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Square-free values of $f(p)$, f cubic

by

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1. Introduction

An integer is said to be *square-free* if it is not divisible by the square d^2 of any integer d greater than 1. It is easy to prove that for $f(x)=mx+a$, $a, m\in\mathbb{Z}$, there are infinitely many integers n such that $mn+a$ is square-free—provided, of course, that $\gcd(a,m)$ is square-free.

For f quadratic, the infinity of integers n such that $f(n)$ is square-free was proved by Estermann [7] in 1931. (Again, there are necessary conditions that have to be fulfilled: f should not have repeated roots (i.e., for $\deg f=2$, f should not be a constant times a square) and $f(x)\not\equiv 0 \bmod q^2$ should have a solution in $\mathbb{Z}/q^2\mathbb{Z}$ for every prime q.)

For f cubic, the fact that there are infinitely many integers n such that $f(n)$ is square-free was proven by Erdős [6]. (See also [15, Chapter IV].) It can be argued that Erdős' proof wittily avoids several underlying issues, some of which are diophantine problems that are far from trivial. Perhaps because of this, Erdős posed the problem of proving that $f(p)$ is square-free for infinitely many *primes* p. The diophantine issues then become unavoidable, and the problem becomes much harder.

The paper [12] settled the issue for f cubic with Galois group Alt(3). Unfortunately, most cubics have Galois group Sym(3).

The present paper solves the problem for all f cubic.

Main theorem. *Let $f\in\mathbb{Z}[x]$ be a cubic polynomial without repeated roots. Then the number of prime numbers $p\leqslant N$ such that $f(p)$ is square-free is*

$$(1+o_f(1))\prod_{q\text{ prime}}\left(1-\frac{\varrho_f(q^2)}{\phi(q^2)}\right)\frac{N}{\log N}+O(1), \tag{1.1}$$

where $\varrho_f(q^2)$ is the number of solutions to $f(x)\equiv 0 \bmod q^2$ in $(\mathbb{Z}/q^2\mathbb{Z})^*$.

Here, as usual, $o_f(1)$ is a quantity that goes to 0 as $N \to \infty$ (at a rate that may depend on $f \in \mathbb{Z}[x]$), whereas $O(1)$ is an absolute constant.

It is easy to show that, if $f(x) \not\equiv 0 \bmod q^2$ has at least one solution in $(\mathbb{Z}/q^2\mathbb{Z})^*$ for every prime q smaller than a constant depending only on f, then the infinite product in (1.1) converges to a non-zero value (see the remark at the end of §2). In other words, we have a necessary and sufficient condition for the product in (1.1) to be non-zero, and this condition is such that it can be checked explicitly in time $O_f(1)$.

The analogous problem—namely, proving that, for a polynomial f of degree k satisfying the necessary conditions as above, there is an infinite number of primes p such that $f(p)$ has no divisors of the form d^{k-1}, $d>1$—was solved by Nair [20] for $k \geqslant 7$. Several cases with $k=3,4,5,6$ were solved in [12]; see the list in [12, (1.3)]. A summary of the proof in this paper appeared previously in [13]. Since then, the cases of $k=5,6$ have been settled by Browning [4, Theorem 2], building in part on arguments by Salberger [21] and Heath-Brown [10]. As a consequence, only the case of polynomials f of degree $k=4$ with Galois group Alt(4) or Sym(4) remains open.

The author's interest in the problem was first sparked by his work on root numbers of elliptic curves. There are indeed many problems in number theory where matters become much simpler technically if one assumes one is working with square-free numbers. This is the natural domain of application of the results in this paper.

1.1. Notation

In this paper, p and q always denote primes. We write $\omega(d)$ for the number of prime divisors of an integer d, and $\tau_k(d)$ for the number of tuples of positive integers $(m_1, ..., m_k)$ such that $d = m_1 m_2 ... m_k$. Given a prime p and a non-zero integer n, the valuation $v_p(n)$ is the largest non-negative integer r such that $p^r | n$. Given positive integers n and m, we write $\gcd(n, m^\infty)$ for $\prod_{p|m} p^{v_p(n)}$. Let $\pi(N)$ be the number of primes $\leqslant N$.

Let K be a number field with Galois group $\mathrm{Gal}(K/\mathbb{Q})$. We write $\omega_K(d)$ for the number of prime ideals dividing d in the number field K. Given a rational prime p unramified in $K/\mathbb{Q}$, we denote by $\mathrm{Frob}_p \subset \mathrm{Gal}(K/\mathbb{Q})$ the Frobenius symbol of p; it is always a conjugacy class in $\mathrm{Gal}(K/\mathbb{Q})$. For $g \in \mathrm{Gal}_f$, we write $\omega_{\mathrm{Cl}(g)}(n)$ for the number of prime divisors $p|n$ such that $\mathrm{Frob}_p = \mathrm{Cl}(g)$, where $\mathrm{Cl}(g)$ is the conjugacy class of g.

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2. Reduction to the problem of large square factors $q^2 \mid f(x)$, q prime

We wish to reduce the problem of estimating the number of primes $p \leqslant N$ such that $f(p)$ is square-free to the problem of bounding from above the number of primes $p \leqslant N$ such that $f(p)$ has a square factor of the form q^2, q prime, $q > N(\log N)^{-\varepsilon}$. If we cared about minimising the error term, this would be a non-trivial problem; see the treatment in [11, §3]. As it happens, the error terms we will get later from other sources will be fairly large anyhow, and thus we can afford to carry out things in this section in a way that is easy and classical. (See [15, Chapter IV] or [9], for instance.)

In what follows, p and q always range over the primes. We have

$$\begin{aligned}|\{p \leqslant N : f(p) \text{ is square-free}\}| &= \big|\big\{p \leqslant N : q^2 \mid f(p) \Rightarrow q \geqslant \tfrac{1}{3}\log N\big\}\big| \\ &\quad + O\big(\big|\big\{p \leqslant N : \text{there exists } q \text{ such that } q^2 \mid f(p) \text{ and } q \geqslant \tfrac{1}{3}\log N\big\}\big|\big).\end{aligned}$$

By the inclusion-exclusion principle and the Bombieri–Vinogradov theorem,

$$\begin{aligned}\big|\big\{p \leqslant N : q^2 \mid f(p) \Rightarrow q \geqslant \tfrac{1}{3}\log N\big\}\big| &= \sum_{\substack{d \\ q \mid d \Rightarrow q < (\log N)/3}} \mu(d)\big|\{p \leqslant N : d^2 \mid f(p)\}\big| \\ &= \sum_{\substack{d \\ q \mid d \Rightarrow q < (\log N)/3}} \mu(d)\varrho_f(d^2)\frac{\pi(N)}{\phi(d^2)} + O\bigg(\frac{N}{(\log N)^{100}}\bigg) \\ &= \prod_{\substack{q \text{ prime} \\ q < (\log N)/3}} \bigg(1 - \frac{\varrho_f(q^2)}{\phi(q^2)}\bigg)\pi(N) + O\bigg(\frac{N}{(\log N)^{100}}\bigg) \\ &= \prod_{q \text{ prime}} \bigg(1 - \frac{\varrho_f(q^2)}{\phi(q^2)}\bigg)\pi(N) + O\bigg(\frac{N}{(\log N)^2}\bigg).\end{aligned}$$

Recall as well that

$$\pi(N) = \frac{N}{\log N} + O\bigg(\frac{N}{(\log N)^2}\bigg)$$

(the prime number theorem).

At the same time,

$$\begin{aligned}
&\big|\{p\leqslant N:\text{there exists } q \text{ such that } q^2\mid f(p) \text{ and } q\geqslant \tfrac{1}{3}\log N\}\big|\\
&\quad\leqslant \big|\{p\leqslant N:\text{there exists } q \text{ such that } q^2\mid f(p) \text{ and } \tfrac{1}{3}\log N\leqslant q<N^{1/3}\}\big|\\
&\qquad+\big|\{p\leqslant N:\text{there exists } q \text{ such that } q^2\mid f(p) \text{ and } N^{1/3}\leqslant q<N(\log N)^{-\varepsilon}\}\big|\\
&\qquad+\big|\{p\leqslant N:\text{there exists } q \text{ such that } q^2\mid f(p) \text{ and } q\geqslant N(\log N)^{-\varepsilon}\}\big|\\
&\quad\leqslant \sum_{(\log N)/3\leqslant q<N^{1/3}} O\Big(\frac{N/\log N}{q(q-1)}\Big)+O\Big(\frac{N}{(\log N)^{100}}\Big)\\
&\qquad+\sum_{N^{1/3}\leqslant q<N(\log N)^{-\varepsilon}} O\Big(\frac{N}{q^2}+1\Big)\\
&\qquad+\big|\{p\leqslant N:\text{there exists } q \text{ such that } q^2\mid f(p) \text{ and } q\geqslant N(\log N)^{-\varepsilon}\}\big|,
\end{aligned}$$

where we have used the Brun–Titchmarsh theorem (or any upper-bound sieve) to justify the second inequality, and where, as per our convention, q ranges only over the primes. The series on the right-hand side sum up to $O(N/(\log N)^2)$ and $O(N/(\log N)^{1+\varepsilon})$, respectively; hence

$$\begin{aligned}
&\big|\{p\leqslant N:\text{there exists } q \text{ such that } q^2\mid f(p) \text{ and } q\geqslant \tfrac{1}{3}\log N\}\big|\\
&\quad\leqslant \big|\{p\leqslant N:\text{there exists } q \text{ such that } q^2\mid f(p) \text{ and } q\geqslant N(\log N)^{-\varepsilon}\}\big|+O\Big(\frac{N}{(\log N)^{1+\varepsilon}}\Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&|\{p\leqslant N: f(p) \text{ is square-free}\}|\\
&\quad=\prod_{q \text{ prime}}\Big(1-\frac{\varrho_f(q^2)}{\phi(q^2)}\Big)\frac{N}{\log N}+O\Big(\frac{N}{(\log N)^{1+\varepsilon}}\Big)\\
&\qquad+\big|\{p\leqslant N:\text{there exists } q \text{ such that } q^2\mid f(p) \text{ and } q\geqslant N(\log N)^{-\varepsilon}\}\big|
\end{aligned}\tag{2.1}$$

for any $\varepsilon>0$.

The only thing that remains is to bound

$$\big|\{p\leqslant N:\text{there exists } q \text{ such that } q^2\mid f(p) \text{ and } q\geqslant N(\log N)^{-\varepsilon}\}\big|.$$

This problem will occupy us in the rest of the paper.

