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Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry

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1. Introduction

The main purpose of this paper is to prove a general result about the geometry of holomorphic line bundles over Kähler manifolds. This result is essentially a partial verification of a conjecture of Tian [30] and Tian has, over many years, highlighted the importance of the question for the existence theory of Kähler–Einstein metrics. We will begin by stating this main result.

We consider data (X,g,J,L,A) where (X,g) is a compact Riemannian manifold of real dimension $2n$, J is a complex structure with respect to which the metric is Kähler, L is a Hermitian line bundle over X and A is a connection on L with curvature $-j\omega$, where ω is the Kähler form. We will often just write X as an abbreviation for this data. We suppose the metric satisfies fixed upper and lower bounds on the Ricci tensor

$$-\tfrac{1}{2}g \leqslant \mathrm{Ric} \leqslant g. \tag{1.1}$$

(The particular bounds we have chosen are just convenient normalisations; any other fixed bounds would do.) For $V,c>0$ let $\mathcal{K}(n,c,V)$ denote the class of all such data such that the volume of X is V and the "non-collapsing" condition

$$\mathrm{Vol}\, B_r \geqslant c\frac{\pi^n}{n!}r^{2n} \tag{1.2}$$

holds. Here B_r is any metric r-ball in X, r is any number less than the diameter of X and the normalising factor $\pi^n/n!$ is the volume of the unit ball in $\mathbb{C}^n$.

The connection induces a holomorphic structure on L and for each positive integer k there is a natural L^2 hermitian metric on the space $H^0(X,L^k)$. Recall that the "density of states" (or Bergman) function $\varrho_{k,X}$ is defined by

$$\varrho_{k,X} = \sum_{\alpha} |s_\alpha|^2,$$

where $\{s_\alpha\}_\alpha$ is any orthonormal basis of $H^0(X, L^k)$. An equivalent definition is that $\varrho_{k,X}(x)$ is the maximum of $|s(x)|^2$ as s runs over the holomorphic sections with L^2 norm 1. Thus, to establish a lower bound on $\varrho_{k,X}(x)$ we have to produce a holomorphic section s with L^2 norm not too large and with $|s(x)|$ not too small. Write

$$\underline{\varrho}(k, X) = \min_{x \in X} \varrho_{k,X}(x).$$

Standard theory, a part of the Kodaira embedding theorem, asserts that for each fixed X we have $\underline{\varrho}(k, X) > 0$ for large enough k. Our main result can be thought of as an extension of this statement which is both *uniform* over $\mathcal{K}(n, c, V)$ and gives a definite lower bound.

THEOREM 1.1. *Given n, c and V, there is an integer k_0 and $b>0$ such that*

$$\underline{\varrho}(k_0, X) \geqslant b^2 \quad \textit{for all } X \in \mathcal{K}(n, c, V).$$

The proof involves a combination of the Gromov–Hausdorff convergence theory—developed by Anderson, Cheeger, Colding, Gromoll, Gromov, Tian and others over the past thirty years or so—and the "Hörmander technique" for constructing holomorphic sections. When $n=2$ the theorem was essentially proved by Tian in [29] and the overall scheme of our proof is similar. We remark that the original conjecture of Tian in [30] is stated for Kähler metrics on Fano manifolds with a uniform positive lower bound on the Ricci curvature, and this amounts to removing the hypothesis on the upper bound of Ricci curvature in the above theorem. This remains an interesting open question to study in the future.

The above theorem provides the foundations for a bridge between the differential geometric convergence theory and algebraic geometry, leading to the following result (as indicated by Tian).

THEOREM 1.2. *Given n, c and V, there is a fixed k_1 and an integer N with the following effect*:

- *Any X in $\mathcal{K}(n, c, V)$ can be embedded in a linear subspace of $\mathbb{CP}^N$ by sections of L^{k_1}.*
- *Let X_j be a sequence in $\mathcal{K}(n, c, V)$ with Gromov–Hausdorff limit X_∞. Then X_∞ is homeomorphic to a normal projective variety W in $\mathbb{CP}^N$. After passing to a subsequence and taking a suitable sequence of projective transformations, we can suppose that the projective varieties $X_j \subset \mathbb{CP}^N$ converge as algebraic varieties to W.*

(More precise statements, and more detailed information, are given in §4 below.)

Many of the ideas and arguments required to derive this are similar to those of Ding and Tian in [14] who considered Fano manifolds with Kähler–Einstein metrics. Then the limit is a "$\mathbb{Q}$-Fano" variety, as Ding and Tian conjectured.

In §2 we review relevant background in convergence theory and complex differential geometry. Given this background, the rest of the proof is essentially self-contained. The proof of Theorem 1.1 is given in §3. We begin by reducing Theorem 1.1 to a "local" statement (Theorem 3.2) involving a point in a Gromov–Hausdorff limit space and attention is then focused on a tangent cone at this point. In §4 we give the proof of Theorem 1.2. We also establish some further relations between the differential geometric and algebro-geometric theories. In §5 we include a more detailed analysis of tangent cones in the 3-dimensional case, showing that these are cones over Sasaki–Einstein orbifolds and discuss the likely picture in the higher-dimensional situation.

Our main interest throughout this paper is in the case when X is a Fano manifold, the metric is Kähler–Einstein with positive Ricci curvature and $L=K_X^{-1}$. Then the Ricci bound (1.1) holds trivially and (as a consequence of Bishop–Gromov monotonicity) the non-collapsing condition (1.2) is automatic for a suitable c, in fact with

$$c=\frac{V(2n-1)!!}{2^{n+1}(2n-1)^n\pi^n}.$$

But the general hypotheses we have made above seem to give the natural context for the discussion here, although the applications outside the Fano case may be limited. In the case of Kähler–Einstein metrics of negative or zero Ricci curvature there is of course a complete existence theory due to Aubin and Yau. It would be interesting to characterise the non-collapsing condition in this situation algebro-geometrically.

We would like to emphasise that this is a "theoretical" paper in the following sense. Our purpose is to establish that some high powers k_0 and k_1 of a positive line bundle have certain good properties uniformly over manifolds in $\mathcal{K}(n,c,V)$ and Gromov–Hausdorff limits thereof. For many reasons one would like to know values of k_0 and k_1 which are, first, explicitly computable and, second, realistic. (That is, not too different from the optimal values which, in reality, yield these good properties.) This paper is theoretical in that we will not attempt to do anything of this kind. Of our two foundations—the Hörmander technique and convergence theory—-the first is quite amenable to explicit estimates but the second is not. So it is unclear whether even in principle one could extract any computable numbers. Of course this is an important question for future research. Given this situation, we have not attempted to make the arguments in §3 and §4 efficient, in the sense that (even if one somehow had effective constructions for the building blocks) our arguments from those building blocks lead to huge, completely unrealistic, numbers. This is connected to certain definite and tractable mathematical questions which we take up briefly again at the very end of the paper, where we formulate a conjectural sharper version of Theorem 1.1 (Conjecture 5.15).

We finish this introduction with some words about the origins of this paper. While the question that we answer in Theorem 1.1 is a central one in the field of Kähler–Einstein geometry, it is not something that the authors have focused on until recently. The main construction in this paper emerged as an off-shoot of a joint project by the first-named author and Xiuxiong Chen, studying the slightly different problem of Kähler–Einstein metrics with cone singularities along a divisor. A companion article by the first-named author and Chen, developing this related theory, will appear shortly. Both authors are very grateful to Chen for discussions of these matters, extending over many years.

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2. Background

2.1. Convergence theory

This subsection is a rapid summary of many formidable results. We will not attempt to give detailed references, but refer to the surveys [4] and [5], and the references therein.

Recall that if Z and W are two compact metric spaces then the Gromov–Hausdorff distance $d_{\mathrm{GH}}(Z,W)$ is the infimum of the numbers δ such that there is a metric on $Z \sqcup W$ extending the given metrics on the components and such that each of Z and W is δ-dense. The starting point of the theory is Gromov's theorem that a sequence of compact m-dimensional Riemannian manifolds (M_j, g_j) with bounded diameter and with Ricci curvature bounded below has a Gromov–Hausdorff convergent subsequence with some limit M_∞, which is a compact metric space. In our situation, with a sequence in $\mathcal{K}(n,c,V)$ the diameter bound follows from the non-collapsing condition (1.2). Passing to this subsequence, we can fix metrics on the disjoint unions $M_j \sqcup M_\infty$ such that M_j and M_∞ are δ_j dense, where $\delta_j \to 0$.

Now suppose that, as in our situation, the Ricci tensors of M_j satisfy fixed upper and lower bounds. Suppose also that a non-collapsing condition (1.2) holds and the volumes are bounded below. Then there is a connected, open, dense subset $M_\infty^{\mathrm{reg}} \subset M_\infty$ which is an m-dimensional $C^{2,\alpha}$-manifold and has a $C^{1,\alpha}$ Riemannian metric g_∞ (for all Hölder exponents $\alpha < 1$). This is compatible with the metric space structure in the following sense. For any compact $K \subset M_\infty^{\mathrm{reg}}$ we can find a number $s > 0$ such that if x_1 and x_2 are points of K with $d(x_1, x_2) \leqslant s$ then $d(x_1, x_2)$ is the infimum of the length of paths in M_∞^{reg} between x_1 and x_2. Moreover, the convergence on this subset is $C^{1,\alpha}$, in the following sense. Given any number $\delta>0$ and compact subset $K\subset M_\infty^{\mathrm{reg}}$ we can find for large enough j open embeddings χ_j of an open neighbourhood of K into M_j such that

(1) the pull-backs by the χ_j of the g_j converge in $C^{1,\alpha}$ over K to g_∞;

(2) $d(x,\chi_j(x))\leqslant\delta$ for all $x\in K$.

The second item here refers to the chosen metric on $M_j\sqcup M_\infty$.

The volume form of the Riemannian metric on the dense set M_∞^{reg} defines a measure on M_∞ and the volume of M_∞ is the limit of the volumes of the M_j. The Hausdorff dimension of the *singular set* $\Sigma=M_\infty\setminus M_\infty^{\mathrm{reg}}$ does not exceed $m-2$.

There is a variant of the theory in which one considers spaces with base points and convergence over bounded distance from the base points. In particular we can take a point $p\in M_\infty$ and any sequence $R_j\to\infty$ and then consider the sequence of based metric spaces given by scaling M_∞ by a factor R_j. The compactness theorem implies that, passing to a subsequence, we get convergence and a fundamental result is that, under our hypotheses, the corresponding limit is a metric cone $C(Y)$—a *tangent cone* of M_∞ at p. Here Y is a metric space which contains a dense open subset Y^{reg} which is a *smooth* $(m-1)$-dimensional Einstein manifold, with Ricci curvature equal to $m-2$, and the metric and Riemannian structures on Y^{reg} are related in a similar way to that above. Likewise for the natural measure on Y. The singular set $\Sigma_Y=Y\setminus Y^{\mathrm{reg}}$ has Hausdorff dimension at most $m-3$. The cone $C(Y)$ has a smooth Ricci-flat metric outside the singular set $\{O\}\cup C(\Sigma_Y)$ (where O is the vertex of the cone) and the convergence of the rescaled metrics is $C^{1,\alpha}$ in the same sense as before. An important numerical invariant of this situation is the volume ratio

$$\varkappa=\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^{m-1})}. \tag{2.1}$$

The Bishop inequality implies that $\varkappa\leqslant 1$ and if a non-collapsing bound like (1.2) holds for the original manifolds M_j we have $\varkappa\geqslant c$.

All of the preceding discussion is in the general Riemannian context. Suppose now that our manifolds are X_j and g_j, and we have additional structures J_j, L_j and A_j as in §1. We define a *polarised limit space* to be a metric limit (X_∞,g_∞) as above, together with extra data as follows:

- a $C^{1,\alpha}$ complex structure J_∞ on the regular set with respect to which the metric is Kähler with 2-form ω_∞;
- a $C^{2,\alpha}$ line bundle L_∞ over the regular set;
- a $C^{1,\alpha}$ connection A_∞ on L_∞ with curvature $-i\omega_\infty$.

(Notice that the integrability theorem for complex structures extends to the $C^{1,\alpha}$ situation [24]. Thus in fact we could say that X_∞^{reg} and L have smooth structures while the convergence is in $C^{1,\alpha}$. But this is largely irrelevant for our purposes.)

We define convergence of a sequence (X_j, J_j, L_j, A_j) to such a polarised limit by requiring that for compact $K \subset X_\infty^{\text{reg}}$ we have maps χ_j as before but in addition so that the pulled back complex structures $\chi_j^*(J_j)$ converge to J_∞ (in $C^{1,\alpha}$) and we have bundle isomorphisms $\widehat{\chi}_j: L_\infty \to \chi_j^*(L_j)$ with respect to which the connections converge to A_∞ in $C^{1,\alpha}$. It is straightforward to extend the compactness theorem to this polarised situation, using the fact that J_j is a covariant-constant tensor with respect to the $C^{0,\alpha}$ Levi-Civita connection of g_j.

Likewise, the regular part of a tangent cone $C(Y)$ at a point in X_∞ has a smooth, Ricci-flat, Kähler metric which is induced from a *Sasaki–Einstein* structure on Y^{reg}. In particular the Kähler form on the smooth part can be written as $\frac{1}{2} i\partial\bar{\partial}|z|^2$, where $|z|$ denotes the distance to the vertex of the cone.

A significant difference in the Kähler case is that the singular sets (both in X_∞ and in Y) are known to have Hausdorff codimension at least 4. (This is conjectured but not established in the real case.) In particular, which will be crucial for us, the codimension is strictly greater than 2 ([6] and [8]).

In the case of primary interest—Kähler–Einstein metrics—we obtain C^∞ convergence on compact subsets of the regular sets. (This is also part of the standard literature.) In addition, if we consider the "Fano case", when L_j is the anticanonical bundle of X_j, then the limit line bundle is just the anticanonical bundle of the regular set in X_∞. The reader may well prefer to restrict to this case. More generally, if we just assume (in addition to (1.1) and (1.2)) that the metrics have constant scalar curvature, then one can still establish this C^∞ convergence, for example using the results of Chen and Weber [9].

2.2. Complex differential geometry: the Hörmander technique

We begin by recalling that, under our hypotheses, there is a uniform Sobolev inequality

$$\|f\|_{L^{2n/(n-1)}} \leqslant C_1 \|\nabla f\|_{L^2} + C_2 \|f\|_{L^2}, \tag{2.2}$$

for functions f on a manifold X in the class $\mathcal{K}(n, c, V)$, where C_1 and C_2 depend only on n, c and V [11]. Here of course we are referring to norms defined by the metric g. When working with the line bundle L^k it will be convenient to use the norms defined by the rescaled metrics kg (for integers $k \geqslant 1$). Thus lengths are scaled by $\sqrt{k}$ and volumes by k^n. We will use the notation $L^{2,\sharp}$, etc. to denote norms defined by these rescaled metrics. Then the scaling weight gives

$$\|f\|_{L^{2n/(n-1),\sharp}} \leqslant C_1 \|\nabla f\|_{L^{2,\sharp}} + C_2 k^{-1/2} \|f\|_{L^{2,\sharp}}. \tag{2.3}$$

So the scaling only helps in the Sobolev inequality. Of course the Ricci tensor $\mathrm{Ric}^\sharp$ of the rescaled metric is bounded between $-(2k)^{-1}g$ and $k^{-1}g$.

PROPOSITION 2.1. (1) *There are constants K_0 and K_1, depending only on n, c and V, such that if X is in $\mathcal{K}(n,c,V)$ and s is a holomorphic section of L^k (for any $k>0$) we have*

$$\|s\|_{L^{\infty,\sharp}} \leqslant K_0\|s\|_{L^{2,\sharp}} \quad \textit{and} \quad \|\nabla s\|_{L^{\infty,\sharp}} \leqslant K_1\|s\|_{L^{2,\sharp}}.$$

(2) *If X is in $\mathcal{K}(n,c,V)$ then for any $k>0$ the Laplacian $\Delta_{\bar\partial}$ on $\Omega^{0,1}(L^k)$ is invertible and $\Delta_{\bar\partial}^{-1}\leqslant 2$.*

In the second item $\Delta_{\bar\partial}=\bar\partial^*\bar\partial+\bar\partial\bar\partial^*$, with adjoints defined using the rescaled metric, and the statement is that, for all ϕ,

$$\langle \Delta_{\bar\partial}^{-1}\phi,\phi\rangle_\sharp \leqslant 2\|\phi\|_{L^{2,\sharp}}^2. \tag{2.4}$$

This proposition summarises results which are well known to workers in the field and which all hinge on various formulae of Bochner–Weitzenbock type. We use the rescaled metrics throughout the discussion. First on C^∞ sections s of L^k we have

$$\nabla^*\nabla s = 2\bar\partial^*\bar\partial s + ns,$$

so when s is holomorphic $\nabla^*\nabla s=ns$ which implies that

$$\Delta|s| \leqslant n|s|, \tag{2.5}$$

where the lack of differentiability of $|s|$ at the zero set is handled in a standard way. (Note that we use the " geometers convention" for the sign of the Laplacian in this paper.) Now the bound on the L^∞ norm follows from the Moser iteration argument applied to this differential inequality, using the uniform Sobolev inequality (see [30]).

