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Geodesics in large planar maps and in the Brownian map

by

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1. Introduction

The main goal of the present work is to study geodesics in the random metric space called the Brownian map, which appears as the scaling limit of several classes of discrete planar maps. We prove in particular that a typical point of the Brownian map is connected by a unique geodesic to the distinguished point called the root. We are also able to give an explicit description of the set of those points that are connected to the root by at least two distinct geodesics. In particular, we obtain that this set is dense in the Brownian map and is homeomorphic to a real tree. Moreover we show that countably many points are connected to the root by three distinct geodesics, but no point is connected to the root by four distinct geodesics. Because of the invariance of the distribution of the Brownian map under uniform re-rooting, similar results hold if we replace the root by a point randomly chosen according to the volume measure on the Brownian map. Although the Brownian map is a singular metric space, there are striking analogies between our analysis and the classical results known in the setting of Riemannian manifolds.

Our main results have direct applications to uniqueness properties for geodesics in discrete planar maps. One indeed conjectures that the Brownian map is the universal scaling limit of discrete random planar maps, in a way similar to the appearance of Brownian motion as the scaling limit of all discrete random paths satisfying mild integrability conditions. If this conjecture is correct, the present work will provide information about the behavior of geodesics in large discrete random maps in a very general setting. The preceding analogy with Brownian motion also suggests that the Brownian map should provide the "right model" for a Brownian random surface.

To motivate our main results, let us start by describing some typical applications to discrete models. Recall that a *planar map* is a proper embedding of a connected

graph in the 2-dimensional sphere $\mathbb{S}^2$. Loops and multiple edges are a priori allowed. The faces of the map are the connected components of the complement of the union of edges. A planar map is *rooted* if it has a distinguished oriented edge called the root edge, whose origin is called the root vertex. Two rooted planar maps are said to be equivalent if the second one is the image of the first one under an orientation-preserving homeomorphism of the sphere, which also preserves the root edge. Two equivalent planar maps are identified. Given an integer $p\geqslant 2$, a $2p$-*angulation* is a planar map where each face has degree $2p$, that is $2p$ adjacent edges (one should count edge sides, so that if an edge lies entirely inside a face it is counted twice). A $2p$-angulation is bipartite, so that it cannot have loops, but it may have multiple edges. We denote by $\mathcal{M}_n^p$ the set of all rooted $2p$-angulations with n faces. Due to the preceding identification, the set $\mathcal{M}_n^p$ is finite.

Let M be a planar map and let $V(M)$ denote the vertex set of M. A *path* in M with length k is a finite sequence $(a_0, a_1, ..., a_k)$ in $V(M)$ such that a_i and a_{i-1} are connected by an edge of the map, for every $i\in\{1, ..., k\}$. The *graph distance* $d_{\mathrm{gr}}(a, a')$ between two vertices a and a' is the minimal k such that there exists a path $\gamma=(a_0, a_1, ..., a_k)$ with $a_0=a$ and $a_k=a'$. A path $\gamma=(a_0, a_1, ..., a_k)$ is called a *discrete geodesic* (from a_0 to a_k) if $k=d_{\mathrm{gr}}(a_0, a_k)$. If $\gamma=(a_0, ..., a_k)$ and $\gamma'=(a'_0, ..., a'_{k'})$ are two paths (possibly with different lengths), we set

$$d(\gamma, \gamma') = \max\{d_{\mathrm{gr}}(a_{i\wedge k}, a'_{i\wedge k'}) : i \geqslant 0\}.$$

Throughout the present work, we fix an integer $p\geqslant 2$, and consider a random $2p$-angulation M_n, which is uniformly distributed over the set $\mathcal{M}_n^p$. We denote the vertex set of M_n by $\mathbf{m}_n=V(M_n)$ and the root vertex of M_n by $\partial_n\in\mathbf{m}_n$. Note that, by Euler's formula, $\#(\mathbf{m}_n)=(p-1)n+2$. For every $a, a'\in\mathbf{m}_n$, we denote by $\mathrm{Geo}_n(a, a')$ the set of all discrete geodesics from a to a' in the map M_n.

PROPOSITION 1.1. *For every* $\delta>0$,

$$\frac{1}{n}\#\{a\in\mathbf{m}_n : \textit{there exist } \gamma, \gamma'\in \mathrm{Geo}_n(\partial_n, a) \textit{ such that } d(\gamma, \gamma')\geqslant \delta n^{1/4}\}\to 0,$$

as $n\to\infty$, *in probability.*

Recall that if $R(M_n)=\max\{d_{\mathrm{gr}}(\partial_n, a) : a\in\mathbf{m}_n\}$ is the radius of the map M_n, it is known that $n^{-1/4}R(M_n)$ converges in distribution to a positive random variable ([13] if $p=2$, and [27], [40] in the general case). Thus typical distances between vertices of M_n are of order $n^{1/4}$. Proposition 1.1 means that when n is large, for a typical vertex a of M_n there is essentially a unique discrete geodesic from the root vertex to a, up to deviations that are small in comparison with the diameter of the map.

One can get a stronger version of Proposition 1.1 by considering approximate geodesics. Fix a non-negative function $\theta(n)$, $n\in\mathbb{N}$, such that $\theta(n)=o(1)$ as $n\to\infty$. An approximate geodesic from a to a' is a path from a to a' whose length is less than $(1+\theta(n))d_{\rm gr}(a,a')$. Denote the set of all approximate geodesics from a to a' in M_n by $\overline{\mathrm{Geo}}_n(a,a')$. Then the statement of Proposition 1.1 still holds if $\mathrm{Geo}_n(\partial_n,a)$ is replaced by $\overline{\mathrm{Geo}}_n(\partial_n,a)$.

Proposition 1.1 is concerned with discrete geodesics from the root vertex to a typical point of $\mathbf{m}_n$. What happens now if we consider exceptional points? To answer this question, fix $\delta>0$, and for every $a,a'\in\mathbf{m}_n$, let $\mathrm{Mult}_{n,\delta}(a,a')$ be the maximal integer k such that there exist k paths $\gamma_1,\dots,\gamma_k\in\mathrm{Geo}_n(a,a')$ with $d(\gamma_i,\gamma_j)\geqslant\delta n^{1/4}$ if $i\neq j$. Define analogously $\overline{\mathrm{Mult}}_{n,\delta}(a,a')$ by replacing discrete geodesics with approximate geodesics.

Proposition 1.2. *We have, for every* $\delta>0$,

$$\lim_{n\to\infty} P(\textit{there exists } a\in\mathbf{m}_n \textit{ such that } \overline{\mathrm{Mult}}_{n,\delta}(\partial_n,a)\geqslant 4)=0.$$

In other words, when n is large, there cannot be more than 3 "macroscopically different" discrete geodesics connecting a point of $\mathbf{m}_n$ to the root vertex. Can the maximal number 3 be attained? The next proposition provides an answer to this question.

Proposition 1.3. *We have*

$$\lim_{\delta\to 0}\liminf_{n\to\infty} P(\textit{there exists } a\in\mathbf{m}_n \textit{ such that } \mathrm{Mult}_{n,\delta}(\partial_n,a)=3)=1.$$

It should be emphasized that the root vertex ∂_n plays no special role in the preceding propositions. These statements remain valid if ∂_n is replaced by a vertex chosen uniformly at random in $\mathbf{m}_n$.

The proofs of Propositions 1.1, 1.2 and 1.3 depend on analogous results concerning the Brownian map. We will now explain how the Brownian map can be obtained as the continuous limit of rescaled uniform $2p$-angulations [24]. We first observe that $(\mathbf{m}_n,d_{\rm gr})$ can be viewed as a random compact metric space, or more precisely as a random variable taking values in the space $\mathbb{K}$ of isometry classes of compact metric spaces. We equip $\mathbb{K}$ with the Gromov–Hausdorff distance $d_{\rm GH}$ (see [11], [18] and §8 below), and the metric space $(\mathbb{K},d_{\rm GH})$ is then Polish. It is proved in [24] that we can find a sequence $\{n_k\}_{k\geqslant 1}$ of values of n converging to ∞ and then construct the random maps M_{n_k} in such a way that along the sequence $\{n_k\}_{k\geqslant 1}$ we have the almost sure convergence

$$(\mathbf{m}_n,\varkappa_p n^{-1/4}d_{\rm gr})\xrightarrow{\text{a.s.}}(\mathbf{m}_\infty,D),\quad \text{as } n\to\infty, \tag{1}$$

in the Gromov–Hausdorff sense (here $\varkappa_p$ is a positive constant depending on p). The limiting random metric space $(\mathbf{m}_\infty,D)$ is called the Brownian map. At this point we

need to comment on the necessity of taking a subsequence in order to get the convergence (1). As will be explained below, the space $\mathbf{m}_\infty$ is obtained as a quotient space of the well-known random real tree called the continuum random tree, by an equivalence relation which is explicitly defined in terms of Brownian labels assigned to the vertices of the continuum random tree. However the metric D, which induces the quotient topology on $\mathbf{m}_\infty$, has not been completely characterized, and it is conceivable that different subsequences might give rise to different limiting metrics D in (1). Still one conjectures that the space $(\mathbf{m}_\infty, D)$ does not depend on the chosen subsequence, nor on the integer p, and that a convergence analogous to (1) holds for more general classes of random planar maps. In the present work, we abusively talk about the Brownian map, but it should be understood that we deal with one of the possible limits in (1) (our terminology is thus different from [29], where the Brownian map refers to the same space $\mathbf{m}_\infty$, but with a specific choice of the distance, which may or may not coincide with D). Despite the lack of uniqueness, the topological or even metric properties of $(\mathbf{m}_\infty, D)$ can be investigated in detail and yield interesting consequences for large planar maps. For instance the non-existence of small "bottlenecks" in large $2p$-angulations was derived in [25] as a consequence of the convergence (1) and the fact that $(\mathbf{m}_\infty, D)$ is homeomorphic to $\mathbb{S}^2$. Our study of geodesics in the Brownian map is motivated in part by the same strategy.

Let us now give a more precise description of the space $\mathbf{m}_\infty$. We use the notation $(\mathcal{T}_\mathbf{e}, d_\mathbf{e})$ for the random rooted real tree known as the *continuum random tree*, which was introduced and studied by Aldous [1], [2]. The notation $\mathcal{T}_\mathbf{e}$ is justified by the fact that $\mathcal{T}_\mathbf{e}$ can be defined as the tree coded by a normalized Brownian excursion $\mathbf{e}=(\mathbf{e}_t)_{0\leqslant t\leqslant 1}$. Precisely, $\mathcal{T}_\mathbf{e}=[0,1]/\!\sim_\mathbf{e}$ is the quotient space of the interval $[0,1]$ for the equivalence relation $\sim_\mathbf{e}$ such that $s\sim_\mathbf{e} t$ if and only if $\mathbf{e}_s=\mathbf{e}_t=\min_{[s\wedge t, s\vee t]}\mathbf{e}_r$. The distance $d_\mathbf{e}(a,b)$ is defined for $a,b\in\mathcal{T}_\mathbf{e}$ by $d_\mathbf{e}(a,b)=\mathbf{e}_s+\mathbf{e}_t-2\min_{[s\wedge t, s\vee t]}\mathbf{e}_r$, where s (resp. t) is an arbitrary representative of a (resp. b) in $[0,1]$ (see §2.3 and §2.4 below). The root $\varrho_\mathbf{e}$ of $\mathcal{T}_\mathbf{e}$ is the equivalence class of 0 in the quotient $[0,1]/\!\sim_\mathbf{e}$, and the uniform probability measure on $\mathcal{T}_\mathbf{e}$ is the image of Lebesgue measure on $[0,1]$ under the canonical projection.

We then introduce Brownian labels assigned to the vertices of the continuum random tree: Conditionally given the tree $\mathcal{T}_\mathbf{e}$, $Z^\mathbf{e}=(Z^\mathbf{e}_a)_{a\in\mathcal{T}_\mathbf{e}}$ is the centered Gaussian process whose distribution is characterized by the properties $Z^\mathbf{e}_{\varrho_\mathbf{e}}=0$ and $E[(Z^\mathbf{e}_a-Z^\mathbf{e}_b)^2]=d_\mathbf{e}(a,b)$ for every $a,b\in\mathcal{T}_\mathbf{e}$.

We are in fact interested in the pair $(\mathcal{T}_{\bar{\mathbf{e}}},(\bar{Z}_a)_{a\in\mathcal{T}_{\bar{\mathbf{e}}}})$, which may be defined as the pair $(\mathcal{T}_\mathbf{e},(Z^\mathbf{e}_a)_{a\in\mathcal{T}_\mathbf{e}})$ conditioned on the event $\{Z^\mathbf{e}_a\geqslant 0 \text{ for all } a\in\mathcal{T}_\mathbf{e}\}$. Some care is needed here, since the latter event has zero probability. The paper [26] provides a simple explicit construction of the pair $(\mathcal{T}_{\bar{\mathbf{e}}},(\bar{Z}_a)_{a\in\mathcal{T}_{\bar{\mathbf{e}}}})$: If a_* denotes the (almost surely unique) vertex of $\mathcal{T}_\mathbf{e}$ such that $Z^\mathbf{e}_{a_*}=\min\{Z^\mathbf{e}_a : a\in\mathcal{T}_\mathbf{e}\}$, then $\mathcal{T}_{\bar{\mathbf{e}}}$ coincides with the tree $\mathcal{T}_\mathbf{e}$ re-rooted at a_*, and

the new labels $\bar{Z}_a$ are obtained by setting $\bar{Z}_a = Z^{\mathbf{e}}_a - Z^{\mathbf{e}}_{a_*}$, so that the label of the new root is still 0. We can again write $\mathcal{T}_{\bar{\mathbf{e}}} = [0,1]/\sim_{\bar{\mathbf{e}}}$, where the equivalence relation $\sim_{\bar{\mathbf{e}}}$ is defined as above from a random continuous function $\bar{\mathbf{e}}$ which has a simple expression in terms of $\mathbf{e}$ (see §2.4 below) and the root $\varrho = \varrho_{\bar{\mathbf{e}}}$ of $\mathcal{T}_{\bar{\mathbf{e}}}$ is the equivalence class of 0 in $[0,1]/\sim_{\bar{\mathbf{e}}}$. If $a, b \in \mathcal{T}_{\bar{\mathbf{e}}}$, we denote by $[a,b]$ the subset of $\mathcal{T}_{\bar{\mathbf{e}}}$ which is the image under the projection $[0,1] \to [0,1]/\sim_{\bar{\mathbf{e}}}$ of the smallest interval $[s,t]$ such that s (resp. t) is a representative of a (resp. b) in $[0,1]$, and $s \leqslant t$. If there exists no such interval $[s,t]$, we take $[a,b] = \mathcal{T}_{\bar{\mathbf{e}}}$ by convention. Informally, $[a,b]$ corresponds to those points of $\mathcal{T}_{\bar{\mathbf{e}}}$ that are visited when going from a to b "around the tree" in "clockwise order" and avoiding the root. We define an equivalence relation on $\mathcal{T}_{\bar{\mathbf{e}}}$ by setting, for every $a, b \in \mathcal{T}_{\bar{\mathbf{e}}}$,

$$a \approx b \quad \text{if and only if} \quad \bar{Z}_a = \bar{Z}_b = \min_{c \in [a,b]} \bar{Z}_c \ \text{ or } \ \bar{Z}_a = \bar{Z}_b = \min_{c \in [b,a]} \bar{Z}_c.$$

The space $\mathbf{m}_\infty$ is then defined as the quotient space $\mathcal{T}_{\bar{\mathbf{e}}}/\approx$. We denote the canonical projection from $\mathcal{T}_{\bar{\mathbf{e}}}$ onto $\mathbf{m}_\infty$ by Π. The metric D induces the quotient topology on $\mathbf{m}_\infty$, and satisfies the bound

$$D(\Pi(a), \Pi(b)) \leqslant \bar{Z}_a + \bar{Z}_b - 2 \min_{c \in [a,b]} \bar{Z}_c \tag{2}$$

for every $a, b \in \mathcal{T}_{\bar{\mathbf{e}}}$. We use the same notation ϱ for the root of $\mathcal{T}_{\bar{\mathbf{e}}}$, and for its equivalence class in $\mathcal{T}_{\bar{\mathbf{e}}}/\approx$, which is a distinguished point of $\mathbf{m}_\infty$.

If (E, δ) is a compact metric space and $x, y \in E$, a *geodesic* or *shortest path* from x to y is a continuous path $\gamma = (\gamma(t))_{0 \leqslant t \leqslant \delta(x,y)}$ such that $\gamma(0) = x$, $\gamma(\delta(x,y)) = y$ and $\delta(\gamma(s), \gamma(t)) = |t - s|$ for every $s, t \in [0, \delta(x,y)]$. The compact metric space (E, δ) is then called a geodesic space if any two points in E are connected by (at least) one geodesic. The space $(\mathbf{m}_n, d_{\mathrm{gr}})$ is not a geodesic space, but it is easy to construct a geodesic space whose Gromov–Hausdorff distance from $(\mathbf{m}_n, d_{\mathrm{gr}})$ is bounded above by 1. From (1) and the fact that Gromov–Hausdorff limits of geodesic spaces are geodesic spaces (see [11, Theorem 7.5.1]), one gets that $(\mathbf{m}_\infty, D)$ is almost surely a geodesic space. Our main goal is to determine the geodesics between the root ϱ and an arbitrary point of $\mathbf{m}_\infty$.

Before stating our main result, we still need to introduce one more notation. We define the *skeleton* $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ by saying that a point a of $\mathcal{T}_{\bar{\mathbf{e}}}$ belongs to $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ if and only if $\mathcal{T}_{\bar{\mathbf{e}}} \setminus \{a\}$ is not connected (informally, $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ is obtained by removing the leaves of the tree $\mathcal{T}_{\bar{\mathbf{e}}}$). It is proved in Proposition 3.3 below that, with probability 1, the restriction of the projection Π to $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ is a homeomorphism, and the Hausdorff dimension of $\Pi(\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}}))$ is equal to 2. We write $\mathrm{Skel}_\infty = \Pi(\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}}))$ to simplify notation. Since the Hausdorff dimension of $\mathbf{m}_\infty$ is equal to 4 almost surely [24], the set Skel_∞ is a relatively small subset of $\mathbf{m}_\infty$. The set Skel_∞ is dense in $\mathbf{m}_\infty$ and from the previous observations it is homeomorphic to a non-compact real tree. Moreover, for every $x \in \mathrm{Skel}_\infty$, the set $\mathrm{Skel}_\infty \setminus \{x\}$ is disconnected.

The following theorem, which summarizes our main contributions, provides a nice geometric interpretation of the set Skel_∞ (see Theorems 7.4 and 7.6 below for more precise statements).

THEOREM 1.4. *The following properties hold almost surely. For all $x \in \mathbf{m}_\infty \setminus \mathrm{Skel}_\infty$, there is a unique geodesic from ϱ to x. On the other hand, for every $x \in \mathrm{Skel}_\infty$, the number of distinct geodesics from ϱ to x is equal to the number of connected components of $\mathrm{Skel}_\infty \setminus \{x\}$. In particular, the maximal number of distinct geodesics from ϱ to a point of $\mathbf{m}_\infty$ is equal to 3.*

Remarks. (i) The invariance of the distribution of the Brownian map under uniform re-rooting (see §8 below) shows that results analogous to Theorem 1.4 hold if one replaces the root by a point z distributed uniformly over $\mathbf{m}_\infty$. Here the word "uniformly" refers to the volume measure λ on $\mathbf{m}_\infty$, which is the image of the uniform probability measure on $\mathcal{T}_{\bar{\mathbf{e}}}$ under the projection Π.

(ii) The last assertion of the theorem easily follows from the previous ones. Indeed, we already noticed that the (unrooted) tree $\mathcal{T}_{\bar{\mathbf{e}}}$ is isometric to the continuum random tree $\mathcal{T}_{\mathbf{e}}$, and standard properties of linear Brownian motion imply that the maximal number of connected components of $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}}) \setminus \{a\}$, when a varies over $\mathcal{T}_{\mathbf{e}}$, is equal to 3. Furthermore there are countably many points a for which this number is 3.

The construction of the Brownian map $(\mathbf{m}_\infty, D)$ as a quotient space of the random tree $\mathcal{T}_{\bar{\mathbf{e}}}$ may appear artificial, even though it is a continuous counterpart of the bijections relating labeled trees to discrete planar maps (see in particular [7]). Theorem 1.4 shows that the skeleton of $\mathcal{T}_{\bar{\mathbf{e}}}$, or rather its homeomorphic image under the canonical projection Π, has an intrinsic geometric meaning: It exactly corresponds to the cut locus of $\mathbf{m}_\infty$ relative to the root ϱ, provided we define this cut locus as the set of all points that are connected to ϱ by at least two distinct geodesics. Note that this definition of the cut locus is slightly different from the one that appears in Riemannian geometry (see e.g. [16]), since the latter does not make sense in our singular setting.

Remarkably enough, the assertions of Theorem 1.4 parallel the known results in the setting of differential geometry, which go back to Poincaré [34]. Myers [33] proved that for a complete analytic 2-dimensional manifold which is homeomorphic to the sphere, the cut locus associated with a given point A is a topological tree, and the number of distinct geodesics joining A to a point M of the cut locus is equal to the number of connected components of the complement of $\{M\}$ in the cut locus (see Hebda [19] and Itoh [21] and the references therein for more recent related results under C^∞ or C^2-regularity assumptions). On the other hand, Shiohama and Tanaka [38] give examples showing that in the (non-differentiable) setting of Alexandrov spaces with curvature bounded

from below, the cut locus may have a fractal structure.

We are able to give explicit formulas for all geodesics from the root to an arbitrary point of $\mathbf{m}_\infty$. Indeed all these geodesics are obtained as simple geodesics, which had already been considered in [29]. The main difficulty in the proof of Theorem 1.4 is to verify that there are no other geodesics from the root. To this end, we define for every point $x \in \mathbf{m}_\infty$ a minimal and a maximal geodesic from the root to x. Loosely speaking, these are defined in such a way that any other geodesic from ϱ to x will lie "between" the minimal and the maximal one: See §4 for more exact statements. Then one needs to check that for a given point $x \in \mathbf{m}_\infty \setminus \mathrm{Skel}_\infty$, the minimal and the maximal geodesic from ϱ to x coincide, and therefore also coincide with the simple geodesic from ϱ to x. For this purpose, the key step is to prove that a minimal (or maximal) geodesic cannot visit a point of Skel_∞, except possibly at its endpoint. The technical estimates of §5 and §6 below are devoted to the proof of this property. Some of these estimates are of independent interest. In particular Corollary 6.2 gives the following uniform estimate on the volume of balls in $(\mathbf{m}_\infty, D)$: For every $\beta \in]0,1[$, there exists a (random) constant K_β such that the volume of any ball of radius r in $\mathbf{m}_\infty$ is bounded above by $K_\beta r^{4-\beta}$, for every $r>0$. In the multifractal formalism, this means that the multifractal spectrum of the volume measure λ on $\mathbf{m}_\infty$ is degenerate.

A rather surprising consequence of our results is the following remarkable confluence property of geodesics. For every $\varepsilon>0$, there exists a (random) constant $\alpha>0$ such that, if γ and γ' are two geodesics starting from the root and with length greater than ε, we have $\gamma(t)=\gamma'(t)$ for every $t \in [0,\alpha]$ (Corollary 7.7). Some calculations related to a discrete version of this property can be found in Bouttier and Guitter [9] in the particular case of quadrangulations.

Let us comment on recent results related to the present work. The idea of studying continuous limits of discrete planar maps appeared in the pioneering paper of Chassaing and Schaeffer [13]. The problem of establishing a convergence of the type (1) was raised by Schramm [37] in the setting of triangulations. Marckert and Mokkadem [29] considered the case of quadrangulations and proved a weak form of the convergence (1). See also [27] for a generalization to Boltzmann distributions on bipartite planar maps. As an important ingredient of our proofs, we use a bijection between bipartite planar maps and certain labeled trees called mobiles, which is due to Bouttier, Di Francesco and Guitter [7]. The recent work of Miermont [30] and Miermont and Weill [32] strongly suggests that a convergence analogous to (1) should hold for planar maps that are not bipartite, such as triangulations. In a recent paper [31], Miermont has obtained, independently of the present work, certain uniqueness results for geodesics in continuous limits of discrete quadrangulations, in a setting which is however different from ours. See also the recent

work of Bouttier and Guitter [8] for a detailed discussion of the number of geodesics connecting two given points in a large planar map, and of exceptional pairs of points that can be linked by "truly" distinct geodesics. In a different but related direction, the papers of Angel and Schramm [5], Angel [4] and Chassaing and Durhuus [12] study various properties of random infinite planar maps that are uniformly distributed in some sense.

To complete this presentation, let us mention that planar maps are important objects in several areas of mathematics and physics. They have been studied extensively in combinatorics since the pioneering work of Tutte [39]. Planar maps, or maps on more general surfaces, have significant geometric and algebraic applications: See the book of Lando and Zvonkin [22]. The interest in planar maps in theoretical physics first arose from their connections with matrix integrals [20], [10]. More recently, planar maps have served as models of random (discrete) surfaces in the theory of 2-dimensional quantum gravity: See in particular the book of Ambjørn, Durhuus, and Jonsson [3]. Bouttier's recent thesis [6] provides a nice account of the relations between the statistical physics of random surfaces and the combinatorics of planar maps.

The paper is organized as follows. §2 contains a detailed presentation of the basic objects which are of interest in this work. In particular we discuss the Bouttier–Di Francesco–Guitter bijection between bipartite planar maps and the labeled trees called mobiles [7], which plays an important role in our arguments. Such bijections between maps and trees were discovered by Cori and Vauquelin [14] and then studied in particular by Schaeffer [36]. Theorem 2.2 restates the main result of [24] in a form convenient for our applications. §3 is devoted to some preliminary results. In §4, we recall the definition of simple geodesics, and we introduce minimal and maximal geodesics. The main result of this section is Proposition 4.1, which shows that the so-called minimal geodesic is indeed a geodesic. §5 proves the key technical estimate (Lemma 5.1). Loosely speaking, this lemma bounds the probability that the range of a minimal geodesic intersects (the image under the canonical projection of) an interval containing the right end of an excursion interval of $\bar{\mathbf{e}}$ with length greater than some fixed $\delta>0$. The estimate of Lemma 5.1 is then used in §6 to prove Proposition 6.3, which shows that the range of a typical minimal geodesic does not intersect Skel_∞. As another ingredient in the proof of Proposition 6.3, we use our uniform estimates for the volume of small balls in $\mathbf{m}_\infty$. §7 contains the proof of our main results Theorems 7.4 and 7.6, from which Theorem 1.4 readily follows. §8 discusses the re-rooting invariance property of the Brownian map. Finally §9 provides applications to large planar maps, and gives the proofs of Propositions 1.1, 1.2 and 1.3. The proof of two technical discrete lemmas is presented in the appendix.