In the meantime, let us note that $\varrho_f(q^2)\leqslant \deg f$ for every q larger than a constant depending only on f (by Hensel's lemma). Hence the infinite product in (2.1) is non-zero provided that $\varrho_f(q^2)<\phi(q^2)$ (i.e., provided that $f(x)\not\equiv 0 \bmod q^2$ has at least one solution in $(\mathbb{Z}/q^2\mathbb{Z})^*$) for every q smaller than a constant depending only on f. If there is a q such that $f(x)\not\equiv 0 \bmod q^2$ has no solutions in $(\mathbb{Z}/q^2\mathbb{Z})^*$, then $f(p)$ can be square-free only when $\gcd(p,q^2)\neq 1$; obviously, $\gcd(p,q^2)\neq 1$ can happen for at most one value of p, namely, $p=q$. (This is where the term $O(1)$ in (1.1) comes from.)

3. Integer points on a typical quadratic twist of an elliptic curve

Consider two points (x_1, y_1) and (x_2, y_2) $(x_i, y_i \in \mathbb{Z})$ on the curve $dy^2 = f(x)$. This is an elliptic curve. It is well known that points with integer coordinates on an elliptic curve tend to repel each other; this was already used in the present context in [11] (see also the earlier work [22]). As was pointed out in [14], two points repel each other more strongly if their coordinates are congruent to each other modulo some large integer. (This is a somewhat intuitive description; we will do things rigorously below.)

In [12], I used this phenomenon on the curve $dy^2 = f(x)$. I first showed by elementary means that most integers $d \leqslant N$ have large factors $d_0 | d$, $d_0 > N^{1-\varepsilon}$, such that d_0 has few prime divisors. It is then the case that the x-coordinates of the points (x, y) on the curve fall into few congruence classes modulo d_0 (because d_0 has few prime divisors). Moreover, by the argument on elliptic curves just given, there can be only few points whose x-coordinates are in a given congruence class modulo d_0 (because d_0 is large, and makes points in such a congruence class repel each other strongly). It follows that there are few points (x, y) $(x, y \in \mathbb{Z},\ 1 \leqslant x, y \leqslant N)$ on the curve $dy^2 = f(x)$, unless d is in some small exceptional set.

We carry out this argument again, largely just by citing [11] and [12].

PROPOSITION 3.1. *Let $f \in \mathbb{Z}[x]$ be a polynomial of degree 3 with no repeated roots. Let d be a square-free integer. Then, for any N, the number of integer solutions $(x, y) \in \mathbb{Z}^2$ to $dy^2 = f(x)$ with $N^{1/2} < x \leqslant N$ is at most*

$$O_f(C^{\omega(d)}), \tag{3.1}$$

where C is an absolute constant.

This bound is an immediate consequence of [22, Theorem A], which is already based on the idea of repelling points (and does not require the condition $N^{1/2} < x \leqslant N$). The alternative proof in [11, Corollary 4.18] provides an explicit value for C by means of sphere-packing bounds [18].

Proof. By [11, Corollary 4.18] (applied with $\varepsilon = \frac{1}{2}$) and any rank bound obtained by descent, e.g., the standard bound in [5, Proposition 7.1] (namely,

$$\operatorname{rank} \leqslant \omega_K(d) - \omega(d) + O_f(1) \leqslant 2\omega(d) + O_f(1),$$

where $K = \mathbb{Q}(\alpha)$ and α is a root of $f(\alpha) = 0$). □

PROPOSITION 3.2. *Let $f \in \mathbb{Z}[x]$ be a polynomial of degree 3 with no repeated roots. Let $d \leqslant X$ be a positive integer. Suppose that d has an integer divisor $d_0 \geqslant X^{1-\varepsilon}$, $\varepsilon > 0$.*

Assume furthermore that $\gcd(d_0, 2\operatorname{Disc} f)=1$. *Then the number of integer solutions* $(x,y)\in\mathbb{Z}^2$ *to* $dy^2=f(x)$ *with* $X^{1-\varepsilon}<x\leqslant X$ *is at most*

$$O_{f,\varepsilon}(e^{O_f(\varepsilon\omega(d))}3^{\omega(d_0)}). \tag{3.2}$$

This bound uses the divisor d_0 in order to increase repulsion in the way outlined above. If a d_0 with few prime divisors is chosen, the bound (3.2) will be much smaller than (3.1).

Proof. This is a special case ($\deg f=3$, $k=2$ and $c=2$) of [12, Proposition 4.3]. □

We now need two lemmas on the integers.

LEMMA 3.3. *Let* $f\in\mathbb{Z}[x]$ *be a polynomial. For any* $A>0$ *and for all but*

$$O_A(N(\log N)^{-A})$$

integers n *between* 1 *and* N, *the number of prime divisors* $\omega(f(n))$ *of* $f(n)$ *is*

$$O_{A,f}(\log\log N).$$

Proof. This is standard. If $f(n)$ has $\geqslant C\log\log N$ prime factors, then it has

$$\geqslant \frac{C}{\deg f}\log\log N$$

prime factors (namely, the $(C/\deg f)\log\log N$ smallest ones) whose product is $\ll_f N$. Their products give us $\geqslant 2^{(C/\deg f)\log\log N}=(\log N)^{C\log 2/\deg f}$ divisors $d\ll_f N$ of $f(n)$. At the same time,

$$\sum_{n\leqslant N}\sum_{\substack{d\leqslant N\\ d|f(n)}}1=\sum_{d\leqslant N}\sum_{\substack{n\leqslant N\\ d|f(n)}}1\leqslant\sum_{d\leqslant N}\left(\frac{N}{d}+1\right)(\deg f)^{\omega(d)}\ll N(\log N)^B,$$

where $B=O_f(1)$. Thus, there can be at most $N(\log N)^{-(C\log 2/\deg f-B)}$ integers $n\leqslant N$ such that $f(n)$ has $\geqslant C\log\log N$ prime factors. We set C so that $(C\log 2)/\deg f-B\geqslant A$ and we are done. □

LEMMA 3.4. *Let* $f\in\mathbb{Z}$ *be a polynomial. For any* $A>0$, $\varepsilon>0$ *and* $m>0$, *it is the case that, for all but* $O_{A,\varepsilon,m}(N(\log N)^{-A})$ *integers* n *between* 1 *and* N, *there is a divisor* $d_1|f(n)$ *such that* $d_1<N^{\varepsilon/2}$, $\omega(f(n)/d_1)<\varepsilon\log\log X$ *and* $\gcd(f(n)/d_1,m)=1$.

Proof. Let $\delta(N)$ be as in [12, Lemma 5.2] with $\frac{1}{4}\varepsilon$ instead of ε. (That is, we let $\delta(N)=(\log N)^{-\varepsilon/4re^{2r}}$, where $r=\deg f$.) Let

$$d_1 = \gcd(f(n), m^\infty) \prod_{\substack{p\mid f(n)\\ p\nmid m\\ p\leqslant N^{\delta(N)}}} p.$$

By definition, $\gcd(f(n)/d_1, m)=1$. Also, by [12, Lemma 5.2], we know that

$$\omega(f(n)/d_1) < \varepsilon \log\log N \quad \text{and} \quad \prod_{\substack{p\mid n\\ p\leqslant N^{\delta(N)}}} p < N^{\varepsilon/4}$$

for all but $O_{A,\varepsilon}(N(\log N)^{-A})$ integers n between 1 and N.

Now,

$$\sum_{n\leqslant N} \gcd(n, m^\infty) \leqslant \sum_{d\mid m^\infty} \sum_{\substack{n\leqslant N\\ d\mid n}} d \leqslant \sum_{\substack{d\mid m^\infty\\ d\leqslant N}} N \leqslant N \prod_{p\mid m} \sum_{\substack{\alpha\geqslant 1\\ p^\alpha\leqslant N}} 1 \ll N(\log N)^{\omega(m)} \ll_{m,\varepsilon} N^{1+\varepsilon/8}.$$

It follows that, for all but $O_{m,\varepsilon}(N^{1-\varepsilon/8})$ integers n between 1 and N,

$$\gcd(f(n), m^\infty) \leqslant N^{\varepsilon/4}.$$

Hence, $d_1 \leqslant N^{\varepsilon/2}$. □

PROPOSITION 3.5. *Let $f\in\mathbb{Z}[x]$ be a polynomial of degree 3 with no repeated roots. Let D be a set of positive integers. Then the total number of integers x with $1\leqslant x\leqslant N$ such that*

$$dy^2 = f(x)$$

for some integer $y\geqslant N(\log N)^{-\varepsilon}$ and some $d\in D$ is at most

$$O_{f,\varepsilon}(|D|(\log N)^{\varepsilon}) + O_{f,A,\varepsilon}(N(\log N)^{-A}) \tag{3.3}$$

for arbitrary A and $\varepsilon>0$.

Proof. Let $\varepsilon>0$ be a small parameter to be set later. If $dy^2=f(x)$ for some integer y and some integer $d<N^{1-\varepsilon/4}$, then $d'(y')^2=f(x)$ for some integer y' and some square-free integer $d'<N^{1-\varepsilon/4}$. By Proposition 3.1 and Lemma 3.3, the total number of $x\leqslant N$ satisfying such an equation is $(\log N)^{O_{f,A}(1)}N^{1-\varepsilon/4}+O_A(N(\log N)^{-A}) \ll_{A,f,\varepsilon} N(\log N)^{-A}$ for A arbitrarily large.

Let, then, $dy^2=f(x)$, $d\geqslant N^{1-\varepsilon/4}$, $x\leqslant N$ and $y\geqslant N(\log N)^{-\varepsilon}$. By Lemma 3.3, we may assume that $\omega(d)\ll_{A,f}\log\log N$ (taking out at most $O_A(N(\log N)^{-A})$ values of x). By

Lemma 3.4, we may assume (taking out at most $O_{A,f,\varepsilon}(N(\log N)^{-A})$ values of x) that there is a $d_1 | f(x)$ such that

$$d_1 < N^{\varepsilon/2}, \quad \omega\left(\frac{f(n)}{d_1}\right) < \varepsilon \log\log N \quad \text{and} \quad \gcd\left(\frac{f(n)}{d_1}, 2\operatorname{Disc}(f)\right) = 1.$$

Let $d_0 = d/\gcd(d, d_1)$. Then $d_0 \geqslant d/N^{\varepsilon/2} > N^{1-3\varepsilon/4}$, $\omega(d_0) < \varepsilon \log\log N$, $\omega(d) \ll_{A,f} \log\log N$ and $\gcd(d_0, 2\operatorname{Disc}(f)) = 1$.

