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On locally constructible spheres and balls

by

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1. Introduction

Ambjørn, Boulatov, Durhuus, Jonsson and others have worked to develop a 3-dimensional analogue of the simplicial quantum gravity theory, as provided for two dimensions by Regge [42]. (See [3] and [43] for surveys.) The discretized version of quantum gravity considers simplicial complexes instead of smooth manifolds; the metric properties are artificially introduced by assigning length a to any edge. (This approach is due to Weingarten [47] and known as "theory of dynamical triangulations".) A crucial path integral over metrics, the "partition function for gravity", is then defined via a weighted sum over all triangulated manifolds of fixed topology. In three dimensions, the whole model is convergent only if the number of triangulated 3-spheres with N facets grows not faster than C^N, for some constant C. But does this hold? How many simplicial spheres are there with N facets, for N large?

This crucial question still represents a major open problem, which was put into the spotlight also by Gromov [19, pp. 156-157]. Its 2-dimensional analogue, however, was answered a long time ago by Tutte [45], [46], who proved that there are asymptotically fewer than $(16/3\sqrt{3}\,)^N$ combinatorial types of triangulated 2-spheres. (By Steinitz' theorem, cf. [49, Lecture 4], this quantity equivalently counts the maximal planar maps on $n\geqslant 4$ vertices, which have $N=2n-4$ faces, and also the combinatorial types of simplicial 3-dimensional polytopes with N facets.)

In the following, the adjective "simplicial" will often be omitted when dealing with balls, spheres or manifolds, as all the regular cell complexes and polyhedral complexes that we consider are simplicial.

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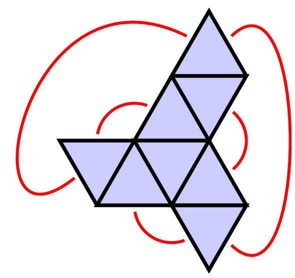


Figure 1. How to get an octahedron from a tree of 8 triangles (i.e., a triangulated 10-gon).

Why are 2-spheres "not so many"? Every combinatorial type of triangulation of the 2-sphere can be generated as follows (see Figure 1): First for some even $N \geqslant 4$ build a tree of N triangles (which combinatorially is the same thing as a triangulation of an $(N+2)$-gon), and then glue edges according to a complete matching of the boundary edges. A necessary condition in order to obtain a 2-sphere is that such a matching is *planar*. Planar matchings and triangulations of $(N+2)$-gons are both enumerated by the Catalan number C_{N+2}, and since the Catalan numbers satisfy a polynomial bound

$$C_N = \frac{1}{N+1}\binom{2N}{N} < 4^N,$$

we get an exponential upper bound for the number of triangulations.

Neither this simple argument nor Tutte's precise count can be easily extended to higher dimensions. Indeed, we have to deal with three different problems when trying to extend results or methods from dimension 2 to dimension 3:

(i) Many combinatorial types of simplicial 3-spheres are not realizable as boundaries of convex 4-polytopes; thus, even though we observe below that there are only exponentially many simplicial 4-polytopes with N facets, the 3-spheres could still be more numerous.

(ii) The counts of combinatorial types according to the number n of vertices and according to the number N of facets are not equivalent any more. We have $3n-10 \leqslant N \leqslant \frac{1}{2}n(n-3)$ by the lower (resp. upper) bound theorem for simplicial 3-spheres. We know that there are more than $2^{n\sqrt[4]{n}}$ 3-spheres [30], [40], but less than $2^{20n\log n}$ types of 4-polytopes with n vertices [1], [17], yet this does not answer the question for a count in terms of the number N of facets.

(iii) While it is still true that there are only exponentially many "trees of N tetrahedra", the matchings that can be used to glue 3-spheres are not planar any more; thus, they could be more than exponentially many. If, on the other hand, we restrict ourselves to "local gluings", we generate only a limited family of 3-spheres, as we will show below.

In the early nineties, new finiteness theorems by Cheeger [12] and Grove, Petersen and Wu [20] yielded a new approach, namely, to count d-manifolds of "fluctuating topology" (not necessarily spheres) but "bounded geometry" (curvature and diameter bounded from above, and volume bounded from below). This allowed Bartocci, Bruzzo, Carfora and Marzuoli [6] to bound, for any d-manifold, the number of triangulations with N or more facets, under the assumption that no vertex had degree higher than a fixed integer. However, for this it is crucial to restrict the topological type: Already for $d=2$, there are more than exponentially many triangulated 2-manifolds of bounded vertex degree with N facets (see [7]).

In 1995, the physicists Durhuus and Jonsson [14] introduced the class of "locally constructible" (LC) 3-spheres. An LC 3-sphere (with N facets) is a sphere obtainable from a tree of N tetrahedra, by identifying pairs of adjacent triangles in the boundary. "Adjacent" means here "sharing at least one edge", and represents a dynamic requirement. Clearly, every 3-sphere is obtainable from a tree of N tetrahedra by matching the triangles in its boundary; according to the definition of LC, however, we are allowed to match only those triangles that *are* adjacent—or that have *become* adjacent by the time of the gluing.

Durhuus and Jonsson proved an exponential upper bound on the number of combinatorially distinct LC spheres with N facets. Based also on computer simulations ([4], see also [11] and [2]) they conjectured that all 3-spheres should be LC. A positive solution of this conjecture would have implied that spheres with N facets are at most C^N, for a constant C—which would have been the desired missing link to implement discrete quantum gravity in three dimensions.

In the present paper, we show that the conjecture of Durhuus and Jonsson has a negative answer: There are simplicial 3-spheres that are not LC. (With this, however, we do not resolve the question whether there are fewer than C^N simplicial 3-spheres on N facets, for some constant C.)

On the way to this result, we provide a characterization of LC simplicial d-complexes which relates the "locally constructible" spheres defined by physicists to concepts that originally arose in topological combinatorics.

Main theorem 1. (Theorem 2.1) *A simplicial d-sphere, $d\geqslant 3$, is* LC *if and only if the sphere after removal of one facet can be collapsed onto a complex of dimension $d-2$. Moreover, the following inclusions between families of simplicial d-spheres hold*:

$$\{\text{vertex decomposable}\} \subsetneq \{\text{shellable}\} \subseteq \{\text{constructible}\} \subsetneq \{\text{LC}\} \subsetneq \{\text{all } d\text{-spheres}\}.$$

We use the hierarchy in conjunction with the following extension and sharpening of Durhuus and Jonsson's theorem (who discussed only the case $d=3$).

Main theorem 2. (Theorem 4.4) *For fixed* $d\geqslant 2$, *the number of combinatorially distinct simplicial* LC d*-spheres with* N *facets grows not faster than* $2^{d^2 N}$.

We will give a proof for this theorem in §4; the same type of upper bound, with the same type of proof, also holds for LC d-balls with N facets.

Already in 1988, Kalai [30] constructed, for every $d\geqslant 4$, a family of more than exponentially many d-spheres on n vertices; Lee [34] later showed that all of Kalai's spheres are shellable. Combining this with Theorems 4.4 and 2.1, we obtain the following asymptotic result.

Corollary. *For fixed* $d\geqslant 4$, *the number of shellable simplicial* d*-spheres grows more than exponentially with respect to the number* n *of vertices*, *but only exponentially with respect to the number* N *of facets.*

The hierarchy of Main theorem 1 is not quite complete: It is still not known whether constructible, non-shellable 3-spheres exist (see [15] and [31]). A shellable 3-sphere that is not vertex-decomposable was found by Lockeberg in his 1977 Ph. D. work (reported in [33, p. 742]; see also [23]). Again, the 2-dimensional case is much simpler and completely solved: All 2-spheres are vertex decomposable (see [41]).

In order to show that not all spheres are LC we study in detail simplicial spheres with a "knotted triangle"; these are obtained by adding a cone over the boundary of a ball with a knotted spanning edge (as in Furch's 1924 paper [16]; see also Bing [9]). Spheres with a knotted triangle cannot be boundaries of polytopes. Lickorish [36] had shown in 1991 the following:

A 3*-sphere with a knotted triangle is not shellable if the knot is at least* 3*-complicated.*

Here "at least 3-complicated" refers to the technical requirement that the fundamental group of the complement of the knot has no presentation with less than four generators. A concatenation of three or more trefoil knots satisfies this condition. In 2000, Hachimori and Ziegler [21], [26] demonstrated that Lickorish's technical requirement is not necessary for his result:

A 3*-sphere with* any *knotted triangle is not constructible.*

In the present work, we re-justify Lickorish's technical assumption, showing that this is exactly what we need if we want to reach a stronger conclusion, namely, a topological obstruction to local constructibility. Thus, the following result is established in order to prove that the last inclusion of the hierarchy in Theorem 2.1 is strict.

Main theorem 3. (Theorem 2.13) *A* 3*-sphere with a knotted triangle is not* LC *if the knot is at least* 3*-complicated.*

The knot complexity requirement is now necessary, as non-constructible spheres with a single or double trefoil knot can still be LC (see Example 2.26 and Remark 2.32).

The combinatorial topology of d-balls and that of d-spheres are of course closely related—our study builds on the well-known connections and also adds new ones.

MAIN THEOREM 4. (Theorems 3.1 and 3.10) *A simplicial d-ball is* LC *if and only if after the removal of a facet it collapses down to the union of the boundary with a complex of dimension at most $d-2$. We have the following hierarchy:*

$$\left\{\begin{matrix}\text{vertex}\\ \text{decomp.}\end{matrix}\right\} \subsetneq \{\text{shellable}\} \subsetneq \{\text{construct.}\} \subsetneq \{\text{LC}\} \subsetneq \left\{\begin{matrix}\text{collapsible onto a}\\ (d-2)\text{-complex}\end{matrix}\right\} \subsetneq \{\text{all } d\text{-balls}\}.$$

All the inclusions of Main theorem 4 hold with equality for simplicial 2-balls. In the case of $d=3$, collapsibility onto a $(d-2)$-complex is equivalent to collapsibility. In particular, we settle a question of Hachimori (see e.g. [22, pp. 54 and 66]) whether all constructible 3-balls are collapsible.

Furthermore, we show in Corollary 3.24 that some collapsible 3-balls do not collapse onto their boundary minus a facet, a property that comes up in classical studies in combinatorial topology (compare [13] and [35]). In particular, a result of Chillingworth can be restated in our language as "if for any geometric simplicial complex Δ the support (union) $|\Delta|$ is a convex 3-dimensional polytope, then Δ is necessarily an LC 3-ball", see Theorem 3.27. Thus any geometric subdivision of the 3-simplex is LC.

1.1. Definitions and Notation

1.1.1. Simplicial regular CW complexes

In the following, we present the notion of "local constructibility" (due to Durhuus and Jonsson). Although in the end we are interested in this notion as applied to finite simplicial complexes, the iterative definition of locally constructible complexes dictates that for intermediate steps we must allow for the greater generality of finite "simplicial regular CW complexes". A CW complex is *regular* if the attaching maps for the cells are injective on the boundary (see e.g. [10]). A regular CW complex is *simplicial* if for every proper face F, the interval $[0, F]$ in the face poset of the complex is boolean. Every simplicial complex (and in particular, any triangulated manifold) is a simplicial regular CW complex.