Let us conclude with a comment about sets of zero probability. As usual in a random

setting, many of the results that are presented in this work hold almost surely, that is outside a set of zero probability. In a few instances, such as Lemma 3.4, this set of zero probability depends on the choice of a parameter $U\in[0,1]$, which corresponds to fixing a point of $\mathbf{m}_\infty$. In such cases, we will always make this dependence clear: Compare Propositions 7.1 and 7.2 below for instance.

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2. The Brownian map

2.1. Real trees

As was already mentioned in the introduction, the Brownian map is defined as a quotient space of a random real tree. We start by discussing real trees in a deterministic setting. A metric space $(\mathcal{T},d)$ is a *real tree* if the following two properties hold for every $a,b\in\mathcal{T}$.

(a) There is a unique isometric map $f_{a,b}$ from $[0,d(a,b)]$ into $\mathcal{T}$ such that $f_{a,b}(0)=a$ and $f_{a,b}(d(a,b))=b$.

(b) If q is a continuous injective map from $[0,1]$ into $\mathcal{T}$, such that $q(0)=a$ and $q(1)=b$, we have $q([0,1])=f_{a,b}([0,d(a,b)])$.

A *rooted real tree* is a real tree $(\mathcal{T},d)$ with a distinguished vertex $\varrho=\varrho(\mathcal{T})$ called the *root*.

Let us consider a rooted real tree $(\mathcal{T},d)$ with root ϱ. To avoid trivialities, we assume that $\mathcal{T}$ has more than one point. For $a\in\mathcal{T}$, the number $d(\varrho,a)$ is called the *generation* of a in the tree $\mathcal{T}$. For $a,b\in\mathcal{T}$, the range of the mapping $f_{a,b}$ in (a) is denoted by $[\![a,b]\!]=[\![b,a]\!]$: This is the line segment between a and b in the tree. We will also use the obvious notation $]\!]a,b[\![$, $[\![a,b[\![$ and $]\!]a,b]\!]$. For every $a\in\mathcal{T}$, $[\![\varrho,a]\!]$ is interpreted as the ancestral line of vertex a.

More precisely we can define a partial order on the tree, called the *genealogical order*, by setting $a\prec b$ if and only if $a\in[\![\varrho,b]\!]$. If $a\prec b$, a is called an *ancestor* of b, and b is a *descendant* of a. If $a,b\in\mathcal{T}$, there is a unique $c\in\mathcal{T}$ such that $[\![\varrho,a]\!]\cap[\![\varrho,b]\!]=[\![\varrho,c]\!]$. We write $c=a\triangle b$ and call c the most recent common ancestor to a and b. Note that $[\![a,b]\!]=[\![a\triangle b,a]\!]\cup[\![a\triangle b,b]\!]$.

The *multiplicity* of a vertex $a\in\mathcal{T}$ is the number of connected components of $\mathcal{T}\setminus\{a\}$. In particular, a is called a *leaf* if it has multiplicity one. The *skeleton* $\mathrm{Sk}(\mathcal{T})$ is the set of all vertices a of $\mathcal{T}$ which are not leaves. Note that $\mathrm{Sk}(\mathcal{T})$ equipped with the induced metric is itself a (non-compact) real tree. A point $a\in\mathrm{Sk}(\mathcal{T})$ is called *simple* if $\mathcal{T}\setminus\{a\}$ has exactly two connected components. If $\mathcal{T}\setminus\{a\}$ has (at least) three connected components, we say that a is a *branching point* of $\mathcal{T}$.

If $a\in\mathcal{T}$, the subtree of descendants of a is denoted by $\mathcal{T}(a)$ and defined by

$$\mathcal{T}(a)=\{b\in\mathcal{T}:a\prec b\}.$$

If $a\neq\varrho$, a belongs to $\mathrm{Sk}(\mathcal{T})$ if and only if $\mathcal{T}(a)\neq\{a\}$.

2.2. Coding compact real trees

Compact real trees can be coded by "contour functions". Let $\sigma>0$ and let g be a continuous function from $[0,\sigma]$ into $[0,\infty[$ such that $g(0)=g(\sigma)=0$. To avoid trivialities, we will also assume that g is not identically zero. For every $s,t\in[0,\sigma]$, we set

$$m_g(s,t)=\min_{r\in[s\wedge t,s\vee t]}g(r),$$

and

$$d_g(s,t)=g(s)+g(t)-2m_g(s,t).$$

It is easy to verify that d_g is a pseudo-metric on $[0,\sigma]$. As usual, we introduce the equivalence relation $s\sim_g t$ if and only if $d_g(s,t)=0$ (or equivalently if and only if $g(s)=g(t)=m_g(s,t)$). The function d_g induces a distance on the quotient space $\mathcal{T}_g:=[0,\sigma]/\sim_g$, and we keep the notation d_g for this distance. We denote by $p_g:[0,\sigma]\to\mathcal{T}_g$ the canonical projection. Clearly p_g is continuous (when $[0,\sigma]$ is equipped with the Euclidean metric and $\mathcal{T}_g$ with the metric d_g), and therefore $\mathcal{T}_g=p_g([0,\sigma])$ is a compact metric space. Moreover, it is easy to verify that the topology induced by d_g coincides with the quotient topology on $\mathcal{T}_g$.

By [15, Theorem 2.1], the metric space $(\mathcal{T}_g,d_g)$ is a (compact) real tree. Furthermore the mapping $g\mapsto\mathcal{T}_g$ is continuous with respect to the Gromov–Hausdorff distance, if the set of continuous functions g is equipped with the supremum distance. We will always view $(\mathcal{T}_g,d_g)$ as a rooted real tree with root $\varrho_g=p_g(0)=p_g(\sigma)$. Note that $d_g(\varrho_g,a)=g(s)$ if $a=p_g(s)$.

If $s,t\in[0,\sigma]$, the property $p_g(s)\prec p_g(t)$ holds if and only if $g(s)=m_g(s,t)$. Suppose that $0\leqslant s<t\leqslant\sigma$. Then $[\![p_g(s),p_g(t)]\!]\subset p_g([s,t])$, and in particular $p_g(s)\mathbin{\triangle}p_g(t)=p_g(r)$ for

any $r\in[s,t]$ such that $p_g(r)=m_g(s,t)$. Such simple remarks will be used without further comment in the forthcoming proofs.

Let $a\in\mathcal{T}_g$ and let $s(a)$ (resp. $t(a)$) denote the smallest (resp. largest) element in $p_g^{-1}(a)$. Then $\mathcal{T}_g(a)=p_g([s(a),t(a)])$. Hence, if g is not constant on any non-trivial interval, a vertex $a\neq\varrho_g$ belongs to $\mathrm{Sk}(\mathcal{T}_g)$ if and only if $s(a)<t(a)$, that is if $p_g^{-1}(a)$ is not a singleton. Moreover, if g does not vanish on $]0,\sigma[$, then $\varrho_g\notin\mathrm{Sk}(\mathcal{T}_g)$. The last two properties will hold a.s. for the (random) functions g that are considered below.

2.3. The Brownian snake

Let g be as in the previous subsection, and also assume that g is Hölder continuous with exponent δ for some $\delta>0$. We first introduce the Brownian snake driven by the function g.

Let $\mathcal{W}$ be the space of all finite paths in $\mathbb{R}$. Here a *finite path* is simply a continuous mapping $w\colon[0,\zeta]\to\mathbb{R}$, where $\zeta=\zeta_{(w)}$ is a non-negative real number called the *lifetime* of w. The set $\mathcal{W}$ is a Polish space when equipped with the distance

$$d_{\mathcal{W}}(w,w')=|\zeta_{(w)}-\zeta_{(w')}|+\sup_{t\geqslant0}|w(t\wedge\zeta_{(w)})-w'(t\wedge\zeta_{(w')})|.$$

The endpoint (or *tip*) of the path w is denoted by $\widehat{w}=w(\zeta_{(w)})$. For every $x\in\mathbb{R}$, we set $\mathcal{W}_x=\{w\in\mathcal{W}:w(0)=x\}$.

The *Brownian snake* driven by g is the continuous random process $(W_s^g)_{0\leqslant s\leqslant\sigma}$ taking values in $\mathcal{W}_0$, whose distribution is characterized by the following properties:

(a) For every $s\in[0,\sigma]$, $\zeta_{(W_s^g)}=g(s)$.

(b) The process $(W_s^g)_{0\leqslant s\leqslant\sigma}$ is time-inhomogeneous Markov, and its transition kernels are specified as follows: If $0\leqslant s\leqslant s'$, then

- $W_{s'}^g(t)=W_s^g(t)$ for every $t\in[0,m_g(s,s')]$, a.s.;
- the random path $(W_{s'}^g(m_g(s,s')+t)-W_{s'}^g(m_g(s,s')))_{0\leqslant t\leqslant g(s')-m_g(s,s')}$ is independent of W_s^g and distributed as a 1-dimensional Brownian motion started at 0 and stopped at time $g(s')-m_g(s,s')$.

Informally, the value W_s^g of the Brownian snake at time s is a random path with lifetime $g(s)$. When $g(s)$ decreases, the path is erased from its tip, and when $g(s)$ increases, the path is extended by adding "little pieces" of Brownian paths at its tip. The path continuity of the process $(W_s^g)_{0\leqslant s\leqslant\sigma}$ (or rather the existence of a modification with continuous sample paths) easily follows from the fact that g is Hölder continuous.

Property (b) implies that if $s\sim_g s'$ then $W_s^g=W_{s'}^g$ a.s., and this property holds simultaneously for all pairs (s,s') outside a single set of zero probability, by a continuity argument. Hence we may view W^g as indexed by the tree $\mathcal{T}_g=[0,\sigma]/\sim_g$. We write

$Z^g_s = \widehat{W}^g_s$ for the endpoint of W_s. According to the preceding remark, we can view Z^g as indexed by the tree $\mathcal{T}_g$: If $a \in \mathcal{T}_g$, we interpret Z^g_a as the spatial position of the vertex a. Then it is not difficult to verify that, for every $r \in [0, d_g(\varrho_g, a)]$, $W^g_a(r)$ is the spatial position of the ancestor of a at generation r.

The process $(Z^g_a)_{a \in \mathcal{T}_g}$ can be viewed as Brownian motion indexed by $\mathcal{T}_g$: Indeed it is a centered Gaussian process such that $Z^g_{\varrho_g} = 0$ and $E[(Z^g_a - Z^g_b)^2] = d_g(a,b)$ for every $a, b \in \mathcal{T}_g$.

We now randomize the coding function g. Let $\mathbf{e} = (\mathbf{e}_t)_{t \in [0,1]}$ be the normalized Brownian excursion, and take $g = \mathbf{e}$ and $\sigma = 1$ in the previous discussion. The random real tree $(\mathcal{T}_\mathbf{e}, d_\mathbf{e})$ coded by $\mathbf{e}$ is the so-called *continuum random tree*. Using the fact that local minima of Brownian motion are distinct, one easily checks that points of $\mathcal{T}_\mathbf{e}$ can have multiplicity at most 3 (equivalence classes for $\sim_\mathbf{e}$ can contain at most three points).

We then consider the process $(W^\mathbf{e}_s)_{s \in [0,1]}$ such that conditionally given $\mathbf{e}$, $(W^\mathbf{e}_s)_{s \in [0,1]}$ is the Brownian snake driven by $\mathbf{e}$. Notice that for every $s \in [0,1]$, $W^\mathbf{e}_s = (W^\mathbf{e}_s(t), 0 \leqslant t \leqslant \mathbf{e}_s)$ is now a random path with a random lifetime $\mathbf{e}_s$. As previously, we write $Z^\mathbf{e}_s = \widehat{W}^\mathbf{e}_s$ for the endpoint of $W^\mathbf{e}_s$. We refer to [23] for a detailed discussion of the Brownian snake driven by a Brownian excursion, and its connections with non-linear partial differential equations.

2.4. Conditioning the Brownian snake

In view of our applications, it is important to consider the process $(W^\mathbf{e}_s)_{s \in [0,1]}$ conditioned on the event

$$W^\mathbf{e}_s(t) \geqslant 0 \quad \text{for every } s \in [0,1] \text{ and } t \in [0, \mathbf{e}_s].$$

Here some justification is needed for the conditioning, since the latter event has probability zero. The paper [26] describes several limit procedures that allow one to make sense of the previous conditioning. These procedures all lead to the same limiting process $\overline{W}$ which can be described as follows from the original process $W^\mathbf{e}$. Set

$$Z_* = \min_{t \in [0,1]} Z^\mathbf{e}_t$$

and let s_* be the (almost surely) unique time in $[0,1]$ such that $Z^\mathbf{e}_{s_*} = Z_*$. The fact that the minimum Z_* is attained at a unique time ([26, Proposition 2.5]) entails that the vertex $p_\mathbf{e}(s_*)$ is a leaf of the tree $\mathcal{T}_\mathbf{e}$. For every $s, t \in [0,1]$, set

$$s \oplus t = \begin{cases} s+t, & \text{if } s+t \leqslant 1, \\ s+t-1, & \text{if } s+t > 1. \end{cases}$$

Then, for every $t\in[0,1]$, we set

- $\bar{\mathbf{e}}_t=\mathbf{e}_{s_*}+\mathbf{e}_{s_*\oplus t}-2m_{\mathbf{e}}(s_*,s_*\oplus t)$;
- $\bar{Z}_t=Z^{\mathbf{e}}_{s_*\oplus t}-Z^{\mathbf{e}}_{s_*}$.

Note that $\bar{Z}_0=\bar{Z}_1=0$ and $\bar{Z}_t>0$ for every $t\in]0,1[$. The function $\bar{\mathbf{e}}$ is continuous on $[0,1]$, positive on $]0,1[$, and such that $\bar{\mathbf{e}}_0=\bar{\mathbf{e}}_1=0$. Hence the tree $\mathcal{T}_{\bar{\mathbf{e}}}$ is well defined, and this tree is isometrically identified with the tree $\mathcal{T}_{\mathbf{e}}$ re-rooted at the (minimizing) vertex $p_{\mathbf{e}}(s_*)$: See [15, Lemma 2.2]. To simplify notation, we will write $\varrho=\varrho_{\bar{\mathbf{e}}}$ for the root of $\mathcal{T}_{\bar{\mathbf{e}}}$.

One easily verifies that $s\sim_{\bar{\mathbf{e}}}t$ if and only if $s_*\oplus s\sim_{\mathbf{e}}s_*\oplus t$, and so $\bar{Z}_t$ only depends on the equivalence class of t in the tree $\mathcal{T}_{\bar{\mathbf{e}}}$. Therefore, we may and will often view $\bar{Z}$ as indexed by vertices of the tree $\mathcal{T}_{\bar{\mathbf{e}}}$.

The conditioned Brownian snake $(\overline{W}_s)_{s\in[0,1]}$ is now defined by the following properties. For every $s\in[0,1]$, $\overline{W}_s$ is the random element of $\mathcal{W}_0$ such that

(a) the lifetime of $\overline{W}_s$ is $\bar{\mathbf{e}}_s$,

(b) we have $\widehat{\overline{W}}_s=\bar{Z}_s$, and more generally, for every $r\in[0,\bar{\mathbf{e}}_s]$, $\overline{W}_s(r)=\bar{Z}_{a_s(r)}$, where $a_s(r)$ is the ancestor of $p_{\bar{\mathbf{e}}}(s)$ at generation r in the tree $\mathcal{T}_{\bar{\mathbf{e}}}$.

To interpret this definition, note that $\bar{Z}$ and $\overline{W}$ can both be viewed as indexed by the tree $\mathcal{T}_{\bar{\mathbf{e}}}$, which is identified with the tree $\mathcal{T}_{\mathbf{e}}$ re-rooted at the minimal spatial position. The new spatial positions $\bar{Z}_a$ on the re-rooted tree are obtained by shifting the original spatial positions $Z^{\mathbf{e}}_a$ in such a way that the position of the new root is still zero, and the path $\overline{W}_a$ just gives the (new) spatial positions along the ancestral line of a in the re-rooted tree. See [26] for more details.

2.5. The Brownian map

To simplify notation we write $\sim$ instead of $\sim_{\bar{\mathbf{e}}}$ in the remaining part of this work. Then $\sim$ is a (random) equivalence relation on $[0,1]$ whose graph is closed.

We now use the process $(\bar{Z}_t)_{t\in[0,1]}$ of the previous subsection to define one more (random) equivalence relation on $[0,1]$. For every $s,t\in[0,1]$, we write

$$s\approx t \quad\text{if and only if}\quad \bar{Z}_s=\bar{Z}_t=\min_{s\wedge t\leqslant r\leqslant s\vee t}\bar{Z}_r.$$

Notice the obvious similarity with the definition of $\sim$ (in fact the equivalence relation $\approx$ is nothing but $\sim_{\bar{Z}}$). From the fact that local minima of $Z^{\mathbf{e}}$ are distinct ([25, Lemma 3.1]), one easily obtains that equivalence classes of $\approx$ contain one, two or three points at most: See the discussion in [25, §2].

We say that $t\in]0,1]$ is a *left-increase time* of $\bar{\mathbf{e}}$ (resp. of $\bar{Z}$) if there exists $\varepsilon\in]0,t]$ such that $\bar{\mathbf{e}}_r\geqslant\bar{\mathbf{e}}_t$ (resp. $\bar{Z}_r\geqslant\bar{Z}_t$) for every $r\in[t-\varepsilon,t]$. We similarly define the notion of

a *right-increase time*. For $t\in]0,1[$, $p_{\bar{\mathbf{e}}}(t)\in \mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ if and only if t is a left-increase or a right-increase time of $\bar{\mathbf{e}}$.

LEMMA 2.1. *With probability* 1, *any point* $t\in]0,1[$ *which is a right-increase or a left-increase time of* $\bar{\mathbf{e}}$ *is neither a right-increase nor a left-increase time of* $\overline{Z}$. *Consequently, it is almost surely true that, for every* $s,t,r\in]0,1[$, *the properties* $s\sim t$ *and* $s\approx r$ *imply that* $s=t$ *or* $s=r$.

In other words, if the equivalence class of $s\in]0,1[$ for $\sim$ is not a singleton, then its equivalence class for $\approx$ must be a singleton, and conversely (we need to exclude the values $s=0$ and $s=1$, since clearly the pair $\{0,1\}$ is an equivalence class for both $\sim$ and $\approx$). Lemma 2.1 is proved in [25] (Lemma 3.2). This lemma plays a very important role in what follows. We will systematically discard the negligible set on which the conclusion of Lemma 2.1 does not hold.

Due to Lemma 2.1, we can define a new equivalence relation $\simeq$ on $[0,1]$ whose graph is the union of the graphs of $\sim$ and $\approx$ respectively: $s\simeq t$ if and only if $s\sim t$ or $s\approx t$.

By definition, the *Brownian map* $\mathbf{m}_\infty$ is the quotient space $[0,1]/\simeq$, equipped with a random distance D which is obtained as a weak limit of the graph distance on approximating discrete maps. We will explain this in greater detail below, but we already record certain properties that can be found in [24]. It is more convenient to view D as a pseudo-distance on $[0,1]$. Precisely, the random process $(D(s,t))_{s,t\in[0,1]}$ is continuous, takes values in $[0,\infty[$, and satisfies the following conditions:

(i) $D(s,t)=D(t,s)$ and $D(r,t)\leqslant D(r,s)+D(s,t)$ for every $r,s,t\in[0,1]$;

(ii) $D(s,t)=0$ if and only $s\simeq t$, for every $s,t\in[0,1]$;

(iii) $D(0,t)=\overline{Z}_t$ for every $t\in[0,1]$;

(iv) $D(s,t)\leqslant \overline{Z}_s+\overline{Z}_t-2\min_{s\wedge t\leqslant r\leqslant s\vee t}\overline{Z}_r$ for every $s,t\in[0,1]$.

Due to (i) and (ii), D induces a distance on the quotient space $\mathbf{m}_\infty=[0,1]/\simeq$, which is still denoted by D. The (random) metric space $([0,1]/\simeq,D)$ appears as a limit of rescaled planar maps, in the sense of the Gromov–Hausdorff convergence: See §2.7 below. The canonical projection from $[0,1]$ onto $\mathbf{m}_\infty$ will be denoted by $\mathbf{p}$. By definition, the *volume measure* λ on $\mathbf{m}_\infty$ is the image of Lebesgue measure on $[0,1]$ under $\mathbf{p}$.

In agreement with the presentation that was given in the introduction above, it is often useful to view the Brownian map as a quotient space of the random real tree $\mathcal{T}_{\bar{\mathbf{e}}}$. Note that the equivalence relation $\approx$ makes sense on $\mathcal{T}_{\bar{\mathbf{e}}}$: If $a,b\in\mathcal{T}_{\bar{\mathbf{e}}}$, $a\approx b$ if and only if there exist a representative s of a and a representative t of b in $[0,1]$ such that $s\approx t$. Then, D induces a pseudo-distance on $\mathcal{T}_{\bar{\mathbf{e}}}$: $D(a,b)=D(s,t)$ if s (resp. t) is any representative of a (resp. b). The quotient space $\mathcal{T}_{\bar{\mathbf{e}}}/\approx$, equipped with D, is a metric space, which is canonically identified with $(\mathbf{m}_\infty,D)$. This slightly different perspective will be useful as

the genealogical structure of $\mathcal{T}_{\bar{\mathbf{e}}}$ plays an important role in our arguments. Note that the equivalence class of any $a\in\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ for $\approx$ is a singleton, by Lemma 2.1.

As in §1, the canonical projection from $\mathcal{T}_{\bar{\mathbf{e}}}$ onto $\mathbf{m}_\infty$ is denoted by Π. Note that $\mathbf{p}=\Pi\circ p_{\bar{\mathbf{e}}}$ (as above $p_{\bar{\mathbf{e}}}$ denotes the canonical projection from $[0,1]$ onto $\mathcal{T}_{\bar{\mathbf{e}}}$). Both projections $\mathbf{p}$ and Π are continuous, when $[0,1]$ is equipped with the usual topology and $\mathcal{T}_{\bar{\mathbf{e}}}$ with the distance $d_{\bar{\mathbf{e}}}$: For the first one, this is a consequence of (iv), and the result for the second one follows because the topology of $\mathcal{T}_{\bar{\mathbf{e}}}$ is the quotient topology. It will be important to carefully distinguish elements of $[0,1]$ from their equivalence classes in $\mathbf{m}_\infty$ or in $\mathcal{T}_{\bar{\mathbf{e}}}$. We will typically use the letters s,t to denote elements of $[0,1]$, a,b for elements of $\mathcal{T}_{\bar{\mathbf{e}}}$ and x,y for elements of $\mathbf{m}_\infty$. The symbol ϱ will stand both for the root of $\mathcal{T}_{\bar{\mathbf{e}}}$ and for the corresponding element in $\mathbf{m}_\infty$, which is just the equivalence class of 0 or of 1.

If $x\in\mathbf{m}_\infty$, property (iii) above implies that $\bar{Z}_t=D(\varrho,x)$ for every $t\in[0,1]$ such that $\mathbf{p}(t)=x$. So $\bar{Z}$ can also be viewed as a random function on $\mathbf{m}_\infty$. We will write indifferently $\bar{Z}_x=\bar{Z}_a=\bar{Z}_s$ and $D(x,y)=D(a,b)=D(s,t)$, when $x,y\in\mathbf{m}_\infty$, $a,b\in\mathcal{T}_{\bar{\mathbf{e}}}$ and $s,t\in[0,1]$ are such that $\mathbf{p}(s)=\Pi(a)=x$ and $\mathbf{p}(t)=\Pi(b)=y$.

2.6. Discrete maps and the Bouttier–Di Francesco–Guitter bijection

Recall that the integer $p\geqslant 2$ is fixed throughout this work, and that $\mathcal{M}_n^p$ denotes the set of all rooted $2p$-angulations with n faces. We will first discuss the Bouttier–Di Francesco–Guitter bijection between $\mathcal{M}_n^p$ and the set of all p-mobiles with n black vertices.

We use the standard formalism for plane trees as found in [24, §2.1]. A *plane tree* τ is a finite subset of the set

$$\mathcal{U}=\bigcup_{n=0}^{\infty}\mathbb{N}^n$$

of all finite sequences of positive integers (including the empty sequence $\varnothing$), which satisfies three obvious conditions: First $\varnothing\in\tau$, then, for every $v=(u_1,\dots,u_k)\in\tau$ with $k\geqslant 1$, the sequence $(u_1,\dots,u_{k-1})$ (the "parent" of v) also belongs to τ, and finally for every $v=(u_1,\dots,u_k)\in\tau$ there exists an integer $k_v(\tau)\geqslant 0$ (the "number of children" of v) such that the vertex $vj:=(u_1,\dots,u_k,j)$ belongs to τ if and only if $1\leqslant j\leqslant k_v(\tau)$. The generation of $v=(u_1,\dots,u_k)$ is denoted by $|v|=k$. The notions of an ancestor and a descendant in the tree τ are defined in an obvious way.

A *p-tree* is a plane tree τ that satisfies the following additional property: For every $v\in\tau$ such that $|v|$ is odd, $k_v(\tau)=p-1$.

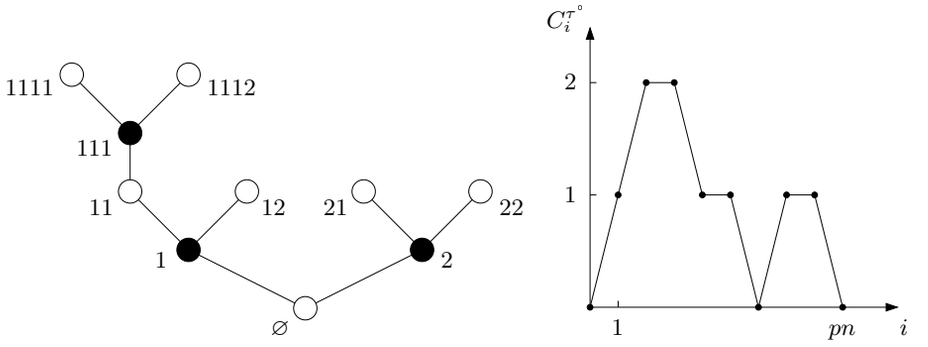

Figure 1. A 3-tree τ and the associated contour function C^{τ° of τ°.

If τ is a p-tree, the vertices v of τ such that $|v|$ is even are called *white* vertices, and vertices v of τ such that $|v|$ is odd are called *black* vertices. We denote by τ° the set of all white vertices of τ and by $\tau^\bullet$ the set of all black vertices. See the left side of Figure 1 for an example of a 3-tree.

