6) we may assume that K is algebraically closed since

$$\Gamma(F)=\Gamma(F_{\widehat{K^{\mathrm{alg}}}})/\mathrm{Gal}(K^{\mathrm{alg}}/K).$$

By Proposition 2.9 the claim is evident for (C5) since $\pi_{\Omega/K}$ induces an isomorphism $\Delta(x,\Gamma(F_\Omega))\xrightarrow{\sim}\Delta(\pi_{\Omega/K}(x),\Gamma(F))$. Finally $\pi_{\Omega/K}$ preserves the slopes so (C6) descends. □

LEMMA 2.13. (Flat directions do not belongs to $\Gamma(F)$) *Let $D\subset X$ be an open virtual disk of radius ϱ. Assume that $F\colon X\to\mathbb{R}_{>0}$ is a function satisfying*

(C1-D) *$\varrho_F(x)>0$ for all $x\in D$;*

(C3-D) *for all $x\in D$ the function L_xF is concave on $]-\infty,\log\varrho[$.*

Then F is constant on D if and only if F is constant on an individual complete segment $\Lambda(x)\cap D$.

Moreover if F is non-constant, then the first break of L_xF arises at $\log\varrho_F(x)$.

Proof. Assume that F is constant on $\Lambda(x)\cap D$. Since D is topologically a tree, for all $x'\in D$ the segment $I:=\Lambda(x)\cap\Lambda(x')\cap D$ is non-empty. So F is constant, with value $F(x)$, on $I\subset\Lambda(x')$. Now condition (C1-D) imply the constancy around x'. So the concavity (C3-D) implies constancy on the whole $\Lambda(x')\cap D$ (concavity implies continuity on $]-\infty,\log\varrho[$). Hence $F(x)=F(x')$. □

As a consequence we have the following result.

PROPOSITION 2.14. (Decreasing on disks) *Assume that F satisfies the properties* (C1), (C2), *and* (C3). *Let D be a virtual disk such that $D\cap\Gamma=\varnothing$. Let x be the boundary point of D, and let b be the germ of segment out of x contained in D (b is oriented out of x). Then D intersects $\Gamma(F)$ if and only if $\partial_bF(x)>0$ (equivalently, $\Gamma(F)\cap D=\varnothing$ if and only if $\partial_bF(x)=0$).* □

PROPOSITION 2.15. (No breaks implies no bifurcations) *Assume that F satisfies* (C1), (C2), (C3), (C5), *and* (C6), *but not necessarily* (C4). *Let $]x,y[\subset X$ be a segment satisfying*

(i) *$]x,y[$ is the analytic skeleton of a virtual open annulus $C(]x,y[)$ in X;*(5)

(ii) *$]x,y[\cap\mathscr{C}(F)=\varnothing$, and $(C(]x,y[)\cap\Gamma)\subseteq]x,y[$;*

(iii) *F has no breaks along $]x,y[$.*

Then $\Gamma(F)$ has no bifurcation points along $]x,y[$, and F is harmonic on $C(]x,y[)$.

Proof. Let $z\in]x,y[$. Each direction b out of z which is not in $]x,y[$ lies inside a disk $D_b\subset C(]x,y[)$ with boundary z. By (ii), Proposition 2.14 holds over D_b, so $\partial_bF(z)\geqslant0$, and $\partial_bF(z)>0$ if and only if $b\in\Gamma(F)$. If z is a bifurcation point of $\Gamma(F)$, this shows that $\sum_{b\notin]x,y[}\partial_bF(z)>0$. But by (C6) we have $dd^cF(z)\leqslant0$, and hence F must have a break along $]x,y[$ at z, contradicting (iii). □

PROPOSITION 2.16. (Finiteness over a disk) *Assume that F satisfies the six properties* (C1)–(C6). *Let $D\subset X$ be an open virtual disk such that $D\cap(\Gamma\cup\mathscr{C}(F))=\varnothing$.*

Then there is a finite number N of bifurcation points of $\Gamma(F)$ inside D.

Moreover, let x be the point at the boundary of D, and let b be the germ of segment out of x contained in D (b is oriented out of x).

Then $N\leqslant\lfloor\partial_bF(x)/\nu_F\rfloor-1$, where $\lfloor r\rfloor$ denotes the largest integer $\leqslant r$.(6)

(5) We recall that the analytic skeleton of an open annulus $\{x\in\mathbb{A}_K^{1,\mathrm{an}}:0<R_1<|T-c|(x)<R_2<\infty\}$ is the set of points without open neighborhoods isomorphic to a virtual open disk.

(6) Recall that $\lfloor\partial_bF(x)/\nu_F\rfloor-1\geqslant0$, because of property (C4).

Proof. By Proposition 2.14 plus (C4), we may assume $\partial_b F(x) \geqslant \nu_F$. By (C2) there is a segment $]y,x[\subset D$ where F has no breaks. By Proposition 2.15, $]y,x[$ is the skeleton Γ_C of a virtual open annulus $C \subset D$ over which $\Gamma(F)$ has no bifurcations. Let $z \in D$ be the first bifurcation point of $\Gamma(F)$ that one encounters proceeding from x towards the interior of D. Let $b_\infty :=]z,x[$, and let $b_1, ..., b_{n_z}$ be the other germs of segment out of z belonging to $\Gamma(F)$ ($b_\infty, b_1, ..., b_{n_z}$ are now all oriented outside z). By super-harmonicity (C6) one has $dd^c F(z) \leqslant 0$, so

$$\sum_{i=1}^{n_z} \partial_{b_i} F(z) \leqslant -\partial_{b_\infty} F(z) = \partial_b F(x). \tag{2.11}$$

By Proposition 2.14, one has $\partial_{b_i} F(z) > 0$ for all $i = 1, ..., n_z$. And, by (C4), for all i one has $\partial_{b_i} F(z) \geqslant \nu_F$. So, since $n_z \geqslant 2$, for all i one has $\partial_{b_i} F(z) \leqslant -\partial_{b_\infty} F(z) - \nu_F = \partial_b F(x) - \nu_F$. Let D_i be the virtual open disk with boundary z containing b_i. Then D_i fulfills the same assumptions as D, but its last slope is now less than $\partial_b F(x) - \nu_F$. We then conclude by induction on $\lfloor \partial_b F(x)/\nu_F \rfloor$. □

THEOREM 2.17. *If* $F: X \to \mathbb{R}_{>0}$ *satisfies the six conditions* (C1)–(C6), *then* F *is finite.*

Proof. Since Γ is finite we are reduced to prove that $\Gamma'(F) := \Gamma(F) \cup \Gamma$ is finite. Moreover, up to replacing Γ by $\Gamma \cup \mathrm{Sat}(\mathscr{C}(F))$, we may assume that $\mathscr{C}(F) \subset \Gamma$. Since $\Gamma(F)$ is directionally finite at its bifurcation points, it is enough to prove that there are a finite number of bifurcation points of $\Gamma'(F)$.

Now, $X \setminus \Gamma$ is a disjoint union of virtual open disks on which we can apply Proposition 2.16. So, by directionally finiteness (C5), we know that for all $x \in \Gamma$ there are a finite number of virtual open disks D with boundary x intersecting $\Gamma(F)$.

Hence we are reduced to proving that there are a finite number of bifurcation points of $\Gamma(F)$ belonging to Γ. The set $\mathscr{C}$ formed by the points in $\mathscr{C}(F)$, the bifurcation points of Γ, and the points in $\Gamma \cap \partial X$, is finite and we can neglect it.

So we have to prove that $\Gamma(F)$ has a finite number of bifurcation points along each connected component $]x,y[$ of $\Gamma \setminus \mathscr{C}$. This follows from Proposition 2.15 and by (C2). □

2.4.1. Assumption (C4) is superfluous

Assumption (C4) is satisfied by the radii of convergence of a differential equation, and it is important for the explicit computation of the number of edges of $\Gamma(F)$ (cf. [NP3]). So we preserve the above claims.

Nevertheless we add the following result derived from Theorem 2.17. Its proof does not involve (C4), which is replaced by a compactness argument.

Theorem 2.18. *Let* $F: X \to \mathbb{R}_{>0}$ *be a function satisfying* (C1), (C2), (C3), (C5), *and* (C6) (*but not necessarily* (C4)). *Then* F *is finite.*

Proof. We prove that $\Gamma'' := \Gamma(F) \cup \Gamma \cup \mathrm{Sat}(\mathscr{C}(F))$ is locally finite in the Berkovich topology of X. Recall that this is an admissible graph in X, so $X \setminus \Gamma''$ is a disjoint union of open disks.

Let $x \in \Gamma''$. Let $V(x)$ be the union of x with all the virtual open disks in X with boundary x on which F is constant. By (C5) and Proposition 2.14, $V(x)$ is an affinoid domain of X on which F is constant.

Let $b_1, ..., b_n$ be the family of germs of segment out of x which are not in $V(x)$; then $b_1, ..., b_n \in \Gamma''$. For all $i = 1, ..., n$ there is $]x, y_i[\in b_i$ which is the skeleton of a virtual open annulus C_i such that $\Gamma'' \cap C_i =]x, y_i[$. By (C2) we can choose $]x, y_i[$ small enough to fulfill the assumptions of Proposition 2.15. Hence $U := V(x) \cup \bigcup_{i=1}^n C_i$ is an open neighborhood of x in X such that $U \cap \Gamma'' = \bigcup_{i=1}^n [x, y_i[$. We proceed so for all $x \in \Gamma''$. Together with the complement of Γ'' in X, this gives a covering of X by open sets whose intersection with Γ'' is a finite graph. Since X is compact, we can extract a finite sub-covering, so Γ'' is finite. □

2.4.2. Non-compact disks and annuli

Let $C(I) = \{x \in \mathbb{A}_K^{1,\mathrm{an}} : |T|(x) \in I\}$ be a (possibly not closed) annulus, or disk if $0 \in I$. Definition 2.1 extends to $X = C(I)$ in an evident way.

In this case $\Gamma(F)$ is finite if there is a compact sub-interval $J \subset I$ (resp. if $0 \in I$, then $0 \in J$) such that $\Gamma(F|_J)$ is finite over $C(J)$, and $\Gamma(F) = \Gamma(F|_{C(J)}) \cup \Gamma_{C(I)}$.

Corollary 2.19. *Let* $F: C(I) \to \mathbb{R}_{>0}$. *Assume that* $\mathscr{C}(F)$ *is finite and contained in* $C(J)$ *for some compact* $J \subset I$. *If* $F|_{C(J)}$ *is finite, and if* F *is log-affine along each connected component of* $I \setminus J$, *then* F *is finite and* $\Gamma(F) = \Gamma(F|_{C(J)}) \cup \Gamma_{C(I)}$.

Proof. Apply Proposition 2.15 over the open annuli that are connected components of $C(I) \setminus C(J)$. □

Example 2.20. (1) Let Γ be a finite admissible graph. The function $x \mapsto \varrho_\Gamma(x)$ satisfies the six properties (C1)–(C6) with respect to Γ, and $\mathscr{C}(\varrho_\Gamma) = \varnothing$. If $I \subseteq \Gamma$ is any segment contained in some $\Lambda(x)$, and if I is oriented towards ∞, then ϱ_Γ is log-affine on I with slope 1. In particular it is super-harmonic in the sense of definition 1.14, and $\Gamma(\varrho_\Gamma) = \Gamma$.

(2) If $F_1, ..., F_n$ are functions satisfying the six properties (C1)–(C6), then so does $\min\{F_1, ..., F_n\}$.

(3) Let $f_1, ..., f_n \in \mathscr{O}(X)$ and $\alpha_1, ..., \alpha_n > 0$. Assume that each f_i has no zeros on $X(\widehat{K^{\mathrm{alg}}})$. Then the function $F(x) := \min_i |f_i|(x)^{-\alpha_i}$ satisfies (C1)–(C6), with $\Gamma = \Gamma_X$,

and $\Gamma(F)=\Gamma_X$. Moreover F is also super-harmonic (cf. Definition 1.14), because so is each function $x\mapsto|f_i(x)|^{-\alpha_i}$.

3. Radii of convergence and statement of main result

We here give the definition of the radii of convergence (3.4), and of the convergence Newton polygon (3.5). We then state our main result (cf. Theorem 3.9) whose proof will be given in the next sections.

3.1. Newton polygons (formal definition)

Let $r\geqslant 1$ be a natural number. Let $v:\{0,1,...,r\}\to\mathbb{R}\cup\{\infty\}$, be any sequence $i\mapsto v_i$ satisfying $v_0=0$. The *Newton polygon* $\mathrm{NP}(v)\subset\mathbb{R}^2$ is the convex hull in $\mathbb{R}^2$ of the family of half-lines $L_v:=\bigcup_{i=0}^{r}\{(x,y)\in\mathbb{R}^2:x=i \text{ and } y\geqslant v_i\}$, i.e. the intersection of all upper half planes $H_{a,b}:=\{(x,y)\in\mathbb{R}^2:y\geqslant ax+b\}$, $a,b\in\mathbb{R}$, containing L_v.

For $i=0,...,r$, the *i-th partial height* of the polygon is the value

$$h_i:=\min\{y\in\mathbb{R}\cup\{\infty\}:(i,y)\in\mathrm{NP}(v)\}. \tag{3.1}$$

If $h:\{0,...,r\}\to\mathbb{R}\cup\{\infty\}$ denotes the function $i\mapsto h_i$, then $\mathrm{NP}(v)=\mathrm{NP}(h)$, and h is the smallest function with this property.

We have $h_i=\sup_{s\in\mathbb{R}}(si+\min_{j=0,\dots,r}(v_j-sj))$. In fact, if $y=sx+q_s$ is the line of slope s which is tangent to $\mathrm{NP}(v)$, then $q_s=\min_{j=0,\dots,r}(v_j-sj)$, and h_i is the supremum of the values of those lines at $x=i$. In particular, since $v_0=0$, for $i=1$ we have $h_1=\min_{i=1,\dots,r}(v_i/i)$.

A *slope sequence* is any non-decreasing sequence $s:\{1,...,r\}\to\mathbb{R}\cup\{\infty\}$ such that $s_1\leqslant...\leqslant s_r$.

The *slope sequence of* $\mathrm{NP}(v)=\mathrm{NP}(h)$ is defined by $s_i:=h_i-h_{i-1}$, $i=1,...,r$, where $s_i=\infty$ if h_i or h_{i-1} is equal to ∞. The slope sequence of $\mathrm{NP}(h)$ determines the function $h_i=s_1+...+s_i$, and hence $\mathrm{NP}(h)$.

If $s_i<s_{i+1}$, or if $i=r$, we say that i is a *vertex* of $\mathrm{NP}(v)$.

Let $s:s_1\leqslant...\leqslant s_r$ be a slope sequence. The *truncated slope sequence* by the constant $C\in\mathbb{R}$ is by definition the sequence $s|_C:=\{s'_i\}_{i=1}^{r}$, where $s'_i:=\min\{s_i,C\}$, for all i.

As a matter of fact, in the sequel we will deal only with truncated slope sequences by a convenient constant $C<\infty$, so we do not have to deal with infinite slopes.

Example 3.1. Let $(F,|\cdot|_F)$ be a valued field and let $P(T):=\sum_{i=0}^{r}a_{r-i}T^i\in F[T]$ be such that $a_0=1$. Let $v_{P,i}:=-\log|a_i|\in\mathbb{R}\cup\{\infty\}$. The *Newton polygon* of $P(T)$ is by definition $\mathrm{NP}(v_P)$.

3.2. Convergence Newton polygon of a differential equation

Let X be an affinoid domain of $\mathbb{A}_K^{1,\mathrm{an}}$. A differential equation over X is a locally free $\mathscr{O}_X$-module $\mathscr{F}$ of finite rank together with a connection $\nabla: \mathscr{F} \to \mathscr{F} \otimes \Omega_X^1$. Let r be the rank of $\mathscr{F}$.