The k-dimensional cells of a regular CW complex C are called *k-faces*; the inclusion-maximal faces are called *facets*, and the inclusion-maximal proper subfaces of the facets are called *ridges*. The *dimension* of C is the largest dimension of a facet; *pure* complexes are complexes where all facets have the same dimension. All complexes that we consider in the following are finite, most of them are pure. A *d-complex* is a d-dimensional complex. Conventionally, the 0-faces are called *vertices*, and the 1-faces *edges*. (In the discrete quantum gravity literature, the $(d-2)$-faces are sometimes called "hinges" or "bones", whereas the edges are sometimes referred to as "links".) If the union $|C|$ of all simplices of C is homeomorphic to a manifold M, then C is a *triangulation* of M; if C is a triangulation of a d-ball or of a d-sphere, we will call C simply a *d-ball* (resp. *d-sphere*). The *dual graph* of a pure d-dimensional simplicial complex C is the graph whose nodes correspond to the facets of C: Two nodes are connected by an arc if and only if the corresponding facets share a $(d-1)$-face.

1.1.2. Knots

All the knots we consider are *tame*, that is, realizable as 1-dimensional subcomplexes of some triangulated 3-sphere. A knot is *m-complicated* if the fundamental group of the complement of the knot in the 3-sphere has a presentation with $m+1$ generators, but no presentation with m generators. By "at least m-complicated" we mean "k-complicated for some $k\geqslant m$". There exist arbitrarily complicated knots: Goodrick [18] showed that the connected sum of m trefoil knots is at least m-complicated.

Another measure of how tangled a knot can be is the bridge index (see e.g. [32, p. 18] for the definition). If a knot has bridge index b, the fundamental group of the knot complement admits a presentation with b generators and $b-1$ relations [32, p. 82]. In other words, the bridge index of an m-complicated knot is at least $m+1$. As a matter of fact, the connected sum of m trefoil knots is m-complicated, and its bridge index is exactly $m+1$ [15].

1.1.3. The combinatorial topology hierarchy

In the following, we review the key properties from the inclusion

$$\{\text{shellable}\} \subsetneq \{\text{constructible}\}$$

valid for all simplicial complexes, and the inclusion

$$\{\text{shellable}\} \subsetneq \{\text{collapsible}\}$$

applicable only for *contractible* simplicial complexes, both known from combinatorial topology (see [10, §11] for details).

Shellability can be defined for pure simplicial complexes as follows:

– every simplex is shellable;

– a d-dimensional pure simplicial complex C which is not a simplex is shellable if and only if it can be written as $C=C_1\cup C_2$, where C_1 is a shellable d-complex, C_2 is a d-simplex, and $C_1\cap C_2$ is a shellable $(d-1)$-complex.

Constructibility is a weakening of shellability, defined by:

– every simplex is constructible;

– a d-dimensional pure simplicial complex C which is not a simplex is constructible if and only if it can be written as $C=C_1\cup C_2$, where C_1 and C_2 are constructible d-complexes and $C_1\cap C_2$ is a constructible $(d-1)$-complex.

Let C be a d-dimensional simplicial complex, not necessarily pure. An *elementary collapse* is the simultaneous removal from C of a pair of faces (σ, Σ) with the following properties:

– $\dim\Sigma=\dim\sigma+1$;

– σ is a proper face of Σ;

– σ is not a proper face of any other face of C.

(The three conditions above are usually abbreviated in the expression "σ is a free face of Σ"; some complexes have no free faces.) If $C':=(C-\Sigma)-\sigma$, we say that the complex C *collapses onto* the complex C'. We also say that the complex C *collapses onto* the complex D, and write $C\searrow D$, if C can be reduced to D by a finite sequence of elementary collapses. Thus a *collapse* refers to a sequence of elementary collapses. A *collapsible* complex is a complex that can be collapsed onto a single vertex.

Since $C':=(C-\Sigma)-\sigma$ is a deformation retract of C, each collapse preserves the homotopy type. In particular, all collapsible complexes are contractible. The converse does not hold in general: For example, the so-called "dunce hat" is a contractible 2-complex without free edges, and thus with no elementary collapse to start with. However, the implication "contractible $\Rightarrow$ collapsible" holds for all 1-complexes, and also for shellable complexes of any dimension.

A connected 2-dimensional complex is collapsible if and only if it does *not* contain a 2-dimensional complex without a free edge. In particular, for 2-dimensional complexes, if $C\searrow D$ and D is not collapsible, then C is also not collapsible. This does not hold anymore for complexes C of dimension larger than 2 [28].

1.1.4. LC pseudomanifolds

By a *d-pseudomanifold* (possibly with boundary) we mean a finite regular CW complex P which is pure d-dimensional, simplicial, and such that each $(d-1)$-dimensional cell belongs to at most two d-cells. The *boundary* of the pseudomanifold P, denoted ∂P, is the smallest subcomplex of P containing all the $(d-1)$-cells of P which belong to exactly one d-cell of P.

According to our definition, a pseudomanifold need not be a simplicial complex; it might be disconnected; and its boundary might not be a pseudomanifold.

Definition 1.1. (Locally constructible pseudomanifold) For $d \geqslant 2$, let C be a pure d-dimensional simplicial complex with N facets. A *local construction* for C is a sequence $T_1, ..., T_N, ..., T_k$, $k \geqslant N$, such that T_i is a d-pseudomanifold for each i and

(1) T_1 is a d-simplex;

(2) if $i \leqslant N-1$, then T_{i+1} is obtained from T_i by gluing a new d-simplex to T_i alongside one of the $(d-1)$-cells in ∂T_i;

(3) if $i \geqslant N$, then T_{i+1} is obtained from T_i by identifying a pair σ, τ of $(d-1)$-cells in the boundary ∂T_i whose intersection contains a $(d-2)$-cell F;

(4) $T_k = C$.

We say that C is *locally constructible*, or *LC*, if a local construction for C exists. With a little abuse of notation, we will call each T_i an *LC pseudomanifold*. We also say that C is locally constructed *along* T, if T is the dual graph of T_N, and thus a spanning tree of the dual graph of C.

The identifications described in item (3) above are operations which are not closed with respect to the class of simplicial complexes. Local constructions where all steps are simplicial complexes produce only a very limited class of manifolds, consisting of d-balls with no interior $(d-3)$-faces. (When in an LC step the identified boundary facets intersect in *exactly* a $(d-2)$-cell, no $(d-3)$-face is sunk into the interior, and the topology stays the same.)

However, since by definition the local construction in the end must arrive at a pseudomanifold C that *is* a simplicial complex, each intermediate step T_i must satisfy severe restrictions: for each $t \leqslant d$,

– distinct t-simplices which are not in the boundary of T_i share at most one $(t-1)$-simplex;

– distinct t-simplices in the boundary of T_i which share more than one $(t-1)$-simplex will need to be identified by the time the construction of C is completed.

Moreover,

– if σ and τ are the two $(d-1)$-cells glued together in the step from T_i to T_{i+1}, σ and τ cannot belong to the same d-simplex of T_i; nor can they belong to two d-simplices which are already adjacent in T_i.

For example, in each step of the local construction of a 3-sphere, no two tetrahedra share more than one triangle. Moreover, any two distinct interior triangles either are disjoint, or they share a vertex, or they share an edge; but they cannot share two edges, nor three; and they also cannot share one edge and the opposite vertex. If we glued together two boundary triangles which belong to adjacent tetrahedra, no matter what we did afterwards, we would not end up with a simplicial complex any more. Roughly speaking,

> *a locally constructible* 3*-sphere is a triangulated* 3*-sphere obtained from a tree of tetrahedra* T_N *by repeatedly identifying two adjacent triangles in the boundary.*

As mentioned, the boundary of a pseudomanifold need not be a pseudomanifold. However, if P is an LC d-pseudomanifold, ∂P is automatically a $(d-1)$-pseudomanifold. Nevertheless, ∂P may be disconnected, and thus, in general, it is not LC.

All LC d-pseudomanifolds are simply connected; in case $d=3$, their topology is controlled by the following result.

THEOREM 1.2. (Durhuus–Jonsson [14]) *Every* LC 3*-pseudomanifold* P *is homeomorphic to a* 3*-sphere with a finite number of "cacti of* 3*-balls" removed.* (*A cactus of* 3*-balls is a tree-like connected structure in which any two* 3*-balls share at most one point.*) *Thus the boundary* ∂P *is a finite disjoint union of cacti of* 2*-spheres. In particular*, *each connected component of* ∂P *is a simply-connected* 2*-pseudomanifold.*

Thus every closed 3-dimensional LC pseudomanifold is a sphere, while for $d>3$ other topological types such as products of spheres are possible (see Benedetti [8]).

2. On LC spheres

In this section, we establish the following hierarchy announced in the introduction.

THEOREM 2.1. *For all* $d\geqslant 3$, *we have the following inclusion relations between families of simplicial* d*-spheres*:

$$\{\text{vertex decomposable}\} \subsetneq \{\text{shellable}\} \subseteq \{\text{constructible}\} \subsetneq \{\text{LC}\} \subsetneq \{\text{all } d\text{-spheres}\}.$$

Proof. The first two inclusions, and strictness of the first one, are known; the third one will follow from Lemma 2.23 and will be shown to be strict by Example 2.26 together with Lemma 2.24; finally, Corollary 2.22 will establish the strictness of the fourth inclusion for all $d\geqslant 3$. □

2.1. Some d-spheres are not LC

Let S be a simplicial d-sphere, $d \geqslant 2$, and T be a spanning tree of the dual graph of S. We denote by K^T the subcomplex of S formed by all the $(d-1)$-faces of S which are not intersected by T.

LEMMA 2.2. *Let S be any d-sphere with N facets. Then for every spanning tree T of the dual graph of S,*

(i) *K^T is a contractible pure $(d-1)$-dimensional simplicial complex with*

$$\tfrac{1}{2}(dN-N+2)$$

facets;

(ii) *for any facet Δ of S, we have that $S-\Delta \searrow K^T$.*

Any collapse of a d-sphere S minus a facet Δ to a complex of dimension at most $d-1$ proceeds along a dual spanning tree T. To see this, fix a collapsing sequence. We may assume that the collapse of $S-\Delta$ is ordered so that the pairs $((d-1)\text{-face}, d\text{-face})$ are removed first. Whenever both the following conditions are met:

(i) σ is the $(d-1)$-dimensional intersection of the facets Σ and Σ' of S;

(ii) the pair (σ, Σ) is removed in the collapsing sequence of $S-\Delta$;

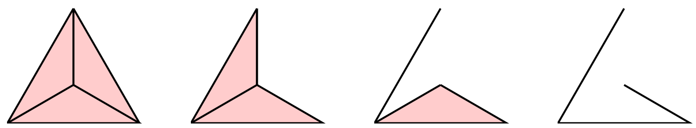
draw an oriented arrow from the center of Σ' to the center of Σ. This yields a directed spanning tree T of the dual graph of S, where Δ is the root. Indeed, T is *spanning* because all d-simplices of $S-\Delta$ are removed in the collapse; it is *connected*, because the only free $(d-1)$-faces of $S-\Delta$, where the collapse can start at, are the proper $(d-1)$-faces of the "missing simplex" Δ; it is *acyclic*, because the center of each d-simplex of $S-\Delta$ is reached by exactly one arrow. We will say that the collapsing sequence *acts along the tree* T (in its top-dimensional part). Thus the complex K^T appears as an intermediate step of the collapse: It is the complex obtained after the $(N-1)$-st pair of faces has been removed from $S-\Delta$.

Definition 2.3. By a *facet-killing sequence* for a d-dimensional simplicial complex C we mean a sequence $C_0, ..., C_t$ of complexes such that $t=f_d(C)$, $C_0=C$ and C_{i+1} is obtained by an elementary collapse which removes a free $(d-1)$-face σ of C_i, together with the unique facet Σ containing σ.