A (rooted) *p-mobile* is a pair $\theta=(\tau,(\ell_v)_{v\in\tau^\circ})$ that consists of a p-tree τ and a collection of integer labels assigned to the white vertices of τ, such that the following properties hold:

(a) $\ell_\varnothing=1$ and $\ell_v\geqslant 1$ for each $v\in\tau^\circ$;

(b) let $v\in\tau^\bullet$, let $v_{(0)}$ be the parent of v and let $v_{(j)}=vj$, $1\leqslant j\leqslant p-1$, be the children of v; then for every $j\in\{0,1,...,p-1\}$, $\ell_{v_{(j+1)}}\geqslant\ell_{v_{(j)}}-1$, where by convention $v_{(p)}=v_{(0)}$.

The left side of Figure 2 gives an example of a 3-mobile. The numbers appearing inside the circles representing white vertices are the labels assigned to these vertices. Condition (b) above means that if one lists the white vertices adjacent to a given black vertex in clockwise order, the labels of these vertices can decrease by at most 1 at each step.

We will now describe the Bouttier–Di Francesco–Guitter bijection between $\mathcal{M}_n^p$ and the set of all p-mobiles with n black vertices. This bijection can be found in [7, §2] in the more general setting of bipartite planar maps. Note that [7] deals with pointed planar maps rather than with rooted planar maps. However, the results described below easily follow from [7].

Let τ be a p-tree with n black vertices and let $k=\#\tau-1=pn$. The depth-first search sequence of τ is the sequence $w_0,w_1,...,w_{2k}$ of vertices of τ which is obtained by induction as follows. First $w_0=\varnothing$, and then for every $i\in\{0,...,2k-1\}$, w_{i+1} is either the first child of w_i that has not yet appeared in the sequence $w_0,...,w_i$, or the parent of w_i if all children of w_i already appear in the sequence $w_0,...,w_i$. It is easy to verify that $w_{2k}=\varnothing$ and that all vertices of τ appear in the sequence $w_0,w_1,...,w_{2k}$ (of course some of them appear more than once).

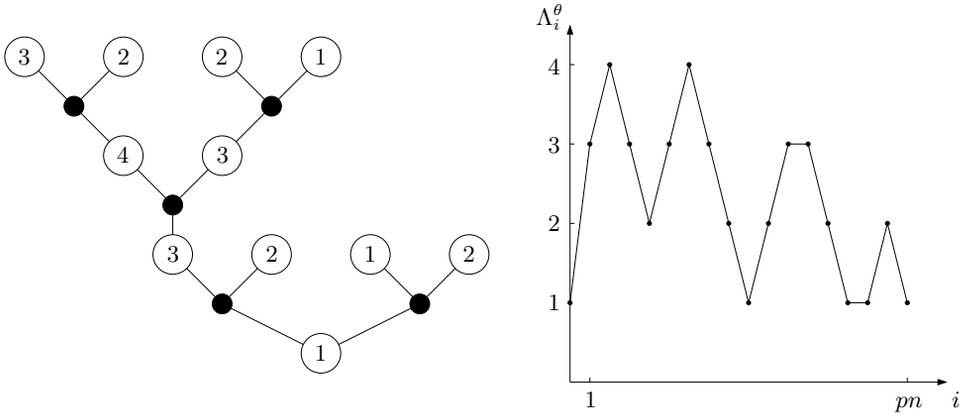

Figure 2. A 3-mobile θ with 5 black vertices and the associated spatial contour function.

The vertices w_i are white when i is even and black when i is odd. The *contour sequence* of τ° is by definition the sequence $v_0, ..., v_k$ defined by $v_i = w_{2i}$ for $i \in \{0, 1, ..., k\}$.

Now let $\theta = (\tau, (\ell_v)_{v \in \tau^\circ})$ be a p-mobile with n black vertices. As previously, denote the contour sequence of τ° by $v_0, v_1, ..., v_{pn}$. Suppose that the tree τ is drawn in the plane as pictured on Figure 3 and add an extra vertex ∂. We associate with θ a rooted $2p$-angulation M with n faces, whose set of vertices is

$$V(M) = \tau^\circ \cup \{\partial\}$$

and whose edges are obtained by the following device: For every $i \in \{0, 1, ..., pn-1\}$,

- if $\ell_{v_i} = 1$, draw an edge between v_i and ∂;
- if $\ell_{v_i} \geqslant 2$, draw an edge between v_i and v_j, where j is the first index in the sequence $i+1, i+2, ..., pn$ such that $\ell_{v_j} = \ell_{v_i} - 1$ (we then say that j is the successor of i, or sometimes that v_j is a successor of v_i—note that a given vertex v can appear several times in the contour sequence and so may have several different successors).

Notice that $v_{pn} = v_0 = \varnothing$ and $\ell_\varnothing = 1$, and that condition (b) in the definition of a p-tree entails that $\ell_{v_{i+1}} \geqslant \ell_{v_i} - 1$ for every $i \in \{0, 1, ..., pn-1\}$. This ensures that whenever $\ell_{v_i} \geqslant 2$ there is at least one vertex among $v_{i+1}, v_{i+2}, ..., v_{pn}$ with label $\ell_{v_i} - 1$. The construction can be made in such a way that edges do not intersect, except possibly at their endpoints: For every vertex v, each index i such that $v_i = v$ corresponds to a "corner" of v, and the associated edge starts from this corner. We refer to [7, §2] for a more detailed description (here we will only need the fact that edges are generated in the way described above). The resulting planar graph M is a $2p$-angulation, which is rooted at the oriented edge between ∂ and $v_0 = \varnothing$, corresponding to $i = 0$ in the previous construction. Each black vertex of τ is associated with a face of the map M. See Figure 3 for the 6-angulation associated with the 3-mobile of Figure 2.

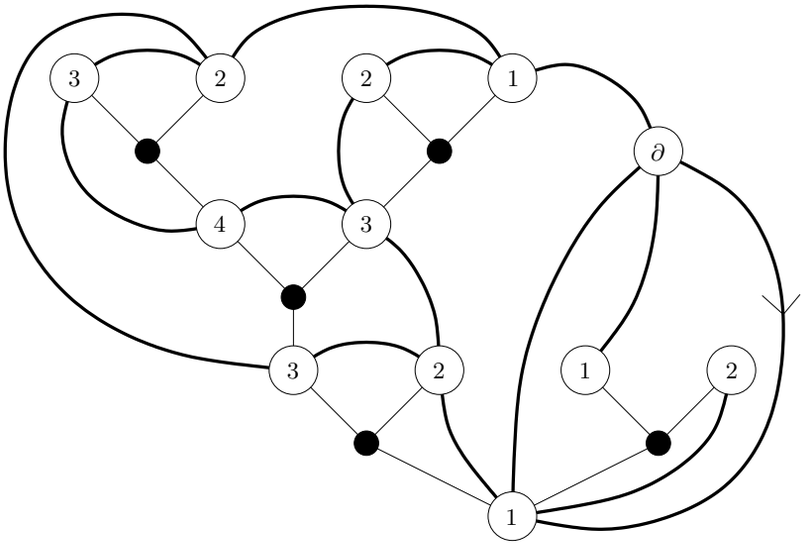

Figure 3. The Bouttier–Di Francesco–Guitter bijection: A rooted 3-mobile with 5 black vertices and the associated rooted 6-angulation with 5 faces. The root of the map is the edge between the vertex ∂ and the root of the tree at the right end of the figure.

The following property, which relates labels on the tree τ° to distances in the planar map M, plays a key role in our applications: For every vertex $v \in \tau^\circ$, the graph distance in M between v and the root vertex ∂ is equal to ℓ_v.

It follows from [7] that the preceding construction yields a bijection between the set $\mathbb{T}_n^p$ of all p-mobiles with n black vertices and the set $\mathcal{M}_n^p$.

The *contour function* of τ° is the discrete sequence $C_0^{\tau^\circ}, C_1^{\tau^\circ}, ..., C_{pn}^{\tau^\circ}$ defined by

$$C_i^{\tau^\circ} = \tfrac{1}{2}|v_i| \quad \text{for every } 0 \leqslant i \leqslant pn.$$

See Figure 1 for an example with $p=n=3$. It is easy to verify that the contour function determines τ°, which in turn determines the p-tree τ uniquely. We will also use the *spatial contour function* of $\theta = (\tau, (\ell_v)_{v\in\tau^\circ})$, which is the discrete sequence $(\Lambda_0^\theta, \Lambda_1^\theta, ..., \Lambda_{pn}^\theta)$ defined by

$$\Lambda_i^\theta = \ell_{v_i} \quad \text{for every } 0 \leqslant i \leqslant pn.$$

From property (b) of the labels and the definition of the contour sequence, it is clear that $\Lambda_{i+1}^\theta \geqslant \Lambda_i^\theta - 1$ for every $0 \leqslant i \leqslant pn-1$ (cf. Figure 2). The pair $(C^{\tau^\circ}, \Lambda^\theta)$ determines θ uniquely.

Define an equivalence relation $\sim_{[\tau]}$ on $\{0,1,...,pn\}$ by setting $i\sim_{[\tau]}j$ if and only if $v_i=v_j$. The quotient space $\{0,1,...,pn\}/\sim_{[\tau]}$ is then obviously identified with τ°. If $i\leqslant j$, the relation $i\sim_{[\tau]}j$ implies that

$$\min_{i\leqslant k\leqslant j} C_k^{\tau^\circ}=C_i^{\tau^\circ}=C_j^{\tau^\circ}.$$

The converse is not true (except if $p=2$) but the conditions $j>i+1$, $C_i^{\tau^\circ}=C_j^{\tau^\circ}$ and

$$C_k^{\tau^\circ}>C_i^{\tau^\circ}\quad\text{for every } k\in]i,j[\cap\mathbb{Z}$$

imply that $i\sim_{[\tau]}j$.

2.7. Convergence towards the Brownian map

For every integer $n\geqslant 1$, let M_n be a random rooted p-angulation, which is uniformly distributed over the set $\mathcal{M}_p^n$, and let $\theta_n=(\tau_n,(\ell_v^n)_{v\in\tau_n^\circ})$ be the random mobile corresponding to M_n via the Bouttier–Di Francesco–Guitter bijection. Then θ_n is uniformly distributed over the set $\mathbb{T}_n^p$ of all p-mobiles with n black vertices. We denote by $C^n=\{C_i^n\}_{0\leqslant i\leqslant pn}$ the contour function of τ_n° and by $\Lambda^n=\{\Lambda_i^n\}_{0\leqslant i\leqslant pn}$ the spatial contour function of θ_n. Recall that the pair (C^n,Λ^n) determines θ_n and thus M_n.

Let $\mathbf{m}_n$ stand for the vertex set of M_n. By the Bouttier–Di Francesco–Guitter bijection, we have the identification

$$\mathbf{m}_n=\tau_n^\circ\cup\{\partial_n\},$$

where ∂_n denotes the root vertex of M_n. The graph distance on $\mathbf{m}_n$ will be denoted by d_n. In particular, if $a,b\in\tau_n^\circ$, $d_n(a,b)$ denotes the graph distance between a and b viewed as vertices in the map M_n.

To simplify notation, we write $\sim_{[n]}$ for the equivalence relation $\sim_{[\tau_n]}$ on $\{0,1,...,pn\}$, so that τ_n° is canonically identified with the quotient $\{0,1,...,pn\}/\sim_{[n]}$. We also write p_n for the canonical projection from $\{0,1,...,pn\}$ onto $\tau_n^\circ=\mathbf{m}_n\setminus\{\partial_n\}$. To be specific, $p_n(i)=v_i^n$, if $v_0^n,v_1^n,...,v_{pn}^n$ denotes the contour sequence of τ_n°. We have $\Lambda_i^n=d_n(\partial_n,p_n(i))$ for every $i\in\{0,...,pn\}$, by the properties recalled in the previous subsection.

If $i,j\in\{0,1,...,pn\}$, we set $d_n(i,j)=d_n(p_n(i),p_n(j))$. By the triangle inequality, we have $|\Lambda_i^n-\Lambda_j^n|\leqslant d_n(i,j)$.

The following theorem restates the main result of [24] in a form convenient for the present work. To simplify notation, we set

$$\lambda_p=\frac{1}{2}\sqrt{\frac{p}{p-1}}\quad\text{and}\quad\varkappa_p=\left(\frac{9}{4p(p-1)}\right)^{1/4}.$$

THEOREM 2.2. *From every sequence of integers converging to ∞, we can extract a subsequence $\{n_k\}_{k\geqslant 1}$ such that the following properties hold. On a suitable probability space, for every integer n belonging to the sequence $\{n_k\}_{k\geqslant 1}$ we can construct the uniformly distributed random p-angulation M_n, in such a way that*

$$\begin{aligned}(\lambda_p n^{-1/2} C^n_{\lfloor pnt\rfloor}, \varkappa_p n^{-1/4}\Lambda^n_{\lfloor pnt\rfloor}, \varkappa_p n^{-1/4} d_n(\lfloor pns\rfloor,\lfloor pnt\rfloor))_{0\leqslant s\leqslant 1, 0\leqslant t\leqslant 1}\\ \xrightarrow{\text{a.s.}} (\bar{\mathbf{e}}_t, \overline{Z}_t, D(s,t))_{0\leqslant s\leqslant 1, 0\leqslant t\leqslant 1}, \quad \text{as } n\to\infty,\end{aligned} \tag{3}$$

where the convergence holds uniformly in $s,t\in[0,1]$ along the sequence $\{n_k\}_{k\geqslant 1}$. In (3)*, $(\bar{\mathbf{e}}_t, \overline{Z}_t)_{0\leqslant t\leqslant 1}$ has the distribution described in* §2.4*, and $(D(s,t))_{0\leqslant s\leqslant 1, 0\leqslant t\leqslant 1}$ is a continuous random process that satisfies properties* (i)–(iv) *stated in* §2.5*. Furthermore, the pointed compact metric spaces*

$$(\mathbf{m}_n, \varkappa_p n^{-1/4} d_n, \partial_n)$$

converge almost surely to $(\mathbf{m}_\infty, D, \varrho)$ in the sense of Gromov–Hausdorff convergence.

Remarks. (a) The convergence of the first two components in (3) does not require the use of a subsequence: See [40, Theorem 3.3]. A subsequence is needed only to get the convergence of the third component via a compactness argument.

(b) The last assertion of the theorem refers to the Gromov–Hausdorff distance on the space of isometry classes of pointed compact metric spaces. The definition of this distance is recalled in §8 below (this definition is not needed in the proof of our main results, and our main tool will be the convergence (3)). The last assertion is then a rather simple consequence of the convergence (3), and the fact that the process D satisfies the above-mentioned properties (i)–(iv): See the proof of Theorem 8.1 below for a sketch of the argument. The key point in the proof of Theorem 2.2 is to verify property (ii) for the pseudo-metric D. See [24] for more details.

We will need a simple application of (3) to the convergence of "discrete snakes" associated with the p-mobiles $(\tau_n, (\ell^n_v)_{v\in\tau_n^\circ})$. Recall that the contour sequence of τ_n° is denoted by $v^n_0, v^n_1, ..., v^n_{pn}$. Then, for every $i\in\{0,1,...,pn\}$, define the finite sequence $W^n_i=(W^n_i(j), 0\leqslant j\leqslant C^n_i)$ by requiring that $W^n_i(j)=\ell^n_{u^n_i(j)}$, where $u^n_i(j)\in\tau_n^\circ$ is the ancestor of v^n_i at generation $2j$ in the tree τ_n. In particular, $W^n_i(C^n_i)=\Lambda^n_i$. Now, if (3) holds, we also have, along the sequence $\{n_k\}_{k\geqslant 1}$,

$$\sup_{0\leqslant s\leqslant 1}\sup_{r\geqslant 0} |\varkappa_p n^{-1/4} W^n_{\lfloor pns\rfloor}(\lfloor \lambda_p^{-1} n^{1/2} r\rfloor \wedge C^n_{\lfloor pns\rfloor}) - \overline{W}_s(r\wedge \bar{\mathbf{e}}_s)| \xrightarrow{\text{a.s.}} 0, \quad \text{as } n\to\infty, \tag{4}$$

where the process $(\overline{W}_s)_{0\leqslant s\leqslant 1}$ is defined from the pair $(\bar{\mathbf{e}}_s, \overline{Z}_s)_{0\leqslant s\leqslant 1}$ as explained at the end of §2.4. The convergence (4) is a consequence of the convergence of the first two

components in (3). A simple way to verify this is to use the homeomorphism theorem of [28]. We leave the details to the reader.

In the remaining part of this work (with the important exception of §9), we choose a sequence $\{n_k\}_{k\geqslant 1}$ as in Theorem 2.2 and we consider only values of n belonging to this sequence. We fix the random maps M_n as in the theorem, and we argue on the set of full probability measure where the convergences (3) and (4) hold, and the process $(D(s,t))_{0\leqslant s\leqslant 1, 0\leqslant t\leqslant 1}$ satisfies the properties (i)–(iv).

3. Preliminary results

In this section, we state and prove a few preliminary facts that will be used in the subsequent proofs.

Recall our notation $\mathcal{T}_{\bar{\mathbf{e}}}(c)$ for the subtree of descendants of a vertex $c\in\mathcal{T}_{\bar{\mathbf{e}}}$. Before we state the first proposition, we make a simple remark concerning the boundary of the set $\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(c))$, when c is a vertex of $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$. We claim that this boundary consists of the point $\Pi(c)$ and the points $\Pi(d)$ for all $d\in\mathcal{T}_{\bar{\mathbf{e}}}(c)$ such that there exists $d'\in\mathcal{T}_{\bar{\mathbf{e}}}\setminus\mathcal{T}_{\bar{\mathbf{e}}}(c)$ with $d'\approx d$. To see this, first note that $\Pi(c)$ belongs to the boundary since the set $\Pi([\![\varrho,c[\![)$ is contained in $\mathbf{m}_\infty\setminus\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(c))$ by Lemma 2.1. Similarly, for every point $d\in\mathcal{T}_{\bar{\mathbf{e}}}(c)$ such that there exists $d'\in\mathcal{T}_{\bar{\mathbf{e}}}\setminus\mathcal{T}_{\bar{\mathbf{e}}}(c)$ with $d'\approx d$, the point $\Pi(d)$ belongs to the boundary. Conversely, let x be a point on the boundary of $\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(c))$, and write $x=\Pi(b)$ with $b\in\mathcal{T}_{\bar{\mathbf{e}}}(c)$. We may assume that $b\neq c$. Then we have $\Pi(b)=\lim_{k\to\infty}\Pi(a_k)$, where $\{a_k\}_{k\geqslant 1}$ is a sequence in $\mathcal{T}_{\bar{\mathbf{e}}}\setminus\mathcal{T}_{\bar{\mathbf{e}}}(c)$. Note that $(\mathcal{T}_{\bar{\mathbf{e}}}\setminus\mathcal{T}_{\bar{\mathbf{e}}}(c))\cup\{c\}$ is compact as the image under $p_{\bar{\mathbf{e}}}$ of the union of two closed subintervals of $[0,1]$, and so we may assume that the sequence $\{a_k\}_{k\geqslant 1}$ converges to $a\in(\mathcal{T}_{\bar{\mathbf{e}}}\setminus\mathcal{T}_{\bar{\mathbf{e}}}(c))\cup\{c\}$. Then $\Pi(b)=\Pi(a)$ and $a\neq c$, since we assumed that $b\neq c$. This gives our claim.

Proposition 3.1. *Almost surely, for every $a,b\in\mathcal{T}_{\bar{\mathbf{e}}}$ and every continuous curve $(\omega(t),0\leqslant t\leqslant T)$ in $\mathbf{m}_\infty$ such that $\omega(0)=\Pi(a)$ and $\omega(T)=\Pi(b)$, we have*

$$\overline{Z}_c\geqslant\min_{0\leqslant t\leqslant T}\overline{Z}_{\omega(t)} \tag{5}$$

for every $c\in[\![a,b]\!]$. Furthermore, if equality holds in (5), then there exists $t\in[0,T]$ such that $\omega(t)=\Pi(c)$.

Proof. By symmetry, we may restrict our attention to the case when $c\in[\![a\triangle b,b]\!]$. When $c=b$ there is nothing to prove. Suppose first that $c\in]\!]a\triangle b,b[\![$. We can also assume that $\Pi(c)\notin\{\omega(t):0\leqslant t\leqslant T\}$. Note that $b\in\mathcal{T}_{\bar{\mathbf{e}}}(c)$ but $a\notin\mathcal{T}_{\bar{\mathbf{e}}}(c)$. Set

$$t_0=\inf\{t\in[0,T]:\omega(t)\in\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(c))\}.$$

Notice that $\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(c))$ is closed as the image under the projection $\mathbf{p}$ of a compact subinterval of $[0,1]$. It follows that $\omega(t_0)\in\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(c))$.

If $t_0=0$, then $\Pi(a)=\omega(0)=\Pi(a')$ for some $a'\in\mathcal{T}_{\bar{\mathbf{e}}}(c)$. Thus $a\approx a'$, which from the definition of the equivalence relation $\approx$ entails $\bar{Z}_c\geqslant\bar{Z}_a=\bar{Z}_{a'}$ (note that $c\in[\![a,a']\!]$ so that, if $a=p_{\bar{\mathbf{e}}}(s)$ and $a'=p_{\bar{\mathbf{e}}}(t)$, the set $\{p_{\bar{\mathbf{e}}}(r):s\wedge t\leqslant r\leqslant s\vee t\}$ contains c). There is even a strict inequality because otherwise we would have $c\approx a$, which is impossible by Lemma 2.1, since c belongs to the skeleton of the tree $\mathcal{T}_{\bar{\mathbf{e}}}$.

If $t_0>0$, then $\omega(t_0)$ belongs to the boundary of $\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(c))$ in $\mathbf{m}_\infty$. We apply the remark preceding the statement of the proposition, noting that $\omega(t_0)\neq\Pi(c)$ since $\Pi(c)\notin\{\omega(t):0\leqslant t\leqslant T\}$. Thus, there exists a point $d\in\mathcal{T}_{\bar{\mathbf{e}}}(c)$ such that $\omega(t_0)=\Pi(d)=\Pi(d')$ for some $d'\in\mathcal{T}_{\bar{\mathbf{e}}}\setminus\mathcal{T}_{\bar{\mathbf{e}}}(c)$. As in the case $t_0=0$, this entails that $\bar{Z}_c\geqslant\bar{Z}_d$, and there is even a strict inequality.

It remains to consider the case $c=a\triangle b$. Then the bound (5) is immediate by a continuity argument. To get the desired conclusion when there is equality in (5), we can assume that $a\neq a\triangle b$ and $b\neq a\triangle b$. We then argue in a similar way as in the case $c\in]\!]a\triangle b,b[\![$, but we replace $\mathcal{T}_{\bar{\mathbf{e}}}(c)$ by $p_{\bar{\mathbf{e}}}([r_1,r_2])$, where $[r_1,r_2]$ is the unique compact subinterval of $[0,1]$ such that $p_{\bar{\mathbf{e}}}(r_1)=p_{\bar{\mathbf{e}}}(r_2)=a\triangle b$, $b\in p_{\bar{\mathbf{e}}}([r_1,r_2])$ and $a\notin p_{\bar{\mathbf{e}}}([r_1,r_2])$. We leave the details to the reader. □

As an immediate consequence of the proposition, we get a useful lower bound for the distance D, which should be compared to the upper bound (2), or equivalently to assertion (iv) in §2.5.

Corollary 3.2. *Almost surely, for every $a,b\in\mathcal{T}_{\bar{\mathbf{e}}}$, we have*

$$D(\Pi(a),\Pi(b))\geqslant\bar{Z}_a+\bar{Z}_b-2\min_{c\in[\![a,b]\!]}\bar{Z}_c.$$

Proof. In agreement with §2.5, we write $D(a,b)=D(\Pi(a),\Pi(b))$. We already noticed that $(\mathbf{m}_\infty,D)$ is a geodesic space and so we can choose a geodesic $\omega=(\omega(t),0\leqslant t\leqslant D(a,b))$ from $\Pi(a)$ to $\Pi(b)$ in $\mathbf{m}_\infty$. By Proposition 3.1,

$$\min_{c\in[\![a,b]\!]}\bar{Z}_c\geqslant\min_{0\leqslant t\leqslant D(a,b)}\bar{Z}_{\omega(t)}.$$

On the other hand, we may choose $t_0\in[0,D(a,b)]$ such that

$$\bar{Z}_{\omega(t_0)}=\min_{0\leqslant t\leqslant D(a,b)}\bar{Z}_{\omega(t)}.$$

By the definition of geodesics, $D(a,b)=D(\omega(0),\omega(t_0))+D(\omega(t_0),\omega(D(a,b)))$. Furthermore,

$$D(\omega(0),\omega(t_0))\geqslant\bar{Z}_a-\bar{Z}_{\omega(t_0)}\quad\text{and}\quad D(\omega(t_0),\omega(D(a,b)))\geqslant\bar{Z}_b-\bar{Z}_{\omega(t_0)},$$

by the triangle inequality. The desired result follows. □

Recall the notation $\mathrm{Skel}_\infty = \Pi(\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}}))$.

PROPOSITION 3.3. *The following properties hold almost surely. The restriction of the projection* Π *to* $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ *is a homeomorphism from* $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ *onto* Skel_∞, *and the Hausdorff dimension of* Skel_∞ *is equal to* 2.

Proof. We already know that the projection Π from $\mathcal{T}_{\bar{\mathbf{e}}}$ onto $\mathbf{m}_\infty$ is continuous, and so is its restriction to $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$. This restriction is also one-to-one by Lemma 2.1. We need to verify that its inverse is continuous in order to get the first assertion. To see this, let $\{x_k\}_{k\geqslant 1}$ be a sequence in Skel_∞ that converges to a point $x_\infty \in \mathrm{Skel}_\infty$, in the sense of the metric D. For every $k \geqslant 1$, we have $x_k = \Pi(a_k)$ with $a_k \in \mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ and similarly $x_\infty = \Pi(a_\infty)$ with $a_\infty \in \mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$. As $\mathcal{T}_{\bar{\mathbf{e}}}$ is compact, we may find a subsequence $\{a_{k_j}\}_{j\geqslant 1}$ that converges to a point $b \in \mathcal{T}_{\bar{\mathbf{e}}}$. By the continuity of Π, we have then $\Pi(b) = x = \Pi(a_\infty)$. Since $a_\infty \in \mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$, Lemma 2.1 shows that this is possible only if $b = a_\infty$. We conclude that the sequence $\{a_k\}_{k\geqslant 1}$ must converge to a_∞ as required.

As for the second assertion, we first observe that $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ is a countable union of sets that are isometric to line segments, so that its Hausdorff dimension is 1. On the other hand, the bound (iv) in §2.5 easily implies that the mapping Π from $(\mathcal{T}_{\bar{\mathbf{e}}}, d_{\bar{\mathbf{e}}})$ onto $(\mathbf{m}_\infty, D)$ is Hölder continuous with exponent $\frac{1}{2} - \varepsilon$, for every $\varepsilon > 0$ (see [24, Lemma 5.1] for a closely related statement). This gives the upper bound $\dim(\mathrm{Skel}_\infty) \leqslant 2$.