We now define the radii of $\mathscr{F}$ at $x \in X$. We fix a field extension $\Omega \in E(\mathscr{H}(x))$ which is algebraically closed, spherically complete, and with value group $|\Omega^\times| = \mathbb{R}_{>0}$. Let $t \in X(\Omega)$ be a Dwork generic point for x, and let $\mathscr{F}|_{D(x,X)}$ be the restriction of $\mathscr{F}_\Omega = \mathscr{F} \widehat{\otimes}_K \Omega$ to $D(x,X) \subset X_\Omega$.

We recall that the radius of $D(x,X)$ is $\varrho_{x,X}$. For all $0 < R \leqslant \varrho_{x,X}$, we denote by $D(x,R) \subset D(x,X)$ the open sub-disk centered at t with radius R, and by

$$\mathrm{Fil}^{\geqslant R}\mathrm{Sol}(\mathscr{F}, t, \Omega) \subset \mathscr{F}|_{D(x,R)}$$

the Ω-vector space of solutions of $\mathscr{F}$ with values in $\mathscr{O}(D(x,R))$, that is the kernel of $\nabla \otimes 1 + 1 \otimes d/dT$ acting on $\mathscr{F}|_{D(x,R)}$. The space $\mathrm{Sol}(\mathscr{F}, t, \Omega)$ of all Taylor solutions of $\mathscr{F}$ around t is given by

$$\mathrm{Sol}(\mathscr{F}, t, \Omega) := \bigcup_{R>0} \mathrm{Fil}^{\geqslant R}\mathrm{Sol}(\mathscr{F}, t, \Omega). \tag{3.2}$$

Since Ω is spherically closed, by a result of Lazard [L], $\mathscr{F}|_{D(x,X)}$ is free. So, once a basis is chosen we have a differential equation $Y' = GY$, $G \in M_r(\mathscr{O}(D(x,X)))$, and hence, by the Cauchy existence theorem, $\mathrm{Sol}(\mathscr{F}, t, \Omega)$ has dimension r over Ω (cf. [DGS, Appendix]). If $\Omega \subseteq \Omega'$, a descent argument (cf. [Ke3, Proposition 6.9.1]) shows that $\mathrm{Fil}^{\geqslant R}\mathrm{Sol}(\mathscr{F}, t, \Omega') = \mathrm{Fil}^{\geqslant R}\mathrm{Sol}(\mathscr{F}, t, \Omega) \widehat{\otimes}_\Omega \Omega'$.

Proposition 3.2. *The filtration is independent of the choice of Ω and t in the following sense. If (t', Ω') is another choice, there exists $\Omega, \Omega' \leqslant \Omega'' \in E(K)$, together with a Galois automorphisms $\sigma \in \mathrm{Gal}^{\mathrm{cont}}(\Omega''/K)$, such that $\sigma(t) = t'$, inducing for all $R \leqslant \varrho_{x,X}$ the identification*

$$\sigma: \mathrm{Fil}^{\geqslant R}\mathrm{Sol}(\mathscr{F}, t, \Omega) \widehat{\otimes}_\Omega \Omega'' \xrightarrow[\sigma]{\sim} \mathrm{Fil}^{\geqslant R}\mathrm{Sol}(\mathscr{F}, t', \Omega'') \widehat{\otimes}_{\Omega'} \Omega''. \tag{3.3}$$

Proof. The existence of σ such that $\sigma(t) = t'$ follows from Lemma 1.5. Since σ is isometric, $\sigma(D^-(t,R)) = D^-(t',R)$ for all $0 < R \leqslant \varrho_{x,X}$. This provides an isomorphism of rings $\sum_{i=0}^\infty a_i (T-t)^i \mapsto \sum_{i=0}^\infty \sigma(a_i)(T-t')^i : \mathscr{O}(D^-(t,R)) \xrightarrow{\sim} \mathscr{O}(D^-(t',R))$, over Ω'', commuting with d/dT. □

Definition 3.3. (Convergence radii) For all $i = 1, ..., r$ we define $\mathcal{R}_i^{\mathscr{F}}(x)$ as the largest value of $R \leqslant \varrho_{x,X}$ such that $\dim_\Omega \mathrm{Fil}^{\geqslant R}\mathrm{Sol}(\mathscr{F}, t, \Omega) \geqslant r - i + 1$. We set

$$H_i^{\mathscr{F}}(x) := \prod_{k=1}^{i} \mathcal{R}_k^{\mathscr{F}}(x)$$

and

$$\mathcal{R}_i(x,\mathscr{F}) := \frac{\mathcal{R}_i^{\mathscr{F}}(x)}{\varrho_{x,X}} \quad \text{and} \quad H_i(x,\mathscr{F}) := \prod_{k=1}^{i} \mathcal{R}_k(x,\mathscr{F}) = \frac{H_i^{\mathscr{F}}(x)}{\varrho_{x,X}^i}. \tag{3.4}$$

We also set $s_i^{\mathscr{F}}(x) := \log \mathcal{R}_i^{\mathscr{F}}(x)$, $h_i^{\mathscr{F}}(x) := s_1^{\mathscr{F}}(x) + ... + s_i^{\mathscr{F}}(x)$, and $h_0^{\mathscr{F}}(x) = 0$.

The polygon $\mathrm{NP}(\log H_i(x,\mathscr{F}))$ is called the convergence Newton polygon and it is denoted by

$$\mathrm{NP}^{\mathrm{conv}}(x,\mathscr{F}). \tag{3.5}$$

Remark 3.4. (1) By Proposition 3.2, the above functions are independent of the choices of t and Ω.

(2) Obviously the definition only depends on the restriction $\mathscr{F}|_{D(x,X)}$, so the same definitions can be given for a differential module over a virtual open disk D, replacing $\varrho_{x,X}$ by the radius of D.

(3) As a consequence, the definition is insensitive on extension of K: for all $\Omega \in E(K)$ and all $y \in X_\Omega$,

$$\mathcal{R}_i(y,\mathscr{F}_\Omega) = \mathcal{R}_i(\pi_{\Omega/K}(y),\mathscr{F}) \quad \text{for all } i = 1, ..., r. \tag{3.6}$$

In particular the assumptions of Proposition 2.12 are satisfied.

(4) Since $y \mapsto \varrho_{y,X}$ is constant on each maximal disk $D(x,X)$, it immediately follows that

$$\Gamma(\mathcal{R}_i(-,\mathscr{F})) = \Gamma(\mathcal{R}_i^{\mathscr{F}}) \quad \text{and} \quad \Gamma(H_i(-,\mathscr{F})) = \Gamma(H_i^{\mathscr{F}}). \tag{3.7}$$

(5) More precisely, $\mathcal{R}_i(-,\mathscr{F})$ and $\mathcal{R}_i^{\mathscr{F}}$ differ by a constant function over each maximal disk $D(x,X)$. Hence if b is a germ of segment out of $x \in X$ we have either $\partial_b \mathcal{R}_i(x,\mathscr{F}) = \partial_b \mathcal{R}_i^{\mathscr{F}}(x)$ if $b \notin \Gamma_X$, or, if b is oriented towards ∞, we have $\partial_b \mathcal{R}_i(x,\mathscr{F}) = \partial_b \mathcal{R}_i^{\mathscr{F}}(x) - 1$.

(6) The dimension $\dim_\Omega \mathrm{Fil}^{\geqslant R} \mathrm{Sol}(\mathscr{F}, t, \Omega)$ is obviously constant on $D(x,R)$. Hence $\mathcal{R}_i(x,\mathscr{F})$ and $\mathcal{R}_i^{\mathscr{F}}(x)$ are constant on $D(x, \mathcal{R}_i^{\mathscr{F}}(x))$, so

$$\max\{\mathcal{R}_i^{\mathscr{F}}(x), r(x)\} \leqslant \varrho_{\mathcal{R}_i^{\mathscr{F}}}(x) = \varrho_{\mathcal{R}_i(-,\mathscr{F})}(x). \tag{3.8}$$

(7) It follows from the definition that if $\mathscr{F}' \subset \mathscr{F}$ is a sub-differential equation, the radii of $\mathscr{F}'$ all appear among the radii of $\mathscr{F}$.

The radii do not behave well by exact sequences, but we have the following result.

PROPOSITION 3.5. *Let $\mathscr{F} = \mathscr{F}_1 \oplus \mathscr{F}_2$ be a direct sum of differential equations over X of ranks r_1 and r_2 respectively. Then, up to permutation,*[7] *for all $x \in X$ one has*

$$\{\mathcal{R}_1^{\mathscr{F}}(x), ..., \mathcal{R}_{r_1+r_2}^{\mathscr{F}}(x)\} = \{\mathcal{R}_1^{\mathscr{F}_1}(x), ..., \mathcal{R}_{r_1}^{\mathscr{F}_1}(x)\} \cup \{\mathcal{R}_1^{\mathscr{F}_2}(x), ..., \mathcal{R}_{r_2}^{\mathscr{F}_2}(x)\}. \tag{3.9}$$

The same holds replacing X by an open disk, or replacing $\mathcal{R}_i^{\mathscr{F}}$ by $\mathcal{R}_i(-,\mathscr{F})$.

[7] If a radius R appears n_i times in $\mathrm{NP}^{\mathrm{conv}}(\mathscr{F}_i, x)$, it is understood that it appears $n_1 + n_2$ times in $\mathrm{NP}^{\mathrm{conv}}(\mathscr{F}, x)$.

Proof. The functor $\mathscr{F}\mapsto\mathrm{Fil}^{\geqslant R}\mathrm{Sol}(\mathscr{F},t,\Omega)$ is additive, so for $R\leqslant\varrho_{x,X}$ we have

$$\mathrm{Fil}^{\geqslant R}\mathrm{Sol}(\mathscr{F},t,\Omega)=\mathrm{Fil}^{\geqslant R}\mathrm{Sol}(\mathscr{F}_1,t,\Omega)\oplus\mathrm{Fil}^{\geqslant R}\mathrm{Sol}(\mathscr{F}_2,t,\Omega).$$

The claim then follows directly from Definition 3.3. □

3.3. Statement of the main result

Definition 3.6. We say that the index i (resp. $\mathcal{R}_i(x,\mathscr{F})$) is

$$\begin{cases} \textit{spectral} \text{ at } x\in X, & \text{if } \mathcal{R}_i^{\mathscr{F}}(x)\leqslant r(x),\\ \textit{solvable} \text{ at } x\in X, & \text{if } \mathcal{R}_i^{\mathscr{F}}(x)=r(x),\\ \textit{over-solvable} \text{ at } x\in X, & \text{if } \mathcal{R}_i^{\mathscr{F}}(x)>r(x).\end{cases} \tag{3.10}$$

We say that the index i is *free of solvability at* x if none of the indices $j\leqslant i$ is solvable. We say that $\mathscr{F}$ is *free of solvability at* x if none of the indices $i=1,...,r$ is solvable at x.

Remark 3.7. From Proposition 2.5 and equation (3.8) it follows that i is spectral at all points of $\Gamma(\mathcal{R}_i(-,\mathscr{F}))$.

Definition 3.8. For all $i=1,...,r$, we set

$$\Gamma_0:=\Gamma_X \quad\text{and}\quad \Gamma_i:=\bigcup_{j=1}^{i}\Gamma(\mathcal{R}_j(-,\mathscr{F})). \tag{3.11}$$

Recall that the index i is a *vertex at* x of $\mathrm{NP}^{\mathrm{conv}}(x,\mathscr{F})$ if $\mathcal{R}_i(x,\mathscr{F})<\mathcal{R}_{i+1}(x,\mathscr{F})$, or if $i=r$.

The main result of this paper is the following theorem.

THEOREM 3.9. *Let $\mathscr{F}$ be a differential module of rank r over X.*

For $i=1,...,r$ the functions $\mathcal{R}_i(-,\mathscr{F})$ and $H_i(-,\mathscr{F})$ (and hence also $s_i^{\mathscr{F}}$, $h_i^{\mathscr{F}}$, $\mathcal{R}_i^{\mathscr{F}}$, and $H_i^{\mathscr{F}}$) are finite.

They satisfy moreover the following properties:

(i) *For all $i=1,...,r$ the i-th partial heights $H_i(-,\mathscr{F})$ and $H_i^{\mathscr{F}}$ both satisfy* (C1), (C2), (C4), (C5) *of* §2.4, *and also* (C3) *with respect to* $\Gamma:=\Gamma_{i-1}$.

(ii) (Integrality) *Let $x\in X$ be a point. Then*

(ii-a) *if i is a vertex of* $\mathrm{NP}^{\mathrm{conv}}(x,\mathscr{F})$, *then for all germs of segment b out of x, we have*

$$\partial_b H_i(x,\mathscr{F}),\partial_b H_i^{\mathscr{F}}(x)\in\mathbb{Z}; \tag{3.12}$$

(ii-b) *if i is not a vertex, one proves by interpolation*([8]) *from* (3.12) *that*

$$\partial_b H_i(x,\mathscr{F}),\partial_b H_i^{\mathscr{F}}(x)\in\mathbb{Z}\cup\frac{1}{2}\mathbb{Z}\cup...\cup\frac{1}{r}\mathbb{Z}. \tag{3.13}$$

([8]) Interpolation means that we proceed as in the proof of point (iv) of Proposition 4.13.

(iii) (Concavity) *Let* $]x,z[\subset\Lambda(x)$ *be an open segment in* X. *Let* $]x,y[:=]x,z[\setminus\Gamma_X$.(9) *For all* $i=1,...,r$ *let* H_i *denote the* i*-th partial height* $H_i(-,\mathscr{F})$ *or* $H_i^{\mathscr{F}}$. *Then*

(iii-a) H_i *is log-concave on each sub-segment of* $]y,z[$ *which is the skeleton of a virtual annulus contained in* X (*cf. Definition* 1.12);

(iii-b) H_i *is log-concave on each sub-segment of* $]x,y[$ *which does not intersect the set*

$$\{\lambda_x(\mathcal{R}_1^{\mathscr{F}}(x)),...,\lambda_x(\mathcal{R}_i^{\mathscr{F}}(x))\};\tag{3.14}$$

also, fixed $\tau\in\{\mathcal{R}_k^{\mathscr{F}}(x):k\leqslant i\}$, *if for all* $k\leqslant i$ *such that* $\mathcal{R}_k^{\mathscr{F}}(x)=\tau$ *the function* $\mathcal{R}_k(-,\mathscr{F})$ (*or equivalently* $\mathcal{R}_k^{\mathscr{F}}$) *is log-concave at* $\log\tau$, *then* H_i *is also log-concave at* $\log\tau$;(10)

(c) H_i *is logarithmically non-increasing on each sub-segment* $I\subset]x,y[$ *on which* i *is free of solvability* (*i.e.* $\mathcal{R}_j^{\mathscr{F}}(x')\neq r(x')$ *for all* $x'\in I$, *and all* $j\leqslant i$).

(iv) (Weak super-harmonicity) *We define inductively a family* $\mathscr{C}_1(\mathscr{F}),...,\mathscr{C}_r(\mathscr{F})\in X\setminus\Gamma_X$ *of finite subsets as*

$$\mathscr{C}_i:=\bigcup_{i=1}^{i}A_i,\tag{3.15}$$

where A_i *is the finite set of points* $x\in X\setminus S_X$ *satisfying the following properties*:

(iv-a) *the index* i *is solvable at* x;

(iv-b) x *is an end-point of* $\Gamma(\mathcal{R}_i(-,\mathscr{F}))$;

(iv-c) $x\in\Gamma(\mathcal{R}_i(-,\mathscr{F}))\cap\Gamma(H_i(-,\mathscr{F}))\cap\Gamma_{i-1}$.

