Acta Math., 207 (2011), 203–254
DOI: 10.1007/s11511-012-0069-3

Random conformal weldings

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1. Introduction

There has been great interest in conformally invariant random curves and fractals in the plane ever since it was realized that such geometric objects appear naturally in statistical mechanics models at the critical temperature [13]. A major breakthrough in the field occurred when O. Schramm [29] introduced the Schramm–Loewner evolution (SLE), a stochastic process whose sample paths are conjectured (and in several cases proved) to be the curves occurring in the physical models. We refer to [30] and [32] for a general overview and some recent work on SLE. The SLE curves come in two varieties: the radial one, where the curve joins a boundary point (say of the disc) to an interior point, and the chordal case, where two boundary points are joined.

SLE describes a curve growing in time: the original curve of interest (say a cluster boundary in a spin system) is obtained as time tends to infinity. In this paper we give a different construction of random curves which is stationary, i.e. the probability measure on curves is directly defined without introducing an auxiliary time. We carry out this construction for closed curves, a case that is not naturally covered by SLE.

Our construction is based on the idea of conformal welding. Consider a Jordan curve Γ bounding a simply connected region Ω in the plane. By the Riemann mapping theorem, there are conformal maps $f_\pm$ mapping the unit disc $\mathbb{D}$ and its complement to Ω and its complement. The map $f_+^{-1}\circ f_-$ extends continuously to the boundary $\mathbb{T}=\partial\mathbb{D}$ of the disc, and defines a homeomorphism of the circle. Conformal welding is the inverse

K. A., A. K. and E. S. were supported by the Academy of Finland, P. J. by NSF, A. K. by ERC and K. A. by the EU-network CODY.

operation where, given a suitable homeomorphism of the circle, one constructs a Jordan curve on the plane (see §2). In fact, in our case the curve is determined up to a Möbius transformation of the plane. Thus random curves (modulo Möbius transformations) can be obtained from random homeomorphisms via welding.

In this paper we introduce a random scale invariant set of homeomorphisms $h_\omega\colon\mathbb{T}\to\mathbb{T}$ and construct the welding curves. The model considered here has been proposed by the second author. The construction depends on a real parameter β (the "inverse temperature") and the maps are a.s. in ω Hölder continuous for $\beta<\beta_c$. For this range of β the welding map will be a.s. well defined. For $\beta>\beta_c$ we expect the map h_ω not to be continuous and no welding to exist. Our curves are closely related to SLE($\varkappa$) for $\varkappa<4$, see the footnote on page 205 and [31]. The case $\beta=\beta_c$, presumably corresponding to SLE(4), is not covered by our analysis.

Since we are interested in random curves that are stochastically self-similar, it is natural to take h with such properties. Our choice for h is constructed by starting with the Gaussian random field X on the circle (see §3 for precise definitions) with covariance

$$\mathbb{E}X(z)X(z')=-\log|z-z'|, \tag{1}$$

where $z,z'\in\mathbb{C}$ have modulus 1. X is just the restriction of the 2-dimensional massless free field (Gaussian free field) on the circle. The exponential of βX gives rise to a random measure τ on the unit circle $\mathbb{T}$, formally given by

$$\text{“}d\tau=e^{\beta X(z)}\,dz\text{”}. \tag{2}$$

The proper definition involves a limiting process

$$\tau(dz)=\lim_{\varepsilon\to0^+}\frac{e^{\beta X_\varepsilon(z)}}{\mathbb{E}e^{\beta X_\varepsilon(z)}}\,dz,$$

where X_ε stands for a suitable regularization of X, see §3.3 below.

Identifying the circle as $\mathbb{T}=\mathbb{R}/\mathbb{Z}=[0,1)$, our random homeomorphism $h\colon[0,1)\to[0,1)$ is defined as

$$h(\theta)=\frac{\tau([0,\theta))}{\tau([0,1))}\quad\text{for } \theta\in[0,1). \tag{3}$$

The main result of this paper can then be summarized as follows.

For $\beta^2<2$ and almost surely in ω, formula (3) *defines a Hölder continuous circle homeomorphism, such that the welding problem has a solution Γ, where Γ is a Jordan curve bounding a domain $\Omega=f_+(\mathbb{D})$ with a Hölder continuous Riemann mapping f_+. For a given ω, the solution is unique up to a Möbius map of the plane. Moreover, the curve Γ is continuous in $\beta\in[0,\sqrt{2}\,)$.*

We refer to §5 (Theorems 5.2 and 5.3) for the exact statement of the main result.

With minor changes our method generalizes to the situation where(1) the random homeomorphism ϕ is replaced by $\phi_+\circ\phi_-^{-1}$, where ϕ_+ and ϕ_- are random circle homeomorphisms having the same distribution as ϕ with parameters β_+ and β_-, respectively, i.e. formally

$$d\phi_\pm \sim e^{\beta_\pm X(z)}dz.$$

In the case where $\phi_\pm$ are independent, we have the following result.

For every pair $\beta_+,\beta_-<\sqrt{2}$ and almost surely in ω, the welding problem for the homeomorphism $\phi_+\circ\phi_-^{-1}$ has a solution $\Gamma=\Gamma_{\beta_+,\beta_-}$, where Γ_{β_+,β_-} is a Jordan curve bounding the domains $\Omega_+=f_+(\mathbb{D})$ and $\Omega_-=f_-(\mathbb{C}\setminus\overline{\mathbb{D}})$, with Hölder continuous Riemann mappings $f_\pm$. For a given ω, the solution is unique up to a Möbius map of the plane and the curves Γ_{β_+,β_-} are continous in β_+ and β_-.

Apart from connection to SLE, the weldings constructed in this paper should be of interest to complex analysts as they form a natural family that degenerates as $\beta\uparrow\sqrt{2}$. It would be of great interest to understand the critical case $\beta=\sqrt{2}$ as well as the low temperature "spin glass phase" $\beta>\sqrt{2}$. It would also be of interest to understand the connection of our weldings to those arising from stochastic flows [2]. In [2] Hölder continuous homeomorphisms are considered, but the boundary behavior of the welding and hence its existence and uniqueness are left open.

In writing the paper we have tried to be generous in providing details on both the function-theoretic and the stochastics notions and tools needed, in order to serve readers with varied backgrounds. The structure of the paper is as follows. §2 contains background material on conformal welding and the geometric-analysis tools we need later on. To be more specific, §2 recalls the notion of conformal welding and explains how the welding problem is reduced to the study of the Beltrami equation. Also we recall a useful method due to Lehto [23] to prove the existence of a solution for a class of non-uniformly elliptic Beltrami equations, and a theorem by Jones and Smirnov [18] that will be used to verify the uniqueness of our welding. Finally we recall the Beurling–Ahlfors extension of circle homeomorphisms to the unit disc. For our purposes we need to estimate carefully the dependence of the dilatation of the extension in a Whitney cube by just using small amount of information of the homeomorphism on a 'shadow' of the cube.

In §3 we introduce the 1-dimensional trace of the Gaussian free field and recall some known properties of its exponential that we will use to define and study the random

(1) Heuristic arguments from Liouville quantum gravity suggest [11], [15] that there might be a more precise relation between SLE and the welding of the homeomorphism $\phi_+^{-1}\circ\phi_-$, which we do not consider here as it would require considerable changes in our argument.

circle homeomorphism. §4 is the technical core of the paper as it contains the main probabilistic estimates we need to control the random dilatation of the extension map. Finally, in §5 things are put together and the a.s. existence and uniqueness of the welding map is proven.

Let us conclude by a remark on notation. We denote by c and C generic constants which may vary between estimates. When the constants depend on parameters such as β we denote this by $C(\beta)$.

Acknowledgements. We thank M. Bauer, D. Bernard, I. Binder, M. Nikula, S. Rohde, S. Smirnov and W. Werner for useful discussions.

Note added in proof. After the manuscript was sent to publication, the works [3] and [31] have appeared. The paper [3] by Airault, Malliavin and Thalmayer continues the work on the welding of stochastic flows initiated in [2]. Sheffield's preprint [31] contains, among other things, confirmation for the close relation between welding of the homeomorphism $\phi_+^{-1}\circ\phi_-$ and SLE, see the footnote on page 205.

2. Conformal welding

In this section we recall for the general readers benefit basic notions and results from geometric analysis that are needed in our work. In particular, we recall the notion of conformal welding, Lehto's method for solving the Beltrami equations, the uniqueness result for weldings due to Jones and Smirnov, and the last subsection contains estimates for the Beurling–Ahlfors extension tailored for our needs.

2.1. Welding and Beltrami equation

One of the main methods for constructing conformally invariant families of (Jordan) curves comes from the theory of *conformal welding*. Put briefly, in this method we glue the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ and the exterior disc $\mathbb{D}_\infty=\{z\in\widehat{\mathbb{C}}:|z|>1\}$ along a homeomorphism $\phi:\mathbb{T}\to\mathbb{T}$, by the identification

$$x\sim y, \quad \text{when } x\in\mathbb{T}=\partial\mathbb{D} \text{ and } y=\phi(x)\in\mathbb{T}=\partial\mathbb{D}_\infty.$$

The problem of welding is to give a natural complex structure to this topological sphere. Uniformizing the complex structure then gives us the curve, the image of the unit circle.

More concretely, given a Jordan curve $\Gamma\subset\widehat{\mathbb{C}}$, let

$$f_+:\mathbb{D}\to\Omega_+ \quad \text{and} \quad f_-:\mathbb{D}_\infty\to\Omega_-$$

be a choice of Riemann mappings onto the components of the complement $\widehat{\mathbb{C}}\setminus\Gamma=\Omega_+\cup\Omega_-$. By Carathéodory's theorem, f_- and f_+ both extend continuously to $\partial\mathbb{D}=\partial\mathbb{D}_\infty$, and thus

$$\phi = f_+^{-1}\circ f_- \tag{4}$$

is a homeomorphism of $\mathbb{T}$. In the welding problem we are asked to invert this process; given a homeomorphism $\phi\colon\mathbb{T}\to\mathbb{T}$ we are to find a Jordan curve Γ and conformal mappings $f_\pm$ onto the complementary domains $\Omega_\pm$ so that (4) holds.

It is clear that the welding problem, when solvable, has natural conformal invariance attached to it; any image of the curve Γ under a Möbius transformation of $\widehat{\mathbb{C}}$ is equally a welding curve. Similarly, if $\phi\colon\mathbb{T}\to\mathbb{T}$ admits a welding, then so do all its compositions with Möbius transformations of the disc. Note, however, that not all circle homeomorphisms admit a welding, for examples see [26] and [34].

The most powerful tool in solving the welding problem is given by the Beltrami differential equation, defined in a domain Ω by

$$\frac{\partial f}{\partial \bar z} = \mu(z)\frac{\partial f}{\partial z} \quad \text{for a.e. } z\in\Omega, \tag{5}$$

where we look for homeomorphic solutions $f\in W^{1,1}_{\mathrm{loc}}(\Omega)$. Here (5) is an elliptic system whenever $|\mu(z)|<1$ almost everywhere, and uniformly elliptic if $\|\mu\|_\infty<1$.

In the uniformly elliptic case, homeomorphic solutions to (5) exist for every coefficient with $\|\mu\|_\infty<1$, and they are unique up to post-composing with a conformal mapping [5, p. 179]. In fact, it is this uniqueness property that gives us a way to produce the welding. To see this, suppose first that

$$\phi = f|_{\mathbb{T}}, \tag{6}$$

where $f\in W^{1,2}_{\mathrm{loc}}(\mathbb{D};\mathbb{D})\cap C(\overline{\mathbb{D}})$ is a homeomorphic solution to (5) in the disc $\mathbb{D}$. Find then a homeomorphic solution to the auxiliary equation

$$\frac{\partial F}{\partial \bar z} = \chi_{\mathbb{D}}(z)\,\mu(z)\frac{\partial F}{\partial z} \quad \text{for a.e. } z\in\mathbb{C}. \tag{7}$$

Now $\Gamma=F(\mathbb{T})$ is a Jordan curve. Moreover, as $\partial_{\bar z}F=0$ for $|z|>1$, we can set $f_-:=F|_{\mathbb{D}_\infty}$ and $\Omega_-:=F(\mathbb{D}_\infty)$ to define a conformal mapping

$$f_-\colon\mathbb{D}_\infty\to\Omega_-.$$

On the other hand, since both f and F solve the equation (5) in the unit disc $\mathbb{D}$, by uniqueness of the solutions we have

$$F(z) = f_+\circ f(z), \quad z\in\mathbb{D}, \tag{8}$$

for some conformal mapping $f_+:\mathbb{D}=f(\mathbb{D})\to\Omega_+:=F(\mathbb{D})$. Finally, on the unit circle,

$$\phi(z)=f|_{\mathbb{T}}(z)=f_+^{-1}\circ f_-(z), \quad z\in\mathbb{T}. \tag{9}$$

Thus we have found a solution to the welding problem, under the assumption (6). That the welding curve Γ is unique up to a Möbius transformation of $\mathbb{C}$ follows from [5, Theorem 5.10.1]; see also Corollary 2.5 below.

To complete this circle of ideas, we need to identify the homeomorphisms $\phi:\mathbb{T}\to\mathbb{T}$ that admit the representation (5), (6) with uniformly elliptic μ in (5). It turns out [5, Lemma 3.11.3 and Theorem 5.8.1] that such ϕ's are precisely the quasisymmetric mappings of $\mathbb{T}$, mappings that satisfy

$$K(\phi):=\sup_{s,t\in\mathbb{R}}\frac{|\phi(e^{2\pi i(s+t)})-\phi(e^{2\pi is})|}{|\phi(e^{2\pi i(s-t)})-\phi(e^{2\pi is})|}<\infty. \tag{10}$$

2.2. Existence in the degenerate case: the Lehto condition

The previous subsection describes an obvious model for constructing random Jordan curves, by first finding random homeomorphisms of the circle and then solving for each of them the associated welding problem. In the present work, however, we are faced with the obstruction that circle homeomorphisms with the exponentiated Gaussian free field as derivative almost surely do not satisfy the quasisymmetry assumption (10). Thus we are forced outside the uniformly elliptic partial differential equations and need to study (5) with degenerate coefficients with only $|\mu(z)|<1$ almost everywhere. We are even outside the much studied class of maps of exponentially integrable distortion, see [5, §20.4.] In such generality, however, the homeomorphic solutions to (5) may fail to exist, or the crucial uniqueness properties of (5) may similarly fail.

In his important work [23], Lehto gave a very general condition in the degenerate setting for the existence of homeomorphic solutions to (5). To recall his result, assume we are given the complex dilatation $\mu=\mu(z)$, and write then

$$K(z)=\frac{1+|\mu(z)|}{1-|\mu(z)|}, \quad z\in\Omega,$$

for the associated distortion function. Note that $K(z)$ is bounded precisely when the equation (5) is uniformly elliptic, i.e. $\|\mu\|_\infty<1$. Thus the question is how strongly $K(z)$ can grow for the basic properties of (5) still to remain true. In order to state Lehto's condition we fix some notation. For given $z\in\mathbb{C}$ and radii $0\leqslant r<R<\infty$, let us denote the corresponding annulus by

$$A(z,r,R):=\{w\in\mathbb{C}:r<|w-z|<R\}.$$

In the Lehto approach, one needs to control the conformal moduli of image annuli in a suitable way. This is done by introducing, for any annulus $A(w,r,R)$ and for the given distortion function K, the following quantity, which we call *the Lehto integral*:

$$L(z,r,R) := L_K(z,r,R) := \int_r^R \frac{1}{\int_0^{2\pi} K(z+\varrho e^{i\theta})\,d\theta}\,\frac{d\varrho}{\varrho}. \tag{11}$$

For the following formulation of Lehto's theorem see [5, p. 584].

THEOREM 2.1. *Suppose μ is measurable and compactly supported with $|\mu(z)|<1$ for almost every $z\in\mathbb{C}$. Assume that the distortion function*

$$K(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$$

is locally integrable, that is

$$K \in L^1_{\mathrm{loc}}(\mathbb{C}), \tag{12}$$

and that for some $R_0>0$ the Lehto integral satisfies

$$L_K(z,0,R_0) = \infty \quad \textit{for all } z\in\mathbb{C}. \tag{13}$$

Then the Beltrami equation

$$\frac{\partial f}{\partial \bar z}(z) = \mu(z)\frac{\partial f}{\partial z}(z) \quad \textit{for a.e. } z\in\mathbb{C} \tag{14}$$

admits a homeomorphic $W^{1,1}_{\mathrm{loc}}$-solution $f\colon\mathbb{C}\to\mathbb{C}$.

As a consequence, the welding extends beyond the class of quasisymmetric functions.

COROLLARY 2.2. *Suppose that $\phi\colon\mathbb{T}\to\mathbb{T}$ extends to a homeomorphism $f\colon\mathbb{C}\to\mathbb{C}$ satisfying* (12)–(14) *together with the condition*

$$K(z) \in L^\infty_{\mathrm{loc}}(\mathbb{D}). \tag{15}$$

Then ϕ admits a welding: there are a Jordan curve $\Gamma\subset\widehat{\mathbb{C}}$ and conformal mappings $f_\pm$ onto the complementary domains of Γ such that

$$\phi(z) = f_+^{-1}\circ f_-(z), \quad z\in\mathbb{T}.$$

Proof. Given the extension $f\colon\mathbb{C}\to\mathbb{C}$, let us again look at the auxiliary equation

$$\frac{\partial F}{\partial \bar z} = \chi_{\mathbb{D}}(z)\mu(z)\frac{\partial F}{\partial z} \quad \text{for a.e. } z\in\mathbb{C}. \tag{16}$$

Since Lehto's condition holds as well for the new distortion function

$$K(z)=\frac{1+|\chi_{\mathbb{D}}(z)\mu(z)|}{1-|\chi_{\mathbb{D}}(z)\mu(z)|},$$

we see from Theorem 2.1 that the auxiliary equation (16) admits a homeomorphic solution $F:\mathbb{C}\to\mathbb{C}$. Arguing as in (6)–(9), it will then be sufficient to show that

$$F(z)=f_+\circ f(z), \quad z\in\mathbb{D},$$

where f_+ is conformal in $\mathbb{D}$. But this is a local question; every point $z\in\mathbb{D}$ has a neighborhood where $K(z)$ is uniformly bounded, by (15). In such a neighborhood the usual uniqueness results to solutions of (5) apply; see [5, p. 179]. Thus f_+ is holomorphic, and as a homeomorphism it is conformal. This proves the claim. □

Consequently, in the study of random circle homeomorphisms $\phi=\phi_\omega$, a key step for the conformal welding of ϕ_ω will be to show that almost surely each such mapping admits a homeomorphic extension to $\mathbb{C}$, where the distortion function satisfies a condition such as (13). In our setting where the derivative of ϕ is given by the exponentiated trace of a Gaussian free field, the extension procedure is described in §2.4 and the appropriate estimates it requires are proven in §4.