To get the corresponding lower bound, we can in fact prove that the Hausdorff dimension of the image under Π of any (non-trivial) line segment in $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ is bounded below by 2. To see this, let I be a non-trivial line segment contained in $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$, and denote Lebesgue measure on I by μ_I. From standard results about Hausdorff measures, it is enough to verify that, for every $\delta > 0$,

$$\iint_{I\times I} \frac{\mu_I(da)\mu_I(db)}{D(a,b)^{2-\delta}} < \infty. \tag{6}$$

This follows from the lower bound on $D(a,b)$ provided by Corollary 3.2. Recall that $\bar{Z}$ has been constructed by shifting the process $Z^{\mathbf{e}}$, which behaves like linear Brownian motion along any segment of $\mathcal{T}_{\mathbf{e}}$. Denoting by B a standard linear Brownian motion, we see that the bound (6) is a consequence of the fact that

$$\iint_{[0,1]^2} \Big(B_s + B_t - 2 \min_{s\leqslant r\leqslant t} B_r \Big)^{-(2-\delta)} \, ds\, dt < \infty \quad \text{a.s.}$$

This property easily follows from a first-moment calculation, using Pitman's theorem ([35, Theorem VI.3.5]), which asserts that the process $R_t := B_t - 2\min_{0\leqslant r\leqslant t} B_r$ is a 3-dimensional Bessel process. □

Our next lemma shows that for any fixed time $U\in]0,1[$, $p_{\bar{\mathbf{e}}}(U)$ does not belong to the skeleton of $\mathcal{T}_{\bar{\mathbf{e}}}$. Although this statement is intuitively clear, it is not so easy to give a precise argument.

LEMMA 3.4. *Let $U\in]0,1[$. Then $p_{\bar{\mathbf{e}}}(U)\notin \mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ and $\mathbf{p}(U)\notin \mathrm{Skel}_\infty$ almost surely.*

Proof. To get the first assertion, we need to prove that for every $\varepsilon\in]0,U\wedge(1-U)[$, we have almost surely $m_{\bar{\mathbf{e}}}(U,U+\varepsilon)<\bar{\mathbf{e}}_U$ and $m_{\bar{\mathbf{e}}}(U-\varepsilon,U)<\bar{\mathbf{e}}_U$. Since the law of the pair $(\mathbf{e}_t, Z^{\mathbf{e}}_t)_{0\leqslant t\leqslant 1}$ is invariant under the time-reversal operation $t\mapsto 1-t$, the same holds for the pair $(\bar{\mathbf{e}}_t, \overline{Z}_t)_{0\leqslant t\leqslant 1}$, and so it is enough to prove the first part of the preceding assertion. Equivalently, we must show that $P(A)=0$, where

$$A=\bigcup_{\varepsilon\in]0,1-U[}\{m_{\bar{\mathbf{e}}}(U,U+\varepsilon)=\bar{\mathbf{e}}_U\}.$$

Now recall the definition of s_* and $\bar{\mathbf{e}}$ in §2.4. It follows from the invariance under re-rooting stated in [26, §2.3] that s_* is independent of $\bar{\mathbf{e}}$ and uniformly distributed over $[0,1]$. In particular, if we fix $\varepsilon_0\in]0,1-U[$, we have

$$(1-U-\varepsilon_0)P(A)=P(\{s_*<1-U-\varepsilon_0\}\cap A).$$

On the other hand, from the definition of $\bar{\mathbf{e}}$ in terms of $\mathbf{e}$, we get

$$\{s_*<1-U-\varepsilon_0\}\cap A\subset A_1\cup A_2,$$

where

$$A_1=\{s_*<1-U-\varepsilon_0\}\cap\Big(\bigcup_{\varepsilon\in]0,\varepsilon_0[}\{m_{\mathbf{e}}(s_*+U,s_*+U+\varepsilon)=\mathbf{e}_{s_*+U}\}\Big)$$

and

$$A_2=\{s_*<1-U-\varepsilon_0\}\cap\{m_{\mathbf{e}}(s_*,s_*+U)=\mathbf{e}_{s_*+U}\}.$$

Let us verify that $P(A_1)=0$. For every rational $q\in]0,1[$, set

$$s_*(q)=\sup\{s\in[0,q]: Z^{\mathbf{e}}_s=\min_{0\leqslant r\leqslant q} Z^{\mathbf{e}}_r\}.$$

Clearly the random variable $s_*(q)$ is measurable with respect to the σ-field generated by $(\mathbf{e}_r, W_r)_{0\leqslant r\leqslant q}$. As a consequence, the random time

$$T_{(q)}=q+(U+s_*(q)-q)^+$$

is a stopping time of the filtration generated by $(\mathbf{e}_r, W_r)_{0\leqslant r\leqslant 1}$. Notice that the process $(\mathbf{e}_t)_{0\leqslant t\leqslant 1}$ is Markovian with respect to this filtration. By a standard property of linear

Brownian motion, we get that, almost surely on the event $\{T_{(q)}<1\}$, we have for every $\varepsilon>0$,

$$m_{\mathbf{e}}(T_{(q)},(T_{(q)}+\varepsilon)\wedge 1)<\mathbf{e}_{T_{(q)}}.$$

Finally, on the event $\{s_*<1-U-\varepsilon_0\}$, we can pick a rational q such that $s_*<q<s_*+U$, and then we have $s_*(q)=s_*$ and $T_{(q)}=s_*+U$. From the preceding observations we see that A_1 does not hold almost surely. A similar argument gives $P(A_2)=0$. It follows that $P(A)=0$, which completes the proof of the first assertion. The second one follows since the equivalence class of any $a\in\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ for $\approx$ is a singleton, by Lemma 2.1. □

Recall from §1 the definition of a geodesic connecting two points of $\mathbf{m}_\infty$. Also recall that $D(\varrho,x)=\bar{Z}_x$ for every $x\in\mathbf{m}_\infty$.

LEMMA 3.5. *Almost surely, for every $x\in\mathbf{m}_\infty$, every geodesic $\omega=(\omega(t))_{0\leqslant t\leqslant\bar{Z}_x}$ from ϱ to x and every interval $[u,v]$ such that $0\leqslant u<v\leqslant\bar{Z}_x$, the range of ω over $[u,v]$ intersects $\mathbf{m}_\infty\setminus\mathrm{Skel}_\infty$.*

Proof. Let $x\in\mathbf{m}_\infty$ and let ω be a geodesic from ϱ to x. Also, let u and v be reals such that $0\leqslant u<v<\bar{Z}_x$. We argue by contradiction and suppose that the set $\{\omega(t):u\leqslant t\leqslant v\}$ is contained in Skel_∞. From Proposition 3.3, we may write $\omega(r)=\Pi(\psi(r))$ for every $r\in[u,v]$, where ψ is a continuous mapping from $[u,v]$ into $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$. This mapping ψ is also one-to-one because ω is a geodesic. Since $\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$ is a (non-compact) real tree, it follows that the range of ψ is the line segment $[\![\psi(u),\psi(v)]\!]$, and moreover ψ is a homeomorphism from $[u,v]$ onto $[\![\psi(u),\psi(v)]\!]$. The fact that $\bar{Z}_{\psi(r)}=r$ for every $r\in[u,v]$ then implies that the mapping $a\mapsto\bar{Z}_a$ is monotone increasing over $[\![\psi(u),\psi(v)]\!]$. However, this is absurd since $\bar{Z}$ has been constructed by shifting the process $Z^{\mathbf{e}}$, and we already observed that $Z^{\mathbf{e}}$ behaves like linear Brownian motion along line segments of the tree $\mathcal{T}_{\mathbf{e}}$, so that it cannot vary monotonically on a line segment of the tree. □

4. Two classes of geodesics

In this section, we will discuss several classes of geodesics connecting the root to a point of $\mathbf{m}_\infty$. This point will be of the form $\mathbf{p}(s)$ for $s\in[0,1]$. All the results of this section are valid outside a set of zero probability (discarding such a set is needed e.g. to apply Lemma 2.1 or Proposition 3.1). It is important to note that this set of zero probability does not depend on s. We will omit the words "almost surely" in the statements of this section.

4.1. Simple geodesics

For every $s\in[0,1]$, we define a mapping $\varphi_s\colon[0,\bar{Z}_s]\to[0,1]$ by setting

$$\varphi_s(t)=\sup\{r\leqslant s:\bar{Z}_r\leqslant t\},\quad 0\leqslant t\leqslant\bar{Z}_s.$$

Clearly, $\mathbf{p}(\varphi_s(0))=\varrho$ and $\mathbf{p}(\varphi_s(\bar{Z}_s))=\mathbf{p}(s)$. Also, $\bar{Z}_{\varphi_s(t)}=t$ and if $0\leqslant t\leqslant t'\leqslant\bar{Z}_s$,

$$D(\varphi_s(t),\varphi_s(t'))\leqslant\bar{Z}_{\varphi_s(t)}+\bar{Z}_{\varphi_s(t')}-2\min_{[\varphi_s(t),\varphi_s(t')]}\bar{Z}_r=t+t'-2t=t'-t.$$

Since

$$D(\varphi_s(t),\varphi_s(t'))\geqslant D(0,\varphi_s(t'))-D(0,\varphi_s(t))=t'-t,$$

we must have $D(\varphi_s(t),\varphi_s(t'))=t'-t$ for every $0\leqslant t\leqslant t'\leqslant\bar{Z}_s$. Thus $(\mathbf{p}(\varphi_s(t)),0\leqslant t\leqslant\bar{Z}_s)$ is a geodesic from ϱ to the point $\mathbf{p}(s)$. We will write

$$\Phi_s(t)=\mathbf{p}(\varphi_s(t))$$

for every $t\in[0,\bar{Z}_s]$ and call Φ_s a *simple* geodesic from ϱ to $\mathbf{p}(s)$ (sometimes we abusively say that φ_s itself is a simple geodesic). We can also give a dual definition of simple geodesics, by setting, for every $s\in[0,1]$,

$$\overline{\varphi}_s(t)=\inf\{r\geqslant s:\bar{Z}_r\leqslant t\},\quad 0\leqslant t\leqslant\bar{Z}_s.$$

It is immediate that $\varphi_s(t)\approx\bar{\varphi}_s(t)$, so that $\mathbf{p}(\bar{\varphi}_s(t))=\mathbf{p}(\varphi_s(t))=\Phi_s(t)$ for every $t\in[0,\bar{Z}_s]$.

Let s and s' be two distinct points in $]0,1[$ such that $\mathbf{p}(s)=\mathbf{p}(s')$. If $s\approx s'$, we have

$$\bar{Z}_s=\bar{Z}_{s'}=\min_{[s\wedge s',s\vee s']}\bar{Z}_r$$

and it readily follows that the geodesics Φ_s and $\Phi_{s'}$ coincide. On the other hand, if $s\sim s'$ (so that $s\approx s'$ does not hold), it is not hard to see that the geodesics Φ_s and $\Phi_{s'}$ differ. We will come back to this later.

It is important to note that a simple geodesic from ϱ to $\mathbf{p}(s)$ does not intersect Skel_∞, except possibly at its endpoint $\mathbf{p}(s)$. Indeed, if $0\leqslant t<\bar{Z}_s$, it is immediate from the definition that $\varphi_s(t)$ is a right-increase time of $\bar{Z}$, and then Lemma 2.1 implies that $\mathbf{p}(\varphi_s(t))\notin\mathrm{Skel}_\infty$. Much of what follows is devoted to proving that a similar property holds for any geodesic connecting the root to another point of $\mathbf{m}_\infty$.

4.2. Minimal geodesics

Let $s\in[0,1]$, and denote by $\mathrm{G}(s)$ the set of all mappings $\gamma\colon[0,\bar{Z}_s]\to[0,s]$ such that $(\mathbf{p}(\gamma(t)),0\leqslant t\leqslant\bar{Z}_s)$ is a geodesic from ϱ to $\mathbf{p}(s)$. Notice that $\mathrm{G}(s)$ is not empty since $\varphi_s\in\mathrm{G}(s)$. We then define a mapping $\gamma_s\colon[0,\bar{Z}_s]\to[0,1]$ by setting

$$\gamma_s(t)=\inf\{\gamma(t):\gamma\in\mathrm{G}(s)\},\quad 0\leqslant t\leqslant\bar{Z}_s.$$

In contrast with φ_s, γ_s only depends on $\mathbf{p}(s)$ and not on the particular choice of a representative of $\mathbf{p}(s)$ in $[0,1]$.

We have $\gamma_s(0)=0$ and $\mathbf{p}(\gamma_s(\bar{Z}_s))=\mathbf{p}(s)$. From the continuity of $\bar{Z}$ and of the distance D, it is also clear that

$$\bar{Z}_{\gamma_s(r)}=D(0,\gamma_s(r))=r \tag{7}$$

and

$$D(\gamma_s(r),s)=\bar{Z}_s-r \tag{8}$$

for every $r\in[0,\bar{Z}_s]$.

It is easy to verify that the mapping $t\mapsto\gamma_s(t)$ is monotone increasing. Indeed, let $t\in]0,\bar{Z}_s]$ and $\gamma\in\mathrm{G}(s)$. Put

$$\widetilde{\gamma}(r)=\begin{cases}\gamma(r), & \text{if } r\in[t,\bar{Z}_s],\\ \varphi_{\gamma(t)}(r), & \text{if } r\in[0,t[.\end{cases}$$

Then, we also have $\widetilde{\gamma}\in\mathrm{G}(s)$. Furthermore, $\widetilde{\gamma}(r)<\gamma(t)$ for every $r\in[0,t[$. From the definition of γ_s, it now follows that $\gamma_s(r)<\gamma_s(t)$ for every $0\leqslant r<t\leqslant\bar{Z}_s$ (note that $\gamma_s(r)=\gamma_s(t)$ is impossible since $\bar{Z}_{\gamma_s(r)}=r\neq t=\bar{Z}_{\gamma_s(t)}$).

To simplify notation, we set $\Gamma_s(t)=\mathbf{p}(\gamma_s(t))$ for every $t\in[0,\bar{Z}_s]$.

PROPOSITION 4.1. *The curve* $(\Gamma_s(t),0\leqslant t\leqslant\bar{Z}_s)$ *is a geodesic from* ϱ *to* $\mathbf{p}(s)$. *It is called the* minimal geodesic *from* ϱ *to* $\mathbf{p}(s)$. (*Sometimes we also say that* γ_s *is a* minimal geodesic.)

The proof of Proposition 4.1 is based on the following lemma. We denote by $\mathrm{G}^*(s)$ the set of all $\gamma\in\mathrm{G}(s)$ such that $\gamma(t)$ is the smallest representative of $\mathbf{p}(\gamma(t))$ for every $t\in[0,\bar{Z}_s]$. Obviously we may replace $\mathrm{G}(s)$ by $\mathrm{G}^*(s)$ in the definition of γ_s.

LEMMA 4.2. *If* γ *and* γ' *belong to* $\mathrm{G}^*(s)$, *then* $\gamma\wedge\gamma'$ *also belongs to* $\mathrm{G}^*(s)$.

Proof. We first claim that the set $\{t\in]0,\bar{Z}_s[:\gamma(t)<\gamma'(t)\}$ is open. To see this, let t_1 be an element of this set. We argue by contradiction and assume that there exists a sequence $\{r_k\}_{k\geqslant1}$ in $]0,\bar{Z}_s[$ such that $r_k\to t_1$ and $\gamma(r_k)\geqslant\gamma'(r_k)$ for every $k\geqslant1$. Via a

compactness argument, we may further assume that $\gamma(r_k)\to u$ and $\gamma'(r_k)\to u'$ as $k\to\infty$, where $0\leqslant u'\leqslant u\leqslant 1$. By the continuity of geodesics, we have

$$\mathbf{p}(u)=\mathbf{p}(\gamma(t_1)) \quad \text{and} \quad \mathbf{p}(u')=\mathbf{p}(\gamma'(t_1)).$$

In particular,

$$\overline{Z}_u=\overline{Z}_{\gamma(t_1)}=t_1=\overline{Z}_{\gamma'(t_1)}=\overline{Z}_{u'}.$$

Since $\gamma'(t_1)$ is the smallest representative of $\mathbf{p}(\gamma'(t_1))=\mathbf{p}(u')$, we have $\gamma'(t_1)\leqslant u'$, and we also know that $\gamma(t_1)<\gamma'(t_1)$ by assumption. Thus we have both $\gamma(t_1)<\gamma'(t_1)\leqslant u'\leqslant u$ and $\mathbf{p}(u)=\mathbf{p}(\gamma(t_1))$. Note that $\gamma(t_1)\approx u$ is impossible, because the previous observations (and the fact that $\overline{Z}_{\gamma(t_1)}=\overline{Z}_{\gamma'(t_1)}$) would imply that we also have $\gamma(t_1)\approx\gamma'(t_1)$ and then $\gamma(t_1)=\gamma'(t_1)$, contradicting our initial assumption. Thus, we have $\gamma(t_1)\sim u$, and the inequalities $\gamma(t_1)<\gamma'(t_1)\leqslant u$ imply that $p_{\bar{\mathbf{e}}}(\gamma'(t_1))$ is a descendant of $p_{\bar{\mathbf{e}}}(\gamma(t_1))$ in the tree $\mathcal{T}_{\bar{\mathbf{e}}}$. Equivalently, $p_{\bar{\mathbf{e}}}(\gamma(t_1))\in]\!]\varrho, p_{\bar{\mathbf{e}}}(\gamma'(t_1))[\![$, and in particular $p_{\bar{\mathbf{e}}}(\gamma(t_1))\in\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$.

Consider first the case when $p_{\bar{\mathbf{e}}}(\gamma(t_1))\in]\!]\varrho, p_{\bar{\mathbf{e}}}(\gamma'(t_1))\triangle p_{\bar{\mathbf{e}}}(s)[\![$. As $p_{\bar{\mathbf{e}}}(\gamma(t_1))\in\mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}})$, Lemma 2.1 shows that the representatives of $\mathbf{p}(\gamma(t_1))$ in $[0,1]$ are exactly the representatives of $p_{\bar{\mathbf{e}}}(\gamma(t_1))$ in $[0,1]$. Let v be the greatest representative of $p_{\bar{\mathbf{e}}}(\gamma(t_1))$ in $[0,s]$. As $\mathbf{p}(u)=\mathbf{p}(\gamma(t_1))$ and $u\in[0,s]$ (at this point, we use in a crucial way the fact that γ takes values in $[0,s]$), we get that $u\leqslant v$. On the other hand, since

$$p_{\bar{\mathbf{e}}}(\gamma(t_1))\in[\![\varrho, p_{\bar{\mathbf{e}}}(\gamma'(t_1))\triangle p_{\bar{\mathbf{e}}}(s)[\![,$$

simple considerations about the genealogy of the tree $\mathcal{T}_{\bar{\mathbf{e}}}$ show that $v<\gamma'(t_1)$. It follows that $u\leqslant v<\gamma'(t_1)\leqslant u'$, which is a contradiction.

Let us turn to the case when $p_{\bar{\mathbf{e}}}(\gamma(t_1))\in[\![p_{\bar{\mathbf{e}}}(\gamma'(t_1))\triangle p_{\bar{\mathbf{e}}}(s), p_{\bar{\mathbf{e}}}(\gamma'(t_1))[\![$. In this case, we apply Proposition 3.1 to the continuous curve $(\mathbf{p}(\gamma'(\overline{Z}_s-t)), 0\leqslant t\leqslant\overline{Z}_s-t_1)$. The assumptions of this proposition are satisfied with $a=p_{\bar{\mathbf{e}}}(s)$, $b=p_{\bar{\mathbf{e}}}(\gamma'(t_1))$ and $c=p_{\bar{\mathbf{e}}}(\gamma(t_1))$. Since

$$\overline{Z}_{\gamma(t_1)}=t_1=\inf\{\overline{Z}_{\gamma'(t)} : t_1\leqslant t\leqslant\overline{Z}_s\},$$

we get that $\mathbf{p}(\gamma(t_1))=\mathbf{p}(\gamma'(t^*))$ for some $t^*\in[t_1,\overline{Z}_s]$. Necessarily $t^*=t_1$ and $\gamma(t_1)=\gamma'(t_1)$, which is again a contradiction. This completes the proof of the claim.

We can now complete the proof of Lemma 4.2. Let us fix $r\in]0,\overline{Z}_s[$ and verify that, for every $t\in[r,\overline{Z}_s[$,

$$D(\gamma\wedge\gamma'(r),\gamma\wedge\gamma'(t))=t-r. \tag{9}$$

Notice that this equality holds for $t=\overline{Z}_s$ since both $\gamma(\overline{Z}_s)$ and $\gamma'(\overline{Z}_s)$ are equal to the smallest representative of $\mathbf{p}(s)$. If $\gamma(r)=\gamma'(r)$, (9) holds because $\mathbf{p}(\gamma(t))$ and $\mathbf{p}(\gamma'(t))$ are

both geodesics. We can thus assume that $\gamma(r)\neq\gamma'(r)$, and by symmetry we may restrict our attention to the case $\gamma(r)<\gamma'(r)$.

We again argue by contradiction and assume that (9) fails for some $t\in]r,\bar{Z}_s[$. We then set

$$t_0=\inf\{t\in]r,\bar{Z}_s[:D(\gamma\wedge\gamma'(r),\gamma\wedge\gamma'(t))\neq t-r\}<\bar{Z}_s.$$

We consider the following three cases:

Case 1. If $\gamma(t_0)=\gamma'(t_0)$, then

$$D(\gamma\wedge\gamma'(r),\gamma\wedge\gamma'(t_0))=t_0-r$$

and, for every $t\in[t_0,\bar{Z}_s]$,

$$D(\gamma\wedge\gamma'(t_0),\gamma\wedge\gamma'(t))=t-t_0.$$

Thus

$$D(\gamma\wedge\gamma'(r),\gamma\wedge\gamma'(t))\leqslant t-r$$

for every $t\in[t_0,\bar{Z}_s]$. The reverse inequality is also clear by writing

$$D(\gamma\wedge\gamma'(r),\gamma\wedge\gamma'(t))\geqslant D(0,\gamma\wedge\gamma'(t))-D(0,\gamma\wedge\gamma'(r))=t-r$$

and we get a contradiction to the definition of t_0.

Case 2. If $\gamma(t_0)<\gamma'(t_0)$, then the claim gives $\varepsilon>0$ such that $\gamma(t)<\gamma'(t)$ for all $t\in[t_0,t_0+\varepsilon[$, and thus

$$D(\gamma\wedge\gamma'(r),\gamma\wedge\gamma'(t))=D(\gamma(r),\gamma(t))=t-r.$$

This again contradicts the definition of t_0.

Case 3. If $\gamma(t_0)>\gamma'(t_0)$, then $t_0>r$ (because $\gamma(r)<\gamma'(r)$) and the claim gives $\varepsilon\in]0,t_0-r[$ such that $\gamma(t)>\gamma'(t)$ for every $t\in]t_0-\varepsilon,t_0+\varepsilon[$. Then, for every $t,t'\in]t_0-\varepsilon,t_0+\varepsilon[$, with $t<t'$,

$$D(\gamma\wedge\gamma'(t),\gamma\wedge\gamma'(t'))=D(\gamma'(t),\gamma'(t'))=t'-t.$$

Fix $t\in]t_0-\varepsilon,t_0[$. By the definition of t_0,

$$D(\gamma\wedge\gamma'(r),\gamma\wedge\gamma'(t))=t-r$$

and so

$$D(\gamma\wedge\gamma'(r),\gamma\wedge\gamma'(t'))\leqslant t'-r$$

for every $t'\in[t_0,t_0+\varepsilon[$. As in the first case above, the last inequality must be an equality, which again contradicts the definition of t_0.

This completes the proof of Lemma 4.2. □

Proof of Proposition 4.1. We need to verify that

$$D(\gamma_s(r),\gamma_s(t)) = t-r \tag{10}$$

for every $0\leqslant r<t\leqslant \bar{Z}_s$. If $r=0$, or if $t=\bar{Z}_s$, (10) follows from (7) and (8). Let $\varepsilon>0$ and let us fix r and t with $0<r<t<\bar{Z}_s$. By the definition of γ_s, we can find $\gamma,\gamma'\in \mathrm{G}^*(s)$ such that

$$\gamma_s(r)\leqslant\gamma(r)\leqslant\gamma_s(r)+\varepsilon \quad \text{and} \quad \gamma_s(t)\leqslant\gamma'(t)\leqslant\gamma_s(t)+\varepsilon.$$

It follows that

$$\gamma_s(r)\leqslant\gamma\wedge\gamma'(r)\leqslant\gamma_s(r)+\varepsilon \quad \text{and} \quad \gamma_s(t)\leqslant\gamma\wedge\gamma'(t)\leqslant\gamma_s(t)+\varepsilon.$$

Choosing ε small, we see that $D(\gamma_s(r),\gamma_s(t))$ can be made arbitrarily close to the quantity $D(\gamma\wedge\gamma'(r),\gamma\wedge\gamma'(t))=t-r$, by Lemma 4.2. This completes the proof of (10). □

In view of our applications, it will also be important to consider a notion that is dual to the notion of a minimal geodesic. For every $s\in[0,1]$, we denote by $\overline{\mathrm{G}}(s)$ the set of all mappings $\gamma\colon[0,\bar{Z}_s]\to[s,1]$ such that $(\mathbf{p}(\gamma(t)),0\leqslant t\leqslant\bar{Z}_s)$ is a geodesic from ϱ to $\mathbf{p}(s)$. We then set

$$\bar{\gamma}_s(t)=\sup\{\gamma(t):\gamma\in\overline{\mathrm{G}}(s)\} \quad \text{and} \quad \bar{\Gamma}_s(t)=\mathbf{p}(\bar{\gamma}_s(t))$$

for every $t\in[0,\bar{Z}_s]$. The very same arguments as in the proof of Proposition 4.1 show that $(\bar{\Gamma}_s(t),0\leqslant t\leqslant\bar{Z}_s)$ is a geodesic from ϱ to $\mathbf{p}(s)$, which is called the *maximal geodesic*.

The next two sections are devoted to a number of technical results about geodesics. Although we concentrate on minimal geodesics, it should be noted that symmetric arguments yield corresponding results for maximal geodesics.

5. The main estimate

In this section we fix $U\in]0,1[$. Recall that γ_U is the minimal geodesic that was introduced in the previous section. We denote by $\mathcal{R}(\gamma_U)$ the range of γ_U:

$$\mathcal{R}(\gamma_U)=\{\gamma_U(t):0\leqslant t\leqslant\bar{Z}_U\}.$$

We also fix $u_0\in\left]0,\frac{1}{2}U\right[$.