Then, for all $x\notin S_X\cup\mathscr{C}_i$ (*cf. equation* (2.8)) *we have*

$$dd^cH_i(x,\mathscr{F})\leqslant 0.\tag{3.16}$$

While, for $x\in S_X\setminus\partial X$, *we have*

$$dd^cH_i(x,\mathscr{F})\leqslant(N_X(x)-2\deg(x))\min\{i,i_x^{\mathrm{sp}}\},\tag{3.17}$$

where $\deg(x)$ *is the degree of* x (*that is, the number of points of* $X_{\widehat{K^{\mathrm{alg}}}}$ *over* $x\in X$); $N_X(x):=\sum_{b\in\Delta(x,\Gamma_X)}m_b$, *where* m_b *is the multiplicity of* b (*cf. Definition* 1.14); *and* $0\leqslant i_x^{\mathrm{sp}}\leqslant r$ *is the largest index of* $\mathscr{F}$ *which is spectral non-solvable at* x.(11)

This is equivalent to saying that $H_i^{\mathscr{F}}$ *is super-harmonic* (*at least*) *at all points* $x\in X\setminus(\mathscr{C}_i\cup\partial X)$.

In particular $\mathcal{R}_1^{\mathscr{F}}$ *is super-harmonic* (*outside* ∂X).

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(9) In other words if D is the largest virtual open disk in X intersecting $]x,z[$, then $]x,y[=D\cap]x,z[$. If $]x,z[\subset\Gamma_X$, it is understood that $]x,y[=\varnothing$.

(10) In particular this happens by definition if $\tau<r(x)$, since $L_x\mathcal{R}_k(-,\mathscr{F})$ is constant on $]-\infty,r(x)]$.

(11) It is understood that $i_x^{\mathrm{sp}}=0$ if and only if all the radii of $\mathscr{F}$ are solvable or over-solvable at x.

(v) (Weak harmonicity of the vertices) *Let* $x\in X\setminus\partial X$. *Then*

(v-a) *if* $x\notin\Gamma(H_i(-,\mathscr{F}))$, *then for all* $b\in\Delta(x)$ *we have* $\partial_b H_i(x,\mathscr{F})=0$, *so* $H_i(-,\mathscr{F})$ *is harmonic at* x;

(v-b) *if* $x\in\Gamma(H_i(-,\mathscr{F}))$, *and if* i *is a vertex free of solvability at* x, *then* (3.16) *and* (3.17) *are equalities. In particular* $H_i^{\mathscr{F}}$ *is harmonic at* x.

The proof of Theorem 3.9 is placed in §6.

As a straightforward generalization of Theorem 3.9 we have the following result.

Corollary 3.10. *Let* $C(I):=\{x:|T|(x)\in I\}$ *be a possibly not closed annulus or disk* (*if* $0\in I$ *one has a disk*). *Let* $\mathscr{F}$ *be a differential module of rank* r *over a differential ring* $\mathscr{O}$.

Then Theorem 3.9 *holds for* $\mathscr{F}$ *in the following cases*:

(i) *if* $\mathscr{O}$ *is the ring of Krasner analytic elements over* $C(I)$ (*cf.* [Ke3, Definition 8.5.1]);

(ii) *if* K *is discretely valued, and* $\mathscr{O}$ *is the ring* $\mathcal{B}(C(I))$ *of bounded analytic functions on* $C(I)$;

(iii) *if* $\mathscr{O}=\mathcal{B}(C(I))$ *or* $\mathscr{O}=\mathscr{O}(C(I))$, *and all* $\mathcal{R}_1(-,\mathscr{F}),...,\mathcal{R}_r(-,\mathscr{F})$ (*or equivalently all* $H_i(-,\mathscr{F})$) *have a finite number of breaks along the skeleton* $\Gamma_{C(I)}=\{x_{0,\varrho}:\varrho\in I\}$.

Moreover, if $\mathscr{O}=\mathcal{B}(C(I))$ *or* $\mathscr{O}=\mathscr{O}(C(I))$, *and if there exists* $i\leqslant r$ *such that all* $\mathcal{R}_1(-,\mathscr{F}),...,\mathcal{R}_i(-,\mathscr{F})$ *have a finite number of breaks along* $\Gamma_{C(I)}$, *then* $\mathcal{R}_1(-,\mathscr{F}),...,\mathcal{R}_i(-,\mathscr{F})$ *are finite.* □

Corollary 3.11. *Assume that* $\mathrm{rank}(\mathscr{F})=1$, *and that* $x\notin\Gamma_X$. *Then* x *is an end-point of* $\Gamma(\mathcal{R}_1(x,\mathscr{F}))$ *if and only if* $\mathcal{R}_1(x,\mathscr{F})$ *is solvable at* x *and* $\partial_{b_\infty}\mathcal{R}_1(x,\mathscr{F})<0$, *where* b_∞ *denotes the germ of segment out of* x *directed towards* ∞ (*and oriented out of* x).

Proof. If $\mathcal{R}_1^{\mathscr{F}}(x)=r(x)$ and $\partial_{b_\infty}\mathcal{R}_1(x,\mathscr{F})<0$, then $\varrho_{\mathcal{R}_1(-,\mathscr{F})}(x)=r(x)$ as a consequence of Lemma 2.13. Hence $x\in\Gamma(\mathcal{R}_1(-,\mathscr{F}))$ by Proposition 2.5. Now x is an end-point of $\Gamma(\mathcal{R}_1(-,\mathscr{F}))$ by Lemma 6.5.

Conversely, by Lemma 2.13 and Proposition 2.14 a boundary point x of $\Gamma(\mathcal{R}_1(-,\mathscr{F}))$ not in Γ_X satisfies $\partial_b\mathcal{R}_1(x,\mathscr{F})=0$ for all $b\neq b_\infty$, and $\partial_{b_\infty}\mathcal{R}_1(x,\mathscr{F})<0$. In particular $\mathcal{R}_1(-,\mathscr{F})$ is not harmonic at x. Hence $\mathcal{R}_1(-,\mathscr{F})$ must be solvable at x by point (iv) of Theorem 3.9. □

Remark 3.12. Assume that K is trivially valued. The field of Laurent formal power series $K((T))$ (resp. Laurent polynomials $K[T,T^{-1}]$) coincides in this case with the ring of analytic functions over $\{T:|T|\in I\}$ for all (open or closed) intervals $I\subseteq]0,1[$ (resp. $I\subseteq\mathbb{R}_{>0}$, with $1\in I$). Analytic functions are always bounded, and point (ii) of Corollary 3.10

holds. Moreover, the radii have no breaks along $]0, x_{0,1}]$, and all differential equations are solvable at $x_{0,1}$. Furthermore, $\omega=1$, and the radii are always explicit by Proposition 4.11. The slopes along $]0, x_{0,1}[$ are also directly related to the formal Newton polygon of $\mathscr{F}$ [Ra], [Ro2], [DMR, pp. 97–107] (see [NP3] for more details).

Remark 3.13. In other terms, if K is spherically complete and $|K|=\mathbb{R}$, Theorem 3 says in particular that the functions $\mathcal{R}_i(-,\mathscr{F})$ are all *definable* in the sense of [HL].

The rest of the paper is devoted to proving Theorem 3.9. The definition of $\mathcal{R}_i^{\mathscr{F}}$ and $\mathcal{R}_i(-,\mathscr{F})$ are stable by scalar extensions of K (cf. Remark 3.4). So we assume the following hypotesis.

Hypothesis 3.14. From now on we assume that K is algebraically closed.

4. Spectral polygons and related results

The ring $\mathscr{O}(X)$ is a principal ideal domain, for which each ideal is generated by a polynomial, and hence there are no non-trivial ideals stable by d/dT. This implies that each coherent $\mathscr{O}_X$-module with connection is free over $\mathscr{O}(X)$ (the proof of [Ke3, Proposition 9.1.2] works). The choice of a basis $e_1, ..., e_r \in \mathscr{F}(X)$ gives an isomorphism $\mathscr{F}(X) \xrightarrow{\sim} \mathscr{O}(X)^r$ in which the connection ∇ becomes of the form

$$\nabla(f_1, ..., f_r)^t = (f_1', ..., f_r')^t - G(f_1, ..., f_r)^t, \tag{4.1}$$

with $G \in M_{r\times r}(\mathscr{O}(X))$, where $\Omega^1_X(X) \xrightarrow{\sim} \mathscr{O}(X)$ via the map $f\, dT \mapsto f$. The matrix G is called the matrix of ∇. In that basis, the fundamental Taylor solution matrix of $\mathscr{F}$ at a point $t \in X(\Omega)$ is

$$Y(T,t) := \sum_{n\geqslant 0} \frac{G_n(t)(T-t)^n}{n!}, \tag{4.2}$$

where G_n is inductively defined by $G_0=\mathrm{Id}$, $G_1=G$, and $G_{n+1}=G_nG+G_n'$. The columns of $Y(T,t)$ form a basis of $\mathrm{Sol}(\mathscr{F}, t, \Omega)$ (cf. definition (3.2)). We set

$$\mathcal{R}^Y(x) := \liminf_{n\to\infty} \left|\frac{G_n}{n!}\right|(x)^{-1/n} = \liminf_{n\to\infty} \left|\frac{G_n(t)}{n!}\right|_{\Omega}^{-1/n}. \tag{4.3}$$

This is a function $\mathcal{R}^Y: X \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$. Clearly $\mathcal{R}_1^{\mathscr{F}}(x) = \min\{\mathcal{R}^Y(x), \varrho_{x,X}\}$, and we set

$$\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}(x) := \min\{\mathcal{R}^Y(x), r(x)\} = \min\{\mathcal{R}_1^{\mathscr{F}}(x), r(x)\}. \tag{4.4}$$

The function $\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}: X \to [0, R_0]$ is called *spectral radius* of $\mathscr{F}$ (and also *generic radius*).

Notice that $\mathcal{R}^Y(x)$ depends on the chosen basis of $\mathscr{F}(X)$, while $\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}(x)$ does not.

LEMMA 4.1. *For all $x \in X$ one has*

$$\varrho_{\mathcal{R}^Y}(x) = \varrho_{\mathcal{R}_1^{\mathscr{F}}}(x) = \varrho_{\mathcal{R}_1(-,\mathscr{F})}(x) \quad \textit{and} \quad \varrho_{\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}}(x) = r(x). \tag{4.5}$$

Proof. We have $Y(T,t) \in \mathrm{GL}_r(\mathscr{O}(D^-(t,\mathcal{R}_1^{\mathscr{F}}(t))))$, and if $|t'-t| < \mathcal{R}_1^{\mathscr{F}}(t)$ one has the cocycle relation $Y(T,t) = Y(T,t')Y(t',t)$ (cf. [CM5]). From this it follows that $\mathcal{R}^Y(t) \geqslant \mathcal{R}^Y(t')$, and by symmetry we have $\mathcal{R}^Y(t) = \mathcal{R}^Y(t')$. Hence $\mathcal{R}^Y$ and $\mathcal{R}_1^{\mathscr{F}}$ are both constant on $D^-(t_x, \mathcal{R}_1^{\mathscr{F}}(x))$.

The claim follows from this fact, together with (2.2) and

$$\mathcal{R}_1^{\mathscr{F}}(x) = \min\{\mathcal{R}^Y(x), \varrho_{x,X}\}. \qquad \square$$

From (4.5) and Lemma 2.3 one immediately has (here r is the function of (1.6))

$$\Gamma(\mathcal{R}^Y) = \Gamma(\mathcal{R}_1^{\mathscr{F}}) = \Gamma(\mathcal{R}_1(-,\mathscr{F})) \quad \text{and} \quad \Gamma(\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}) = \Gamma(r) = X. \tag{4.6}$$

PROPOSITION 4.2. (Concavity and transfer theorems) *If $x_1(f) \leqslant x_2(f)$ for all $f \in \mathscr{O}(X)$, then*

$$\mathcal{R}^Y(x_1) \geqslant \mathcal{R}^Y(x_2) \quad \textit{and} \quad \mathcal{R}_1^{\mathscr{F}}(x_1) \geqslant \mathcal{R}_1^{\mathscr{F}}(x_2). \tag{4.7}$$

Moreover $\mathcal{R}^Y$ and $\mathcal{R}_1^{\mathscr{F}}$ satisfy property (C3) *of* §2.4 *with respect to $\Gamma = \Gamma_X$. If $I \subseteq [0, R_0]$ is an interval with interior $\mathring{I}$ and if the open annulus $\{T : |T - t_x| \in \mathring{I}\}$ is contained in $X_{\mathscr{H}(x)}$, then $\mathcal{R}^{\mathscr{F}}$ and $\mathcal{R}(-,\mathscr{F})$ are log-concave on I.*

Proof. All the claims for $\mathcal{R}^Y$ immediately follow from (4.3) which is lim inf of super-harmonic functions (and hence log-concave along I). For $\mathcal{R}_1^{\mathscr{F}}$, the claims follow from the equality $\mathcal{R}_1^{\mathscr{F}}(x) = \min\{\mathcal{R}^Y(x), \varrho_{x,X}\}$. More precisely, the bounds (4.7) hold since one has $\varrho_{x_1,X} = \varrho_{x_2,X}$ (cf. Remark 1.10). $\square$

4.1. Spectral radius and spectral norm of the connection

Let $(F, |\cdot|_F) \in E(K)$ and let V be a finite-dimensional F-vector space. A norm $|\cdot|_V$ on V compatible with $|\cdot|_F$ is a map $|\cdot|_V : V \to \mathbb{R}_{\geqslant 0}$ such that (i) $|v|_V = 0$ if and only if $v = 0$; (ii) $|v - v'|_V \leqslant \max\{|v|_V, |v'|_V\}$ for all $v, v' \in V$; (iii) $|fv|_V = |f|_F |v|_V$ for all $f \in F$ and $v \in V$.

If $T : V \to V$ is a bounded $\mathbb{Z}$-linear operator, we define the *norm* and the *spectral norm* of T by

$$|T|_V := \sup_{v \neq 0} \frac{|T(v)|_V}{|v|_V} \quad \text{and} \quad |T|_{\mathrm{Sp},V} := \lim_{s\to\infty} |T^s|_V^{1/s}. \tag{4.8}$$

One proves that the limit exists, and that $|T|_{\mathrm{Sp},V}$ only depends on $|\cdot|_F$ and not on the choice of $|\cdot|_V$ compatible with $|\cdot|_F$ (cf. [Ke3, Definition 6.1.3]).

Let $\omega{:=}\lim_{n\to\infty}|n!|^{1/n}$. If the restriction of $|\cdot|$ to the sub-field of rational numbers $\mathbb{Q}$ is p-adic (resp. trivial), then $\omega{=}|p|^{1/(p-1)}$ (resp. $\omega{=}1$).

If x is not of type 1, then $(\mathscr{H}(x),x){=}(\mathscr{M}(X),x)^{\wedge}$ is the completion of the fraction field $\mathscr{M}(X)$ of $\mathscr{O}(X)$ with respect to the norm x. The following lemma proves that the derivation d/dT is continuous, and hence it extends by continuity to $\mathscr{H}(x)$. Recall that $K{=}\widehat{K^{\mathrm{alg}}}$.