Actually, in §5, when proving our main theorem, we need to present a variant of Lehto's argument where it will be enough to estimate the Lehto integral only at a suitable countable set of points $z\in\mathbb{T}$. We also utilize there the fact that the extension of our random circle homeomorphism ϕ satisfies (15). In verifying the Hölder continuity of the ensuing map, we shall apply a useful estimate (Lemma 2.3 below) that estimates the geometric distortion of an annulus under a quasiconformal map.

Given a bounded (topological) annulus $A\subset\mathbb{C}$, with E being the bounded component of $\mathbb{C}\setminus A$, we denote by $D_O(A):=\operatorname{diam}(A)$ the outer diameter, and by $D_I(A):=\operatorname{diam}(E)$ the inner diameter of A.

LEMMA 2.3. *Let f be a quasiconformal mapping on the annulus $A(w,r,R)$, with distortion function K_f. It then holds that*

$$\frac{D_O(f(A(w,r,R)))}{D_I(f(A(w,r,R)))}\geqslant\frac{1}{16}e^{2\pi^2L_{K_f}(w,r,R)}.$$

Proof. Recall first that for a rigid annulus $A=A(w,r,R)$, we define its conformal modulus by

$$\operatorname{mod}(A)=2\pi\log\frac{R}{r},$$

while for any topological annulus A, one sets

$$\operatorname{mod}(A) = \operatorname{mod}(g(A)),$$

where g is a conformal map of A onto a rigid annulus. Then we have [5, Corollary 20.9.2] the following basic estimate for the modulus of the image annulus in terms of the Lehto integrals:

$$\operatorname{mod}(f(A(w,r,R))) \geqslant 2\pi L_{K_f}(w,r,R). \tag{17}$$

On the other hand, by combining [35, Lemma 7.38 and Corollary 7.39] and [4, Exercise 5.68 (16)] we obtain for any bounded topological annulus $A \subset \mathbb{C}$,

$$\frac{1}{16} e^{\pi \operatorname{mod}(A)} \leqslant \frac{D_O(A)}{D_I(A)}.$$

Put together, the desired estimate follows. □

2.3. Uniqueness of the welding

An important issue of the welding is its uniqueness, that the curve Γ is unique up to composing with a Möbius transformation of $\widehat{\mathbb{C}}$. As the above argument indicates, this is essentially equivalent to the uniqueness of solutions to the appropriate Beltrami equations, up to a Möbius transformation. However, in general the assumptions of Theorem 2.1 alone are much too weak to imply this.

In fact, in our case the uniqueness of solutions to the Beltrami equation (16) is equivalent to the conformal removability of the curve $F(\mathbb{T})$. Recall that a compact set $B \subset \widehat{\mathbb{C}}$ is *conformally removable* if every homeomorphism of $\widehat{\mathbb{C}}$ which is conformal off B is conformal in the whole sphere, and hence a Möbius transformation.

It follows easily that e.g. images of circles under quasiconformal mappings, i.e. homeomorphisms satisfying (5), with $\|\mu\|_\infty < 1$, are conformally removable, while Jordan curves of positive area are never conformally removable.

For general curves the removability is a deep problem; no characterization of conformally removable Jordan curves is known to this date. What saves us in the present work is that we have the remarkable result of Jones and Smirnov in [18] available. We will not need their result in its full generality, as the following special case will be sufficient for our purposes.

THEOREM 2.4. (Jones–Smirnov [18]) *Let $\Omega \subset \widehat{\mathbb{C}}$ be a simply connected domain such that the Riemann mapping $\psi: \mathbb{D} \to \Omega$ is α-Hölder continuous for some $\alpha > 0$.*

Then the boundary $\partial\Omega$ is conformally removable.

Adapting this result to our setting, we obtain the following result.

COROLLARY 2.5. *Suppose that* $\phi\colon\mathbb{T}\to\mathbb{T}$ *is a homeomorphism that admits a welding*

$$\phi(z) = f_+^{-1}\circ f_-(z), \quad z\in\mathbb{T},$$

where $f_\pm$ *are conformal mappings of* $\mathbb{D}$ *and* $\mathbb{D}_\infty$, *respectively, onto complementary Jordan domains* $\Omega_\pm$.

Assume that f_- (*or* f_+) *is* α-*Hölder continuous on the boundary* $\partial\mathbb{D}_\infty=\mathbb{T}$. *Then the welding is unique*: *any other welding pair* (g_+, g_-) *of* ϕ *is of the form*

$$g_\pm = \Phi\circ f_\pm,$$

where $\Phi\colon\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ *is a Möbius transformation.*

Proof. Suppose we have Riemann mappings $g_\pm$ onto complementary Jordan domains such that

$$g_+^{-1}\circ g_-(z) = \phi(z) = f_+^{-1}\circ f_-(z), \quad z\in\mathbb{T}.$$

Then the formula

$$\Psi(z) = \begin{cases} g_+\circ f_+^{-1}(z), & \text{if } z\in f_+(\mathbb{D}),\\ g_-\circ f_-^{-1}(z), & \text{if } z\in f_-(\mathbb{D}_\infty)\end{cases}$$

defines a homeomorphism of $\widehat{\mathbb{C}}$ which is conformal outside $\Gamma=f_\pm(\mathbb{T})$. From the Jones–Smirnov theorem we see that Ψ extends conformally to the entire sphere, and thus it is a Möbius transformation. □

As we shall see in Theorem 5.1, for circle homeomorphisms ϕ with the exponentiated Gaussian free field as derivative, the solutions F to the auxiliary equation (16) will be Hölder continuous almost surely. Then $f_-=F|_{\mathbb{D}_\infty}$ is a Riemann mapping onto a complementary component of the welding curve of $\phi=\phi_\omega$. It follows that almost surely $\phi=\phi_\omega$ admits a welding curve $\Gamma=\Gamma_\omega$ which is unique, up to composing with a Möbius transformation.

2.4. Extension of the homeomorphism

In this section we discuss in detail suitable methods for extending homeomorphisms $\phi\colon\mathbb{T}\to\mathbb{T}$ to the unit disc; by reflecting across $\mathbb{T}$, the map then extends to $\mathbb{C}$. Extensions of homeomorphisms $h\colon\mathbb{R}\to\mathbb{R}$ of the real line are convenient to describe, and it is not difficult to find constructions that sufficiently well respect the conformally invariant features of h. Given a homeomorphism $\phi\colon\mathbb{T}\to\mathbb{T}$ on the circle, we hence represent it in the form

$$\phi(e^{2\pi ix}) = e^{2\pi ih(x)}, \tag{18}$$

where $h:\mathbb{R}\to\mathbb{R}$ is a homeomorphism of the line with $h(x+1)=h(x)+1$. We may assume that $\phi(1)=1$, with $h(0)=0$.

We will now extend the 1-periodic mapping h to the upper (or lower) half-plane so that it becomes the identity map at large height. Then a conjugation to a mapping of the disc is easily done. For the extension we use the classical Beurling–Ahlfors extension [10] modified suitably far away from the real axis.

Thus, given a homeomorphism $h:\mathbb{R}\to\mathbb{R}$ such that

$$h(x+1)=h(x)+1,\quad x\in\mathbb{R},\quad \text{with } h(0)=0, \tag{19}$$

we define our extension F as follows. For $0<y<1$ let

$$F(x+iy)=\frac{1}{2}\int_0^1 (h(x+ty)+h(x-ty))\,dt+i\int_0^1 (h(x+ty)-h(x-ty))\,dt. \tag{20}$$

Then $F=h$ on the real axis, and F is a continuously differentiable homeomorphism.

Moreover, by (19), it follows that for $y=1$,

$$F(x+i)=x+i+c_0,$$

where $c_0=\int_0^1 h(t)\,dt-\frac{1}{2}\in\left[-\frac{1}{2},\frac{1}{2}\right]$. Thus, for $1\leqslant y\leqslant 2$, we set

$$F(z)=z+(2-y)c_0, \tag{21}$$

and finally have an extension of h with the extra properties

$$F(z)\equiv z \qquad \text{when } y=\operatorname{Im} m(z)\geqslant 2, \tag{22}$$

$$F(z+k)=F(z)+k,\quad k\in\mathbb{Z}. \tag{23}$$

The original circle mapping admits a natural extension to the disc,

$$\Psi(z)=\exp\left(2\pi i F\left(\frac{\log z}{2\pi i}\right)\right),\quad z\in\mathbb{D}. \tag{24}$$

From (18) and (23) we see that this is a well-defined homeomorphism of the disc with $\Psi|_{\mathbb{T}}=\phi$ and $\Psi(z)\equiv z$ for $|z|\leqslant e^{-4\pi}$. The distortion properties are not altered under this locally conformal change of variables,

$$K(z,\Psi)=K(w,F),\quad z=e^{2\pi i w},\ w\in\mathbb{H}, \tag{25}$$

so we will reduce all distortion estimates for Ψ to the corresponding ones for F. Since F is conformal for $y>2$, it suffices to restrict the analysis to the strip

$$S=\mathbb{R}\times[0,2]. \tag{26}$$

To estimate $K(w,F)$ we introduce some notation. Let

$$\mathcal{D}_n=\{[k2^{-n},(k+1)2^{-n}]:k\in\mathbb{Z}\}$$

be the set of all dyadic intervals of length 2^{-n} and write

$$\mathcal{D}=\{\mathcal{D}_n:n\geqslant 0\}.$$

Consider the measure

$$\tau([a,b])=h(b)-h(a).$$

For a pair of intervals $\mathbf{J}=\{J_1,J_2\}$ let us introduce the following quantity

$$\delta_\tau(\mathbf{J})=\frac{\tau(J_1)}{\tau(J_2)}+\frac{\tau(J_2)}{\tau(J_1)}. \tag{27}$$

If J_1 and J_2 are the two halves of an interval I, then $\delta_\tau(\mathbf{J})$ measures the local doubling properties of the measure τ. In such a case we define $\delta_\tau(I)=\delta_\tau(\mathbf{J})$. In particular, (10) holds for the circle homeomorphism $\phi(e^{2\pi ix})=e^{2\pi ih(x)}$ if and only if the quantities $\delta_\tau(I)$ are uniformly bounded, for all (not necessarily dyadic) intervals I.

The local distortion of the extension F will be controlled by sums of the expressions $\delta_\tau(\mathbf{J})$ in the appropriate scale. For this, let us pave the strip S by Whitney cubes $\{C_I\}_{I\in\mathcal{D}}$ defined by

$$C_I=\{(x,y):x\in I \text{ and } 2^{-n-1}\leqslant y\leqslant 2^{-n}\}$$

for $I\in\mathcal{D}_n$, $n>0$, and $C_I=I\times\left[\frac{1}{2},2\right]$ for $I\in\mathcal{D}_0$. Given an $I\in\mathcal{D}_n$ let $j(I)$ be the union of I and its neighbors in $\mathcal{D}_n$ and

$$\mathcal{J}(I):=\{\mathbf{J}=(J_1,J_2):J_1,J_2\in\mathcal{D}_{n+5} \text{ and } J_1,J_2\subset j(I)\}. \tag{28}$$

We then define

$$K_\tau(I):=\sum_{\mathbf{J}\in\mathcal{J}(I)}\delta_\tau(\mathbf{J}). \tag{29}$$

With these notions we have the basic geometric estimate for the distortion function, in terms of the boundary homeomorphism.

THEOREM 2.6. *Let $h:\mathbb{R}\to\mathbb{R}$ be a 1-periodic homeomorphism and let $F:\mathbb{H}\to\mathbb{H}$ be its extension. Then, for each $I\in\mathcal{D}$,*

$$\sup_{z\in C_I}K(z,F)\leqslant C_0K_\tau(I), \tag{30}$$

with a universal constant C_0.

Proof. The distortion properties of the Beurling–Ahlfors extension are well studied in the existing litterature, but none of these works gives directly Theorem 2.6, as the main point for us is the linear dependence on the local distortion $K_\tau(I)$. The most elementary extension operator is due to Jerison and Kenig [17] (see also [5, §5.8]), but for this extension the linear dependence fails.

For the reader's convenience we sketch a proof of the theorem. We will modify the approach of Reed [27], and start with a simple lemma.

LEMMA 2.7. *For each dyadic interval*

$$I=[k2^{-n},(k+1)2^{-n}],$$

with left half $I_1=\left[k2^{-n},\left(k+\frac{1}{2}\right)2^{-n}\right]$ *and right half* $I_2=\overline{I\setminus I_1}$, *we have*

$$\frac{1}{1+\delta_\tau(I)}|\tau(I)|\leqslant|\tau(I_j)|\leqslant\frac{\delta_\tau(I)}{1+\delta_\tau(I)}|\tau(I)|,$$

$j=1,2$, *with*

$$\frac{1}{|I|}\int_I(h(t)-h(k2^{-n}))\,dt\leqslant\frac{3\delta_\tau(I)}{1+3\delta_\tau(I)}|\tau(I)|$$

and

$$\frac{1}{|I|}\int_I(h((k+1)2^{-n})-h(t))\,dt\leqslant\frac{3\delta_\tau(I)}{1+3\delta_\tau(I)}|\tau(I)|.$$

Proof. The definition of $\delta_\tau(I)$ gives the first estimate. As $h(t)\leqslant h\left(\left(k+\frac{1}{2}\right)2^{-n}\right)$ on the left half and $h(t)\leqslant h((k+1)2^{-n})$ on the right half of I, we have

$$\frac{1}{|I|}\int_I(h(t)-h(k2^{-n}))\,dt\leqslant\left(\frac{1}{2}\frac{\delta_\tau(I)}{1+\delta_\tau(I)}+\frac{1}{2}\right)|\tau(I)|\leqslant\frac{3\delta_\tau(I)}{1+3\delta_\tau(I)}|\tau(I)|.$$

The last estimate follows similarly. □

To continue with the proof of Theorem 2.6, the pointwise distortion of the extension F is easy to calculate explicitly, and we obtain [10], [27] the following estimate, sharp up to a multiplicative constant,

$$K(x+iy,F)\leqslant\left(\frac{\alpha(x,y)}{\beta(x,y)}+\frac{\beta(x,y)}{\alpha(x,y)}\right)\left[\frac{\tilde{\alpha}(x,y)}{\alpha(x,y)}+\frac{\tilde{\beta}(x,y)}{\beta(x,y)}\right]^{-1},\tag{31}$$

where

$$\alpha(x,y)=h(x+y)-h(x),\quad \beta(x,y)=h(x)-h(x-y)$$

and

$$\tilde{\alpha}(x,y)=h(x+y)-\frac{1}{y}\int_x^{x+y}h(t)\,dt,\quad \tilde{\beta}(x,y)=\frac{1}{y}\int_{x-y}^{x}h(t)\,dt-h(x-y).$$

Now the argument of Reed [27, pp. 461–464], combined with Lemma 2.7 and its estimates, precisely shows that $K(x+iy,F)\leqslant 24\max\delta_\tau(\tilde I)$, where $\tilde I$ runs over the intervals with endpoints contained in the set

$$\big\{x, x\pm\tfrac14 y, x\pm\tfrac12 y, x\pm y\big\}. \tag{32}$$

Thus, for example, if we fix $k\in\mathbb{Z}$ and $n\in\mathbb{N}$, we get for the corner point $z=k2^{-n}+i2^{-n}$ of the Whitney cube C_I the estimate

$$K(k2^{-n}+i2^{-n},F)\leqslant 24\sum_{\mathbf{J}}\delta_\tau(\mathbf{J}),\quad \mathbf{J}=(J_1,J_2),\ J_1,J_2\in\mathcal{D}_{n+3}, \tag{33}$$

where $J_1,J_2\subset j(I)$ as above. For a general point $z=x+iy\in C_I$, we have to take a few more generations of dyadic intervals. Here $\big[x,x+\frac14 y\big]$ has length at least 2^{-n-3}. On the other hand, for any (non-dyadic) interval $\tilde I$ with $2^{-m}\leqslant|\tilde I|<2^{-m+1}$, one observes that it contains a dyadic interval of length 2^{-m-1} and is contained inside a union of at most three dyadic intervals of length 2^{-m}. By this manner, one estimates

$$\delta_\tau(\tilde I)\leqslant\sum_{\mathbf{J}}\delta_\tau(\mathbf{J}),\quad \text{where } \mathbf{J}=(J_1,J_2),\ J_k\in\mathcal{D}_{m+2} \text{ and } J_k\cap\tilde I\neq\varnothing,\ k=1,2.$$

Choosing the endpoints of $\tilde I$ from the set in (32) then gives the bound (30). Note that the estimates hold also for $n=0$, since by (21) we have $K(z,F)\leqslant\frac54$ whenever $y\geqslant 1$. Hence the proof of Theorem 2.6 is complete. □

3. Exponential of GFF and random homeomorphisms of $\mathbb{T}$

3.1. Trace of the Gaussian free field

Let us recall that the 2-dimensional Gaussian free field (in other words, the massless free field) Y in the plane has the covariance

$$\mathbb{E}Y(x)Y(x')=\log\frac{1}{|x-x'|},\quad x,x'\in\mathbb{R}^2.$$

Actually, the definition of this field in the whole plane has to be done carefully, because of the blowup of the logarithm at infinity. However, the definition of the trace $X:=Y|_{\mathbb{T}}$ on the unit circle $\mathbb{T}$ avoids this problem, since it is formally obtained by requiring (in the convenient complex notation) that

$$\mathbb{E}X(z)X(z')=\log\frac{1}{|z-z'|},\quad z,z'\in\mathbb{T}. \tag{34}$$

The above definition needs to be made precise. In order to serve also readers with less background in non-smooth stochastic fields, let us first recall the definition of Gaussian random variables with values in the space of distributions $\mathcal{D}'(\mathbb{T})$. An element in $F\in\mathcal{D}'(\mathbb{T})$ is real-valued if it takes real values on real-valued test functions. Identifying $\mathbb{T}$ with $[0,1)$, a real-valued F may be written as

$$F=a_0+\sum_{n=1}^{\infty}(a_n\cos 2\pi nt+b_n\sin 2\pi nt),$$

with real coefficients satisfying $|a_n|,|b_n|=O(n^a)$ for some $a\in\mathbb{R}$. Conversely, every such Fourier series converges in $\mathcal{D}'(\mathbb{T})$.