LEMMA 5.1. *Let* $\delta\in\big]0,\frac{1}{2}u_0\big[$ *and* $\eta>0$. *For every* $u\in]0,U[$, *let* $B_{\eta,u}$ *denote the event*

$$B_{\eta,u}=\Big\{\bar{\mathbf{e}}_u>\eta,\ \inf_{\eta/2\leqslant t\leqslant\bar{\mathbf{e}}_u}\overline{W}_u(t)>\eta\Big\}.$$

Then, there exists a constant $C=C(p,U,u_0,\eta,\delta)$ *such that, for every* $u\in[u_0,U[$, $v\in]u,U]$ *and* $\alpha>0$,

$$P\Big[\{\mathcal{R}(\gamma_U)\cap]u,v]\neq\varnothing\}\cap B_{\eta,u}\cap\Big\{\inf_{u-\delta\leqslant t\leqslant u}\bar{\mathbf{e}}_t>\bar{\mathbf{e}}_u-\alpha\Big\}\Big]\leqslant C\alpha P\big[\mathcal{R}(\gamma_U)\cap]u,v]\neq\varnothing\big]. \quad (11)$$

The remaining part of this section is devoted to the proof of Lemma 5.1. To simplify notation, we will write $\gamma=\gamma_U$ and $\Gamma=\Gamma_U$ in this proof. The underlying idea is to apply the Markov property in reverse time to the process $(\bar{\mathbf{e}}_r,\overline{W}_r)_{0\leqslant r\leqslant 1}$ at time u. Notice that the event $B_{\eta,u}$ only depends on $(\bar{\mathbf{e}}_u,\overline{W}_u)$. Assuming that the event $\{\mathcal{R}(\gamma)\cap]u,v]\neq\varnothing\}$ is also measurable with respect to the σ-field generated by $(\bar{\mathbf{e}}_r,\overline{W}_r)_{u\leqslant r\leqslant 1}$, we would need to bound the conditional probability of the event

$$\Big\{\inf_{u-\delta\leqslant t\leqslant u}\bar{\mathbf{e}}_t>\bar{\mathbf{e}}_u-\alpha\Big\}$$

given that σ-field. The latter conditional probability is bounded above by $C\alpha$ as would be the case if $\bar{\mathbf{e}}$ were replaced by a linear Brownian path.

The above line of reasoning is not easy to implement, mainly because the required measurability property of the event $\{\mathcal{R}(\gamma)\cap]u,v]\neq\varnothing\}$ seems difficult to establish. Instead, we will use a discrete version of the preceding ideas, and we will rely on the convergence (3) to derive the bound of Lemma 5.1 from our discrete estimates. This explains why the constant C in Lemma 5.1 depends on p. Assuming that the distribution of the Brownian map is unique, or more precisely that the distribution of the limiting triplet $(\bar{\mathbf{e}},\overline{Z},D)$ in (3) is independent of p, the constant C in Lemma 5.1 should of course not depend on p.

The proof of Lemma 5.1 requires several intermediate lemmas. Before stating the first of these lemmas, we start with a few simple remarks.

We already noticed that the minimal geodesic $(\gamma(r),0\leqslant r\leqslant\overline{Z}_U)$ is monotone increasing. Moreover, for every $r\in]0,\overline{Z}_U]$ we have $\mathbf{p}(\gamma(r-))=\Gamma(r-)=\Gamma(r)=\mathbf{p}(\gamma(r))$ by continuity, and the definition of the minimal geodesic shows that we must have $\gamma(r-)=\gamma(r)$. The mapping $r\mapsto\gamma(r)$ is thus left-continuous on $]0,U]$.

Then let $r_1,r_2,\ldots$ be a sequence dense in $]0,\infty[$. We have for every $u<v$,

$$\{\mathcal{R}(\gamma)\cap]u,v]\neq\varnothing\}=\lim_{k\to\infty}\bigcup_{i=1}^{k}\{u<\gamma(r_i)\leqslant v\}.$$

Here and below we make the convention that $\gamma(r)=\infty$ if $r>\overline{Z}_U$.

Thus, if we can prove that a bound analogous to (11) holds when the event

$$\{\mathcal{R}(\gamma)\cap]u,v]\neq\varnothing\}$$

is replaced by an event of the type

$$\bigcup_{i=1}^{k}\{u<\gamma(r_i)\leqslant v\},$$

with a constant C independent of k, then a monotone passage to the limit will give us the desired result.

Our first technical lemma will relate events of the form $\{\gamma(r)\leqslant u\}$ to other events that are more suitable for the discrete approximations that we will use to derive our estimates. For every $u\in]0,1[$, we denote by $\mathrm{Fr}([0,u])$ the set of all $v\in[u,1]$ such that $\mathbf{p}(v)=\mathbf{p}(v')$ for some $v'\in[0,u]$. Note that $\mathrm{Fr}([0,u])$ is a closed subset of $[0,1]$. Furthermore, the ancestral line of $p_{\overline{\mathbf{e}}}(u)$ is contained in $p_{\overline{\mathbf{e}}}(\mathrm{Fr}([0,u]))$.

In the following three lemmas, we fix $u\in]0,U[$ and $r>0$.

LEMMA 5.2. *For every $\varepsilon>0$, let $A_\varepsilon(r,u)$ stand for the event on which there exist an integer $q\geqslant 1$ and points $s_0,s_1,...,s_q\in[u,U]$ such that $s_0=U$, $s_q\in\mathrm{Fr}([0,u])$ and, for every $i\in\{0,1,...,q-1\}$,*

$$\overline{Z}_{s_i}\geqslant r, \tag{12}$$

$$D(s_i,s_{i+1})\leqslant\varepsilon, \tag{13}$$

and, for every $i\in\{0,1,...,q-2\}$,

$$D(s_i,s_{i+1})\leqslant\overline{Z}_{s_i}-\overline{Z}_{s_{i+1}}+\frac{\varepsilon}{q}. \tag{14}$$

Then,

$$\{\gamma(r)\leqslant u\}\subset\bigcap_{\varepsilon>0}A_\varepsilon(r,u)\quad a.s.$$

Remark. The inclusion in the conclusion of the lemma can in fact be replaced by an equality. See Lemma 5.4 below.

Proof. Let $\varepsilon>0$ and let us verify that $\{\gamma(r)\leqslant u\}\subset A_\varepsilon(r,u)$ a.s. We set

$$r_0=\sup\{t\in[0,\overline{Z}_U]:\gamma(t)\leqslant u\}.$$

If $r_0=\overline{Z}_U$, then $\gamma(\overline{Z}_U)\leqslant u$ by left-continuity, and since $\mathbf{p}(U)=\mathbf{p}(\gamma(\overline{Z}_U))$ we get that $U\in\mathrm{Fr}([0,u])$. In this case, we simply take $q=1$ and $s_0=s_1=U$, noting that $\overline{Z}_U\geqslant r$ on the event $\{\gamma(r)\leqslant u\}$ (recall that we made the convention that $\gamma(r)=\infty$ if $r>\overline{Z}_U$).

So we may assume that $r_0<\overline{Z}_U$. We observe that $\gamma(r_0+)\in\mathrm{Fr}([0,u])$. Indeed we have $\gamma(r_0)\leqslant u$, $\gamma(r_0+)\geqslant u$ and $\mathbf{p}(\gamma(r_0))=\mathbf{p}(\gamma(r_0+))$ by continuity.

We then set

$$s_0=U,\quad s_1=\gamma(\overline{Z}_U-\varepsilon),\quad \ldots,\quad s_{q-1}=\gamma(\overline{Z}_U-(q-1)\varepsilon),$$

where q is the first integer such that $\overline{Z}_U-q\varepsilon\leqslant r_0$. We finally set $s_q=\gamma(r_0+)$. Then $s_q\geqslant u$ and $s_i\in[u,U]$ for every $i\in\{0,1,\ldots,q\}$ because γ is non-decreasing. Conditions (13) and (14) clearly hold: In (14) we have even $D(s_i,s_{i+1})=\overline{Z}_{s_i}-\overline{Z}_{s_{i+1}}$. Moreover the condition $\gamma(r)\leqslant u$ implies that $r\leqslant r_0$ and thus $\overline{Z}_{s_i}\geqslant\overline{Z}_{\gamma(r_0)}\geqslant\overline{Z}_{\gamma(r)}=r$ for every $i\in\{0,1,\ldots,q\}$. □

We now want to use the convergence (3) in order to get discrete approximations of the sets $A_\varepsilon(r,u)$. For every integer n, and for $0\leqslant k\leqslant k'\leqslant pn$, we denote by $\mathcal{G}^n(k,k')$ the σ-field generated by the variables (C_i^n,Λ_i^n), $k\leqslant i\leqslant k'$.

Fix $k\in\{0,1,\ldots,pn\}$. If $i,j\in\{k,k+1,\ldots,pn\}$, then the distance $d_n(i,j)$ is in general not measurable with respect to the σ-field $\mathcal{G}^n(k,pn)$. However, we may define a "modified distance" $\tilde{d}_n^k(i,j)$ for which this will be true: To this end, we restrict our attention to edges of the map $\mathbf{m}_n$ that are generated between steps k and $pn-1$ of the Bouttier–Di Francesco–Guitter bijection of §2.6. To give a more precise definition, recall from §2.6 the notion of a successor (with respect to the mobile $(\tau_n,(\ell_v^n)_{v\in\tau_n^\circ})$), and say by convention that the successor of any integer $i\in\{0,1,\ldots,pn\}$ such that the corresponding vertex in τ_n° has label 1 is $pn+1$. Then, for $i,j\in\{k,k+1,\ldots,pn\}$, $\tilde{d}_n^k(i,j)$ is the minimal integer $\ell\geqslant0$ for which there exists a sequence $i_0,j_0,i_1,j_1,\ldots,i_\ell,j_\ell$ of integers in $\{k,k+1,\ldots,pn,pn+1\}$, such that

- $i_0=i$ and $j_\ell=j$;
- for every $0\leqslant m\leqslant\ell$, either $i_m\neq pn+1$ and $j_m\neq pn+1$, or $i_m=j_m=pn+1$;
- $i_m\sim_{[n]}j_m$ for every $0\leqslant m\leqslant\ell$ such that $i_m\neq pn+1$;
- for all $1\leqslant m\leqslant\ell$, either i_m is the successor of j_{m-1} or j_{m-1} is the successor of i_m.

By convention, $\tilde{d}_n^k(i,j)=\infty$ if $i\notin\{k,k+1,\ldots,pn\}$ or $j\notin\{k,k+1,\ldots,pn\}$. It is then easy to verify that $\tilde{d}_n^k(i,j)$ is measurable with respect to $\mathcal{G}^n(k,pn)$. Furthermore,

$$\tilde{d}_n^k(i,j)\geqslant d_n(i,j).$$

Recall our notation p_n for the canonical projection from $\{0,1,\ldots,pn\}$ onto τ_n° (if $v_0^n,v_1^n,\ldots,v_{pn}^n$ denotes the contour sequence of the tree τ_n°, then $p_n(i)=v_i^n$). We denote by $\mathrm{Fr}_n([0,k])$ the set of all $j\in\{k,k+1,\ldots,pn\}$ for which at least one of the following two conditions holds:

(a) there exists $i\in\{0,1,\ldots,k\}$ such that $d_n(p_n(i),p_n(j))\in\{0,1\}$;

(b) $\Lambda_i^n\geqslant\Lambda_j^n$ for every $k\leqslant i\leqslant j$.

One can then verify that, for every $j\in\{k,k+1,...,pn\}$, the event $\{j\in\mathrm{Fr}_n([0,k])\}$ is measurable with respect to the σ-field $\mathcal{G}^n(k,pn)$. In particular, the properties of the contour function imply that the event $\{p_n(j)\in\{p_n(0),...,p_n(k)\}\}$ is measurable for the σ-field generated by $(C_i^n, k\leqslant i\leqslant pn)$. The event

$$\{\text{there exists } i\in\{0,...,k\} \text{ such that } d_n(p_n(i),p_n(j))=1\}$$

is not $\mathcal{G}^n(k,pn)$-measurable in general, but its union with the event where (b) holds is $\mathcal{G}^n(k,pn)$-measurable.

To simplify notation, we write $\tilde{d}_n=\tilde{d}_n^{\lfloor pnu\rfloor}$. Recall that $d_n\leqslant\tilde{d}_n$. We can now define our discrete approximation of the set $A_\varepsilon(r,u)$. We let $A_\varepsilon^n(r,u)$ denote the event on which there exists an integer $q_n\geqslant 1$ and points $s_0^n,...,s_{q_n}^n$ in $\{\lfloor pnu\rfloor,...,\lfloor pnU\rfloor\}$ such that $s_0^n=\lfloor pnU\rfloor$, $s_{q_n}^n\in\mathrm{Fr}_n([0,\lfloor pnu\rfloor])$ and for every $i\in\{0,1,...,q_n-1\}$,

$$\Lambda^n_{s_i^n}\geqslant\varkappa_p^{-1}(r-\varepsilon)n^{1/4}, \tag{15}$$

$$\tilde{d}_n(s_i^n,s_{i+1}^n)\leqslant 2\varkappa_p^{-1}\varepsilon n^{1/4}, \tag{16}$$

and, for every $i\in\{0,1,...,q_n-2\}$,

$$\tilde{d}_n(s_i^n,s_{i+1}^n)\leqslant\Lambda^n_{s_i^n}-\Lambda^n_{s_{i+1}^n}+\frac{2\varkappa_p^{-1}\varepsilon}{q_n}n^{1/4}. \tag{17}$$

It is important to observe that the event $A_\varepsilon^n(r,u)$ is measurable with respect to the σ-field $\mathcal{G}^n(\lfloor pnu\rfloor,pn)$.

Lemma 5.3. *For every* $\varepsilon\in\left]0,\frac{1}{10}r\right[$,

$$A_\varepsilon(r,u)\subset\liminf_{n\to\infty}A_\varepsilon^n(r,u)\quad a.s.$$

Proof. On the event $A_\varepsilon(r,u)$, we can choose $s_0,s_1,...,s_q\in[u,U]$ as specified in the statement of Lemma 5.2, so that in particular properties (12), (13) and (14) hold. Since $s_q\in\mathrm{Fr}([0,u])$, we can find $s'_q\in]0,u]$ such that $\mathbf{p}(s_q)=\mathbf{p}(s'_q)$. We set $\bar{s}_i^n=\lfloor pns_i\rfloor$ for every $i\in\{0,1,...,q-1\}$, and $\bar{s}_q^n=\lfloor pns'_q\rfloor-1<\lfloor pnu\rfloor$. Then $\bar{s}_0^n=\lfloor pnU\rfloor$ and the convergence (3) ensures that a.s. for all sufficiently large n, for every $i\in\{0,1,...,q-1\}$,

$$\Lambda^n_{\bar{s}_i^n}\geqslant\varkappa_p^{-1}(r-\varepsilon)n^{1/4}, \tag{18}$$

$$d_n(\bar{s}_i^n,\bar{s}_{i+1}^n)\leqslant 2\varkappa_p^{-1}\varepsilon n^{1/4}, \tag{19}$$

and, for every $i\in\{0,1,...,q-2\}$,

$$d_n(\bar{s}_i^n,\bar{s}_{i+1}^n)\leqslant\Lambda^n_{\bar{s}_i^n}-\Lambda^n_{\bar{s}_{i+1}^n}+\frac{2\varkappa_p^{-1}\varepsilon}{q}n^{1/4}. \tag{20}$$

Consider the first index $i_0 \in \{0,1,...,q-1\}$ such that $d_n(\bar{s}^n_{i_0}, \bar{s}^n_{i_0+1}) < \tilde{d}_n(\bar{s}^n_{i_0}, \bar{s}^n_{i_0+1})$. This index exists since $\tilde{d}_n(\bar{s}^n_{q-1}, \bar{s}^n_q) = \infty$, because $\bar{s}^n_q < \lfloor pnu \rfloor$. Then choose a discrete geodesic ω_n from $p_n(\bar{s}^n_{i_0})$ to $p_n(\bar{s}^n_{i_0+1})$ in the map $\mathbf{m}_n$. By (18), (19) and our assumption $\varepsilon < \frac{1}{10} r$, it is clear that this geodesic does not visit the root vertex of $\mathbf{m}_n$. Furthermore, at least one point on the geodesic ω_n must belong to $\{p_n(0), ..., p_n(\lfloor pnu \rfloor)\}$ (otherwise we would have $d_n(\bar{s}^n_{i_0}, \bar{s}^n_{i_0+1}) = \tilde{d}_n(\bar{s}^n_{i_0}, \bar{s}^n_{i_0+1})$).

We let b_n be the first point on the geodesic ω_n that belongs to $\{p_n(0), ..., p_n(\lfloor pnu \rfloor)\}$. We put $s^n_{i_0+1} = \bar{s}^n_{i_0}$ if $b_n = p_n(\bar{s}^n_{i_0})$, and otherwise, we choose $s^n_{i_0+1} \in \{\lfloor pnu \rfloor, ..., pn\}$ such that $p_n(s^n_{i_0+1})$ is the point preceding b_n on the geodesic ω_n. In both cases it is clear that $s^n_{i_0+1} \in \mathrm{Fr}_n([0, \lfloor pnu \rfloor])$.

We also set $q_n = i_0 + 1$ and $s^n_i = \bar{s}^n_i$ for every $i \in \{0, 1, ..., i_0\}$. Then we have

$$\tilde{d}_n(s^n_{i_0}, s^n_{i_0+1}) = d_n(s^n_{i_0}, s^n_{i_0+1}).$$

Indeed, this equality is trivial if $b_n = p_n(\bar{s}^n_{i_0})$, and otherwise there exists a geodesic from $p_n(s^n_{i_0})$ to $p_n(s^n_{i_0+1})$ such that for every point a of this geodesic all representatives of a lie in $\{\lfloor pnu \rfloor, ..., pn\}$. It then follows from (18)–(20) that the points $s^n_0, s^n_1, ..., s^n_{q_n}$ satisfy (15)–(17), so that $A^n_\varepsilon(r,u)$ holds for n large. This completes the proof. □

LEMMA 5.4. *For every $\varepsilon > 0$, set*

$$\tilde{A}_\varepsilon(r,u) = \limsup_{n\to\infty} A^n_\varepsilon(r,u).$$

Then,

$$\{\gamma(r) \leqslant u\} = \bigcap_{\varepsilon>0} A_\varepsilon(r,u) = \bigcap_{\varepsilon>0} \tilde{A}_\varepsilon(r,u) \quad \textit{a.s.}$$

Proof. From Lemma 5.3, we have that $A_\varepsilon(r,u) \subset \tilde{A}_\varepsilon(r,u)$ a.s. if $\varepsilon < \frac{1}{10} r$. Then it follows from Lemma 5.2 that

$$\{\gamma(r) \leqslant u\} \subset \bigcap_{\varepsilon>0} A_\varepsilon(r,u) \subset \bigcap_{\varepsilon>0} \tilde{A}_\varepsilon(r,u) \quad \text{a.s.}$$

Thus we only need to prove that

$$\bigcap_{\varepsilon>0} \tilde{A}_\varepsilon(r,u) \subset \{\gamma(r) \leqslant u\} \quad \text{a.s.}$$

From now on we assume that the event $\tilde{A}_\varepsilon(r,u)$ holds for every ε belonging to a (fixed) sequence decreasing to 0. Fix one value of ε in this sequence. By the definition of the set $\tilde{A}_\varepsilon(r,u)$ we can find, for every n belonging to a (random) sequence converging

to ∞, a discrete path $\gamma^\varepsilon_{(n)}=(\gamma^\varepsilon_{(n)}(i),0\leqslant i\leqslant L^\varepsilon_{(n)})$ taking values in $\{0,1,...,pn\}$, such that the following properties hold:

- $\gamma^\varepsilon_{(n)}$ starts from $\lfloor pnU\rfloor$ and ends at a point $z_n\in\mathrm{Fr}_n([0,\lfloor pnu\rfloor])$;
- the length $L^\varepsilon_{(n)}$ is bounded above by $\Lambda^n_{\lfloor pnU\rfloor}-\Lambda^n_{z_n}+6\varkappa_p^{-1}\varepsilon n^{1/4}$;
- any point on the path $\gamma^\varepsilon_{(n)}$ lies within d_n-distance at most $2\varkappa_p^{-1}\varepsilon n^{1/4}$ from a point z of $\{\lfloor pnu\rfloor,...,\lfloor pnU\rfloor\}$ such that $\Lambda^n_z\geqslant\varkappa_p^{-1}(r-\varepsilon)n^{1/4}$;
- $d_n(\gamma^\varepsilon_{(n)}(i),\gamma^\varepsilon_{(n)}(j))\leqslant|j-i|$ for every $i,j\in\{0,1,...,L^\varepsilon_{(n)}\}$;
- if $y=\gamma^\varepsilon_{(n)}(i)$ and $y'=\gamma^\varepsilon_{(n)}(j)$, with $0\leqslant i<j\leqslant L^\varepsilon_{(n)}$, then

$$d_n(y,y')\leqslant\Lambda^n_y-\Lambda^n_{y'}+10\varkappa_p^{-1}\varepsilon n^{1/4}. \tag{21}$$

The path $\gamma^\varepsilon_{(n)}$ is constructed by choosing the points $s^n_0,s^n_1,...,s^n_{q_n}$ as in the definition of $A^n_\varepsilon(r,u)$ and then concatenating discrete geodesics (relative to d_n) between s^n_{i-1} and s^n_i for $1\leqslant i\leqslant q_n$. The preceding properties of $\gamma^\varepsilon_{(n)}$ follow from the properties stated in the definition of $A^n_\varepsilon(r,u)$, using also the fact that $d_n\leqslant\tilde d_n$ and the bound $|\Lambda^n_y-\Lambda^n_{y'}|\leqslant d_n(y,y')$ for every $y,y'\in\{0,1,...,pn\}$ (see §2.7). In particular, (21) follows from (16) and (17).

Using (3) and extracting a diagonal subsequence, we may assume that along a sequence $\{n'_j\}_{j\geqslant1}$ of values of n converging to ∞, $\varkappa_p n^{-1/4}L^\varepsilon_{(n)}$ converges to $L^\varepsilon\leqslant\overline{Z}_U+6\varepsilon$, $(pn)^{-1}z_n$ converges to z_∞ and for every rational $a\in[0,L^\varepsilon[$,

$$(pn)^{-1}\gamma^\varepsilon_{(n)}(\lfloor\varkappa_p^{-1}n^{1/4}a\rfloor)$$

converges to a number $\gamma^\varepsilon(a)\in[0,1]$. Then, for all rationals $a,b\in[0,L^\varepsilon[$, the convergence (3) gives

$$\begin{aligned}D(\gamma^\varepsilon(a),\gamma^\varepsilon(b))&=\lim_{j\to\infty}\varkappa_p(n'_j)^{-1/4}d_{n'_j}(\gamma^\varepsilon_{(n'_j)}(\lfloor\varkappa_p^{-1}(n'_j)^{1/4}a\rfloor),\gamma^\varepsilon_{(n'_j)}(\lfloor\varkappa_p^{-1}(n'_j)^{1/4}b\rfloor))\\&\leqslant|a-b|.\end{aligned}$$

Set $\omega^\varepsilon(a)=\mathbf{p}(\gamma^\varepsilon(a))$ for every rational $a\in[0,L^\varepsilon[$. By the preceding bound, the mapping $a\mapsto\omega^\varepsilon(a)$, from $[0,L^\varepsilon[\cap\mathbb{Q}$ into $\mathbf{m}_\infty$, is 1-Lipschitz. It can thus be extended to a 1-Lipschitz path from $[0,L^\varepsilon]$ into $\mathbf{m}_\infty$, which we still denote by ω^ε. Clearly, $\omega^\varepsilon(0)=\mathbf{p}(U)$ and $\omega^\varepsilon(L^\varepsilon)=\mathbf{p}(z_\infty)$.

Using the fact that $z_n\in\mathrm{Fr}_n([0,\lfloor pnu\rfloor])$, it is not hard to verify that $z_\infty\in\mathrm{Fr}([0,u])$, and thus $\omega^\varepsilon(L^\varepsilon)\in\mathbf{p}(\mathrm{Fr}([0,u]))$. From the bound $L^\varepsilon_{(n)}\leqslant\Lambda^n_{\lfloor pnU\rfloor}-\Lambda^n_{z_n}+6\varkappa_p^{-1}\varepsilon n^{1/4}$, we also get

$$L^\varepsilon\leqslant\overline{Z}_U-\overline{Z}_{\omega^\varepsilon(L^\varepsilon)}+6\varepsilon.$$

Moreover (21) gives

$$D(\omega^\varepsilon(t),\omega^\varepsilon(t'))\leqslant\overline{Z}_{\omega^\varepsilon(t)}-\overline{Z}_{\omega^\varepsilon(t')}+10\varepsilon$$

for every $0\leqslant t\leqslant t'\leqslant L^\varepsilon$. Finally, any point of the path ω^ε lies within D-distance at most 2ε from a point x of $\mathbf{p}([u,U])$ such that $\overline{Z}_x\geqslant r-\varepsilon$. In particular, $\overline{Z}_{\omega^\varepsilon(t)}\geqslant r-3\varepsilon$ for every $t\in[0,L^\varepsilon]$.

The preceding construction can be made for every ε belonging to a sequence decreasing to 0. Again via a compactness argument, we can extract a subsequence of values of ε along which L^ε converges to L and the 1-Lipschitz paths $(\omega^\varepsilon(t\wedge L^\varepsilon),0\leqslant t\leqslant L)$ converge uniformly to a limiting path $(\omega^0(t),0\leqslant t\leqslant L)$. The path ω^0 satisfies the following properties:

- ω^0 is 1-Lipschitz, starts from $\mathbf{p}(U)$ and ends at a point $\omega^0(L)\in\mathbf{p}(\mathrm{Fr}([0,u]))$;
- for every $0\leqslant t\leqslant t'\leqslant L$,

$$D(\omega^0(t),\omega^0(t'))\leqslant\overline{Z}_{\omega^0(t)}-\overline{Z}_{\omega^0(t')};\tag{22}$$

- for every $t\in[0,L]$, $\omega^0(t)\in\mathbf{p}([0,U])$ and $\overline{Z}_{\omega^0(t)}\geqslant r$;
- $L\leqslant\overline{Z}_U-\overline{Z}_{\omega^0(L)}$.