The first-derivative bound is obtained in a similar way. Changing notation slightly, for a holomorphic section s with $\bar\partial s=0$ we write $\nabla s=\partial s$, where

$$\partial\colon \Omega^{p,q}(L^k) \longrightarrow \Omega^{p+1,q}(L^k)$$

is defined using the connection. Since $\partial^2=0$, we have

$$\Delta_\partial \partial s = \partial\Delta_\partial s,$$

where $\Delta_\partial=\partial^*\partial+\partial\partial^*$. Then for a holomorphic section s, $\Delta_\partial s=\nabla^*\nabla s=ns$ and

$$\Delta_\partial(\partial s) = n\partial s.$$

Now the Bochner–Weitzenbock formula comparing Δ_∂ and $\nabla^*\nabla$ on $\Omega^{1,0}(L^k)$ has the form (see for example [22], (1.4.31) applied for $E=T^{(1,0)}X\otimes L^k$, and (1.4.49a) and (1.5.13) applied for $E=L^k$)

$$2\Delta_\partial=\nabla^*\nabla+n-2+\mathrm{Ric}^\sharp,$$

so

$$(\nabla^*\nabla(\partial s),\partial s)\leqslant\big(n+\tfrac{3}{2}\big)|\partial s|^2.$$

It follows that

$$\Delta|\partial s|\leqslant\big(n+\tfrac{3}{2}\big)|\partial s|,$$

and the Moser argument applies as before. Notice that, with some labour, the constants K_0 and K_1 could be computed explicitly in terms of n, c and V.

For the second item in the proposition we need a Bochner–Weizenbock formula on $\Omega^{0,1}(L^k)$, i.e., sections of the bundle $\overline{T}^*\otimes L^k$. We decompose the covariant derivative on this bundle into (1,0) and (0,1) parts: $\nabla=\nabla'+\nabla''$. Then the formula we want is (see for example [22, (1.4.63)])

$$\Delta_{\bar\partial}=(\nabla'')^*\nabla''+\mathrm{Ric}^\sharp+1. \tag{2.6}$$

Given this, we have, in the operator sense, $\Delta_{\bar\partial}\geqslant\frac{1}{2}$ since $\mathrm{Ric}^\sharp\geqslant-\frac{1}{2}$ from which the invertibility and bound on the inverse follow immediately. An efficient way to derive (2.6) is to make the identification

$$\Omega^{0,1}(L^k)=\Omega^{n,1}(K_X^{-1}\otimes L^k),$$

under which ∇'' becomes identified with

$$\partial^*:\Omega^{n,1}(K_X^{-1}\otimes L^k)\longrightarrow\Omega^{n-1,1}(K_X^{-1}\otimes L^k).$$

The formula (2.6) then becomes a special case of the Kodaira–Nakano formula ([19, p. 154]), using the fact that the Ricci form is the curvature of K_X^{-1}.

With this background in place we move on to recall a version of the "Hörmander" construction of holomorphic sections. Suppose we have the following data:

- a (non-compact) manifold U, a base point $u_*\in U$ and an open neighbourhood $D\Subset U$ of u_*;
- a C^∞ Hermitian line bundle $\Lambda\to U$;
- a complex structure J and a Kähler metric g on U with Kähler form Ω;
- a connection A on Λ having curvature $-i\Omega$.

We use this connection to define a $\bar\partial$-operator on sections of Λ, and hence a holomorphic structure.

We define a "Property (H)" which this data might have. Fix any $p>2n$.

Property (H). There is a number $C>0$ and a compactly supported section σ of $\Lambda\to U$ such that the following hold:

(H1) $\|\sigma\|_{L^2}<(2\pi)^{n/2}$;

(H2) $|\sigma(u_*)|>\frac{3}{4}$;

(H3) for any smooth section τ of Λ over a neighbourhood of $\overline{D}$ we have

$$|\tau(u_*)|\leqslant C(\|\bar{\partial}\tau\|_{L^p(D)}+\|\tau\|_{L^2(D)});$$

(H4) $\|\bar{\partial}\sigma\|_{L^2}<\min\{1/8C\sqrt{2},(2\pi)^{n/2}/10\sqrt{2}\}$;

(H5) $\|\bar{\partial}\sigma\|_{L^p(D)}<1/8C$.

Many of the specific numbers here are arbitrary but it is convenient to fix some definite numbers.

We have the following result.

Lemma 2.2. *Property* (H) *is open with respect to variations in* (g,J,A) (*for fixed* (U,D,u_*,Λ)) *and the topology of convergence in* C^0 *on compact subsets of* U.

Notice first that for any choice of data there is *some* constant C for which the bound in (H3) holds. This follows from the elliptic estimate

$$\|\tau\|_{L^p_1(D_0)}\leqslant C_3(\|\bar{\partial}\tau\|_{L^p(D)}+\|\tau\|_{L^2(D)}), \tag{2.7}$$

(in the more usual form one uses $\|\tau\|_{L^p(D)}$ on the right-hand side, but here the above estimate follows from a simple interpolation) and the Sobolev inequality

$$|\tau(u_*)|\leqslant C_4\|\tau\|_{L^p_1(D_0)}.$$

Here $D_0\subset D$ is some interior domain containing u_*. We can write the $\bar{\partial}$-operator on functions for a perturbed complex structure as $\bar{\partial}+\mu\partial$, where μ is a "Beltrami differential". Similarly, if the variation of the connection is given by a 1-form a, then the perturbed $\bar{\partial}$-operator on sections can be written as

$$(\bar{\partial}+\mu\partial)+(a''+\mu a'),$$

where $a=a'+a''$ is the decomposition into type. It follows that if μ and a are small in C^0 then the perturbation of the $\bar{\partial}$ operator is small in the $L^p_1\to L^p$ operator norm and it is then clear that the inequality in the third item holds for the perturbed operator, with a slightly larger constant C. For the perturbed structure we use the same section σ, so the first and second items are automatic. Then it is also clear that, for sufficiently small perturbations, the bounds in the fourth and fifth items (with a slightly larger constant C) are also preserved, since we impose strict inequality.

For a connection A on a line bundle L write $A^{\otimes k}$ for the induced connection on L^k. The following proposition—basically well known—will provide the core of our proof of Theorem 1.1.

Proposition 2.3. *Suppose that* (X,g,J,L,A) *is in* $\mathcal{K}(n,c,V)$ *and* (U,D,u_*) *are as above. Suppose that* $\chi\colon U\to X$ *is an open embedding and the data*

$$\chi^*(J),\quad \chi^*(kg),\quad \chi^*(L^k)\quad \textit{and}\quad \chi^*(A^{\otimes k})$$

has Property (H). *Then there is a holomorphic section* s *of* $L^k\to X$ *with* $L^{2,\sharp}$ *norm at most* $\frac{11}{10}(2\pi)^{n/2}$ *and with* $|s(x)|\geqslant\frac14$ *at all points* x *a distance* (*in the scaled metric*) *less than* $(4K_1')^{-1}$ *from* $\chi(u_*)$. *Here* $K_1'=\frac{11}{10}(2\pi)^{n/2}K_1$, *and* K_1 *is the constant in Proposition* 2.1.

To prove this we transport the section σ using the map χ and regard it as a smooth section of L^k over X, extending by zero. The norms we considered over U match up with the $\sharp$-norms over X. We write $s=\sigma-\tau$ where $\tau=\bar\partial^*\Delta_{\bar\partial}^{-1}\bar\partial\sigma$. By simple Hodge theory, we have $\bar\partial s=0$. Now

$$\|\tau\|^2_{L^{2,\sharp}}=\langle\Delta_{\bar\partial}^{-1}\bar\partial\sigma,\bar\partial\bar\partial^*\Delta_{\bar\partial}^{-1}\bar\partial\sigma\rangle=\langle\Delta_{\bar\partial}^{-1}\bar\partial\sigma,\bar\partial\sigma\rangle,$$

since $\bar\partial\bar\partial\sigma=0$. Thus

$$\|\tau\|_{L^{2,\sharp}}\leqslant\sqrt2\|\bar\partial\sigma\|_{L^{2,\sharp}}\leqslant\min\biggl\{\frac{1}{8C},\frac{(2\pi)^{n/2}}{10}\biggr\}. \tag{2.8}$$

Hence in particular

$$\|s\|_{L^{2,\sharp}}\leqslant\|\sigma\|_{L^{2,\sharp}}+\|\tau\|_{L^{2,\sharp}}\leqslant\tfrac{11}{10}(2\pi)^{n/2}.$$

Now work over the image $\chi(D)$. Applying item (H3) to the section τ and using (H4) and (H5), we get $|\tau(\chi(u_*))|\leqslant\frac14$, so $|s(\chi(u_*))|\geqslant\frac12$. By the derivative bound, $|s|$ exceeds $\frac14$ at points a distance less than $(4K_1')^{-1}$ from $\chi(u_*)$.

To sum up we have the following result.

Proposition 2.4. *Suppose that* U, D, u_* *and* Λ *are as above and data* g_0, J_0 *and* A_0 *has Property* (H). *Then there is some* $\psi>0$ *with the following effect. Suppose that* (X,g_X,J_X,L,A_X) *is in* $\mathcal{K}(n,c,V)$. *If we can find* $k>0$, *an open embedding* $\chi\colon U\to X$ *and a bundle isomorphism* $\widehat\chi\colon\Lambda\to\chi^*(L^k)$ *such that*

$$\|\chi^*(J_X)-J_0\|_U,\|\chi^*(kg_X)-g_0\|_U,\|\chi^*(A_X^{\otimes k})-A\|_U\leqslant\psi,$$

then there is a holomorphic section s *of* $L^k\to X$ *with* $L^{2,\sharp}$ *norm at most* $\frac{11}{10}(2\pi)^{n/2}$ *and with* $|s(x)|\geqslant\frac14$ *at all points* x *a distance* (*in the scaled metric*) *less than* $(4K_1')^{-1}$ *from* $\chi(u_*)$.

This is just a direct combination of Lemma 2.2 and Proposition 2.3. (Here we use the notation $\|\cdot\|_U$ to indicate the C^0-norm over U.)

To illustrate this, take the case when U is the ball of radius $R>2$ in $\mathbb{C}^n$ with the standard flat metric and standard Kähler form Ω_0. Let Λ be the trivial holomorphic line bundle with metric $\exp\bigl(-\frac{1}{2}|z|^2\bigr)$ so the trivialising section, σ_0 say, has norm $\exp\bigl(-\frac{1}{4}|z|^2\bigr)$ and the induced connection A_0 has curvature $-i\Omega_0$ as required. Let u_* be the origin and D be the unit ball. Let β_R be a standard cut-off function of $|z|$, equal to 1 when $|z|\leqslant\frac{1}{2}R$ and vanishing when $|z|\geqslant\frac{9}{10}R$. Define $\sigma=\beta_R\sigma_0$. Then we have $\bar\partial\sigma=(\bar\partial\beta_R)\sigma_0$. The L^2 norm of σ is slightly less than $(2\pi)^{n/2}$, and $|\sigma(0)|=1$. The section σ is holomorphic over D, so we get (H5) and there certainly is some constant C as in item (H3) of Property (H), independent of R. It is clear that, because of the exponential decay, we can fix R so that item (H4) is satisfied. So we have a set of data satisfying Property (H). Now let x be a point in some X in $\mathcal{K}(n,c,V)$. Since the ball is simply connected, $U(1)$ connections over it are determined up to isomorphism by their curvature tensors. It is then clear that, when k is sufficiently large, we can find a map χ with $\chi(0)=x$ and such that the pull-back of kg_X, J_X and $A_X^{\otimes k}$ differs by an arbitrarily small amount from the model g_0, J_0 and A_0. Then we construct a holomorphic section of $L^k\to X$, of controlled L^2 norm and of a definite positive size on a definite neighbourhood of x.

Remark 2.5. There are many possible variants of our Property (H) which will end up having the same effect. In particular one can avoid the L^p theory. In the context we work in, we have a first-derivative bound as in Proposition 2.1 (1), and it is easy to show using this that the L^2 norm of τ controls $|\tau(\chi(u_*))|$.

3. Proof of Theorem 1.1

3.1. Reduction to the local case

We begin with a simple observation.

LEMMA 3.1. *For any integer $\mu\geqslant 1$ and any k we have*

$$\varrho_{\mu k,X}(x)\geqslant(K_0^2k^n)^{1-\mu}\varrho_{k,X}(x)^\mu,$$

where K_0 is the constant in the C^0-bound of Proposition 2.1.

Transforming the C^0-bound to the *unscaled* norms gives, for any holomorphic section of L^k,

$$\|s\|_{L^\infty}\leqslant K_0k^{n/2}\|s\|_{L^2}.$$

Write $\varrho=\varrho_{k,X}(x)$, so there is a section s with L^2 norm 1 and with $|s(x)|^2=\varrho$. Then s^μ is a holomorphic section of $L^{k\mu}$ with

$$|s^\mu(x)|^2=\varrho^\mu\quad\text{and}\quad\|s^\mu\|_{L^2}^2\leqslant\|s\|_{L^\infty}^{2\mu-2}\|s\|_{L^2}^2\leqslant K_0^{2\mu-2}k^{n(\mu-1)},$$

from which the result follows.

We will use this several times below. In the context of our remarks in the introduction, note that when μ is large this gives a rather poor estimate compared with what one would hope to be true, but it suffices for our purposes.

THEOREM 3.2. *Let p be a point in a space X_∞ which is a Gromov–Hausdorff limit of manifolds in $\mathcal{K}(n,c,V)$. There are real numbers $b(p), r(p)>0$ and integers $k(p)$ and $j(p)$ with the following effect. Suppose X_j in $\mathcal{K}(n,C,V)$ has Gromov–Hausdorff limit X_∞. Then there is some $k \leqslant k(p)$ such that for all $j \geqslant j(p)$, if x is a point in X_j with $d(x,p) \leqslant r(p)$ then $\varrho_{k,X_j}(x) \geqslant b(p)^2$.*

Here, as before, we assume we have fixed metrics on the $X_j \sqcup X_\infty$.

PROPOSITION 3.3. *Theorem 3.2 implies Theorem 1.1.*

LEMMA 3.4. *Let X_∞ be a limit space then, assuming the truth of Theorem 3.2, there is an integer k_{X_∞} and a $b_{X_\infty}>0$ such that if $X_j \in \mathcal{K}(n,C,V)$ has Gromov–Hausdorff limit X_∞ then for sufficiently large j we have $\underline{\varrho}(k_{X_\infty}, X_j) \geqslant b^2_{X_\infty}$.*

Proof. We first use the compactness of X_∞. The $\frac{1}{2}r(p)$-balls centred at points p cover X_∞ so we can find a finite subcover by balls of radius $\frac{1}{2}r(p_\alpha)$ centred at points $p_\alpha \in X_\infty$. Let r be the minimum of the $r(p_\alpha)$. Let j be large enough that for any $x \in X_j$ there is a point $x_\infty \in X_\infty$ with $d(x,x_\infty) \leqslant \frac{1}{4}r$. In addition, suppose that $j \geqslant \max_\alpha j(p_\alpha)$. Then x_∞ lies in the $\frac{1}{2}r(p_\alpha)$-ball centred at p_α for some α and hence $d(x,p_\alpha) < \frac{3}{4}r(p_\alpha)$. Now Theorem 3.2 states that there are $k(p_\alpha)$ and $b(p_\alpha)$ such that for a suitable $k_\alpha \leqslant k(p_\alpha)$ we have $\varrho_{k_\alpha,X_j}(x) \geqslant b(p_\alpha)^2$. Take k_{X_∞} to be the least integer such that each integer less than or equal to each $k(p_\alpha)$ divides k_{X_∞}. Then Lemma 3.1 implies that a positive lower bound on any $\underline{\varrho}(k_\alpha, X_j)$ gives a positive lower bound on $\underline{\varrho}(k_{X_\infty}, X_j)$ and the lemma follows. □

Proof of Proposition 3.3. The same argument, using Lemma 3.1, shows that, given the statement of Lemma 3.4, there are for each integer $\mu \geqslant 1$ numbers $b_\mu > 0$ (depending only on X_∞) such that $\underline{\varrho}(\mu k_{X_\infty}, X_j) \geqslant b^2_\mu$ once j is sufficiently large. Now we prove Theorem 1.1 (assuming Theorem 3.2) by contradiction. If Theorem 1.1 is false then there are $X_{j,s} \in \mathcal{K}(n,C,V)$ such that $\underline{\varrho}(s!, X_{j,s})$ tends to zero for fixed s as $j \to \infty$. By Gromov's compactness theorem, there is no loss in supposing that, for each fixed s, the $X_{j,s}$ converge to some limit X_s as $j \to \infty$. Taking a subsequence $s(\nu)$ we can suppose also that the $X_{s(\nu)}$ converge to X_∞. For large enough ν the integer k_{X_∞} divides $s(\nu)!$; say $s(\nu)! = m(\nu) k_{X_\infty}$. Now choose $j(\nu)$ so large that $X_{j(\nu),s(\nu)}$ converge to X_∞ as $\nu \to \infty$ and also so that $\underline{\varrho}(s(\nu)!, X_{j(\nu),s(\nu)}) < b^2_{m(\nu)}$. This gives a contradiction. □

3.2. Proof of Theorem 3.2

3.2.1. Cut-offs

To begin we fix some sequence $k_\nu\to\infty$ so that the scalings of the based space (X_∞,p) by $\sqrt{k_\nu}$ converge to a tangent cone $C(Y)$. For a while we focus attention on this cone. Write $|z|$ for the distance from the vertex. Let $\Sigma_Y\subset Y$ be the singular set and $Y^{\mathrm{reg}}=Y\setminus\Sigma_Y$.