Let $(\Omega,\mathcal{F},\mathbb{P})$ stand for a probability space. A map $X:\Omega\to\mathcal{D}'(\mathbb{T})$ is a (real-valued) centered $\mathcal{D}'(\mathbb{T})$-valued Gaussian if for every (real-valued) $\psi\in C_0^\infty(\mathbb{T})$ the map

$$\omega\longmapsto\langle X(\omega),\psi\rangle$$

is a centered Gaussian on Ω. Here $\langle\cdot,\cdot\rangle$ refers to the standard distributional duality. Alternatively, one may define such a random variable by requiring that a.s.

$$X(\omega)=A_0(\omega)+\sum_{n=1}^{\infty}(A_N(\omega)\cos 2\pi nt+B_n(\omega)\sin 2\pi nt),$$

where A_n and B_n are centered Gaussians satisfying $\mathbb{E}A_n^2,\mathbb{E}B_n^2=O(n^a)$ for some $a\in\mathbb{R}$. The random variable X is stationary if and only if the coefficients $A_0,A_1,...,B_1,B_2,...$ are independent.

Due to Gaussianity, the distribution of X is uniquely determined by the knowledge of the covariance operator $C_X:C^\infty(\mathbb{T})\to\mathcal{D}'(\mathbb{T})$, where

$$\langle C_X\psi_1,\psi_2\rangle:=\mathbb{E}\langle X(\omega),\psi_1\rangle\langle X(\omega),\psi_2\rangle.$$

In case the covariance operator has an integral kernel, we use the same symbol for the kernel, and in this case for almost every $z\in\mathbb{T}$ one has

$$(C_X\psi)(z)=\int_{\mathbb{T}}C_X(z,w)\psi(w)m(dw),$$

where m stands for the normalized Lebesgue measure on $\mathbb{T}$. Most of the above definitions and statements carry over directly on $\mathcal{S}'(\mathbb{R})$-valued random variables, but the above knowledge is enough for our purposes.

The exact definition of (34) is understood in the above sense.

Definition 3.1. The trace X of the 2-dimensional GFF (Gaussian free field) on $\mathbb{T}$ is a centered $\mathcal{D}'(\mathbb{T})$-valued Gaussian random variable such that its covariance operator has the integral kernel

$$C_X(z,z') = \log\frac{1}{|z-z'|}, \quad z,z'\in\mathbb{T}.$$

Observe that in the identification $\mathbb{T}=[0,1)$ the covariance of X takes the form

$$C_X(t,u) = \log\frac{1}{2\sin\pi|t-u|} \quad \text{for } t,u\in[0,1). \tag{35}$$

The existence of such a field is most easily established by writing down the Fourier expansion:

$$X = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(A_n\cos 2\pi nt + B_n\sin 2\pi nt), \quad t\in[0,1), \tag{36}$$

where all the coefficients $A_n, B_n\sim N(0,1)$, $n\geqslant 1$, are independent standard Gaussians. Writing X as

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(\alpha_n z^n + \bar{\alpha}_n\bar{z}^n)$$

with $|z|=1$ and $\alpha=\frac{1}{2}(A+iB)$, it is readily checked that it has the stated covariance.

What makes the trace X of the 2-dimensional GFF particularly natural for the circle homeomorphisms is its invariance properties, that X is Möbius invariant *modulo constants*. To see this, note that the covariance $C(z,z')=\log(1/|z-z'|)$ satisfies the transformation rule

$$C(g(z),g(z')) = C(z,z') + A(z) + B(z'),$$

where A (resp. B) is independent of z' (resp. z), whence the last two terms vanish in integration against mean-zero test-functions.

It is well known that, with probability 1, $X(\omega)$ is not an element in $L^1(\mathbb{T})$ (or a measure on $\mathbb{T}$), but it just barely fails to be a function-valued field. Namely, if $\varepsilon>0$ and one considers the ε-smoothened field $(1-\Delta)^{-\varepsilon}X$, one computes that this field has a Hölder-continuous covariance, whence its realization belongs to $C(\mathbb{T})$ almost surely. This follows from the following fundamental result of Dudley which we will use repeatedly.

THEOREM 3.2. *Let $(Y_t)_{t\in T}$ be a centered Gaussian field indexed by the set T, where T is a compact metric space with distance d. Define the* (*pseudo*)*distance d' on T by setting $d'(t_1,t_2)=(\mathbb{E}|Y_{t_1}-Y_{t_2}|^2)^{1/2}$ for $t_1,t_2\in T$. Assume that $d'\colon T\times T\to\mathbb{R}$ is continuous. For $\delta>0$ denote by $N(\delta)$ the minimal number of balls of radius δ in the d'-metric needed to cover T. If*

$$\int_0^1\sqrt{\log N(\delta)}\,d\delta<\infty, \tag{37}$$

then Y has a continuous version, i.e. almost surely the map $T\ni t\mapsto Y_t$ is continuous.

For a proof we refer to [1, Theorem 1.3.5] or [19, Chapter 15, Theorem 4]. The second result we will need is an inequality due to C. Borell and, independently, to B. Tsirelson, I. Ibragimov and V. Sudakov. According to the inequality, the tail of the supremum is dominated by a Gaussian tail:

$$\mathbb{P}\Big(\sup_{t\in T}|Y_t|>u\Big)\leqslant Ae^{Bu-u^2/2\sigma_T^2}, \tag{38}$$

where $\sigma_T:=\max_{t\in T}(\mathbb{E}Y_t^2)^{1/2}$, and the constants A and B depend on (T,d'), see [1, §2.1]. We shall also need an explicit quantitative version of this inequality in the special case where T is an interval.

LEMMA 3.3. *Let $T=[x_0,x_0+\ell]$, and suppose that the covariance is Lipschitz continuous with constant L, i.e. $\mathbb{E}|Y_t-Y_{t'}|^2\leqslant L|t-t'|$ for $t,t'\in T$. Assume also that $Y_{t_0}\equiv 0$ for a $t_0\in T$. Then*

$$\mathbb{P}\Big(\sup_{t\in T}|Y_t|>\sqrt{L\ell}u\Big)\leqslant c(1+u)e^{-u^2/2},$$

where c is a universal constant.

Proof. The result is essentially due to Samorodnitsky [28] and Talagrand [33]. It is a direct consequence of [1, Theorem 4.1.2], since after scaling it is possible to assume that $L=1=\ell$, and then $\sigma_T\leqslant 1$ and $N(\varepsilon)\leqslant 1/\varepsilon^2$. □

3.2. White noise expansion

The Fourier series expansion (36) is often not the most suitable representation of X for explicit calculations. Instead, we shall apply a representation that uses white noise in the upper half-plane, due to Bacry and Muzy [6]. The white noise representation is very convenient, since it allows one to consider correlation between different scales both on the stochastic side and on $\mathbb{T}$ in a flexible and geometrically transparent manner. Moreover, as we define the exponential of the field X in the next subsection we are then able to refer to known results in [6] and elsewhere.

To commence with, let λ stand for the hyperbolic area measure in the upper half-plane $\mathbb{H}$,

$$\lambda(dx\,dy)=\frac{dx\,dy}{y^2}.$$

Denote by w a white noise in $\mathbb{H}$ with respect to the measure λ. More precisely, w is a centered Gaussian process indexed by Borel sets $A\in\mathcal{B}_f(\mathbb{H})$, where

$$\mathcal{B}_f(\mathbb{H}):=\Big\{A\subset\mathbb{H}\ \text{Borel}:\lambda(A)<\infty\ \text{and}\sup_{(x,y),(x',y')\in A}|x'-x|<\infty\Big\},$$

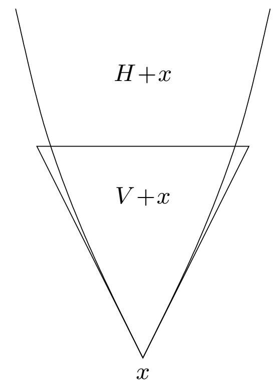

Figure 1. White noise dependence of the fields $H(x)$ and $V(x)$.

i.e. Borel sets of finite hyperbolic area and finite width, and with the covariance structure

$$\mathbb{E}(w(A_1)w(A_2)) = \lambda(A_1 \cap A_2), \quad A_1, A_2 \in \mathcal{B}_f(\mathbb{H}).$$

We shall need a periodic version of w, which can be identified with a white noise on $\mathbb{T}\times\mathbb{R}_+$. Thus, define W as the centered Gaussian process, also indexed by $\mathcal{B}_f(\mathbb{H})$, and with covariance

$$\mathbb{E}(W(A_1)W(A_2)) = \lambda\biggl(A_1 \cap \bigcup_{n\in\mathbb{Z}} (A_2+n)\biggr).$$

We will represent the trace X using the following random field $H(x)$. Consider the wedge shaped region

$$H := \left\{ (x,y) \in \mathbb{H} : -\frac{1}{2} < x < \frac{1}{2} \text{ and } y > \frac{2}{\pi} \tan|\pi x| \right\}$$

and formally set

$$H(x) := W(H+x), \quad x \in \mathbb{R}/\mathbb{Z};$$

see Figure 1. The reader should think about the y-axis as parametrizing the spatial scale. Roughly, the white noise at level y contributes to $H(x)$ in that spatial scale.

To define H rigorously we introduce a short distance cutoff parameter $\varepsilon>0$ and, given any $A\in\mathcal{B}_f(\mathbb{H})$, let $A_\varepsilon := \{(x,y)\in A : y>\varepsilon\}$. Then set

$$H_\varepsilon(x) := W(H_\varepsilon + x). \tag{39}$$

According to Dudley's Theorem 3.2, one may pick a version of the white noise W in such a way that the map

$$(0,1)\times\mathbb{R} \ni (\varepsilon, x) \longmapsto H_\varepsilon(x)$$

is continuous. In the limit $\varepsilon\to 0^+$, we nicely recover X.

LEMMA 3.4. *One may assume that the version of the white noise is chosen so that for any* $\varepsilon>0$ *the map* $x\mapsto H_\varepsilon(x)$ *is continuous, and as* $\varepsilon\to0^+$ *it converges in* $\mathcal{D}'(\mathbb{T})$ *to a random field* H. *Moreover*

$$H\sim X+G,$$

where $G\sim N(0,2\log 2)$ *is a (scalar) Gaussian factor, independent of* X.

Proof. Observe first that we may compute formally (as $H(\cdot)$ is not well defined pointwise) for $t\in(0,1)$,

$$\mathbb{E}H(0)H(t)=\lambda(H\cap(H+t))+\lambda(H\cap(H+t-1)).$$

The first term in the right-hand side can be computed as

$$\lambda(H\cap(H+t))=2\int_{t/2}^{1/2}\biggl(\int_{2(\tan\pi x)/\pi}^{\infty}\frac{dy}{y^2}\biggr)dx=\pi\int_{t/2}^{1/2}(\cot\pi x)\,dx=\log\frac{1}{\sin\frac{1}{2}\pi t}.$$

Hence we obtain by symmetry that

$$\mathbb{E}H(0)H(t)=\log\frac{1}{\sin\frac{1}{2}\pi t}+\log\frac{1}{\sin\frac{1}{2}\pi(1-t)}=2\log 2+\log\frac{1}{2\sin\pi t}. \tag{40}$$

The stated relation between X and H follows immediately from this as soon as we prove the rest of the theorem. Observe that the covariance of the smooth field $H_\varepsilon(\cdot)$ on $\mathbb{T}$ converges to the above pointwise for $t\neq0$. A computation (that e.g. applies the fact that the singularity of the kernel of the operator $(1-\Delta)^{-\delta}$ is of order $|x-y|^{2\delta-1}$) shows that for any $\delta>0$ the covariance of the field

$$[0,1]\times[0,1)\ni(\varepsilon,x)\longmapsto(1-\Delta)^{-\delta}H_\varepsilon(x):=H_{\varepsilon,\delta}(x)$$

(at $\varepsilon=0$ one applies the covariance computed in (40)) is Hölder continuous on the compact set $[0,1]\times\mathbb{T}$, whence Dudley's theorem yields the existence of a continuous version on that set, especially $H_{\varepsilon,\delta}(\cdot)\to H_{0,\delta}(\cdot)$ in $C(\mathbb{T})$, and hence in $\mathcal{D}'(\mathbb{T})$. By applying $(1-\Delta)^{\delta}$ on both sides we obtain the stated convergence. Especially, we see that the convergence takes place in any of the Zygmund spaces $C^{-\delta}(\mathbb{T})$, with $\delta>0$. □

The logarithmic singularity in the covariance of $H(x)$ is produced by the asymptotic shape of the region H near the real axis. It will often be convenient to work with the following auxiliary field, which is geometrically slightly easier to tackle while for small scales it does not distinguish between w and its periodic counterpart W. Thus, consider this time the triangular set

$$V:=\bigl\{(x,y)\in\mathbb{H}:-\tfrac{1}{4}<x<\tfrac{1}{4}\text{ and }2|x|<y<\tfrac{1}{2}\bigr\}, \tag{41}$$

and let $V_\varepsilon(x)=W(V_\varepsilon+x)$ (see Figure 1). The existence of the limit $V(x):=\lim_{\varepsilon\to 0^+} V_\varepsilon(\cdot)$ is established just like for H, and we get the covariance

$$\mathbb{E}V(x)V(x') = \log\frac{1}{2|x-x'|}+2|x-x'|-1 \tag{42}$$

for $|x-x'|\leqslant\frac{1}{2}$ (while for $|x-x'|>\frac{1}{2}$ the periodicity must be taken into account).

Since the regions H and V have the same slope at the real axis, the difference $H(\cdot)-V(\cdot)$ is a quite regular field.

LEMMA 3.5. *Set*

$$\xi := \sup_{\substack{x\in[0,1)\\ \varepsilon\in(0,1/2]}} |V_\varepsilon(x)-H_\varepsilon(x)|.$$

Then, almost surely, $\xi<\infty$. *Moreover,* $\mathbb{E}e^{a\xi}<\infty$ *for all* $a>0$.

Proof. We may write for $\varepsilon\in\left[0,\frac{1}{2}\right]$,

$$V_\varepsilon(x)-H_\varepsilon(x)=T_\varepsilon(x)-G(x),$$

where $T_\varepsilon(x)$ and $G(x)$ are constructed as $V_\varepsilon(x)$ out of the sets

$$G:=\left\{(x,y)\in H: y\geqslant\tfrac{1}{2}\right\} \quad\text{and}\quad T:=V\setminus\left\{(x,y)\in H: y<\tfrac{1}{2}\right\}.$$

Observe first that $G(x)$ is independent of ε and it clearly has a Lipschitz covariance in x. Thus, by Dudley's theorem and (38), almost surely the map $G(\cdot)\in C(\mathbb{T})$ and, moreover, the tail of $\|G(\cdot)\|_{C(\mathbb{T})}$ is dominated by a Gaussian, whence its exponential moments are finite.

In a similar manner, the exponential integrability of

$$\sup_{\substack{x\in[0,1)\\ \varepsilon\in[0,1]}} |T_\varepsilon(x)|$$

is deduced from Dudley's theorem and (38) as soon as we verify that there is an exponent $\alpha>0$ such that for any $|x-x'|\leqslant\frac{1}{2}$ we have

$$\mathbb{E}|T_\varepsilon(x)-T_{\varepsilon'}(x')|^2\leqslant c(|x-x'|+|\varepsilon-\varepsilon'|)^\alpha. \tag{43}$$

In order to verify this it is enough to change one variable at a time. Observe first that if $1>\varepsilon>\varepsilon'\geqslant 0$, then

$$\mathbb{E}|T_\varepsilon(x)-T_{\varepsilon'}(x)|^2=\lambda(\{(x,y)\in T:\varepsilon'<y<\varepsilon\})\leqslant\int_{\varepsilon'}^{\varepsilon} cx^3\,dx\leqslant c'|\varepsilon'-\varepsilon|,$$

where we applied the inequality

$$0 \leqslant \frac{t}{2} - \frac{\arctan \frac{1}{2}\pi t}{\pi} \leqslant 2t^3.$$

Next we estimate the dependence on x. Set $z := |x-x'| \leqslant \frac{1}{2}$. We note that for any $y_0 \in \left(0, \frac{1}{2}\right)$ the linear measure of the intersection $\{(x,y): y=y_0\} \cap (T\Delta(T+z))$ is bounded by $\min\{2z, 4y_0^3\}$. Hence, by the definition of T_ε and the fact that for $z=|x-x'|\leqslant\frac{1}{2}$ the periodicity of W has no effect on estimating T, we obtain

$$\begin{aligned}\mathbb{E}|T_\varepsilon(x)-T_\varepsilon(x')|^2 &\leqslant \mathbb{E}|T_0(x)-T_0(x')|^2 = \lambda(T\Delta(T+z)) \\ &\leqslant 2z\int_{z^{1/3}}^{1/2} \frac{dy}{y^2} + \int_0^{z^{1/3}} \frac{4y^3}{y^2}\,dy \leqslant cz^{2/3},\end{aligned}$$

which finishes the proof of the lemma. □

3.3. Exponential of X and the random homeomorphism h

We are now ready to define the exponential of the free field discussed in the introduction and use it to define the random circle homeomorphisms.