By the last property and the triangle inequality,

$$L\leqslant D(\varrho,\mathbf{p}(U))-D(\varrho,\omega^0(L))\leqslant D(\mathbf{p}(U),\omega^0(L)).$$

Since ω^0 is 1-Lipschitz, we also have the reverse inequality $D(\mathbf{p}(U),\omega^0(L))\leqslant L$, and thus

$$D(\mathbf{p}(U),\omega^0(L))=L,$$

which implies that ω^0 is a geodesic from $\mathbf{p}(U)$ to $\omega^0(L)$. For similar reasons, the inequality in (22) must be an equality, and in particular

$$D(\varrho,\mathbf{p}(U))=D(\varrho,\omega^0(L))+D(\omega^0(L),\mathbf{p}(U)).$$

Since $\omega^0(L)\in\mathbf{p}(\mathrm{Fr}([0,u]))$, we can choose $t_1\in[0,u]$ such that $\mathbf{p}(t_1)=\omega^0(L)$. Due to the property in the last display, we can concatenate ω^0 with the (time-reversed) simple geodesic Φ_{t_1}, in order to get a geodesic from $\mathbf{p}(U)$ to ϱ. Write $\widetilde{\omega}$ for the time-reversal of the latter geodesic. Then $\widetilde{\omega}$ is a geodesic from ϱ to $\mathbf{p}(U)$, which takes values in $\mathbf{p}([0,U])$. Hence we can find a mapping $\widetilde{\gamma}\colon[0,\overline{Z}_U]\to[0,U]$ such that $\widetilde{\omega}(t)=\mathbf{p}(\widetilde{\gamma}(t))$ for every $t\in[0,\overline{Z}_U]$. We can impose $\widetilde{\gamma}(\overline{Z}_{\omega^0(L)})=t_1\leqslant u$. However, by the definition of $\gamma=\gamma_U$, we have $\gamma(t)\leqslant\widetilde{\gamma}(t)$ for every $t\in[0,\overline{Z}_U]$, and thus in particular $\gamma(\overline{Z}_{\omega^0(L)})\leqslant\widetilde{\gamma}(\overline{Z}_{\omega^0(L)})\leqslant u$. On the other hand, we know that $\overline{Z}_{\omega^0(L)}\geqslant r$, and by the monotonicity of γ we conclude that $\gamma(r)\leqslant u$. This completes the proof of Lemma 5.4. □

Before we proceed to the proof of Lemma 5.1, we need one more estimate. Recall from §2.7 the notation $(W_i^n)_{0\leqslant i\leqslant pn}$ for the "discrete snake" associated with the p-mobile $(\tau_n,(\ell_v^n)_{v\in\tau_n^\circ})$.

LEMMA 5.5. *Let $\delta\in\left]0,\frac{1}{2}u_0\right[$ and $\eta,\eta'>0$. There exists a constant*

$$K=K(p,U,u_0,\eta,\eta',\delta)$$

such that, for every $u\in[u_0,U[$ and $\alpha>0$, we have for all sufficiently large n,

$$P\Big(\inf_{\lfloor pn(u-\delta)\rfloor\leqslant i\leqslant\lfloor pnu\rfloor} C_i^n\geqslant C^n_{\lfloor pnu\rfloor}-\alpha n^{1/2}\,\Big|\,\mathcal{G}^n(\lfloor pnu\rfloor,pn)\Big)\leqslant K\alpha$$

on the event

$$\Big\{C^n_{\lfloor pnu\rfloor}\geqslant\eta n^{1/2},\inf_{\eta n^{1/2}/2\leqslant j\leqslant C^n_{\lfloor pnu\rfloor}} W^n_{\lfloor pnu\rfloor}(j)\geqslant\eta' n^{1/4}\Big\}.$$

The proof of Lemma 5.5 is postponed to the appendix.

Proof of Lemma 5.1. As we already noticed after the statement of Lemma 5.1, it is enough to establish the bound

$$\begin{aligned}P\Big[\Big(\bigcup_{i=1}^k\{u<\gamma(r_i)\leqslant v\}\Big)\cap B_{\eta,u}\cap\Big\{\inf_{u-\delta\leqslant t\leqslant u}\bar{\mathbf{e}}_t>\bar{\mathbf{e}}_u-\alpha\Big\}\Big]&\\ \leqslant C\alpha P\Big[\bigcup_{i=1}^k\{u<\gamma(r_i)\leqslant v\}\Big],&\end{aligned}\tag{23}$$

where $r_1,...,r_k$ are fixed positive numbers, and the constant C only depends on p, U, u_0, η and δ.

By Lemma 5.4, we have for every $i\in\{1,...,k\}$,

$$\mathbf{1}_{\{\gamma(r_i)\leqslant u\}}=\lim_{\varepsilon\to0}\mathbf{1}_{A_\varepsilon(r_i,u)}=\lim_{\varepsilon\to0}\mathbf{1}_{\tilde A_\varepsilon(r_i,u)}\quad\text{a.s.}$$

and consequently

$$\mathbf{1}_{\{u<\gamma(r_i)\leqslant v\}}=\lim_{\varepsilon\to0}(\mathbf{1}_{A_\varepsilon(r_i,v)}-\mathbf{1}_{\tilde A_\varepsilon(r_i,u)})^+\quad\text{a.s.}$$

So we have

$$\mathbf{1}\Big\{\bigcup_{i=1}^k\{u<\gamma(r_i)\leqslant v\}\Big\}=\lim_{\varepsilon\to0}\sup_{1\leqslant i\leqslant k}(\mathbf{1}_{A_\varepsilon(r_i,v)}-\mathbf{1}_{\tilde A_\varepsilon(r_i,u)})^+\quad\text{a.s.}$$

and a symmetric argument gives

$$\mathbf{1}\Big\{\bigcup_{i=1}^k\{u<\gamma(r_i)\leqslant v\}\Big\}=\lim_{\varepsilon\to0}\sup_{1\leqslant i\leqslant k}(\mathbf{1}_{\tilde A_\varepsilon(r_i,v)}-\mathbf{1}_{A_\varepsilon(r_i,u)})^+\quad\text{a.s.}$$

We will verify that, for every $\varepsilon>0$,

$$\begin{aligned} E\Big[\sup_{1\leqslant i\leqslant k}(\mathbf{1}_{A_\varepsilon(r_i,v)}-\mathbf{1}_{\tilde{A}_\varepsilon(r_i,u)})^+\mathbf{1}\Big\{\inf_{u-\delta\leqslant t\leqslant u}\bar{\mathbf{e}}_t>\bar{\mathbf{e}}_u-\alpha\Big\}\mathbf{1}_{B_{\eta,u}}\Big] \\ \leqslant C\alpha E\Big[\sup_{1\leqslant i\leqslant k}(\mathbf{1}_{\tilde{A}_\varepsilon(r_i,v)}-\mathbf{1}_{A_\varepsilon(r_i,u)})^+\Big], \end{aligned} \tag{24}$$

where $C=\lambda_p^{-1}K(p,U,u_0,\lambda_p^{-1}\eta,\varkappa_p^{-1}\eta,\delta)$ with the notation of Lemma 5.5. By passing to the limit $\varepsilon\to0$, we see that (23) follows from (24).

By Lemmas 5.3 and 5.4, we have a.s. for $i\in\{1,...,k\}$,

$$\begin{aligned} (\mathbf{1}_{A_\varepsilon(r_i,v)}-\mathbf{1}_{\tilde{A}_\varepsilon(r_i,u)})^+ &\leqslant \Big(\liminf_{n\to\infty}\mathbf{1}_{A^n_\varepsilon(r_i,v)}-\limsup_{n\to\infty}\mathbf{1}_{A^n_\varepsilon(r_i,u)}\Big)^+ \\ &\leqslant \liminf_{n\to\infty}(\mathbf{1}_{A^n_\varepsilon(r_i,v)}-\mathbf{1}_{A^n_\varepsilon(r_i,u)})^+. \end{aligned}$$

Therefore,

$$\sup_{1\leqslant i\leqslant k}(\mathbf{1}_{A_\varepsilon(r_i,v)}-\mathbf{1}_{\tilde{A}_\varepsilon(r_i,u)})^+\leqslant\liminf_{n\to\infty}\sup_{1\leqslant i\leqslant k}(\mathbf{1}_{A^n_\varepsilon(r_i,v)}-\mathbf{1}_{A^n_\varepsilon(r_i,u)})^+\quad\text{a.s.}$$

Then, using (3), (4) and Fatou's lemma, the left-hand side of (24) is bounded above by

$$\begin{aligned} \liminf_{n\to\infty}E\Big[\sup_{1\leqslant i\leqslant k}(\mathbf{1}_{A^n_\varepsilon(r_i,v)}-\mathbf{1}_{A^n_\varepsilon(r_i,u)})^+ \\ \times\mathbf{1}\Big\{\inf_{\lfloor pn(u-\delta)\rfloor\leqslant i\leqslant\lfloor pnu\rfloor}C^n_i\geqslant C^n_{\lfloor pnu\rfloor}-\lambda_p^{-1}\alpha n^{1/2}\Big\}\mathbf{1}_{B^n_{\eta,u}}\Big], \end{aligned}$$

where

$$B^n_{\eta,u}=\Big\{C^n_{\lfloor pnu\rfloor}\geqslant\lambda_p^{-1}\eta n^{1/2},\inf_{\lambda_p^{-1}\eta n^{1/2}/2\leqslant j\leqslant C^n_{\lfloor pnu\rfloor}}W^n_{\lfloor pnu\rfloor}(j)\geqslant\varkappa_p^{-1}\eta n^{1/4}\Big\}.$$

Now note that both $B^n_{\eta,u}$ and the variable

$$\sup_{1\leqslant i\leqslant k}(\mathbf{1}_{A^n_\varepsilon(r_i,v)}-\mathbf{1}_{A^n_\varepsilon(r_i,u)})^+$$

are measurable with respect to $\mathcal{G}^n(\lfloor pnu\rfloor,pn)$. Hence, we can apply Lemma 5.5 and we get that the left-hand side of (24) is bounded above by

$$\lambda_p^{-1}K\alpha\liminf_{n\to\infty}E\Big[\sup_{1\leqslant i\leqslant k}(\mathbf{1}_{A^n_\varepsilon(r_i,v)}-\mathbf{1}_{A^n_\varepsilon(r_i,u)})^+\Big],$$

where $K=K(p,U,u_0,\lambda_p^{-1}\eta,\varkappa_p^{-1}\eta,\delta)$. So, in order to get (24), it only remains to verify that the latter lim inf is bounded above by

$$E\Big[\sup_{1\leqslant i\leqslant k}(\mathbf{1}_{\tilde{A}_\varepsilon(r_i,v)}-\mathbf{1}_{A_\varepsilon(r_i,u)})^+\Big].$$

To this end, we just have to note that

$$\begin{aligned}\limsup_{n\to\infty}\sup_{1\leqslant i\leqslant k}(\mathbf{1}_{A^n_\varepsilon(r_i,v)}-\mathbf{1}_{A^n_\varepsilon(r_i,u)})^+ &= \sup_{1\leqslant i\leqslant k}\limsup_{n\to\infty}(\mathbf{1}_{A^n_\varepsilon(r_i,v)}-\mathbf{1}_{A^n_\varepsilon(r_i,u)})^+\\ &\leqslant \sup_{1\leqslant i\leqslant k}\Big(\limsup_{n\to\infty}\mathbf{1}_{A^n_\varepsilon(r_i,v)}-\liminf_{n\to\infty}\mathbf{1}_{A^n_\varepsilon(r_i,u)}\Big)^+\\ &\leqslant \sup_{1\leqslant i\leqslant k}(\mathbf{1}_{\tilde{A}_\varepsilon(r_i,v)}-\mathbf{1}_{A_\varepsilon(r_i,u)})^+.\end{aligned}$$

This completes the proof of Lemma 5.1. □

6. Uniform estimates for the volume of balls

Our next goal is to get uniform bounds for the volume of small balls in $\mathbf{m}_\infty$. The main ingredient of the proof will be a bound on the moments of the quantities $\overline{\mathcal{J}}([0,\varepsilon])$, where $\varepsilon>0$, and $\overline{\mathcal{J}}$ is the occupation measure of $\overline{Z}$, which is the random measure on $[0,\infty[$ defined by

$$\overline{\mathcal{J}}(A)=\int_0^1 \mathbf{1}_A(\overline{Z}_s)\,ds$$

for every Borel subset A of $[0,\infty[$.

We will rely on certain estimates under the infinite excursion measure $\bar{\mathbb{N}}_0$ of the conditioned Brownian snake. Let us briefly recall the definition of $\bar{\mathbb{N}}_0$. More details can be found in [26].

Let $C([0,\infty[,\mathcal{W}_0)$ stand for the space of all continuous functions from $\mathbb{R}_+$ into $\mathcal{W}_0$, and denote the canonical process on this space by $(W_s)_{s\geqslant 0}$. Also let ζ_s denote the lifetime of W_s, for every $s\geqslant 0$. For $\omega\in C([0,\infty[,\mathcal{W}_0)$, set $\sigma(\omega)=\sup\{s\geqslant 0:\zeta_s(\omega)>0\}$.

Denote by $\bar{\mathbb{N}}_0^{(1)}$ the probability measure on $C([0,\infty[,\mathcal{W}_0)$ which is the law of the process $(\overline{W}_{s\wedge 1})_{s\geqslant 0}$. Note that $\sigma=1$ $\bar{\mathbb{N}}_0^{(1)}$-a.s. We can now use scaling transformations to define a probability measure $\bar{\mathbb{N}}_0^{(a)}$, for every $a>0$, in the following way. For every $\mu>0$, we set

$$\begin{aligned}\zeta^{(\mu)}_s &= \mu^2\zeta_{s/\mu^4}, && \text{for } s\geqslant 0,\\ W^{(\mu)}_s(t) &= \mu W_{s/\mu^4}(t/\mu^2), && \text{for } s\geqslant 0 \text{ and } 0\leqslant t\leqslant \zeta^{(\mu)}_s.\end{aligned}$$

By definition, $\bar{\mathbb{N}}_0^{(a)}$ is the law of $(W_s^{(a^{1/4})})_{s\geqslant 0}$ under $\bar{\mathbb{N}}_0^{(1)}$. Clearly, we have $\sigma=a$ $\bar{\mathbb{N}}_0^{(a)}$-a.s. We may now set

$$\bar{\mathbb{N}}_0=\int_0^\infty \frac{1}{2\sqrt{2\pi a^5}}\,\bar{\mathbb{N}}_0^{(a)}\,da. \tag{25}$$

LEMMA 6.1. *For every integer $k\geqslant 2$, there exists a constant $c_k<\infty$ such that, for every $\varepsilon>0$,*

$$\bar{\mathbb{N}}_0\left(\left(\int_0^\sigma \mathbf{1}_{\{\widehat{W}_s\leqslant\varepsilon\}}\,ds\right)^k\right)=c_k\varepsilon^{4k-6}.$$

Consequently, for every integer $k\geqslant 1$ and every $\delta\in]0,1]$, there exists a constant $c_{k,\delta}<\infty$ such that, for every $\varepsilon>0$,

$$E[\overline{\mathcal{J}}([0,\varepsilon])^k]\leqslant c_{k,\delta}\varepsilon^{4k-\delta}. \tag{26}$$

Remark. It should be possible to replace $\varepsilon^{4k-\delta}$ by ε^{4k} in (26): See [24, p. 669] for the case $k=1$.

Proof. Recall the notation $W^{(\mu)}$ introduced before the lemma. Then the law of $(W_s^{(\mu)})_{s\geqslant 0}$ under $\bar{\mathbb{N}}_0$ is $\mu^6\bar{\mathbb{N}}_0$. This is easily seen from the decomposition formula (25). It follows that, for every $\varepsilon>0$,

$$\begin{aligned}\bar{\mathbb{N}}_0\left(\left(\int_0^\sigma \mathbf{1}_{\{\widehat{W}_s\leqslant\varepsilon\}}\,ds\right)^k\right)&=\varepsilon^{-6}\bar{\mathbb{N}}_0\left(\left(\int_0^{\sigma^{(\varepsilon)}} \mathbf{1}_{\{\widehat{W}_s^{(\varepsilon)}\leqslant\varepsilon\}}\,ds\right)^k\right)\\&=\varepsilon^{-6}\bar{\mathbb{N}}_0\left(\left(\int_0^{\varepsilon^4\sigma} \mathbf{1}_{\{\widehat{W}_{s/\varepsilon^4}\leqslant 1\}}\,ds\right)^k\right)\\&=\varepsilon^{4k-6}\bar{\mathbb{N}}_0\left(\left(\int_0^{\sigma} \mathbf{1}_{\{\widehat{W}_s\leqslant 1\}}\,ds\right)^k\right).\end{aligned}$$

So, setting

$$c_k=\bar{\mathbb{N}}_0\left(\left(\int_0^\sigma \mathbf{1}_{\{\widehat{W}_s\leqslant 1\}}\,ds\right)^k\right),$$

we need to verify that $c_k<\infty$ if $k\geqslant 2$.

To this end we will use Theorem 5.1 in [26]. We need to introduce some notation. A *marked tree* is a pair $\theta=(\tau,(h_v)_{v\in\tau})$, where τ is a plane tree (in the formalism of §2.6), and, for every vertex $v\in\tau$, h_v is a non-negative real number, which is interpreted as the length of the branch associated with v. We then say that τ is the *discrete skeleton* of θ. The plane tree τ is called *binary* if every vertex of τ has either 0 or 2 children, and vertices without children are called *leaves*.

The uniform measure on the set of all (binary) marked trees with k leaves is defined by

$$\int F(\theta)\Lambda_p(d\theta)=\sum_{\tau\in\mathbf{A}_k^{\mathrm{bin}}}\int_{\mathbb{R}_+^\tau} F(\tau,(h_v)_{v\in\tau})\prod_{v\in\tau}dh_v,$$

where $\mathbf{A}_k^{\mathrm{bin}}$ is the set of all binary plane trees with k leaves. Let $r\geqslant 0$ and let

$$\theta=(\tau,(h_v)_{v\in\tau})$$

be a marked tree. We can combine the branching structure of θ with spatial displacements given by 9-dimensional Bessel processes, to get random variables $\overline{V}_a$, $a\in\tau$, which are defined under a probability measure Q_r^θ and constructed as follows. First consider a 9-dimensional Bessel process $R^\varnothing=(R_t^\varnothing)_{t\geqslant 0}$ started from r, and set $\overline{V}_\varnothing=R^\varnothing_{h_\varnothing}$. Then, conditionally given $R^\varnothing$, consider two other independent 9-dimensional Bessel processes R^1 and R^2, both started from $\overline{V}_\varnothing$, and set $\overline{V}_1=R^1_{h_1}$ and $\overline{V}_2=R^2_{h_2}$. We can easily continue the construction by induction. See [26, §5] for more details.

As a direct consequence of [26, Theorem 5.1], we have $c_k=2^{k-1}k!\tilde{d}_k$, where

$$\tilde{d}_k=\int Q_0^\theta\Big(\Big(\prod_{a\in I(\theta)}\overline{V}_a^4\Big)\Big(\prod_{a\in L(\theta)}\overline{V}_a^{-4}\mathbf{1}_{\{\overline{V}_a\leqslant 1\}}\Big)\Big)\Lambda_k(d\theta).$$

Here $L(\theta)$ stands for the set of all leaves of the discrete skeleton τ of θ, and $I(\theta)$ is the set of all other vertices of τ. If $k\geqslant 2$, we can decompose the binary tree θ at its first node to obtain that

$$\tilde{d}_k=E_0^{(9)}\Big[\int_0^\infty R_t^4\Big(\sum_{j=1}^{k-1}\tilde{d}_j(R_t)\tilde{d}_{k-j}(R_t)\Big)\,dt\Big],$$

where R denotes a 9-dimensional Bessel process that starts from r under the probability measure $P_r^{(9)}$, and for every $j\geqslant 1$ and $r\geqslant 0$,

$$\tilde{d}_j(r)=\int Q_r^\theta\Big(\Big(\prod_{a\in I(\theta)}\overline{V}_a^4\Big)\Big(\prod_{a\in L(\theta)}\overline{V}_a^{-4}\mathbf{1}_{\{\overline{V}_a\leqslant 1\}}\Big)\Big)\Lambda_j(d\theta).$$

Let us prove by induction that for every integer $j\geqslant 1$ there exists a constant M_j such that, for every $r\geqslant 0$,

$$\tilde{d}_j(r)\leqslant\begin{cases}M_j(r^{-2}\wedge r^{-7}), & \text{if } j=1,\\ M_j(1\wedge r^{-7}), & \text{if } j\geqslant 2.\end{cases}\tag{27}$$

If $j=1$,

$$\tilde{d}_1(r)=E_r^{(9)}\Big[\int_0^\infty R_t^{-4}\mathbf{1}_{\{R_t\leqslant 1\}}\,dt\Big]=\int_{\mathbb{R}^9}G_9(y_r,z)|z|^{-4}\mathbf{1}_{\{|z|\leqslant 1\}}\,dz,$$

where $G_9(y,z)=\alpha|z-y|^{-7}$ is the Green function of 9-dimensional Brownian motion, and y_r denotes an arbitrary point in $\mathbb{R}^9$ such that $|y_r|=r$. From the preceding explicit formula, straightforward estimates give the bound (27) when $j=1$. Let $\ell\geqslant 2$ and assume that the bound (27) holds for $j=1,\dots,\ell-1$. Using again a decomposition at the first node, we get

$$\begin{aligned}\tilde{d}_\ell(r)&=E_r^{(9)}\Big[\int_0^\infty R_t^4\Big(\sum_{j=1}^{\ell-1}\tilde{d}_j(R_t)\tilde{d}_{\ell-j}(R_t)\Big)\,dt\Big]\\&\leqslant\sum_{j=1}^{\ell-1}M_jM_{\ell-j}E_r^{(9)}\Big[\int_0^\infty R_t^4(R_t^{-2}\wedge R_t^{-7})^2\,dt\Big]\\&=\widetilde{M}_\ell E_r^{(9)}\Big[\int_0^\infty(1\wedge R_t^{-10})\,dt\Big]\end{aligned}$$

for some constant $\widetilde{M}_\ell$. We have then

$$E_r^{(9)}\left[\int_0^\infty (1\wedge R_t^{-10})\,dt\right]=\int_{\mathbb{R}^9} G_9(y_r,z)(1\wedge|z|^{-10})\,dz=\alpha\int_{\mathbb{R}^9}|z-y_r|^{-7}(1\wedge|z|^{-10})\,dz.$$

Simple estimates show that the last integral is bounded above by a constant times $1\wedge r^{-7}$, which gives (27) for $j=\ell$.

If $k\geqslant 2$, then $\tilde{d}_k=\tilde{d}_k(0)<\infty$, which completes the proof of the first assertion of Lemma 6.1.

The bound (26) now comes as an easy consequence. Using the decomposition (25) and a scaling argument, we get

$$\overline{\mathbb{N}}_0^{(1)}\left(\left(\int_0^1 \mathbf{1}_{\{\widehat{W}_s\leqslant\varepsilon\}}\,ds\right)^k\right)\leqslant c'_k\varepsilon^{4k-6}.$$

with another constant c'_k. Since $\overline{\mathcal{J}}([0,\varepsilon])$ has the distribution of

$$\int_0^1 \mathbf{1}_{\{\widehat{W}_s\leqslant\varepsilon\}}\,ds$$

under $\overline{\mathbb{N}}_0^{(1)}$, it follows that

$$E[\overline{\mathcal{J}}([0,\varepsilon])^k]\leqslant c'_k\varepsilon^{4k-6}.$$

This holds for every integer $k\geqslant 2$, so that a simple application of Hölder's inequality yields (26). □

If $x\in\mathbf{m}_\infty$ and $\varepsilon>0$, we denote by $B_D(x,\varepsilon)$ the closed ball of radius ε centered at x in the metric space $(\mathbf{m}_\infty,D)$. Recall our notation λ for the volume measure on $\mathbf{m}_\infty$ (cf. §2.5).

Corollary 6.2. *Let $\delta\in]0,1]$ and*

$$S=\sup_{\varepsilon>0}\Big(\sup_{x\in\mathbf{m}_\infty}\frac{\lambda(B_D(x,\varepsilon))}{\varepsilon^{4-\delta}}\Big).$$

Then $E[S^k]<\infty$ for every integer $k\geqslant 1$.

Proof. Let $k\geqslant 1$ and $\varepsilon>0$. The same argument as in the proof of [24, Lemma 6.2] gives the bound

$$E\left[\int_{\mathbf{m}_\infty}\lambda(B_D(x,\varepsilon))^k\lambda(dx)\right]\leqslant E[\lambda(B_D(\varrho,\varepsilon))^k]=E[\overline{\mathcal{J}}([0,\varepsilon])^k]$$

(the inequality is in fact an equality, see Theorem 8.1 below). Using Lemma 6.1, we thus get

$$E\left[\int_{\mathbf{m}_\infty}\lambda(B_D(x,\varepsilon))^k\lambda(dx)\right]\leqslant c_{k,1}\varepsilon^{4k-1}.$$

Let $r>0$. If we suppose that there exists a point $x_0 \in \mathbf{m}_\infty$ such that $\lambda(B_D(x_0,\varepsilon)) \geqslant r\varepsilon^{4-\delta}$, then we have for every $x \in B_D(x_0,\varepsilon)$,

$$\lambda(B_D(x,2\varepsilon)) \geqslant \lambda(B_D(x_0,\varepsilon)) \geqslant r\varepsilon^{4-\delta},$$

and thus

$$\int_{\mathbf{m}_\infty} \lambda(B_D(x,2\varepsilon))^k \lambda(dx) \geqslant (r\varepsilon^{4-\delta})^{k+1}.$$

It follows that

$$\begin{aligned} P(&\text{there exists } x_0 \in \mathbf{m}_\infty \text{ such that } \lambda(B_D(x_0,\varepsilon)) \geqslant r\varepsilon^{4-\delta}) \\ &\leqslant (r\varepsilon^{4-\delta})^{-k-1} E\left[\int_{\mathbf{m}_\infty} \lambda(B_D(x,2\varepsilon))^k \lambda(dx)\right] \\ &\leqslant c_{k,1}(r\varepsilon^{4-\delta})^{-k-1}(2\varepsilon)^{4k-1} \\ &= 2^{4k-1} c_{k,1} r^{-k-1} \varepsilon^{(k+1)\delta-5}. \end{aligned} \tag{28}$$

We apply this bound with $\varepsilon = 2^{-j}$, $j \in \mathbb{Z}_+$, and k chosen so large that $(k+1)\delta \geqslant 6$. Setting

$$S_j = \sup_{x \in \mathbf{m}_\infty} \frac{\lambda(B_D(x,2^{-j}))}{(2^{-j})^{4-\delta}},$$

we get from (28) that, for every $r>0$,

$$P(S_j > r) \leqslant \bar{c}_k r^{-k-1} 2^{-j},$$

where $\bar{c}_k = 2^{4k-1} c_{k,1}$. It follows that, for every $r>1$,

$$P(S > r) \leqslant \sum_{j=0}^{\infty} P(S_j > 2^{-(4-\delta)} r) \leqslant 2^{(4-\delta)(k+1)+1} \bar{c}_k r^{-k-1}.$$

Since this holds for every (large enough) integer k, the result of the corollary readily follows. □

We will now state and prove the key proposition that motivated the technical lemmas of this section and of the previous one. We denote by LI the set of all left-increase times of $\bar{\mathbf{e}}$, and by RI the set of all its right-increase times. Note that

$$p_{\bar{\mathbf{e}}}(\mathrm{LI}) = p_{\bar{\mathbf{e}}}(\mathrm{RI}) = \mathrm{Sk}(\mathcal{T}_{\bar{\mathbf{e}}}) \cup \{\varrho\}.$$

If $U \in]0,1[$ is fixed, then Lemma 3.4 shows that $U \notin \mathrm{LI} \cup \mathrm{RI}$ a.s. Recall from §4.2 the notation γ_U and $\bar{\gamma}_U$. Also recall the notation $\mathcal{R}(\gamma_U)$ for the range of γ_U. As usual $\overline{\mathcal{R}(\gamma_U)}$ denotes the closure of $\mathcal{R}(\gamma_U)$.