The only information about the singular set which we need is contained in the following proposition. This is very likely a standard fact but the proof is quite short so we include it.

PROPOSITION 3.5. *For any $\eta>0$ there is a function g on Y, smooth on Y^{reg}, supported in the η-neighbourhood of Σ_Y, equal to 1 on some neighbourhood of Σ_Y, and with*

$$\|\nabla g\|_{L^2}\leqslant\eta.$$

Recall that Y has dimension $2n-1$. For clarity in this proof we write $N=2n-1$. We first claim that there are fixed numbers $\bar{c},\underline{c}>0$ such that for $r\leqslant 1$ and any metric ball B_r in Y we have

$$\underline{c}r^N\leqslant\mathrm{Vol}(B_r)\leqslant\bar{c}r^N. \tag{3.1}$$

To see this, we first notice that using the non-collapsing condition (1.2) and the Bishop inequality in the original manifolds, a similar bound for the volume of balls in $C(Y)$ follows from the volume convergence theorem [7]. Then one can obtain (3.1) from the elementary geometry of a metric cone. We know that Σ_Y is a compact set of Hausdorff dimension strictly less than $N-2$. By the definition of Hausdorff dimension we can find a number $\lambda\in(0,N-2)$ with the following property. For any $\varepsilon>0$ there is a cover of Σ_Y by a finite number of balls $B_{r_j/2}(p_j)$ such that

$$\sum_j r_j^{N-2-\lambda}<\varepsilon. \tag{3.2}$$

We write $B_j=B_{r_j}(p_j)$ so, in an obvious notation, the cover is by the balls $\frac{1}{2}B_j$. By the Vitali argument we can suppose that the balls $\frac{1}{10}B_j$ are disjoint. We take $\varepsilon<1$, so for each j we obviously have $r_j\leqslant\varepsilon^{1/(N-2-\lambda)}<1$.

Let $\phi(t)$ be a standard cut-off function, vanishing for $t\geqslant 2$, equal to 1 when $t\leqslant 1$ and with derivative bounded by 2. Define

$$f_j(y)=\phi(r_j^{-1}d(y,p_j)).$$

Thus f_j is supported in $2B_j$ and equal to 1 in B_j. This function need not be smooth but it is Lipschitz and differentiable almost everywhere, with $|\nabla f_j|\leqslant 2r_j^{-1}$. Set $f=\sum_j f_j$.

Let $\Psi(t)$ be a cut-off function, equal to 1 when $t \geqslant \frac{9}{10}$, with $\Psi(0)=0$ and with derivative bounded by 2. Put $g_0=\Psi\circ f$. Then g_0 is equal to 1 on a neighbourhood of Σ_Y and is supported in the $2\varepsilon^{1/(N-2-\lambda)}$-neighbourhood of Σ_Y. Also we have

$$\|\nabla g_0\|_{L^2} \leqslant 2\|\nabla f\|_{L^2}.$$

We claim that $\|\nabla f\|_{L^2}^2 \leqslant C_5\varepsilon$ for some fixed C_5, depending only on $\underline{c}$ and $\overline{c}$. Given this claim we can make $\|\nabla g_0\|_{L^2}$ as small as we please and then finally approximate g_0 by a smooth function g to achieve our result. (Note that this approximation only involves working over a compact subset of Y^{reg}.)

To establish the claim, divide the index set into subsets

$$I_\alpha = \{j : 2^{-\alpha-1} \leqslant r_j < 2^{-\alpha}\},$$

for $\alpha \geqslant 0$. A simple packing argument, using the fact that the balls $\frac{1}{10}B_j$ are disjoint, shows that there is a fixed number C_6 with the following property. If $k\in I_\alpha$ then for each fixed $\beta\leqslant\alpha$ there are at most C_6 balls B_j with $j\in I_\beta$ which intersect B_k. Now we have

$$\|\nabla f\|_{L^2}^2 \leqslant \sum_{j,k} \int_Y |\nabla f_j|\,|\nabla f_k|\; d\mathrm{Vol}.$$

Thus

$$\|\nabla f\|_{L^2}^2 \leqslant 2 \sum_{\substack{j,k \\ r_k \leqslant r_j}} \int_Y |\nabla f_j|\,|\nabla f_k|\; d\mathrm{Vol}.$$

For fixed k there are at most $C_6(1+\log_2 r_k^{-1})$ terms which contribute to this last sum. For each term $|\nabla f_j| \leqslant 2r_j^{-1} \leqslant 2r_k^{-1}$ and the integrand is supported on the ball $2B_k$. So for fixed k the contribution to the sum is bounded by

$$8C_6(1+\log_2 r_k^{-1})r_k^{-2}\overline{c}(2r_k)^N.$$

Hence, summing over k,

$$\|\nabla f\|_{L^2}^2 \leqslant 2^{N+3}C_6\overline{c}\sum_k r_k^{N-2}(\log_2 r_k^{-1}+1).$$

We can find a number C_7 such that for $t\geqslant 1$ we have $1+\log t \leqslant C_7 t^\lambda$. Thus

$$\|\nabla f\|_{L^2}^2 \leqslant 2^{N+3}C_6C_7\overline{c}\sum_k r_k^{N-2-\lambda} \leqslant 2^{N+3}C_6C_7\overline{c}\varepsilon.$$

We pick some base point y_0 in Y^{reg}. We will need four parameters ϱ, ε, δ and R in our basic construction, where ϱ, ε and δ will be "small" and R "large". In particular $\delta \ll \varrho \ll 1 \ll R$.

First we fix ϱ so that $\exp\left(-\frac{1}{4}\varrho^2\right) \geqslant \frac{3}{4}$ and $\varrho \leqslant (16K_1')^{-1}$, where K_1' is the constant in Proposition 2.3. We take $u_* = \varrho y_0 \in C(Y)$, with the obvious notation. Fix any neighbourhood D of u_* whose closure does not meet the singular set in $C(Y)$. For any ε let Y_ε be the set of points of distance greater than ε from Σ_Y. Let $U_{\varepsilon,\delta,R}$ be the set of points z in $C(Y_\varepsilon)$ such that $\delta < |z| < R$. We choose the parameters so that $U_{\varepsilon,\delta,R}$ contains the closure of D. We consider a smooth compactly supported cut-off function β on $U_{\varepsilon,\delta,R}$. For such a function we set

$$E_\beta = \int_Y e^{-|z|^2/2} |\nabla \beta|^2 \, d\mathrm{Vol}\,.$$

LEMMA 3.6. *For any given $\zeta > 0$ we can choose ε, δ, R and a compactly supported function β as above such that*

- $\beta = 1$ *on* D;
- $E_\beta \leqslant \zeta$.

Proof. To see this we take $\beta = \beta_\delta \beta_R \beta_\varepsilon$, where

- β_δ is a standard cut-off function of $|z|$, equal to 1 for $|z| > 2\delta$;
- β_R is likewise a standard cut-off function of $|z|$, equal to 1 for $|z| < \frac{1}{2}R$;
- $\beta_\varepsilon = 1 - g \circ \varpi$ where g is a function on $C(Y)$ of the kind constructed in Proposition 3.5 and ϖ is the radial projection from the cone minus the vertex to Y.

Then the lemma follows from elementary calculations. ☐

3.2.2. The topological obstruction

Recall that the metric on the regular part of the cone has the form $\frac{1}{2} i \partial \bar{\partial} |z|^2$. So, just as in the case of $\mathbb{C}^n$, we have a line bundle Λ_0 with connection A_0, curvature the Kähler form Ω_0 and a holomorphic section σ_0 with $|\sigma_0| = \exp\left(-\frac{1}{4}|z|^2\right)$. Then $\sigma = \beta \sigma_0$ is holomorphic on D. Note that $\|\sigma\|_{L^2}$ will now be slightly less than $\varkappa^{1/2}(2\pi)^{n/2}$ where $\varkappa \leqslant 1$ is the volume ratio as in (2.1).

As we explained, there certainly is some constant C giving the elliptic estimate (H3) and we use Lemma 3.6 to choose ε, δ and R so that this set of data has Property (H).

The parameters ϱ, δ, ε and R are now all *fixed*. We set $U = U_{\varepsilon,\delta,R}$.

Consider next a C^0-small perturbation (g, J) of the metric and complex structure (g_0, J_0), and hence a perturbation Ω of Ω_0. We suppose that $-i\Omega$ is the curvature of a unitary connection A on a bundle Λ. If we can choose a bundle isomorphism between Λ and Λ_0 such that, under this isomorphism, the connection A is a small perturbation of A_0, then we can apply Proposition 2.4 to conclude that the data (J,Ω,A) also has Property (H) (for suitably small perturbations). The difficulty is that if $H_1(U,\mathbb{Z})\neq 0$, a connection on a line bundle is not determined by its curvature. Said in another way, we consider the line bundle $\Lambda\otimes\Lambda_0^*$ with the connection a induced from A and A_0. The curvature of a is small but a need not be close to a trivial flat connection. There is no real loss of generality in supposing that Y_ε has smooth boundary (because we can always replace it by a slightly enlarged domain). Write $\underline{\nu}$ for the normal vector field on the boundary. We want to recall some Hodge theory on this manifold with boundary. Fix $p>2n$.

PROPOSITION 3.7. (1) *The infimum of the L^2 norm on the closed 2-forms in a cohomology class defines a norm on $H^2(Y_\varepsilon,\mathbb{R})$.*

(2) *Let $\mathcal{H}^1$ denote the set of 1-forms α on $\overline{Y}_\varepsilon$ with $d\alpha=0$, $d^*\alpha=0$ and with $(\alpha,\underline{\nu})=0$ on the boundary. Then the natural map from $\mathcal{H}^1$ to $H^1(Y_\varepsilon,\mathbb{R})$ is an isomorphism.*

(3) *If F is any exact 2-form on $\overline{Y}_\varepsilon$, there is a unique 1-form α such that $d^*\alpha=0$, $d\alpha=F$, $(\alpha,\underline{\nu})=0$ and α is L^2-orthogonal to $\mathcal{H}^1$. We have, for some fixed constant C_8, $\|\alpha\|_{L^p_1}\leqslant C_8\|F\|_{L^p}$.*

These are fairly standard results. The first item follows from the fact that the L^2 extension of the image of d is closed. The second asserts the unique solubility of the Neumann boundary value problem for the Laplacian on functions on Y_ε. The existence and uniqueness of α in the third item is similar. The L^p estimate in the third item follows from general theory of elliptic boundary value problems, see [32, Theorem 5.1, p. 77] for a detailed treatment of this case. (Note that in our application the subtleties of the boundary value theory could be avoided by working on a slightly larger domain. Then we can reduce to easier interior estimates. Alternatively one can adjust the setup to reduce to the standard Hodge theory over a compact “double”.)

Write $a|$ for the restriction of the connection a to the restricted bundle over Y_ε. A consequence of item (1) is that there is some number $C_9>0$ such that any closed 2-form F over Y_ε which represents an integral cohomology class and satisfies $\|F\|_{L^2}\leqslant C_9$ is exact. In particular we can apply this to the curvature $F_{a|}=-i(\Omega-\Omega_0)$ of the connection $a|$, using the fact that this represents 2π times an integral class. (Here we are considering Y_ε as embedded in U in the obvious way.) Thus there is a $C_{10}>0$ such that if $\|\Omega-\Omega_0\|_U\leqslant C_{10}$ we can apply item (3) of Proposition 3.7 to write $F_{a|}=d\alpha$ over Y_ε for a small $\alpha=\alpha(a)$. More precisely, α is small in L^p_1 and so in C^0 by Sobolev embedding. Then $a|-\alpha$ is a flat connection on the restriction of $\Lambda\otimes\Lambda_0^*$ to Y_ε. This flat connection is determined up to isomorphism by its holonomy: a homomorphism from $H_1(Y_\varepsilon,\mathbb{Z})$ to S^1.

Fix a direct sum decomposition of $H_1(Y_\varepsilon,\mathbb{Z})$ into torsion and free subgroups. Then we get

$$\mathrm{Hom}(H_1(Y_\varepsilon,\mathbb{Z}),S^1)=G\times T,$$

where G is a finite abelian group and $T=H^1(Y_\varepsilon,\mathbb{R})/H^1(Y_\varepsilon,\mathbb{Z})$ is a torus. (We will write the group structures multiplicatively.) Thus for our connection a with suitably small curvature we get two invariants $g(a)\in G$ and $\tau(a)\in T$. If both vanish, then the restriction of the connection to Y_ε is close to the trivial flat connection. When a is the connection induced from A and A_0 as above we write $g(A,A_0)$ and $\tau(A,A_0)$.

Proposition 3.8. *We can find a neighbourhood W of the identity in T and a number $\psi>0$ to the following effect. If (g,J,A) is a set of data on U with*

- $$\|g-g_0\|_U\leqslant\psi \quad \textit{and} \quad \|J-J_0\|_U\leqslant\psi,$$
- $g(A,A_0)=1$,
- $\tau(A,A_0)\in W$,

then (g,J,A) has Property (H).

This is straightforward. The hypotheses and the above discussion imply that, for small W and ψ, there is a trivialisation of $\Lambda\otimes\Lambda_0^*$ over Y_ε in which the connection form is small in L^p_1 and hence in C^0. Then extend this to a trivialisation over U by parallel transport along rays. In this trivialisation the radial derivative of the connection form is given by a component of the curvature, and so is controlled by ψ. From another point of view this trivialisation is a bundle isomorphism between Λ and Λ_0 under which A is a small perturbation of A_0.

Let m_1 be the order of G. Thus for any $g\in G$ we have $g^{m_1}=1$. Fix a slightly smaller neighbourhood $W'\Subset W$ of the identity in T. By Dirichlet's theorem we can find an m_2 such that for any $\tau\in T$ there is a power τ^q which lies in W' where $1\leqslant q\leqslant m_2$. Write $m=m_1m_2$. Now return to our connection $a|$ on the bundle $\Lambda\otimes\Lambda_0^*$ over Y_ε. Recall that for integer t we write $a|^{\otimes t}$ for the induced connection on $\Lambda^t\otimes\Lambda_0^{-t}$ over Y_ε. Suppose that $\|F(a|)\|_U\leqslant C_{10}/m$. Then for $1\leqslant t\leqslant m$ the invariants $g(a|^{\otimes t})$ and $\tau(a|^{\otimes t})$ are defined and we have the following result.

Proposition 3.9. *We can choose t with $1\leqslant t\leqslant m$ such that*

$$g(a|^{\otimes t})=1 \quad \textit{and} \quad \tau(a|^{\otimes t})\in W'.$$

With m fixed as above, write

$$\widetilde{U}=U(m^{-1/2}\delta,\varepsilon,R).$$

For integers t with $1\leqslant t\leqslant m$ let $\mu_t\colon U\to\widetilde{U}$ be the map $\mu_t(z)=t^{-1/2}z$ (in obvious notation). Thus $\mu_t^*(t\Omega_0)=\Omega_0$.