LEMMA 4.3. *Let $x{\in}\mathbb{A}_K^{1,\mathrm{an}}$ be a point of type 2, 3, or 4. Then the operator norm of $(d/dT)^n$ satisfies*

$$\left|\left(\frac{d}{dT}\right)^n\right|_{\mathscr{H}(x)}=\frac{|n!|}{r(x)^n}\quad \textit{and}\quad \left|\frac{d}{dT}\right|_{\mathrm{Sp},\mathscr{H}(x)}=\frac{\omega}{r(x)}. \tag{4.9}$$

Proof. Let $t{\in}D(x)$ be a Dwork generic point for x (cf. §1.0.1). The Taylor expansion at $t{\in}X_\Omega$ gives an injective isometric map of $\mathscr{H}(x)$ into the ring $\mathcal{B}(D(x))$ of bounded functions over $D(x){=}D^-(t,r(x)){\subset}X_\Omega$ commuting with d/dT. The image of $f{\in}\mathscr{H}(x)$ is $\sum_{i\geqslant 0}f^{(i)}(t)(T-t)^i/i!$ and

$$x(f)=x_{t,0}(f_\Omega)=x_{t,r(x)}(f_\Omega)=\sup_{i\geqslant 0}\left|\frac{f^{(i)}(t)}{i!}\right|\cdot r(x)^i.$$

It is well known that

$$\left|\left(\frac{d}{dT}\right)^n\right|_{\mathcal{B}(D(x))}=\frac{|n!|}{r(x)^n},$$

and this implies that

$$\left|\left(\frac{d}{dT}\right)^n\right|_{\mathscr{H}(x)}\leqslant\frac{|n!|}{r(x)^n}.$$

Now, for all $c{\in}K$, one has

$$|n!|=\left|\left(\frac{d}{dT}\right)^n(T-c)^n\right|(x)\leqslant\left|\left(\frac{d}{dT}\right)^n\right|_{\mathscr{H}(x)}|T-c|(x)^n.$$

Hence we find that

$$\left|\left(\frac{d}{dT}\right)^n\right|_{\mathscr{H}(x)}\geqslant\sup_{c\in K}\frac{|n!|}{|t-c|^n}=\frac{|n!|}{r(x)^n},$$

by Lemma 1.1 (because $K{=}\widehat{K^{\mathrm{alg}}}$). □

PROPOSITION 4.4. *Let $x{\in}\mathbb{A}_K^{1,\mathrm{an}}$ be a point of type 2, 3, or 4. Let $(\mathscr{F},\nabla)$ be a differential module over $\mathscr{H}(x)$ endowed with a norm compatible with $|\cdot|(x)$. Then*

$$\omega|\nabla|^{-1}_{\mathrm{Sp},\mathscr{F}}=\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}(x). \tag{4.10}$$

Proof. A direct computation gives that (cf. Proposition 1.3 in [CD] and Lemma 6.2.5 in [Ke3])

$$|\nabla|_{\mathrm{Sp},\mathscr{F}} = \max\left\{\limsup_{n\to\infty} |G_n|(x)^{1/n}, \left|\frac{d}{dT}\right|_{\mathrm{Sp},\mathscr{H}(x)}\right\}, \tag{4.11}$$

where G_n is the matrix of (4.3). By Lemma 4.3, we have $|d/dT|_{\mathrm{Sp},\mathscr{H}(x)}=\omega/r(x)$. □

Remark 4.5. If K is not algebraically closed we still have

$$\left|\frac{d}{dT}\right|_{\mathrm{Sp},\mathscr{H}(x)} = \frac{\omega}{r(x)} \quad \text{and} \quad \frac{|n!|}{r(x,K)^n} \leqslant \left|\left(\frac{d}{dT}\right)^n\right|_{\mathscr{H}(x)} \leqslant \frac{|n!|}{r(x)^n},$$

where $r(x,K):=\min_{c\in X(K)} |t_x-c|_\Omega$. The proof in this case is more involved, and unnecessary for our purposes.

4.2. Spectral Newton polygon of a differential module

Let $x\in X$ be a point of type 2, 3, or 4. By Proposition 4.4, it follows that $\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}(x)$ only depends on the restriction of $\mathscr{F}$ to the differential field $(\mathscr{H}(x), d/dT)$. We now define higher spectral radii following [Ke3]. Let $\mathscr{F}$ be a differential module of rank r over $(\mathscr{H}(x), d/dT)$. Let

$$0 = \mathrm{M}_0 \subset \mathrm{M}_1 \subset \ldots \subset \mathrm{M}_n = \mathscr{F} \tag{4.12}$$

be a Jordan–Hölder sequence of $\mathscr{F}$. This means that for all k, $\mathrm{N}_k:=\mathrm{M}_k/\mathrm{M}_{k-1}$ has no non-trivial strict differential sub-modules.

Let r_k be the rank of N_k, and let $R_k:=\mathcal{R}_1^{\mathrm{N}_k,\mathrm{sp}}(x)$. Perform a permutation of the indices in order to have $R_1\leqslant\ldots\leqslant R_n$. Let $s^{\mathscr{F},\mathrm{sp}}(x)$: $s_1^{\mathscr{F},\mathrm{sp}}(x)\leqslant\ldots\leqslant s_r^{\mathscr{F},\mathrm{sp}}(x)$ be the slope sequence obtained from $\log R_1\leqslant\ldots\leqslant\log R_n$ by counting the slope $\log(R_k)$ with multiplicity r_k:

$$s^{\mathscr{F},\mathrm{sp}}(x)\colon \underbrace{\log R_1 = \ldots = \log R_1}_{r_1 \text{ times}} \leqslant \underbrace{\log R_2 = \ldots = \log R_2}_{r_2 \text{ times}} \leqslant \ldots \leqslant \underbrace{\log R_n = \ldots = \log R_n}_{r_n \text{ times}}. \tag{4.13}$$

We set $\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x):=\exp(s_i^{\mathscr{F},\mathrm{sp}}(x))$. For all $i=1,\ldots,r$ set $h_0^{\mathscr{F},\mathrm{sp}}(x)=0$ and

$$h_i^{\mathscr{F},\mathrm{sp}}(x) := s_1^{\mathscr{F},\mathrm{sp}}(x)+\ldots+s_i^{\mathscr{F},\mathrm{sp}}(x).$$

We say that the *spectral Newton polygon* is the polygon $\mathrm{NP}^{\mathrm{sp}}(x,\mathscr{F}):=\mathrm{NP}(h^{\mathscr{F},\mathrm{sp}}(x))$.

If $\mathscr{F}$ is a differential equation over X, as for $\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}(x)$ (cf. Definition (4.10)), we extend the definition of $\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}$ to the whole X by setting $\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x)=0$ for all x of type 1.

Theorem 4.6. ([Ke3, Theorem 11.3.2, Remarks 11.3.4 and 11.6.5]) *Let $\mathscr{F}$ be a differential module over X. Let $I\subseteq X$ be an (open/closed/semi-open) segment. The following properties hold:*([12])

(i) *The functions $\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}$ and $H_i^{\mathscr{F},\mathrm{sp}}$ satisfy properties* (C2) *and* (C4) *of* §2.4 *along I.*

If moreover I is the skeleton of a virtual annulus in X, then $H_i^{\mathscr{F},\mathrm{sp}}$ is log-concave along I.

(ii) *Points* (ii) (a) *and* (ii) (b) *of Theorem* 3.9 *hold replacing $H_i^{\mathscr{F}}$ by $H_i^{\mathscr{F},\mathrm{sp}}$.*

(iii) *Assume that I is the skeleton of a virtual* open *annulus in X. Assume that $i\leqslant r$ is a spectral non-solvable index at $x\in I$. Then $\partial_b H_i^{\mathscr{F},\mathrm{sp}}(x)=0$ for all but a finite number of germs of segment b out of x, and $H_i^{\mathscr{F},\mathrm{sp}}$ is super-harmonic at x. If moreover i is a vertex at x of the spectral Newton polygon $\mathrm{NP}^{\mathrm{sp}}(x,\mathscr{F})$, then $H_i^{\mathscr{F},\mathrm{sp}}$ is harmonic at x.*

(iv) *If I is contained in an open virtual disk $D\subset X$, and if the index i is spectral non-solvable at $x\in I$, then $H_i^{\mathscr{F},\mathrm{sp}}$ is non-increasing along an open sub-segment J of I containing x.* □

Proposition 4.7. *For all $i=1,...,r$ we have $\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x)=\min\{\mathcal{R}_i^{\mathscr{F}}(x),r(x)\}$. In particular $\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}=\mathcal{R}_i^{\mathscr{F}}$ along $\Gamma(\mathcal{R}_i^{\mathscr{F}})$ by Remark* 3.7.

Proof. If x is a point of type 1, there is nothing to prove. If x is a point of type 2, 3, or 4, $\mathscr{F}$ admits a decomposition separating the spectral radii $\{\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x)\}_i$ (cf. [Ro2] or [Ke3, Theorem 10.6.2]). Now there is also a decomposition of $\mathscr{F}$ separating the radii of convergence of the Taylor solutions at a Dwork generic point t of x that are smaller than or equal to $r(x)$ (cf. [Ro1]).([13]) Both these decompositions behave well by exact sequences. So we may assume that $\mathscr{F}$ satisfies $\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}(x)=...=\mathcal{R}_r^{\mathscr{F},\mathrm{sp}}(x)$, and that its Taylor solutions at t all have the same radius of convergence. The claim then follows from the case $i=1$ (cf. (4.4) plus (4.10)). □

Remark 4.8. (1) By Remark 3.7, for all $\varrho\geqslant\varrho_{x,X}$ we have

$$(\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}\circ\lambda_x)(\varrho)=(\mathcal{R}_i^{\mathscr{F}}\circ\lambda_x)(\varrho).$$

In general, for all $\varrho\geqslant 0$ we have $r(\lambda_x(\varrho))=\max\{\varrho,r(x)\}$, so Proposition 4.7 gives

$$(\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}\circ\lambda_x)(\varrho)=\min\{\max\{r(x),\varrho\},(\mathcal{R}_i^{\mathscr{F}}\circ\lambda_x)(\varrho)\}. \tag{4.14}$$

([12]) The claim of [Ke3, Theorem 11.3.2] is given for segments free of points of type 4, but around a point x of type 4 we can extend the ground field K to turn x into a point $\sigma_{\Omega/K}(x)$ of type 2, and use Remark 4.9 to replace I by $\sigma_{\Omega/K}(I)$. Also the claim of [Ke3] is given for I being the skeleton of an annulus in X. The claim however holds for *the closure* of the skeleton of an open annulus, if the matrix G of ∇ has bounded coefficients on the annulus. This gives our claims by considering a subdivision of I by segments whose interiors are skeletons of open annuli in X.

([13]) The classical proofs of these decomposition results are given for a point of type 2, but they extend smoothly to all points of type 2, 3, or 4 (cf. [NP3] for more details). The key ingredient is the fact that $|d/dT|_{\mathrm{Sp},\mathscr{H}(x)}=\omega/r(x)$.

(2) In the case $i=1$, $\varrho\mapsto(\mathcal{R}_1^{\mathscr{F}}\circ\lambda_x)(\varrho)$ is moreover log-concave and log-decreasing for $\varrho\in[0,\varrho_{x,X}]$, and since $\varrho\mapsto\max\{r(x),\varrho\}$ is log-convex for all $\varrho\in[0,R_0]$, then

(i) If $\mathcal{R}_1^{\mathscr{F}}(x)\leqslant r(x)$, then $(\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}\circ\lambda_x)(\varrho)=(\mathcal{R}_1^{\mathscr{F}}\circ\lambda_x)(\varrho)$ for all $\varrho\in[0,R_0]$;

(ii) If $\mathcal{R}_1^{\mathscr{F}}(x)>r(x)$, then $(\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}\circ\lambda_x)(\varrho)=(\mathcal{R}_1^{\mathscr{F}}\circ\lambda_x)(\varrho)$ for all $\varrho\in[\mathcal{R}_1^{\mathscr{F}}(x),R_0]$.

In particular, if $(\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}\circ\lambda_x)(\varrho)=(\mathcal{R}_1^{\mathscr{F}}\circ\lambda_x)(\varrho)$ for some ϱ, the same equality holds for all $\varrho'\geqslant\varrho$.

(3) The index i is spectral at $\lambda_x(\varrho)$, for all $\varrho\geqslant\mathcal{R}_i^{\mathscr{F}}(x)$. Indeed, by point (6) of Remark 3.4, if $\mathcal{R}_i^{\mathscr{F}}(y)>r(y)$ for some $y=\lambda_x(\varrho)$, then $D^-(t_x,\mathcal{R}_i^{\mathscr{F}}(x))=D^-(t_y,\mathcal{R}_i^{\mathscr{F}}(y))$, so $\mathcal{R}_i^{\mathscr{F}}(x)<\varrho$.

So for all $x\in X$, and all $\varrho\geqslant 0$, we have

$$(\mathcal{R}_i^{\mathscr{F}}\circ\lambda_x)(\varrho)=\begin{cases}\mathcal{R}_i^{\mathscr{F}}(x), & \text{if } \varrho\in[0,\mathcal{R}_i^{\mathscr{F}}(x)],\\ (\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}\circ\lambda_x)(\varrho), & \text{if } \varrho\geqslant\mathcal{R}_i^{\mathscr{F}}(x).\end{cases} \tag{4.15}$$

Thus, $\mathcal{R}_i^{\mathscr{F}}\circ\lambda_x$ and $\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}\circ\lambda_x$ differ by at most two slopes over $[0,\mathcal{R}_i^{\mathscr{F}}(x)]$ (the slopes of $\max\{r(x),\varrho\}$) that can only be 0 or 1 for both radii.

(4) This shows that Theorem 4.6 implies (C2) and (C4) for $\mathcal{R}_i^{\mathscr{F}}$, and hence also for $\mathcal{R}_i(-,\mathscr{F})$, it implies moreover points (ii) and (iii) of Theorem 3.9.

Remark 4.9. The function $\mathcal{R}_i^{\mathscr{F}_\Omega,\mathrm{sp}}$ is not constant over the fiber $\pi_{\Omega/K}^{-1}(x)$, while $\mathcal{R}_i^{\mathscr{F}}$ is constant on it. It follows from Proposition 4.7 that, for all $y\in\pi_{\Omega/K}^{-1}(x)$, we have $\mathcal{R}_i^{\mathscr{F}_\Omega,\mathrm{sp}}(y)=\min\{\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x),r_\Omega(y)\}$. In particular, $\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x)=\mathcal{R}_i^{\mathscr{F}_\Omega,\mathrm{sp}}(\sigma_{\Omega/K}(x))$.

We also recall the following fundamental result.

THEOREM 4.10. ([Ke3, Theorem 12.4.1]) *Let $\mathscr{F}$ be a differential equation of rank r over a disk $\mathscr{O}(D^-(c,\varrho))$, $c\in K$, and $\varrho>0$. Assume that for some $i\leqslant r$ there exists $\varepsilon>0$ such that $h_{i-1}^{\mathscr{F},\mathrm{sp}}$ is constant along $]x_{c,\varrho-\varepsilon},x_{c,\varrho}[$, and moreover $s_{i-1}^{\mathscr{F},\mathrm{sp}}(x_{c,\varrho'})<s_i^{\mathscr{F},\mathrm{sp}}(x_{c,\varrho'})$ for all $\varrho'\in]\varrho-\varepsilon,\varrho[$. Then $\mathscr{F}=\mathscr{F}_{\geqslant i}\oplus\mathscr{F}_{<i}$, where*

(i) *the ranks of $\mathscr{F}_{<i}$ and $\mathscr{F}_{\geqslant i}$ are $i-1$ and $r-i+1$, respectively*;

(ii) *for all $k=1,...,i-1$ one has $s_k^{\mathscr{F}_{<i},\mathrm{sp}}(x_{c,\varrho'})=s_k^{\mathscr{F},\mathrm{sp}}(x_{c,\varrho'})$ for all $\varrho'\in]\varrho-\varepsilon,\varrho[$*;

(iii) *for all $k=i,...,r$ one has $s_{k-i+1}^{\mathscr{F}_{\geqslant i},\mathrm{sp}}(x_{c,\varrho'})=s_k^{\mathscr{F},\mathrm{sp}}(x_{c,\varrho'})$ for all $\varrho'\in]\varrho-\varepsilon,\varrho[$.* □

4.3. Spectral Newton polygon of a differential operator

Let $\mathcal{L}:=\sum_{i=0}^r g_{r-i}(T)(d/dT)^i$ be a differential operator with $g_0=1$ and $g_i\in\mathscr{O}(X)$. We set

$$v_i^{\mathcal{L},\mathrm{sp}}:=-\log(\omega^{-i}|g_i(x)|). \tag{4.16}$$

We define the *spectral Newton polygon* of $\mathcal{L}$ as $\mathrm{NP}(\mathcal{L},x):=\mathrm{NP}(v^{\mathcal{L},\mathrm{sp}})$.