By stationarity, the covariance

$$\gamma_H(\varepsilon) := \operatorname{Cov}(H_\varepsilon(x)) = \mathbb{E}|H_\varepsilon(x)|^2$$

is independent of x, as is the quantity $\gamma_V(\varepsilon)$ defined analogously. Fix $\beta>0$ (this parameter could be thought of as an "inverse temperature"). Directly from the definitions, for any x and any bounded Borel function g on $[0,1)$, the processes

$$\varepsilon \longmapsto e^{\beta H_\varepsilon(x)-\beta^2\gamma_H(\varepsilon)/2}, \tag{44}$$

$$\varepsilon \longmapsto \int_0^1 e^{\beta H_\varepsilon(u)-\beta^2\gamma_H(\varepsilon)/2} g(u)\,du \tag{45}$$

are L^1-martingales with respect to *decreasing* $\varepsilon \in \left(0, \frac{1}{2}\right]$, whence they converge almost surely. Especially, the L^1-norm stays bounded and the Fourier coefficients of the density $e^{\beta H_\varepsilon(x)-\beta^2\gamma_H(\varepsilon)/2}$ converge as $\varepsilon\to 0^+$.

Now comparing these expressions with (2) and Lemma 3.4, we are led to the exact definition of our desired exponential

$$\text{“}d\tau = e^{\beta X(z)}\,dz\text{”}.$$

Indeed, by the weak*-compactness of the set of bounded positive measures, we have the existence of the almost sure limit measure([2])

$$\text{a.s.} \quad \lim_{\varepsilon\to 0^+} e^{\beta H_\varepsilon(x)-\beta^2\gamma_H(\varepsilon)/2} e^{-\beta G}\, \frac{dx}{2^{\beta^2}} =: \tau(dx) \quad \text{w}^* \text{ in } \mathcal{M}(\mathbb{T}), \tag{46}$$

where $\mathcal{M}(\mathbb{T})$ stands for bounded Borel measures on $\mathbb{T}$ and $G \sim N(0, 2\log 2)$ is a Gaussian (scalar) random variable.

In a similar manner one deduces the existence of the almost sure limit

$$\lim_{\varepsilon\to 0^+} e^{\beta V_\varepsilon(x)-\beta^2\gamma_V(\varepsilon)/2}\, dx \overset{\text{w}^*}{=}: \nu(dx). \tag{47}$$

Lemma 3.5 and stationarity immediately yield the following result.

LEMMA 3.6. *There are versions of τ and ν on a common probability space, together with an almost surely finite and positive random variable G_1, with $\mathbb{E}G_1^a<\infty$ for all $a\in\mathbb{R}$, so that for all Borel sets B one has*

$$\frac{1}{G_1}\tau(B) \leqslant \nu(B) \leqslant G_1 \tau(B).$$

Observe that the random variable G_1 is independent of the set B. Thus, the measures are a.s. comparable.

Limit measures of above type, i.e. measures that are obtained as martingale limits of products (discrete, or continuous as in our case) of exponentials of independent Gaussian fields have been extensively studied in the literature. The study of "multiplicative chaos" starts with Kolmogorov and Yaglom, various versions of multiplicative cascade models were advocated by Mandelbrot [24] and others, and Kahane (also together with Peyrière) made fundamental contributions to the rigorous mathematical theory, see [20], [21] and [22]. We shall make use of these works, and [6], in particular, which study in detail random measures closely related to our ν. We refer the reader to the papers of Barral and Mandelbrot [7]–[9] for a thorough treatment of multifractal measures in terms of the hidden cascade like structure.

For us the key points in constructing and understanding the random circle homeomorphism are the following properties of the measure τ and its variant ν.

([2]) Observe that the limit measure is weak*-measurable in the sense that for any $f\in C(\mathbb{T})$ the integral $\int_{\mathbb{T}} f(t)\,\tau(dt)$ is a well-defined random variable. In this paper all our random measures on $\mathbb{T}$ are measurable (i.e. they are measure-valued random variables) in this sense. A simple limiting argument then shows that e.g. $\tau(I)$ is a random variable for any interval $I\subset\mathbb{T}$.

THEOREM 3.7. (i) *Assume that* $\beta<\sqrt{2}$. *There are* $a_1=a_1(\beta)$, $a_2=a_2(\beta)>0$ *and an almost surely finite random constant* $c=c(\omega,\beta)>0$ *such that for all subintervals* $I\subset[0,1)$ *we have*

$$\frac{1}{c(\omega,\beta)}|I|^{a_1}\leqslant\tau(I)\leqslant c(\omega,\beta)|I|^{a_2}.$$

Especially, almost surely τ *is non-atomic and non-trivial on every subinterval.*

(ii) *Assume that* $\beta<\sqrt{2}$. *Then for every subinterval* $I\subset[0,1)$ *the measure* τ *satisfies*

$$\tau(I)\in L^p(\omega),\quad p\in\left(-\infty,\frac{2}{\beta^2}\right). \tag{48}$$

(iii) *Let* $p\in(1,2)$ *be fixed and set*

$$D_p:=\left\{\beta=\beta_1+i\beta_2:\frac{p}{2}\beta_1^2+\frac{p}{2(p-1)}\beta_2^2<1\right\}.$$

Then there is a version of τ *such that almost surely for every subinterval* $I\subset[0,1)$ *the map* $\beta\mapsto\tau(I)$ *extends to an analytic function in* D_p *with the moment bound*

$$\mathbb{E}|\tau(I)|^p\leqslant c(S)|I|^{\zeta_p(\beta)}\quad \textit{for } \beta\in S, \tag{49}$$

where $S\subset D_p$ *is any compact subset. Here the* (*complex*) *multifractal spectrum is given by the function*

$$\zeta_p(\beta):=p-\tfrac{1}{2}((p^2-p)\beta_1^2+p\beta_2^2)>1\quad \textit{for } \beta=\beta_1+i\beta_2\in D_p.$$

(iv) *One can replace* τ *by the measure* ν *in the statements* (i)–(iii).

Proof. We shall make use of one more auxiliary field, which (together with its exponential) is described in detail in [6].(3) Define

$$U:=\left\{(x,y)\in\mathbb{H}:-\tfrac{1}{2}<x<\tfrac{1}{2}\text{ and }2|x|<y\right\},$$

and for $x\in\mathbb{R}$ let $U(x)=w(U+x)$. Here note in particular that w is a non-periodic white noise.

The covariance of $U(\cdot)$ is easily computed (see [6, equation (25), p. 458]), and we obtain

$$\mathbb{E}U(x)U(x')=\log\frac{1}{\min\{y,1\}},\quad \text{where } y:=|x-x'|. \tag{50}$$

As before define the cutoff field $U_\varepsilon(x)=w(U_\varepsilon+x)$. Then U_ε is (locally) very close to our field $V_\varepsilon(\cdot)$. Indeed, let I be an interval of length $|I|=\frac{1}{2}$. Then $V(\cdot)|_I$ is equal in law

(3) U_0 corresponds to the simple case of log-normal multifractal random measure, see [6, equation (28), p. 462], and $T=1$ in [6, equation (15), p. 455].

to $w(\cdot+V)|_I$, since the periodicity of the white noise W will not enter. Thus we may realize $U_\varepsilon|_I$ and $V_\varepsilon|_I$ for $\varepsilon\in\left(0,\frac{1}{2}\right)$ in the same probability space so that

$$U_\varepsilon - V_\varepsilon := Z = w\bigl(x+\bigl\{(x,y)\in U : y>\tfrac{1}{2}\bigr\}\bigr).$$

We may again apply Dudley's theorem and equation (38) to the random variable

$$\xi_1 := \sup_{\substack{x\in I\\ \varepsilon\in(0,1/2]}} |V_\varepsilon(x)-U_\varepsilon(x)| < \infty \quad \text{a.s.} \tag{51}$$

Especially, $\mathbb{E}e^{a\xi_1}<\infty$ for all $a>0$. In a similar manner as for the measures τ and ν one deduces the existence of the almost sure limit

$$\lim_{\varepsilon\to 0^+} e^{\beta U_\varepsilon(x)-\beta^2\gamma_U(\varepsilon)/2}\,dx =: \eta(dx), \tag{52}$$

where the limit takes place locally weak* on the space of locally finite Borel measures on the real axis. By letting $G_2:=e^{a\xi_1}$, we thus have an analogue of Lemma 3.6,

$$\frac{1}{G_2}\tau(B) \leqslant \nu(B) \leqslant G_2\tau(B) \tag{53}$$

for all $B\subset I$, and the auxiliary variable G_2 satisfies $\mathbb{E}G_2^p<\infty$ for all $p\in\mathbb{R}$. As an aside, note that we cannot have (53) for the full interval $I=[0,1]$, as V is 1-periodic, while U is not.

Now, for proving the theorem, by (51) and Lemma 3.5 it is enough to check the corresponding claims (i)–(iii) for the random measure η, as one may clearly assume that $|I|\leqslant\frac{1}{2}$.

With this reduction in mind, we start with claim (ii), which in the case of positive moments is due to Kahane (see [22] and [20]). Bacry and Muzy [6, Appendix D] gave a proof for the measure η by adapting the argument of Kahane and Peyrière [22] (who considered a cascade model). In Appendix A we discuss the case of complex β which, as a consequence, gives a self-contained proof for the positive moments.

Finiteness of negative moments is stated in [9, Theorem 5.5]. For the reader's convenience we include the details in Appendix B, following the lines of [25] that considers a cascade model. The non-degeneracy of the measure τ is based on L^p-martingale estimates ($p>1$) for $\tau(I)$. At the critical point $\beta=\sqrt{2}$ the L^p bounds blow up for any $p>1$. In fact, one may show that for $\beta\geqslant\sqrt{2}$ the measure τ degenerates almost surely.

For the claim (iii), the fact (49) for η and $0<\beta<\sqrt{2}$ is [6, Theorem 4]. In this case (49) is actually a direct consequence of the exact scaling law (54) below. The observation that the dependence $\beta\mapsto\eta(I)$ extends analytically into a suitable open subset is due to

Barral [7]. The complex multifractal spectrum exponent $\zeta_p(\beta)$ is not explicitely computed there, and for that reason we include a proof of (49) in Appendix A.

In order to treat the rightmost inequality in (i), choose $p\in(1,2/\beta^2)$ and let $a_2>0$ be so small that $b:=\zeta_p(\beta)-pa_2>1$. Chebyshev's inequality in combination with (49) yields that $\mathbb{P}(\eta(I)>|I|^{a_2})\lesssim|I|^b$. In particular, $\sum_I \mathbb{P}(\eta(I)>|I|^{a_2})<\infty$, where one sums over the dyadic subintervals of $[0,1)$. The same holds true if one sums over the dyadic subintervals shifted by their half-length. This observation in combination with the Borel–Cantelli lemma yields the desired upper estimate in (i).

In turn, the finiteness of negative moments, together with a direct computation that uses the exact scaling law (54) below, yields

$$\mathbb{E}\eta(I)^p = C(p,\beta)|I|^{\zeta_p(\beta)} \quad \text{for all } p\in\left(-\infty,\frac{2}{\beta^2}\right)$$

with $\zeta_p(\beta)=p-\frac{1}{2}\beta^2(p^2-p)$. Set $r=-\zeta_{-1}(\beta)>0$. By Chebyshev's inequality, we get

$$\mathbb{P}(\eta(I)<|I|^{1+2r}) = \mathbb{P}(\eta(I)^{-1}>|I|^{-1-2r}) \lesssim |I|^{1+2r}|I|^{-r} = |I|^{1+r}.$$

The argument for the lower bound in (i) is then concluded as in the case of the upper bound, and one may choose $a_2=1+2r$. □

Note that the exact scaling law of the measure η we used in the above proof is given in [6, Theorem 4]. Indeed, for any $\varepsilon,\lambda\in(0,1)$ one has the equivalence of laws

$$U_{\varepsilon\lambda}(\lambda\,\cdot)|_{[0,1]} \sim G_\lambda + U_\varepsilon|_{[0,1]},$$

where $G_\lambda\sim N(0,\log(1/\lambda))$ is a Gaussian independent of U. Therefore, one has the equivalence of laws for measures on $[0,1]$,

$$\eta(\lambda\,\cdot) \sim \lambda e^{\beta G_\lambda+\log(\lambda)\beta^2/2}\eta, \tag{54}$$

and hence scale invariance of the ratios

$$\frac{\eta([\lambda x,\lambda y])}{\eta([\lambda a,\lambda b])} \sim \frac{\eta([x,y])}{\eta([a,b])}. \tag{55}$$

In turn, the exact scaling law of τ is best described in terms of Möbius transformations of the circle. We do not state it, as we do not need it later on.

To finish this section, we are now able to define our circle homeomorphism h.

Definition 3.8. Assume that $\beta^2<2$. The random homeomorphism $\phi:\mathbb{T}\to\mathbb{T}$ is obtained by setting

$$\phi(e^{2\pi i x}) = e^{2\pi i h(x)}, \tag{56}$$

where we let

$$h(x) = h_\beta(x) = \frac{\tau([0,x])}{\tau([0,1])} \quad \text{for } x\in[0,1), \tag{57}$$

and extend periodically over $\mathbb{R}$.

Theorem 3.7 (i) shows that ϕ is indeed a well-defined homeomorphism almost surely. Moreover, we have the following consequence.

COROLLARY 3.9. *Assume that $\beta^2<2$. Then almost surely both ϕ and its inverse map ϕ^{-1} are Hölder continuous.*

Remark 3.10. As an aside, let us note that defining τ_ε as in the left-hand side of equation (46), we have $\lim_{\varepsilon\to 0^+}\tau_\varepsilon=0$ for $\beta^2\geqslant 2$. However, it is a natural conjecture that letting h_ε to be given by (57) with τ replaced by τ_ε, the limit for h_ε exists in a suitable (quite weak) sense as $\varepsilon\to 0^+$ also for $\beta^2\geqslant 2$. Indeed, the normalized measure in equation (57) appears in the physics literature as the Gibbs measure of a random energy model for logarithmically correlated energies [12], [14], [16] and $\beta^2>2$ corresponds to a low temperature "spin glass" phase. However, we do not expect the limiting h to be continuous if $\beta^2>2$.

4. Probabilistic estimates for Lehto integrals

4.1. Notation and statement of the main estimate

We will now study the Lehto integral of equation (11) for the random homeomorphism constructed in the previous section. As explained in §2.4, it suffices to work in the infinite strip $S=\mathbb{R}\times[0,2]$, where the extension F of the random homeomorphism h is non-trivial. We use the bound (30) for the (random) pointwise distortion $K=K(z,F)$ of this extension, and hence it turns out convenient to define K_τ in the upper half-plane by setting

$$K_\tau(z):=K_\tau(I) \quad \text{whenever } z\in C_I. \tag{58}$$

A lower bound for the Lehto integral (11) is then obtained by replacing K there by K_τ. We similarly define $K_\nu(z)$ for $z\in\mathbb{H}$, via the modified Beurling–Ahlfors extension of the periodic homeomorphism defined by the measure ν.

It turns out that we only need to control Lehto integrals centered at the real axis and with some (arbitrarily small, but fixed) outer radius. For this purpose fix (large) $p\in\mathbb{N}$ and choose $\varrho=2^{-p}$, where the final choice of p will be done in §4.3 below.

Our main probabilistic estimate is the following result.

THEOREM 4.1. *Let $w_0\in\mathbb{R}$ and let $\beta<\sqrt{2}$. Then there exists $b>0$ and $\varrho_0>0$ together with $\delta(\varrho)>0$ such that for positive $\varrho<\varrho_0$ and $\delta<\delta(\varrho)$ the Lehto integral satisfies the estimates*

$$\mathbb{P}(L_{K_\nu}(w_0,\varrho^N,2\varrho)<N\delta)\leqslant\varrho^{(1+b)N}, \quad N\in\mathbb{N}. \tag{59}$$

Observe that the estimates in the theorem are in terms of K_ν instead of K_τ, which is the majorant for the distortion of the extension of the actual homeomorphism. However, this discrepancy will easily be taken care of later on in the proof of Theorem 5.1 using the bounds in Lemma 3.5. The proof of the theorem will occupy most of the present section, namely §§4.2–4.4 below. Finally, we consider the almost sure integrability of the distortion in §4.5.