As $y \geqslant N(\log N)^{-\varepsilon}$ and $d = f(x)/y^2$, we have $d \leqslant C_f N(\log N)^{2\varepsilon}$ for some constant C_f. We apply Proposition 3.2 with $X = C_f N(\log N)^{2\varepsilon}$. (The condition $d_0 \geqslant X^{1-\varepsilon}$ is fulfilled by $d_0 > N^{1-3\varepsilon/4}$ provided N is larger than a constant $c_{f,\varepsilon}$ depending only on f and ε; we may assume that N is larger than $c_{f,\varepsilon}$ because conclusion (3.3) is otherwise trivial.) We obtain that the number of integer solutions to $dy^2 = f(x)$ is at most

$$\ll_{f,\varepsilon} |D| e^{O_{f,A}(\varepsilon) \log\log N} 3^{\varepsilon \log\log N}$$

(taking out at most $\ll_{A,f,\varepsilon} N(\log N)^{-A} + X^{1-\varepsilon} \ll_{A,f,\varepsilon} N(\log N)^{-A}$ values of x). For ε small enough in terms of f, A and ε, this is $\leqslant |D|(\log N)^{\varepsilon}$, as desired. □

In view of (3.3), what remains is to show that we can eliminate most possible values of d in $dq^2 = f(p)$, $p \leqslant N$, $q \geqslant N(\log N)^{-\varepsilon}$, where we allow ourselves to take out first a proportion $o(1)$ of all possible values of $p \leqslant N$.

4. Typical properties of $f(q)$ and $d = f(p)/y^2$

Les α be a root of $f(\alpha) = 0$. Let $\mathrm{Gal}_f = \mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$. For $g \in \mathrm{Gal}_f$, let $\omega_{\mathrm{Cl}(g)}(n)$ be the number of prime divisors $p | n$ unramified in $\mathbb{Q}(\alpha)/\mathbb{Q}$ such that $\mathrm{Frob}_p = \mathrm{Cl}(g)$. Let $\alpha_{\mathrm{Cl}(g)}$ be the number of fixed points of any representative g of $\mathrm{Cl}(g)$, considered as a permutation map on the roots of $f(x) = 0$ in $\mathbb{C}$. It is a standard fact that $\alpha_{\mathrm{Cl}(g)}$ equals the number of roots $x \in \mathbb{Z}/p\mathbb{Z}$ of $f(x) \equiv 0 \bmod p$ for any p unramified in $\mathbb{Q}(\alpha)/\mathbb{Q}$ such that $\mathrm{Frob}_p = \mathrm{Cl}(g)$.

As is usual, we write the number of points on the curve $y^2 = f(x) \bmod p$ as $p + 1 - a_p$, where a_p is an integer.

Our aim in this section is to show that, for a proportion $1 + o(1)$ of all primes $q \leqslant N$,

(a)

$$\omega_{\mathrm{Cl}(g)}(f(q)) = (\alpha_{\mathrm{Cl}(g)} + o(1)) \frac{|\mathrm{Cl}(g)|}{\mathrm{Gal}_f} \log\log N$$

for every $g \in \mathrm{Gal}_f$, and

(b)

$$\sum_{p \leqslant z} \frac{a_p}{p} \left(\frac{f(q)}{p}\right) = (1 + o(1)) \sum_{p \leqslant z} \frac{-a_p^2}{p^2}$$

for $1/o(1) \leqslant z \leqslant N^{\delta}$, where $\delta > 0$ is smaller than a constant.

Here (a) is unsurprising; it is clear that the probability of $p|f(q)$ for p fixed and q prime and random is α_f/p, and the sum

$$\sum_{\substack{p\leqslant N\\ \mathrm{Frob}_p=\mathrm{Cl}(g)}} \frac{1}{p} \quad \text{is} \quad (1+o(1))\frac{|\mathrm{Cl}(g)|}{|\mathrm{Gal}_f|}\log\log N$$

by Chebotarev's density theorem.

As for statement (b), the number of points on $y^2=f(x) \bmod p$ is

$$p+1-a_p=\sum_{x\in\mathbb{Z}/p\mathbb{Z}}\left(1+\left(\frac{f(x)}{p}\right)\right)=p+\sum_{x\in\mathbb{Z}/p\mathbb{Z}}\left(\frac{f(x)}{p}\right).$$

(Here and throughout the paper, $(\cdot/\cdot)$ stands for the Jacobi symbol.) Hence, the expected value of $(f(q)/p)$ for p fixed and q prime and random should be

$$\frac{1}{p-1}\left(1-a_p-\left(\frac{f(0)}{p}\right)\right)=-\frac{a_p}{p}+\text{error term}.$$

Thus, the expected value of

$$\sum_{p\leqslant z}\frac{a_p}{p}\left(\frac{f(q)}{p}\right) \quad \text{should be about} \quad \sum_{p\leqslant z}\frac{-a_p^2}{p^2}.$$

As elsewhere in this paper, we will carry out our arguments as is customary in analytic number theory, inspired by the probabilistic reasoning detailed above. (Alternatively, one could start by proving probabilistic statements and deduce statements in number theory from them, as in [12, §5 and §6]. That option generally takes more space and work.)

Part of the point in estimating $\omega_{\mathrm{Cl}(g)}(f(q))$ and $(f(q)/p)$ is that neither quantity changes much when $f(p)$ is divided by the square of a prime: if $d=f(q)/y^2$, y being a prime, then

$$\begin{gathered}\omega_{\mathrm{Cl}(g)}(f(p))-1\leqslant\omega_{\mathrm{Cl}(g)}(d)\leqslant\omega_{\mathrm{Cl}(g)}(f(p))\\ \left(\frac{d}{p}\right)=\left(\frac{f(q)}{p}\right) \quad \text{for } p\neq y.\end{gathered} \tag{4.1}$$

Therefore, what follows will help us to later determine what form any d satisfying $dy^2=f(q)$ must take, where y can be any prime and q can be any prime $\leqslant N$ outside a set of density 0.

We will prove both (a) and (b) using, in essence, bounds on variances and Chebyshev's inequality.

LEMMA 4.1. *Let $f \in \mathbb{Z}[x]$ be a polynomial irreducible over $\mathbb{Q}[x]$. Let $g \in \mathrm{Gal}_f$. Let $z = z(N)$ be such that $\lim_{N\to\infty} z(N) = \infty$ and $z < N^{1/4-\varepsilon}$, $\varepsilon > 0$. Then*

$$\sum_{\substack{p \leqslant z \\ p \text{ unramified} \\ \mathrm{Frob}_p = \mathrm{Cl}(g) \\ p | f(q)}} 1 = (\alpha_{\mathrm{Cl}(g)} + o_f(1)) \frac{|\mathrm{Cl}(g)|}{|\mathrm{Gal}_f|} \log\log z$$

for a proportion $1 + o_{f,\varepsilon}(1)$ of all primes $q \leqslant N$.

The proof will not be very different from Turán's classical proof that the average number of prime divisors of an integer $\leqslant N$ is $\sim \log\log N$.

Proof. In what follows, our sums over p range only over primes p unramified in $\mathbb{Q}(\alpha)/\mathbb{Q}$, α being a root of f, whereas our sums over q range over all primes. We will give a variance bound, i.e., we will show that

$$V = \sum_{q \leqslant N} \left| \sum_{\substack{p \leqslant z \\ \mathrm{Frob}_p = \mathrm{Cl}(g) \\ p | f(q)}} 1 - R \right|^2 \tag{4.2}$$

is small, where

$$R = \sum_{\substack{p \leqslant z \\ \mathrm{Frob}_p = \mathrm{Cl}(g)}} \frac{\alpha_{\mathrm{Cl}(g)}}{p}.$$

Expanding (4.2), we get

$$\begin{aligned} V = R^2 \pi(N) - 2R \sum_{\substack{p \leqslant z \\ \mathrm{Frob}_p = \mathrm{Cl}(g)}} \sum_{\substack{q \leqslant N \\ p | f(q)}} 1 \\ + \sum_{\substack{p_1 \leqslant z \\ \mathrm{Frob}_{p_1} = \mathrm{Cl}(g) \\ p_1 \neq p_2}} \sum_{\substack{p_2 \leqslant z \\ \mathrm{Frob}_{p_2} = \mathrm{Cl}(g)}} \sum_{\substack{q \leqslant N \\ p_1 p_2 | f(q)}} 1 + \sum_{\substack{p \leqslant z \\ \mathrm{Frob}_p = \mathrm{Cl}(g)}} \sum_{\substack{q \leqslant N \\ p | f(q)}} 1. \end{aligned} \tag{4.3}$$

Now

$$\begin{aligned} \sum_{\substack{p \leqslant z \\ \mathrm{Frob}_p = \mathrm{Cl}(g)}} \sum_{\substack{q \leqslant N \\ p | f(q)}} 1 = \sum_{\substack{p \leqslant z \\ \mathrm{Frob}_p = \mathrm{Cl}(g)}} |\{x \in (\mathbb{Z}/p\mathbb{Z})^* : f(x) = 0 \bmod p\}| \frac{\pi(N)}{\phi(p)} \\ + O\left(z + \sum_{p \leqslant z} \max_{\substack{a \bmod p \\ \gcd(a,p)=1}} \left(|\{q \leqslant N : q \equiv a \bmod p\}| - \frac{\pi(N)}{\phi(p)} \right) \right). \end{aligned}$$

By the Bombieri–Vinogradov theorem (as in [2, Theorem 0]),

$$\sum_{m\leqslant N^{1/2-\delta}} \max_{\substack{a \bmod m\\ \gcd(a,m)=1}} \left(|\{q\leqslant N : q\equiv a \bmod m\}| - \frac{\pi(N)}{\phi(m)}\right) \ll_{A,\delta} \frac{N}{(\log N)^A}$$

for all $A,\delta>0$. We also have $|\{x\in(\mathbb{Z}/p\mathbb{Z})^* : f(x)=0\}|=\alpha_{\mathrm{Cl}(g)}$ for all (unramified) p with $\mathrm{Frob}_p=\mathrm{Cl}(g)$. Hence

$$\sum_{\substack{p\leqslant z\\ \mathrm{Frob}_p=\mathrm{Cl}(g)}} \sum_{\substack{q\leqslant N\\ p\mid f(q)}} 1 = \pi(N) \sum_{\substack{p\leqslant z\\ \mathrm{Frob}_p=\mathrm{Cl}(g)}} \frac{\alpha_{\mathrm{Cl}(g)}}{p-1} + O_A(N(\log N)^{-A})$$
$$= \pi(N)\left(O(1) + \sum_{\substack{p\leqslant z\\ \mathrm{Frob}_p=\mathrm{Cl}(g)}} \frac{\alpha_{\mathrm{Cl}(g)}}{p}\right) = \pi(N)(R+O(1)).$$