PROPOSITION 6.3. *For every fixed $U\in]0,1[$, we have*

$$\overline{\mathcal{R}(\gamma_U)}\cap \mathrm{LI}=\varnothing \quad \textit{and} \quad \overline{\mathcal{R}(\bar\gamma_U)}\cap \mathrm{RI}=\varnothing$$

almost surely.

Proof. Fix $k\in\mathbb{N}$, and denote by N_k the number of those intervals of the form $]i2^{-k},(i+1)2^{-k}]$, $0\leqslant i\leqslant 2^k-1$, that intersect $\mathcal{R}(\gamma_U)$.

We also fix $u_0\in\big]0,\frac{1}{2}U\big[$, $\eta>0$ and $\delta\in\big]0,\frac{1}{2}u_0\big[$. Let $\beta\in\big]0,\frac{1}{16}\big[$, and choose $\xi\in\big]0,\frac{1}{2}\big[$ such that

$$\xi>1-\big(\tfrac{1}{4}-\beta\big)(3-\beta). \tag{29}$$

We denote by N_k^* the number of those intervals $]i2^{-k},(i+1)2^{-k}]$ that intersect $\mathcal{R}(\gamma_U)$ and satisfy the following additional properties:

(a) $i2^{-k}\geqslant u_0$;

(b) $\inf_{i2^{-k}-\delta\leqslant t\leqslant i2^{-k}}\bar{\mathbf{e}}_t>\bar{\mathbf{e}}_{i2^{-k}}-(2^{-k})^\xi$;

(c) $\bar{\mathbf{e}}_{i2^{-k}}>\eta$ and $\inf_{\eta/2\leqslant t\leqslant\bar{\mathbf{e}}_{i2^{-k}}}\overline{W}_{i2^{-k}}(t)>\eta$.

Applying Lemma 5.1 with $u=i2^{-k}$ and $v=(i+1)2^{-k}\wedge U$, for every integer i such that $u_0\leqslant i2^{-k}<U$, we get the bound

$$E[N_k^*]\leqslant C(2^{-k})^\xi E[N_k], \tag{30}$$

with a constant C that depends only on p, U, u_0, η and δ.

We now need to bound $E[N_k]$. For every $\varepsilon>0$, set

$$\Omega(\varepsilon)=\sup_{\substack{r,t\in[0,1]\\|r-t|\leqslant\varepsilon}} D(r,t).$$

Recall from §2.5 the bound

$$D(r,t)\leqslant \bar Z_r+\bar Z_t-2\inf_{r\wedge t\leqslant u\leqslant r\vee t}\bar Z_u, \tag{31}$$

and also recall the construction of $\bar Z$ by shifting the process $Z^{\mathbf{e}}$, which conditionally given $\mathbf{e}$ has the distribution of the centered Gaussian process with covariance function $m_{\mathbf{e}}(r,t)$. Using the bound (31) together with standard chaining arguments, one easily gets the estimate

$$P(\Omega(\varepsilon)>\varepsilon^{1/4-\beta})=o(\varepsilon) \tag{32}$$

as $\varepsilon\to0$ (compare with [24, Lemma 5.1]).

Set $\varepsilon_k=2^{-k(1/4-\beta)}$ to simplify notation and write

$$\mathcal{R}_{\varepsilon_k}(\gamma_U)=\{s\in[0,1]:D(r,s)\leqslant\varepsilon_k \text{ for some } r\in\mathcal{R}(\gamma_U)\}$$

for the tubular neighborhood of radius ε_k of $\mathcal{R}(\gamma_U)$, with respect to the pseudo-metric D. Then, on the event $\{\Omega(2^{-k})\leqslant\varepsilon_k\}$, any interval of the form $]i2^{-k},(i+1)2^{-k}]$ that intersects $\mathcal{R}(\gamma_U)$ is contained in $\mathcal{R}_{\varepsilon_k}(\gamma_U)$, and so we have on the same event

$$\lambda(\mathbf{p}(\mathcal{R}_{\varepsilon_k}(\gamma_U)))\geqslant N_k 2^{-k}.$$

Thus

$$E[N_k]\leqslant 2^k E[\lambda(\mathbf{p}(\mathcal{R}_{\varepsilon_k}(\gamma_U)))]+2^k P[\Omega(2^{-k})>\varepsilon_k].$$

By (32), the second term of the sum is $o(1)$ as $k\to\infty$. In order to bound the first term, we cut the geodesic γ_U into slices of length ε_k: Precisely we observe that $\mathbf{p}(\mathcal{R}_{\varepsilon_k}(\gamma_U))$ is contained in the union of the balls $B_D(\mathbf{p}(\gamma_U(i\varepsilon_k)),2\varepsilon_k)$ for $i\in\{0,1,...,\lfloor\varepsilon_k^{-1}\bar{Z}_U\rfloor\}$. It follows that

$$\begin{aligned}E[\lambda(\mathbf{p}(\mathcal{R}_{\varepsilon_k}(\gamma_U)))]&\leqslant E\Big[(\varepsilon_k^{-1}\bar{Z}_U+1)\sup_{x\in\mathbf{m}_\infty}\lambda(B_D(x,2\varepsilon_k))\Big]\\&\leqslant\varepsilon_k^{-1}E[(\bar{Z}_U+1)^2]^{1/2}E\Big[\sup_{x\in\mathbf{m}_\infty}\lambda(B_D(x,2\varepsilon_k))^2\Big]^{1/2}=O(\varepsilon_k^{3-\beta})\end{aligned}$$

as $k\to\infty$, by Corollary 6.2. Finally, we have

$$E[N_k]=O(2^{k(1-(1/4-\beta)(3-\beta))})$$

and, using (29) and (30), it follows that

$$E[N_k^*]\to 0,\quad \text{as } k\to\infty.$$

Thus $P(N_k^*\geqslant 1)\to 0$ as $k\to\infty$.

To complete the proof, suppose that there exists a point $t_0\in\overline{\mathcal{R}(\gamma_U)}\cap\mathrm{LI}$, and suppose in addition that we have the following properties:

(a′) $t_0>u_0$;

(b′) $\bar{\mathbf{e}}_t\geqslant\bar{\mathbf{e}}_{t_0}$ for every $t\in]t_0-2\delta,t_0[$;

(c′) $\bar{\mathbf{e}}_{t_0}>\eta$ and $\inf_{\eta/2\leqslant r\leqslant\bar{\mathbf{e}}_{t_0}}\overline{W}_{t_0}(r)>\eta$.

By Lemma 3.4, we must have $t_0<U$ (outside a set of zero probability depending on U). For every $k\geqslant 1$, let $i(k)$ be the unique index such that $t_0\in]i(k)2^{-k},(i(k)+1)2^{-k}]$. If k is large enough, both indices $i=i(k)$ and $i=i(k)+1$ will satisfy the properties (a)–(c) listed above (for (b), we use (b′) together with the fact that the function $t\mapsto\bar{\mathbf{e}}_t$ is Hölder continuous with exponent $\frac{1}{2}-\varepsilon$, for every $\varepsilon>0$). Moreover, $\mathcal{R}(\gamma_U)$ intersects $]i(k)2^{-k},(i(k)+1)2^{-k}]$, or possibly $](i(k)+1)2^{-k},(i(k)+2)2^{-k}]$ if $t_0=(i(k)+1)2^{-k}$, and so we conclude that $N_k^*\geqslant 1$ for k large enough. Since $P(N_k^*\geqslant 1)$ tends to 0 as $k\to\infty$, we get that, with probability 1, there exists no point $t_0\in\overline{\mathcal{R}(\gamma_U)}\cap\mathrm{LI}$ which satisfies the

additional properties (a′)–(c′). We apply this result to rationals u_0, η and δ that can be made arbitrarily small and we obtain the first assertion of the proposition.

The case of the maximal geodesic is treated in exactly the same manner. All lemmas of §5 can be extended to cover this case, with some minor changes due to the lack of symmetry in the construction of the map M_n from the pair (C^n, Λ^n). We leave the details to the reader. □

7. The main results

For a fixed $U \in]0,1[$, we have defined the simple geodesic Φ_U, the minimal geodesic Γ_U and the maximal geodesic $\bar{\Gamma}_U$. The next proposition will imply that these three geodesics coincide a.s.

PROPOSITION 7.1. *Let* $U \in]0,1[$.

(i) *Almost surely, for every mapping* $\varphi\colon [0,\bar{Z}_U] \to [0,U]$ *such that* $(\mathbf{p}(\varphi(t)), 0 \leqslant t \leqslant \bar{Z}_U)$ *is a geodesic from* ϱ *to* $\mathbf{p}(U)$*, we have*

$$\mathbf{p}(\varphi(t)) = \Gamma_U(t)$$

for every $t \in [0,\bar{Z}_U]$*. In particular,* $\Phi_U = \Gamma_U$ *a.s.*

(ii) *Almost surely, for every mapping* $\varphi\colon [0,\bar{Z}_U] \to [U,1]$ *such that* $(\mathbf{p}(\varphi(t)), 0 \leqslant t \leqslant \bar{Z}_U)$ *is a geodesic from* ϱ *to* $\mathbf{p}(U)$*, we have*

$$\mathbf{p}(\varphi(t)) = \bar{\Gamma}_U(t)$$

for every $t \in [0,\bar{Z}_U]$*. In particular,* $\Phi_U = \bar{\Gamma}_U$ *a.s.*

Proof. Let us prove part (i) of the proposition. By the definition of the minimal geodesic, we have $\gamma_U(t) \leqslant \varphi(t)$ for every $t \in [0,\bar{Z}_U]$. We argue by contradiction and assume that there exists $r_0 \in]0,\bar{Z}_U[$ such that $\gamma_U(r_0) < \varphi(r_0)$ and $\mathbf{p}(\gamma_U(r_0)) \neq \mathbf{p}(\varphi(r_0))$. Then we can find $s \in]\gamma_U(r_0), \varphi(r_0)[$ such that

$$\bar{Z}_s < r_0 = \bar{Z}_{\gamma_U(r_0)} = \bar{Z}_{\varphi(r_0)},$$

because otherwise we would have $\gamma_U(r_0) \approx \varphi(r_0)$. Since the mapping $t \mapsto \gamma_U(t)$ is non-decreasing and left-continuous, there exists $r_1 \in [r_0,\bar{Z}_U]$ such that

$$\begin{aligned} &\gamma_U(t) > s, \quad \text{if } t \in]r_1,\bar{Z}_U], \\ &\gamma_U(t) \leqslant s, \quad \text{if } t \in [0,r_1]. \end{aligned}$$

Note that $r_1=\bar{Z}_U$ does not occur a.s., because this would imply $\gamma_U(\bar{Z}_U)\leqslant s<U$ together with $\mathbf{p}(U)=\mathbf{p}(\gamma_U(\bar{Z}_U))$ and $\bar{Z}_s<\bar{Z}_U$: This is only possible if $U\sim\gamma_U(\bar{Z}_U)$, which contradicts Lemma 3.4.

Thus $r_1<\bar{Z}_U$, and we have $\mathbf{p}(\gamma_U(r_1))=\mathbf{p}(\gamma_U(r_1+))$ by continuity. Since

$$\gamma_U(r_1)\leqslant s\leqslant\gamma_U(r_1+)\quad\text{and}\quad\bar{Z}_s<r_0\leqslant r_1=\bar{Z}_{\gamma_U(r_1)}=\bar{Z}_{\gamma_U(r_1+)},$$

the property $\gamma_U(r_1)\approx\gamma_U(r_1+)$ does not hold. Hence, we must have $\gamma_U(r_1)\sim\gamma_U(r_1+)$, and it follows that $p_{\bar{\mathbf{e}}}(\gamma_U(r_1))=p_{\bar{\mathbf{e}}}(\gamma_U(r_1+))$ is an ancestor of $p_{\bar{\mathbf{e}}}(s)$ in $\mathcal{T}_{\bar{\mathbf{e}}}$. This means in particular that $\gamma_U(r_1+)$ is a left-increase point of $\bar{\mathbf{e}}$, which is impossible by Proposition 6.3. This contradiction completes the proof of the first statement in (i). By applying this statement to $\varphi=\varphi_U$, we get that $\Phi_U=\Gamma_U$ a.s.

Part (ii) of the proposition is proved in a similar way. In particular, the identity $\Phi_U=\bar{\Gamma}_U$ a.s. is obtained by taking $\varphi=\bar{\varphi}_U$. □

PROPOSITION 7.2. *Proposition* 7.1 (i) *holds simultaneously for all* $U\in]0,1[\setminus\mathrm{LI}$ *outside a single set of zero probability. Similarly*, *Proposition* 7.1 (ii) *holds simultaneously for all* $U\in]0,1[\setminus\mathrm{RI}$ *outside a single set of zero probability.*

Proof. The assertions of Proposition 7.1 hold simultaneously for every rational number $U\in]0,1[$ outside a single set of zero probability. From now on, we argue outside this set, and we deal only with part (i) of Proposition 7.1.

Set

$$H=\{U\in]0,1]:\mathbf{p}(U)=\mathbf{p}(U')\text{ for some }U'<U\},$$

so that $\mathrm{LI}\subset H$ in particular. If $U\in H\setminus\mathrm{LI}$, then there exists $U'<U$ such that $U'\approx U$, and $U'\in]0,1[\setminus H$. Note that $\bar{Z}_r\geqslant\bar{Z}_{U'}=\bar{Z}_U$ for every $r\in[U',U]$. It is then obvious that if φ is as in the first statement of Proposition 7.1, we have $\varphi(r)\in[0,U']$ for every $r\in[0,\bar{Z}_U[$. Putting $\varphi'(r)=\varphi(r)$ if $r\in[0,\bar{Z}_U[$ and $\varphi'(\bar{Z}_U)=U'$, we get a function $\varphi':[0,\bar{Z}_{U'}]\to[0,U']$ such that $(\mathbf{p}(\varphi'(t)),0\leqslant t\leqslant\bar{Z}_{U'})$ is a geodesic from ϱ to $\mathbf{p}(U')$. So if we can prove that $\mathbf{p}(\varphi'(t))=\Gamma_{U'}(t)$ for every $t\in[0,\bar{Z}_{U'}]$, we will get the desired conclusion for φ since $\mathbf{p}(\varphi(t))=\mathbf{p}(\varphi'(t))$ and $\Gamma_U(t)=\Gamma_{U'}(t)$ for every $t\in[0,\bar{Z}_U]$.

Due to the previous observations, we may restrict our attention to the case $U\in]0,1[\setminus H$, which we consider from now on. Let $\varepsilon\in]0,\bar{Z}_U[$. Then we can find $s\in]0,U[$ such that the following properties hold:

(a) $\bar{\mathbf{e}}_t>\bar{\mathbf{e}}_s$ for every $t\in]s,U]$, and thus $p_{\bar{\mathbf{e}}}(s)\in]\!]\varrho,p_{\bar{\mathbf{e}}}(U)[\![$;

(b) $\bar{Z}_t\geqslant\bar{Z}_U-\varepsilon$ for every $t\in[s,U]$;

(c) $\min_{a\in[\![p_{\bar{\mathbf{e}}}(s),p_{\bar{\mathbf{e}}}(U)]\!]}\bar{Z}_a<\bar{Z}_s$.

In (a), we use the fact that $U\notin\mathrm{LI}$. We can impose condition (c) because the mapping $a\mapsto\bar{Z}_a$ is not monotone on any non-trivial line segment of the tree $\mathcal{T}_{\bar{\mathbf{e}}}$.

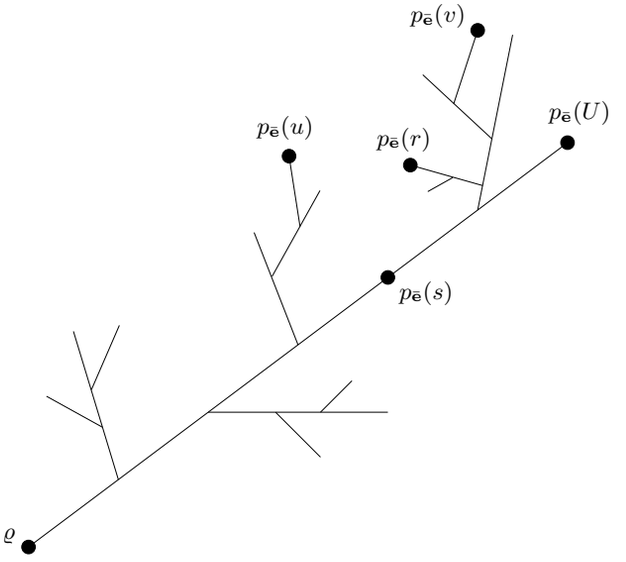

Figure 4. Illustration of the proof of Proposition 7.2. We have $u<s<v$, $u\approx v$ and $\Phi_r(\bar{Z}_u)=\mathbf{p}(u)$ for every rational $r\in]s,v[$.

Let φ be as in the first statement of Proposition 7.1. Since $U\notin H$ and φ takes values in $[0,U]$, it is clear that $\varphi(\bar{Z}_U)=U$. Set

$$t^*=\inf\{t\in[0,\bar{Z}_U]:\varphi(t)\geqslant s\}.$$

Let u be an accumulation point of $\varphi(t)$ along a sequence of values of t belonging to $[0,t^*[$ and converging to t^*. By the definition of t^*, we have $u\in[0,s]$, and we also have $\mathbf{p}(u)=\mathbf{p}(\varphi(t^*))$ by continuity. Similarly, we can construct a sequence $\{t_k\}_{k\geqslant 1}$ in $[t^*,\bar{Z}_U]$ such that $\varphi(t_k)\geqslant s$ for every k, and $t_k\to t^*$ as $k\to\infty$. Let v be an accumulation point of the sequence $\varphi(t_k)$, so that $v\in[s,U]$ and again $\mathbf{p}(v)=\mathbf{p}(\varphi(t^*))=\mathbf{p}(u)$.

We claim that $v>s$. Indeed, if $v=s$, the curve $(\mathbf{p}(\varphi(t)),t^*\leqslant t\leqslant\bar{Z}_U)$ starts from $\mathbf{p}(s)$ and ends at $\mathbf{p}(U)$, and then (c), together with the fact that $\bar{Z}_{\mathbf{p}(\varphi(t))}\geqslant\bar{Z}_{\mathbf{p}(\varphi(t^*))}=\bar{Z}_s$ for every $t\in[t^*,\bar{Z}_U]$, gives a contradiction with Proposition 3.1.

The preceding observations imply that $u\approx v$. Indeed we saw that $\mathbf{p}(u)=\mathbf{p}(v)$, and $u\sim v$ is impossible by (a) since $u\leqslant s<v$.

To complete the proof, choose a rational $r\in]s,v[$. As $u<r<v$ and $u\approx v$, the definition of the simple geodesic shows that $\Phi_r(\bar{Z}_v)=\mathbf{p}(v)=\mathbf{p}(\varphi(t^*))$. Since $\varphi(t)\in[0,s]$ for every $t\in[0,t^*[$ (by the definition of t^*), the mapping $\bar{\varphi}$ defined by

$$\bar{\varphi}(t)=\begin{cases}\varphi_r(t), & \text{if } t^*\leqslant t\leqslant\bar{Z}_r,\\ \varphi(t), & \text{if } 0\leqslant t<t^*,\end{cases}$$

takes values in $[0,r]$ and is such that $(\mathbf{p}(\bar\varphi(t)), 0\leqslant t\leqslant \overline{Z}_r)$ is a geodesic from ϱ to $\mathbf{p}(r)$. As we assumed that the assertions of Proposition 7.1 hold for every rational, we obtain that $\mathbf{p}(\varphi(t))=\Phi_r(t)$ for every $t\in[0,t^*]$. From (b), we get $t^*=\overline{Z}_v\geqslant \overline{Z}_U-\varepsilon$ and also $\Phi_r(t)=\Phi_U(t)$ for $t<\overline{Z}_U-\varepsilon$. Therefore $\mathbf{p}(\varphi(t))=\Phi_U(t)$ for every $t\in[0,\overline{Z}_U-\varepsilon[$. Since ε can be taken arbitrarily small, we have that $\mathbf{p}(\varphi(t))=\Phi_U(t)$ for every $t\in[0,\overline{Z}_U]$. By taking $\varphi=\gamma_U$, we also see that $\Gamma_U=\Phi_U$, which completes the proof. □

For every $U\in]0,1[$, we set

$$\mathcal{L}_U = p_{\bar{\mathbf{e}}}^{-1}([\![\varrho, p_{\bar{\mathbf{e}}}(U)]\!])$$

and

$$\mathcal{R}_U = \Big\{s\in[0,U]: \overline{Z}_s = \min_{s\leqslant r\leqslant U} \overline{Z}_r\Big\}\cup\Big\{s\in[U,1]: \overline{Z}_s = \min_{U\leqslant r\leqslant s} \overline{Z}_r\Big\}.$$

Then $\mathcal{L}_U$ and $\mathcal{R}_U$ are closed. Furthermore, it is easy to verify that $\mathbf{p}(\mathcal{R}_U)$ is just the range of the simple geodesic Φ_U. It then follows from the discussion at the end of §4.1 that $\mathbf{p}(\mathcal{L}_U)\cap\mathbf{p}(\mathcal{R}_U)=\{\varrho,\mathbf{p}(U)\}$. This property holds simultaneously for all $U\in]0,1[$ outside a single set of zero probability.

We also set

$$O_U^1 = [0,U]\setminus(\mathcal{R}_U\cup\mathcal{L}_U) \quad\text{and}\quad O_U^2 = [U,1]\setminus(\mathcal{R}_U\cup\mathcal{L}_U).$$

Then O_U^1 and O_U^2 are open and disjoint, and it is not hard to verify that $\mathbf{p}^{-1}(\mathbf{p}(O_U^1))=O_U^1$ and $\mathbf{p}^{-1}(\mathbf{p}(O_U^2))=O_U^2$. From the definition of the quotient topology, it follows that $\mathbf{p}(O_U^1)$ and $\mathbf{p}(O_U^2)$ are two disjoint open subsets of $\mathbf{m}_\infty$.

LEMMA 7.3. *Almost surely, for every $U\in]0,1[$ and every geodesic $(\omega(t),0\leqslant t\leqslant\overline{Z}_U)$ from ϱ to $\mathbf{p}(U)$, the set $\{\omega(t):0<t<\overline{Z}_U\}$ does not intersect $\mathbf{p}(\mathcal{L}_U)$.*

Proof. We argue by contradiction and assume that there exists $t_0\in]0,\overline{Z}_U[$ such that $\omega(t_0)\in\mathbf{p}(\mathcal{L}_U)$. Then $\omega(t_0)\in\mathbf{p}(\mathcal{L}_U)\setminus\mathbf{p}(\mathcal{R}_U)$. Since $\mathbf{p}(\mathcal{R}_U)$ is closed, a continuity argument shows that $\omega(t)\notin\mathbf{p}(\mathcal{R}_U)$ if $t>t_0$ is sufficiently close to t_0. Also Lemma 3.5 implies that we can find values of $t>t_0$ arbitrarily close to t_0 such that $\omega(t)\notin\mathbf{p}(\mathcal{L}_U)$. It follows that we can find a connected component $]t_1,t_2[$ of the open set

$$\{t\in]t_0,\overline{Z}_U[:\omega(t)\notin\mathbf{p}(\mathcal{R}_U\cup\mathcal{L}_U)\}$$

such that $\omega(t_1)\in\mathbf{p}(\mathcal{L}_U)$. Since $\omega(]t_1,t_2[)$ is connected, we have either $\omega(]t_1,t_2[)\subset\mathbf{p}(O_U^1)$ or $\omega(]t_1,t_2[)\subset\mathbf{p}(O_U^2)$. We assume for definiteness that $\omega(]t_1,t_2[)\subset\mathbf{p}(O_U^1)$.

Let a_1 be the unique element of $\mathcal{T}_{\bar{\mathbf{e}}}$ such that $\Pi(a_1)=\omega(t_1)$, and let $r_1<U$ denote the smallest element of $[0,1]$ such that $p_{\bar{\mathbf{e}}}(r_1)=a_1$. Recall that $\mathcal{T}_{\bar{\mathbf{e}}}(a_1)$ denotes the subtree of all descendants of a_1. As a simple consequence of Proposition 3.1, we have

$$\omega([t_1,\overline{Z}_U])\subset\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(a_1)).$$

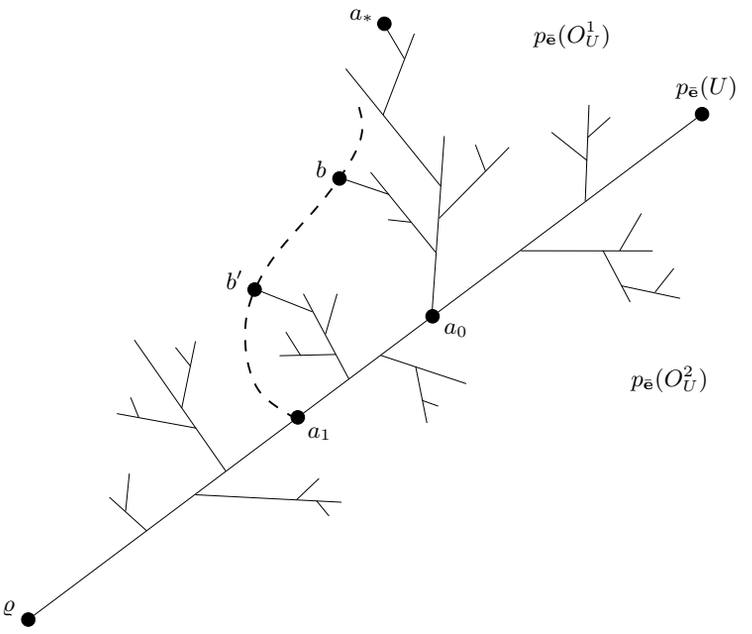

Figure 5. Illustration of the proof of Lemma 7.3. The geodesic ω visits a_1, then stays in the (image under Π of the) subtree $\mathcal{T}_{\bar{\mathbf{e}}}(a_1)$. It hits the smaller subtree $\mathcal{T}_{\bar{\mathbf{e}}}(a_0)$ at time T at the point $\Pi(b)=\Pi(b')$.