Let $s^{\mathcal{L},\mathrm{sp}}(x):s_1^{\mathcal{L},\mathrm{sp}}(x)\leqslant...\leqslant s_r^{\mathcal{L},\mathrm{sp}}(x)$ be its slope sequence. For $i=1$ we have

$$s_1^{\mathcal{L},\mathrm{sp}}(x)=\log\Bigl(\omega \min_{i=1,\ldots,r}|g_i(x)|^{-1/i}\Bigr). \tag{4.17}$$

We define as usual $\mathcal{R}_i^{\mathcal{L},\mathrm{sp}}(x):=\exp(s_i^{\mathcal{L},\mathrm{sp}}(x))$ and $H_i^{\mathcal{L},\mathrm{sp}}(x):=\exp(h_i^{\mathcal{L},\mathrm{sp}}(x))$.

PROPOSITION 4.11. (Small radii, cf. [Y], [CM5, Theorem 6.2], and [Ke3, Theorem 6.5.3]) *Let $x\in X$ be a point of type* 2, 3, *or* 4, *and let $(\mathscr{F},\nabla)$ be the differential module over $(\mathscr{H}(x),d/dT)$ attached to $\mathcal{L}$. Then $\exp(s_i^{\mathcal{L},\mathrm{sp}}(x))<\omega r(x)$ if and only if $\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x)<\omega r(x)$, and in this case we have*

$$\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x)=\exp(s_i^{\mathcal{L},\mathrm{sp}}(x)). \tag{4.18}$$

□

Remark 4.12. If $g_r\neq 0$, then $s_i^{\mathcal{L},\mathrm{sp}}(x), h_i^{\mathcal{L},\mathrm{sp}}(x)<\infty$. This will be the case of major interest, indeed the case where $g_r=0$ reduces to a lower degree, since we have a factorization $\mathcal{L}=\mathcal{L}_1\circ(d/dT)$ for some $\mathcal{L}_1$.

PROPOSITION 4.13. *Assume that $g_0=1$, $g_r\neq 0$, and that for all i, the function g_i is either equal to* 0, *or it has no zeros on X. Then the following are true*:

(i) *For all $i=0,...,r$ the function $x\mapsto H_i^{\mathcal{L},\mathrm{sp}}(x)\in\mathbb{R}$ satisfies the six properties* (C1)–(C6) *with respect to $\Gamma:=\Gamma_X$ and $\mathscr{C}(H_i^{\mathcal{L},\mathrm{sp}}):=\partial X$. It is hence finite by Theorem* 2.17.

(ii) *For all $i=0,...,r$ one has $\Gamma(h_i^{\mathcal{L},\mathrm{sp}})=\Gamma(H_i^{\mathcal{L},\mathrm{sp}})=\Gamma(s_i^{\mathcal{L},\mathrm{sp}})=\Gamma(\mathcal{R}_i^{\mathcal{L},\mathrm{sp}})=\Gamma_X$.*

(iii) *Assume that $x\in X$ is a point of type* 2, 3, *or* 4, *and that i is a vertex of* $\mathrm{NP}(\mathcal{L},x)$ (*i.e.* $i=r$ *or* $s_i^{\mathcal{L},\mathrm{sp}}(x)<s_{i+1}^{\mathcal{L},\mathrm{sp}}(x)$). *Then $\partial_b H_i^{\mathcal{L},\mathrm{sp}}(x)\in\mathbb{Z}$, and, if $x\notin\partial X$, $H_i^{\mathcal{L},\mathrm{sp}}$ is harmonic at x.*

(iv) *For all $i=1,...,r$ the slopes of $h_i^{\mathcal{L},\mathrm{sp}}$ and $s_i^{\mathcal{L},\mathrm{sp}}$ belong to $\mathbb{Z}\cup\frac{1}{2}\mathbb{Z}\cup...\cup\frac{1}{r}\mathbb{Z}$.*

Proof. Since every g_i has no zeros on X the functions $x\mapsto|g_i(x)|$ are constant on every maximal disk $\mathrm{D}(x,X)$. Hence (ii) holds. If i is a vertex, then

(iii) (a) over all germs of segment $b:=[x,y[$ out of x one has $H_i^{\mathcal{L},\mathrm{sp}}=\omega^i|g_i|^{-1}$;

(iii) (b) $\partial_b H_i^{\mathcal{L},\mathrm{sp}}(x)=\partial_b(x\mapsto|g_i(x)|^{-1})\in\mathbb{Z}$;

(iii) (c) if $x\in X\setminus\partial X$, then $H_i^{\mathcal{L},\mathrm{sp}}$ is harmonic at x.

The rest is straightforward (see for example [Ke3, Theorem 11.2.1]). Namely (iv) is deduced by (iii), by interpolation. This means that if $i_1<i<i_2$ are the vertices of the polygon that are closest to i at x, we define

$$G(y):=h_{i_1}^{\mathcal{L},\mathrm{sp}}(y)+(i-i_1)\frac{h_{i_2}^{\mathcal{L},\mathrm{sp}}(y)-h_{i_1}^{\mathcal{L},\mathrm{sp}}(y)}{i_2-i_1}.$$

Then G is super-harmonic at x, $G(x)=h_i(x)$, and $G\geqslant h_i^{\mathcal{L},\mathrm{sp}}$ around x. So $h_i^{\mathcal{L},\mathrm{sp}}$ is super-harmonic by Lemma 1.15. □

Remark 4.14. To deal with the case in which some g_i is not invertible, it is enough to replace X by a sub-affinoid on which each g_i is either zero or invertible.

Remark 4.15. Let $s'_i(x){:=}\min\{s_i^{\mathcal{L},\mathrm{sp}}(x),\log(\omega r(x))\}$ be the truncated slope sequence.

The partial heights of the corresponding polygon satisfy (C2) and (C4), and the property (C3) with respect to $\Gamma{:=}\Gamma_X$. Of course (as for $\mathcal{R}_1^{\mathscr{F},\mathrm{sp}}(x)$) the constancy skeleton of each partial height is equal to X.

If i is a vertex of the truncated polygon, then the slopes of the ith partial height belong to $\mathbb{Z}$, and property (iv) of Proposition 4.13 holds. While (iii) (c) and super-harmonicity only hold for ith partial heights corresponding to indices i satisfying

$$s'_i(x) < \log(\omega r(x)).$$

4.4. Localization to a sub-affinoid

Let $\mathscr{F}$ be a differential module over X, and let $X'\subseteq X$ be a sub-affinoid domain. The polygon $\mathrm{NP}^{\mathrm{sp}}(x,\mathscr{F})$ only depends on the restricted module $\mathscr{F}(x){=}\mathscr{F}\widehat{\otimes}\mathscr{H}(x)$, so it is invariant by restriction to X'. Conversely, $\mathrm{NP}^{\mathrm{conv}}(x,\mathscr{F})$ is not: the radii change by localization. Hence the following proposition is not a direct consequence of Remark 2.8. The claim is given for the function $\mathcal{R}_i^{\mathscr{F}}$, and an immediate translation gives the analogous statement for $\mathcal{R}_i(-,\mathscr{F})$ (cf. Remark 3.4).

PROPOSITION 4.16. *Let $X'\subseteq X$ be a sub-affinoid. Then the following are true*:

(i) *For all $i{=}1,...,r$ and all $x'\in X'$ one has $\mathcal{R}_i^{\mathscr{F}|_{X'}}(x'){=}\min\{\mathcal{R}_i^{\mathscr{F}}(x'),\varrho_{x',X'}\}$, and*

$$\Gamma(X',\mathcal{R}_i^{\mathscr{F}|_{X'}}) = (\Gamma(X,\mathcal{R}_i^{\mathscr{F}})\cap X')\cup\Gamma_{X'}. \tag{4.19}$$

(ii) *$\mathcal{R}_i^{\mathscr{F}}$ is directionally finite at $x'\in X'$ (cf. (C5)) if and only if $\mathcal{R}_i^{\mathscr{F}|_{X'}}$ is directionally finite at x'.*

(iii) *If $\Gamma_{X'}\subseteq\Gamma(X,\mathcal{R}_i^{\mathscr{F}})$, then for all $x'\in X'$ one has*

$$\mathcal{R}_i^{\mathscr{F}|_{X'}}(x') = \mathcal{R}_i^{\mathscr{F}}(x') \quad \text{and} \quad H_i^{\mathscr{F}|_{X'}}(x') = H_i^{\mathscr{F}}(x'). \tag{4.20}$$

In particular, if X' is an affinoid neighborhood of x' in X, then $H_i^{\mathscr{F}}$ is super-harmonic (resp. harmonic) at x' if and only if so is $H_i^{\mathscr{F}|_{X'}}$.

(iv) *Assume that X' is an affinoid neighborhood of x' in X', and that $\mathcal{R}_i^{\mathscr{F}}(x'){<}\varrho_{x',X'}$. Then for all $j{=}1,...,i$ and all $b\in\Delta(x')$ one has $\partial_b\mathcal{R}_j^{\mathscr{F}}(x'){=}\partial_b\mathcal{R}_j^{\mathscr{F}|_{X'}}(x')$, and $\partial_b H_j^{\mathscr{F}}(x'){=}\partial_b H_j^{\mathscr{F}|_{X'}}(x')$. Hence $H_j^{\mathscr{F}}$ is super-harmonic (resp. harmonic) at x' if and only if so is $H_j^{\mathscr{F}|_{X'}}$.*

Proof. (i) and (ii) The relation $\mathcal{R}_i^{\mathscr{F}|_{X'}}(x')=\min\{\mathcal{R}_i^{\mathscr{F}}(x'),\varrho_{x',X'}\}$ follows from Definition 3.3. This, together with (3.8), gives $\varrho_{\mathcal{R}_i^{\mathscr{F}|_{X'}}}(x')=\min\{\varrho_{\mathcal{R}_i^{\mathscr{F}}}(x'),\varrho_{x',X'}\}$. This implies (4.19), and hence (ii) follows.

(iii) Assume that $\Gamma_{X'}\subseteq\Gamma(X,\mathcal{R}_i^{\mathscr{F}})$. Then by point (iii) of Proposition 2.5, for all $j\leqslant i$ we have

$$\mathcal{R}_j^{\mathscr{F}}(x')\leqslant\mathcal{R}_i^{\mathscr{F}}(x')\leqslant\varrho_{\mathcal{R}_i^{\mathscr{F}}}(x')=\varrho_{\Gamma(X,\mathcal{R}_i^{\mathscr{F}})}(x')\leqslant\varrho_{\Gamma_{X'}}(x')=\varrho_{x',X'}, \tag{4.21}$$

so $\mathcal{R}_j^{\mathscr{F}|_{X'}}(x')=\mathcal{R}_j^{\mathscr{F}}(x')$ for all $x'\in X'$. Thus $H_j^{\mathscr{F}|_{X'}}(x')=H_j^{\mathscr{F}}(x')$ for all $x'\in X'$.

(iv) We have $\mathcal{R}_j^{\mathscr{F}|_{X'}}(x')=\min\{\mathcal{R}_j^{\mathscr{F}}(x'),\varrho_{x',X'}\}=\mathcal{R}_j^{\mathscr{F}}(x')$, since $\mathcal{R}_j^{\mathscr{F}}(x')\leqslant\mathcal{R}_i^{\mathscr{F}}(x')<\varrho_{x',X'}$. Moreover, this remains true by continuity over each germ of segment out of x' (cf. Remark 4.8). □

4.5. Base change by a matrix in the fraction field $\mathscr{M}(X)$ of $\mathscr{O}(X)$

Let $\mathscr{M}(X)$ denote the fraction field of $\mathscr{O}(X)$, and let $H\in GL_r(\mathscr{M}(X))$. Replacing X by a sub-affinoid X' having conveniently small holes around the zeros and poles of $H(T)$ and of $H(T)^{-1}$ we obtain $H,H^{-1}\in GL_r(\mathscr{O}(X'))$. If $x\in X$ is a given point of type 2, 3, or 4, then X' can be chosen as an affinoid neighborhood of x in X, because the zeros and poles are K-rational (recall that $K=\widehat{K^{\mathrm{alg}}}$).

4.5.1. Reduction to a cyclic module

Let $r:=\mathrm{rk}(\mathscr{F})$ be the rank of $\mathscr{F}$. By the cyclic vector theorem (cf. [Ka]) one finds a cyclic basis of $\mathscr{F}(X)\otimes_{\mathscr{O}(X)}\mathscr{M}(X)$ in which $\mathscr{F}$ is represented by an operator

$$\mathcal{L}:=\sum_{i=0}^{r}g_{r-i}(T)\left(\frac{d}{dT}\right)^i,$$

with $g_i\in\mathscr{M}(X)$ for all i, and $g_0=1$.

The operator $\mathcal{L}$ represents simultaneously the connection of all differential modules $\mathscr{F}(x)=\mathscr{F}\otimes_{\mathscr{O}(X)}\mathscr{H}(x)$ for all $x\in X$ of type 2, 3, or 4. If $H(T)\in\mathscr{M}(X)$ is the base change matrix, one can chose $X'\subseteq X$ as indicated in §4.5. In order to fulfill Proposition 4.13, we can further restrict X' in order that none of the g_i has poles nor zeros on it.

By Proposition 4.16, the restriction of $\mathscr{F}$ to X' does not affect the finiteness. If moreover $\Gamma_{X'}\subseteq\Gamma(\mathcal{R}_i^{\mathscr{F}})$, the super-harmonicity of $H_i^{\mathscr{F}}$ is also preserved.

5. Push-forward by Frobenius maps

We here recall and slightly generalize some results about Frobenius maps coming from [Ke3] (cf. also [Ch1], [CD], [Pon], and [Ba]). We study the behavior of $dd^c H_i^{\mathscr{F}}(x)$ by Frobenius descent. In [Ke3] this is done for an annulus, here we generalize it to an affinoid domain of $\mathbb{A}_K^{1,\mathrm{an}}$. Throughout §5, we assume that K is of mixed characteristic $(0,p)$, with $p>0$. Recall that $K=\widehat{K^{\mathrm{alg}}}$.

5.1. Frobenius map

Let T and $\widetilde{T}$ be two variables. The ring morphism $\varphi^{\#}: K[T]\to K[\widetilde{T}]$ sending $f(T)$ into $f(\widetilde{T}^p)$, defines a morphism $\varphi: \mathbb{A}_K^{1,\mathrm{an}}\to\mathbb{A}_K^{1,\mathrm{an}}$. If $t\in\mathbb{A}_K^{1,\mathrm{an}}(\Omega)$ is a Dwork generic point for $x\in\mathbb{A}_K^{1,\mathrm{an}}$, then t^p is a Dwork generic point for $\varphi(x)$. Indeed for all $f\in K[T]$ one has $\varphi(x)(f)=x(f(\widetilde{T}^p))=|f(t^p)|_\Omega$.