Similarly, we have

$$\sum_{\substack{p_1\leqslant z\\ \mathrm{Frob}_{p_1}=\mathrm{Cl}(g)}} \sum_{\substack{p_2\leqslant z\\ \mathrm{Frob}_{p_2}=\mathrm{Cl}(g)}} \sum_{\substack{q\leqslant N\\ p_1p_2\mid f(q)}} 1 = \pi(N) \sum_{\substack{p_1\leqslant z\\ \mathrm{Frob}_{p_1}=\mathrm{Cl}(g)}} \sum_{\substack{p_2\leqslant z\\ \mathrm{Frob}_{p_2}=\mathrm{Cl}(g)}} \frac{\alpha_{\mathrm{Cl}(g)}}{(p_1-1)(p_2-1)}$$
$$+ O_A(N(\log N)^{-A}) = \pi(N)(R+O(1))^2.$$

Hence, we conclude from (4.3) that

$$V = R^2\pi(N) - 2R(R+O(1))\pi(N) + \pi(N)(R+O(1))^2 + \pi(N)(R+O(1)) = O(R\pi(N)).$$

Now, if

$$\left| \sum_{\substack{p\leqslant z\\ \mathrm{Frob}_p=\mathrm{Cl}(g)\\ p\mid f(q)}} 1 - R \right| > \delta R \tag{4.4}$$

for some $q\leqslant N$ and $\delta>0$, then that value makes a contribution greater than δ^2R^2 to (4.2). Hence there are at most

$$\frac{O(R)\pi(N)}{\delta^2R^2} = O\left(\frac{1}{\delta^2R}\pi(N)\right)$$

primes $q\leqslant N$ for which (4.4) is the case. By the Chebotarev density theorem,

$$R = (1+o_f(1))|\mathrm{Cl}(g)|\alpha_{\mathrm{Cl}(g)}\log\log z.$$

Thus $R\to\infty$ as $N\to\infty$, and so the statement of the lemma follows. □

We will need a large-sieve lemma of a rather standard kind.

LEMMA 4.2. *For any N and any $\varepsilon>0$,*

$$\sum_{r\leqslant N^{1/2-\varepsilon}} \sum_{\substack{\chi \bmod r\\ \chi \text{ primitive}}} \left| \sum_{\substack{q\leqslant N\\ q \text{ prime}}} \chi(q)\right|^2 \ll_\varepsilon \frac{N^2}{(\log N)^2}. \tag{4.5}$$

This is a special case of [17, Problem 7.19].

Proof. By the triangle inequality, the square root of the left-hand side of (4.5) is at most

$$\sqrt{\sum_{r\leqslant N^{1/2-\varepsilon}} \sum_{\substack{\chi \bmod r\\ \chi \text{ primitive}}} \left| \sum_{\substack{q\leqslant \sqrt{N}\\ q \text{ prime}}} \chi(q)\right|^2}$$

(which is $\ll\sqrt{N^{1-2\varepsilon}(\sqrt{N}/\log N)^2}\ll N/\log N$) plus the square-root of

$$\sum_{r\leqslant N^{1/2-\varepsilon}} \sum_{\substack{\chi \bmod r\\ \chi \text{ primitive}}} \left| \sum_{\substack{\sqrt{N}<q\leqslant N\\ q \text{ prime}}} \chi(q)\right|^2. \tag{4.6}$$

By [1, Theorem 8] with $Q=\sqrt{N}$, (4.6) is at most

$$\frac{1}{\log(\sqrt{N}/N^{1/2-\varepsilon})}(N+Q^2) \sum_{\substack{\sqrt{N}<q\leqslant N\\ q \text{ prime}}} 1 \ll_\varepsilon \frac{N^2}{(\log N)^2}. \qquad \square$$

LEMMA 4.3. *Let $f\in\mathbb{Z}[x]$ be a polynomial irreducible over $\mathbb{Q}[x]$. For every prime p, write $p+1-a_p$ for the number of points in $\mathbb{P}^2(\mathbb{Z}/p\mathbb{Z})$ on the curve $y^2=f(x)$. Let $z=z(N)$ be such that $z<N^{1/4-\varepsilon}$, $\varepsilon>0$ and*

$$\lim_{N\to\infty} \sum_{p\leqslant z} \frac{a_p^2}{p^2} = \infty.$$

Then, for a proportion $1+o(1)$ of all primes $q\leqslant N$ as $N\to\infty$,

$$\sum_{p\leqslant z} \frac{a_p}{p}\left(\frac{f(q)}{p}\right) = (1+o(1)) \sum_{p\leqslant z} \frac{-a_p^2}{p^2}, \tag{4.7}$$

where the implied constants depend only on ε.

Again, the proof will proceed by a variance bound.

Proof. Define

$$V = \sum_{q \leqslant N} \Bigg(\sum_{p \leqslant z} \frac{a_p}{p} \bigg(\Big(\frac{f(q)}{p} \Big) + \frac{a_p}{p} \bigg) \Bigg)^2, \tag{4.8}$$

where, as per our convention, q ranges only over the primes. Changing the order of summation, we obtain

$$V = \sum_{p_1 \leqslant z} \frac{a_{p_1}}{p_1} \sum_{p_2 \leqslant z} \frac{a_{p_2}}{p_2} \sum_{q \leqslant N} \bigg(\Big(\frac{f(q)}{p_1} \Big) + \frac{a_{p_1}}{p_1} \bigg) \bigg(\Big(\frac{f(q)}{p_2} \Big) + \frac{a_{p_2}}{p_2} \bigg). \tag{4.9}$$

Expanding, we see that

$$\begin{aligned} V = (R^2 + O(R))\pi(N) + 2R \sum_{p \leqslant z} \frac{a_p}{p} \sum_{a \bmod p} \Big(\frac{f(a)}{p} \Big) |\{q \leqslant N : q \equiv a \bmod p\}| \\ + \sum_{\substack{p_1 \leqslant z \\ p_1 \neq p_2}} \sum_{p_2 \leqslant z} \frac{a_{p_1}}{p_1} \frac{a_{p_2}}{p_2} \sum_{a \bmod p_1 p_2} \Big(\frac{f(a)}{p_1 p_2} \Big) |\{q \leqslant N : q \equiv a \bmod p_1 p_2\}|, \end{aligned} \tag{4.10}$$

where $R = \sum_{p \leqslant z} a_p^2/p^2$ and $\pi(N)$ is the number of primes $\leqslant N$. (The term $O(R)\pi(N)$ comes from the diagonal terms $p_1 = p_2$ left out of the triple sum on the second line.)

We wish to approximate $|\{q \leqslant N : q \equiv a \bmod p\}|$ (and $|\{q \leqslant N : q \equiv a \bmod p_1 p_2\}|$) by $\pi(N)/\phi(p) = \pi(N)/(p-1)$ for a coprime to p (and, respectively, by $\pi(N)/\phi(p_1 p_2)$ for a coprime to $p_1 p_2$). Now the absolute value of

$$\sum_{p \leqslant z} \frac{a_p}{p} \Bigg(\sum_{\substack{a \bmod p \\ p \nmid a}} \Big(\frac{f(a)}{p} \Big) \bigg| |\{q \leqslant N : q \equiv a \bmod p\}| - \frac{\pi(N)}{p-1} \bigg| + \Big(\frac{f(0)}{p} \Big) |\{q \leqslant N : q \equiv 0 \bmod p\}| \Bigg)$$

is at most

$$\sum_{p \leqslant z} \bigg| \frac{a_p}{p} \bigg| \sum_{\substack{a \bmod p \\ p \nmid a}} \bigg| |\{q \leqslant N : q \equiv a \bmod p\}| - \frac{\pi(N)}{p-1} \bigg| + \sum_{p \leqslant z} \bigg| \frac{a_p}{p} \bigg|.$$

By the trivial bound $|a_p| \ll p$, the second sum is $O(z)$ (and thus will be negligible). We apply the Cauchy–Schwarz inequality twice to obtain that the first sum is at most

$$\sqrt{\sum_{p \leqslant z} \frac{a_p^2}{p^2}} \sqrt{\sum_{p \leqslant z} (p-1) \sum_{\substack{a \bmod p \\ p \nmid a}} \bigg| |\{q \leqslant N : q \equiv a \bmod p\}| - \frac{\pi(N)}{p-1} \bigg|^2}. \tag{4.11}$$

The expression under the first square root is now R, which is $\ll \log z \ll \log N$. By a brief calculation, the expression under the second square root equals

$$\sum_{p\leqslant z} \sum_{\substack{\chi \bmod p \\ \chi \text{ non-principal}}} |S(\chi)|^2 \tag{4.12}$$

for $S(\chi)=\sum_{q\leqslant N}\chi(q)$, where q runs over the primes, as usual. By Lemma 4.2 (with $\varepsilon=\frac{1}{2}$), (4.12) is $O(\pi(N)^2)$. Hence (4.11) is at most $O(\sqrt{R}\pi(N))$. Therefore

$$\begin{aligned}\sum_{p\leqslant z}\frac{a_p}{p}\sum_{a \bmod p}\left(\frac{f(a)}{p}\right)&|\{q\leqslant N : q\equiv a \bmod p\}| \\ &= \sum_{p\leqslant z}\frac{a_p}{p}\sum_{\substack{a \bmod p \\ p\nmid a}}\left(\frac{f(a)}{p}\right)\frac{\pi(N)}{p-1}+O_A(\sqrt{R}\pi(N)).\end{aligned}$$

Now,

$$\begin{aligned}\sum_{\substack{a \bmod p \\ p\nmid a}}\left(\frac{f(a)}{p}\right) &= \sum_{a \bmod p}\left(\frac{f(a)}{p}\right)-\left(\frac{f(0)}{p}\right) \\ &= \sum_{a \bmod p}|\{y\in\mathbb{Z}/p\mathbb{Z} : y^2=f(a)\}|-p-\left(\frac{f(0)}{p}\right) \\ &= (p+1-a_p)+O(1)-p-\left(\frac{f(0)}{p}\right) \\ &= -a_p+O(1),\end{aligned} \tag{4.13}$$

where the implied constant is absolute. Thus

$$\begin{aligned}\sum_{p\leqslant z}\frac{a_p}{p}\sum_{\substack{a \bmod p \\ p\nmid a}}\left(\frac{f(a)}{p}\right)\frac{\pi(N)}{p-1} &= \pi(N)\sum_{p\leqslant z}\frac{a_p}{p}\frac{1}{p-1}(-a_p+O(1)) \\ &= \pi(N)\left(\sum_{p\leqslant z}\frac{-a_p^2}{p^2}+O\left(\sum_{p\leqslant z}\left(\frac{a_p^2}{p^3}+\frac{a_p}{p^2}\right)\right)\right) \\ &= \pi(N)(-R+O(1)),\end{aligned}$$

where we use the Weil bound $|a_p|\ll\sqrt{p}$ in the last step.