We next fix the notation that will be used for the rest of the present section, and explain the philosophy behind the theorem. Given w_0 we may choose the dyadic intervals in Theorem 2.6 as w_0+I. Then, by stationarity, we may assume that $w_0=0$. Let S_r denote the circle of radius $r>0$ with center at the origin. Define (with slight abuse), for $r\leqslant 2\varrho$,

$$K_\nu(r) := \sum_{I:C_I\cap S_r\neq\varnothing} |I| K_\nu(I), \tag{60}$$

and observe that

$$L_{K_\nu}(0,\varrho^N,2\varrho) \geqslant c\sum_{n=1}^{N} M_n, \tag{61}$$

where

$$M_n = \int_{\varrho^n}^{2\varrho^n} \frac{dr}{K_\nu(r)}. \tag{62}$$

Thus, in order to prove the theorem, it is enough to verify for $\beta<\sqrt{2}$ that for small enough $\varrho>0$ and $0<\delta<\delta(\varrho)$ one has

$$\mathbb{P}\biggl(\sum_{n=1}^{N} M_n < N\delta\biggr) \leqslant \varrho^{(1+b)N}. \tag{63}$$

If the summands M_j in (63) were independent, the estimate would follow easily from basic large deviation estimates. However, they are far from being independent. Nevertheless, by the geometry of the setup in the white noise upper half-plane, one expects that there is some kind of exponential decay of dependence, but due to the complicated structure of the Lehto integrals we need to go through a non-trivial technical analysis in order to be able to get hold on the exponential decay.

4.2. Correlation structure of the M_j's

In this section we will study how the random variables M_n are correlated with each other. As one can easily gather from the representation of the field ν in terms of the white noise, all of the variables M_n with $n=1,2,...$ are correlated with each other. Our basic strategy is to estimate M_n from below by the quantity

$$M_n' = m_n s_n \sigma_n$$

(see (84) below), where the random variables m_n depend only on the white noise on the scale $\sim\varrho^n$ and form an independent set. The variables s_n will provide an estimate of *upscale correlations*, i.e. the dependence of M_n' on the white noise over the larger spatial scales $\{(x,y):|x|\gtrsim\varrho^{n-1}\}$. In turn, the variables σ_n measure the *downscale correlations* that corresponds to the dependence of M_n' on the white noise over $\{(x,y):|x|\lesssim\varrho^{n+1}\}$. It turns out that the downscale correlations are harder to estimate.

We start with the upscale correlations and introduce some terminology. For a Borel set $S\subset\mathbb{H}$ let $\mathcal{B}_S$ be the σ-algebra generated by the randoms variables $W(A)$, where A runs over Borel subsets $A\subset S$. We will call a $\mathcal{B}_S$ measurable random variable for short S measurable. Let

$$V_I := \bigcup_{x\in I}(V+x),$$

where we recall that V is given by (41). Then $\nu(I)/\nu(J)$ is $V_{I\cup J}$ measurable and, by (29), we see that $K_\nu(I)$ is $V_{j(I)}$ measurable (recall that $j(I)$ denotes the union of I with its neighboring dyadic intervals). From (60) we deduce that M_n is V_{B_n} measurable, where $B_n:=B(0,4\varrho^n)$. Indeed, the Whitney cubes C_I that intersect the annulus

$$A_n := B(0,2\varrho^n)\setminus B(0,\varrho^n)$$

have $I\subset B(0,2\varrho^n)$ and thus $j(I)\subset B(0,4\varrho^n)$.

We now decompose $V(\,\cdot\,)|_{B_n}$ to scales using the white noise. In general, for $0\leqslant\varepsilon<\varepsilon'$, let

$$V(x,\varepsilon,\varepsilon') := W((V+x)\cap\{\varepsilon<y<\varepsilon'\}). \tag{64}$$

Set, for $n\geqslant1$,

$$\psi_n(x) = V(x,0,\varrho^{n-1/2}) \tag{65}$$

and, for $k\geqslant0$,

$$\zeta_k(x) = V(x,\varrho^{k+1/2},\varrho^{k-1/2}). \tag{66}$$

Letting

$$\Lambda_n = \{z\in\mathbb{H} : y\leqslant\varrho^{n-1/2}\}, \tag{67}$$

we see that in any open set U the field ψ_n is $\left(\bigcup_{y\in U}V_y\right)\cap\Lambda_n$ measurable. In a similar way, $\zeta_k(x)$ is $V_x\cap(\Lambda_k\setminus\Lambda_{k+1})$ measurable, and, since these regions are disjoint, the field V decomposes into a sum of independent fields

$$V = \psi_n+\sum_{k=0}^{n-1}\zeta_k := \psi_n+z_n. \tag{68}$$

Let ν_n be the measure defined as ν but with V replaced by ψ_n. Inserting the second decomposition in (68) to the measure ν we have, for any $I,J\subset B_n$,

$$\frac{\nu(I)}{\nu(J)} \leqslant \frac{\nu_n(I)}{\nu_n(J)} \frac{\sup_{x\in B_n} e^{\beta z_n(x)}}{\inf_{x\in B_n} e^{\beta z_n(x)}}. \tag{69}$$

The first decomposition in (68) then gives

$$\frac{\sup_{x\in B_n} e^{\beta z_n(x)}}{\inf_{x\in B_n} e^{\beta z_n(x)}} \leqslant e^{\sum_{k=0}^{n-1} t_{n,k}} := s_n^{-1}, \tag{70}$$

where

$$t_{n,k} := \log \frac{\sup_{x\in B_n} e^{\beta\zeta_k(x)}}{\inf_{x\in B_n} e^{\beta\zeta_k(x)}}. \tag{71}$$

Thus, if we let

$$\mathcal{M}_n = \int_{\varrho^n}^{2\varrho^n} \frac{dr}{K_{\nu_n}(r)}, \tag{72}$$

we arrive at the following lower bound for M_n:

$$M_n \geqslant \mathcal{M}_n s_n. \tag{73}$$

This is the desired decoupling upscale. Note that the fields ζ_k become more regular as k decreases. This will lead to the following result.

Proposition 4.2. *The random variables $t_{n,k}$ satisfy*

$$\mathbb{P}(t_{n,k} > u\varrho^{(n-k)/2-1/4}) \leqslant ce^{-u^2/c}, \quad k=0,...,n-1, \tag{74}$$

where c is independent of ϱ, n and k. Moreover, $t_{n,k}$ and $t_{n',k'}$ are independent if $k\neq k'$.

The proof of this proposition is postponed to §4.4 below.

The decoupling downscale is done to the random variables $\mathcal{M}_n$ in (72). Obviously $\mathcal{M}_n$ and $\mathcal{M}_m$ are dependent. However, as in (60), most of the terms $K_{n,I}:=K_{\nu_n}(I)$ are independent of $\mathcal{M}_m$ if $m>n$. The few which are not we will process further in a moment.

So let us first look at the dependence of the $K_{n,I}$ on the white noise. For $U\subset\mathbb{R}$ set $V_U^n:=V_U\cap\Lambda_n$. Then $K_{n,I}$ is $V_{j(I)}^n$ measurable and $\mathcal{M}_m$ is $V_{B_m}^m$ measurable. Some drawing will convince the reader that if $\operatorname{dist}(j(I),0)$ is not too small $K_{n,I}$ and $\mathcal{M}_m$ are independent for $m>n$. Indeed, consider the ball $B_n'=B(0,2\varrho^{n+1/2})$ so that $B_{n+1}\subset B_n'\subset B_n$. The regions $V_{B_n\setminus B_n'}^n$ are disjoint (see Figure 2). Thus the σ-algebras $\mathcal{B}_{V_{B_n}^n\setminus V_{B_n'}^n}$ are independent of each other for $n=1,2,\dots$.

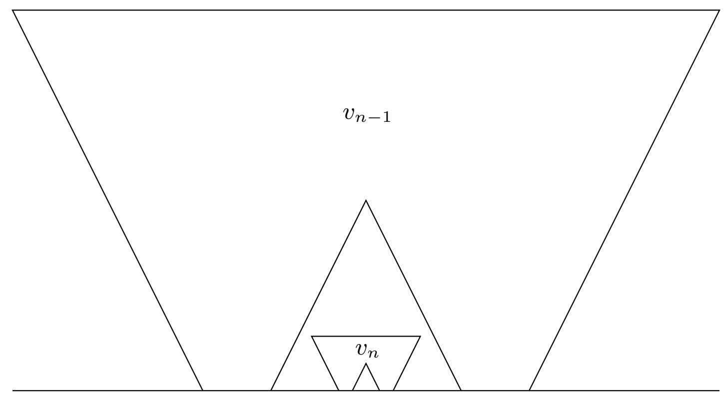

Figure 2. A schematic picture of the regions $V^n_{B_n\setminus B'_n}:=v_n$, where the m_n are measurable.

Let $\mathcal{I}_n$ be the set of $I\in\mathcal{D}$ such that the Whitney cube C_I intersects the annulus $A(0,\varrho^n,2\varrho^n)$ and $j(I)\cap B'_n\neq\varnothing$ (some drawing shows such $I\in\mathcal{D}_{np+j}$ for $j=0,\pm1$). Moreover, for each fixed $r\in(\varrho^n,2\varrho^n)$ let $\mathcal{I}_n(r)$ consist of those intervals I for which $C_I\cap S_r\neq\varnothing$ and $j(I)\cap B'_n=\varnothing$. By (60) we then have

$$K_{\nu_n}(r)\leqslant\varrho^n\biggl(\sum_{I\in\mathcal{I}_n}K_{n,I}+\sum_{I\in\mathcal{I}_n(r)}\varrho^{-n}|I|K_{n,I}\biggr):=\varrho^n(L_n+L_n(r)),\quad r\in(\varrho^n,2\varrho^n).\tag{75}$$

Thus inserting (75) into (72) we get

$$\mathcal{M}_n\geqslant\int_{\varrho^n}^{2\varrho^n}\frac{1}{L_n(r)+L_n}\,\varrho^{-n}\,dr.\tag{76}$$

The term $L_n(r)$ in the integrand (76) is independent of $\mathcal{M}_m$, $m>n$. However L_n is not and we will decouple it now. From (75) and (29) we get

$$L_n\leqslant\sum_{\mathbf{J}}\delta_{\nu_n}(\mathbf{J}),\tag{77}$$

where the sum runs over a set of $\mathbf{J}=(J_1,J_2)$ with $J_k\in\bigcup_{j=0,\pm1}\mathcal{D}_{np+5+j}$ and $J_k\subset B_n$, $k=1,2$. In particular

$$|J_k\setminus B'_n|\geqslant2^{-np-7}=2^{-7}\varrho^n,\quad k=1,2.\tag{78}$$

The sum in (77) has an n-independent number of terms (with multiplicities).

Next, estimate $\delta_{\nu_n}(\mathbf{J})$ in terms of a $V^n_{B_n\setminus B'_n}$ measurable term and perturbation:

$$\begin{aligned}\delta_{\nu_n}(\mathbf{J})&=\frac{\nu_n(J_1\setminus B'_n)+\nu_n(J_1\cap B'_n)}{\nu_n(J_2\setminus B'_n)+\nu_n(J_2\cap B'_n)}+\frac{\nu_n(J_2\setminus B'_n)+\nu_n(J_2\cap B'_n)}{\nu_n(J_1\setminus B'_n)+\nu_n(J_1\cap B'_n)}\\&\leqslant\frac{\nu_n(J_1\setminus B'_n)+\nu_n(J_1\cap B'_n)}{\nu_n(J_2\setminus B'_n)}+\frac{\nu_n(J_2\setminus B'_n)+\nu_n(J_2\cap B'_n)}{\nu_n(J_1\setminus B'_n)}\\&=\delta_{\nu_n}(J_1\setminus B'_n,J_2\setminus B'_n)+\frac{\nu_n(J_1\cap B'_n)}{\nu_n(J_2\setminus B'_n)}+\frac{\nu_n(J_2\cap B'_n)}{\nu_n(J_1\setminus B'_n)}.\end{aligned}$$

Then decompose the perturbation further downscale

$$\nu_n(J_k \cap B'_n) = \sum_{m=n+1}^{\infty} \nu_n(J_k \cap (B'_{m-1} \setminus B'_m)), \quad k=1,2,$$

and (recalling (28)) define

$$L_{n,n} = \sum_{\substack{(J_1,J_2)\in\mathcal{J}(I)\\ I\in\mathcal{I}_n}} \delta_{\nu_n}(J_1 \setminus B'_n, J_2 \setminus B'_n) \tag{79}$$

and

$$L_{n,m} = \sum_{\substack{(J_1,J_2)\in\mathcal{J}(I)\\ I\in\mathcal{I}_n}} \left(\frac{\nu_n(J_1 \cap (B'_{m-1} \setminus B'_m))}{\nu_n(J_2 \setminus B'_n)} + \frac{\nu_n(J_2 \cap (B'_{m-1} \setminus B'_m))}{\nu_n(J_1 \setminus B'_n)} \right) \tag{80}$$

for $m \geqslant n+1$. Then

$$L_n \leqslant \sum_{m=n}^{\infty} L_{n,m}.$$

Defining

$$m_n = \int_{\varrho^n}^{2\varrho^n} \frac{1}{1+L_n(r)+L_{n,n}} \varrho^{-n}\, dr \tag{81}$$

and using the inequality

$$L_n(r)+L_n \leqslant (1+L_n(r)+L_{n,n}) \left(1+\sum_{m=n+1}^{\infty} L_{n,m}\right),$$

we get from (76) that

$$\mathcal{M}_n \geqslant m_n \sigma_n, \tag{82}$$

with

$$\sigma_n := \left(1+\sum_{m=n+1}^{\infty} L_{n,m}\right)^{-1} \tag{83}$$

Combining this with (73) we arrive at the desired bound of M_n in terms of random variables localized in the white noise:

$$M_n \geqslant m_n s_n \sigma_n := M'_n. \tag{84}$$

PROPOSITION 4.3. (i) *The random variables m_n are $V^n_{B_n \setminus B'_n}$ measurable*, $0 \leqslant m_n \leqslant 1$, *and they form an independent set. Moreover*,

$$\mathbb{P}(m_n \leqslant x) \leqslant Cx \quad \textit{for } x>0,$$

where C is independent of ϱ and n.

(ii) *There exist* $a>0$, $q>1$ *and* $C<\infty$ (*independent of* n, m *and* ϱ) *such that for all* $m>n\geqslant 1$ *the random variable* $L_{n,m}$ *satisfies the estimate*

$$\mathbb{P}(L_{n,m}>\lambda)\leqslant C\lambda^{-q}\varrho^{(m-n-1/2)(1+a)}. \tag{85}$$

Moreover, $L_{n,m}$ *is* $V^n_{B_n\setminus B'_m}$ *measurable. Especially,* $L_{n,m}$ *and* $L_{n',m'}$ *are independent if* $n>m'$ *or* $n'>m$.

The proof of this proposition is postponed to §4.4.

4.3. Law of large numbers and proof of Theorem 4.1

Here we prove our main probabilistic estimate assuming Propositions 4.2 and 4.3. By (84), we need to consider

$$P_N:=\mathbb{P}\biggl(\sum_{n=1}^N M'_n<N\delta\biggr)=\mathbb{E}\chi\biggl(\sum_{n=1}^N m_ns_n\sigma_n\leqslant\delta N\biggr)=:\mathbb{E}\chi_{D_N}, \tag{86}$$

where

$$D_N:=\biggl\{\omega:\sum_{n=1}^N m_ns_n\sigma_n\leqslant\delta N\biggr\}.$$

For the sake of notational clarity, we used above (and will often use later on) the shorthand $\chi(A)$ for the characteristic function χ_A. In order to obtain the desired bound for P_N, we insert suitable auxiliary characteristic functions in the expectation. Define

$$\chi_n:=\prod_{m=n+1}^{\infty}\chi(L_{n,m}\leqslant 2^{n-m}\delta^{-1/4})\prod_{m=0}^{n-1}\chi\bigl(t_{n,m}\leqslant 2^{m-n}\log\tfrac{1}{2}\delta^{-1/4}\bigr):=\prod_{m\neq n}\chi_{n,m}. \tag{87}$$

On the support of χ_n we have

$$\sum_{m=n+1}^{\infty}L_{n,m}\leqslant\delta^{-1/4},$$

and thus (for $\delta<1$, say)

$$\sigma_n\geqslant\tfrac{1}{2}\delta^{1/4}.$$

Similarly, $\sum_{m=0}^{n-1}t_{n,m}\leqslant\log\frac{1}{2}\delta^{-1/4}$, and so

$$s_n\geqslant 2\delta^{1/4}.$$

Insert next

$$1=\prod_{n=1}^N(\chi_n+(1-\chi_n)):=\prod_{n=1}^N(\chi_n+\chi_n^c)$$

in the expectation in (86), and expand to get

$$P_N = \sum_{A\subset\{1,\dots,N\}} \mathbb{E}\chi_{D_N}\chi_A\chi^c_{A^c},$$

where $\chi_A=\prod_{n\in A}\chi_n$ and $\chi^c_{A^c}=\prod_{n\in A^c}\chi^c_n$. On the support of $\chi_{D_N}\chi_A\chi^c_{A^c}$ one has

$$N\delta \geqslant \sum_{n=1}^N m_n s_n \sigma_n \geqslant \delta^{1/2}\sum_{n\in A} m_n,$$

so

$$P_N \leqslant \sum_{|A|>\alpha N} \mathbb{E}\chi\biggl(\sum_{n\in A} m_n \leqslant \delta^{1/2}N\biggr) + \sum_{|A|\leqslant\alpha N}\mathbb{E}\chi^c_{A^c}, \tag{88}$$

where we choose $\alpha:=\frac{1}{8}\min\{1,a\}$, with a taken from Proposition 4.3 (ii). Observe that α is independent of ϱ, δ and N.