We now describe the image of a point of type $x_{t,\varrho}$. For all $\sigma>0$ and $\varrho,\varrho'\geqslant 0$ we set

$$\phi(\sigma,\varrho):=\max\{\varrho^p,|p|\sigma^{p-1}\varrho\}=\begin{cases}\varrho^p, & \text{if } \varrho\geqslant\omega\sigma,\\ |p|\sigma^{p-1}\varrho, & \text{if } \varrho\leqslant\omega\sigma,\end{cases} \tag{5.1}$$

$$\psi(\sigma,\varrho'):=\min\left\{(\varrho')^{1/p},\frac{\varrho'}{|p|\sigma^{p-1}}\right\}=\begin{cases}(\varrho')^{1/p}, & \text{if } \varrho'\geqslant\omega^p\sigma^p,\\ \dfrac{\varrho'}{|p|\sigma^{p-1}}, & \text{if } \varrho'\leqslant\omega^p\sigma^p.\end{cases} \tag{5.2}$$

For σ fixed, ϕ and ψ are increasing functions of ϱ such that $\phi(\sigma,\psi(\sigma,\varrho'))=\varrho'$ and $\psi(\sigma,\phi(\sigma,\varrho))=\varrho$. In the sequel of this section by convention of notation we set

$$\varrho'=\phi(\sigma,\varrho)\quad\text{and}\quad\varrho=\psi(\sigma,\varrho'). \tag{5.3}$$

Proposition 5.1. *Let $c\in K$ and $\varrho>0$. Then*

$$\varphi(x_{c,\varrho})=x_{c^p,\phi(|c|,\varrho)}\quad\textit{and}\quad\varphi^{-1}(x_{c^p,\varrho'})=\{x_{\alpha c,\psi(|c|,\varrho')}\}_{\alpha^p=1}. \tag{5.4}$$

In particular, if $\varrho\geqslant\omega|c|$, $\varphi^{-1}(x_{c^p,\varrho'})$ has an individual point, otherwise it has p distinct points. □

The following proposition describes the image and the inverse image by φ of a disk. We mainly apply this to generic disks, so the center of the disk will be denoted by t, and all disks are Ω-rational.

Proposition 5.2. *Let $t\in\Omega$ and $\varrho,\varrho'\geqslant 0$ be such that $\varrho=\psi(|t|,\varrho')$ and $\varrho'=\phi(|t|,\varrho)$. Then the following holds*:

(i) *One has the equalities*

$$\varphi(D^-(t,\varrho))=D^-(t^p,\varrho')\quad\textit{and}\quad\varphi^{-1}(D^-(t^p,\varrho'))=\bigcup_{\alpha^p=1}D^-(\alpha t,\varrho). \tag{5.5}$$

(ii) *For all* $\alpha\in\boldsymbol{\mu}_p(K)$ *the morphism* $\varphi^{\#}_{\alpha,\varrho}\colon\mathscr{O}(D^-(t^p,\varrho'))\to\mathscr{O}(D^-(\alpha t,\varrho))$ *is injective and isometric in the following sense*: *for all* $f\in\mathscr{O}(D^-(t^p,\varrho'))$ *and all* $\eta<\varrho'$ *one has*

$$|f|_{t^p,\eta}=|\varphi^{\#}_{\alpha,\varrho}(f)|_{\alpha t,\psi(|t|,\eta)}. \tag{5.6}$$

(iii) *If* $\varrho'\leqslant\omega^p|t|^p$ *then, for all* $\alpha\in\boldsymbol{\mu}_p(K)$, $\varphi^{\#}_{\alpha,\varrho}$ *is an isomorphism of rings* (*satisfying* (5.6)).

(iv) *If* $\omega^p|t|^p<\varrho'$, *then* $\varphi^{\#}_{\varrho}:=\varphi^{\#}_{\alpha,\varrho}$ *is independent of* α. *Moreover* $\boldsymbol{\mu}_p(K)$ *acts on* $\mathscr{O}(D^-(t,\varrho))$ *by* $\alpha(f)(\widetilde{T}):=f(\alpha\widetilde{T})$, *and* $\varphi^{\#}_{\varrho}(\mathscr{O}(D^-(t^p,\varrho')))=\mathscr{O}(D^-(t,\varrho))^{\boldsymbol{\mu}_p(K)}$. □

The morphism φ induces a K-linear isometric inclusion $\varphi^{\#}\colon\mathscr{H}(\varphi(x))\to\mathscr{H}(x)$.

Corollary 5.3. *Let* $c\in K$, *let* $\varrho\geqslant0$, *and let* $x=x_{c,\varrho}$. *If* $\varrho\neq\omega|c|$, *then the morphism* $\varphi\colon\mathbb{A}^{1,\mathrm{an}}_K\to\mathbb{A}^{1,\mathrm{an}}_K$ *provides a bijection*

$$\varphi\colon\Delta(x)\xrightarrow{\sim}\Delta(\varphi(x)). \tag{5.7}$$

If $\varrho=\omega|c|$, *then* (5.7) *is surjective. The inverse image of the germ of segment out of* $\varphi(x)$ *directed towards* ∞ *has a single element, while the inverse image of each other germ of segment out of* $\varphi(x)$ *is formed by* p *distinct germs of segment out of* x. □

Proposition 5.4. *Let* $c\in K$, $x=x_{c,\varrho}$ *and* $\varrho\geqslant0$. *Then*

(i) *if* $\varrho<\omega|c|$, *then* $[\mathscr{H}(x_{c,\varrho})\colon\mathscr{H}(\varphi(x_{c,\varrho}))]=1$;

(ii) *if* $\varrho>\omega|c|$, *then* $[\mathscr{H}(x_{c,\varrho})\colon\mathscr{H}(\varphi(x_{c,\varrho}))]=p$.

If X *is an affinoid domain of* $\mathbb{A}^{1,\mathrm{an}}_K$, *the same relations hold by density replacing* $\mathscr{H}(x)$ *and* $\mathscr{H}(\varphi(x))$ *by the local rings* $\mathscr{O}_{X,x}$ *and* $\mathscr{O}_{X^p,\varphi(x)}$, *respectively.* □

Remark 5.5. If $x=x_{c,\varrho}$ fulfills the assumptions of Proposition 5.4 (i), then so does also $\varphi(x)$ with respect to $\varphi^2(x)$. This is no longer true if we are in the situation (ii). Namely, if $x=x_{c,\varrho}$ satisfies $\varrho>\omega^{1/p^n}|c|$, then for all $k=1,...,n$, $\varphi^k(x)=x_{c^{p^k},\varrho^{p^k}}$ satisfies $\varrho^{p^k}>\omega|c|^{p^k}$, and $[\mathscr{H}(\varphi^k(x))\colon\mathscr{H}(\varphi^{k+1}(x))]=p$. While $[\mathscr{H}(\varphi^{n+1}(x))\colon\mathscr{H}(\varphi^{n+2}(x))]=1$.

5.2. Behavior of spectral non-solvable radii by Frobenius push-forward

The map $\varphi^{\#}\colon\mathscr{O}(X^p)\to\mathscr{O}(X)$ satisfies

$$\left(\frac{d/d\widetilde{T}}{p\widetilde{T}^{p-1}}\right)\circ\varphi^{\#}=\varphi^{\#}\circ\frac{d}{dT}.$$

Let $\mathscr{F}$ be a finite free module of rank r over $\mathscr{O}(X)$, and let $\nabla\colon\mathscr{F}\to\mathscr{F}$ be a connection with respect to $d/d\widetilde{T}$. The push-forward of $(\mathscr{F},\nabla)$ is the $(\mathscr{O}(X^p),d/dT)$-differential module

$$\left(\mathscr{F},\frac{1}{p\widetilde{T}^{p-1}}\nabla\right),$$

obtained by considering $\mathscr{F}$ as an $\mathscr{O}(X^p)$-module via $\varphi^{\#}$, so that

$$\frac{1}{p\widetilde{T}^{p-1}}\nabla\colon \mathscr{F}\longrightarrow\mathscr{F}$$

is a connection with respect to d/dT. We will denote it by

$$(\varphi_*\mathscr{F},\varphi_*\nabla):=\left(\mathscr{F},\frac{1}{p\widetilde{T}^{p-1}}\nabla\right).$$

Let $x\in X$ be a point of type 2, 3, or 4. Spectral non-solvable radii of $\mathscr{F}$ at x only depend on its restriction to $\mathscr{H}(x)$. We study separately the two cases of Proposition 5.4.

We firstly consider the situation (i) of Proposition 5.4, where $x=x_{c,\varrho}$, with $\varrho<\omega|c|$. In this case $\varphi^{\#}\colon\mathscr{H}(\varphi(x))\to\mathscr{H}(x)$ is an isomorphism of fields, and hence the scalar extension and the restriction of scalars functors are equivalences of categories. By Proposition 5.2 the radii of all sub-disks of the generic disk $D(x)$ are multiplied by $|p|\,|t|^{p-1}$, where t is a Dwork generic point for x.(14) So for all $i=1,...,r$ we have

$$\mathcal{R}_i^{\varphi_*(\mathscr{F}),\mathrm{sp}}(x)=|p|\,|t|^{p-1}\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x). \tag{5.8}$$

In the situation (ii) of Proposition 5.4, where $x=x_{c,\varrho}$, with $\varrho>\omega|c|$, the map $\varphi^{\#}\colon\mathscr{H}(\varphi(x))\to\mathscr{H}(x)$ is a field extension of degree p. The situation is then regulated by the following results.

PROPOSITION 5.6. *Let $x\in\mathbb{A}_K^{1,\mathrm{an}}$ be a point of type 2, 3, or 4 of the form $x=x_{c,\varrho}$, with $c\in K$, and $\varrho>\omega|c|$. Let $\mathscr{F}$ be a differential module over $\mathscr{H}(x)$ of rank r.*

Define $0\leqslant i_1(x)\leqslant r$ as the index satisfying(15)

$$\mathcal{R}_{i_1(x)}^{\mathscr{F},\mathrm{sp}}(x)\leqslant\omega|t|<\mathcal{R}_{i_1(x)+1}^{\mathscr{F},\mathrm{sp}}(x). \tag{5.9}$$

Then, up to a permutation, the list (with multiplicities) of the spectral radii of $\varphi_*(\mathscr{F})$ is given by

$$\bigcup_{1\leqslant i\leqslant i_1(x)}\{\underbrace{|p|\,|t|^{p-1}\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x),...,|p|\,|t|^{p-1}\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x)}_{p\text{ times}}\}\bigcup_{i_1(x)<i\leqslant r}\{\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x)^p,\underbrace{\omega^p|t|^p,...,\omega^p|t|^p}_{p-1\text{ times}}\}. \tag{5.10}$$

Proof. The proof follows [Ke3, Theorem 10.5.1], with slight modifications. □

(14) Note that $x=x_{c,\varrho}$, and hence $|t|=|\widetilde{T}|(x)=\max\{|c|,\varrho\}=\max\{|c|,r(x)\}$, since $r(x)=\varrho$.
(15) It is understood that $i_1(x)=0$ if and only if $\mathcal{R}_i^{\mathscr{F},\mathrm{sp}}(x)>\omega|t|$ for all i.

Corollary 5.7. *We maintain the assumptions of Proposition 5.6. Let* i_x^{sp} (*resp.* $i_1(x)$) *be the largest index satisfying* $\mathcal{R}_i^{\mathscr{F}}(x)<r(x)$ (*resp.* $\mathcal{R}_i^{\mathscr{F}}(x)\leqslant\omega\max\{|c|,r(x)\}$ *as in* (5.9)).

For all $i\in\{1,...,r\}$ *we define*(16)

$$\phi(i,x):=\begin{cases} pi, & \text{if } 1\leqslant i<i_1(x),\\ (p-1)r+i, & \text{if } i_1(x)\leqslant i\leqslant r,\end{cases}\qquad d_i(x):=\begin{cases} i, & \text{if } 1\leqslant i<i_1(x),\\ r, & \text{if } i_1(x)\leqslant i\leqslant r,\end{cases}$$

and

$$\ell_{i,x}(\widetilde{T}):=(p\widetilde{T}^{(p-1)})^{d_i(x)}\in\mathscr{O}(X).$$

Let $i\in\{1,...,i_x^{\mathrm{sp}}\}$. *Then*

$$|\ell_{i,x}|(x)H_i^{\mathscr{F}}(x)=H_{\phi(i,x)}^{\varphi_*\mathscr{F}}(\varphi(x))^{1/p}. \tag{5.11}$$

Proof. Write

$$\begin{aligned} s_1^{\mathscr{F}}(x)\leqslant...\leqslant s_{i_1(x)}^{\mathscr{F}}(x)\leqslant\log(\omega|t|)<s_{i_1(x)+1}^{\mathscr{F}}(x)\\ \leqslant...\leqslant s_{i_x^{\mathrm{sp}}}^{\mathscr{F}}(x)<\log\varrho\leqslant s_{i_x^{\mathrm{sp}}+1}^{\mathscr{F}}(x)\leqslant...\leqslant s_r^{\mathscr{F}}(x).\end{aligned}$$

The behavior of spectral non-solvable radii is given by Proposition 5.6, so we have

$$\begin{aligned} s^{\varphi_*\mathscr{F}}(\varphi(x))\colon &\underbrace{\log(|p||t|^{p-1})+s_1^{\mathscr{F}}(x)=...=\log(|p||t|^{p-1})+s_1^{\mathscr{F}}(x)}_{p\text{ times}}\leqslant...\\ &\leqslant\underbrace{\log(|p||t|^{p-1})+s_{i_1(x)}^{\mathscr{F}}(x)=...=\log(|p||t|^{p-1})+s_{i_1(x)}^{\mathscr{F}}(x)}_{p\text{ times}}\\ &\leqslant\underbrace{\log(\omega^p|t|^p)=...=\log(\omega^p|t|^p)}_{(p-1)(r-i_1(x))\text{ times}}<ps_{i_1(x)+1}^{\mathscr{F}}(x)\leqslant...\leqslant ps_{i_x^{\mathrm{sp}}}^{\mathscr{F}}(x)<p\log\varrho\leqslant...,\end{aligned}\tag{5.12}$$

where t is a Dwork generic point for x. Hence, for all $i<i_1(x)$, we have

$$h_{pi}^{\varphi_*\mathscr{F}}(\varphi(x))=ph_i^{\mathscr{F}}(x)+pi\log(|p||t|^{p-1}).$$

And if $i_1(x)\leqslant i\leqslant i_x^{\mathrm{sp}}$, then

$$\begin{aligned} h_{(p-1)r+i}^{\varphi_*\mathscr{F}}(\varphi(x))&=ph_i^{\mathscr{F}}(x)+pi_1(x)\log(|p||t|^{p-1})+(p-1)(r-i_1(x))\log(\omega^p|t|^p)\\ &=ph_i^{\mathscr{F}}(x)+pr\log(|p||t|^{p-1}).\end{aligned}$$

This proves (5.11). □

Remark 5.8. The proof shows also that if i is a spectral non-solvable index, then i is a vertex of $\mathrm{NP}^{\mathrm{conv}}(x,\mathscr{F})$ (i.e. $i=r$ or $s_i^{\mathscr{F}}(x)<s_{i+1}^{\mathscr{F}}(x)$) if and only if $\phi(i,x)$ is a vertex of $\mathrm{NP}^{\mathrm{conv}}(\varphi(x),\varphi_*(\mathscr{F}))$.

(16) It is understood that if $i_1(x)=i_x^{\mathrm{sp}}$ then $\phi(i,x)=pi$ and $d_i(x)=i$ for all $i\in\{1,...,i_x^{\mathrm{sp}}\}$.

5.3. Behavior of the (spectral non-solvable) slopes by Frobenius push-forward

We now study the behavior of the slopes of the radii along the germs of segment out of a point.