If C is a d-complex, and D is a lower-dimensional complex such that $C \searrow D$, there exists a facet-killing sequence $C_0, ..., C_t$ for C such that $C_t \searrow D$. In other words, the collapse of C onto D can be rearranged so that the pairs $((d-1)\text{-face}, d\text{-face})$ are removed first. In particular, for any d-complex C, the following are equivalent:

(i) there exists a facet-killing sequence for C;

(ii) there exists a k-complex D with $k \leqslant d-1$ such that $C \searrow D$.



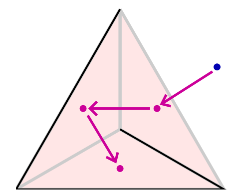


Figure 2. ABOVE. A facet-killing sequence of $S-\Delta$, where S is the boundary of a tetrahedron (d=2), and Δ one of its facets. RIGHT. The 1-complex K^T onto which $S-\Delta$ collapses, and the directed spanning tree T along which the collapse above acts.

What we argued above can be rephrased as follows.

PROPOSITION 2.4. *Let S be a d-sphere and Δ be a d-simplex of S. Let C be a k-dimensional simplicial complex, with $k \leqslant d-2$. Then,*

$$S-\Delta \searrow C \quad \Longleftrightarrow \quad \textit{there exists } T \textit{ such that } K^T \searrow C.$$

The right-hand side in the equivalence of Proposition 2.4 does not depend on the chosen Δ. So, for any d-sphere Δ, either $S-\Delta$ is collapsible for every Δ, or $S-\Delta$ is not collapsible for any Δ.

One more convention: by a *natural labeling* of a rooted tree T on n vertices we mean a bijection $b: V(T) \to \{1, ..., n\}$ such that if v is the root then $b(v)=1$, and if v is not the root then there exists a unique vertex w adjacent to v such that $b(w) < b(v)$.

We are now ready to link the LC concept with collapsibility. Take a d-sphere S, a facet Δ of S and a rooted spanning tree T of the dual graph of S, with root Δ. Since S is given, fixing T is really the same as fixing the manifold T_N in the local construction of S; and at the same time, fixing T is the same as fixing K^T.

Once T, T_N and K^T have been fixed, to describe the first part of a local construction of S (that is, $T_1, ..., T_N$) we just need to specify the order in which the tetrahedra of S have to be added, which is the same as to give a natural labeling of T. Besides, natural labelings of T are in bijection with collapses $S-\Delta \searrow K^T$ (the ith facet to be collapsed is the node of T labeled $i+1$; see Proposition 2.4).

What if we do not fix T? Suppose S and Δ are fixed. Then the previous reasoning yields a bijection among the following sets:

(i) the set of all facet-killing sequences of $S-\Delta$;

(ii) the set of "natural labelings" of spanning trees of S, rooted at Δ;

(iii) the set of the first parts $(T_1, ..., T_N)$ of local constructions for S, with $T_1 = \Delta$.

Can we understand also the second part of a local construction "combinatorially"? Let us start with a variant of the "facet-killing sequence" notion.

Definition 2.5. A *pure facet-massacre* of a pure d-dimensional simplicial complex P is a sequence $P_0, ..., P_t$ of (pure) complexes such that $t = f_d(P)$, $P_0 = P$ and P_{i+1} is obtained by P_i removing:

(a) a free $(d-1)$-face σ of P_i, together with the unique facet Σ containing σ, and

(b) all inclusion-maximal faces of dimension smaller than d which are left after the removal of type (a) or, recursively, after removals of type (b).

In other words, the (b) step removes lower-dimensional facets until one obtains a pure complex. Since $t = f_d(P)$, P_t has no facets of dimension d left, nor inclusion-maximal faces of smaller dimension; hence P_t is empty. The other P_i's are pure complexes of dimension d. Notice that the step $P_i \mapsto P_{i+1}$ is not a collapse, and does not preserve the homotopy type in general. Of course $P_i \mapsto P_{i+1}$ can be "factorized" in an elementary collapse followed by a removal of a finite number of k-faces, with $k < d$. However, this factorization is not unique, as the next example shows.

Example 2.6. Let P be a full triangle. P admits three different facet-killing collapses (each edge can be chosen as a free face), but it admits only one pure facet-massacre, namely $P, \varnothing$.

LEMMA 2.7. *Let P be a pure d-dimensional simplicial complex. Every facet-killing sequence of P naturally induces a unique pure facet-massacre of P. All pure facet-massacres of P are induced by some* (*possibly more than one*) *facet-killing sequence.*

Proof. The map consists in taking a facet-killing sequence $C_0, ..., C_t$, and "cleaning up" the C_i by recursively killing the lower-dimensional inclusion-maximal faces. As the previous example shows, this map is not injective. It is surjective essentially because the removed lower-dimensional faces are of dimension "too small to be relevant". In fact, their dimension is at most $d-1$, hence their presence can interfere only with the freeness of faces of dimension at most $d-2$; so the list of all removals of the form $((d-1)$-face, d-face$)$ in a facet-massacre yields a facet-killing sequence. □

THEOREM 2.8. *Let S be a d-sphere*; *fix a spanning tree T of the dual graph of S. The second part of a local construction for S along T corresponds bijectively to a facet-massacre of K^T.*

Proof. Fix S and T; T_N and K^T are determined by this. Let us start with a local construction $(T_1, ..., T_{N-1},) T_N, ..., T_k$ for S along T. Topologically, $S = T_N/\sim$, where $\sim$ is the equivalence relation determined by the gluing (two distinct points of T_N are equivalent if and only if they will be identified in the gluing). Moreover, $K^T = \partial T_N/\sim$, by the definition of K^T.

Define $P_0 := K^T = \partial T_N/\!\sim$, and $P_j := \partial T_{N+j}/\!\sim$. We leave it to the reader to verify that $k-N$ and $f_d(K^T)$ are the same integer (see Lemma 2.2), which from now on is called D. In particular $P_D = \partial T_k/\!\sim = \partial S/\!\sim = \varnothing$.

In the first LC step, $T_N \mapsto T_{N+1}$, we remove a free ridge r from the boundary, together with the unique pair σ', σ'' of facets of ∂T_N sharing r. At the same time, r and the newly formed face σ are sunk into the interior. This step $\partial T_N \mapsto \partial T_{N+1}$ naturally induces an analogous step $\partial T_{N+j}/\!\sim \mapsto \partial T_{N+j+1}/\!\sim$, namely, the removal of r and of the (unique!) $(d-1)$-face σ containing it.

In the jth LC step, $\partial T_{N+j} \mapsto \partial T_{N+j+1}$, we remove a ridge r from the boundary, together with a pair σ', σ'' of facets sharing r; moreover, we sink into the interior a lower-dimensional face F if and only if we have just sunk into the interior all faces containing F. The induced step from $\partial T_{N+j}/\!\sim$ to $\partial T_{N+j+1}/\!\sim$ is precisely a "facet-massacre" step.

For the converse, we start with a "facet-massacre" $P_0, ..., P_D$ of K^T, and we have $P_0 = K_T = \partial T_N/\!\sim$. The unique $(d-1)$-face σ_j killed in passing from P_j to P_{j+1} corresponds to a unique pair of (adjacent!) $(d-1)$-faces σ'_j, σ''_j in ∂T_{N+j}. Gluing them together is the LC move that transforms T_{N+j} into T_{N+j+1}. □

Remark 2.9. Summing up:

– The first part of a local construction along a tree T corresponds to a facet-killing collapse of $S-\Delta$ (which ends in K^T).

– The second part of a local construction along a tree T corresponds to a pure facet-massacre of K^T.

– A single facet-massacre of K^T corresponds to many facet-killing sequences of K^T.

– By Proposition 2.4, there exists a facet-killing sequence of K^T if and only if K^T collapses onto some $(d-2)$-dimensional complex C. This C is necessarily contractible, like K^T.

So S is locally constructible along T if and only if K^T collapses onto some $(d-2)$-dimensional contractible complex C, if and only if K^T has a facet-killing sequence. What if we do not fix T?

THEOREM 2.10. *Let S be a d-sphere, $d \geqslant 3$. Then the following are equivalent*:

(1) *S is* LC;

(2) *for some spanning tree T of S, K^T is collapsible onto some $(d-2)$-dimensional* (*contractible*) *complex C*;

(3) *there exists a $(d-2)$-dimensional* (*contractible*) *complex C such that for every facet Δ of S, $S-\Delta \searrow C$*;

(4) *for some facet Δ of S, $S-\Delta$ is collapsible onto a $(d-2)$-dimensional* (*contractible*) *complex C.*

Proof. S is LC if and only if it is LC along some tree T; thus (1) $\Leftrightarrow$ (2) follows from Remark 2.9. Besides, (2) $\Rightarrow$ (3) follows from the fact that $S-\Delta \searrow K^T$ (Lemma 2.2), where K^T is independent of the choice of Δ. The implication (3) $\Rightarrow$ (4) is trivial. To show (4) $\Rightarrow$ (2), take a collapse of $S-\Delta$ onto some $(d-2)$-complex C; by Lemma 2.4, there exists some tree T (along which the collapse acts) so that $S-\Delta \searrow K^T$ and $K^T \searrow C$. □

Corollary 2.11. *Let S be a 3-sphere. Then the following are equivalent*:

(1) *S is* LC;

(2) *K^T is collapsible for some spanning tree T of the dual graph of S*;

(3) *$S-\Delta$ is collapsible for every facet Δ of S*;

(4) *$S-\Delta$ is collapsible for some facet Δ of S.*

Proof. This follows from the previous theorem, together with the fact that all contractible 1-complexes are collapsible. □

We are now in the position to exploit results by Lickorish about collapsibility.

Theorem 2.12. (Lickorish [36]) *Let $\mathfrak{L}$ be a knot on m edges in the* 1*-skeleton of a simplicial* 3*-sphere S. Suppose that $S-\Delta$ is collapsible*, *where Δ is some tetrahedron in $S-\mathfrak{L}$. Then $|S|-|\mathfrak{L}|$ is homotopy equivalent to a connected cell complex with one* 0*-cell and at most m* 1*-cells. In particular*, *the fundamental group of $|S|-|\mathfrak{L}|$ admits a presentation with m generators.*

Now assume that a certain sphere S containing a knot $\mathfrak{L}$ is LC. By Corollary 2.11, $S-\Delta$ is collapsible, for any tetrahedron Δ not in the knot $\mathfrak{L}$. Hence by Lickorish's criterion the fundamental group $\pi_1(|S|-|\mathfrak{L}|)$ admits a presentation with m generators.

Theorem 2.13. *Any* 3*-sphere with a* 3*-complicated* 3*-edge knot is not* LC. *More generally*, *a* 3*-sphere with an m-gonal knot cannot be* LC *if the knot is at least m-complicated.*

Example 2.14. As in the construction of the classical "Furch–Bing ball" [16, p. 73], [9, p. 110], [50], we drill a hole into a finely triangulated 3-ball along a triple pike dive of three consecutive trefoils; we stop drilling one step before destroying the property of having a ball (see Figure 3). If we add a cone over the boundary, the resulting sphere has a three edge knot which is a connected sum of three trefoil knots. By Goodrick [18] the connected sum of m copies of the trefoil knot is at least m-complicated. So, this sphere has a knotted triangle, the fundamental group of whose complement has no presentation with three generators. Hence, S cannot be LC.

From this we get a negative answer to the Durhuus–Jonsson conjecture.