Let us consider the two sums on the right-hand side of (88) in turn. For the first one we use independence: if $m_A:=\sum_{n\in A} m_n$ then

$$P(m_A<\delta^{1/2}N)\leqslant e^{\delta^{1/2}tN}\mathbb{E}e^{-tm_A} = e^{\delta^{1/2}tN}\prod_{n\in A}\mathbb{E}e^{-tm_n}. \tag{89}$$

By Proposition 4.3 (i),

$$\mathbb{E}e^{-tm_n}\leqslant Cx+e^{-tx}\leqslant 2e^{-tx(t)}, \tag{90}$$

where the auxiliary variable $x=x(t)$ is chosen so that $Cx(t)=e^{-tx(t)}$. Here $x(t)\to 0$ and $tx(t)\to\infty$ as $t\to\infty$. Thus, assuming δ small enough and taking $t=t(\delta)$ such that

$$x(t)=\frac{2\delta^{1/2}}{\alpha},$$

in the case $|A|\geqslant\alpha N$ the right-hand side of (89) is bounded by $2^N e^{-\delta^{1/2}t(\delta)N}$, where $\delta^{1/2}t(\delta)\to\infty$ as $\delta\to 0^+$. Hence

$$\sum_{|A|>\alpha N}\mathbb{E}\chi\biggl(\sum_{n\in A} m_n\leqslant \delta^{1/2}N\biggr)\leqslant 2^N e^{-g(\delta)N}, \tag{91}$$

where $g(\delta)\to\infty$ as $\delta\to 0^+$.

For the second sum in (88) we need to bound

$$\mathbb{E}\chi^c_B := \mathbb{E}\prod_{n\in B}(1-\chi_n)$$

for $|B|\geqslant(1-\alpha)N$. For that purpose, we shall make use of the elementary identity

$$1-\prod_{j=1}^\infty(1-a_j) = \sum_{j=1}^\infty a_j\prod_{r=1}^{j-1}(1-a_r), \tag{92}$$

valid for any sequence $\{a_j\}_{j\geqslant 1}$ with $a_j\in[0,1]$ for all $j\geqslant 1$. Recall equation (87) and set $\chi^c_{n,m}:=1-\chi_{n,m}$. We also set $\chi^c_{n,m}:=0$ for $m<0$. For any fixed n arrange the variables $\chi^c_{n,m}$ with $m\in\mathbb{Z}$ into a sequence in some order, and apply the identity (92) to write

$$1-\chi_n=1-\prod_{\substack{m\in\mathbb{Z}\\ m\neq n}}(1-\chi^c_{n,m})=\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}}\chi^c_{n,n+l}\widetilde{\chi}_{n,l}, \tag{93}$$

with certain variables $\widetilde{\chi}_{n,l}$ satisfying $0\leqslant\widetilde{\chi}_{n,l}\leqslant 1$. Set

$$\chi^+_n:=\sum_{l>0}\chi^c_{n,n+l}\widetilde{\chi}_{n,l}\quad\text{and}\quad\chi^-_n:=\sum_{l<0}\chi^c_{n,n+l}\widetilde{\chi}_{n,l}. \tag{94}$$

Then $\chi^\pm_n\leqslant 1$ (since $\chi^+_n+\chi^-_n=1-\chi_n$) and

$$\chi^\pm_n\leqslant\sum_{\pm l>0}\chi^c_{n,n+l}. \tag{95}$$

We may then estimate

$$\begin{aligned}\prod_{n\in B}(1-\chi_n)&=\prod_{n\in B}(\chi^+_n+\chi^-_n)=\sum_{\{s_n=\pm\}_{n\in B}}\prod_{n\in B}\chi^{s_n}_n\\ &\leqslant\sum_{s:N_+>(1-2\alpha)N}\prod_{n:s_n=+}\chi^+_n+\sum_{s:N_+\leqslant(1-2\alpha)N}\prod_{n:s_n=-}\chi^-_n,\end{aligned} \tag{96}$$

where N_+ is the number of n in the set B such that $s_n=+$. We estimate the expectations of the two products on the right-hand side in turn.

For the first product, let $D\subset\{1,...,N\}$ with $p:=|D|\geqslant(1-2\alpha)N$. List the elements of D as $n_1<n_2<...<n_p$. Then, as $0\leqslant\chi^+_{n_j}\leqslant 1$,

$$\mathbb{E}\chi^+_{n_1}...\chi^+_{n_p}\leqslant\sum_{l_1>0}\mathbb{E}\chi^c_{n_1,n_1+l_1}\chi^+_{n_2}...\chi^+_{n_p}\leqslant\sum_{l_1>0}\mathbb{E}\chi^c_{n_1,n_1+l_1}\chi^+_{n_{i_2}}...\chi^+_{n_p}, \tag{97}$$

where n_{i_2} is the smallest n_j larger than n_1+l_1. Iterating we get

$$\mathbb{E}\chi^+_{n_1}...\chi^+_{n_p}\leqslant\sum_{r=1}^p\sum_{(l_1,...,l_r)}\mathbb{E}\prod_{j=1}^r\chi^c_{n_{i_j},n_{i_j}+l_j}, \tag{98}$$

where $n_{i_{j+1}}$ is the smallest n_j larger than $n_{i_j}+l_j$, and $n_{i_1}=n_1$. Since the intervals $[n_j,n_j+l_j]$ cover the set D, the r-tuples $(l_1,...,l_r)$ in the above sum satisfy

$$\sum_{j=1}^r l_j\geqslant p-r. \tag{99}$$

Next, by Proposition 4.3 (ii), the factors in the product in (98) are independent, and thus

$$\mathbb{E}\prod_{j=1}^{r}\chi^{c}_{n_{i_j},n_{i_j}+l_j}=\prod_{j=1}^{r}\mathbb{E}\chi^{c}_{n_{i_j},n_{i_j}+l_j}.$$

From (85) and (87) we deduce that

$$\mathbb{E}\chi^{c}_{n_j,n_j+l_j}\leqslant C(\varrho)\delta^{q/4}(2^q\varrho^{1+a})^{l_j},$$

whereby

$$\mathbb{E}\chi^{+}_{n_1}\dots\chi^{+}_{n_p}\leqslant\sum_{r=1}^{p}\sum_{(l_1,\dots,l_r)}(C(\varrho)\delta^{q/4})^r(2^q\varrho^{1+a})^{\sum_{j=1}^{r}l_j}. \tag{100}$$

Using (99), we see that the right-hand side is bounded by

$$\varrho^{(1+a/2)p}\sum_{r=1}^{p}(C(\varrho)\delta^{q/4})^r\sum_{(l_1,\dots,l_r)}(2^q\varrho^{a/2})^{\sum_{j=1}^{r}l_j}.$$

For an upper bound, drop the constraints on l_j to bound (100) by

$$\varrho^{(1+a/2)p}\sum_{r=1}^{p}(C(\varrho)\delta^{q/4})^r\biggl(\sum_{l=1}^{\infty}(2^q\varrho^{a/2})^l\biggr)^r.$$

Choosing first ϱ small enough and then $\delta\leqslant\delta(\varrho)$, this is bounded by

$$C(\varrho)\delta^{1/4}\varrho^{(1+a/2)p}\leqslant C(\varrho)\delta^{1/4}\varrho^{(1+a/2)(1-2\alpha)N}\leqslant C(\varrho)\delta^{1/4}\varrho^{(1+2b)N}$$

with a constant $b>0$ by our choice of α. The expectation of the first sum in equation (96) is then bounded by

$$C(\varrho)2^N\delta^{1/4}\varrho^{(1+2b)N}. \tag{101}$$

Consider finally the second sum in equation (96). We proceed as for the first sum, this time considering a set $D\subset\{1,\dots,N\}$ with elements $n_1>n_2>\dots>n_p$ with $p\geqslant\alpha N$. Now we write $\chi^{-}_{n_1}\leqslant\sum_{l_1>0}\chi^{c}_{n_1,n_1-l_1}$ and end up with the analogue of equation (98):

$$\mathbb{E}\chi^{-}_{n_1}\dots\chi^{-}_{n_p}\leqslant\sum_{r=1}^{p}\sum_{(l_1,\dots,l_r)}\prod_{j=1}^{r}\mathbb{E}\chi^{c}_{n_{i_j},n_{i_j}-l_j}, \tag{102}$$

where $n_{i_{j+1}}$ is the largest n_j smaller than $n_{i_j}-l_j$ and $n_{i_1}=n_1$, and this time Proposition 4.2 was used for independence. From the same proposition we also get

$$\mathbb{E}\chi^{c}_{n,n-l}\leqslant ce^{-c2^{-2l}\varrho^{-l+1/2}(\log\delta)^2}.$$

For small enough ϱ we have $2^{-2l}\varrho^{-l+1/2}\geqslant(l+\varrho^{-1/8})\varrho^{-1/8}$ for all $l\geqslant 1$. Hence

$$\prod_{j=1}^{r}\mathbb{E}\chi^{c}_{n_{i_j},n_{i_j}-l_j}\leqslant c^r\exp\biggl(-c(\log\delta)^2\varrho^{-1/8}\biggl(r\varrho^{-1/8}+\sum_{j=1}^{r}l_j\biggr)\biggr).$$

As $\varrho<1$, by (99) we also have

$$r\varrho^{-1/8}+\sum_{j=1}^{r}l_j\geqslant\frac{1}{2}\biggl(p+\sum_{j=1}^{r}l_j\biggr).$$

Thus

$$\prod_{j=1}^{r}\mathbb{E}\chi^{c}_{n_{i_j},n_{i_j}-l_j}\leqslant\biggl(\exp\frac{c\varrho^{-1/8}(\log\delta)^2p}{2}\biggr)c^r\exp\biggl(-c(\log\delta)^2\varrho^{-1/8}\sum_{j=1}^{r}l_j\biggr).$$

Now recall that $p\geqslant\alpha N$, take δ small enough, and proceed as above by summing first over the l_j's, and then performing a geometric sum over r in order to conclude that the second sum in (96) has the upper bound

$$2\exp\biggl(-\frac{c\varrho^{-1/8}(\log\delta)^2N\alpha}{2}\biggr).$$

For small δ this is by far dominated by the bound (101), and therefore

$$\mathbb{E}\chi^{c}_{B}\leqslant 2^{N+1}\delta^{1/4}\varrho^{(1+2b)N}. \tag{103}$$

Going back to equation (88), and recalling (91) with (101) and (103), we conclude that, for $\delta\leqslant\delta(\varrho)$,

$$P_N\leqslant 2^{2N+2}\delta^{1/4}\varrho^{(1+2b)N}, \tag{104}$$

which gives the claim of Theorem 4.1.

4.4. Proofs of the propositions

We will now prove Propositions 4.2 and 4.3 of §4.2, describing the statistics of m_n, $L_{n,m}$ and $t_{n,k}$. We start by noting that the random measures $\nu_n(\cdot)$ and $\varrho^{n-1}\nu_1(\varrho^{1-n}\cdot)$ are equal in law. Especially, the m_n are independent and identically distributed, so that it suffices to study m_1. Similarly $\zeta_k|_{B_n}$ equals in law with $\zeta_1|_{B_{n-k+1}}$ and thus $t_{n,k}$ equals $t_{n-k+1,1}$ in law. The value $k=0$ is slightly different, but it can be treated exactly in the same manner as the case $k\geqslant 1$. Finally, $L_{n,m}$ and $L_{1,m-n+1}$ are equal in law.

We first need the following lemma. Actually, only the statement (105) is needed for the proofs of Propositions 4.2 and 4.3. The stronger claim (106) will be necessary only later in the proof of Theorem 5.3.

Lemma 4.4. *There exist* $q, q_1>1$ *and* $C>0$ (*each independent of* ϱ) *such that for all intervals* $J, I\subset\left[-\frac{1}{4},\frac{1}{4}\right]$ *satisfying* $|J|\leqslant 2|I|$, *and with mutual distance at most* $100|I|$, *one has*

$$\mathbb{P}\left(\frac{\nu(J)}{\nu(I)}>\lambda\right)\leqslant C\lambda^{-q}\left(\frac{|J|}{|I|}\right)^{q_1}. \tag{105}$$

Even a stronger statement is true: *given* $\beta_{\max}\in(0,\sqrt{2}\,)$, *one may choose* $q=q(\beta_{\max})$, $q_1=q_1(\beta_{\max})>1$ *and* $C=C(\beta_{\max})>0$ *so that*

$$\mathbb{P}\left(\sup_{\beta\in(0,\beta_{\max})}\frac{\nu(J)}{\nu(I)}>\lambda\right)\leqslant C\lambda^{-q}\left(\frac{|J|}{|I|}\right)^{q_1}. \tag{106}$$

Proof. We use the comparison (53) with the measure η in order to estimate

$$\frac{\nu(J)}{\nu(I)}\leqslant G_2\frac{\eta(J)}{\eta(I)}, \tag{107}$$

where we recall that all the moments of the variable G_2 are finite. Next, in case $|I|\leqslant\frac{1}{100}$, we may scale further by using the exact scaling law (55), and apply the translation invariance of η to deduce that $\eta(J)/\eta(I)\sim\eta(J')/\eta(I')$, where now $I', J'\subset[0,1]$ with $\frac{1}{100}\leqslant|I'|$ and $|J'|\leqslant|J|/|I|\leqslant 100|J'|$. In case $|I|\geqslant\frac{1}{100}$, no scaling is needed.

In this situation, if $r<\infty$ it follows from Theorem 3.7, or (130) in Appendix B, that $\eta(I')^{-1}\in L^r$ uniformly with respect to I'. We can thus fix exponents $1<q<\tilde q<p<2/\beta^2$ and get, by (107), Hölder's inequality and Theorem 3.7,

$$\left\|\frac{\nu(J)}{\nu(I)}\right\|_q\leqslant C\left\|\frac{\eta(J)}{\eta(I)}\right\|_{\tilde q}=C\left\|\frac{\eta(J')}{\eta(I')}\right\|_{\tilde q}\leqslant C\|\eta(J')\|_p\leqslant C\left(\frac{|J|}{|I|}\right)^{\zeta(p)/p}, \tag{108}$$

where $\zeta(p)>1$. The constant C depends only on the exponents q, $\tilde q$ and p. Thus

$$\mathbb{P}\left(\frac{\eta(J)}{\eta(I)}>\lambda\right)\leqslant C\lambda^{-q}\left(\frac{|J|}{|I|}\right)^{q\zeta(p)/p}. \tag{109}$$

The desired bound (105) follows by choosing the exponent $q>1$ close enough to p in order to ensure that $q_1:=q\zeta(p)/p>1$.

In order to consider the maximal estimate (106), choose $p>1$ and $\varepsilon_0>0$ small enough so that $[0,\beta_{\max}]+\overline{B(0,3\varepsilon_0)}\subset D_p$. A standard application of the Cauchy integral formula and Theorem 3.7 (iii) and (iv) yield that

$$\mathbb{E}\Big|\sup_{\beta\in(0,\beta_{\max})}\nu(I)\Big|^p\leqslant c|I|^{q_2}\quad\text{and}\quad\mathbb{E}\Big|\sup_{\beta\in(0,\beta_{\max})}\frac{d}{d\beta}\nu(I)\Big|^p\leqslant c|I|^{q_2}, \tag{110}$$

where $q_2=q_2(p,\beta_{\max})>1$ and $c=c(p,\beta_{\max})$. In turn, when considering the needed estimate for the negative moments,

$$\mathbb{E}\Big(\sup_{\beta\in(0,\beta_{\max})}\nu(I)^{-r}\Big)<\infty \quad \text{for all } r>1, \tag{111}$$

we are not interested in the explicit dependence on $|I|$. We compute by applying Theorem 3.7 (ii), Hölder's inequality and (110),

$$\begin{aligned}\mathbb{E}\Big(\sup_{\beta\in(0,\beta_{\max})}\nu(I)^{-r}\Big) &\leqslant \mathbb{E}\nu(I)^{-r}+r\mathbb{E}\int_0^{\beta_{\max}}\left|\frac{d}{d\beta}\nu(I)\right|\nu(I)^{-r-1}\,d\beta\\ &\lesssim C+\left(\mathbb{E}\left|\sup_{\beta\in(0,\beta_{\max})}\frac{d}{d\beta}\nu(I)\right|^p\right)^{1/p}\left(\int_0^{\beta_{\max}}\mathbb{E}\nu(I)^{-(r+1)p'}\,d\beta\right)^{1/p'}.\end{aligned}$$

Note above that, by the argument of Appendix B, for each t, the expectation

$$\mathbb{E}\nu(I)^{-t}=\mathbb{E}\nu_\beta(I)^{-t}$$

is locally bounded for $\beta\in[0,\sqrt{2}\,)$.

The proof of the lemma is now finished, exactly as in case of (105), by applying the estimates (110) and (111), and noting that in the scaling argument the maximal analogue of the variable G_2 is just the original variable G_2 corresponding to the value $\beta=\beta_{\max}$. □

Let us then discuss m_1. Observe that the denominator of the integrand in (81) can be dominated as

$$1+L_{1,1}+L_1(r)\leqslant 1+L_{1,1}+\sum_{m=0}^{\infty}2^{-m}k_m(r), \tag{112}$$

where for $r\in(\varrho,2\varrho)$ and $m\geqslant 0$ one sets

$$k_m(r):=\sum_{I\in\mathcal{D}_{p+m}}K_{1,I}1_{C_I\cap S_r\neq\varnothing}. \tag{113}$$

For any fixed $r\in(\varrho,2\varrho)$ the sum (113) has at most four non-zero terms.