We maintain the notation of §5.2. If $x\in X$ is a point of type 2, 3, or 4, the local ring $\mathscr{O}_{X,x}$ is a differential field. For $b\in\Delta(x)$, the slopes $\partial_b\mathcal{R}_1^{\mathscr{F}}(x),...,\partial_b\mathcal{R}_{i_x^{\mathrm{sp}}}^{\mathscr{F}}(x)$ of the spectral non-solvable radii of $\mathscr{F}$ only depend on its restriction to $\mathscr{O}_{X,x}$. Indeed spectral radii are stable by localization, and if an index i is spectral non-solvable at x, by continuity (cf. Theorem 4.6) it remains spectral non-solvable over b. As above we distinguish the two situations of Proposition 5.4.

If we are in the situation (i) of Proposition 5.4, then φ is a trivial covering of disks (cf. Proposition 5.2). So, if b is a germ of segment out of x, then for all $i=1,...,i_x^{\mathrm{sp}}$ we have

$$\partial_b\mathcal{R}_i^{\mathscr{F}}(x)=\partial_{\varphi(b)}\mathcal{R}_i^{\varphi_*(\mathscr{F})}(\varphi(x)) \quad \text{and} \quad \partial_b H_i^{\mathscr{F}}(x)=\partial_{\varphi(b)}H_i^{\varphi_*(\mathscr{F})}(\varphi(x)). \tag{5.13}$$

The Laplacians are then naturally identified.

In the situation of the point (ii) of Proposition 5.4, the map $\varphi^{\#}\colon\mathscr{O}_{X^p,\varphi(x)}\to\mathscr{O}_{X,x}$ is a field extension of degree p. Let $b=]x,y[$ be a germ of segment out of x. We now compare the slopes $\partial_b H_i^{\mathscr{F}}$ with those of $\partial_b H_{\phi(i,x)}^{\varphi_*\mathscr{F}}$. We can restrict $]x,y[$ in order that the function $z\mapsto i_1(z)$ is constant over $b=]x,y[$. We call the corresponding quantities $i_1(b)$, $\phi(i,b)$, $d_i(b)$, and $\ell_{i,b}$. For $i\leqslant i_x^{\mathrm{sp}}$ we have

$$\partial_{\varphi(b)}H_{\phi(i,b)}^{\varphi_*(\mathscr{F})}(\varphi(x))=\partial_b H_i^{\mathscr{F}}(x)+\partial_b|\ell_{i,b}|(x).\text{[17]} \tag{5.14}$$

Notice that we may have $\phi(i,x)\neq\phi(i,b)$, namely this can only happen if $\mathcal{R}_{i_1(x)}^{\mathscr{F}}(x)=\omega|t|$. However it follows from (5.12) that, if $i\leqslant i_x^{\mathrm{sp}}$ is a *vertex* at x of the convergence Newton polygon, then $\phi(i,x)=\phi(i,b)$ for all $b\in\Delta(x)$. The same happens for $\ell_{i,x}$. We then denote them by $\phi(i)$ and ℓ_i.

Assume that $i\leqslant i_x^{\mathrm{sp}}$ is a vertex of $\mathrm{NP}^{\mathrm{conv}}(x,\mathscr{F})$. Then, for all $b\in\Delta(x)$, we have

$$\partial_{\varphi(b)}H_{\phi(i)}^{\varphi_*(\mathscr{F})}(\varphi(x))=\partial_b H_i^{\mathscr{F}}(x)+\partial_b|\ell_i|(x). \tag{5.15}$$

By (5.7), the directions out of x coincide with those out of $\varphi(x)$. Moreover ℓ_i is harmonic, so the Laplacians are also identified (once we will prove that the sum defining the Laplacian is finite):

$$dd^c H_i^{\mathscr{F}}(x)=dd^c H_{\phi(i)}^{\varphi_*(\mathscr{F})}(\varphi(x)). \tag{5.16}$$

[17] The natural parametrization (1.13) of $\varphi(b)$ multiplies the distances by p, while the exponent $1/p$ of $H_{\phi(i,b)}^{\varphi_*(\mathscr{F})}(\varphi(x))$ divides the result by p. So globally we have (5.14).

In the following proposition we allow for the case where the valuation of K is trivial on $\mathbb{Z}$.

Proposition 5.9. *Let $x\in X$ be a point of type* 2, 3, *or* 4. *If the index i is free of solvability at x* (*cf. Definition* 3.6), *then*

(1) *the slopes of $\mathcal{R}_1^{\mathscr{F}}, ..., \mathcal{R}_i^{\mathscr{F}}$ and of $H_1^{\mathscr{F}}, ..., H_i^{\mathscr{F}}$ are zero for all but a finite number of germs of segment out of x*;

(2) *$H_1^{\mathscr{F}}, ..., H_i^{\mathscr{F}}$ are super-harmonic at x and satisfy properties* (iii) *and* (iv) *of Proposition* 4.13 *around x* (*cf. also Remark* 4.15);

(3) *in particular if i is a vertex of $\mathrm{NP}^{\mathrm{conv}}(x,\mathscr{F})$, then $H_i^{\mathscr{F}}(x)$ is harmonic at x.*

Proof. Over-solvable radii are constant around x by (3.8), so they do not play any role, and we may assume that $i\leqslant i_x^{\mathrm{sp}}$. Assume first that K is of mixed characteristic $(0,p)$, with $p>0$. The radii $\mathcal{R}_i^{\mathscr{F}}$ are insensitive to scalar extension, so replacing K by a larger field we can assume that x is of type 2, and by a translation we may assume $x=x_{c,\varrho}$, with $c=0$. This guarantees that for all $k\geqslant 0$, $\varphi^k(x)$ satisfies the situation (ii) of Proposition 5.4 (cf. Remark 5.5). We apply Frobenius push-forward several times in order that $\mathcal{R}_{\phi^n(i)}^{\varphi_*^n\mathscr{F}}(\varphi^n(x))<\omega|t^{p^n}|$ (which is the assumption of Proposition 4.11). By continuity (cf. Remark 4.8) this assumption remains satisfied along all directions out of $\varphi^n(x)$. Now, by point (iv) of Proposition 4.16, and by §4.5.1, we can localize and pass to a cyclic basis without affecting the super-harmonicity, nor the directional finiteness. In a cyclic basis the radii are explicitly intelligible by Propositions 4.11 and 4.13. Now, by (5.14), for all $b\in\Delta(x)=\Delta(\varphi^n(x))$ the slope $\partial_b H_i^{\mathscr{F}}(x)$ appears among those in the family $\{\partial_{\varphi^n(b)} H_j^{\varphi_*^n\mathscr{F}}(\varphi^n(x))\}_j$. For almost all $b\in\Delta(x)$ these slopes are all zero by Propositions 4.11 and 4.13, so (i) holds. Moreover to prove the other statements we may assume, by interpolation, that i is a vertex at x of $\mathrm{NP}^{\mathrm{conv}}(\mathscr{F})$ (cf. the proof of Proposition 4.13), so we can use (5.16) to reduce to Propositions 4.11 and 4.13.

The case where the valuation is trivial on $\mathbb{Z}$ is much easier. Indeed $\omega=1$, and we may immediately apply Propositions 4.11 and 4.13, without involving any Frobenius machinery. □

6. Proof of the main Theorem 3.9

The properties of Theorem 3.9 are invariant by scalar extension of the ground field K. So, from now on, we assume that K is algebraically closed, and spherically complete.

By Remark 3.4, $\mathcal{R}_i^{\mathscr{F}}$ and $\mathcal{R}_i(-,\mathscr{F})$ are closely related. Indeed the function $x\mapsto\varrho_{x,X}$ is continuous, locally constant outside Γ_X, and with slope 1 on each segment in Γ_X oriented towards ∞. As is clearly stated in Theorem 3.9, each assertion about $\mathcal{R}_i(-,\mathscr{F})$

and $H_i(-,\mathscr{F})$ is equivalent to an assertion about $\mathcal{R}_i^{\mathscr{F}}$ and $H_i^{\mathscr{F}}$. In the following we prove those assertions for $\mathcal{R}_i^{\mathscr{F}}$ and $H_i^{\mathscr{F}}$ since the super-harmonicity and localization properties are more easy.

We begin by describing the link between the graphs of the partial heights and those of the radii.

Remark 6.1. For $i\leqslant r$, let $\boldsymbol{\mathcal{R}}_i^{\mathscr{F}}: X\to\mathbb{R}^i$ be the function defined by

$$\boldsymbol{\mathcal{R}}_i^{\mathscr{F}}(x) := (\mathcal{R}_1^{\mathscr{F}}(x), ..., \mathcal{R}_i^{\mathscr{F}}(x)). \tag{6.1}$$

Define analogously $\boldsymbol{H}_i^{\mathscr{F}}$, $\boldsymbol{s}_i^{\mathscr{F}}$, and $\boldsymbol{h}_i^{\mathscr{F}}$. Clearly $\varrho_{\boldsymbol{\mathcal{R}}_i^{\mathscr{F}}}(x)=\min_{j=1,\dots,i}\varrho_{\mathcal{R}_i^{\mathscr{F}}}(x)$, so that

$$\Gamma(\boldsymbol{\mathcal{R}}_i^{\mathscr{F}}) = \bigcup_{j=1}^{i}\Gamma(\mathcal{R}_j^{\mathscr{F}}(x)) = \Gamma_i. \tag{6.2}$$

Hence the finiteness of $\boldsymbol{\mathcal{R}}_r^{\mathscr{F}}$ is equivalent to the finiteness of all $\mathcal{R}_i^{\mathscr{F}}$. The same holds for $\boldsymbol{H}_i^{\mathscr{F}}$, $\boldsymbol{s}_i^{\mathscr{F}}$, and $\boldsymbol{h}_i^{\mathscr{F}}$.

The maps $\boldsymbol{\mathcal{R}}_i^{\mathscr{F}}$ and $\boldsymbol{H}_i^{\mathscr{F}}$ are the exponentials of $\boldsymbol{s}_i^{\mathscr{F}}$ and $\boldsymbol{h}_i^{\mathscr{F}}$, respectively, and the exponential map is injective, so we are reduced to prove the finiteness of $\boldsymbol{s}_i^{\mathscr{F}}$ and $\boldsymbol{h}_i^{\mathscr{F}}$. The functions $\boldsymbol{s}_i^{\mathscr{F}}$ and $\boldsymbol{h}_i^{\mathscr{F}}$ are related by the bijective map $\boldsymbol{h}_i^{\mathscr{F}}(x)=U\boldsymbol{s}_i^{\mathscr{F}}(x)$, where $U\in\mathrm{GL}_r(\mathbb{Z})$ is the matrix $U=(u_{i,j})$ with $u_{i,j}=1$ if $i\geqslant j$ and $u_{i,j}=0$ otherwise. This proves that for all $i=1,...,r$ (cf. Definition 3.8)

$$\Gamma_i = \Gamma(\boldsymbol{\mathcal{R}}_i^{\mathscr{F}}) = \Gamma(\boldsymbol{H}_i^{\mathscr{F}}) = \Gamma(\boldsymbol{h}_i^{\mathscr{F}}) = \Gamma(\boldsymbol{s}_i^{\mathscr{F}}). \tag{6.3}$$

In particular the finiteness of all the partial heights is equivalent to that of all the radii.

The proof of Theorem 3.9 consists in showing that $H_i^{\mathscr{F}}$ satisfy properties (C1)–(C6) in §2.4, plus the other claims of the theorem. We prove them by induction on i.

We already know, by Remark 4.8, that (C1), (C2), and (C4) hold for $\mathcal{R}_i^{\mathscr{F}}$ and $H_i^{\mathscr{F}}$, and moreover that points (ii) and (iii) of Theorem 3.9 follow from the analogous claims for spectral radii (cf. Theorem 4.6). Finally, point (v) of Theorem 3.9 is proved in point (iii) of Proposition 5.9.

It remains to prove the weak super-harmonicity (i.e. point (iv) of Theorem 3.9), together with (C3) and (C5). This will imply the finiteness by Theorem 2.17.

More precisely we prove, by induction on i, that $H_i^{\mathscr{F}}$ satisfies (C3), (C5), and (C6), with respect to $\Gamma:=\Gamma_{i-1}$ (where $\Gamma_0:=\Gamma_X$), and $\mathscr{C}(H_i^{\mathscr{F}}):=\mathscr{C}_i$ (which is finite by induction, by definition (b) and (c) of point (iv) of Theorem 3.9).

By Proposition 4.2, we know that $H_1^{\mathscr{F}}=\mathcal{R}_1^{\mathscr{F}}$ satisfies (C3) with respect to $\Gamma:=\Gamma_X$.

6.1. Property (C3) for $H_i^{\mathscr{F}}$

Let $D\subset X$ be an open disk on which $\mathcal{R}_1^{\mathscr{F}},...,\mathcal{R}_{i-1}^{\mathscr{F}}$ are constant with values $R_1,...,R_{i-1}$, i.e. $D\cap\Gamma_{i-1}=\varnothing$. Let $b_0:=1$, and if $1\leqslant k\leqslant i-1$ let $b_k:=\prod_{j=1}^{k} R_j$. Then $H_i^{\mathscr{F}}=b_{i-1}\mathcal{R}_i^{\mathscr{F}}$ over D. The functions $H_i^{\mathscr{F}}$ and $\mathcal{R}_i^{\mathscr{F}}$ then have the same properties over D. In particular

$$\Gamma(H_i^{\mathscr{F}})\cap D=\Gamma(\mathcal{R}_i^{\mathscr{F}})\cap D. \tag{6.4}$$

The following proposition asserts that $\mathcal{R}_i^{\mathscr{F}}$ coincides over D with the first radius $\mathcal{R}_1^{\mathscr{F}_{\geqslant i}}$ of a certain sub-module $\mathscr{F}_{\geqslant i}\subseteq\mathscr{F}|_D$ coming from Theorem 4.10.

This is the crucial property for the induction in the proof of Theorem 3.9. It constitutes a generalization to higher radii of the transfer principle (cf. Proposition 4.2).

PROPOSITION 6.2. (Transfer principle) *With the above setting, two situations are possible over D:*

(i) *The function $\mathcal{R}_i^{\mathscr{F}}$ is also constant on D;*

(ii) *$\mathcal{R}_{i-1}^{\mathscr{F}}(x)<\mathcal{R}_i^{\mathscr{F}}(x)$ for all $x\in D$, and one has a decomposition $\mathscr{F}|_D=\mathscr{F}_{\geqslant i}\oplus\mathscr{F}_{<i}$ as in Theorem 4.10 satisfying moreover $\mathcal{R}_i^{\mathscr{F}}(x)=\mathcal{R}_1^{\mathscr{F}_{\geqslant i}}(x)$ for all $x\in D$.*
In particular $\mathcal{R}_i^{\mathscr{F}}$ and $H_i^{\mathscr{F}}$ satisfy (C3) *with respect to $\Gamma:=\Gamma_{i-1}$, and they both satisfy all the properties of a first radius of convergence outside Γ_{i-1}.*

Proof. Assume $\mathcal{R}_i^{\mathscr{F}}$ non-constant over D. In this case we have $\mathcal{R}_k^{\mathscr{F}}=\mathcal{R}_k^{\mathscr{F}|_D}$ over D for all $k\leqslant i$. Indeed, by (3.8), the non-constancy gives for all $x\in D$, $\mathcal{R}_k^{\mathscr{F}}(x)\leqslant\mathcal{R}_i^{\mathscr{F}}(x)\leqslant\varrho_{\mathcal{R}_i^{\mathscr{F}}}(x)<\varrho$, where ϱ is the radius of D. Hence $\mathcal{R}_k^{\mathscr{F}|_D}(x)=\min\{\mathcal{R}_k^{\mathscr{F}}(x),\varrho\}=\mathcal{R}_k^{\mathscr{F}}(x)$.

The functions $H_i^{\mathscr{F}}$ and $\mathcal{R}_i^{\mathscr{F}}$ have the same slopes over D because $H_i^{\mathscr{F}}=b_{i-1}\mathcal{R}_i^{\mathscr{F}}$. So $\mathcal{R}_i^{\mathscr{F}}$ satisfies the concavity property (b) of point (iii) of Theorem 3.9 which we have already proved.