Corollary 2.15. *Not all simplicial* 3*-spheres are* LC.

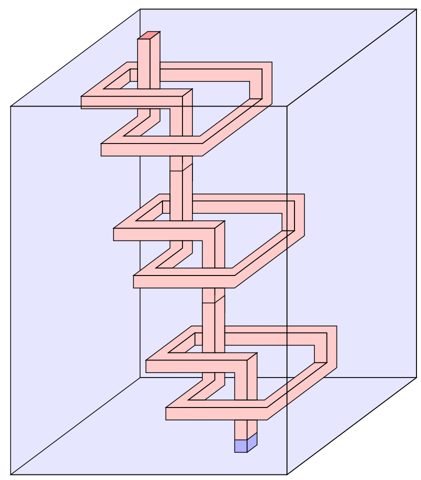


Figure 3. Furch–Bing ball with a (corked) tubular hole along a triple-trefoil knot. The cone over the boundary of this ball is a sphere which is *not* LC.

Lickorish proved also a higher-dimensional statement, basically by taking successive suspensions of the 3-sphere in Example 2.14.

THEOREM 2.16. (Lickorish [36]) *For each* $d \geqslant 3$, *there exists a* PL d*-sphere* S *such that* $S - \Delta$ *is not collapsible for any facet* Δ *of* S.

To exploit our Theorem 2.10 we need a sphere S such that $S - \Delta$ is not even collapsible to a $(d-2)$-complex. To establish that such a sphere exists, we strengthen Lickorish's result.

Definition 2.17. Let K be a d-manifold, A be an r-simplex in K and $\hat{A}$ be the barycenter of A. Consider the barycentric subdivision $\mathrm{sd}(K)$ of K. The *dual* A^* of A is the subcomplex of $\mathrm{sd}(K)$ given by all flags

$$A \subset A_0 \subset \ldots \subset A_r,$$

where $r = \dim A$ and $\dim A_{i+1} = \dim A_i + 1$ for each i.

A^* is a cone with apex $\hat{A}$, and thus collapsible. If K is PL (see e.g. Hudson [29] for the definition), we can say more:

LEMMA 2.18. ([29, Lemma 1.19]) *Let* K *be a* PL d*-manifold* (*without boundary*), *and let* A *be a simplex in* K *of dimension* r. *Then*

(i) A^* *is a* $(d-r)$*-ball, and*

(ii) *if* A *is a face of an* $(r+1)$*-simplex* B, *then* B^* *is a* $(d-r-1)$*-subcomplex of* ∂A^*.

We have observed in Lemma 2.2 that for any d-sphere S and any facet Δ, the ball $S-\Delta$ is collapsible onto a $(d-1)$-complex: in other words, via collapses one can always get *one* dimension down. To get *two* dimensions down is not so easy: our Theorem 2.10 states that $S-\Delta$ is collapsible onto a $(d-2)$-complex precisely when S is LC.

This "number of dimensions down you can get by collapsing" can be related to the minimal presentations of certain homotopy groups. The idea of the next theorem is that if one can get k dimensions down by collapsing a manifold minus one facet, then the $(k-1)$-th homotopy group of the complement of any $(d-k)$-subcomplex of the manifold cannot be too complicated to present.

THEOREM 2.19. *Let t and d be such that $0\leqslant t\leqslant d-2$, and let K be a* PL *d-manifold* (*without boundary*). *Suppose that $K-\Delta$ collapses onto a t-complex, for some facet Δ of K. Then, for each t-dimensional subcomplex $\mathfrak{L}$ of K, the homotopy group*

$$\pi_{d-t-1}(|K|-|\mathfrak{L}|)$$

has a presentation with $f_t(\mathfrak{L})$ generators, while $\pi_i(|K|-|\mathfrak{L}|)$ is trivial for $i<d-t-1$.

Proof. As usual, we assume that the collapse of $K-\Delta$ is ordered so that

– first all pairs ($(d-1)$-face, d-face) are collapsed;

– then all pairs ($(d-2)$-face, $(d-1)$-face) are collapsed;

⋮

– finally, all pairs (t-face, $(t+1)$-face) are collapsed.

Let us put together all the faces that appear above, maintaining their order, to form a single list of simplices

$$A_1, ..., A_{2M}.$$

In such a list, A_1 is a free face of A_2, A_3 is a free face of A_4 with respect to the complex $(K-A_1)-A_2$, and so on. In general, A_{2i-1} is a face of A_{2i} for each i, and in addition, if $j>2i$, A_{2i-1} is not a face of A_j.

We set $X_0=A_0:=\hat{\Delta}$ and define a finite sequence $X_1, ..., X_M$ of subcomplexes of $\operatorname{sd}(K)$ by

$$X_j := \bigcup_{\substack{i\in\{0,...,2j\}\\ A_i\notin\mathfrak{L}}} A_i^* \quad \text{for } j\in\{1,...,M\}.$$

None of the A_{2i}'s can be in $\mathfrak{L}$, because $\mathfrak{L}$ is t-dimensional and $\dim A_{2i}\geqslant\dim A_{2M}=t+1$. However, exactly $f_t(\mathfrak{L})$ of the A_{2i-1}'s are in $\mathfrak{L}$. Consider how X_j differs from X_{j-1}. There are two cases:

- If A_{2j-1} is not in $\mathfrak{L}$, then

$$X_j = X_{j-1}\cup A_{2j-1}^*\cup A_{2j}^*.$$

By Lemma 2.18, setting $r = \dim A_{2j-1}$, A^*_{2j-1} is a $(d-r)$-ball which contains in its boundary the $(d-r-1)$-ball A^*_{2j}. Thus $|X_j|$ is just $|X_{j-1}|$ with a $(d-r)$-cell attached via a cell in its boundary, and such an attachment does not change the homotopy type.

- If A_{2j-1} is in $\mathfrak{L}$, then

$$X_j = X_{j-1} \cup A^*_{2j}.$$

As this occurs only when $\dim A_{2j-1} = t$, we have that $\dim A_{2j} = t+1$ and $\dim A^*_{2j} = d-t-1$; hence $|X_j|$ is just $|X_{j-1}|$ with a $(d-t-1)$-cell attached via its whole boundary.

Only in the second case does the homotopy type of $|X_j|$ change at all, and this second case occurs exactly $f_t(\mathfrak{L})$ times. Since X_0 is one point, it follows that X_M is homotopy equivalent to a bouquet of $f_t(\mathfrak{L})$ many $(d-t-1)$-spheres.

Now let us list by (weakly) decreasing dimension the faces of K that do not appear in the previous list $A_1, ..., A_{2M}$. We name the elements of this list

$$A_{2M+1}, A_{2M+2}, ..., A_F$$

(where $F+1$ is the number of non-empty faces of K).

Correspondingly, we recursively define a new sequence of subcomplexes of $\mathrm{sd}(K)$ setting $Y_0 := X_M$ and

$$Y_h := \begin{cases} Y_{h-1}, & \text{if } A_{2M+h} \in \mathfrak{L}, \\ Y_{h-1} \cup A^*_{2M+h}, & \text{otherwise.} \end{cases}$$

Since $\dim A_{2M+h} \leqslant \dim A_{2M+1} = t$, we have that $|Y_h|$ is just $|Y_{h-1}|$ with possibly a cell of dimension at least $d-t$ attached via its whole boundary. Let us consider the homotopy groups of the Y_h's : Recall that Y_0 was homotopy equivalent to a bouquet of $f_t(\mathfrak{L})$ $(d-t-1)$-spheres. Clearly, for all h,

$$\pi_j(Y_h) = 0 \quad \text{for each } j \in \{1, ..., d-t-2\}.$$

Moreover, the higher-dimensional cell attached to $|Y_{h-1}|$ to get $|Y_h|$ corresponds to the addition of relators to a presentation of $\pi_{d-t-1}(Y_{h-1})$ to get a presentation of $\pi_{d-t-1}(Y_h)$. This means that for all h the group $\pi_{d-t-1}(Y_h)$ is generated by (at most) $f_t(\mathfrak{L})$ elements.

The conclusion follows from the fact that, by construction, Y_{F-2M} is the subcomplex of $\mathrm{sd}(K)$ consisting of all simplices of $\mathrm{sd}(K)$ that have no vertex in $\mathrm{sd}(\mathfrak{L})$; and one can easily prove (see [36, Lemma 1]) that such a complex is a deformation retract of $|K| - |\mathfrak{L}|$. □

COROLLARY 2.20. *Let S be a* PL *d-sphere with a $(d-2)$-dimensional subcomplex $\mathfrak{L}$. If the fundamental group of $|S| - |\mathfrak{L}|$ has no presentation with $f_{d-2}(\mathfrak{L})$ generators, then S is not* LC.

Proof. Set $t=d-2$ in Theorem 2.19, and apply Theorem 2.10. □

COROLLARY 2.21. *Fix an integer* $d\geqslant 3$. *Let* S *be a* 3*-sphere with an* m*-gonal knot in its* 1*-skeleton, so that the knot is at least* $(m2^{d-3})$*-complicated. Then the* $(d-3)$*-rd suspension of* S *is a* PL d*-sphere which is not* LC.

Proof. Let S' be the $(d-3)$-rd suspension of S, and let $\mathfrak{L}'$ be the subcomplex of S' obtained taking the $(d-3)$-rd suspension of the m-gonal knot $\mathfrak{L}$. Since $|S|-|\mathfrak{L}|$ is a deformation retract of $|S'|-|\mathfrak{L}'|$, they have the same homotopy groups. In particular, the fundamental group of $|S'|-|\mathfrak{L}'|$ has no presentation with $m2^{d-3}$ generators. Now $\mathfrak{L}'$ is $(d-2)$-dimensional, and

$$f_{d-2}(\mathfrak{L}')=2^{d-3}f_1(\mathfrak{L})=m2^{d-3},$$

whence we conclude via Corollary 2.20, since all 3-spheres are PL (and the PL property is maintained by suspensions). □

COROLLARY 2.22. *For every* $d\geqslant 3$, *not all* PL d*-spheres are* LC.

Theorem 2.19 can be used in connection with the existence of 2-knots, that is, 2-spheres embedded in a 4-sphere in a knotted way (see Kawauchi [32, p. 190]), to see that there are many non-LC 4-spheres beyond those that arise by suspension of 3-spheres. Thus, being "non-LC" is not simply induced by classical knots.

2.2. Many spheres are LC

Next we show that all constructible manifolds are LC.

LEMMA 2.23. *Let* C *be a* d*-pseudomanifold. If* C *can be split in the form* $C=C_1\cup C_2$, *where* C_1 *and* C_2 *are* LC d*-pseudomanifolds and* $C_1\cap C_2$ *is a strongly connected* $(d-1)$*-pseudomanifold, then* C *is* LC.

Proof. Notice first that $C_1\cap C_2=\partial C_1\cap\partial C_2$. In fact, every ridge of C belongs to at most two facets of C, and hence every $(d-1)$-face σ of $C_1\cap C_2$ is contained in exactly one d-face of C_1 and in exactly one d-face of C_2.

Each C_i is LC; let us fix a local construction for each of them, and call T_i the tree along which C_i is locally constructed. Choose some $(d-1)$-face σ in $C_1\cap C_2$, which thus specifies a $(d-1)$-face in the boundary of C_1 and of C_2. Let C' be the pseudomanifold obtained attaching C_1 to C_2 along the two copies of σ. C' can be locally constructed along the tree obtained by joining T_1 and T_2 by an edge across σ: Just redo the same moves of the local constructions of the C_i's. So C' is LC.