For $m\geqslant 0$ denote by $\mathcal{H}_m$ the set of all pairs $\mathbf{J}=(J_1,J_2)$ that contribute to $k_m(r)$ in (113) for some $r\in(\varrho,2\varrho)$. To estimate $\delta_{\nu_1}(\mathbf{J})$, we may scale by the factor $\varrho^{-1/2}$ in order to consider instead the identically distributed quantity $\nu(J_1')/\nu(J_2')$, where now $J_1',J_2'\subset\big[-\frac14,\frac14\big]$. Thus Lemma 4.4 applies. As we additionally have $|J_1|=|J_2|$, there is $q>1$ and a constant $C>0$ such that

$$\mathbb{P}(\delta_{\nu_1}(\mathbf{J})>R)\leqslant CR^{-q} \quad \text{for all } \mathbf{J}\in\bigcup_{m=0}^{\infty}\mathcal{H}_m. \tag{114}$$

Choose next $\alpha>0$ and $\gamma\in(0,1)$ such that

$$4\alpha\sum_{m=0}^{\infty}2^{m(\gamma-1)}\leqslant 1$$

together with $\gamma q>1$. Fix $R>0$. We observe that by these choices

$$\delta_{\nu_1}(\mathbf{J})\leqslant\alpha 2^{\gamma m}R \text{ for all } \mathbf{J}\in\mathcal{H}_m,\ m\geqslant 0 \quad\Longrightarrow\quad L_1(r)\leqslant R \text{ for all } r\in(\varrho,2\varrho).$$

Since we have the obvious estimate $\#\mathcal{H}_m\leqslant c2^m$ for the number of pairs in $\mathcal{H}_m$, by combining the above implication with the uniform estimate (114), one may estimate

$$\begin{aligned}\mathbb{P}(L_1(r(\sigma))>R \text{ for some } r\in(\varrho,2\varrho)) &\leqslant\sum_{m=0}^{\infty}\sum_{\mathbf{J}\in\mathcal{H}_m}\mathbb{P}(\delta_{\nu_1}(\mathbf{J})>\alpha 2^{\gamma m}R)\\ &\leqslant CR^{-q}\sum_{m=0}^{\infty}c2^m2^{-q\gamma m}\leqslant CR^{-q}.\end{aligned}$$

In a similar vain we may apply Lemma 4.4 to immediately obtain the corresponding tail estimate for $L_{1,1}$. Indeed, by (79) this depends only on a finite (ϱ-independent) number of ratios $\delta_{\nu_1}(I_1,I_2)$, with $I_1,I_2\subset[-4\varrho,4\varrho]$ and $|I_1|,|I_2|\geqslant 2^{-7}\varrho$; see (78). Putting things together, we obtain (for $R>1$, say) the bound

$$\mathbb{P}\left(m_1<\frac{1}{R}\right)\leqslant CR^{-q}\leqslant CR^{-1}, \tag{115}$$

where C is independent of ϱ.

Consider next $L_{n,m}$ with $m>n$ and use $L_{n,m}\sim L_{1,m-n+1}$. By (80), $L_{1,m-n+1}$ is bounded from above by a sum of terms (with ϱ-independent upper bound for their number)

$$\frac{\nu_1(J)}{\nu_1(I)},$$

where $2^{-8}\varrho\leqslant|I|\leqslant 2^{-4}\varrho$ and $|J|\leqslant\varrho^{m-n+1/2}$, and in addition $I,J\subset[-4\varrho,4\varrho]$. The constant C above is independent of m, n and ϱ. Via scaling the desired bound (85) is now a direct consequence of Lemma 4.4, as we observe that $|J|/|I|\leqslant C\varrho^{m-n-1/2}$.

Finally we turn to $t_{n,1}$ given in (71). By scaling, we may take the sup and the inf over $x\in B_n\cap\mathbb{R}$ of $e^{\beta\tilde{\psi}}$, where $\tilde{\psi}:=\psi(\cdot,\varrho^{3/2},\varrho^{1/2})$, and we may replace there $\tilde{\psi}$ by $\hat{\psi}:=\tilde{\psi}(\cdot)-\tilde{\psi}(0)$. The covariance of $\hat{\psi}$ is clearly $c\varrho^{-3/2}$-Lipschitz and the length of the interval $B_n\cap\mathbb{R}$ is $8\varrho^n$. Lemma 3.3 yields that

$$\mathbb{P}(|\tilde{\psi}|>\lambda c\varrho^{n/2-3/4})\leqslant C(1+\lambda)e^{-\lambda^2/2},$$

which finishes the proof of the remaining Proposition 4.2.

4.5. Integrability of K_ν

In the next section we shall also make use of the the following observation.

LEMMA 4.5. *Let* $\beta<\sqrt{2}$. *Then almost surely* $K_\nu\in L^1([0,1]\times[0,2])$.

Proof. Recall that $S=\mathbb{R}\times[0,2]$ is tiled by the Whitney squares C_I. By definition, on such a square K_ν is a finite sum of ratios $\nu(J_1)/\nu(J_2)$ with $|J_1|=|J_2|\leqslant 2^{-4}$ and of controlled mutual distance as in Lemma 4.4. Thus, for $|J_k|$ small enough J_k lie on a common interval of length $\frac{1}{2}$ and we have a uniform bound for $\mathbb{E}\nu(J_1)/\nu(J_2)\leqslant\|\nu(J_1)/\nu(J_2)\|_q$, $q<2/\beta^2$, from Lemma 4.4 (or more directly from (108)). For the finitely many J_k not fitting to such an interval we use again (108). Hence there is also a uniform bound for $\mathbb{E}K_\nu(I)$ and one obtains

$$\mathbb{E}\int_{[0,1]\times[0,2]} K_\nu\,dz\leqslant\sum_{I\subset D([0,1])}|C_I|\mathbb{E}K_\nu(I)\leqslant C\sum_I|C_I|<\infty. \qquad \square$$

5. Conclusion of the proof

In this final section we give a precise formulation to our main result as a theorem and prove it using the work done in the previous sections. In order to make the setup clear, let us recall that our random circle homeomorphism was defined in §3 via formulae (56) and (57). Its extension to the unit disc is constructed by the method described in §2.4, and formula (24) in particular.

The welding method described in §2 requires estimates for the Lehto integral of the distortion function in $\mathbb{D}$. Theorem 2.6 reduces these bounds to the boundary function, and here the crucial estimates are provided by our Theorem 4.1 in §4.

THEOREM 5.1. *Assume that* $\beta^2<2$. *Let* $\phi:\mathbb{T}\to\mathbb{T}$ *be the random circle homeomorphism from Definition* 3.8, *and let* $\Psi:\mathbb{D}\to\mathbb{D}$ *be its extension as in* (20)–(24). *Let*

$$\mu=\mu_\Psi:=\frac{\partial_{\bar z}\Psi}{\partial_z\Psi}$$

be the complex dilatation of the extension on $\mathbb{D}$, *and set* $\mu=0$ *outside* $\mathbb{D}$.

Then almost surely there exists a (random) homeomorphic $W^{1,1}_{\mathrm{loc}}$*-solution* $f:\mathbb{C}\to\mathbb{C}$ *to the Beltrami equation*

$$\partial_{\bar z}f=\mu\,\partial_z f,\quad \text{a.e. in } \mathbb{C}, \tag{116}$$

that satisfies the normalization $f(z)=z+o(1)$ *as* $z\to\infty$. *Moreover, there exists* $\alpha>0$ *such that the restriction* $f:\mathbb{T}\to\mathbb{C}$ *is a.s.* α*-Hölder continuous.*

Proof. We sketch the proof along the lines of [5, Theorem 20.9.4], to which presentation we refer for further details and background.

For any integer $n\geqslant 1$ choose $N_n=[\varrho^{-(1+b/2)n}]\in\mathbb{N}$, where b is as in Theorem 4.1. Set

$$\zeta_{n,k}:=e^{2\pi ik/N_n}\quad \text{for } k=1,...,N_n.$$

Write also $G_n:=\{\zeta_{n,1},...,\zeta_{n,N_n}\}$. Thus the distance on $\mathbb{T}$ to the set G_n is bounded by $\pi/N_n\sim\varrho^{(1+b/2)n}$.

For a given $n\geqslant 1$ and $k\in\{1,...,N_n\}$ let us denote by $A_{n,k}$ the event

$$A_{n,k}=\left\{\omega: L_{K_\nu}\left(\frac{k}{N_n},\varrho^n,2\varrho\right)<n\delta\right\},$$

and set $A_n=\bigcup_{k=1}^{N_n}A_{n,k}$. Note that here we consider Lehto integrals in the half-plane. Theorem 4.1 combined with stationarity yields that

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n)\leqslant\sum_{n=1}^{\infty}\sum_{k=1}^{N_n}\mathbb{P}(A_{n,k})\leqslant\sum_{n=1}^{\infty}N_n\varrho^{(1+b)n}\leqslant\sum_{n=1}^{\infty}\varrho^{bn/2}<\infty.$$

Borel–Cantelli's lemma yields that almost every ω belongs to the complement of the event

$$\bigcup_{n>n_0(\omega)}A_n.$$

Also, we obtain by Lemma 3.6 that

$$K_\tau\leqslant E^2K_\nu,$$

where almost surely $E<\infty$.

From Theorem 2.6 and (58) we see that $K(z,F)$, the distortion of the extension of h, is bounded by a constant times $K_\tau(z)$. Hence Lemma 4.5 implies that almost surely

$$\int_{[0,1]\times[0,2]}K(z,F)\,dz\leqslant C_0\int_{[0,1]\times[0,2]}K_\tau\,dz\leqslant C_0E^2\int_{[0,1]\times[0,2]}K_\nu\,dz<\infty.$$

We may thus forget the probabilistic setup by fixing an event $\omega_0\in\Omega$ so that we are in the following situaton: we are given the complex dilatation μ on $\mathbb{D}$, so that the distortion

$$K=\frac{1+|\mu|}{1-|\mu|}$$

satisfies pointwise

$$K(e^{2\pi iz})\leqslant C_0K_\tau(z)\leqslant C_0E(\omega_0)^2K_\nu(z),\quad z\in\mathbb{H}.$$

Further, from the definition in (24) we have $K\equiv 1$ for $|z|\leqslant e^{-4\pi}$. Also, $K_\nu\in L^1\cap L^\infty_{\mathrm{loc}}$ on the square $[0,1]\times(0,2]$, and for each $n\geqslant n_0$ and $k\in\{1,...,N_n\}$ it holds that

$$L_{K_\tau}\left(\frac{k}{N_n},\varrho^n,2\varrho\right)\geqslant E(\omega_0)^{-2}L_{K_\nu}\left(\frac{k}{N_n},\varrho^n,2\varrho\right)\geqslant n\delta E(\omega_0)^{-2}=:n\delta'.$$

We next proceed as in the standard proof of Lehto's theorem by approximating μ by e.g. the sequence

$$\mu_l:=\frac{l}{l+1}\mu,\quad l\in\mathbb{N}.$$

Let f_l denote the corresponding normalized solution of the Beltrami equation with coefficient μ_l, i.e. with the asymptotics $f_l(z)=z+o(1)$ as $z\to\infty$. Then every f_l is a quasiconformal homeomorphism of $\mathbb{C}$.

To show that (116) has a homeomorphic $W^{1,1}$-solution, we need to control the approximations f_l. For this we first apply [5, Lemma 20.2.3], which tells that the inverse maps $g_l=f_l^{-1}$ have the following modulus of continuity:

$$|g_l(z)-g_l(w)|\leqslant\frac{16\pi^2}{\log(e+1/|z-w|)}\left(|z|^2+|w|^2+\int_{\mathbb{D}}\frac{1+|\mu_l(\zeta)|}{1-|\mu_l(\zeta)|}\,d\zeta\right),\quad z,w\in\mathbb{C}.$$

Here the integrals are uniformly bounded as

$$\frac{1+|\mu_l(\zeta)|}{1-|\mu_l(\zeta)|}\leqslant K(\zeta)\leqslant C_0K_\tau(z),\quad \zeta=e^{2\pi iz},$$

and $K_\tau\in L^1([0,1]\times[0,2])$. Thus the inverse maps $g_l=f_l^{-1}$ form an equicontinuous family.

In order to check the equicontinuity of the family $\{f_l\}_{l\geqslant 1}$ itself, we first consider a point $z\in\mathbb{D}$. Writing $2a=1-|z|$, observe that K is bounded in $B(z,a)$ and, as

$$K_l:=K(\cdot\,,f_l)\leqslant K,$$

we have, for any $l\geqslant 1$ and $u\in\left(0,\tfrac{1}{2}a\right)$,

$$L_{K_l}(z,u,1)\geqslant L_K(z,u,a)\geqslant\frac{1}{\|K\|_{L^\infty(B(z,a))}}\log\frac{a}{u}\to\infty,\quad\text{as } u\to 0^+.$$

Moreover, by Koebe's theorem or [5, Corollary 2.10.2], we obtain

$$f(2\mathbb{D})\subset 5\mathbb{D}. \tag{117}$$

Thus, $\operatorname{diam}(f_l(B(z,1)))\leqslant 5$, which may be combined with Lemma 2.3 to obtain

$$\operatorname{diam}(f_l(B(z,u)))\to 0,\quad\text{as } u\to 0^+,\quad\text{uniformly in } l.$$

This proves the equicontinity at interior points $z\in\mathbb{D}$. Equicontinuity at exterior points follows e.g. from Koebe's theorem.

In order to next consider the uniform behavior on $\mathbb{T}$, note that it suffices to prove local equicontinuity on points of $[0,1]$ for the family

$$F_l(z)=f_l(e^{2\pi iz}),\quad l\in\mathbb{N}.$$

We first estimate the diameter of the image $F_l(B(k/N_n,\varrho^n))$, assuming that $n\geqslant n_0$. Applying the fact that $\operatorname{diam}(F_l(B(k/N_n,2\varrho)))\leqslant\operatorname{diam}(f_l(B(\zeta_{n,k},1)))\leqslant 5$ and using this together with Lemma 2.3, we obtain

$$\operatorname{diam}(F_l(B(k/N_n,\varrho^n)))\leqslant\operatorname{diam}(F_l(B(k/N_n,2\varrho)))16e^{-2\pi^2n\delta'}\leqslant 80e^{-nc'}. \tag{118}$$

From these estimates we get the required equicontinuity. Namely, working now on the circle $\mathbb{T}$, since the set G_n is evenly spread on $\mathbb{T}$, the balls $B(\zeta_{n,k},\varrho^{n+1})$ cover a ϱ^{n+2}-neighborhood of $\mathbb{T}$ in such a way that any two points that are in this neighborhood, with distance not exceeding ϱ^{n+2}, lie in the same ball. Since this holds for every $n\geqslant n_0$, we infer from (118) that there are $\varepsilon_0>0$ and $\alpha>0$ such that, uniformly in l,

$$|f_l(z)-f_l(w)|\leqslant C|z-w|^\alpha \text{ if } |z|=1,\quad 1-\varepsilon_0\leqslant|w|\leqslant 1+\varepsilon_0 \quad\text{and}\quad |z-w|\leqslant\varepsilon_0. \tag{119}$$

One may actually take $\alpha=c'/\log(1/\varrho)$. This clearly yields equicontinuity at the points of $\mathbb{T}$, and hence on $\widehat{\mathbb{C}}$. We may now pass to a limit and one obtains $W^{1,1}$-homeomorphic solution $f(z)=\lim_{l\to\infty}f_l(z)$ to the Beltrami equation as in [5, p. 585].

At the same time the estimate (119) shows that $f\colon\mathbb{T}\to\mathbb{C}$ is Hölder continuous. Since f is analytic outside the disc, with $f(z)=z+o(1)$ at infinity, in fact it follows that f is Hölder continuous on $\mathbb{C}\setminus\mathbb{D}$. ☐

Collecting the results established, we now arrive at the main theorem of this paper.

THEOREM 5.2. *Let $\phi=\phi_\omega$ be the random circle homeomorphism, with the exponentiated* GFF *as derivative, as defined in* (56) *and* (57).

Then, for $\beta^2<2$ and almost surely in ω, the mapping ϕ admits a conformal welding. That is, there are a random Jordan curve

$$\Gamma=\Gamma_{\omega,\beta} \tag{120}$$

and conformal mappings $f_\pm$ onto the complementary domains of Γ such that $\phi=f_+^{-1}\circ f_-$ on $\mathbb{T}$.

Moreover, almost surely in ω, the Jordan curve Γ in (120) *is unique, up to composing with a Möbius transformation $\Phi=\Phi_\omega$ of the Riemann sphere.*

Proof. We argue as in §2. Extend ϕ to a homeomorphism $\Psi:\mathbb{D}\to\mathbb{D}$ and using this define the complex dilatation $\mu(z)$ as in Theorem 5.1. Applying the theorem, we then find a homeomorphic solution f to the auxiliary equation (116). This is conformal outside the disc, so we set $f_-=f|_{\mathbb{C}\setminus\mathbb{D}}$. Inside the disc, $K(z,f)$ is locally bounded, so the uniqueness of the Beltrami equation gives $f(z)=f_+\circ\Psi(z)$, $z\in\mathbb{D}$, where f_+ is a conformal homeomorphism defined on $\mathbb{D}$. Since the boundary $\partial f_+(\mathbb{D})=\partial f_-(\mathbb{C}\setminus\mathbb{D})=f(\mathbb{T})=\Gamma$ is a Jordan curve, $f_\pm$ extend to $\mathbb{T}$, where we have

$$\phi=f_+^{-1}\circ f_-.$$

Finally, according to the proof of Theorem 5.1, f_- is Hölder continuous in $\mathbb{C}\setminus\mathbb{D}$, and thus the uniqueness of the welding curve follows from the Jones–Smirnov Theorem 2.4. □

Once the random families of curves $\Gamma_{\omega,\beta}$ have been constructed, it is natural to enquire for their dependence on the parameter β. In fact, according to Theorem 3.7 (iii), the circle homeomorphisms ϕ depend analytically on β. More precisely, in any compact subinterval of $[0,\sqrt{2}\,)$, the dependence $\beta\mapsto\tau(I)$, and hence $\beta\mapsto\phi(x)$, is analytic when the measures are computed using the same *fixed realization* of the white noise W in the upper half-plane.

We show that, in the same manner, the welding curves themselves depend continuously on β. This should be compared with the analogous question for SLE curves which is still open.

THEOREM 5.3. *Almost surely the random welding curve is continuous in the parameter* $\beta\in(0,\sqrt{2}\,)$.

Proof. After the maximal estimate (106), the proof follows closely the proof of Theorem 5.2, and hence we only sketch the argument. By (106) in Lemma 4.4, we may run through the proofs of Propositions 4.2 and 4.3, and obtain the analoguous maximal versions. Actually, for Proposition 4.2, this is simple since the dependence of $t_{n,k}$ on β is linear. In turn, for Proposition 4.3, one uses fully the bound (106).