Let us now estimate the sum in the second line of (4.10). Since the only primes q not coprime to p_1 or p_2 are $q=p_1$ and $q=p_2$, the contribution of the terms with $\gcd(a,p_1p_2)\neq 1$ is at most

$$2\sum_{p_1\leqslant z}\sum_{p_2\leqslant z}\frac{a_{p_1}}{p_1}\frac{a_{p_2}}{p_2}\ll z,$$

which is negligible. We write

$$\sum_{\substack{p_1\leqslant z\\ p_1\neq p_2}}\sum_{p_2\leqslant z}\frac{a_{p_1}}{p_1}\frac{a_{p_2}}{p_2}\sum_{\substack{a \bmod p_1p_2\\ \gcd(a,p_1p_2)=1}}\left(\frac{f(a)}{p_1p_2}\right)|\{q\leqslant N: q\equiv a \bmod p_1p_2\}| \tag{4.14}$$

$$\begin{aligned}
&=\sum_{\substack{p_1\leqslant z\\ p_1\neq p_2}}\sum_{p_2\leqslant z}\frac{a_{p_1}}{p_1}\frac{a_{p_2}}{p_2}\\
&\quad\times\sum_{\substack{a \bmod p_1p_2\\ \gcd(a,p_1p_2)=1}}\left(\frac{f(a)}{p_1p_2}\right)\frac{\pi(N)}{\phi(p_1p_2)}+O\Bigg(\sum_{\substack{p_1\leqslant z\\ p_1\neq p_2}}\sum_{p_2\leqslant z}\frac{|a_{p_1}|}{p_1}\frac{|a_{p_2}|}{p_2}\sum_{\substack{a \bmod p_1p_2\\ \gcd(a,p_1p_2)=1}}\Delta_{a,p_1p_2}\Bigg)\\
&\quad+\sum_{\substack{p_1\leqslant z\\ p_1\neq p_2}}\sum_{p_2\leqslant z}\frac{a_{p_1}}{p_1}\frac{a_{p_2}}{p_2}\sum_{\substack{a \bmod p_1p_2\\ \gcd(a,p_1p_2)=1}}\left(\frac{f(a)}{p_1p_2}\right)\left(\frac{1}{\phi(p_1)}|\{q\leqslant N: q\equiv a \bmod p_2\}|-\frac{\pi(N)}{\phi(p_1p_2)}\right)\\
&\quad+\sum_{\substack{p_1\leqslant z\\ p_1\neq p_2}}\sum_{p_2\leqslant z}\frac{a_{p_1}}{p_1}\frac{a_{p_2}}{p_2}\sum_{\substack{a \bmod p_1p_2\\ \gcd(a,p_1p_2)=1}}\left(\frac{f(a)}{p_1p_2}\right)\left(\frac{1}{\phi(p_2)}|\{q\leqslant N: q\equiv a \bmod p_1\}|-\frac{\pi(N)}{\phi(p_1p_2)}\right),
\end{aligned}$$

where

$$\begin{aligned}
\Delta_{a,p_1p_2}&=|\{q\leqslant N: q\equiv a \bmod p_1p_2\}|-\frac{1}{\phi(p_1)}|\{q\leqslant N: q\equiv a \bmod p_2\}|\\
&\quad-\frac{1}{\phi(p_2)}|\{q\leqslant N: q\equiv a \bmod p_1\}|+\frac{1}{\phi(p_1p_2)}\pi(N).
\end{aligned}$$

The first sum on the right-hand side of (4.14) is the main term; by (4.13), it equals

$$\pi(N)\sum_{\substack{p_1\leqslant z\\ p_1\neq p_2}}\sum_{p_2\leqslant z}\frac{a_{p_1}}{p_1}\frac{a_{p_2}}{p_2}\frac{(-a_{p_1}+O(1))(-a_{p_2}+O(1))}{\phi(p_1p_2)}=\pi(N)(R^2+O(R)).$$

By the Cauchy–Schwarz inequality, the second sum in (4.14) (the sum within $O(...)$) is at most

$$\sqrt{\sum_{\substack{p_1\leqslant z\\ p_1\neq p_2}}\sum_{p_2\leqslant z}\frac{a_{p_1}^2}{p_1^2}\frac{a_{p_2}^2}{p_2^2}}\sqrt{\sum_{\substack{p_1\leqslant z\\ p_1\neq p_2}}\sum_{p_2\leqslant z}\left|\sum_{\substack{a \bmod p_1p_2\\ \gcd(a,p_1p_2)=1}}\Delta_{a,p_1p_2}\right|^2}. \tag{4.15}$$

The expression under the first square root is $\leqslant R^2$. By another application of the Cauchy–

Schwarz inequality and a brief calculation (cf. [1, §2, Theorem 5]), we get

$$\begin{aligned}\Big|\sum_{\substack{a \bmod p_1p_2\\ \gcd(a,p_1p_2)=1}} \Delta_{a,p_1p_2}\Big|^2 &\leqslant \phi(p_1p_2) \sum_{\substack{a \bmod p_1p_2\\ \gcd(a,p_1p_2)=1}} |\Delta_{a,p_1p_2}|^2\\ &= \phi(p_1p_2) \sum_{\substack{a \bmod p_1p_2\\ \gcd(a,p_1p_2)=1}} |\{q\leqslant N : q\equiv a \bmod p_1p_2\}|^2\\ &\quad -\phi(p_1) \sum_{\substack{a \bmod p_1\\ \gcd(a,p_1)=1}} |\{q\leqslant N : q\equiv a \bmod p_1\}|^2\\ &\quad -\phi(p_2) \sum_{\substack{a \bmod p_2\\ \gcd(a,p_2)=1}} |\{q\leqslant N : q\equiv a \bmod p_2\}|^2+\pi(N)^2\\ &= \sum_{\substack{\chi \bmod p_1p_2\\ \chi \text{ primitive}}} |S(\chi)|^2.\end{aligned}$$

We apply Lemma 4.2, and obtain that (4.15) is $\ll_\varepsilon \sqrt{R^2}\sqrt{\pi(N)^2}=R\pi(N)$.

By (4.13), the next to last line of (4.14) is

$$\begin{aligned}&\sum_{p_1\leqslant z} \frac{a_{p_1}}{p_1}\frac{-a_{p_1}+O(1)}{p_1-1} \sum_{\substack{p_2\leqslant z\\ p_2\neq p_1}} \sum_{\substack{a \bmod p_2\\ p_2\nmid a}} \left(|\{q\leqslant N : q\equiv a \bmod p_2\}|-\frac{\pi(N)}{p_2-1}\right)\\ &\quad\leqslant \left(-\sum_{p_1\leqslant z}\frac{a_{p_1}^2}{p^2}+O(1)\right) \sum_{p_2\leqslant z} \sum_{\substack{a \bmod p_2\\ p_2\nmid a}} \left||\{q\leqslant N : q\equiv a \bmod p_2\}|-\frac{\pi(N)}{p_2-1}\right|.\end{aligned}$$

The first factor is $-R+O(1)$, whereas the second factor was already shown before to be $O(\sqrt{R}\pi(N))$. Hence the next to last line of (4.14) is $O(R^{3/2}\pi(N))$. Obviously the same is true of the last line of (4.14).

Putting everything together, we see that (4.10) has become

$$\begin{aligned}V &= (R^2+R)\pi(N)+2R(-R\pi(N)+O(\sqrt{R}\pi(N)))+(R^2+O_\varepsilon(R^{3/2}))\pi(N)\\ &= O_\varepsilon(R^{3/2}\pi(N)).\end{aligned}$$

Now, if

$$\left|\left(\sum_{p\leqslant z}\frac{a_p}{p}\left(\frac{f(q)}{p}\right)\right)-(-R)\right| > \delta R \tag{4.16}$$

for some $q\leqslant N$ and $\delta>0$, then that value of q makes a contribution greater than δ^2R^2 to V (see (4.8)).

Hence there are at most

$$\ll \frac{R^{3/2}\pi(N)}{\delta^2 R^2} = \frac{\pi(N)}{\delta^2\sqrt{R}}$$

primes $q \leqslant N$ for which (4.16) is the case. As $\lim_{N\to\infty} R = \infty$, we see that

$$\frac{\pi(N)}{\delta^2\sqrt{R}} = o_{\delta,\varepsilon}(\pi(N))$$

for any $\delta > 0$. Since δ is arbitrarily small, the statement of the lemma follows. □

5. Rarity of typical twists: large deviations and higher moments

We have seen (Lemmas 4.1 and 4.3, plus (4.1)) that, if q is a prime $\leqslant N$ lying outside a set containing a proportion $o(1)$ of all primes $\leqslant N$, and $dy^2 = f(q)$, where y is a prime, then d has some special properties:

(a) $\omega_{\mathrm{Cl}(g)}(d)$ must be of roughly a given size for each $g \in \mathrm{Gal}_f$, and

(b)

$$\sum_{p\leqslant z} \frac{a_p}{p}\left(\frac{d}{p}\right) \sim -\sum_{p\leqslant z} \frac{a_p^2}{p^2}, \tag{5.1}$$

i.e., d will have a slight tendency to be a quadratic residue mod p when a_p is negative, and a non-residue when a_p is positive.

We will see in this section that only a small minority of all integers $d \ll N(\log N)^{2\varepsilon}$ satisfy these properties. Here "small minority" actually means

$$\text{"fewer than } O((\log N)^{-(1+\delta)})\text{"},$$

where $\delta > 0$ is fixed. This will be crucial later.

Let us first examine how one would bound separately the number of integers satisfying (a) and the number of integers satisfying (b), i.e., equation (5.1). (We will eventually have to bound the number of integers satisfying both (a) and (b).)

One way of bounding $|\{d \ll N(\log N)^{2\varepsilon} : d \text{ satisfies (a)}\}|$ is to translate large-deviation estimates from probability theory. This was the approach followed in [12]. Here we will follow what would look like a more familiar approach to an analytic number theorist, though its content is essentially the same: we will bound expressions of the form

$$\sum_d e^{\sum_i \alpha_i \omega_{S_i}(d)}, \tag{5.2}$$

where $\alpha_i \in \mathbb{R}$ will be chosen at will, $\omega_{S_i} = \{p \in S_i : p|d\}$ and S_i is a set of primes (in our case, all unramified primes with Frob_p equal to a fixed element of the Galois group). The

bounds will be the same as those given by large-deviation theory—in particular, there will be relative entropies in the exponents.

How should we bound $|\{d \ll N(\log N)^{2\varepsilon} : d \text{ satisfies (5.1)}\}|$? A variance bound would not be good enough for our purposes. If we could truly handle reduction modulo distinct primes as so many independent random variables, we would use an exponential moment bound. As mutual independence does not truly hold, we will use instead a high moment, i.e., we will bound

$$\sum_d \left(\sum_{p \leqslant z} \frac{a_p}{p} \left(\frac{d}{q} \right) \right)^{2k} \tag{5.3}$$

for k large.