Figure 4. Gluing the simplicial 3-balls along the shaded 2-dimensional subcomplex gives an LC, non-constructible 3-pseudomanifold.

If $C_1 \cap C_2$ consists of one simplex only, then $C' \equiv C$ and we are already done. Otherwise, by the strongly connectedness assumption, the facets of $C_1 \cap C_2$ can be labeled $0, ..., m$, so that

(i) the facet labeled by 0 is σ;

(ii) each facet labeled by $k \geqslant 1$ is adjacent to some facet labeled j with $j < k$.

Now, for each $i \geqslant 1$, glue together the two copies of the facet i inside C'. All these gluings are *local* because of the labeling chosen, and we eventually obtain C. Thus, C is LC. $\square$

Since all constructible simplicial complexes are pure and strongly connected [10], we obtain for simplicial d-pseudomanifolds that

$$\{\text{constructible}\} \subseteq \{\text{LC}\}.$$

The previous containment is strict: Let C_1 and C_2 be two LC simplicial 3-balls on 7 vertices consisting of 7 tetrahedra, as indicated in Figure 4. (The 3-balls are cones over the subdivided triangles on their fronts.)

Glue them together in the shaded strongly connected subcomplex in their boundary (which uses 5 vertices and 4 triangles). The resulting simplicial complex C, on 9 vertices and 14 tetrahedra, is LC by Lemma 2.23, but the link of the top vertex is an annulus, and hence not LC. In fact, the complex C is not constructible, since the link of the top vertex is not constructible. Also, C is not 2-connected, it retracts to a 2-sphere. So, LC d-pseudomanifolds are not necessarily $(d-1)$-connected. Since all constructible d-complexes are $(d-1)$-connected, and every constructible d-pseudomanifold is either a d-sphere or a d-ball [25, Proposition 1.4, p. 374], the previous argument produces many examples of d-pseudomanifolds with boundary which are LC but not constructible.

None of these examples, however, will be a sphere (or a ball). We will prove in Theorem 3.16 that there are LC 3-balls which are not constructible; we show now that for d-spheres, for every $d \geqslant 3$, the containment $\{\text{constructible}\} \subseteq \{\text{LC}\}$ is strict.

LEMMA 2.24. *Suppose that a 3-sphere* $\overline{S}$ *is* LC *but not constructible. Then for all* $d \geqslant 3$, *the* $(d-3)$*-rd suspension of* $\overline{S}$ *is a* d*-sphere which is also* LC *but not constructible.*

Proof. Whenever S is an LC sphere, $v*S$ is an LC $(d+1)$-ball. (The proof is straightforward from the definition of "local construction".) Thus the suspension $(v*S)\cup(w*S)$ is also LC by Lemma 2.23. On the other hand, the suspension of a non-constructible sphere is a non-constructible sphere [26, Corollary 2]. □

Of course, we should show that the 3-sphere $\overline{S}$ in the assumption of Lemma 2.24 really exists. This will be established in Example 2.26, using Corollary 2.11 as follows.

LEMMA 2.25. *Let B be a 3-ball, v be an external point and $B\cup v*\partial B$ be the 3-sphere obtained by adding to B a cone over its boundary. If B is collapsible, then $B\cup v*\partial B$ is* LC.

Proof. By Corollary 2.11, and since B is collapsible, all we need to prove is that $(B\cup v*\partial B)-(v*\sigma)$ collapses onto B, for some triangle σ in the boundary of B.

As all 2-balls are collapsible, and $\partial B-\sigma$ is a 2-ball, there is some vertex P in ∂B such that $\partial B-\sigma\searrow P$. This naturally induces a collapse of $v*\partial B-v*\sigma$ onto $\partial B\cup v*P$, according to the correspondence

$$\sigma \text{ is a free face of } \Sigma \quad\Longleftrightarrow\quad v*\sigma \text{ is a free face of } v*\Sigma.$$

Collapsing the edge $v*P$ down to P, we get that $v*\partial B-v*\sigma\searrow\partial B$.

In the collapse given here, the pairs of faces removed are all of the form $(v*\sigma, v*\Sigma)$; thus, the $(d-1)$-faces in ∂B are removed together with subfaces (and not with superfaces) in the collapse. This means that the freeness of the faces in ∂B is not needed; so when we glue back B the collapse $v*\partial B-v*\sigma\searrow\partial B$ can be read off as $B\cup v*\partial B-v*\sigma\searrow B$. □

Example 2.26. In [37], Lickorish and Martin described a collapsible 3-ball B with a knotted spanning edge. This was also obtained independently by Hamstrom and Jerrard [27]. The knot is an arbitrary 2-bridge index knot (for example, the trefoil knot). Merging B with the cone over its boundary, we obtain a knotted 3-sphere $\overline{S}$ which is LC (by Lemma 2.25; see also [36]) but not constructible (because it is knotted; see [22, p. 54] or [26]).

Remark 2.27. In his 1991 paper [36, p. 530], Lickorish announced (for a proof see [7, pp. 100–103]) that "with a little ingenuity" one can get a sphere S with a 2-complicated triangular knot (the double trefoil), such that $S-\Delta$ is collapsible. Such a sphere is LC by Corollary 2.11. See Remark 2.32.

Example 2.28. The triangulated knotted 3-sphere $S^3_{13,56}$ realized by Lutz [38] has 13 vertices and 56 facets. Since it contains a 3-edge trefoil knot in its 1-skeleton, $S^3_{13,56}$ cannot be constructible, according to Hachimori and Ziegler [26].

Let $B_{13,55}$ be the 3-ball obtained removing the facet $\Delta=\{1,2,6,9\}$ from $S^3_{13,56}$. Let σ be the triangle $\{2,6,9\}$. Then $B^3_{13,55}$ collapses to the 2-disk $\partial\Delta-\sigma$ (F. H. Lutz, personal communication; see [7, pp. 106–107]). All 2-disks are collapsible. In particular, $B^3_{13,55}$ is collapsible, so $S^3_{13,56}$ is LC.

Corollary 2.29. *For each $d\geqslant 3$, not all* LC *d-spheres are constructible. In particular, a knotted 3-sphere can be* LC (*but is not constructible*) *if the knot is just* 1-*complicated or* 2-*complicated.*

The knot in the 1-skeleton of the ball B in Example 2.26 consists of a path on the boundary of B together with a "spanning edge", that is, an edge in the interior of B with both extremes on ∂B. This edge determines the knot, in the sense that any other path on ∂B between the two extremes of this edge closes it up into an equivalent knot. For these reasons such an edge is called a *knotted spanning edge*. More generally, a *knotted spanning arc* is a path of edges in the interior of a 3-ball, such that both extremes of the path lie on the boundary of the ball, and any boundary path between these extremes closes it into a knot. (According to this definition, the relative interior of a knotted spanning arc is allowed to intersect the boundary of the 3-ball; this is the approach of Hachimori and Ehrenborg in [15].)

The Example 2.26 can then be generalized by adopting the idea that Hamstrom and Jerrard used to prove their "Theorem B" [27, p. 331], as follows.

Theorem 2.30. *Let K be any* 2-*bridge knot* (*e.g. the trefoil knot*). *For any positive integer m, there exists a collapsible* 3-*ball B_m with a knotted spanning arc of m edges, such that the knot is the connected union of m copies of K.*

Proof. By the work of Lickorish–Martin [37] (see also [27] and Example 2.26), there exists a collapsible 3-ball B with a knotted spanning edge $[x,y]$, the knot being K. So if $m=1$ we are already done.

Otherwise, take m copies $B^{(1)},...,B^{(m)}$ of the ball B and glue them all together by identifying the vertex $y^{(i)}$ of $B^{(i)}$ with the vertex $x^{(i+1)}$ of $B^{(i+1)}$, for each i in $\{1,...,m-1\}$. The result is a cactus of 3-balls C_m. By induction on m, it is easy to see that a cactus of m collapsible 3-balls is collapsible. To obtain a 3-ball from C_m, we thicken the junctions between the 3-balls by attaching $m-1$ square pyramids with apex $y^{(i)}\equiv x^{(i+1)}$. Each pyramid can be triangulated into two tetrahedra to make the final complex simplicial. Let B_m be the resulting 3-ball. All the spanning edges of the $B^{(i)}$'s are concatenated in B_m to yield a knotted spanning arc of m edges, the knot being equivalent to the connected union of m copies of K. Moreover, the "extra pyramids" introduced can be collapsed away. This yields a collapse of the ball B_m onto the complex C_m, which is collapsible. □

Corollary 2.31. *A 3-sphere with an m-complicated $(m+2)$-gonal knot can be* LC.

Proof. Let $S_m = B_m \cup (v * \partial B_m)$, where B_m is the 3-ball constructed in the previous theorem. By Lemma 2.25, S_m is LC. The spanning arc of m edges is closed up in v to form an $(m+2)$-gon. □

Remark 2.32. The bound given by Corollary 2.31 can be improved: In fact, for each positive integer m there exists an LC 3-sphere with an $(m+1)$-complicated $(m+2)$-gonal knot. The proof is rather long, so we preferred to omit it, referring the reader to [7, pp. 100–103].

The spheres mentioned in Corollary 2.31 and Remark 2.32 are not vertex decomposable, not shellable and not constructible, because of the following result about the bridge index.

Theorem 2.33. (Ehrenborg, Hachimori, Shimokawa, [15], [25]) *Suppose that a 3-sphere (or a 3-ball) S contains a knot of m edges.*

– *If the bridge index of the knot exceeds $\frac{1}{3}m$, then S is not vertex decomposable*;

– *If the bridge index of the knot exceeds $\frac{1}{2}m$, then S is not constructible.*

The bridge index of a t-complicated knot is at least $t+1$. So, if a knot is at least $\lfloor \frac{1}{3}m \rfloor$-complicated, its bridge index automatically exceeds $\frac{1}{3}m$. Thus, Ehrenborg–Hachimori–Shimokawa's theorem, the results of Hachimori and Ziegler in [26], the previous examples, and our present results blend into the following new hierarchy.

Theorem 2.34. *A 3-sphere with a non-trivial knot consisting of*

3 *edges*, 1-*complicated*	*is not constructible*, *but can be* LC;
3 *edges*, 2-*complicated*	*is not constructible*, *but can be* LC;
3 *edges*, 3-*complicated or more*	*is not* LC;
4 *edges*, 1-*complicated*	*is not vertex dec.*, *but can be shellable*;
4 *edges*, 2- *or* 3-*complicated*	*is not constructible*, *but can be* LC;
4 *edges*, 4-*complicated or more*	*is not* LC;
5 *edges*, 1-*complicated*	*is not vertex dec.*, *but can be shellable*;
5 *edges*, 2-, 3- *or* 4-*complicated*	*is not constructible*, *but can be* LC;
5 *edges*, 5-*complicated or more*	*is not* LC;
6 *edges*, 1-*complicated*	*can be vertex decomposable*;
6 *edges*, 2-*complicated*	*is not vertex dec.*, *but can be* LC;
6 *edges*, 3-, 4 *or* 5-*complicated*	*is not constructible*, *but can be* LC;
6 *edges*, 6-*complicated or more*	*is not* LC;
⋮	⋮

m edges, *k-complicated*, $k \geqslant \lfloor \frac{1}{3}m \rfloor$	*is not vertex decomposable*;
m edges, *k-complicated*, $k \geqslant \lfloor \frac{1}{2}m \rfloor$	*is not constructible*;
m edges, *k-complicated*, $k \leqslant m-1$	*can be* LC;
m edges, *k-complicated*, $k \geqslant m$	*is not* LC.