Our model structures g_0, J_0, Λ_0 and A_0 are defined over $\widetilde{U}$. Now consider deformed structures J, Ω, Λ and A as before but which are also defined over $\widetilde{U}$. Suppose that

$$\|g-g_0\|_{\widetilde{U}}\leqslant\tilde{\psi}\quad\text{and}\quad\|J-J_0\|_{\widetilde{U}}\leqslant\tilde{\psi},$$

where $\|\cdot\|_{\widetilde{U}}$ here denotes C^0 norms over $\widetilde{U}$. For integers t as above, let (g_t,J_t,Λ_t,A_t) be the data over U given by pulling back $(tg,J,\Lambda^t,A^{\otimes t})$ using the map μ_t. It is clear that if $\tilde{\psi}$ is sufficiently small then for every t we have

$$\|g_t-g_0\|_U\leqslant\psi\quad\text{and}\quad\|J_t-J_0\|_U\leqslant\psi.$$

It is also clear that, if $\tilde{\psi}$ is sufficiently small, then the invariants $g(A_0,A_t)$ and $\tau(A_0,A_t)$ are defined.

Proposition 3.10. *If $\tilde{\psi}$ is sufficiently small then we can choose $t\leqslant m$ so that $g(A_0,A_t)=1$ and $\tau(A_0,A_t)\in W$.*

We choose t according to Proposition 3.9, so that $g(a|^{\otimes t})=1$ and $\tau(a|^{\otimes t})\in W'$.

Write $\tau(a|^{\otimes t})=\tau$. Thus τ can be regarded as a small element of $H^1(Y_\varepsilon,\mathbb{R})$. It follows from our setup that there is a trivialisation of the bundle $\Lambda^t\otimes\Lambda_0^{-t}$ over Y_ε in which the connection $a|^{\otimes t}$ is represented by a C^0-small connection form. Extend this trivialisation to $\widetilde{U}$ using parallel transport along rays. As above, in the proof of Proposition 3.8, the radial derivative of the connection form in this trivialisation is given by the curvature $F_{a^{\otimes t}}$ and it follows easily that if $\tilde{\psi}$ is sufficiently small then in the induced trivialisation the pull-back $\mu_t^*(a^{\otimes t})$ restricted to Y_ε has a C^0-small connection form. In particular, given that $W'\Subset W$ we can, by fixing $\tilde{\psi}$ sufficiently small, ensure that the "τ-invariant" of this connection lies in W and the "g-invariant" is 1. Now the fact that $\mu_t^*(A_0^{\otimes t})$ is isomorphic to A_0 yields the result stated.

We sum up in the following way.

Proposition 3.11. *We can choose $\tilde{\psi}>0$ to the following effect. Suppose g, J, Λ and A are structures as above over $\widetilde{U}$. Suppose that $\|g-g_0\|_{\widetilde{U}},\|J-J_0\|_{\widetilde{U}}\leqslant\tilde{\psi}$. Then we can find an integer t with $1\leqslant t\leqslant m$ such that the data $(\mu_t^*(tg),\mu_t^*(J),\mu_t^*(\Lambda^t),\mu_t^*(A^{\otimes t}))$ over U has Property* (H).

3.2.3. Completion of proof

With this lengthy discussion involving the tangent cone in place, we return to the limit space X_∞. Recall that we have a sequence of scalings $\sqrt{k_\nu}$. We consider embeddings

$\chi_\nu:\widetilde{U}\to X_\infty^{\mathrm{reg}}$. Given such a χ_ν we write J^ν for the pull-back of the complex structure on X_∞^{reg} and g^ν for the pull-back of k_ν times the metric.

PROPOSITION 3.12. *There is a k_ν so that we can find an embedding χ_ν as above, such that*

- $$\tfrac{1}{2}k_\nu^{-1/2}|z|\leqslant d(p,\chi_\nu(z))\leqslant 2k_\nu^{-1/2}|z|;$$
- $$\|J^\nu-J_0\|_{\widetilde{U}},\|g^\nu-g_0\|_{\widetilde{U}}\leqslant\tfrac{1}{2}\tilde{\psi}.$$

This follows easily from the general assertions in §2.1 about convergence. We now fix this k_ν and define $k(p)=mk_\nu$ and $r(p)=\varrho k(p)^{-1/2}$. We write $\chi_{k_\nu}=\chi$.

Let $X_j\in\mathcal{K}(n,C,V)$ be a sequence converging to X_∞. We fix distance functions on $X_\infty\sqcup X_j$. We consider embeddings $\chi^j:\widetilde{U}\to X_j$. Given such maps we write g_j and J_j for the pull backs of the metric and complex structure, Λ_j for the pull-back of $L_j^{k_\nu}$ and A_j for the pulled back connection.

PROPOSITION 3.13. *For large enough j we can choose χ^j with the following two properties*:

- $d(\chi^j(z),\chi(z))\leqslant\frac{1}{100}\varrho k(p)^{-1/2}$;
- $\|g_j-g_0\|_{\widetilde{U}},\|J_j-J_0\|_{\widetilde{U}}\leqslant\tilde{\psi}$.

Again this follows from our general discussion of convergence.

Fix j large enough, as in Proposition 3.13. We apply Proposition 3.11 to find a t such that the pull-back by μ_t of the data $(tg_j,J_j,\Lambda_j^t,A_j^{\otimes t})$ has Property (H) over U. Now write $k=tk_\nu$, so $k\leqslant k(p)$. We apply Proposition 2.4 to construct a holomorphic section s of $L_j^k\to X_j$, with a fixed bound on the $L^{2,\sharp}$ norm and with $|s(x)|\geqslant\frac{1}{4}$ at points x with $d^\sharp(x,\chi^j(t^{-1/2}u_*))<(4K_1')^{-1}$. Here we are writing $d^\sharp$ for the scaled metric, so in terms of the original metric the condition is $d(x,\chi^j(t^{-1/2}u_*))<k^{-1/2}(4K_1')^{-1}$.

To finish, suppose $q\in X_j$ has $d(q,p)\leqslant r(p)$. By construction $r(p)\leqslant\varrho k^{-1/2}$. Note also that if we set $p'=\chi^j(t^{-1/2}u_*)$, then

$$d(q,p')\leqslant d(q,p)+d(p,\chi(t^{-1/2}u_*))+d(\chi(t^{-1/2}u_*),\chi^j(t^{-1/2}u_*))\leqslant 4\varrho k^{-1/2}.$$

This means that $d^\sharp(q,p')\leqslant 4\varrho$ which is less than $(4K_1')^{-1}$ by our choice of ϱ.

4. Connections with algebraic geometry

The consequences of Theorem 1.1, for the relation between algebro-geometric and differential geometric limits, could be summarised by saying that things work out in the way that one might at first sight guess at. As we have mentioned before, the proofs of many of the statements, given Theorem 1.1, have been outlined by Tian in [31]. Thus we view this section, broadly speaking, as an opportunity to attempt a careful exposition of the material.

4.1. Proof of Theorem 1.2

LEMMA 4.1. *There are numbers N_k, depending only on n, c, V and k, such that for any X in $\mathcal{K}(n,c,V)$ we have* $\dim H^0(X,L^k)\leqslant N_k+1$.

We work in the rescaled metric. Given $\varepsilon>0$ we can choose a maximal set of points x_j in X such that the distance between any two is at least ε. Then the 2ε-balls with these centres cover X and the $\frac{1}{2}\varepsilon$-balls are disjoint. Consider the evaluation map

$$\mathrm{ev}\colon H^0(X,L^k)\longrightarrow \bigoplus_j L^k_{x_j}.$$

We first show that if ε is sufficiently small then this map is injective. For if it is not injective there is a holomorphic section s with $L^{2,\sharp}$ norm 1 vanishing at all the x_j. Since the 2ε-balls cover X, we get $\|s\|_{L^\infty}\leqslant 2K_1\varepsilon$. This gives a contradiction to $\|s\|_{L^{2\sharp}}=1$ if ε is small enough. On the other hand since the $\frac{1}{2}\varepsilon$-balls are disjoint, the non-collapsing condition gives an upper bound on the number of the points x_j which completes the proof.

In fact the estimate one gets by this argument is

$$N_k+1=\frac{2^{4n}K_1^{2n}V^{n+1}n!}{c\pi^n}k^{n^2+n},$$

which is very poor compared with the asymptotics

$$\dim H^0(X,L^k)\sim(2\pi)^{-n}Vk^n$$

for a fixed X, as $k\to\infty$.

For our purposes there is no loss of generality in supposing that the k_0 of Theorem 1.1 is 1. Then the sections of L^k define a regular map of X for all k. Suppose we choose isometric embeddings

$$\phi_k\colon H^0(X,L^k)^*\longrightarrow\mathbb{C}^{N_k+1},$$

using the L^2 norm on the left-hand side and the fixed standard Hermitian form on the right. Then we get projective varieties

$$V(X,\phi_k)\subset\mathbb{CP}^{N_k},$$

and holomorphic maps

$$T_k: X \longrightarrow V(X, \phi_k).$$

Of course T_k depends on the choice of ϕ_k which is arbitrary, but any two choices differ by the action of the unitary group $U(N_k+1)$. The fact that this group is *compact* will mean that in the end the choice of ϕ_k will not be important. Since L is ample, T_k is always a finite map. Soon we will reduce to the case when T_k is generically one-to-one but we do not need to assume that yet, so T_k could be a multiple cover of an n-dimensional variety. In any case we get, by straightforward arguments, a fixed upper bound on the *degree* of $V(X, \phi_k)$ (depending on k, n and V).

By standard general principles there is a system of morphisms of projective varieties, for integer λ,

$$f_\lambda: V(X, \phi_{\lambda k}) \longrightarrow V(X, \phi_k),$$

with $f_{\lambda\mu}=f_\lambda \circ f_\mu$ and $f_\lambda T_{\lambda k}=T_k$. (The morphism f_λ can be viewed as induced by the linear map which is the transpose of $s^\lambda(H^0(X, L^k)) \to H^0(X, L^{\lambda k})$ composed with the inverse of the Veronese map.)

Now we bring in the crucial lower bound provided by Theorem 1.1.

LEMMA 4.2. *Taking $k_0=1$, the map $T_1: X \to V(X, \phi_1)$ has derivative bounded by $K_1 N_1^{1/2} b^{-1}$, where b is the lower bound in Theorem 1.1 and K_1 is the constant in the first-derivative estimate.*

Here we are referring to the "operator norm" of the derivative, regarded as a map from the tangent space of X at a point, with the given metric g, to the tangent space of $\mathbb{CP}^{N_1}$ with the standard Fubini–Study metric.

The proof of the lemma comes directly from the definitions. Given a point $x \in X$ we can choose an orthonormal basis of sections $s_0, s_1, ..., s_N$ with $s_j(x)=0$ for $j>0$ and $|s_0(x)|=B \geqslant b$. There is no loss of generality in supposing that ϕ_1 maps the dual basis to the first $N+1$ basis vectors in $\mathbb{C}^{N_1+1}$. Fix a unitary isomorphism of the fibre L_x with $\mathbb{C}$. Then the derivative of each s_j, for $j>0$, can be regarded as an element of the cotangent space of X at x. Identifying the tangent space of $\mathbb{CP}^{N_1}$ at $(1, 0, ..., 0)$ with $\mathbb{C}^{N_1}$ in the standard way, the derivative of T at x is represented by

$$s_0^{-1}(\partial s_1, ..., \partial s_N, ..., 0),$$

and the lemma follows.

Using Lemma 3.1, we get similar universal bounds on the derivatives of all maps T_k, for suitable constants which we do not need to keep track of.

Now suppose that X_j is a sequence in $\mathcal{K}(n, c, v)$ with the Gromov–Hausdorff limit being a polarised limit space X_∞. For each fixed k we choose $\phi_{k,j}$, so we have a sequence of projective varieties $V(X_j, \phi_{k,j})$ of bounded dimension and degree. By standard results (compactness of the Chow variety) we can, choosing a subsequence, suppose that for each k these converge in the algebro-geometric sense to a limit W_k. (More precisely, we can suppose that for each k the $V(X_j, \phi_{k,j})$ have fixed degree and dimension and converge as points in the Chow variety parameterising algebraic cycles of that type. Then we take W_k to be the corresponding reduced algebraic set.) It follows easily from the compactness of $U(N_k+1)$ that W_k is independent, up to projective unitary transformations, of the choice of maps $\phi_{k,j}$.

LEMMA 4.3. *After perhaps passing to a subsequence of the X_j, for each k the maps $T_k: X_j \to V(X_j, \phi_{k,j})$ extend by continuity to a continuous map $T_k: X_\infty \to W_k$, holomorphic on X_∞^{reg}.*

More precisely, what we mean is that we suppose we have fixed metrics on the $X_j \sqcup X_\infty$, and then for all $\varepsilon > 0$ we can find $\delta > 0$ so that the distance in the projective space between $T_k(x)$ and $T_k(y)$ is less than ε if $d(x,y) < \delta$.

The proof of the lemma is very easy using the equicontinuity of the maps T_k on the X_j. The limit map T_k on X_∞ is unique up to unitary transformations preserving W_k and the possible existence of such maps is the only reason that we may need to pass to a subsequence.

In the next subsection we will collect some further analytical results which will give a much clearer view of the situation. Then we return to discussing the relation between X_∞ and the W_k further in §4.3.

4.2. More analysis

Recall that we have a uniform C^0 estimate (Proposition 2.1) for holomorphic sections of $L^k \to X$, for any $X \in \mathcal{K}(n,c,V)$. We will now extend this to a polarised limit space X_∞.

LEMMA 4.4. *If s is a bounded holomorphic section of L^k over X_∞^{reg} then*

$$\|s\|_{L^\infty} \leqslant K_0 \|s\|_{L^{2,\sharp}}.$$

Here of course we are writing $L^k \to X_\infty^{\mathrm{reg}}$ for the limiting line bundle and we are defining the $L^{2,\sharp}$ norm with the rescaled metric.

Proof. We prove the lemma by contradiction. The argument is very similar to our main construction in §3. Suppose there is a holomorphic section s with $\|s\|_{L^{2,\sharp}} = 1$ and $\|s\|_{L^\infty} = B$, and there is a point $p \in X_\infty^{\mathrm{reg}}$ with $|s(p)| = K_0 + \lambda$ for some $\lambda > 0$. Choose a neighbourhood D of p which lies inside X_∞^{reg}. There is some constant C so that an estimate like that in (H3) of Property (H) holds. The singular set in X_∞ has Hausdorff codimension strictly bigger than 2, so by the argument of Proposition 3.5 we can construct a cut-off function β equal to 1 over D and with $\|\nabla\beta\|_{L^2}$ as small as we like. In particular we can make this much smaller than $\lambda B^{-1}C^{-1}$. When j is large we can choose maps χ_j from a neighbourhood of the support of β into X_j and lifts $\widehat{\chi}_j$ so that the structures match up as closely as we please. Transport βs by these maps to a section of $L^k\to X_j$ and adjust to get a holomorphic section s_j just as in §3. Then when j is large enough, by arguments just like those in the proof of Proposition 2.3 one can show that $\|s_j\|_{L^{2,\sharp}}$ is as close to 1 as we like while $\|s_j\|_{L^\infty}$ exceeds $K_0+\frac{1}{2}\lambda$. This clearly contradicts the estimate in Proposition 2.1. □

Now we *define* $H^0(X_\infty, L^k)$ to be the space of bounded holomorphic sections over the regular part. Let $S\subset\mathbb{R}$ be the set

$$S=\left\{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...,\frac{1}{j},...\right\},$$

and for integers k let $S_k\subset S$ be the subset $\{0,k^{-1},(k+1)^{-1},...\}$. We are regarding S as a topological space, so any sequence tending to zero would do equally well. Let

$$\mathcal{X}=\bigsqcup_{j=1,2,...,\infty} X_j.$$

Thus there is a map of sets $\pi:\mathcal{X}\to S$ which takes X_j to j^{-1} for $j=1,...,\infty$. The distance functions on $X_j\sqcup X_\infty$ define a natural topology on $\mathcal{X}$ such that π is continuous.

Now let

$$\mathcal{H}=\bigsqcup_{j=1,2,...,\infty} H^0(X_j,L^k),$$

taking the above definition in the case $j=\infty$. There is an obvious map of sets $\varpi:\mathcal{H}\to S$. We put a topology on $\mathcal{H}$ by saying that sections are close if they are close when compared by maps χ_j and $\widehat{\chi}_j$, as above.