As we said, we would actually like to bound the number of integers $d \ll N(\log N)^{2\varepsilon}$ satisfying both (a) and (b) (i.e., equation (5.1)). Getting an estimate that combines information from both sources is, as we shall see, a technically delicate task, to be achieved by the *enveloping* use of a sieve.

The following lemma will allow us to work with small primes only without much of a loss in our estimates.

LEMMA 5.1. *For any $A>0$, $\varepsilon>0$ and every N, there is a $z=z(N,A,\varepsilon)$ with*

$$\log\log z > (1-\varepsilon)\log\log N$$

and $z<N^{\varepsilon}$ such that, for all but $O_{A,\varepsilon}(N(\log N)^{-A})$ integers n between 1 and N,

(a) $\prod_{p|n : p \leqslant z} p^{v_p(n)} < N^{\varepsilon}$,

(b) $\omega(n) - \sum_{p|n : p \leqslant z} 1 < \varepsilon \log\log z$.

Proof. Apply [12, Lemma 5.2] with $f(x)=x$ and $\frac{1}{2}\varepsilon$ instead of ε; let $z=N^{\delta(N)}$. Then

$$\log\log z = \log\log N - \log\log \delta(N) > \left(1-\tfrac{1}{2}\varepsilon\right)\log\log N.$$

Furthermore, $z=N^{o_{A,\varepsilon}(1/\log\log N)}<N^{\varepsilon}$ if (as we may assume) N is larger than a constant depending on A and ε.

By conclusion (a) in [12, Lemma 5.2], $\prod_{p|n : p \leqslant z} p < N^{\varepsilon/2}$. It is also the case that the largest square factor in n is $\leqslant N^{\varepsilon/2}$ for all but $O(N^{1-\varepsilon/4})$ integers between 1 and N. Part (a) of the statement follows. Conclusion (b) in [12, Lemma 5.2] implies that $\omega(n) - \sum_{p|n : p \leqslant z} 1 < \frac{1}{2}\varepsilon \log\log N$; since $\log\log z > \left(1-\frac{1}{2}\varepsilon\right)\log\log N \geqslant \frac{1}{2}\log\log N$, part (b) of the statement follows immediately. □

The next lemma is both elementary and of a very classical type.

LEMMA 5.2. *Let S be a set of primes; define $S_z=\{p\in S:p\leqslant z\}$. Assume that*

$$\sum_{p\in S_z}\frac{1}{p}\leqslant\beta\log\log z+C,$$

C being a constant. Let N_z denote the set of all positive integers that are products of primes in S_z alone. Let $\eta\geqslant 1$. Then

$$\sum_{\substack{n\in N_z\\ \omega(n)\geqslant\eta\beta\log\log z}}\frac{1}{n}\ll_{C,\eta}(\log z)^{\beta(\eta-\eta\log\eta)}.$$

The lemma would still be true for $\eta<1$ positive, but the exponent on the right would no longer be optimal.

Proof. Recall that $\sum_{n\geqslant 1}1/n^2=\pi^2/6$. For any $\alpha>0$,

$$\left(\frac{\pi^2}{6}\right)^\alpha\prod_{p\in S_z}\left(1+\frac{1}{p}\right)^\alpha\geqslant\prod_{p\in S_z}\left(1+\frac{1}{p}+\frac{1}{p^2}+\ldots\right)^\alpha\geqslant\prod_{p\in S_z}\left(1+\frac{\alpha}{p}+\frac{\alpha}{p^2}+\ldots\right)=\sum_{n\in N_z}\frac{\alpha^{\omega(n)}}{n}.$$

Hence,

$$\sum_{\substack{n\in N_z\\ \omega(n)\geqslant\eta\beta\log\log z}}\frac{1}{n}\leqslant\frac{1}{\alpha^{\eta\beta\log\log z}}\sum_{n\in N_z}\frac{\alpha^{\omega(n)}}{n}\leqslant\frac{1}{\alpha^{\eta\beta\log\log z}}\left(\frac{\pi^2}{6}\right)^\alpha\prod_{p\in S_z}\left(1+\frac{1}{p}\right)^\alpha$$

$$\ll_{C,\alpha}\frac{1}{\alpha^{\eta\beta\log\log z}}e^{\alpha\beta\log\log z}=(\log z)^{(\alpha-\eta\log\alpha)\beta}.$$

To minimise $\alpha-\eta\log\alpha$, we set $\alpha=\eta$. Then $(\log z)^{(\alpha-\eta\log\alpha)\beta}=(\log z)^{\beta(\eta-\eta\log\eta)}$. □

LEMMA 5.3. *Let S and S' be sets of primes with*

(1) *$S\subset S'$;*

(2) *$\sum_{p\in S:p\leqslant z}1/p\leqslant\beta\log\log z+C$ for all $z>e$, where C is a constant;*

(3) *$\sum_{n\leqslant z:p|n\Rightarrow p\in S'}1/n\geqslant C'(\log z)^{\beta'}$ for all $z>e$, where C' is a constant.*

Let N be a positive integer, and $\eta>1$. Let B be the set of all integers $n\leqslant N$ having at least $\eta\beta\log\log N$ divisors in S, but no divisors in $S'\setminus S$. Then, for all $\varepsilon>0$ and every $A>0$, there is a sequence of non-negative reals $\{b_n\}_{n\leqslant N}$ such that

(a) *$b_n\leqslant\tau_5(n)$ for every nC;*

(b) *$|\{n\in B:b_n<1\}|\ll_{A,\varepsilon}N/(\log N)^A$;*

(c) *$\sum_{n\leqslant N}b_n\ll_{C,C',\eta}N/(\log N)^{(1-\varepsilon/4)(\beta'+(\beta-\varepsilon/4)(\eta\log\eta-\eta))}$;*

(d)

$$\sum_{\substack{n\leqslant N\\ n\equiv a\bmod m}}b_n=\frac{1}{\phi(m)}\sum_{\substack{n\leqslant N\\ \gcd(n,m)=1}}b_n+O_\varepsilon(N^\varepsilon)$$

for every $m\leqslant N^{1-\varepsilon}$ and every a coprime to m.

The sequence b_n is a variant of what is sometimes called an *enveloping sieve*; here, as per (b), the sequence b_n almost "envelops" (i.e., majorises the characteristic function of) B, but not quite.

Proof. Let z be as in Lemma 5.1 with $\frac{1}{4}\varepsilon$ instead of ε; in particular, $z<N^{\varepsilon/4}$. Let λ_d, $d\leqslant N^{\varepsilon/2}$, be the weights in Selberg's sieve[1] when used to sieve out prime factors $p\leqslant N^{\varepsilon/4}$ in S. (Here we are using λ_d to denote the sequence of non-negative reals λ_d (where $\lambda_d=0$ for $d>N^{\varepsilon/2}$) obtained by the identity $\sum_{d|m}\lambda_d=\left(\sum_{d|m}\varrho_d\right)^2$, where ϱ_d is as in, say, [8, (7.15)]. In particular, $\lambda_1=1$ and $|\lambda_d|\leqslant 1$ for all d. Note that some other texts use an opposite convention, exchanging the roles of λ_d and ϱ_d.)

Define

$$b_n = \sum_{\substack{m|n\\ m\in N_z(S)\\ \omega(m)\geqslant(\eta-\varepsilon/4)\beta\log\log z\\ m\leqslant N^{\varepsilon/4}}} \sum_{\substack{d|n/m\\ d\in N_z(S')}} \lambda_d, \tag{5.4}$$

where, for a set P of primes, $N_z(P)$ is the set of all positive integers that are products of primes in $\{p\in P : p\leqslant z\}$ alone.

Since $\lambda_d\leqslant\tau_3(d)$, conclusion (a) is immediate. Let $n\in B$. Then

$$b_n \geqslant \sum_{\substack{m|n\\ m\in N_z(S)\\ \omega(m)\geqslant(\eta\beta-\varepsilon/4)\log\log z\\ m\leqslant N^{\varepsilon/4}\\ p|n/m\Rightarrow p\notin S}} 1,$$

since the condition $p|n/m\Rightarrow p\notin S$ ensures (given that n has no divisors in $S'\setminus S$, due to $n\in B$) that the inner sum in (5.4) has λ_1 (which equals 1) as its only term. By $n\in B$ and the definition of B, n has at least $\eta\beta\log\log z$ divisors in S. Hence b_n can be less than 1 only if, for $m=\prod_{p|n,p\in S:p\leqslant z}p^{v_p(n)}$, either $m>N^{\varepsilon/4}$, or $\omega(n)-\omega(m)>\frac{1}{4}\varepsilon\log\log z$. By Lemma 5.1, at most $O_{A,\varepsilon}(N(\log N)^{-A})$ satisfy either statement (where $A>0$ is arbitrary). Hence conclusion (b) holds.

Now

$$\sum_{n\leqslant N} b_n = \sum_{\substack{m\leqslant N^{\varepsilon/4}\\ m\in N_z(S)\\ \omega(m)\geqslant(\eta\beta-\varepsilon/4)\log\log z}} \sum_{n\leqslant N/m} \sum_{\substack{d|n\\ d\in N_z(S')}} \lambda_d.$$

[1] Brun's (non-pure) sieve or the Iwaniec–Rosser sieve (as in [8, §6] and [8, §11], respectively) would do just as well as Selberg's sieve in this context. In fact, it would do slightly better, in that the subscript in (a) would go down from 5 to 3.

By the main result on the Selberg sieve (see, e.g., [8, Theorem 7.1], with $a_n=1$ for all $n\leqslant N/m$ and $a_n=0$ for $n>N/m$)

$$\sum_{n\leqslant N/m}\sum_{\substack{d\mid n\\ d\in N_z(S')}}\lambda_d=\left(\prod_{\substack{p\in S'\\ p\leqslant z}}\frac{1}{1-1/p}\right)^{-1}\frac{N}{m}+O\Big(\sum_{d<N^{\varepsilon/2}}\tau_3(d)\Big)$$
$$\leqslant\left(\sum_{\substack{d\leqslant N^{\varepsilon/4}\\ d\in N_z(S')}}\frac{1}{d}\right)^{-1}\frac{N}{m}+O_\varepsilon(N^{3\varepsilon/4}).$$

By condition (3) and $z<N^{\varepsilon/4}$, we know that $\sum_{d\leqslant N^{\varepsilon/4},d\in N_z(S')}1/d\gg_{C'}(\log z)^{\beta'}$. Thus,

$$\sum_{n\leqslant N}b_n\ll_{C'}\frac{N}{(\log z)^{\beta'}}\sum_{\substack{m\leqslant N^{\varepsilon/4}\\ m\in N_z(S)\\ \omega(m)\geqslant(\eta\beta-\varepsilon/4)\log\log z}}\frac{1}{m}+O_\varepsilon(N^\varepsilon).$$

We now apply Lemma 5.2, and conclude that

$$\sum_{n\leqslant N}b_n\ll_{C,C',\eta}\frac{N}{(\log z)^{\beta'-(\beta-\varepsilon/4)(\eta-\eta\log\eta)}}.$$

Lemma 5.1 assures us that $\log\log z>\big(1-\frac14\varepsilon\big)\log\log N$, and so $\log z>(\log N)^{1-\varepsilon/4}$. We thus obtain conclusion (c).