The same conclusions are valid for 3-balls which contain a knot, up to replacing the word "LC", wherever it occurs, with the word "collapsible". (See Lemma 2.25, Corollary 3.12 and [26].)

One may also derive from Zeeman's theorem ("any PL simplicial ball admits a collapsible subdivision" [48, Chapter III, Theorem 4]) that any 3-sphere will become LC after a suitable subdivision. On the other hand, there is no fixed number r of barycentric subdivisions that is sufficient to make *all* 3-spheres LC. (For this use sufficiently complicated knots, together with Theorem 2.13.)

3. On LC balls

The combinatorial topology of d-balls and that of d-spheres are intimately related: Removing any facet Δ from a d-sphere S we obtain a d-ball $S-\Delta$, and adding a cone over the boundary of a d-ball B we obtain a d-sphere S_B. We do have a combinatorial characterization of LC d-balls, which we will reach in Theorem 3.10; it is a bit more complicated, but otherwise analogous to the characterization of LC d-spheres as given in Main theorem 1.

THEOREM 3.1. *For simplicial d-balls*, *we have the following hierarchy*:

$$\left\{\begin{matrix}\text{vertex}\\ \text{decomp.}\end{matrix}\right\} \subsetneq \{\text{shellable}\} \subsetneq \{\text{construct.}\} \subsetneq \{\text{LC}\} \subsetneq \left\{\begin{matrix}\text{collapsible onto a}\\ (d-2)\text{-complex}\end{matrix}\right\} \subsetneq \{\text{all } d\text{-balls}\}.$$

Proof. The first two inclusions are known. We have already seen that all constructible complexes are LC (Lemma 2.23). Every LC d-ball is collapsible onto a $(d-2)$-complex by Corollary 3.11.

Let us see next that all inclusions are strict for $d=3$: For the first inclusion this follows from Lockeberg's example of a 4-polytope whose boundary is not vertex decomposable. For the second inclusion, take Ziegler's non-shellable ball from [50], which is constructible by construction. A non-constructible 3-ball which is LC will be provided by Theorem 3.16. A collapsible 3-ball which is not LC will be given in Theorem 3.23. Finally, Bing and Goodrick showed that not every 3-ball is collapsible [9], [18].

To show that the inclusions are strict for all $d \geqslant 3$, we argue as follows. For the first four inclusions we get this from the case $d=3$, since

– cones are always collapsible,

– the cone $v*B$ is vertex decomposable (resp. shellable, constructible) if and only if B is,

– and in Proposition 3.25 we will show that $v*B$ is LC if and only if B is.

For the last inclusion and $d \geqslant 3$, we look at the d-balls obtained by removing a facet from a non-LC d-sphere. These exist by Corollary 2.21; they do not collapse onto a $(d-2)$-complex by Theorem 2.10. □

3.1. Local constructions for d-balls

We begin with a relative version of the notions of "facet-killing sequence" and "facet massacre", which we introduced in §2.1.

Definition 3.2. Let P be a pure d-complex. Let Q be a proper subcomplex of P, either pure d-dimensional or empty. A *facet-killing sequence* of (P,Q) is a sequence $P_0, ..., P_t$ of simplicial complexes such that $t = f_d(P) - f_d(Q)$, $P_0 = P$ and P_{i+1} is obtained from P_i removing a pair (σ, Σ) such that σ is a free $(d-1)$-face of Σ which does not lie in Q (which also implies that $\Sigma \notin Q$).

It is easy to see that P_t has the same d-faces as Q. The version of facet killing sequences given in Definition 2.3 is a special case of this one, namely the case when Q is empty.

Definition 3.3. Let P be a pure d-dimensional simplicial complex. Let Q be either the empty complex, or a pure d-dimensional proper subcomplex of P. A *pure facet-massacre* of (P,Q) is a sequence $P_0, ..., P_t$ of (pure) complexes such that $t = f_d(P) - f_d(Q)$, $P_0 = P$, and P_{i+1} is obtained from P_i removing

(a) a pair (σ, Σ) such that σ is a free $(d-1)$-face of Σ which does not lie in Q, and

(b) all inclusion-maximal faces of dimension smaller than d which are left after the removal of type (a) or, recursively, after removals of type (b).

Necessarily $P_t = Q$ (and when $Q = \varnothing$ we recover the notion of facet-massacre of P introduced in Definition 2.5). It is easy to see that a step $P_i \mapsto P_{i+1}$ can be factorized (not in a unique way) into an elementary collapse followed by a removal of faces of dimensions smaller than d which makes P_{i+1} a pure complex. Thus, a single pure facet-massacre of (P,Q) corresponds to many facet-killing sequences of (P,Q).

We will apply both definitions to the pair $(P,Q) = (K^T, \partial B)$, where K^T is defined for balls as follows.

Definition 3.4. If B is a d-ball with N facets, and T is a spanning tree of the dual graph of B, define K^T as the subcomplex of B formed by all $(d-1)$-faces of B which are not hit by T.

LEMMA 3.5. *Under the previous notation,*

(i) K^T *is a pure* $(d-1)$*-dimensional simplicial complex, containing* ∂B *as a subcomplex*;

(ii) K^T *has* $D+\frac{1}{2}b$ *facets, where* $D:=\frac{1}{2}(dN-N+2)$ *and* b *is the number of facets in* ∂B;

(iii) $B-\Delta \searrow K^T$ *for any* d*-simplex* Δ *of* B;

(iv) K^T *is homotopy equivalent to a* $(d-1)$*-dimensional sphere.*

We introduce another convenient piece of terminology.

Definition 3.6. Let B be a simplicial d-ball. A *seepage* is a $(d-1)$-dimensional subcomplex C of B whose $(d-1)$-faces are exactly given by the boundary of B.

A seepage is not necessarily pure; actually there is only one pure seepage, namely ∂B itself. Since K^T contains ∂B, a collapse of K^T onto a seepage must remove all the $(d-1)$-faces of K^T which are not in ∂B: this is what we called a facet-killing sequence of $(K^T, \partial B)$.

PROPOSITION 3.7. *Let* B *be a* d*-ball, and* Δ *be a* d*-simplex of* B. *Let* C *be a seepage of* ∂B. *Then,*

$$B-\Delta \searrow C \quad \Longleftrightarrow \quad \text{there exists } T \text{ such that } K^T \searrow C.$$

Proof. The proof is analogous to that of Proposition 2.4. The crucial assumption is that no face of ∂B is removed in the collapse (since all boundary faces are still present in the final complex C). □

If we fix a spanning tree T of the dual graph of B, we then have a one-to-one correspondence between the following sets:

(i) the set of collapses $B-\Delta \searrow K^T$;

(ii) the set of "natural labelings" of T, where Δ is labeled by 1;

(iii) the set of the first parts $(T_1, ..., T_N)$ of local constructions for B, with $T_1=\Delta$.

THEOREM 3.8. *Let* B *be a* d*-ball. Fix a facet* Δ *and a spanning tree* T *of the dual graph of* B, *rooted at* Δ. *The second part of a local construction for* B *along* T *corresponds bijectively to a facet-massacre of* $(K^T, \partial B)$.

Proof. Let us start with a local construction $(T_1, ..., T_{N-1},)T_N, ..., T_k$ for B along T. Topologically, $B=T_N/\sim$, where $\sim$ is the equivalence relation determined by the gluing, and $K^T=\partial T_N/\sim$.

K^T has $D+\frac{1}{2}b$ facets (see Lemma 3.5), and all of them, except the b facets in the boundary, represent gluings. Thus we have to describe a sequence $P_0, ..., P_t$ with

$t{=}D-\frac{1}{2}b$. But the local construction $(T_1, ..., T_{N-1},)T_N, ..., T_k$ produces B (which has b facets in the boundary) from T_N (which has $2D$ facets in the boundary, cf. Lemma 4.1) in $k-N$ steps, each removing a pair of facets from the boundary. So, $2D-2(k-N)=b$, which implies that $k-N=t$.

Define $P_0 := K_T = \partial T_N/\sim$ and $P_j := \partial T_{N+j}/\sim$. In the first LC step, $T_N \mapsto T_{N+1}$, we remove a free ridge r from the boundary, together with the unique pair σ', σ'' of facets of ∂T_N sharing r. At the same time, r and the newly formed face σ are sunk into the interior; so obviously neither σ nor r will appear in ∂B. This step $\partial T_N \mapsto \partial T_{N+1}$ naturally induces an analogous step $\partial T_{N+j}/\sim \mapsto \partial T_{N+j+1}/\sim$, namely, the removal of r and of the unique $(d-1)$-face σ containing it, with r not in ∂B.

The rest is analogous to the proof of Theorem 2.8. □

Thus, B can be locally constructed along a tree T if and only if K^T collapses onto some seepage. What if we do not fix the tree T or the facet Δ?

Lemma 3.9. *Let B be a d-ball, let σ be a $(d-1)$-face in the boundary ∂B, and let Σ be the unique facet of B containing σ. Let C be a subcomplex of B. If C contains ∂B, the following are equivalent*:

(1) $B-\Sigma \searrow C$;

(2) $(B-\Sigma)-\sigma \searrow C-\sigma$;

(3) $B \searrow C-\sigma$.

Theorem 3.10. *Let B be a d-ball. Then the following are equivalent*:

(1) *B is* LC;

(2) *K^T collapses onto some seepage C for some spanning tree T of the dual graph of B*;

(3) *there exists a seepage C such that $B-\Delta \searrow C$ for every facet Δ of B*;

(4) *$B-\Delta \searrow C$ for some facet Δ of B and for some seepage C*;

(5) *there exists a seepage C such that $B \searrow C-\sigma$ for every facet σ of ∂B*;

(6) *$B \searrow C-\sigma$ for some facet σ of ∂B and for some seepage C.*

Proof. The equivalences (1) ⇔ (2) ⇔ (3) ⇔ (4) are established analogously to the proof of Theorem 2.10. Finally, Lemma 3.9 implies that (3) ⇒ (5) ⇒ (6) ⇒ (4). □

Corollary 3.11. *Every* LC *d-ball collapses onto a $(d-2)$-complex.*

Proof. By Theorem 3.10, the ball B collapses onto the union of the boundary of B minus a facet with some $(d-2)$-complex. The boundary of B minus a facet is a $(d-1)$-ball; thus it can be collapsed down to dimension $d-2$, and the additional $(d-2)$-complex will not interfere. □

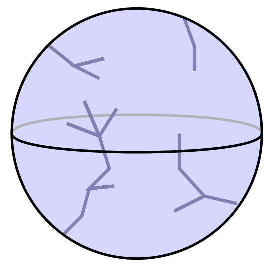


Figure 5. A seepage of a 3-ball.

Corollary 3.12. *Let B be a 3-ball. Then the following are equivalent*:

(1) B *is* LC;

(2) $K^T \searrow \partial B$ *for some spanning tree T of the dual graph of B*;

(3) $B - \Delta \searrow \partial B$ *for every facet Δ of B*;

(4) $B - \Delta \searrow \partial B$ *for some facet Δ of B*;

(5) $B \searrow \partial B - \sigma$ *for every facet σ of ∂B*;

(6) $B \searrow \partial B - \sigma$ *for some facet σ of ∂B.*

Proof. When B has dimension 3, any seepage C of ∂B is a 2-complex containing ∂B, plus some edges and vertices. If a complex homotopy equivalent to S^2 collapses onto C, then C is also homotopy equivalent to S^2; thus C can only be ∂B with some trees attached (see Figure 5), which implies that $C \searrow \partial B$. □

Corollary 3.13. *All* LC 3*-balls are collapsible.*

Proof. If B is LC, it collapses to some 2-ball $\partial B - \sigma$, but all 2-balls are collapsible. □

Corollary 3.14. *All constructible* 3*-balls are collapsible.*

For example, Ziegler's ball, Grünbaum's ball and Rudin's ball are collapsible (see [50]).