Lemma 4.5. *For sufficiently large j the restriction of $\varpi:\mathcal{H}\to S$ to $S_j\subset S$ is a vector bundle.*

(Note that this is for fixed k: for different values of k one might a priori have to take different values of j.)

The proof uses much the same construction as in Lemma 4.4. The content of the statement is that, for large enough j, we can define linear isomorphisms

$$Q_j: H^0(X_\infty,L^k)\longrightarrow H^0(X_j,L^k)$$

such that $Q_j(s)$ tends to s as $j\to\infty$, in the sense above. We choose a family of compactly supported cut-off functions β_j on X_∞^{reg} with the following properties:

- the compact sets $\beta_j^{-1}(1)$ give an exhaustion of X_∞^{reg};
- the support of β_j is contained in the domain of a map χ_j, with a bundle lift $\widehat{\chi}_j$, under which the structures compare with a small error η_j with $\eta_j\to 0$ as $j\to\infty$;
- $\|\nabla\beta_j\|_{L^{2,\sharp}}\to 0$ as $j\to\infty$; in particular $\|\nabla\beta_j\|_{L^{2,\sharp}}$ can be taken very small compared with K_0^{-1}.

Then for any holomorphic section $s\in H^0(X_\infty, L^k)$ we transport $\beta_j s$ to X_j using χ_j and project to get an element $Q_j(s)\in H^0(X_j, L^k)$ in the familiar way. Our standard argument in §2 shows that $Q_j(s)$ can be made as close as we please to s, when compared using the maps χ_j and $\widehat{\chi}_j$. (That is, the hypotheses imply that the L^2 norm of $\bar\partial(\widehat{\chi}_j(\beta_j s))$ tends to zero with j and the fundamental Hörmander estimate shows that the same is true for the L^2 norms of $\widehat{\chi}_j(\beta_j s-Q_j(s))$ which leads to corresponding estimates for all derivatives over interior domains in the regular set.) In particular, for j large enough, for any $s\in H^0(X_\infty, L^k)$ we have $\|Q_j(s)\|_{L^2}\geqslant\frac{1}{2}\|s\|_{L^2}$, and this shows that Q_j is injective, for large j. (Note that the point of establishing Lemma 4.4 first is that the bounds we require on $\|\nabla\beta_j\|_{L^2}$ do not depend on s, but only on K_0.) To prove surjectivity we argue by contradiction. If Q_j is not surjective we can find $s_j\in H^0(X_j, L^k)$ of $L^{2,\sharp}$ norm 1 and $L^{2,\sharp}$-orthogonal to the image of Q_j. Passing to a subsequence and taking a limit as $j\to\infty$ we get a section $s_\infty\in H^0(X_\infty, L^k)$. The C^0 estimate shows that s_∞ has $L^{2,\sharp}$ norm 1 and we easily get a contradiction to the fact that s_j is orthogonal to $Q_j(s_\infty)$ for all j.

Our reason for formulating things in this way is that it is natural to consider families $\pi\colon\mathcal{X}\to B$ over a general base. Here we want the fibres of π to be either smooth manifolds in $\mathcal{K}(n,c,V)$ or polarised Gromov–Hausdorff limits of such manifolds, and we want the topology on $\mathcal{X}$ to be compatible with the Gromov–Hausdorff distance in an obvious way. It is not hard to set up the definitions, and the proof of Lemma 4.5 shows that, if B is connected, there is a "direct image" which is a vector bundle over B. However there does not seem much point in developing the theory in detail since in the end, after we have proved Theorem 1.2, this construction can be obtained from the standard algebraic geometry direct image.

We now turn to the problem of separating points.

Proposition 4.6. *Suppose X_∞ is a polarised limit space and $\varrho>0$. We can find a k such that if $p_1, p_2\in X_\infty$ are points with $d(p_1,p_2)>\varrho$ then the map $T_k\colon X_\infty\to\mathbb{CP}^{N_k}$ takes p_1 and p_2 to distinct points in $\mathbb{CP}^{N_k}$.*

The proof is a small extension of our main argument in §3. By a compactness argument, it suffices to find a k which works for a fixed pair of distinct points p_1 and p_2.

We choose a sequence X_j from $\mathcal{K}(n,c,V)$ converging to X_∞. We can find a sequence $k_\nu \to \infty$ such that after rescaling X_∞ by $\sqrt{k_\nu}$ at each of the points we get convergence to tangent cones $C(Y_1)$ and $C(Y_2)$ and we construct U_1, U_2, etc. in each case. We then choose k so that we get maps $\chi_s: U_s \to X_j$ as in §3. Clearly we can also suppose that $\chi_1(U_1)$ and $\chi_2(U_2)$ are disjoint. (For this we will need to take $\sqrt{k}$ large compared with ϱ^{-1}.) Then we get holomorphic sections s_1 and s_2 of $L^k \to X_j$ with fixed $L^{2,\sharp}$ norm and such that $|s_j| \geqslant \frac{1}{2}$ say at points close to p_j. Consider the section s_1 at points in X_j close to the image of χ_2. Recall that $s_1 = \sigma_1 - \tau_1$, where σ_1 vanishes on the image of χ_2 and the $L^{2,\sharp}$ norm of τ_1 can be made as we please by our original choice of parameters. Let $u_* \in D \subset U_2$ be the base point. Since τ_1 is holomorphic over $\chi_2(D)$ the size of $\tau_1(\chi_2(u_*))$ can be controlled by the L^2 norm of τ_1 over $\chi_2(D)$. Thus, by a suitable choice of original parameters (depending only on knowledge of Y_1 and Y_2), we can arrange that $|s_1(x)| = |\tau_1(x)| \leqslant \frac{1}{100}$, say, for points x close to $\chi_2(u_*)$. Taking the limit as $j \to \infty$ we get sections $s_1, s_2 \in H^0(X_\infty, L^k)$ with $|s_j(p_j)| \geqslant \frac{1}{2}$ and $|s_j(p_k)| \leqslant \frac{1}{100}$ for $j \neq k$.

PROPOSITION 4.7. *Given a compact set $K \subset X_\infty^{\mathrm{reg}}$ we can find an integer $m(K)$ such that, for $k \geqslant m(K)$, any point $x \in K$ and any tangent vector v at x, there is a holomorphic section $s \in H^0(X_\infty, L^k)$ with $s(x) = 0$ and the derivative of s along v not zero.*

Proof. This is another straightforward application of the Hörmander technique. First for $a \in \mathbb{C}^{n+1}$ we take our model section σ_a to be the Gaussian section σ_0 on $\mathbb{C}^n$ multiplied by an appropriate holomorphic function of the form $f_a(z) = a_0 + \sum_{j=1}^n a_j z_j$, and follow the same construction as in §2. In a small neighbourhood that we are interested in we can control the derivative of the error term τ_a by its $L^{2,\sharp}$ norm (which is small from the construction) since $\tau_a = \sigma_a - s_a$ is essentially holomorphic there. So we obtain a linear map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ by sending a to $(s_a(x), \partial s_a(x))$, where we have fixed a unitary isomorphism between L_x and $\mathbb{C}$, and between $T_x^{*(1,0)} X$ and $\mathbb{C}^n$. This map could be made as close to the identity map as we like by taking k big (depending only on K), and this clearly implies the statement of the proposition. There are other possible ways to prove the proposition, for example, one can apply the L^2 estimate for singular metrics (see [13], and also [22, Theorem B.4.6]). □

4.3. Recap

We can go back to the discussion of §4.1 and state things in a much clearer way. For a given k we can suppose that all the spaces $H^0(X_j, L^k)$ have the same dimension and identify them with $H^0(X_\infty, L^k)$ as in Lemma 4.5. Also it is clear that the bound in Theorem 1.1 also holds on X_∞. In the usual way, the sections in $H^0(X_\infty, L^k)$ define a holomorphic map from X_∞^{reg} to $\mathbb{P}(H^0(X_\infty, L^k)^*)$. We fix a basis in $H^0(X_\infty, L^k)$ so that we can say that we map to $\mathbb{CP}^N$. From the proof of Lemma 4.5 it follows that the derivative estimate as in Proposition 2.1 also holds for sections in $H^0(X_\infty, L^k)$. The same argument as in Lemma 4.2 gives a bound on the derivative of this map so it has a unique continuous extension to X_∞. Pulling back the hyperplane bundle by this map (in the case $k=1$) defines an extension of the line bundle L to X_∞ (at this stage, as a topological bundle). Theorem 1.1 implies that the original metric on L is uniformly equivalent to the metric pulled back from the hyperplane bundle. (The two Hermitian metrics differ by a factor $e^{-\varrho(x)}$ (see for example [22, Theorem 5.1.3]) and the upper bound of ϱ follows easily from Proposition 2.1.) The convergence of the maps $T_k: X_j \to \mathbb{CP}^N$ to $T_k: X_\infty \to \mathbb{CP}^N$ over the regular part is completely clear because of the way we chose our identifications of $H^0(X_j, L^k)$ and $H^0(X_\infty, L^k)$. The algebraic set W_k is the image $T_k(X_\infty)$. It is also clear that we have a system of morphisms $f_\lambda: W_{k\lambda} \to W_k$ such that $f_{\lambda\mu} = f_\lambda \circ f_\mu$ and

$$T_k = f_\lambda \circ T_{k\lambda}: X_\infty \longrightarrow W_k.$$

Consider any collection of sets W_k, for integers $k \geqslant 1$, and maps $f_\lambda: W_{k\lambda} \to W_k$ with $f_{\lambda\mu} = f_\lambda \circ f_\mu$. Then we can form the limit set

$$W_\leftarrow \subset \prod_{k=1}^{\infty} W_k$$

given by sequences $(w_1, w_2, \dots)$ such that $f_\lambda(w_{k\lambda}) = w_k$ for all k and λ. If we have another set X_∞ and maps $T_k: X_\infty \to W_k$ compatible with the f_λ then we get an induced map from X_∞ to $W_\leftarrow$. In our situation, Proposition 4.6 implies that this map is a bijection, so what we know at this stage is that we can recover the Gromov–Hausdorff limit algebro-geometrically (at least as a set) in this way.

It is interesting to compare this with [15] and [16], where the first-named author made a different attack on the same kind of problem. This attack was made in the absence of Theorem 1.1, and the cost of that absence was that one got a system like the f_λ but only of *rational* maps (or "web of descendants" in the language of [15]). The core of the problem was that, without something like Theorem 1.1, one does not know that the W_k are irreducible. This difficulty is also explained by Tian in [31]. The construction of [15] should probably best be thought of as an attempt to define the Gromov–Hausdorff limit as a "limit" of algebraic sets or schemes (in the sense of $\lim_\leftarrow$) in this fashion. (From a more algebraic point of view the limiting process we conceive of here is related to considering rings that are not finitely generated.) But, having now Theorem 1.1, we can take a simpler and more direct path (in the context of manifolds satisfying the hypotheses (1.1) and (1.2)). However it seems likely that related ideas on the algebraic side may play a role in the future in the study of constant scalar curvature Kähler metrics (lacking (1.1) and (1.2)). In this direction, see the recent work of Székelyhidi [28].

4.3.1. Completion of proof of Theorem 1.2

LEMMA 4.8. *For each k, the algebraic set W_k is irreducible.*

This is crucial, as we indicated above, but the proof is easy. The set X_∞^{reg} is dense in X_∞ so its image is dense in W_k. Thus we can choose a point $x_0 \in X_\infty^{\mathrm{reg}}$ so that $T_k(x_0)$ lies in a unique component U of W_k. Suppose there is a point w in W_k which is not in U. Then we can find a polynomial P of degree λ, say, so that P vanishes on U but not at w. Regarding P as a section of a line bundle we can suppose $|P(w)|=1$. Now P also defines holomorphic sections σ_j of $L^{\lambda k}$ over X_j for each j (including $j=\infty$) which satisfy a fixed L^∞ bound (because of the equivalence of the metrics on the line bundle). By construction the section σ_∞ vanishes in a neighbourhood of x_0 and so by analytic continuation and the fact that the regular set is dense and connected it vanishes identically. It follows from the L^∞ bound on σ_j, the general estimate in Proposition 2.1 and convergence on compact subsets of the regular set, that $\|\sigma_j\|_{L^\infty}$ tends to 0 as $j \to \infty$. But this contradicts the fact that $|P(w)|=1$ (again using the equivalence of the two metrics on L^k).

(Notice that in this proof we do use the fact that X_∞^{reg} has an analytic, not just $C^{2,\alpha}$, structure.)

Recall that we have compatible maps $T_k: X_\infty \to W_k$ and $f_\lambda: W_{\lambda k} \to W_k$. Proposition 4.6 implies that the T_k asymptotically separate points, in the sense that the induced map from X_∞ to $\lim_{\leftarrow} W_k$ is injective. What we want to show now is that in fact there is some fixed k for which this is true.

LEMMA 4.9. *We can find a k so that all fibres of $T_k: X_\infty \to W_k$ are finite.*

First Proposition 4.7 implies that we can choose k so that the map T_k is an embedding in a neighbourhood U of some point $x \in X_\infty^{\mathrm{reg}}$. Then, by Proposition 4.6, we may shrink U even more to arrange that $T_k^{-1}(w)$ is a single point for w in the open subset $T_k(U)$ of W_k. As usual, we may as well suppose that this happens for $k=1$ and hence for all k. Thus all maps $f_\lambda: W_{\lambda k} \to W_k$ are generically one-to-one. Our main theorem (Theorem 1.1) and the first-derivative estimate imply that there is a number $r>0$ so that for any $X \in \mathcal{K}(n,c,V)$ and any point $x \in X$ there is a holomorphic section of L with $L^{2,\sharp}$ norm equal to 1 and with a definite positive lower bound on the ball of radius r about x. Taking limits, one sees that the same also holds for the limit space X_∞ and $H^0(X_\infty, L)$. Choose k in accordance with Proposition 4.6 taking $\varrho = \frac{1}{2}r$ say. Thus, if p_1 and p_2 are two points in the same fibre $F = T_k^{-1}(w)$ of $T_k: X_\infty \to W_k$, the distance between them is less than $\frac{1}{2}r$. In other words, the fibre F is contained in the $\frac{1}{2}r$-ball about p_1, so there is a section $s \in H^0(X_\infty, L)$ of L which does not vanish on F. By construction, F maps by $T_{k\lambda}$ onto $F_\lambda = f_\lambda^{-1}(w)$ for any $f_\lambda: W_{\lambda k} \to W_k$. The section $s^{\lambda k} \in H^0(X_\infty, L^{\lambda k})$

defines one component of $T_{k\lambda}$, so the fact that s does not vanish on F implies that F_λ lies in the corresponding affine subspace. Since F_λ is a compact algebraic set it must be finite. Thus all maps $f_\lambda: W_{k\lambda} \to W_k$ have finite fibres. Let $N(w)$ be the number of local irreducible components of W_k at w. As f_λ is generically one-to-one, the number of points in $f_\lambda^{-1}(w)$ is at most $N(w)$. It follows then that the number of points in $T_k^{-1}(w)$ is also finite, and in fact bounded by $N(w)$.

Proposition 4.10. *We can find a k so that T_k is injective.*

As usual we may as well suppose that the value of k in the previous Lemma is 1. Thus $T_1: X_\infty \to W_1 \subset \mathbb{CP}^{N_1}$ has finite fibres. For any given point $w_1 \in W_1$ we can find a k such that $T_1^{-1}(w_1)$ is mapped injectively to W_k by T_k. It is clear then that there is a decomposition of W_1 into a finite number of quasi-projective subvarieties Z_α such that $T_1^{-1}(Z_\alpha)$ is a disjoint union of n_α copies of Z_α. Pick points $z_\alpha \in Z_\alpha$. If for some α some T_k separates the points $T_1^{-1}(z_\alpha)$, then it is clear that T_k separates points in $T_1^{-1}(z)$ for generic $z \in Z_\alpha$. Now the proposition follows from a simple induction argument, using induction on the maximal dimension of a Z_α with $n_\alpha > 1$ and the number of components Z_α with this maximal dimension.

We have now achieved our main goal—the central statement in Theorem 1.2. We have a continuous bijection $T_k: X_\infty \to W_k$ which is a homeomorphism, since the spaces are compact. As usual we may as well suppose that this k is 1, so all T_k are homeomorphisms.

Recall that we denote the differential geometric singular set, the complement of X_∞^{reg} by Σ. Let $S_k \subset W_k$ denote the algebro-geometric singular set.