Lastly, for every r and every a coprime to r,

$$\sum_{\substack{n\leqslant N\\ n\equiv a \bmod r}}b_n=\sum_{\substack{m\leqslant N^{\varepsilon/4}\\ m\in N_z(S)\\ \omega(m)\geqslant(\eta\beta-\varepsilon/4)\log\log z}}\sum_{\substack{d\leqslant N^{\varepsilon/2}\\ d\in N_z(S')}}\lambda_d\sum_{\substack{n\leqslant N/md\\ n\equiv a \bmod r}}1$$
$$=\sum_{\substack{m\leqslant N^{\varepsilon/4}\\ m\in N_z(S)\\ \omega(m)\geqslant(\eta\beta-\varepsilon/4)\log\log z}}\sum_{\substack{d\leqslant N^{\varepsilon/2}\\ d\in N_z(S')}}\lambda_d\left(\frac{1}{\phi(r)}\sum_{\substack{n\leqslant N/md\\ \gcd(n,r)=1}}1+O(1)\right)$$
$$=\frac{1}{\phi(r)}\sum_{\substack{n\leqslant N\\ \gcd(n,r)=1}}b_n+O\Big(\sum_{m\leqslant N^{\varepsilon/4}}\sum_{d\leqslant N^{\varepsilon/4}}\lambda_d\Big)$$
$$=\frac{1}{\phi(r)}\sum_{\substack{n\leqslant N\\ \gcd(n,r)=1}}b_n+O_\varepsilon(N^\varepsilon),$$

i.e., conclusion (d) holds. □

We begin by an easy application of Lemma 5.2 to the case already treated in [12]. We do this both for contrast with a later application (the proof of Proposition 5.5, which uses the divergence of $\sum_p a_p^2/p^2$ and where the sieve does play an enveloping role) and to make the paper relatively self-contained.

LEMMA 5.4. *Let $K/\mathbb{Q}$ be a cubic extension of $\mathbb{Q}$ with Galois group* Alt(3). *Let S be the set of unramified primes that split completely in $K/\mathbb{Q}$. Let V be the set of integers $n \leqslant N$ such that n has at least $(1+o(1)) \log\log N$ divisors in S, and n is not divisible by any unramified primes outside S. Then, for every $\varepsilon > 0$,*

$$|V| \ll_{K,\varepsilon} \frac{N}{(\log N)^{(1-\varepsilon)\log 3}}.$$

Proof. Let S' be the set of all unramified primes. Note that conditions (1) and (3) in Lemma 5.3 are clear, and condition (2) holds by the Chebotarev density theorem and partial summation. By the conclusions (b) and (c) in that lemma, applied with $A=2$,

$$|V| \leqslant O_\varepsilon\left(\frac{N}{(\log N)^A}\right) + \sum_{n \leqslant N} b_n \ll_\varepsilon \frac{N}{(\log N)^{(1-\varepsilon)(1+(1/3)(3\log 3-3))}} = \frac{N}{(\log N)^{(1-\varepsilon)\log 3}}. \quad \square$$

The following is the more difficult case.

PROPOSITION 5.5. *Let $K/\mathbb{Q}$ be a cubic extension of $\mathbb{Q}$ with Galois group* Sym(3). *Let S be the set of unramified primes that split completely in $K/\mathbb{Q}$; let S' be the set of unramified primes that either split completely or are inert in $K/\mathbb{Q}$. For every prime p, let a_p be such that $|a_p| \leqslant 2\sqrt{p}$ and, for $z = e^{(\log N)/2\log\log N}$,*

$$\left|\sum_{p \leqslant z} \frac{a_p^2}{p^2}\right| = (1+o(1)) \log\log z. \tag{5.5}$$

Let V be the set of integers $n \leqslant N$ such that (a) *n has at least $\left(\frac{1}{2}+o(1)\right)\log\log N$ divisors in S,* (b) *n has no divisors in $S' \setminus S$,* (c) *n satisfies*

$$\left|\sum_{p \leqslant z} \frac{a_p}{p}\left(\frac{n}{p}\right)\right| \geqslant (1+o(1)) \log\log z \tag{5.6}$$

for z as above. Then, for every $\varepsilon > 0$,

$$|V| \ll_{K,\varepsilon} \frac{N}{(\log N)^{(1+\log 3)/2-\varepsilon}},$$

where the implied constant depends on K, ε and the implied constants in (a), (b), (5.5) *and* (5.6).

Proof. We first verify that S and S' satisfy conditions (1)–(3) of Lemma 5.3. Condition (1) is obvious. Condition (2) holds with $\beta=\frac{1}{6}$ by the Chebotarev density theorem. Condition (3) holds for related reasons: as in (say) the proof of [11, Lemma 4.10], we can write

$$\begin{aligned}&\prod_{p\in S'}\frac{1}{1-p^{-s}}\\&\quad=\prod_p\Big(\frac{1}{1-p^{-s}}\Big)\Big(\prod_{p\notin S'}\Big(\frac{1}{1-p^{-s}}\Big)\prod_{p\in S'\setminus S}\Big(\frac{1}{1-p^{-s}}\Big)^3\Big)^{-1}\Big(\prod_{p\in S'\setminus S}\Big(\frac{1}{1-p^{-s}}\Big)^6\Big)^{1/2}\\&\quad=L_1(s)\zeta(s)\zeta_{K/\mathbb{Q}}(s)^{-1}\zeta_{L/\mathbb{Q}}(s)^{1/2},\end{aligned}$$

where $L_1(s)$ is holomorphic and bounded on $\{s:\mathrm{Re}(s)>\frac{1}{2}+\varepsilon\}$ and L is the Galois closure of K. Since ζ, $\zeta_{K/\mathbb{Q}}$ and $\zeta_{L/\mathbb{Q}}$ each have a pole of order 1 at $s=1$, we obtain

$$\sum_{\substack{n\leqslant z\\ p|n\Rightarrow p\in S'}}\frac{1}{n}\sim C(\log z)^{1-1+1/2}=C(\log z)^{1/2},$$

for some constant C, by contour integration or a real Tauberian theorem (e.g., a Hardy–Littlewood Tauberian theorem, [19, Theorem 5.11]; there is no need for a complex Tauberian theorem here).

Apply Lemma 5.3. By conclusion (b), we will find it enough to bound $\sum_{n\in V}b_n$ from above: $|V|$ will exceed this sum by at most $O_A(N/(\log N)^A)$, where we can set A as large as needed. For any k, (5.6) ensures that

$$\begin{aligned}\sum_{n\in V}b_n&\leqslant\Big(\max_{n\in V}\sum_{p\leqslant z}\frac{a_p}{p}\Big(\frac{n}{p}\Big)\Big)^{-2k}\sum_{n\in V}b_n\Big(\sum_{p\leqslant z}\frac{a_p}{p}\Big(\frac{n}{p}\Big)\Big)^{2k}\\&\leqslant\frac{1}{((1+o(1))\log\log z)^{2k}}\sum_{n\leqslant N}b_n\Big(\sum_{p\leqslant z}\frac{a_p}{p}\Big(\frac{n}{p}\Big)\Big)^{2k}.\end{aligned}\tag{5.7}$$

The following amounts to a proof of a special case of Khinchin's inequality, generalised to the case of random variables that are only approximately independent. First, we have

$$\sum_{n\leqslant N}b_n\Big(\sum_{p\leqslant z}\frac{a_p}{p}\Big(\frac{n}{p}\Big)\Big)^{2k}=\sum_{p_1,\dots,p_{2k}\leqslant z}\frac{a_{p_1}}{p_1}\dots\frac{a_{p_{2k}}}{p_{2k}}\sum_{n\leqslant N}b_n\Big(\frac{n}{p_1}\Big)\dots\Big(\frac{n}{p_{2k}}\Big).\tag{5.8}$$

Set $m=p_1p_2\dots p_{2k}$ and assume $m\leqslant N$. Using conclusions (a) and (d) in Lemma 5.3, we

get

$$\begin{aligned}\sum_{n\leqslant N} b_n\left(\frac{n}{p_1}\right)\dots\left(\frac{n}{p_{2k}}\right) &= \sum_{\substack{a \bmod m\\ \gcd(a,m)=1}}\left(\frac{a}{p_1}\right)\dots\left(\frac{a}{p_{2k}}\right)\sum_{\substack{n\leqslant N\\ n\equiv a \bmod m}} b_n + O\Big(\sum_{n\leqslant m} b_n\Big)\\ &= \sum_{\substack{a \bmod m\\ \gcd(a,m)=1}}\left(\frac{a}{p_1}\right)\dots\left(\frac{a}{p_{2k}}\right)\frac{1}{\phi(m)}\sum_{\substack{n\leqslant N\\ \gcd(n,m)=1}} b_n\\ &\quad + O_\varepsilon(N^\varepsilon)\sum_{a \bmod m} 1 + O\Big(\sum_{n\leqslant m}\tau_5(m)\Big)\\ &= \sum_{\substack{n\leqslant N\\ \gcd(n,m)=1}} b_n \sum_{a \bmod m}\left(\frac{a}{p_1}\right)\dots\left(\frac{a}{p_{2k}}\right)\frac{1}{\phi(m)} + O_\varepsilon(N^\varepsilon m),\end{aligned}$$

provided that $z^{2k}\leqslant N$. If there is a p appearing an odd number of times in $p_1, p_2, ..., p_{2k}$, the sum

$$\sum_{a \bmod m}\left(\frac{a}{p_1}\right)\dots\left(\frac{a}{p_{2k}}\right)$$

vanishes. On the other hand, given a multiset S consisting of k not necessarily distinct primes, the number of distinct tuples $(p_1, p_2, ..., p_{2k})$ such that every prime p appearing exactly ℓ times in S appears exactly 2ℓ times in $p_1, p_2, ..., p_{2k}$ is at most $(2k)!/2^k k!$ times the number of tuples $(q_1, q_2, ..., q_k)$ such that $S=\{q_1, q_2, ..., q_k\}$. (This is so by the crude bound $(2r)!\geqslant 2r!$ for $r\geqslant 1$.) Hence, going back to (5.8) and using conclusion (c) in Lemma 5.3, we obtain

$$\begin{aligned}\sum_{n\leqslant N} b_n\Big(\sum_{p\leqslant z}\frac{a_p}{p}\left(\frac{n}{p}\right)\Big)^{2k} &\leqslant \frac{(2k)!}{2^k k!}\sum_{q_1,\dots,q_k\leqslant z}\frac{a_{q_1}^2}{q_1^2}\dots\frac{a_{q_k}^2}{q_k^2}\sum_{n\leqslant N} b_n\\ &\quad + O_\varepsilon\Big(\sum_{p_1,\dots,p_{2k}\leqslant z}\frac{a_{p_1}}{p_1}\dots\frac{a_{p_{2k}}}{p_{2k}}N^\varepsilon p_1 p_2\dots p_{2k}\Big)\\ &\ll_{f,\varepsilon} \frac{(2k)!}{2^k k!}\frac{N}{(\log N)^{(1-\varepsilon/4)(1/2+(1/6-\varepsilon/4)(3\log 3-3))}}\Big(\sum_{p\leqslant z}\frac{a_p^2}{p^2}\Big)^k\\ &\quad + O_\varepsilon\Big(\Big(\sum_{p\leqslant z}\frac{2\sqrt{p}}{p}\Big)^k N^\varepsilon z^{2k}\Big)\\ &\leqslant \frac{(2k)!}{2^k k!}\frac{N}{(\log N)^{(1-\varepsilon)(\log 3)/2}}(1+o(1))^k(\log\log z)^k + O_\varepsilon(N^\varepsilon z^{3k}).\end{aligned}$$