The argument of §4.3 now yields an estimate for the Lehto integral which is uniform in $\beta\in(0,\beta_{\max})$: for positive $\varrho<\varrho_0(\beta_{\max})$ and $\delta<\delta(\varrho,\beta_{\max})$, with $\beta_{\max}<\sqrt{2}$,

$$\mathbb{P}(L_{K_\nu}(w_0,\varrho^N,2\varrho)<N\delta \text{ for all } \beta\in(0,\beta_{\max}))\leqslant\varrho^{(1+b)N}. \tag{121}$$

In addition, we obtain a maximal version of Lemma 4.5:

$$\sup_{\beta\in(0,\beta_{\max})} K_\nu\in L^1([0,1]\times[0,2]). \tag{122}$$

Let us fix a realization of the white noise so that the estimates (121) and (122) hold true for all $\beta_{\max}<\sqrt{2}$, with the understanding that the appropriate constants may depend on $\beta_{\max}$. Assume that $\beta_j\to\beta_\infty\in(0,\sqrt{2})$. Let us denote by $f_{\pm,j}$ (resp. μ_j and K_j) the corresponding conformal maps (resp. random dilatation and distortion in $\mathbb{D}$) obtained in Theorem 5.2.

First of all, Theorem 2.6 yields a local uniform boundedness of K_∞ and $\sup_{j\geqslant 1} K_j$ in $\mathbb{D}$. These facts together with (121) and (122) yield equicontinuity for the maps $f_{\pm,j}$ and their inverses (and even uniform local Hölder continuity for $f_{-,j}$).

Next, since the random homeomorphism ϕ depends continuously on β, the complex dilatations μ_j converge pointwise to μ_∞. At this stage a well-known argument (see e.g. the proof of [5, Theorem 20.9.4]) shows that f_j has a subsequence that converges to a solution of the Beltrami equation with dilatation μ_∞. In particular, one applies the estimate (122) to obtain uniform integrability for the derivatives of the solutions.

Finally, the uniqueness of the welding in the present situation is inherited for the normalized solutions of the Beltrami equation, and this yields the local uniform convergence for the whole sequences $f_{\pm,j}$ to $f_{\pm,\infty}$, as was to be shown. □

Remark 5.4. One may easily verify that the distortion function K of our Beurling–Ahlfors extension has a.s. the property $K\notin L^s(\mathbb{D})$, where the exponent $s=s(\beta)\to 2$ as $\beta\uparrow\sqrt{2}$. Especially, this shows that our random setting lies far from the setup where the general (deterministic) theory of existence and uniqueness for solutions to degenerate Beltrami equations still works. In fact, even the optimal deterministic results roughly require an exponential type integrability of the distortion $K(z)=(1+|\mu|)/(1-|\mu|)$, for details see [5, Theorems 20.3.1 and 20.5.2].

Finally, we state and prove the generalization of the previous two theorems for a composition of two independent copies of our random homeomorphism discussed in the introduction. Since the argument is essentially identical, we only sketch the needed changes in the proof.

THEOREM 5.5. *Let $0\leqslant\beta_+,\beta_-<\sqrt{2}$. Assume that ϕ_+ and ϕ_- are two independent copies of the random homeomorphism* (56) *with parameters $\beta_\pm$. Then, almost surely in ω, the welding problem for the homeomorphism $\phi_+\circ\phi_-^{-1}$ has a solution $\Gamma=\Gamma_{\beta_+,\beta_-}$, where Γ_{β_+,β_-} is a Jordan curve bounding the domains $\Omega_+=f(\mathbb{D})$ and $\Omega_-=f_-(\mathbb{D}_\infty)$, with Hölder continuous Riemann mappings $f_\pm$. For a given ω, the solution is unique up to a Möbius map of the plane and the curves Γ_{β_+,β_-} are continous in β_+ and β_-.*

Proof. Let us extend the circle homeomorphism ϕ_+ inside the unit disc by the Beurling–Ahlfors extension as in §2.4, and we now denote by ϕ_+ also the extension. Moreover, let μ_+ stand for the dilatation of ϕ_+ in $\mathbb{D}$. In a similar manner, define the

Beurling–Ahlfors extension of ϕ_- into $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$ by $\phi_-(1/\bar{z})=1/\overline{\phi_-(z)}$, and denote by μ_- its dilatation. Set

$$\mu(z)=\begin{cases}\mu_+(z), & \text{if } z\in\mathbb{D},\\ \mu_-(z), & \text{if } z\in\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}.\end{cases}$$

Due to our specific choice of the extension, μ has a compact support, see (22) and (24). Since the estimates for the Lehto integral of the distortion function

$$K(z)=\frac{1+|\mu|}{1-|\mu|}$$

in the new situation are identical to those presented in Theorem 4.1, the proof of Theorem 5.1 carries through with only notational changes, yielding as before a solution to the Beltrami equation

$$\frac{\partial F}{\partial \bar{z}}(z)=\mu(z)\frac{\partial F}{\partial z}(z) \quad \text{for almost every } z\in\mathbb{C},$$

with the normalization $F(z)-z=o(1)$ as $z\to\infty$. In addition, $F|_{\mathbb{T}}$ is Hölder continuous.

Next we define the analytic maps $f_+:\mathbb{D}\to\mathbb{C}$ and $f_-:\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}\to\overline{\mathbb{C}}$ via

$$F(z)=f_+\circ\phi_+(z),\ z\in\mathbb{D}, \quad \text{and} \quad F(z)=f_-\circ\phi_-(z),\ z\in\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}.$$

Especially, one obtains the key formula

$$f_+\circ\phi_+=f_-\circ\phi_- \quad \text{on } \mathbb{T}, \tag{123}$$

which shows that the maps $f_\pm$ solve the stated welding problem. Uniqueness of the solution is deduced as before by observing that $f_+|_{\mathbb{T}}$ is a Hölder map since $f_+=F\circ\phi_+^{-1}$, where ϕ_+^{-1} is Hölder continuous according to Corollary 3.9. Finally, the proof of Theorem 5.3 can be repeated in order to verify the stated continuity with respect to $\beta_\pm$. In the argument one needs to additionally observe that almost surely ϕ_+^{-1} is uniformly Hölder continuous with respect to β_+, when the parameter is varying over any compact subinterval of $[0,\sqrt{2}\,)$. In turn, this follows from the proof of the lower bound in Theorem 3.7 (i) combined with a quantitative version of the inequality (111), obtained easily from the scaling law (54). □

Appendix A. Analytic dependence

The analyticity of the dependence of multifractal random measures on the parameter β was established by Barral in [7]. We will prove the analyticity of $\beta\mapsto\eta(I)$ (the proofs for $\beta\mapsto\tau(I)$ and $\beta\mapsto\nu(I)$ are analogous) and the statements formulated in Theorem 3.7 (iii) via the techniques of that paper. To start with, we may clearly assume that I is dyadic, more specifically $I=[0,2^{-m}]$ with $m\geqslant 1$. We apply the notation of §3, and denote, for integers $n\geqslant 1$, by $\mathcal{F}_n$ the σ-algebra generated by the restriction of the white noise w to $\{(x,y):y>2^{-n}\}$.

Define

$$U(x,n):=w((U+x)\cap\{y>2^{-n}\}),$$

and set, for $\beta\in\mathbb{C}$ and $x\in\mathbb{R}$,

$$f_n(x,\beta):=e^{\beta U(x,n)-\beta^2\gamma_U(n)/2}, \tag{124}$$

where $\gamma_U(n)=\mathrm{Cov}(U(x,n))=1+n\log 2$. Then $f_n(x,\beta)$ is $\mathcal{F}_n$-measurable, and, by definition, $\{f_n(x,\beta)\}_{n\geqslant 1}$ is a complex martingale sequence with respect to the filtration $\{\mathcal{F}_n\}_{n\geqslant 1}$. By considering the real part in the exponent, we compute, for $\beta=\beta_1+i\beta_2$,

$$\mathbb{E}|f_n(\beta,x)|^p=c(\beta,p)2^{n((p^2-p)\beta_1^2+p\beta_2^2)/2}\quad \text{with } p\geqslant 1, \tag{125}$$

where $c(\beta,p)=e^{((p^2-p)\beta_1^2+p\beta_2^2)/2}$. The proof below basically uses only this fact together with the underlying hidden cascade-like structure of the hyperbolic white noise.

Set

$$X_n:=\int_I f_n(x,\beta)\,dx=\int_0^{2^{-m}} f_n(x,\beta)\,dx. \tag{126}$$

Obviously $\{X_n\}_{n\geqslant 1}$ is a complex martingale sequence. Let

$$g_{n+1}(x,\beta)=-1+\frac{f_{n+1}(x,\beta)}{f_n(x,\beta)}. \tag{127}$$

Then the $g_n(x,\beta)$, $n\in\mathbb{N}$, are independent. In particular, $g_{n+1}(x,\beta)$ is independent of $\mathcal{F}_n$. Let also $I_{n,j}:=[(j-1)2^{-n},j2^{-n}]$ for $1\leqslant j\leqslant 2^n$.

Assume next that $n\geqslant m$ and write

$$\begin{aligned} X_{n+1}-X_n&=\int_0^{2^{-m}} f_n(x,\beta)g_{n+1}(x,\beta)\,dx\\ &=\Biggl(\sum_{\substack{j=1\\ j\text{ odd}}}^{2^{n-m}}+\sum_{\substack{j=1\\ j\text{ even}}}^{2^{n-m}}\Biggr)\int_{I_{n,j}} f_n(x,\beta)g_{n+1}(x,\beta)\,dx=:Y+Y'. \end{aligned}$$

Conditioned on $\mathcal{F}_n$, the quantities $g_{n+1}(\cdot,\beta)|_{I_{n,j}}$, for odd j, are independent by construction. Hence the Burkholder–Gundy inequality and the simple estimate $\|\cdot\|_{\ell^2}\leqslant\|\cdot\|_{\ell^p}$

imply that

$$\begin{aligned}\mathbb{E}(|Y|^p|\mathcal{F}_n) &\lesssim \mathbb{E}\Bigg(\Bigg(\sum_{\substack{j=1\\ j \text{ odd}}}^{2^{n-m}}\bigg|\int_{I_{n,j}} f_n(x,\beta)g_{n+1}(x,\beta)\,dx\bigg|^2\Bigg)^{p/2}\Bigg|\mathcal{F}_n\Bigg)\\ &\leqslant \mathbb{E}\Bigg(\sum_{j=1}^{2^{n-m}}\bigg|\int_{I_{n,j}} f_n(x,\beta)g_{n+1}(x,\beta)\,dx\bigg|^p\mathcal{F}_n\Bigg).\end{aligned}$$

Taking expectations and applying Hölder's inequality inside the integral yields

$$\begin{aligned}\mathbb{E}|Y|^p &\lesssim \sum_{j=1}^{2^{n-m}}|I_{n,j}|^{p-1}\int_{I_{n,j}}\mathbb{E}|f_n(x,\beta)g_{n+1}(x,\beta)|^p\,dx\\ &\lesssim \sum_{j=1}^{2^{n-m}}|I_{n,j}|^p(\mathbb{E}|f_n(0,\beta)|^p+\mathbb{E}|f_{n+1}(0,\beta)|^p)\\ &\lesssim 2^{n-m}2^{-np}2^{n((p^2-p)\beta_1^2+p\beta_2^2)/2}.\end{aligned}$$

A similar estimate holds for Y' and, by recalling that $|I|=2^{-m}$, it follows that

$$\|X_{n+1}-X_n\|_p \leqslant |I|^{\zeta_p(\beta)/p}2^{-(n-m)(\zeta_p(\beta)-1)/p} \quad \text{for } n\geqslant m. \tag{128}$$

In turn, by applying directly (125) together with Hölder's inequality in (126), we obtain

$$\|X_m\|_p \leqslant |I|^{\zeta_p(\beta)/p}. \tag{129}$$

As the geometric convergence of the estimate (128) is uniform in any compact subset of D_p, we get by a standard application of Cauchy's integral formula and Borel–Cantelli's lemma, the a.s. limit

$$X := \lim_{n\to\infty} X_n,$$

and the convergence is locally uniform with respect to the parameter $\beta\in D_p$. Hence, X is a.s. analytic in D_p and, by summing up the estimates (128) and (129), we obtain in any compact subset $\beta\in S\subset D_p$ the estimate

$$\|X\|_p \leqslant C(S)|I|^{\zeta_p(\beta)/p},$$

as was to be shown.

Appendix B. Negative moments

Here we prove the finiteness of all negative moments for the measure η, defined in the proof of Theorem 3.7. One first needs to verify non-degeneracy of η over any subinterval ([20], [21]; see also [6, Theorems 1 and 2]), and to this end we recall the functions f_n and g_n from (124) and (127). We may write η as the a.s. limit

$$\eta(dx) = \text{w}^*\text{-}\lim_{n\to\infty} f_n(x,\beta;\omega)\,dx = \text{w}^*\text{-}\lim_{n\to\infty} \prod_{j=0}^{n} (1+g_j)\,dx,$$

where the densities $g_j = g_j(x,\beta,\omega)$ are independent. Moreover, $\mathbb{E}f_n(x)=1$ for each x and n, and $f_n(x)$ are a.s. bounded from below by a positive constant.

Let I be a dyadic subinterval and set $Y_n(\omega) := \int_I f_n(x,\omega)\,dx$. By Kolmogorov's 0-1 law, the probability for $\lim_{n\to\infty} Y_n = 0$ is either zero or one. The second alternative can be ruled out by observing that $\mathbb{E}Y_n = |I|$ for all n and that $\{Y_n\}_{n\geqslant 1}$ is an L^p-martingale with $p>1$, according to Theorem 3.7.

We are now ready to start the proof of

$$\mathbb{E}\eta(I)^{-q} \leqslant C < \infty, \quad 0 < q < \infty, \tag{130}$$

for $\beta^2 < 2$. Here the constant $C = C(q,|I|,\beta)$ depends only on the exponent q, the length $|I|$ and the parameter β. Fix $t>0$. Define, for $\varepsilon>0$, the set $U_{\varepsilon,t} := \{(x,y)\in U : \varepsilon < y \leqslant t\}$. As in (52) one deduces the existence of the limit measure

$$\eta_t(dx) := \lim_{\varepsilon\to 0^+} e^{\beta U_{\varepsilon,t}(x) - \beta^2 \operatorname{Cov}(U_{\varepsilon,t})/2}\,dx. \tag{131}$$

Set $M := \eta_{1/2}([0,1])$, $M_1 := \eta_{1/8}\big(\big[0,\tfrac14\big]\big)$ and $M_2 := \eta_{1/8}\big(\big[\tfrac34,1\big]\big)$. By scaling and translation invariance the random variables M_1, M_2 and M are identically distributed. Moreover, by comparing the exponents as in the proof of Lemma 3.5, we see that

$$M \geqslant B(M_1+M_2), \tag{132}$$

where

$$B := e^{\inf_{x\in[0,1]} \beta U_{1/8,1/2}(x) - \beta^2 \operatorname{Cov}(U_{1/8,1/2})/2}$$

has all moments finite. By construction, the random variables M_1, M_2 and B are independent.

Similarly, by comparing η and $\eta_{1/2}$, we see that it is enough to prove that

$$\mathbb{E}M^{-q} < \infty \quad \text{for } q > 0. \tag{133}$$

We first prove this for small values of q. To this end, consider for $s>0$ the Laplace transform

$$\Psi_M(s) := \mathbb{E}e^{-sM} \leqslant \mathbb{E}e^{-sB(M_1+M_2)} \leqslant \mathbb{E}\Psi_{M_1}(sB)\Psi_{M_2}(sB) = \mathbb{E}\Psi_M(sB)^2. \tag{134}$$

Since especially $\mathbb{E}B^{-1}<\infty$, we may estimate $\mathbb{P}(B<1/s)\leqslant c/s$. By substituting s^2 in place of s in (134) and applying this inequality, we obtain

$$\Psi_M(s^2) \leqslant \frac{c}{s} + \Psi_M^2(s), \tag{135}$$

where one may assume that $c\geqslant 2$.

Set $f(s):=(c/s^{1/2}+\Psi_M(s))$. Then (135) yields

$$f(s^2) = \frac{c}{s} + \Psi_M(s^2) \leqslant f^2(s). \tag{136}$$

Since $\Psi_M(s)\to 0$ as $s\to\infty$ (while $\mathbb{P}(M=0)=0$), we may choose $s_0>0$ with $\Psi_M(s_0)\leqslant\frac{1}{2}$, whence (136) iterates to $f(s_0^{2^k})\leqslant 2^{-2^k}$ for $k\geqslant 1$. Together with the monotonicity of f, this yields $\delta>0$ such that $f(s)\leqslant cs^{-\delta}$ for $s>0$, especially $\Psi_M(s)\leqslant cs^{-\delta}$.

We obtain that

$$\mathbb{E}M^{-\delta/2} = c\int_0^\infty \mathbb{E}e^{-sM} s^{\delta/2-1}\,ds < \infty.$$

In order to cover all values of q in (133) we employ a simple bootstrapping argument. Assume that $\mathbb{E}M^{-q}<\infty$ for some $q>0$. By applying the inequality between the arithmetic and geometric mean, the independence of B, M_1 and M_2, and the fact that B has all negative moments finite, we may estimate

$$\mathbb{E}M^{-2q} \leqslant \mathbb{E}(B(M_1+M_2))^{-2q} \leqslant c\mathbb{E}(M_1M_2)^{-q} = c(\mathbb{E}M^{-q})^2 < \infty. \tag{137}$$

By induction, this finishes the proof.

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Received September 8, 2009
Received in revised form August 3, 2010