Hence, if $c\in D$ is a K-rational point, and if $R:=\mathcal{R}_i^{\mathscr{F}}(c)$, then along the segment $]x_{c,R},x_{c,\varrho}[$ we must have $\mathcal{R}_i^{\mathscr{F}}>\mathcal{R}_{i-1}^{\mathscr{F}}$, because $\mathcal{R}_i^{\mathscr{F}}$ is concave on it, while $\mathcal{R}_{i-1}^{\mathscr{F}}$ is constant on D.

Now, by (4.15), $\mathcal{R}_i^{\mathscr{F}}$ is spectral along $]x_{c,R},x_{c,\varrho}[$, and hence $\mathcal{R}_{i-1}^{\mathscr{F}}$ is spectral non-solvable on it. So, by Theorem 4.10, there is a unique direct sum decomposition $\mathscr{F}|_D=\mathscr{F}_{\geqslant i}\oplus\mathscr{F}_{<i}$ such that for all $x\in]x_{c,R},x_{c,\varrho}[$ one has $\mathcal{R}_{k-i+1}^{\mathscr{F}_{\geqslant i}}(x)=\mathcal{R}_k^{\mathscr{F}}(x)$ for $k=i,...,r$, and $\mathcal{R}_k^{\mathscr{F}_{<i}}(x)=\mathcal{R}_k^{\mathscr{F}}(x)$ for $k=1,...,i-1=\mathrm{rank}(\mathscr{F}_{<i})$.

We now prove that these equalities hold for all $x\in D$, the claim will then follow.

By Proposition 3.5 the convergence radii of $\mathscr{F}$ at x are the union (with multiplicities) of those of $\mathscr{F}_{\geqslant i}$ and of $\mathscr{F}_{<i}$. Therefore it is enough to prove that for all $x\in D$ one has $\mathcal{R}_{i-1}^{\mathscr{F}_{<i}}(x)<\mathcal{R}_1^{\mathscr{F}_{\geqslant i}}(x)$.

LEMMA 6.3. *Let $x\in D$. If $\mathcal{R}_{i-1}^{\mathscr{F}}(x)<\mathcal{R}_1^{\mathscr{F}_{\geqslant i}}(x)$, then $\mathcal{R}_{i-1}^{\mathscr{F}_{<i}}(x)<\mathcal{R}_1^{\mathscr{F}_{\geqslant i}}(x)$.*

Proof. Since $\mathscr{F}_{\geqslant i}\subseteq\mathscr{F}$, the assumption implies that

$$\{\mathcal{R}_1^{\mathscr{F}_{\geqslant i}}(x),...,\mathcal{R}_{r-i+1}^{\mathscr{F}_{\geqslant i}}(x)\}\subset\{\mathcal{R}_i^{\mathscr{F}}(x),...,\mathcal{R}_r^{\mathscr{F}}(x)\}.$$

Moreover, by point (7) of Remark 3.4, the two multi-sets must coincide since they are equipotent. By difference, this implies that $\mathcal{R}_k^{\mathscr{F}_{<i}}(x)=\mathcal{R}_k^{\mathscr{F}}(x)$ for all $k\leqslant i-1$. □

We now prove that $\mathcal{R}_{i-1}^{\mathscr{F}}(x)<\mathcal{R}_1^{\mathscr{F}_{\geqslant i}}(x)$ for all $x\in D$. Since the radii are insensitive to scalar extensions of K, we may assume that x is K-rational.

Now $\mathcal{R}_1^{\mathscr{F}_{\geqslant i}}$ is log-concave with non-positive log-slopes along $[x,x_{c,\varrho}[$. Then, for all ϱ' close enough to ϱ, one has $\mathcal{R}_1^{\mathscr{F}_{\geqslant i}}(x)\geqslant\mathcal{R}_1^{\mathscr{F}_{\geqslant i}}(x_{c,\varrho'})=\mathcal{R}_i^{\mathscr{F}}(x_{c,\varrho'})>\mathcal{R}_{i-1}^{\mathscr{F}}(x_{c,\varrho'})=\mathcal{R}_{i-1}^{\mathscr{F}}(x)$, as desired. □

Remark 6.4. By Proposition 2.14, we have $\Gamma(\mathcal{R}_i^{\mathscr{F}})\cap D\neq\varnothing$ if and only if $\partial_b\mathcal{R}_i^{\mathscr{F}}(x)>0$, where x is the point at the boundary of D, and b is the germ of segment out of x lying in D oriented inside D.

6.2. Finiteness (C5) and super-harmonicity (C6)

The following crucial lemma describes the locus of points where $\mathcal{R}_i^{\mathscr{F}}$ is solvable or over-solvable.

Lemma 6.5. *If $\mathcal{R}_i^{\mathscr{F}}(x)\geqslant r(x)$, then either $x\notin\Gamma(\mathcal{R}_i^{\mathscr{F}})$ or, if $x\in\Gamma(\mathcal{R}_i^{\mathscr{F}})$, the following conditions hold*:

(i) *If $x\in\Gamma(\mathcal{R}_i^{\mathscr{F}})\setminus\Gamma_{i-1}$, then x is a boundary point of $\Gamma(\mathcal{R}_i^{\mathscr{F}})$*;

(ii) *If $x\in\Gamma_{i-1}\cap\Gamma(\mathcal{R}_i^{\mathscr{F}})$, then $\Delta(x,\Gamma(\mathcal{R}_i^{\mathscr{F}}))\subseteq\Delta(x,\Gamma_{i-1})$.*

Proof. Assume that $x\in\Gamma(\mathcal{R}_i^{\mathscr{F}})$. It is enough to prove that $\mathcal{R}_i^{\mathscr{F}}$ is constant on each open disk $D\subset X$ with boundary x such that $D\cap\Gamma_{i-1}=\varnothing$. Let $c\in D$ be a rational point. By Proposition 6.2, the function $\mathcal{R}_i^{\mathscr{F}}$ enjoys concavity properties on D, so $\mathcal{R}_i^{\mathscr{F}}(c)\geqslant\mathcal{R}_i^{\mathscr{F}}(x)\geqslant r(x)$. Since $r(x)$ coincides with the radius of D, this means that $\mathcal{R}_i^{\mathscr{F}}$ is constant on D by (3.8). □

The following statement will be the base case of our induction in the proof of Theorem 3.9.

Proposition 6.6. *Theorem 3.9 holds for $\mathcal{R}_1^{\mathscr{F}}$.*

Proof. By Proposition 2.14, to prove directional finiteness (C5) we shall prove that $\partial_b\mathcal{R}_1^{\mathscr{F}}(x)=0$ for all but a finite number of germs of segment out of x. This follows from Propositions 5.9 if the index $i=1$ is not solvable at x, and from Lemma 6.5 if $i=1$ is solvable at x.

The super-harmonicity properties (iii) and (iv) of Theorem 3.9 for $\mathcal{R}_1^{\mathscr{F}}$ follow again from Proposition 5.9 if the index $i=1$ is not solvable at x.

Otherwise, if $i=1$ is solvable at x, then we have three cases:

- If $x\notin\Gamma(\mathcal{R}_1^{\mathscr{F}})$ there is nothing to prove;
- If $x\in\Gamma(\mathcal{R}_1^{\mathscr{F}})\setminus\Gamma_X$, then x is a boundary point of $\Gamma(\mathcal{R}_1^{\mathscr{F}})$ by Lemma 6.5, in this case the super-harmonicity is just the concavity property (C3) of Proposition 6.2;
- If $x\in\Gamma_X$, then $\Gamma(\mathcal{R}_1^{\mathscr{F}})=\Gamma_X$ around x, by Lemma 6.5. Furthermore the function $y\mapsto\mathcal{R}_1^{\mathscr{F}}(y)$ is bounded by $y\mapsto\varrho_{y,X}$ around x, and the two functions are equal at x. So $\mathcal{R}_1^{\mathscr{F}}$ is super-harmonic at x, by Lemma 1.15. □

The following two propositions conclude the proof of Theorem 3.9.

PROPOSITION 6.7. *If $H_1^{\mathscr{F}},...,H_{i-1}^{\mathscr{F}}$ are finite, then $H_i^{\mathscr{F}}$ is directionally finite* (C5)*.*

Proof. Let $x\in\Gamma(H_i^{\mathscr{F}})$ be a bifurcation point. We have to prove that there are a finite number of open disks $D\subset X$ with boundary x such that $D\cap\Gamma(H_i^{\mathscr{F}})\neq\varnothing$. Since Γ_{i-1} is finite, there are a finite number of such disks intersecting it, so we can neglect them.

By (6.4) (cf. also Remark 6.1), we have $\Gamma(\mathcal{R}_i^{\mathscr{F}})\setminus\Gamma_{i-1}=\Gamma(H_i^{\mathscr{F}})\setminus\Gamma_{i-1}$. So we can replace $H_i^{\mathscr{F}}$ by $\mathcal{R}_i^{\mathscr{F}}$, and apply Proposition 6.2 to have the properties of a first radius over each open disk D with boundary x such that $\Gamma_{i-1}\cap D=\varnothing$. In particular by Proposition 2.14, $H_i^{\mathscr{F}}$ is constant over such a disk D if and only if $\partial_b H_i^{\mathscr{F}}(x)=0$, where b is the germ of segment out of x inside D.

As in the proof of Proposition 6.6, directional finiteness (C5) is then a consequence of Proposition 5.9 (if i is not solvable at x) and Lemma 6.5 (if i is solvable at x). □

PROPOSITION 6.8. *If $H_1^{\mathscr{F}},...,H_{i-1}^{\mathscr{F}}$ satisfy Theorem 3.9, then so does $H_i^{\mathscr{F}}$.*

Proof. It remains to prove that $H_i^{\mathscr{F}}$ satisfies the super-harmonicity property (iv) of Theorem 3.9. This will guarantee that $H_i^{\mathscr{F}}$ fulfill the assumptions (C1)–(C6) of Theorem 2.17 with respect to $\Gamma:=\Gamma_{i-1}$ and $\mathscr{C}(H_i^{\mathscr{F}}):=\mathscr{C}_i$. Notice that $\mathscr{C}_i\subseteq\Gamma_{i-1}$ is finite by (b) and (c) of point (iv) in Theorem 3.9.

If i is free of solvability then we deduce the super-harmonicity from Proposition 5.9.

It remains to prove that if $x\in X\setminus(\mathscr{C}_i\cup\partial X)$, and if some index $j\leqslant i$ is solvable at x, then $H_i^{\mathscr{F}}$ is super-harmonic at x.

Since $\mathscr{C}_{i-1}\subset\mathscr{C}_i$, by induction $H_{i-1}^{\mathscr{F}}$ is super-harmonic at $x\notin\mathscr{C}_i\cup\partial X$. We then write

$$H_i^{\mathscr{F}}=H_{i-1}^{\mathscr{F}}\mathcal{R}_i^{\mathscr{F}}. \tag{6.5}$$

If $x\notin\Gamma_{i-1}$, then $H_i^{\mathscr{F}}$ enjoys the properties of a first radius of convergence outside Γ_{i-1} by Proposition 6.2, so it is super-harmonic at x by Proposition 6.6.

If $x\notin\Gamma(H_i^{\mathscr{F}})$, then $H_i^{\mathscr{F}}$ is constant around x (and hence harmonic at x).

If $x \in \Gamma(H_i^{\mathscr{F}}) \cap \Gamma_{i-1}$, we now prove that $\mathcal{R}_i^{\mathscr{F}}$ is super-harmonic at x. By (6.5) this will imply that $H_i^{\mathscr{F}}$ is super-harmonic at x.

If i is over-solvable at x, or if $x \notin \Gamma(\mathcal{R}_i^{\mathscr{F}})$, then $\mathcal{R}_i^{\mathscr{F}}$ is constant around x, and hence it is super-harmonic at x.

It remains to check the case where i is solvable at x (i.e. $\mathcal{R}_i^{\mathscr{F}}(x) = r(x)$) and

$$x \in \Gamma(H_i^{\mathscr{F}}) \cap \Gamma_{i-1} \cap \Gamma(\mathcal{R}_i^{\mathscr{F}}). \tag{6.6}$$

We have to prove that $\mathcal{R}_i^{\mathscr{F}}$ is super-harmonic at the points of that graph that are not in $\mathscr{C}_{i-1}$ nor in the boundary of $\Gamma(\mathcal{R}_i^{\mathscr{F}})$. As observed, these points are finite in number because this admissible graph is finite by induction.

By Lemma 6.5 we have the inclusion $\Delta(x, \Gamma(\mathcal{R}_i^{\mathscr{F}})) \subseteq \Delta(x, \Gamma_{i-1})$. So $\Gamma(\mathcal{R}_i^{\mathscr{F}})$ is finite around x. Now, since x is not a boundary point of $\Gamma(\mathcal{R}_i^{\mathscr{F}})$, the function $\varrho_{\Gamma(\mathcal{R}_i^{\mathscr{F}})}: X \to \mathbb{R}$ is super-harmonic at x (as in point (1) of Example 2.20). Moreover, by (3.8), and by point (i) of Proposition 2.5, for all $b \in \Delta(x)$ we have, respectively,

$$\partial_b \mathcal{R}_i^{\mathscr{F}}(x) \leqslant \partial_b \varrho_{\Gamma(\mathcal{R}_i^{\mathscr{F}})}(x) \quad \text{and} \quad \mathcal{R}_i^{\mathscr{F}}(x) = \varrho_{\Gamma(\mathcal{R}_i^{\mathscr{F}})}(x) = r(x). \tag{6.7}$$

The function $\mathcal{R}_i^{\mathscr{F}}$ is then super-harmonic by Lemma 1.15. □

7. Notes

A first proof of the harmonicity properties is due to Robba [Ro3], [Ro4], for rank-1 differential equations with rational coefficients. He obtained the harmonicity of the radius function by expressing its slopes by means of the index (cf. [Ro3, Theorem 4.2, p. 201]), and then deducing the harmonicity from the additivity of the indices (cf. [Ro3, Proposition 4.5, p. 207]).[18]

In a recent paper of Kedlaya [Ke2, §5] there is a proof of the finiteness of a certain function related to the radii of a differential equation over a surface. It is a function on a Berkovich closed unit disk over $K := k((z))$, where k is a trivially valued field. The definitions of [Ke2] are given ad hoc to deal with a closed disk and there are discrepancies with those of this paper, especially for the definition of the skeleton of a function (which is defined in [Ke2] in term of the slopes). It turns out that the two definitions eventually coincide over a closed disk (by Lemma 2.13), and in fact certain techniques of this paper are not far from those of [Ke2] and [Ke3].

A proof of the finiteness of the first radius function have been given by Christol [Ch2] for differential equations of rank 1 with polynomial coefficients. The proof uses an explicit

[18] The principle is used in [Ro4, top of p. 50] to construct the so-called Robba exponentials.

formula for $\mathcal{R}_1^{\mathscr{F}}$ that we have contributed to realize (cf. the introduction of [Ch2]). The generalization of such a formula to rank-1 differential equations with arbitrary coefficients is the object of a forthcoming paper. This has been the starting point of the present paper.

After the first version of the present paper and of [NP2] had appeared, another proof of the finiteness of the controlling graphs have been obtained in [Ke4]. Kedlaya obtains there a shorter derivation of our proof, based on the same methods. No description of the super-harmonicity locus is given.([19]) On the other hand, he obtains a deep result showing that the end-points of the controlling graphs cannot be of type 4.

We also quote the remarkable result of André about the semi-continuity of the irregularity for meromorphic connections in a relative context [A2]. With our notation the irregularity is (related to) the slope $\partial_b H_r^{\mathscr{F}}(x)$: it is the derivative of the height of the convergence Newton polygon.

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