Thus, by (5.7),

$$\sum_{n\in V} b_n \leqslant \frac{((1+o(1))\log\log z)^k}{((1+o(1))\log\log z)^{2k}}\frac{(2k)!}{2^k k!}\frac{N}{(\log N)^{(1-\varepsilon)(\log 3)/2}}+O_\varepsilon(N^\varepsilon z^{3k})$$
$$\ll \frac{e^{-k}(2k)^k}{((1+o(1))\log\log N)^k}\frac{N}{(\log N)^{(1-\varepsilon)(\log 3)/2}}+O_\varepsilon(N^\varepsilon e^{3k(\log N)/2\log\log N}).$$

We set $k=\frac{1}{2}\log\log N$, and obtain

$$\sum_{n\in V} b_n \ll \frac{(\log N)^{-1/2}(2k)^k}{(1+o(1))^{(\log\log N)/2}(2k)^k}\frac{N}{(\log N)^{(1-\varepsilon)(\log 3)/2}}+O(N^{3/4+\varepsilon})$$
$$\ll_\varepsilon \frac{N}{(\log N)^{(1+\log 3)/2-\varepsilon}}+O(N^{3/4+\varepsilon}).$$ □

6. Modularity. Conclusion.

It remains to estimate $\sum_{p\leqslant z} a_p^2/p^2$, where, as usual, we define a_p by letting $p+1-a_p$ be the number of (projective) points mod p on the curve $y^2=f(x)$. Our estimate will be based on the fact that the Rankin–Selberg L-function $L_{f\otimes f}$ has a pole at $s=2$.

LEMMA 6.1. *Let $f\in\mathbb{Z}[x]$ be a cubic polynomial irreducible over $\mathbb{Q}[x]$. For every prime p, write $p+1-a_p$ for the number of points in $\mathbb{P}^2(\mathbb{Z}/p\mathbb{Z})$ on the curve $y^2=f(x)$. Then, as $x\to\infty$,*

$$\sum_{p\leqslant x}\frac{a_p^2}{p^2} = (1+o_f(1))\log\log x.$$

Proof. By the modularity of elliptic curves ([24], [23], [3]), there is a primitive cusp form f of weight 2 and level N such that $f(z)=\sum_{n=1}^\infty a_n n^{1/2}e(nz)$. The Rankin–Selberg L-function

$$L(f\otimes\bar f,s)=\sum_{n=1}^\infty |a_n|^2 n^{-s-1}=\sum_{n=1}^\infty a_n^2 n^{-s-1}=L(f\otimes f,s)$$

([16, (13.49)], where $a(n)=n^{-1/2}a_n$) then has a simple pole at $s=1$ (the residue given by [16, (13.52)] is non-zero). Its Euler product decomposition is

$$L(f\otimes f,s)=\prod_p (1+p^{-s})(1-\alpha_p^2 p^{-s})^{-1}(1-p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1}$$
$$=\frac{1}{\zeta(2s)}\prod_p (1-p^{-s})^{-2}(1-\alpha_p^2 p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1},$$

where α_p and β_p are the reals satisfying $\alpha_p+\beta_p=a_p/\sqrt{p}$ and $\alpha_p\beta_p=1$.

Now

$$\begin{aligned}-\frac{L'(f\otimes f,s)}{L(f\otimes f,s)} &= (-\log L(f\otimes f,s))'\\ &= 2\frac{\zeta'(2s)}{\zeta(2s)}+\sum_p (\log p)\sum_{m=1}^{\infty} p^{-ms}(2+\alpha_p^{2m}+\beta_p^{2m})\\ &= \sum_p (\log p)a_p^2 p^{-s}+G(s),\end{aligned}$$

where $G(s)$ is holomorphic for $\mathrm{Re}(s)>\frac{1}{2}$.

Because $L(f\otimes f,s)$ has a simple pole at $s=1$, the function $-L'(f\otimes f,s)/L(f\otimes f,s)$ has a simple pole with residue 1 at 1. It is now enough to apply a Tauberian theorem of Hardy–Littlewood type [19, Theorem 5.11]; we obtain

$$\sum_{n\leqslant x}\frac{(\log p)a_p^2}{p^2}\sim \log x,$$

which, by partial summation, gives

$$\sum_{n\leqslant x}\frac{a_p^2}{p^2}\sim \log\log x,$$

as desired. □

Proof of the main theorem. By (2.1), it is enough to show that

$$|\{p\leqslant N : \text{there exists } q \text{ such that } q^2\mid f(p) \text{ and } q\geqslant N(\log N)^{-\varepsilon}\}| = o\left(\frac{N}{\log N}\right)$$

for some $\varepsilon>0$ independent of N. (Recall that p and q both denote primes.) If f is reducible, the problem reduces to that with f replaced by each of its irreducible factors g (since $p^2\mid f(n)$ for any prime p not dividing the discriminant $\mathrm{Disc}(f)$ implies $p^2\mid g(n)$ for some irreducible factor g of f) and then, since $\deg g\leqslant 2$, we have the problem solved by Estermann [7] (use simply [12, Lemma 6.2]).

We may thus assume that f is an irreducible polynomial. We may also assume without loss of generality that the leading coefficient of f is positive. Let α be a root of $f(x)=0$. Define $K=\mathbb{Q}(\alpha)/\mathbb{Q}$.

Let $N'=\max_{n\leqslant N} f(n)/(N(\log N)^{-\varepsilon})^2$. Clearly $N'\sim c_f N(\log N)^{2\varepsilon}$, where c_f is the leading coefficient of f. Let $z=e^{(\log N')/2\log\log N'}$. Let S be the set of unramified primes that split completely in $K/\mathbb{Q}$. By Lemma 4.1, the number of primes in S dividing $f(p)$ is

$$\begin{cases}(3+o_f(1))\frac{1}{6}\log\log z=\left(\frac{1}{2}+o_f(1)\right)\log\log z, & \text{if } \mathrm{Gal}_{K/\mathbb{Q}}=\mathrm{Sym}(3),\\ (3+o_f(1))\frac{1}{3}\log\log z=(1+o_f(1))\log\log z, & \text{if } \mathrm{Gal}_{K/\mathbb{Q}}=\mathrm{Alt}(3),\end{cases}$$

for all but $o_f(N/\log N)$ primes $p \leqslant N$. (A prime p that splits completely has Frob_p equal to $\{e\}$, where e is the identity element of the Galois group.) The number of primes in S dividing $f(p)/q^2$ differs from this by at most 1, and thus is also

$$\begin{cases} \left(\frac{1}{2}+o_f(1)\right)\log\log z, & \text{if } \mathrm{Gal}_{K/\mathbb{Q}} = \mathrm{Sym}(3), \\ (1+o_f(1))\log\log z, & \text{if } \mathrm{Gal}_{K/\mathbb{Q}} = \mathrm{Alt}(3). \end{cases}$$

Note that no unramified prime inert in $K/\mathbb{Q}$ can divide $f(p)$ (and thus no such prime can divide $f(p)/q^2$).

Suppose first that $\mathrm{Gal}_{K/\mathbb{Q}}=\mathrm{Alt}(3)$. Lemma 5.4 (applied with N' instead of N) gives us that there are at most

$$O_{f,\varepsilon}\left(\frac{N}{(\log N)^{\log 3-4\varepsilon}}\right)$$

possible values of $d=f(p)/q^2$, where p ranges across the primes $p\leqslant N$, with $o_f(N/\log N)$ primes excluded. Let D be the set of such values d.

Suppose now that $\mathrm{Gal}_{K/\mathbb{Q}}=\mathrm{Sym}(3)$. By Lemma 6.1,

$$\sum_{p\leqslant z}\frac{a_p^2}{p^2} = (1+o_f(1))\log\log z;$$

we can thus apply Lemma 4.3, and obtain that, for all but $o_f(N/\log N)$ primes $p\leqslant N$,

$$\sum_{p'\leqslant z}\frac{a_{p'}}{p'}\left(\frac{f(p)/q^2}{p'}\right) = O(1)+\sum_{p'\leqslant z}\frac{a_{p'}}{p'}\left(\frac{f(p)}{p'}\right) = -(1+o(1))\log\log z.$$

Proposition 5.5 (applied with N' instead of N) now gives us that there are at most

$$O_{f,\varepsilon}\left(\frac{N}{(\log N)^{(1+\log 3)/2-3\varepsilon}}\right)$$

possible values of $d=f(p)/q^2$, where p ranges across the primes $p\leqslant N$, with $o_f(N/\log N)$ primes excluded. Let D be the set of such values d.

We now use Proposition 3.5, and obtain that the numbers of integers (prime or not) $1\leqslant x\leqslant N$ such that $dq^2=f(x)$ for some $d\in D$ and some integer $q\geqslant N(\log N)^{-\varepsilon}$ is at most $O_{f,\varepsilon}(N/(\log N)^{\log 3-4\varepsilon})$ (if $\mathrm{Gal}_{K/\mathbb{Q}}=\mathrm{Alt}(3)$) or at most $O_{f,\varepsilon}(N/(\log N)^{(1+\log 3)/2-4\varepsilon})$ (if $\mathrm{Gal}_{K/\mathbb{Q}}=\mathrm{Sym}(3)$). Since $\log 3>1$ and $\frac{1}{2}(1+\log 3)>1$, we are done. □

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