Remark 3.15. The locally constructible 3-balls with N facets are precisely the 3-balls which admit a "special collapse", namely such that after the first elementary collapse, in the next $N-1$ collapses, no triangle of ∂B is collapsed away. Such a collapse acts along a dual (directed) tree of the ball, whereas a generic collapse acts along an acyclic graph that might be disconnected.

One could argue that maybe "special collapses" are not that special: Perhaps every collapsible 3-ball has a collapse that removes only one boundary triangle in its top-dimensional phase? This is not so: We will produce a counterexample in the next subsection (Theorem 3.23).

THEOREM 3.16. *For every $d\geqslant 3$, not all* LC *d-balls are constructible.*

Proof. If B is a non-constructible d-ball and v is a new vertex, then $v*B$ is a non-constructible $(d+1)$-ball. Also, it is easy to see that if B is LC then $v*B$ is also LC (cf. Proposition 3.25). Therefore, it suffices to prove the claim for $d=3$.

In Example 2.28 we described the 3-ball $B_{13,55}$ which collapses onto its boundary minus a facet. By Corollary 3.12, $B_{13,55}$ is LC. At the same time, $B_{13,55}$ contains a 3-edge trefoil knot, which prevents $B_{13,55}$ from being constructible [26, Theorem 1]. □

3.2. 3-balls without interior vertices.

Here we show that a simplicial 3-ball with all vertices on the boundary cannot contain any knotted spanning edge if it is LC, but might contain some if it is collapsible. We use this fact to establish our hierarchy for d-balls (Theorem 3.1).

Let us fix some notation first. Recall that by Theorem 1.2, each connected component of the boundary of a simplicial LC 3-pseudomanifold is homeomorphic to a simply-connected union of 2-spheres, any two of which share at most one point. Let us call the points shared by two or more spheres in the boundary of an LC 3-pseudomanifold *pinch points.*

Definition 3.17. (Steps of types (i)–(ix) in LC constructions) Any admissible step in a local construction of a 3-pseudomanifold falls into one of the following nine types:

(i) attaching a tetrahedron along a triangle;
(ii) identifying two boundary triangles which share exactly 1 edge;
(iii) identifying two boundary triangles which share 1 edge and the opposite vertex;
(iv) identifying two b. t. which share 2 edges that meet in a pinch point;
(v) identifying two b. t. which share 2 edges that do not meet in a pinch point;
(vi) identifying two b. t. which share 3 edges, all of whose vertices are pinch points;
(vii) identifying two b. t. which share 3 edges, two of whose vertices are pinch points;
(viii) identifying two b. t. which share 3 edges, one of whose vertices is a pinch point;
(ix) identifying two b. t. which share 3 edges, none of whose vertices is a pinch point.

For example, the first $N-1$ steps of any local construction of a 3-pseudomanifold with N tetrahedra are all of type (i); the last step in the local construction of a 3-sphere is necessarily of type (ix).

Table 1 summarizes the distinguished effects of the steps. The asterisk recalls that a type (iii) step *almost* disconnects the boundary, by pinching it in a point.

Now, let B be an LC 3-ball *without* interior vertices. Steps of type (v), (vii), (viii) and (ix) sink one, one, two and three vertices into the interior, respectively, so they

step type	number of interior vertices	number of connected components of the boundary
(i)	+0	+0
(ii)	+0	+0
(iii)	+0	+0 (∗)
(iv)	+0	+1
(v)	+1	+0
(vi)	+0	+3
(vii)	+1	+2
(viii)	+2	+0
(ix)	+3	−1

Table 1.

cannot occur in the local construction of B. Furthermore, any identification of type (iv) or (vi) increases the number of connected components in the boundary; hence it must be followed by at least one step of type (ix), which destroys a connected component of the boundary. Yet (ix) is forbidden, so no identification of type (iv) or (vi) can occur. Finally, the "pinching step" (iii) needs to be followed by one of the steps (vi), (vii), (viii) and (ix) in order to restore the ball topology—but such steps are forbidden. This leads us to the following lemma.

Lemma 3.18. *Let B be an* LC 3*-pseudomanifold. The following are equivalent*:

(1) *in some local construction for B all steps are of type* (i) *or* (ii);

(2) *in every local construction for B all steps are of type* (i) *or* (ii);

(3) *B is a* 3*-ball without interior vertices.*

We will use Lemma 3.18 to obtain examples of non-LC 3-balls. We already know that non-collapsible balls are not LC, by Corollary 3.13: so a 3-ball with a knotted spanning edge cannot be LC if the knot is the sum of two or more trefoil knots. (See also Bing [9] and Goodrick [18].) What about balls with a spanning edge realizing a single trefoil knot?

Proposition 3.19. *An* LC 3*-ball without interior vertices does not contain any knotted spanning edge.*

Proof. An LC 3-ball B without interior vertices is obtained from a tree of tetrahedra via local gluings of type (ii), by Lemma 3.18. A tree of tetrahedra has no interior edge. Each type-(ii) step preserves the existing spanning edges (because it does not sink vertices into the interior), and creates one more spanning edge e, clearly unknotted (because the other two edges of the sunk triangle form a boundary path which "closes up" the edge e onto an S^1 bounding a disk inside B). It is easy to verify that the subsequent type-(ii) steps leave such an edge e spanning and unknotted. □

Remark 3.20. The presence of knots/knotted spanning edges is not the only obstruction to local constructibility. Bing's thickened house with two rooms [9], [24] is a 3-ball B with all vertices on the boundary, so that every interior triangle of B has at most one edge on the boundary ∂B. Were B LC, every step in its local construction would be of type (ii) (by Lemma 3.18); in particular, the last triangle to be sunk into the interior of B would have exactly two edges on the boundary of B. Thus Bing's thickened house with two rooms cannot be LC, even if it does not contain a knotted spanning edge.

Example 3.21. Furch's 3-ball [16, p. 73], [9, p. 110] can be triangulated without interior vertices (see e.g. [24]). Since it contains a knotted spanning edge, by Proposition 3.19 Furch's ball is not LC.

Remark 3.22. In [21, Lemma 2], Hachimori claimed that any 3-ball C obtained from a constructible 3-ball C' via a type-(ii) step is constructible. This would imply by Lemma 3.18 that all LC 3-balls without interior vertices are constructible, which is stronger than Proposition 3.19, since constructible 3-balls do not contain knotted spanning edges [26, Lemma 1]. Unfortunately, Hachimori's proof [21, p. 227] is not satisfactory: If $C'=C'_1\cup C'_2$ is a constructible decomposition of C', and C_i is the subcomplex of C with the same facets as C'_i, $C=C_1\cup C_2$ need not be a constructible decomposition for C. (For example, if the two glued triangles both lie on $\partial C'_1$, and if the two vertices which the triangles do not have in common lie in $C'_1\cap C'_2$, then $C_1\cap C_2$ is not a 2-ball, and one of C_1 and C_2 is not a 3-ball.)

At present we do not know whether Hachimori's claim is true: Does C' admit a different constructible decomposition which survives the type-(ii) step? On this depends the correctness of the algorithm [21, p. 227], [22, p. 101] to test *constructibility* of 3-balls without interior vertices by cutting them open along triangles with exactly two boundary edges. However, we point out that Hachimori's algorithm can be validly used to decide the *local constructibility* of 3-balls without interior vertices: In fact, by Lemma 3.18, the algorithm proceeds by reversing the LC-construction of the ball.

We can now move on to complete the proof of our Theorem 3.1. Inspired by Proposition 3.19, we show that a *collapsible* 3-ball without interior vertices may contain a knotted spanning edge. Our construction is a tricky version of Lickorish–Martin's (see Example 2.26).

Theorem 3.23. *Not all collapsible* 3*-balls are* LC.

Proof. Start with a large $m\times m\times 1$ pile of cubes, triangulated in the standard way, and take away two distant cubes, leaving only their bottom squares X and Y. The 3-

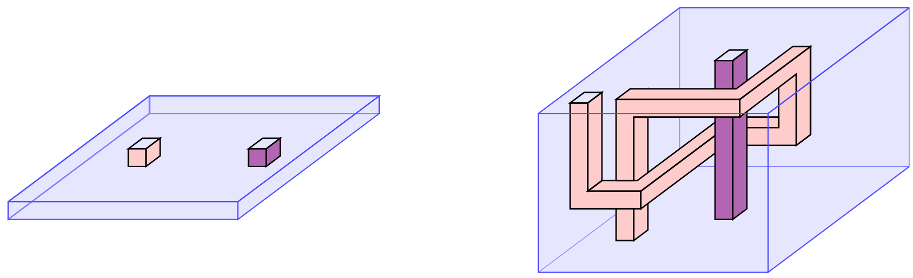


Figure 6. C and C' are obtained from a 3-ball drilling away two tubular holes, and then "corking" the holes on the bottom with 2-dimensional membranes.

complex C obtained can be collapsed vertically onto its square basis; in particular, it is collapsible, and has no interior vertices.

Let C' be a 3-ball with two tubular holes drilled away, but where (1) each hole has been corked at a bottom with a 2-disk, and (2) the tubes are disjoint but intertwined, so that a closed path that passes through both holes and between these traverses the top (resp. bottom) face of C' yields a trefoil knot (see Figure 6).

C and C' are homeomorphic. Any homeomorphism induces a collapsible triangulation on C' with no interior vertices. X and Y correspond via the homeomorphism to the corking membranes of C', which we will call correspondingly X' and Y'. To get from C' to a ball with a knotted spanning edge we will carry out two more steps:

(i) create a single edge $[x', y']$ which goes from X' to Y';

(ii) thicken the "bottom" of C' a bit, so that C' becomes a 3-ball and $[x', y']$ becomes an interior edge (even if its extremes are still on the boundary).

We perform both steps by adding cones over 2-disks to the complex. Such steps preserve collapsibility, but in general they produce interior vertices; thus we choose "specific" disks with few interior vertices.

(i) Provided m is large enough, one finds a "nice" strip $F_1, ..., F_k$ of triangles on the bottom of C', such that $F_1 \cup ... \cup F_k$ is a disk without interior vertices, F_1 has a single vertex x' in the boundary of X', while F_k has a single vertex y' in the boundary of Y', and the whole strip intersects $X' \cup Y'$ only in x' and y'. Then we add a cone to C', setting

$$C_1 \ := \ C' \cup (y' * (F_1 \cup ... \cup F_{k-1})) \, .$$

(An explicit construction of this type is carried out in [26, pp. 164-165].) Thus one obtains a collapsible 3-complex C_1 with no interior vertex, and with a direct edge from X' to Y'.

(ii) Let R be a 2-ball inside the boundary of C_1 which contains in its interior the 2-complex $X' \cup Y' \cup [x', y']$, and such that every interior vertex of R lies either in X' or in Y'. Take a new point z' and define $C_2 := C_1 \cup (z' * R)$.