Lemma 4.11. *We can choose k so that T_k^{-1} maps S_k to Σ.*

Proof. Of course it is equivalent to say that T_k maps X_∞^{reg} to smooth points of W_k. The proof is similar to that of the previous lemma. It follows from Proposition 4.7 that for any given compact subset $K \subset X_\infty^{\mathrm{reg}}$ we can choose k so that T_k maps K into the smooth points of W_k. On the other hand the singular set S_1 has a finite number of irreducible components. If there is a component which meets $T_1(X_\infty^{\mathrm{reg}})$ we choose one of maximal dimension, say V. Thus there is a point $x \in X_\infty^{\mathrm{reg}}$ with $T_1(x) \in V$. We apply Proposition 4.7 with $K = \{x\}$ to find a k such that $T_k(x)$ lies in the smooth set of W_k. Using the relation $T_k = f_k^{-1} \circ T_1$ and the definition of f_k, one easily sees that if a point $x \in X_\infty^{\mathrm{reg}}$ is mapped by T_1 into the smooth part of W_1, then it is also mapped by T_k into the smooth part of W_k. Then it is clear that the number of irreducible components of S_k is strictly less than for S_1, and the proof is completed by induction. $\square$

As usual we can suppose that the k in Lemma 4.11 is 1. In the next subsection we will show that, at least for Kähler–Einstein limits, the singular sets match up, but we do not need to use this fact.

LEMMA 4.12. *We can choose a k such that W_k is a normal variety.*

Suppose W_1 is not normal. Let $\nu: \widehat{W}_1 \to W_1$ be the normalisation. Thus ν is a finite map (cf. [20, Exercise II.3.8]) which is bijective outside the singular set S_1 of W_1. It is a general fact (cf. [20, Exercise III.5.7]) that the pull-back $\mathcal{L}=\nu^*(\mathcal{O}(1))$ is an ample line bundle on $\widehat{W}_1$, so we can choose k such that sections of $\mathcal{L}^k$ define a projective embedding of $\widehat{W}_1$ in $\mathbb{P}$, say. The map $T_1: X_\infty^{\mathrm{reg}} \to W_1$ maps into the smooth part and so lifts to $\widehat{T}_1: X_\infty^{\mathrm{reg}} \to \widehat{W}_1$. Clearly the pull-back of $\mathcal{L}$ to X_∞^{reg} by this map is identified with our polarising bundle L. Moreover, Theorem 1.1 implies that the metrics on the bundle agree up to a bounded factor. So the sections of $\mathcal{L}^k$ over $\widehat{W}_1$ define bounded sections of L^k over X_∞^{reg}, that is, elements of $H^0(X_\infty, L^k)$. Write $U \subset H^0(X_\infty, L^k)$ for the image of this map from $H^0(\widehat{W}_1, \mathcal{L}^k)$. These sections define a map α from X_∞^{reg} to $\mathbb{P}$ and the definitions mean that this is just the composition of $\widehat{T}_1$ with the above projective embedding of $\widehat{W}_1$. The subspace U contains the kth powers of sections in $H^0(X_\infty, L)$ which uniformly generate the fibres, so we have a first-derivative estimate on the map α. Hence α extends to a Lipschitz map, which we also call α, from X_∞ to $\mathbb{P}$ with image $\widehat{W}_1$. Let Z be the intersection of the smooth part of $\widehat{W}_1$ with $\alpha(\Sigma)$. The Lipschitz bound implies that the Hausdorff dimension of Z is at most $2n-4$ and it follows that any local holomorphic function defined on the complement of Z extends holomorphically over Z [26]. This means that $H^0(X_\infty, L^k)$ can be identified with bounded holomorphic sections of the hyperplane bundle over the smooth part of $\widehat{W}_1$. But it is a basic general fact about a normal variety (cf. [23]) that its structure sheaf can be defined by bounded holomorphic functions on the smooth part. So the subspace U is in fact the whole of $H^0(X_\infty, L^k)$. Thus α is exactly T_k and W_k is $\widehat{W}_1$, and hence normal.

To complete the story, we have the following result.

LEMMA 4.13. *If W_1 is normal, then W_k is the embedding of W_1 defined by sections of $\mathcal{O}(k)$.*

This follows from the same argument as above. One simply notices that the normality of W_1 implies that $H^0(X_\infty, L^k)$ could be identified with $H^0(W_1, \mathcal{O}(k))$.

We have now almost completed the proof of Theorem 1.2. For any given polarised limit space X_∞ we can choose a k so that $H^0(X_\infty, L^k)$ represents X_∞ as a normal variety, and if X_j is a sequence converging to X_∞ in the Gromov–Hausdorff sense we can choose a convergent sequence of embeddings. (Notice that the only reason for passing to a subsequence in the statement of Theorem 1.2 is that we can have different polarisations on the same Riemannian limit space.) The last point is to show that there is a single k_1 which works for all X_∞. But this follows from Gromov compactness and the easy fact that if k has the desired property for X_∞ it does also for all limit spaces sufficiently close to X_∞, in the Gromov–Hausdorff sense. To see this we argue as follows. The preceding discussion shows that we can fix k so that for any limit space X'_∞ sufficiently close to X_∞ in the Gromov–Hausdorff metric the map T_k is defined on X'_∞ and is generically one-to-one. If we know that the the image $W'_k = T_k(X'_\infty)$ is a normal variety then it follows just as above that it is isomorphic to W'_{ak} for any $a>0$ and hence T_k is an embedding of X'_∞. Now we may assume that the Chow point of the image $T_k(X'_\infty)$ is close to that of $T_k(X_\infty)$ in the appropriate Chow variety *Chow*. If we knew that the set representing normal varieties is open in *Chow* then it would follow that W'_k is normal, for X'_∞ sufficiently close in the Gromov–Hausdorff metric to X_∞. However, while we feel that it should be true, we have not found this statement about the openness of the normality condition in the literature and it does not seem completely straightforward to prove. To get around this point we argue as follows. It is stated in the literature that normality is open in a flat family (see for example [18, Appendix E]). Lemma 4.1 gives a bound on the Hilbert polynomial of W'_k in that $\dim H^0(W_k, \mathcal{O}(a)) \leqslant C(a)$ for some $C(a)$ independent of X'_∞ and it follows that we can suppose that W'_k has the same Hilbert polynomial as W_k. Thus we can lift the Chow point of W'_k to a point in a fixed Hilbert scheme and deduce what we need from the openness of normality in the flat family over the Hilbert scheme.

To spell out a little more the consequences of Theorem 1.2, observe that now that we are considering embeddings, the degree of W is determined by k_1 and V. So (for theoretical purposes) we can operate in a fixed quasi-projective Chow variety $\mathcal{T}$ parameterising normal n-dimensional subvarieties of the given degree in a suitable large projective space $\mathbb{CP}^N$. "Algebro-geometric convergence" of X_j to X_∞ means convergence in $\mathcal{T}$. There is a universal variety $\mathcal{U} \to \mathcal{T}$ and, by general facts ([20, Theorem III.9.11]), this is a flat family. So we see that if X_j converge to X_∞ in the Gromov–Hausdorff sense, then X_j and $W = X_\infty$ can be realised as fibres in a flat family. So, for example, the Hilbert polynomials of X_j and $W = X_\infty$ are the same.

There are different ways of going about the proofs of Theorem 1.2. We mention one elegant alternative, based on a result from the thesis of Chi Li [21, Proposition 7]. This in turn depends upon results of Siu and Skoda. For X in $\mathcal{K}(n,c,V)$ let R_X be the graded ring

$$R_X = \bigoplus_l H^0(X, L^l).$$

Then from standard theory we know that R_X is finitely generated and $X = \mathrm{Proj}(R_X)$. Assuming the lower bound in Theorem 1.1, Li proves an effective form of finite generation in the sense that, if σ_j is an orthonormal basis in the finite-dimensional space $\bigoplus_{k=0}^{(n+2)k_0} H^0(X, L^k)$, then the σ_j generate R_X and for each l there is a number B_l such that any element of L^2 norm 1 in $H^0(X, L^l)$ can be expressed as a polynomial in the σ_j with coefficients bounded by B_l. It follows easily that for a polarised limit space X_∞ the graded ring

$$R_{X_\infty} = \bigoplus_l H^0(X_\infty, L^l)$$

is finitely generated. Then we can immediately define the algebraic variety W as

$$\mathrm{Proj}(R_{X_\infty}).$$

Of course there is still some work to do in checking the properties of W.

4.4. Further results

We will now restrict attention to the case when X_∞ is the limit of Kähler–Einstein manifolds $X_j \in \mathcal{K}(n, c, V)$ with Ricci curvature 1, $-\frac{1}{2}$ or 0. We suppose that $L = K_X^{-1}$ or $L = K_X^2$ in the first and second situations, and in the third situation we suppose that the manifolds are Calabi–Yau, so we have fixed holomorphic n forms Θ_j over X_j with $\Theta_j \wedge \overline{\Theta}_j$ being the volume form. For brevity we just call this "the Kähler–Einstein case".

PROPOSITION 4.14. *In the Kähler–Einstein case the map $T: X_\infty \to W$ takes the differential geometric limit singular set to the algebro-geometric singular set.*

The argument in the previous subsection implies that T maps the smooth set in X_∞ to the regular set in W. So we need to show that if $T(p)$ is a smooth point of W, then the limit metric on X_∞ is also smooth at p. Denote by ω_j the Kähler–Einstein metric on X_j, and by ω_j' the induced Fubini–Study metric. Then, we have $\omega_j' = \omega_j + i\partial\bar{\partial}\phi_j$ with $\phi_j = \log \varrho(\omega_j)$. By Theorem 1.1 and Proposition 2.1, there is a constant $C_1 > 0$ such that $\|\phi_j\|_{L^\infty} + \|\nabla_{\omega_j}\phi_j\|_{L^\infty} \leqslant C_1$ for all j. By Lemma 4.2, there is a constant $C_2 > 0$ with the uniform bound $\omega_j' \leqslant C_2\omega_j$. Now, write $\mathrm{Ric}(\omega_j') = \lambda\omega_j' + i\partial\bar{\partial}f_j$, where λ is 1, $-\frac{1}{2}$ or 0. So, with suitable normalisation of f_j, we have the equation

$$\omega_j^n = e^{f_j + \lambda\phi_j}(\omega_j')^n. \tag{4.1}$$

Then, it is not hard to see that $\int_{X_j} f_j^2 \omega_j'^n \, d\mathrm{Vol} \leqslant C_3$ for some constant $C_3 > 0$. Now, for any p in W^{reg}, we choose a small neighbourhood $B(p, \delta) \subset W^{\mathrm{reg}}$. Then, there are corresponding points $p_j \in X_j$ such that $B(p_j, \delta)$ converges smoothly to $B(p, \delta)$ in $\mathbb{CP}^{N_k}$. By standard elliptic estimates applied to the equation $\Delta_{\omega_j'} f_j = S(\omega_j') - \lambda n$ (where $S(\omega_j')$ denotes the scalar curvature function of ω_j'), we see that $\|f_j\|_{C^1(B(p_j, \delta/2), \omega_j')}$ is uniformly bounded. Then, by (4.1), there is $C_4 > 0$ such that $C_4^{-1}\omega_j \leqslant \omega_j' \leqslant C_4\omega_j$ in $B\big(p_j, \frac{1}{2}\delta\big)$. In $B\big(p_j, \frac{1}{2}\delta\big)$ we have $|d\phi_j|_{\omega_j'} \leqslant C_4 |d\phi_j|_{\omega_j} \leqslant C_4 C_1$, and thus, with respect to the metric

ω_j', the right-hand side of (4.1) has a uniform C^1 bound. Therefore we can apply the Evans–Krylov theory (see for example [3]) to conclude that $|\phi_j|$ has a uniform $C^{2,\alpha}$ bound in $B\big(p_j, \frac{1}{4}\delta\big)$. Then standard arguments show that all covariant derivatives of ϕ_j (with respect to ω_j') are uniformly bounded, so the Kähler–Einstein metrics ω_j converge smoothly in a neighbourhood of p.

Proposition 4.15. *In the Kähler–Einstein case, the algebro-geometric limit W has log-terminal singularities.*

By general theory, what the statement really means is that for any singular point x in W, there is a neighbourhood U, an integer $m>0$, and a nowhere zero holomorphic section Θ of K^m over $W^{\mathrm{reg}}\cap U$ with $\int_{W^{\mathrm{reg}}\cap U}(\Theta\wedge\overline{\Theta})^{1/m}<\infty$. We first consider the cases $L=K_X^2$ and $L=K_X^{-1}$. By Theorem 1.1 and Proposition 2.1, we have shown that, for any $x\in W$, there is a neighbourhood U of x, a constant $C>0$, and a section s of L^k (k is the constant in Lemma 4.12) over $X_\infty\setminus\Sigma=W^{\mathrm{reg}}$ with $\|s\|_{L^2}=1$ and $C^{-1}\leqslant|s(y)|^2\leqslant C$ for any $y\in W^{\mathrm{reg}}\cap U$. Here the norm is taken with respect to the Kähler–Einstein metric. When $L=K_X^2$, we set $\Theta=s$ and $m=2k$, then

$$\int_{W^{\mathrm{reg}}\cap U}(\Theta\wedge\overline{\Theta})^{1/m}=\int_{W^{\mathrm{reg}}\cap U}|s|^{1/k}\,d\mathrm{Vol}\leqslant C^{1/2k}\,\mathrm{Vol}(W).$$

When $L=-K_X$, we define $\Theta=s^*$ and $m=k$, where s^* is the dual section of s. So

$$|s^*|=|s|^{-1}.$$

Then

$$\int_{W^{\mathrm{reg}}\cap U}(\Theta\wedge\overline{\Theta})^{1/m}=\int_{W^{\mathrm{reg}}\cap U}|s^*|^{2/k}\,d\mathrm{Vol}\leqslant C^{1/k}\,\mathrm{Vol}(W).$$

In the Calabi–Yau case, since Θ_j has norm 1 everywhere, we easily see that there is a limit holomorphic volume form Θ on $X_\infty\setminus\Sigma=W^{\mathrm{reg}}$ with norm 1. This means that

$$\int_{W^{\mathrm{reg}}}\Theta\wedge\overline{\Theta}=\mathrm{Vol}(W)<\infty.$$

Remark. The limit metric ω_∞ on W^{reg} defines a singular Kähler–Einstein metric on W, in the sense of pluri-potential theory (cf. [17]). This is an immediate consequence of the above results on W, and here we make a brief explanation. The readers are referred to [17] for more details on the relevant notions etc. Adopting the notation in the proof of Proposition 4.14, since ϕ_j has a uniform Lipschitz bound (by Theorem 1.1 and Proposition 2.1), the limit ϕ_∞ is continuous on W (and smooth on W^{reg}). Let ω_∞' be the restriction of the Fubini–Study metric on W, and f_∞' be its Ricci potential, normalised so that

$$(\omega_\infty'+i\partial\bar\partial\phi_\infty)^n=e^{f_\infty'+\lambda\phi_\infty}(\omega_\infty')^n$$

holds on the smooth part W^{reg}. The uniform bound on ϕ_∞ means that we can view the left-hand side as a global Monge–Ampère measure on W, so that the singular locus has zero measure. The fact that W is log-terminal implies that the term $e^{f'_\infty}$ is in $L^p(W,(\omega'_\infty)^n)$ for some $p>1$ ([17, Lemma 6.4]), and the right-hand side also defines a global measure on W. These properties fit exactly into the framework of [17].

5. Structure of 3-dimensional tangent cones

In this section we make a more detailed study of the structure of the tangent cones occurring in the previous sections, in particular we prove the following result.

THEOREM 5.1. *In complex dimension* 3, *the link Y of any tangent cone of the Gromov–Hausdorff limit X_∞ is a* 5-*dimensional Sasaki–Einstein orbifold.*

As before, we write $Y=Y^{\mathrm{reg}}\cup\Sigma$, where Y^{reg} is the smooth part and Σ is the singular part. We view Y as the radius-1 link in $C(Y)$. For any $q\in\Sigma$, any tangent cone of $C(Y)$ at q splits at least one line, so by general theory (see for example [5]) it must have the form $\mathbb{C}\times(\mathbb{C}^2/\Gamma)$ for some $\Gamma\in U(2)$. Moreover, Γ depends only on q. So the tangent cone of Y at q is $\mathbb{R}\times(\mathbb{C}^2/\Gamma)$.

LEMMA 5.2. *Y^{reg} is geodesically convex in Y.*

For any two points p and q in Y, it is a general fact that a minimising geodesic in $C(Y)$ connecting p and q must be of the form $(r(t),\gamma(t))$, where $\gamma(t)$ is a geodesic in Y, r is a universal function of $d_Y(p,q)$ and t is determined by elementary trigonometry. By a recent result of Colding–Naber [10] we know that $C(Y^{\mathrm{reg}})$ is geodesically convex in $C(Y)$, so the lemma follows.

As usual there is a Reeb field $\xi=Jr\partial/\partial r$ on $C(Y^{\mathrm{reg}})$ which is holomorphic, Killing, of unit length, and tangent to Y^{reg}. For any $p\in Y^{\mathrm{reg}}$ we denote by $p(t)$ the integral curve $\exp(t\xi)p$. For $|t|$ sufficiently small, $p(t)$ defines a geodesic segment in Y^{reg}.

LEMMA 5.3. *For any $p_1,p_2\in Y^{\mathrm{reg}}$, if $p_1(t)$ and $p_2(t)$ are both defined on some interval $[0,T]$, then $f(t)=d(p_1(t),p_2(t))$ is independent of t.*

By Lemma 5.2, for any $t\in[0,T]$ the minimising geodesic γ connecting $p_1(t)$ and $p_2(t)$ lies in Y^{reg}. So there is $\varepsilon>0$ such that the curve $\gamma_s=\exp(s\xi)\gamma$ is in Y^{reg} for $s\in[0,\varepsilon]$. Clearly the length of γ_s is independent of s. Thus $f(t)$ is a decreasing function. Replacing ξ by $-\xi$, one sees that f is also an increasing function. Thus f is constant.

PROPOSITION 5.4. *ξ generates a* 1-*parameter group of isometric actions on Y.*

Fix any point p in Y^{reg}, and choose a convex embedded ball $B(p,r)$ in Y^{reg}. We claim that $B(p(t),r)\cap\Sigma=\varnothing$ for all t. For otherwise there is a $T>0$ such that $B(p(t),r)\cap\Sigma=\varnothing$ for $t\in[0,T]$, but $\partial B(p(T),r)\cap\Sigma$ is non-empty. Choose a point q in this intersection. Let $\gamma:[0,1]\to Y$ be the radial geodesic connecting $p(T)$ and q, and let $p_j=\gamma(1-2^{-j})$. Then $B_j=B(p_j,2^{-j}r)\subset B(p(T),r)$, and $d(p_j,q)=2^{-j}$. Consider the pointed sequence $(Y,2^jd_Y,q)$. By assumption we know, as j tends to infinity by passing to a subsequence, that this converges to a tangent cone $Y_q=\mathbb{R}\times\mathbb{C}^2/\Gamma$. Then the rescaled balls 2^jB_j converge to a ball $B(p_\infty,r)$ in Y_q and $d(p_\infty,0)=r$. But B_j is isometric to a ball in $B(p,r)$ and so have uniformly bounded geometry, and thus 2^jB_j converges to a flat ball B_∞. Moreover, by Lemma 5.3, the distance between any two points in B_∞ is realised by the length of a geodesic within B_∞. Clearly this cannot happen on Y_q.

By the claim, the isometric action $\exp(t\xi)$ is well defined on Y^{reg} for all t. Then we can extend the action to an isometric action on Y: given $p\in\Sigma$, we pick a Cauchy sequence $p_j\in Y^{\mathrm{reg}}$ converging to q; for any t, $p_j(t)=\exp(t\xi)$ is also a Cauchy sequence in Y^{reg}, so there is a unique limit $p(t)$. We define $\exp(t\xi)p=p(t)$. Clearly $\exp(t\xi)$ is distance-preserving. Moreover, $\exp(t\xi)$ preserves both Y^{reg} and Σ.

We denote by $\psi(t)=\exp(t\xi)$, $t\in\mathbb{R}$, the above 1-parameter group action. Then we have the following result.

LEMMA 5.5. *There is no point in Y fixed by ψ.*

If q is a fixed point, then clearly $q\in\Sigma$. Choose a tangent cone $Y_q=\mathbb{R}\times(\mathbb{C}^2/\Gamma)$ at q. The action of ψ induces a 1-parameter group of isometric actions on Y_q, which fixes the origin. On the other hand, on the smooth part of Y_q, the corresponding infinitesimal action is given by a Killing field of constant length. Clearly such a Killing field cannot have zeroes. This is a contradiction.

Now we are ready to conclude the following result.

PROPOSITION 5.6. *Σ is a disjoint union of finitely many periodic orbits of ψ.*

Proof. Fix any $q\in\Sigma$. Since it is not a fixed point of ψ, we can choose a neighbourhood $B_r(q)$ such that any path-connected component of the intersection of an orbit of ψ with $\overline{B_r(q)}$ is compact. Let O_q be one path-connected component of $\psi(q)$ in $B_r(q)$. We claim that, for $s>0$ sufficiently small, $\Sigma\cap B_s(q)=O_q\cap B_s(q)$. If not, then there is a sequence $p_j\in(B_r(q)\setminus O_q)\cap\Sigma$ converging to q. We can choose q_j on the path-connected component of the orbit of p_j in $\overline{B_r(q)}$ which has least distance to q. Then $s_j=d(q,q_j)>0$. For j sufficiently large we have $d(q_j,O_q)=s_j$. Now consider the rescaled pointed sequence $(B_r(q),s_j^{-1}d_Y,q)$. As $j\to\infty$, by passing to a subsequence, this converges to $\mathbb{R}\times(\mathbb{C}^2/\Gamma)$. Moreover, O_q converges to $\mathbb{R}\times\{0\}$ and q_j converges to q_∞, which has distance 1 to

$\mathbb{R}\times\{0\}$. But q_j is singular for all j, so q_∞ is also singular. This yields a contradiction. Then the proposition follows from the claim and an obvious compactness argument. □

Now we pick a point q in Σ. Choose a neighbourhood U_q of q such that $\Sigma\cap U_q=O_q$ consists of exactly one component. Then one can take a local quotient of U_q by ψ, and obtain a 4-dimensional (incomplete) metric ball $B(q,200)$ with an isolated singularity q. Moreover, the tangent cones at q are all isometric to $\mathbb{C}^2/\Gamma$ for a unique $\Gamma\subset U(2)$. The metric g on the smooth part $B(q,200)\setminus\{q\}$ is Kähler–Einstein. We write $B=B(q,100)$, and $B^*=B(q,100)\setminus\{q\}$. Denote by $\widehat{B}$ the standard ball of radius 100 in $\mathbb{C}^2/\Gamma$, and let $\widehat{B}^*=B\setminus\{0\}$.

THEOREM 5.7. *There is a diffeomorphism $F\colon\widehat{B}^*\to B^*$ such that F^*g extends to a smooth orbifold Riemannian metric on $\widehat{B}$.*

Given this theorem, it is not hard to prove Theorem 5.1. On the local quotient B we have an orbifold chart $\{z^j\}_j$ with Kähler metric $\omega=i\partial\bar\partial\phi$. We pull back the coordinate $\{z^j\}_j$ to U_q. Let η be the contact form associated with the Sasaki structure on the smooth part $U_q^0=U_q\setminus O_q$. Then the 1-form $\eta'=\eta-2\operatorname{Im}(\partial_z\phi)$ is closed. Clearly $H^1(U_q^0,\mathbb{R})=0$, so $\eta'=dx$ for some function x. Then it is easy to see that $\xi=\partial/\partial x$, in the coordinates (x,z^1,z^2). This gives rise to an orbifold chart for U_q. The compatibility condition between the orbifold charts follows easily from the local action ψ.

Theorem 5.7 is certainly well known, being due to Anderson [1], Bando–Kasue–Nakajima [2], and Tian [29]. We include a proof here for the reader's convenience. For simplicity of notation we assume Γ is trivial, and the proof is the same for a general Γ. For any $a_1<a_2$, we set

$$A(a_1,a_2)=\{p\in B: a_1<d(p,q)<a_2\}\quad\text{and}\quad \hat A(a_1,a_2)=\{x\in\mathbb{C}^2: a_1<|x|<a_2\}.$$

Since any tangent cone at p is isometric to $\mathbb{C}^2/\Gamma$, by general results of Anderson and Colding, there is $\delta\in\left(0,\frac{1}{10}\right)$ such that, for r sufficient small, there is an embedding

$$\phi_r\colon \hat A(1-\delta,100+\delta)\longrightarrow B(q,200)$$

such that $(1-\varepsilon(r))|x|\leqslant r^{-1}d(q,\phi_r(x))\leqslant(1+\varepsilon(r))|x|$ and $|r^{-2}\phi_r^*g-g_0|_{C^4}\leqslant\varepsilon(r)$, with $\varepsilon(r)$ being a monotone function that goes to 0 as r tends to 0. Here and from now on, the norm of a quantity defined on an annulus in $\mathbb{R}^4$ is always taken with respect to the Euclidean metric. Then we readily see that, for all $r<s<1$, there is a deformation retract from $A(r,1)$ to $A(s,1)$, and B is homeomorphic to $\widehat{B}$. The proof of Theorem 5.7 is divided into four steps.

Step I. (C^0 chart)

To construct a chart so that g is continuous we need to glue together the above almost Euclidean annuli in a controllable way. This is elementary and we begin with the following lemma.

LEMMA 5.8. *For $\varepsilon>0$ sufficiently small, there is a constant $K(\varepsilon)>0$ which goes to zero as ε tends to zero, such that for any smooth map $\phi\colon \hat{A}(30,80)\to\mathbb{R}^4$ with*

$$\|\phi^*g_0-g_0\|_{C^4(\hat{A}(30,80))}\leqslant\varepsilon,$$

there is an isometry P of $\mathbb{R}^4$ such that $\|P\circ\phi-\mathrm{Id}\|_{C^3(\hat{A}(40,70))}\leqslant K(\varepsilon)$.

Assume the statement fails, then there is a constant $\tau>0$, a sequence $\varepsilon_j\to0$, and maps $\phi_j\colon \hat{A}(30,80)\to\mathbb{R}^4$ with $\|\phi_j^*g_0-g_0\|_{C^4(\hat{A}(30,80))}\leqslant\varepsilon_j$, but for any isometry P we have $\|P\circ\phi_j-\mathrm{Id}\|_{C^3(\hat{A}(40,70))}\geqslant\tau$. Then ϕ_j converges to a map ϕ_∞ in $C^3(\hat{A}(40,70))$ such that $\phi_\infty^*g_0=g_0$. So ϕ_∞ is an isometry of $\mathbb{R}^4$. Since $\|\phi_\infty^{-1}\circ\phi_j-\mathrm{Id}\|_{C^3(\hat{A}(40,70))}$ converges to zero as j goes to infinity, we arrive at a contradiction.

LEMMA 5.9. *Suppose the two maps*

$$f_0\colon \hat{A}(1,100)\longrightarrow B(q,200)\quad \textit{and}\quad f_1\colon \hat{A}(1-\delta,100+\delta)\longrightarrow B(q,200)$$

satisfy, for $j=0,1$ and some $r>0$,

$$(1-\varepsilon)|x|\leqslant\frac{10^j}{r}d(q,f_j(x))\leqslant(1+\varepsilon)|x|\quad \textit{and}\quad \left\|\frac{10^{2j}}{r^2}f_j^*g-g_0\right\|_{C^4}\leqslant\varepsilon$$

on $\hat{A}(10^j,10^{j+1})$. Then there is a constant $G=G(\varepsilon)$ with $\lim_{\varepsilon\to0}G(\varepsilon)=0$, a rotation $R\in O(4)$, and a map $f\colon \hat{A}(10^{-1},100)\to B(q,200)$ with $f(x)=f_0(x)$ on $\hat{A}(9,100)$, $f(x)=f_1(10R^{-1}(x))$ on $\hat{A}(10^{-1},2)$, and $\|r^{-2}f^*g-g_0\|_{C^2}\leqslant C(\varepsilon)$ on $\hat{A}(10^{-1},100)$.

By the obvious scaling invariance, we may assume $r=1$. Let $D=\mathrm{Im}(f_0)\cap\mathrm{Im}(f_1)$. Since ε is small, we may assume that $\hat{A}(3,8)$ is contained in $f_0^{-1}(D)$. Then there is a constant C_1 independent of ε such that the map $\psi=10^{-1}f_1^{-1}\circ f_0\colon \hat{A}(3,8)\to\mathbb{R}^4$ satisfies

$$\|\psi^*g_0-g_0\|_{C^4}\leqslant C_1\varepsilon\quad\text{and}\quad(1-3\varepsilon)|x|\leqslant|\psi(x)|\leqslant(1+3\varepsilon)|x|.$$

By Lemma 5.8, there is an isometry P of $\mathbb{R}^4$ such that $\|P\circ\psi-\mathrm{Id}\|_{C^3}\leqslant K(C_1\varepsilon)$ on $\hat{A}(4,7)$. We write $P(x)=R(x+\xi)$ for a rotation R and a translation ξ. Then it is easy to see that $\|R\circ\psi-\mathrm{Id}\|_{C^3}\leqslant C_2(\varepsilon)$ with $\lim_{\varepsilon\to0}C_2(\varepsilon)=0$, and $R(\hat{A}(1-\delta,100+\delta))$ contains $\hat{A}(1,100)$. Choose a cut-off function $\chi(x)$ on $\hat{A}(1,100)$ with $\chi(x)=1$ for $|x|\leqslant5$ and $\chi(x)=0$ for $|x|\geqslant6$. Using the map f_0 we get a corresponding cut-off function on $B(q,200)$, still denoted by χ. Then $\|\chi\|_{C^4_g}\leqslant C_3$ for a constant C_3 independent of ε. Clearly $\chi(p)=0$ when $p\notin \mathrm{Im}(f_1)$ and $\chi(p)=1$ when $p\notin\mathrm{Im}(f_0)$. Define $h:\mathrm{Im}\, f_0\cup\mathrm{Im}\, f_1\to\mathbb{R}^4$ sending p to $10^{-1}\chi(p)R\circ f_1^{-1}(x)+(1-\chi(p))f_0^{-1}(x)$. Then, for ε sufficiently small, we have $h=f_0^{-1}$ on $A(8,100)$, $h=10^{-1}R\circ f_1^{-1}$ on $A(10^{-1},3)$, and $\|h^*g_0-g\|_{C^2_g}\leqslant C(\varepsilon)$ with $\lim_{\varepsilon\to 0} C(\varepsilon)=0$. Define $f=h^{-1}$. Then $f(x)$ meets the required properties.

PROPOSITION 5.10. *There is a diffeomorphism $F:\widehat{B}^*\to B^*$ such that F^*g extends to a C^0 metric tensor over B.*

Since the problem is local, we may assume for all $r\leqslant 1$ that the above map ϕ_r exists and $\varepsilon(1)$ is as small as we like. For simplicity we set $\phi_k=\phi_{10^{-k}}$ and $\varepsilon_k=\varepsilon(10^{-k})$. Now we first define $F_0(x)=\phi_0(x)$ on $\hat{A}(1,100)$. Inductively suppose F_k is defined on $\hat{A}(10^{-k},10^{-k+2})$ satisfying $F_k(x)=\phi_k\circ R_k^{-1}(10^k x)$ on $\hat{A}(10^{-k},20\cdot 10^{-k})$ for some rotation $R_k\in O(4)$. We apply Lemma 5.9 to the two maps $\phi_k\circ R_k^{-1}$ and ϕ_{k+1}, with $r=10^{-k}$ and $\varepsilon=\max\{\varepsilon_{k-1},\varepsilon_k\}$, and obtain a map f_{k+1} defined on $\hat{A}\big(\frac{1}{10},100\big)$ satisfying (2). Then we define $F_{k+1}(x)$ to be $f_{k+1}(10^k x)$ on $\hat{A}(10^{-k-1},10^{-k+1})$. By Lemma 5.9, we see that all the F_k's match together to a map F from $\widehat{B}^*$ to $B(q,200)$, and we can modify F slightly near $\partial\widehat{B}$ so that the image is exactly B^*. It is easy to see that

$$\|F^*g-g_0\|_{L^\infty(\hat{A}(10^{-k},10^{-k+1}))}=\|10^{2k}f_{k+1}^*g-g_0\|_{L^\infty(\hat{A}(10,100))}\leqslant G(\max\{\varepsilon_{k-1},\varepsilon_k\}),$$

and F^*g extends to a continuous metric tensor over B.

Step II. (Curvature bound)

Now we may assume that g is a C^0 metric on $B=\widehat{B}$.