As $z'*R$ collapses onto R, it is easy to verify that C_2 is a collapsible 3-ball with a knotted spanning edge $[x', y']$. By Proposition 3.19, C_2 is not LC. □

COROLLARY 3.24. *There exists a collapsible* 3*-ball B such that for any boundary facet σ, the ball B does not collapse onto $\partial B - \sigma$.*

Theorem 3.23 can be extended to higher dimensions by taking cones. In fact, even though the link of an LC complex need not be LC, the link of an LC closed star is indeed LC.

PROPOSITION 3.25. *Let C be a d-pseudomanifold and v be a new point. Then C is* LC *if and only if $v*C$ is* LC*.*

Proof. The implication "if C is LC, then $v*C$ is LC" is straightforward. For the converse, assume T_i and T_{i+1} are intermediate steps in the local construction of $v*C$, so that passing from T_i to T_{i+1} we glue together two adjacent d-faces σ' and σ'' of ∂T_i. Let F be any $(d-1)$-face of T_i. If F does not contain v, then F is in the boundary of $v*C$, so $F \in \partial T_{i+1}$. Therefore, F cannot belong to the intersection of σ' and σ'', which is sunk into the interior of T_{i+1}.

So, every $(d-1)$-face in the intersection $\sigma' \cap \sigma''$ must contain the vertex v. This implies that $\sigma' = v*S'$ and $\sigma'' = v*S''$, with S' and S'' being *distinct* $(d-1)$-faces. S' and S'' must share some $(d-2)$-face, otherwise σ' and σ'' would not be adjacent. So from a local construction of $v*C$ we can read off a local construction of C. □

COROLLARY 3.26. *For every $d \geqslant 3$, not all collapsible d-balls are* LC*.*

Proof. All cones are collapsible. If B is a non-LC d-ball, then $v*B$ is a non-LC $(d+1)$-ball by Proposition 3.25. □

We conclude this chapter observing that Chillingworth's theorem, "every geometric triangulation of a convex 3-dimensional polytope is collapsible", can be strengthened as follows.

THEOREM 3.27. (Chillingworth [13]) *Every* 3*-ball embeddable as a convex subset of the Euclidean* 3*-space $\mathbb{R}^3$ is* LC*.*

Proof. The argument of Chillingworth for collapsibility runs showing that

$$B \searrow \partial B - \sigma,$$

where σ is any triangle in the boundary of B. Now Theorem 3.12 ends the proof. □

Thus any subdivided 3-simplex is LC. If Hachimori's claim is true (see Remark 3.22), then any subdivided 3-simplex with all vertices on the boundary is also constructible. (So far we can only exclude the presence of knotted spanning edges in it, see Lemma 3.18.) However, a subdivided 3-simplex might be non-shellable even if it has all vertices on the boundary (Rudin's ball is an example).

4. Upper bounds on the number of LC d-spheres.

For fixed $d \geqslant 2$ and a suitable constant C which depends on d, there are less than C^N combinatorial types of LC d-spheres with N facets. Our proof for this fact is a d-dimensional version of the main theorem of Durhuus and Jonsson [14], and allows us to determine an explicit constant C, for any d. It consists of two different phases:

(1) we observe that there are less trees of d-simplices than planted plane d-ary trees, which are counted by order-d Fuss–Catalan numbers;

(2) we count the number of "LC matchings" according to ridges in the tree of simplices.

4.1. Counting the trees of d-simplices.

We will here establish that there are less than

$$C_d(N) := \frac{1}{(d-1)N+1}\binom{dN}{N}$$

trees of N d-simplices.

LEMMA 4.1. *Every tree of N d-simplices has $(d-1)N+2$ boundary facets of dimension $d-1$ and $N-1$ interior faces of dimension $d-1$.*

It has $\frac{1}{2}d((d-1)N+2)$ faces of dimension $d-2$, all of which lie in the boundary.

By *rooted* tree of simplices we mean a tree of simplices B together with a distinguished facet δ of ∂B, whose vertices have been labeled $1, ..., d$. Rooted trees of d-simplices are in bijection with "planted plane d-ary trees", that is, plane rooted trees such that every non-leaf vertex has exactly d (left-to-right ordered) sons; cf. [39].

PROPOSITION 4.2. *There is a bijection between rooted trees of N d-simplices and planted plane d-ary trees with N non-leaf vertices, which in turn are counted by the Fuss–Catalan numbers $C_d(N)$. Thus, the number of combinatorially-distinct trees of N d-simplices satisfies*

$$\frac{1}{(d-1)N+2}\,\frac{1}{d!}\,C_d(N) \leqslant \#\{\textit{trees of } N \ d\textit{-simplices}\} \leqslant C_d(N).$$

Proof. Given a rooted tree of d-simplices with a distinguished facet δ in its boundary, there is a unique extension of the labeling of the vertices of δ to a labeling of all the vertices by labels $1,...,d+1$, such that no two adjacent vertices get the same label. Thus each d-simplex receives all $d+1$ labels exactly once.

Now, label each $(d-1)$-face by the unique label that none of its vertices has. With this we get an edge-labeled rooted d-ary tree whose non-leaf vertices correspond to the N d-simplices; the root corresponds to the d-simplex which contains δ, and the labeled edges correspond to all the $(d-1)$-faces other than δ. We get a plane tree by ordering the down-edges at each non-leaf vertex left to right according to the label of the corresponding $(d-1)$-face.

The whole process is easily reversed, so that we can get a rooted tree of d-simplices from an arbitrary planted plane d-ary tree.

There are exactly $C_d(N)$ planted plane d-ary trees with N interior vertices (see e.g. Aval [5]; the integers $C_2(N)$ are the "Catalan numbers", which appear in many combinatorial problems, see e.g. Stanley [44, Exercise 6.19]). Any tree of N d-simplices has exactly $(d-1)N+2$ boundary facets, so it can be rooted in exactly $((d-1)N+2)d!$ ways, which however need not be inequivalent. This explains the first inequality claimed in the lemma. Finally, combinatorially-inequivalent trees of d-simplices also yield inequivalent rooted trees, whence the second inequality follows. □

COROLLARY 4.3. *The number of trees of N d-simplices, for N large, is bounded by*

$$\binom{dN}{N} \sim \left(d\left(\frac{d}{d-1}\right)^{d-1}\right)^{N} < (de)^{N}.$$

4.2. Counting the matchings in the boundary.

We know from the previous section that there are exponentially many trees of N d-simplices. Our goal is to find an exponential upper bound for the LC spheres obtainable by a matching of adjacent facets in the boundary of one fixed tree of simplices.

THEOREM 4.4. *Fix $d\geqslant 2$. The number of combinatorially distinct* LC *d-spheres* (*or* LC *d-balls*) *with N facets*, *for N large*, *is not larger than*

$$\left(d\left(\frac{d}{d-1}\right)^{d-1} 2^{(2d^2-d)/3}\right)^{N}.$$

Proof. Let us fix a tree of N d-simplices B. We adopt the word "couple" to denote a pair of facets in the boundary of B which are glued to one another during the local construction of S.

Let us set $D:=\frac{1}{2}(2+N(d-1))$, which is an integer. By Lemma 4.1, the boundary of the tree of N d-simplices contains $2D$ facets, so each perfect matching is just a set of D pairwise disjoint couples. We are going to partition every perfect matching into "rounds". The first round will contain couples which are adjacent in the boundary of the tree of simplices. Recursively, the $(i+1)$-th round will consist of all pairs of facets that *become* adjacent only after a pair of facets are glued together in the ith round.

Selecting a pair of adjacent facets is the same as choosing the ridge between them; and by Lemma 4.1, the boundary contains dD ridges. Thus the first round of identifications consists in choosing n_1 ridges out of dD, where n_1 is some positive integer. After each identification, at most $d-1$ new ridges are created; so, after this first round of identifications, there are at most $(d-1)n_1$ new pairs of adjacent facets.

In the second round, we identify $2n_2$ of these newly adjacent facets: as before, it is a matter of choosing n_2 ridges, out of the at most $(d-1)n_1$ just created ones. Once this is done, at most $(d-1)n_2$ ridges are created. And so on.

We proceed this way until all the $2\,D$ facets in the boundary of B have been matched (after f steps, say). Clearly $n_1+...+n_f=D$, and since the n_i's are positive integers, $f\leqslant D$ must hold. This means there are at most

$$\sum_{f=1}^{D} \sum_{\substack{n_1+...+n_f=D\\ n_1,...,n_f\geqslant 1\\ n_{i+1}\leqslant (d-1)n_i}} \binom{dD}{n_1}\binom{(d-1)n_1}{n_2}\binom{(d-1)n_2}{n_3}\cdots\binom{(d-1)n_{f-1}}{n_f}$$

possible perfect matchings of $(d-1)$-simplices in the boundary of a tree of N d-simplices.

We sharpen this bound by observing that not all ridges may be chosen in the first round of identifications. For example, we should exclude those ridges which belong to just two d-simplices of B. An easy double-counting argument reveals that in a tree of d-simplices, the number of ridges belonging to at least three d-simplices is less than or equal to $\frac{1}{3}N\binom{d+1}{2}$. So in the upper bound above we may replace the first factor $\binom{dD}{n_1}$ with the smaller factor

$$\binom{\frac{1}{3}N\binom{d+1}{2}}{n_1}.$$

To bound the sum from above, we use $\binom{n}{k}\leqslant 2^n$ and $n_1+...+n_{f-1}<n_1+...+n_f=D$, while ignoring the conditions $n_{i+1}\leqslant(d-1)n_i$. Thus we obtain the upper bound

$$2^{(N/3)\binom{d+1}{2}+(N/2)(d-1)^2+(d-1)}\sum_{f=1}^{D}\binom{D-1}{f-1}=2^{(N/3)(2d^2-d)+(d-1)}.$$

The factor 2^{d-1} is asymptotically negligible. Thus the number of ways to fold a tree of N d-simplices into a sphere via a local construction sequence is smaller than $2^{(2d^2-d)N/3}$.

Combining this with Proposition 4.2, we conclude the proof for the case of d-spheres. We leave the adaptation of the proof for d-balls (or general LC d-pseudomanifolds) to the reader. □

The upper bound of Theorem 4.4 can be simplified in many ways. For example, for $d \geqslant 16$, it is smaller than $4^{d^2 N/3}$. From Theorem 4.4 we obtain explicit upper bounds:

(i) there are less than 216^N LC 3-spheres with N facets;

(ii) there are less than 6117^N LC 4-spheres with N facets;

and so on.

We point out that these upper bounds are not sharp, as we overcounted both on the combinatorial side and on the algebraic side. When $d=2$, Tutte's upper bound is asymptotically 3.08^N, whereas the one given by our formula is 16^N. When $d=3$, however, our constant is smaller than what follows from Durhuus–Jonsson's original argument:

(i) we improved the matchings-bound from 384^N to 32^N;

(ii) for the count of trees of tetrahedra we obtain an essentially sharp bound of 6.75^N. (The value implicit in the Durhuus–Jonsson argument [14, p. 184] is larger, since one has to take into account that different trees of tetrahedra can have the same unlabeled dual graph.)

COROLLARY 4.5. *For any fixed* $d \geqslant 2$, *there are exponential lower and upper bounds for the number of* LC d*-spheres on* N *facets.*

Proof. We have just obtained an upper bound; we also get a lower bound from Proposition 4.2 and Corollary 4.3, since the boundary of a tree of $(d+1)$-simplices is a stacked d-sphere, and for $d \geqslant 2$ the stacked d-sphere determines the tree of $(d+1)$-simplices uniquely. □

We know very little about the number of LC d-spheres with N facets when d is not constant and N is relatively small (say, bounded by a polynomial) in terms of d, and whether the LC condition is crucial for that. Compare Kalai [30].

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