LEMMA 5.11. *We have*

$$\int_{B^*}|\mathrm{Rm}(g)|^2\,d\mathrm{Vol}_g<\infty.$$

Let $A_\pm$ be the connection induced by the Levi-Civita connection of g on $\Lambda_g^\pm$. The Einstein condition implies that A_+ is self-dual and A_- anti-self-dual with respect to g. Thus

$$|\mathrm{Rm}(g)|^2\,d\mathrm{Vol}_g=\mathrm{Tr}(F_{A_+}\wedge F_{A_+}-F_{A_-}\wedge F_{A_-}).$$

By the tangent cone condition, we can easily find a smooth family of spheres S_r in B^* with the property that, as r tends to zero, $(S_r,r^{-2}g)$ converges smoothly to the round sphere in $\mathbb{R}^4$, and the restriction to $(S_r,r^{-2}g)$ of the connection $A_\pm$ converges to the trivial flat connection. Then, for any $s<r$,

$$\int_{A(s,r)}\mathrm{Tr}\,F_{A_+}\wedge F_{A_+}=\mathrm{CS}(A_+,S_r)-\mathrm{CS}(A_+,S_s)\quad(\mathrm{mod}\ \mathbb{Z}),$$

where $\mathrm{CS}(A,M)=\int_M dA\wedge A+\frac{2}{3}A\wedge A\wedge A$ is the Chern–Simons invariant of a connection A over a 3-manifold M, defined modulo $\mathbb{Z}$. By assumption, $\mathrm{CS}(A_+,S_r)=\mathrm{CS}(A_+,S_r/r)\to 0$ as $r\to 0$. We choose r so small that for any $s\leqslant r$ we have $|\mathrm{CS}(A_+,S_r)|\leqslant\frac{1}{8}$ modulo $\mathbb{Z}$. So $\int_{A(s,r)}\operatorname{Tr}F_{A_+}\wedge F_{A_+}$ is in $\left[-\frac{1}{4},\frac{1}{4}\right]$ modulo $\mathbb{Z}$, and on the other hand it clearly depends continuously on s, and thus the integral is uniformly bounded for all $s<r$. One can similarly deal with A_-. Together this implies that $\int_{B^*}|\mathrm{Rm}(g)|^2\,d\mathrm{Vol}_g$ is finite.

Proposition 5.12. *For any $k\geqslant 0$, $|\nabla_g^k\mathrm{Rm}(g)|$ is uniformly bounded in B^*.*

Since the metric g is C^0 equivalent to the flat metric g_0, the Sobolev space $W^{1,p}$ is the same with respect to both metrics, and the Moser iteration works for the operator $\Delta=\Delta_g$. Here again we use the geometers' convention for the sign. By the Bochner formula, there is a constant $C_1>0$ such that

$$\Delta|\mathrm{Rm}(g)|\leqslant C_1|\mathrm{Rm}(g)|^2,$$

which is on the borderline of applying Moser iteration. Due to Bando–Kasue–Nakajima [2, Corollary 4.10], there is an improved Kato's inquality, namely, there are $C_2>0$ and $\delta\in(0,1)$ such that

$$\Delta|\mathrm{Rm}(g)|^{1-\delta}\leqslant C_2|\mathrm{Rm}(g)|^{2-\delta}.$$

Let $u=|\mathrm{Rm}(g)|^{1-\delta}$ and $f=|\mathrm{Rm}(g)|$. Then we can apply [27, Lemma 2.1] with $q=1/(1-\delta)$ and $q_0=1/2(1-\delta)$ to conclude that $|\mathrm{Rm}(g)|$ is in $W^{1,2}$. By Sobolev embedding we see $|\mathrm{Rm}(g)|\in L^4$. Also that $|\nabla\mathrm{Rm}(g)|\in L^2$ implies that the inequality $\Delta|\mathrm{Rm}(g)|\leqslant C_1|\mathrm{Rm}(g)|^2$ holds weakly on the whole ball B. Then we can apply the standard Moser iteration to conclude that $|\mathrm{Rm}(g)|$ is uniformly bounded. Now consider $|\nabla\mathrm{Rm}(g)|$. For any $p\in B^*$ with $d(p,q)=r\leqslant\frac{1}{2}$, the rescaled ball $r^{-1}B\left(p,\frac{1}{2}r\right)$ has uniformly bounded geometry, so standard elliptic regularity for the Einstein equation then implies that $|\nabla\mathrm{Rm}(g)|\leqslant C_3r^{-1}$, for some constant $C_3>0$. Thus $|\nabla\mathrm{Rm}(g)|\in L^3$. By the Bochner formula again there is a constant $C_4>0$ such that

$$\Delta|\nabla\mathrm{Rm}(g)|\leqslant C_4|\mathrm{Rm}(g)|\,|\nabla\mathrm{Rm}(g)|.$$

Let $u=|\nabla\mathrm{Rm}(g)|$ and $f=C_4|\mathrm{Rm}(g)|$, and applying [27, Lemma 2.1] with $q=1$ and $q_0=\frac{3}{4}$, we get $|\nabla\mathrm{Rm}(g)|\in W^{1,2}$. Thus the inequality holds weakly on B and by Moser iteration $|\nabla\mathrm{Rm}(g)|$ is uniformly bounded. Then similarly one can prove the bound for higher covariant derivatives of the curvature tensor.

Step III. ($C^{1,\alpha}$ chart)

To construct a coordinate chart so that g is $C^{1,\alpha}$, we shall use Rauch's comparison theorem, following [2]. The following lemma is a direct consequence of the tangent cone condition (by using the maps ϕ_r).

LEMMA 5.13. *There is a sequence* $\varepsilon_j \to 0$ *and a sequence of smooth embeddings* f_j *from* S^3 *to* B *with the following properties*:

(1) $d_{\mathrm{GH}}(S_j, \partial B(j^{-1})) \leqslant j^{-1}\varepsilon_j$, *where* $S_j = f_j(S^3)$;

(2) $\|j^2 f_j^* g - h_0\|_{C^4_{h_0}} \leqslant \varepsilon_j$, *where* h_0 *is the standard round metric on* S^3;

(3) $\|j^{-1} A_{S_j} + \mathrm{Id}\,\|_{C^3_{h_0}} \leqslant \varepsilon_j$, *where* $A_{S_j}: TS_j \to TS_j$ *is the shape operator.*

PROPOSITION 5.14. *There is a* C^3 *diffeomorphism* $F: B^* \to B^*$ *such that* $F^* g$ *extends to a* $C^{1,1}$ *metric tensor on* B.

We define $F_j: S^3 \times [j^{-1}, 1] \to B^*$ by sending (x, t) to $\exp_{f_j(x)}((t - j^{-1})N(x))$, where $N(x)$ is the outward normal vector at $f_j(x)$. Consider a Jacobi field $J(t)$ along a geodesic $\gamma_x(t) = F_j(x, t)$. Then, since the curvature of g is uniformly bounded, by the Rauch comparison theorem there are constants $C_1 > 0$ and $\delta > 0$ independent of x and j such that $C_1^{-1} |J(j^{-1})|_g \leqslant |J(t)|_g \leqslant C_1 j |J(j^{-1})|_g$ for $t \in [j^{-1}, \delta]$. For simplicity of notation, we may assume that $\delta = 1$. So, for j large enough, F_j has no critical points in $[j^{-1}, 1]$. Indeed F_j is a diffeomorphism. For otherwise there would be a geodesic loop $\sigma(s)$, $s \in [0, T]$, which is perpendicular to S_j when $s = 0$ and $s = T$. It is then easy to see this cannot happen for sufficiently large j, by passing to a tangent cone.

Now we write $F_j^* g = dt^2 + t^2 h_j(t)$. We first notice that $|d_g(0, F_j(x, t)) - t| \leqslant j^{-1}\varepsilon_j$. We derive estimates for $g_j(t)$. Given a unit tangent vector ξ at $x \in S^3$, let $J(t)$ be the Jacobi field along $\gamma_x(t)$ with $J(j^{-1}) = df_j(\xi)$ and $\dot{J}(j^{-1}) = A_{S_j}(J(j^{-1}))$. Then $J(t) = dF_{j\,(x,t)}(\xi)$. Clearly,

$$\big||J(j^{-1})|_g - j^{-1}\big| \leqslant j^{-1}\varepsilon_j \quad \text{and} \quad |\dot{J}(j^{-1}) - jJ(j^{-1})|_g \leqslant 2\varepsilon_j.$$

Let $\{e_1(t), ..., e_n(t) = \dot{\gamma}_x(t)\}$ be an orthonormal frame of parallel vector fields along $\gamma_x(t)$ such that $J(j^{-1}) = |J(j^{-1})|_g e_1$. Under the decomposition $J(t) = \sum_\alpha J_\alpha(t) e_\alpha(t)$, we have

$$\ddot{J}_\alpha(t) + \sum_\beta R_{\alpha n \beta n}(\gamma_x(t)) J_\beta(t) = 0,$$

where $R_{\alpha n \beta n} = R(e_\alpha, e_n, e_\beta, e_n)$. From the above discussion, we have that $|J(t)| \leqslant 2C_1$ for $t \in [j^{-1}, 1]$. So it is easy to see that there is a constant $C_2 > 0$ such that

$$\big||J(t)|_g - t\big| \leqslant C_2(j^{-1} + \varepsilon_j t + t^3).$$

Thus

$$\|h_j(t) - h_0\|_{L^\infty_{h_0}} \leqslant C_2(j^{-1} t^{-1} + \varepsilon_j + t^2).$$

Next take a unit tangent vector X at x, we vary $J(j^{-1})$ so that $\nabla^0_X J(j^{-1}) = 0$ at x, and extend X to a unit tangent vector field in a neighbourhood U of x in S^3. We may also view X as a tangent vector field on $U\times[j^{-1},1]$. Now we differentiate the Jacobi field equation, and similar arguments as above yield

$$|\nabla_X J(t)|_g \leqslant C_3(j^{-1}+\varepsilon_j t+t^3),$$

for a constant $C_3>0$. This implies that there is a constant $C_4>0$ such that

$$t^{-1}|\nabla^0(h_j(t)-h_0)|_{h_0} \leqslant C_4(j^{-1}t^{-2}+\varepsilon_j t^{-1}+t).$$

Similarly one can get bounds on higher derivatives of $h_j(t)-h_0$. The point is that for a fixed $\tau>0$, as $j\to\infty$, we know that $F_j(x,t)$ converges in C^3 to a limit $F^\tau_\infty(x,t)$ on $S^3\times[\tau,1]$. Then we can let $\tau\to 0$ and obtain a limit $F: S^3\times(0,1]\to B^*$ with the property that $d_g(0,F(x,t))=t$ and

$$t^{-2}\|F^*g-g_0\|_{C^0_{g_0}}+t^{-1}\|F^*g-g_0\|_{C^1_{g_0}}+\|F^*g-g_0\|_{C^2_{g_0}} \leqslant C_5$$

for some constant $C_5>0$. This implies that F^*g extends to a $C^{1,\alpha}$ metric on B.

Step IV. (C^∞ chart)

Now we may assume that g is a $C^{1,\alpha}$ metric on B. Notice that the metric g is also Kähler, and the compatible almost complex structure J is $C^{1,\alpha}$ in B. Thus, by the integrability theorem [24], modifying by a $C^{2,\alpha'}$ diffeomorphism, with $\alpha'<\alpha$, we may assume that J is the standard complex structure near the origin. So, in a small ball B_ε, the Kähler form of g is of the form $\omega=i\partial\bar\partial\phi$ for a real-valued function ϕ with regularity $C^{3,\alpha'}$. The Kähler–Einstein equation on B^*_ε has the form

$$(i\partial\bar\partial\phi)^2=e^{-\lambda\phi+h}\omega_0^2,$$

where h is a pluri-harmonic function on B^*_ε and ω_0 is the standard Kähler form on $\mathbb{C}^2$. By the Hartogs theorem, h extends smoothly to B_ε. Then the standard elliptic regularity implies that ϕ and hence g is smooth on B_ε. This finishes the proof of Theorem 5.7.

5.1. Further discussion

We can use this detailed description of the link Y in the 3-dimensional case, to get a more precise understanding of the " topological obstruction" of §3.2.2. A representation $\alpha: \pi_1(Y\setminus\Sigma)\to S^1$ defines a covering of $Y\setminus\Sigma$ and it is clear that the metric completion of this is again an orbifold $\widetilde{Y}$ with a metric of Ricci curvature $2n-1$. It is then clear that the usual proof of the Myers theorem extends to show that $\widetilde{Y}$ is compact, so the representation maps to a finite group. Thus $\pi_1(Y\setminus\Sigma)$ is also finite and the torus T in the discussion of §3.2.2 is in this case trivial. (Of course the set Y_ε can be assumed to be homotopy equivalent to $Y\setminus\Sigma$.) Moreover it is also clear that the usual proof of the Bishop theorem extends to this case to show that the volume of $\widetilde{Y}$ cannot exceed that of S^{2n-1}. Thus, the order of the cover is bounded by $\varkappa^{-1}$, where $\varkappa$ is the volume ratio, and hence by c^{-1}. Let $D=D(c)$ be the least integer such that all integers less than or equal to c^{-1} divide D. Then we see that the power α^D of any such representation must be trivial. Hence, if from the beginning of the discussion in §3, we consider powers L^{Dk}, then we never encounter the topological obstruction. The point here of course is that D is determined in a simple explicit way by c, which, in turn, in the Fano case, is known explicitly. In many practical cases of interest D is not too large.

We expect that in fact the same will be true in higher dimensions (with the same $D(c)$). Of course we do not expect that the singularities will always be of orbifold type, but it seems likely that the Bishop theorem can still be extended to the metric completion of a covering, as above. There is a slightly weaker statement which should be easier to prove. Let y be a point in the singular set Σ_Y of a $(2n-1)$-dimensional link Y. Let B be a sufficiently small ball about y and $B^{\mathrm{reg}}\subset B$ be the regular set. Suppose that we have found a number E such that for all such points (in all tangent cones of all limits of manifolds in $\mathcal{K}(n,c,V)$) the homology group $H_1(B^{\mathrm{reg}},\mathbb{Z})$ has order bounded by E. Let α be a representation of $\pi_1(Y\setminus\Sigma)$ as above. Then, in the covering defined by α^E, the pre-image of B^{reg} is a disjoint union of copies of B^{reg}. In this situation it is straightforward to apply recent results of Colding and Naber [10] to show that the regular set in the metric completion $\widetilde{Y}$ is geodesically convex, and then to extend the Bishop argument to this case. Thus we see that, if from the beginning of the discussion in §3 we consider powers L^{DEk} then we never encounter the topological obstruction. Arguing by induction on dimension, it seems likely that in fact the number $E=D^{n-2}$ will have the property stated above so, for this weaker statement, we would consider powers $L^{D^{n-1}k}$. But, in fact, it seems to us most likely that these higher powers of D are not required.

In this direction we make the following conjecture, which (if true) would be a substantial sharpening of Theorem 1.1.

Conjecture 5.15. *For any n, c, V and $\eta<1$ there is a number $k_0(n,c,V,\eta)$ such that if $k\geqslant k_0$ then for any $X\in\mathcal{K}(n,c,V)$ we have*

$$\eta\left(\frac{kD}{2\pi}\right)^n \leqslant \varrho_{kD,X} \leqslant \frac{1}{\eta c}\left(\frac{kD}{2\pi}\right)^n,$$

with $D=D(c)$ as above.

To put this in context, recall that for a *fixed* X the standard asymptotics is

$$\varrho_{k,X} \sim \left(\frac{k}{2\pi}\right)^n \quad \text{as } k\to\infty.$$

This essentially follows from the fact that on $\mathbb{C}^n$ we have $\varrho=(2\pi)^{-n}$. The conjectural lower bound here is a uniform version of this over $\mathcal{K}(n,c,V)$, provided we work over multiples of D. On the other hand the corresponding upper bound

$$\varrho_{kD,X} \leqslant \frac{1}{\eta}\left(\frac{kD}{2\pi}\right)^n$$

almost certainly fails, because at the vertex of a cone $C(Y)$ we have $\varrho=\varkappa^{-1}(2\pi)^{-n}$, where $\varkappa\geqslant c$ is the volume ratio. This is why we believe that the plausible upper bound should include the extra factor c^{-1}. In a similar way, if in fact we do encounter the topological obstruction of §3.2.2 in some limit space, then it seems it would not be true that there is a lower bound on $\varrho_{k,X}$ for *all* sufficiently large k, since the twisting of the line bundle will force ϱ to be small as we approach the singularity. This phenomenon—that near to a singularity ϱ gets larger or smaller depending on divisibility—is similar to the orbifold situation considered by Dai–Liu–Ma [12] and Ross–Thomas [25].

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