Indeed, suppose that there exists $t_1'\in]t_1,\overline{Z}_U[$ such that $\omega(t_1')\notin\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(a_1))$, and let $a_1'\in\mathcal{T}_{\bar{\mathbf{e}}}\setminus\mathcal{T}_{\bar{\mathbf{e}}}(a_1)$ be such that $\omega(t_1')=\Pi(a_1')$. Then $a_1\in[\![a_1'\triangle p_{\bar{\mathbf{e}}}(U),p_{\bar{\mathbf{e}}}(U)]\!]$, and by applying Proposition 3.1 to the curve $(\omega(t),t_1'\leqslant t\leqslant\overline{Z}_U)$, we get $\overline{Z}_{\omega(t_1)}\geqslant\min\{\overline{Z}_{\omega(t)}:t_1'\leqslant t\leqslant\overline{Z}_U\}$, which is a contradiction.

Fix $t^*\in]t_1,t_2[$, and choose $a^*\in\mathcal{T}_{\bar{\mathbf{e}}}(a_1)$ such that $\Pi(a^*)=\omega(t^*)$. Set $a_0=a^*\triangle p_{\bar{\mathbf{e}}}(U)$. Again, a simple application of Proposition 3.1 shows that $a_0\neq a_1$, and therefore $a_0\in[\![a_1,p_{\bar{\mathbf{e}}}(U)]\!]$. If r_0 is the smallest representative of a_0 in $[0,1]$, we have thus $r_1<r_0$. Set

$$T=\inf\{t\geqslant t_1:\omega(t)\in\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(a_0))\}.$$

We have $T>t_1$ because $\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(a_0))$ is closed in $\mathbf{m}_\infty$, and $\omega(t_1)=\Pi(a_1)\notin\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(a_0))$. Moreover $T\leqslant t^*$ because $a^*\in\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(a_0))$. Thus we have $t_1<T\leqslant t^*<t_2$. Furthermore, we can write $\omega(T)=\Pi(b)$ for some $b\in\mathcal{T}_{\bar{\mathbf{e}}}(a_0)$. Note that $\omega(T)$ belongs to the boundary of $\Pi(\mathcal{T}_{\bar{\mathbf{e}}}(a_0))$, and $\omega(T)\neq\Pi(a_0)$ (because $\{\omega(t):t_1<t<t_2\}$ does not intersect $\mathbf{p}(\mathcal{L}_U)$). As noticed in §3, this implies the existence of $b'\in\mathcal{T}_{\bar{\mathbf{e}}}\setminus\mathcal{T}_{\bar{\mathbf{e}}}(a_0)$ such that $b\approx b'$.

Let s be the unique representative of b in $[0,1]$, so that $\mathbf{p}(s)=\Pi(b)=\omega(T)$. Since $b\in\mathcal{T}_{\bar{\mathbf{e}}}(a_0)$ and $b\neq a_0$, we have $r_0<s$. On the other hand, for every $t\in]t_1,T[$, we know that

$\omega(t)\in\mathbf{m}_\infty\setminus\Pi(\mathcal{T}_{\tilde{\mathbf{e}}}(a_0))$, and $\omega(t)\in\mathbf{p}(O_U^1)$. It follows that, for every $t\in]t_1,T[$, we can find $\psi(t)\in[0,r_0]$ such that $\omega(t)=\mathbf{p}(\psi(t))$.

We also set $\psi(T)=s$, and for every $t\in[0,t_1]$, we put $\psi(t)=\varphi_{r_1}(t)$. The mapping $(\psi(t),0\leqslant t\leqslant T)$ is such that $(\mathbf{p}(\psi(t)),0\leqslant t\leqslant T)$ is a geodesic from ϱ to $\omega(T)$, and $\psi(t)\in[0,\psi(T)]$ for every $t\in[0,T]$. Since $\psi(T)=s\notin\mathrm{LI}$ (by Lemma 2.1, since the equivalence class of s for $\approx$ is not a singleton), the first statement of Proposition 7.2 entails that $(\mathbf{p}(\psi(t)),0\leqslant t\leqslant T)$ coincides with the simple geodesic Φ_s. This is a contradiction, since $\mathbf{p}(\psi(t_1))=\omega(t_1)\in\mathrm{Skel}_\infty$ and a simple geodesic cannot visit a point of Skel_∞, except possibly at its endpoint.

In the case when $\omega(]t_1,t_2[)\subset\mathbf{p}(O_U^2)$, the proof is exactly similar, but we now use the second statement of Proposition 7.2. □

We now come to one of our main results.

THEOREM 7.4. *Almost surely, for every $x\in\mathbf{m}_\infty\setminus\mathrm{Skel}_\infty$, there is a unique geodesic from ϱ to x, which is the simple geodesic Φ_s for an arbitrary choice of $s\in[0,1]$ such that $\mathbf{p}(s)=x$.*

Proof. Let $x\in\mathbf{m}_\infty\setminus\mathrm{Skel}_\infty$ and let $U\in]0,1[$ be such that $\mathbf{p}(U)=x$ (we exclude the trivial case $x=\varrho$). Notice that, for every $r\in[0,\overline{Z}_U]$, there is exactly one point y in $\mathbf{p}(\mathcal{R}_U)$ such that $\overline{Z}_y=r$, and this point is $y=\Phi_U(r)$. In order to prove the theorem, we need only prove that any geodesic from ϱ to x takes values in $\mathbf{p}(\mathcal{R}_U)$.

So let $(\omega(t),0\leqslant t\leqslant\overline{Z}_U)$ be a geodesic from ϱ to $x=\mathbf{p}(U)$. From Lemma 7.3, we already know that $(\omega(t),0<t<\overline{Z}_U)$ does not intersect $\mathbf{p}(\mathcal{L}_U)$. We argue by contradiction and suppose that the range of ω intersects $\mathbf{m}_\infty\setminus\mathbf{p}(\mathcal{R}_U)$. Let $]t_1,t_2[$ be a connected component of the open set

$$\{s\in]0,\overline{Z}_U[:\omega(s)\in\mathbf{m}_\infty\setminus(\mathbf{p}(\mathcal{R}_U)\cup\mathbf{p}(\mathcal{L}_U))\}.$$

Then, $\omega(t_1)\in\mathbf{p}(\mathcal{R}_U)$, and $\omega(t_2)\in\mathbf{p}(\mathcal{R}_U)$. By the remark of the beginning of the proof, we have that $\omega(t_1)=\Phi_U(t_1)$ and $\omega(t_2)=\Phi_U(t_2)$. Also, as in the preceding proof, we have that $\omega(]t_1,t_2[)\subset\mathbf{p}(O_U^1)$ or $\omega(]t_1,t_2[)\subset\mathbf{p}(O_U^2)$, and we assume for definiteness that $\omega(]t_1,t_2[)\subset\mathbf{p}(O_U^1)$. For every $t\in]t_1,t_2[$, let $\psi(t)$ be the smallest representative of $\omega(t)$ in $[0,1]$. Also, let $\psi(t_2)$ be the largest representative of $\omega(t_2)$. Finally set $\psi(t)=\varphi_U(t)$ for $t\in[0,t_1]$. Then $\psi(t)\leqslant U$ for every $t\in[0,t_2[$, whereas $\psi(t_2)\geqslant U$. So we see that ψ takes values in $[0,\psi(t_2)]$ and $(\mathbf{p}(\psi(t)),0\leqslant t\leqslant t_2)$ is a geodesic from ϱ to $\mathbf{p}(\psi(t_2))$. In order to apply Proposition 7.2, we still need to verify that $\psi(t_2)\notin\mathrm{LI}$. This is clear if $t_2<\overline{Z}_U$ (because the equivalence class of $\psi(t_2)$ for $\approx$ has at least two representatives) and if $t_2=\overline{Z}_U$ this follows from our assumption $x\notin\mathrm{Skel}_\infty$. Proposition 7.2 now shows that $\mathbf{p}(\psi(t))$ must coincide with $\Phi_{\psi(t_2)}(t)$ for every $t\in[0,t_2]$. This is a contradiction, since the range of the simple geodesic $\Phi_{\psi(t_2)}$ is contained in $\mathbf{p}(\mathcal{R}_U)$. □

COROLLARY 7.5. *Almost surely, for λ-almost every $x\in\mathbf{m}_\infty$, there is a unique geodesic from ϱ to x.*

Proof. By Lemma 3.4, $\lambda(\mathrm{Skel}_\infty)=0$ a.s., so that the statement follows from Theorem 7.4. □

We then consider the case of vertices belonging to Skel_∞. Such a vertex x can be written uniquely as $x=\Pi(a)$ with $a\in\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})$, and we denote the multiplicity of a in $\mathcal{T}_{\mathbf{e}}$ by $m(x)$. By Proposition 3.3, $m(x)$ is also the number of connected components of $\mathrm{Skel}_\infty\setminus\{x\}$.

THEOREM 7.6. *Almost surely, for every $x\in\mathrm{Skel}_\infty$, there are exactly $m(x)$ distinct geodesics from ϱ to x. These geodesics are the simple geodesics Φ_s for all $s\in[0,1]$ such that $\mathbf{p}(s)=x$.*

Proof. Let $x\in\mathrm{Skel}_\infty$, and let $(\omega(t),0\leqslant t\leqslant\overline{Z}_x)$ be a geodesic from ϱ to x. By Lemma 3.5, we can find a sequence $\{r_k\}_{k\geqslant1}$ in $[0,\overline{Z}_x]$ such that $r_k\to\overline{Z}_x$, as $k\to\infty$, and $\omega(r_k)\in\mathbf{m}_\infty\setminus\mathrm{Skel}_\infty$ for every k. Choose $s_k\in[0,1]$ such that $\mathbf{p}(s_k)=\omega(r_k)$. By extracting a subsequence if necessary, we may assume that $s_k\to s\in[0,1]$ as $k\to\infty$, and we have then $\mathbf{p}(s)=x$. On the other hand, for every $k\geqslant1$, Theorem 7.4 and the fact that $\omega(r_k)\in\mathbf{m}_\infty\setminus\mathrm{Skel}_\infty$ imply that we have $\omega(t)=\Phi_{s_k}(t)$ for every $t\in[0,r_k]$. Passing to the limit $k\to\infty$ we get $\omega(t)=\Phi_s(t)$ for every $t\in[0,\overline{Z}_x[$.

Conversely, for every $s\in[0,1]$ such that $\mathbf{p}(s)=x$, Φ_s is a geodesic from ϱ to x, and the geodesics obtained in this way are distinct. Indeed, consider the case when $m(x)=2$. Then there exist two reals s_1 and s_2 such that $0<s_1<s_2<1$, $p_{\mathbf{e}}(s_1)=p_{\mathbf{e}}(s_2)$ and $x=\mathbf{p}(s_1)=\mathbf{p}(s_2)$. By Lemma 2.1, we must have

$$\min_{s_1\leqslant s\leqslant s_2}\overline{Z}_s<\overline{Z}_{s_1}=\overline{Z}_{s_2}.$$

It follows that, if $r>0$ is sufficiently small, $\varphi_{s_2}(\overline{Z}_x-r)$ is a point of $]s_1,s_2[$ which is not equivalent to any point of $[0,s_1]$. Thus the geodesics Φ_{s_1} and Φ_{s_2} are distinct. The same argument applies when $m(x)=3$. This completes the proof. □

A consequence of the previous results is the fact that if x and x' are two points of $\mathbf{m}_\infty$ distinct from the root, and if ω (resp. ω') is a geodesic from ϱ to x (resp. from ϱ to x'), then ω and ω' must coincide over a small time interval. We state this confluence property of geodesics in a slightly more precise form.

COROLLARY 7.7. *Almost surely, for every $\eta>0$, there exists $\alpha\in]0,\eta[$ such that the following holds. Let $x,x'\in\mathbf{m}_\infty$ be such that $D(\varrho,x)\geqslant\eta$ and $D(\varrho,x')\geqslant\eta$, and let ω (resp. ω') be a geodesic from ϱ to x (resp. to x'). Then, $\omega(t)=\omega'(t)$ for all $t\in[0,\alpha]$.*

Proof. If $\eta>0$ is given, we can choose $\varepsilon>0$ such that $\bar{Z}_s<\eta$ for all $s\in[0,\varepsilon]\cup[1-\varepsilon,1]$. We then set

$$\alpha=\inf_{s\in[\varepsilon,1-\varepsilon]}\bar{Z}_s.$$

Notice that $\alpha>0$, since $\bar{Z}_s>0$ for every $s\in]0,1[$.

Let x, x', ω and ω' be as in the corollary. By the previous two theorems, we can write $\omega=\Phi_r$ and $\omega'=\Phi_{r'}$, where $r,r'\in[0,1]$ are such that $x=\mathbf{p}(r)$ and $x'=\mathbf{p}(r')$. In particular, $\bar{Z}_r=D(\varrho,x)\geqslant\eta$ and $\bar{Z}_{r'}=D(\varrho,x')\geqslant\eta$. Our choice of ε now ensures that both r and r' belong to $[\varepsilon,1-\varepsilon]$. But then, from the definition of the functions φ_r and $\varphi_{r'}$, it is immediate that $\varphi_r(t)=\varphi_{r'}(t)$ for every $t\in[0,\alpha[$. □

Remark. Our results show that all geodesics starting from the root are simple geodesics. It follows that a point x of $\mathbf{m}_\infty$ which is a relative interior point of a geodesic from the root must be of the form $x=\mathbf{p}(s)$, for some $s\in[0,1[$ which is a right-increase time of $\bar{Z}$. One easily checks that the set of all such points x has Hausdorff dimension 1, and so is a very small subset of $\mathbf{m}_\infty$, which has Hausdorff dimension 4. This may be compared to the work of Zamfirescu [41], who proved, in the sense of Baire's category, that on "most" convex surfaces, a typical point is not an interior point of any geodesic segment.

8. Gromov–Hausdorff distances and the invariance under re-rooting

The previous sections were devoted to the study of geodesics connecting the root of $\mathbf{m}_\infty$ to another point. Similar results hold if we replace the root $\varrho=\mathbf{p}(0)$ by a random point chosen according to the volume measure λ. This follows from the invariance of the distribution of the Brownian map under uniform re-rooting, which will be discussed below.

Let us first recall some basic definitions concerning pointed metric spaces and the Gromov–Hausdorff convergence. Let $k\geqslant0$ be an integer. A *k-pointed* compact metric space is a triplet $(E,d,(a_1,a_2,...,a_k))$, where (E,d) is a compact metric space and $(a_1,a_2,...,a_k)$ is a k-tuple of (not necessarily distinct) distinguished points of E. A 0-pointed compact metric space is just a compact metric space, and when $k=1$ we say *pointed* rather than 1-pointed. We denote the space of all isometry classes of k-pointed compact metric spaces by $\mathbb{K}^{(k)}$ (and in agreement with the introduction above, we set $\mathbb{K}=\mathbb{K}^{(0)}$). The *Gromov–Hausdorff distance* on $\mathbb{K}^{(k)}$ is then defined by

$$\begin{aligned}&d_{\mathrm{GH}}((E,d,(a_1,...,a_k)),(E',d',(a'_1,...,a'_k)))\\&\qquad=\inf_{(F,\delta),\phi,\phi'}\Big\{\delta_H(\phi(E),\phi'(E'))\vee\max_{1\leqslant i\leqslant k}\delta(\phi(a_i),\phi'(a'_i))\Big\},\end{aligned}$$

where the infimum is over all choices of the metric space (F,δ) and the isometric embeddings $\phi\colon E\to F$ and $\phi'\colon E'\to F$, and δ_H stands for the usual Hausdorff distance between compact subsets of F. Then d_{GH} is a distance on $\mathbb{K}^{(k)}$, and the space $(\mathbb{K}^{(k)},d_{\mathrm{GH}})$ is Polish. We equip $\mathbb{K}^{(k)}$ with the associated Borel σ-field.

An equivalent definition of the Gromov–Hausdorff distance can be given in terms of correspondences. Let $(E,d,(a_1,...,a_k))$ and $(E',d',(a'_1,...,a'_k))$ be two k-pointed compact metric spaces. A *correspondence* $\mathcal{C}$ between (E,d) and (E',d') is a subset of the Cartesian product $E\times E'$ such that for every $x\in E$ there exists $y\in E'$ such that $(x,y)\in\mathcal{C}$, and conversely. The *distortion* of $\mathcal{C}$ is defined by

$$\mathrm{dist}(\mathcal{C})=\sup\{|d(x,y)-d'(x',y')|:(x,x')\in\mathcal{C}\text{ and }(y,y')\in\mathcal{C}\}.$$

Then $d_{\mathrm{GH}}((E,d,(a_1,...,a_k)),(E',d',(a'_1,...,a'_k)))$ equals half the infimum of the quantities $\mathrm{dist}(\mathcal{C})$ when $\mathcal{C}$ varies over correspondences between E and E' that contain all the pairs (a_i,a'_i) for $1\leqslant i\leqslant k$. See [11, Theorem 7.3.25] for a proof in the case $k=0$, which is easily extended.

As was already stated in Theorem 2.2, we have the almost sure convergence in $(\mathbb{K}^{(1)},d_{\mathrm{GH}})$,

$$(\mathbf{m}_n,\varkappa_p n^{-1/4}d_n,\partial_n)\xrightarrow{\text{a.s.}}(\mathbf{m}_\infty,D,\varrho),\quad\text{as } n\to\infty. \tag{33}$$

The next theorem gives a precise form of the invariance of the distribution of the Brownian map under uniform re-rooting.

THEOREM 8.1. *Let F be a non-negative measurable function on $\mathbb{K}^{(1)}$. Then*

$$E\left[\int F(\mathbf{m}_\infty,D,x)\lambda(dx)\right]=E[F(\mathbf{m}_\infty,D,\varrho)].$$

More generally, for every integer $k\geqslant 2$ and every non-negative measurable function F on $\mathbb{K}^{(k)}$,

$$E\left[\int F(\mathbf{m}_\infty,D,(x_1,...,x_k))\lambda(dx_1)\dots\lambda(dx_k)\right]$$
$$=E\left[\int F(\mathbf{m}_\infty,D,(\varrho,x_1,...,x_{k-1}))\lambda(dx_1)\dots\lambda(dx_{k-1})\right].$$

Proof. We deal only with the first statement of the theorem. The second one can be proved in a similar manner. Let U be a random variable that is uniformly distributed over $[0,1]$ and independent of the sequence of random maps $\{M_n\}_{n\geqslant 1}$ (and in particular of $(\mathbf{m}_\infty,D)$). The first assertion of the theorem will follow from the fact that the law of $(\mathbf{m}_\infty,D,\mathbf{p}(U))$ coincides with the law of $(\mathbf{m}_\infty,D,\varrho)$.

To establish this fact, we will use a similar invariance property for the discrete maps M_n and then pass to the limit $n\to\infty$ using (33). We start with some simple combinatorial considerations. We let $\mathcal{M}_p^{n,*}$ denote the set of all pairs (M,e), where M is a rooted $2p$-angulation with n faces ($M\in\mathcal{M}_p^n$) and e is a distinguished oriented edge of M. If $(M,e)\in\mathcal{M}_p^{n,*}$, we define $\Gamma(M,e)\in\mathcal{M}_p^n$ by saying that $\Gamma(M,e)$ is the "same" map as M but re-rooted at the oriented edge e. Then it is easy to verify that the image under Γ of the uniform distribution over $\mathcal{M}_p^{n,*}$ is the uniform distribution over $\mathcal{M}_p^n$.

Now consider the random $2p$-angulation M_n. The Bouttier–Di Francesco–Guitter bijection of §2.6 allows us to associate with each $i\in\{0,1,...,pn-1\}$ a unique non-oriented edge e_i of M_n, and in this way we get a bijection between $\{0,1,...,pn-1\}$ and the set of all (non-oriented) edges of M_n. Set $U_n=\lfloor pnU\rfloor$, in such a way that U_n is uniformly distributed over $\{0,1,...,pn-1\}$. Choose one of the two possible orientations of the edge e_{U_n} with equal probabilities, independently of U and of the sequence of maps $\{M_n\}_{n\geqslant 1}$, and denote the resulting oriented edge by $\tilde{e}_{U_n}$. Also set $\widetilde{M}_n=\Gamma(M_n,\tilde{e}_{U_n})$. Note that the vertex set and the edges of $\widetilde{M}_n$ are the same as those of M_n, but the root is different. By the combinatorial considerations of the beginning of the proof, $\widetilde{M}_n$ has the same distribution as M_n. Writing $\tilde{\partial}_n$ for the root vertex of $\widetilde{M}_n$, we thus have

$$(\mathbf{m}_n,\varkappa_p n^{-1/4}d_n,\tilde{\partial}_n)\overset{(\mathrm{d})}{=}(\mathbf{m}_n,\varkappa_p n^{-1/4}d_n,\partial_n). \tag{34}$$

Note that both sides of (34) are viewed as random variables with values in $\mathbb{K}^{(1)}$.

In view of (33) and (34), the proof of the theorem will be complete if we can verify that

$$(\mathbf{m}_n,\varkappa_p n^{-1/4}d_n,\tilde{\partial}_n)\xrightarrow{\text{a.s.}}(\mathbf{m}_\infty,D,\mathbf{p}(U)),\quad \text{as } n\to\infty. \tag{35}$$

To this end, let us first recall the proof of (33) in [24]. We define a correspondence $\mathcal{C}_n$ between $(\mathbf{m}_n,\varkappa_p n^{-1/4}d_n)$ and $(\mathbf{m}_\infty,D)$ by declaring that a pair $(a,b)\in\mathbf{m}_n\times\mathbf{m}_\infty$ belongs to $\mathcal{C}_n$ if and only if $a=\partial_n$ and $b=\varrho$, or if there exists $t\in[0,1]$ such that $a=p_n(\lfloor pnt\rfloor)$ and $b=\mathbf{p}(t)$. Then it easily follows from the convergence (3) that $\mathrm{dist}(\mathcal{C}_n)$ converges to 0 a.s. as $n\to\infty$ (see [24], but note that the presentation there is slightly different because [24] does not deal with pointed spaces—the same argument however goes through without change). We may slightly enlarge the correspondence $\mathcal{C}_n$ by including also those pairs $(a,b)\in\mathbf{m}_n\times\mathbf{m}_\infty$ for which there exists $t\in[0,1]$ such that $d_n(a,p_n(\lfloor pnt\rfloor))=1$ and $b=\mathbf{p}(t)$. If $\mathcal{C}'_n$ denotes the enlarged correspondence, the distortion $\mathrm{dist}(\mathcal{C}'_n)$ still converges to 0 a.s. as $n\to\infty$. However, from our construction of the root edge of $\widetilde{M}_n$, it is clear that we have $d_n(\tilde{\partial}_n,p_n(\lfloor pnU\rfloor))\leqslant 1$. Therefore $(\tilde{\partial}_n,\mathbf{p}(U))\in\mathcal{C}'_n$, and the fact that $\mathrm{dist}(\mathcal{C}'_n)$ converges to 0 also yields our claim (35). This completes the proof. □

As a by-product of the preceding proof, we get the following variant of (33), which will be useful in the next section.

PROPOSITION 8.2. *For every n we can construct a random vertex V_n which is uniformly distributed over $\mathbf{m}_n$, in such a way that*

$$(\mathbf{m}_n, \varkappa_p n^{-1/4} d_n, (\partial_n, V_n)) \xrightarrow{\text{a.s.}} (\mathbf{m}_\infty, D, (\varrho, \mathbf{p}(U))), \quad \text{as } n \to \infty, \tag{36}$$

where the random variable U is uniformly distributed over $[0,1]$ and independent of the sequence of random maps $\{M_n\}_{n\geqslant 1}$, and the convergence holds in the space $(\mathbb{K}^{(2)}, d_{\mathrm{GH}})$.

Proof. The argument for (35) also gives the convergence

$$(\mathbf{m}_n, \varkappa_p n^{-1/4} d_n, (\partial_n, \tilde{\partial}_n)) \xrightarrow{\text{a.s.}} (\mathbf{m}_\infty, D, (\varrho, \mathbf{p}(U))), \quad \text{as } n \to \infty, \tag{37}$$

which holds in the space $(\mathbb{K}^{(2)}, d_{\mathrm{GH}})$. This does not immediately give the desired result, because $\tilde{\partial}_n$ is not uniformly distributed over $\mathbf{m}_n$. However, we may set

$$N_n = \lfloor ((p-1)n+2)U \rfloor$$

and let V_n denote the N_n-th vertex of τ_n° in lexicographical order, with the convention that $V_n = \partial_n$ if $N_n = 0$. Then, V_n has the property stated in the proposition, and furthermore

$$n^{-1/4} d_n(V_n, \tilde{\partial}_n) \xrightarrow{\text{a.s.}} 0, \quad \text{as } n \to \infty.$$

See the proof of [24, Proposition 3.5] for a justification of this convergence. The result of the proposition now follows from (37). □

As was mentioned above, Theorem 8.1 allows one to get information about geodesics starting from a point chosen uniformly in $\mathbf{m}_\infty$ rather than from the root ϱ. We illustrate this in the following corollary. If $x, x' \in \mathbf{m}_\infty$, we denote by $\mathrm{Geo}(x \to x')$ the set of all geodesics from x to x'.

COROLLARY 8.3. *The following properties hold almost surely*:

(i) $\lambda \otimes \lambda\{(x,x') : \#\,\mathrm{Geo}(x \to x') > 1\} = 0$;

(ii) $\lambda(\{x \in \mathbf{m}_\infty : \textit{there exists } x' \in \mathbf{m}_\infty \textit{ such that } \#\,\mathrm{Geo}(x \to x') \geqslant 4\}) = 0$;

(iii) $\lambda(\{x \in \mathbf{m}_\infty : \textit{there exists } x' \in \mathbf{m}_\infty \textit{ such that } \#\,\mathrm{Geo}(x \to x') = 3\}) = 1$.

Remark. Property (i) of the corollary should be compared to [31, Theorem 3].

Proof. We start by proving (i). Let U and U' be two independent random variables which are uniformly distributed over $[0,1]$, and such that the pair (U, U') is independent of the sequence $\{M_n\}_{n\geqslant 1}$. The statement in (i) is equivalent to proving that

$$\#\,\mathrm{Geo}(\mathbf{p}(U) \to \mathbf{p}(U')) = 1 \quad \text{a.s.}$$

However, as a simple consequence of Theorem 8.1, we have

$$(\mathbf{m}_\infty, D, (\varrho, \mathbf{p}(U))) \overset{(\mathrm{d})}{=} (\mathbf{m}_\infty, D, (\mathbf{p}(U), \mathbf{p}(U'))),$$

where both sides of the equality are random variables with values in $\mathbb{K}^{(2)}$. Note that the set of all 2-pointed spaces $(E, d, (x, x'))$ such that there exists a unique geodesic from x to x' is a measurable subset of $\mathbb{K}^{(2)}$. Indeed, if $\delta>0$ is fixed, the set of all $(E, d, (x, x'))$ such that there exist two geodesics γ and γ' from x to x' with $\max\{d(\gamma(t), \gamma'(t)\} \geqslant \delta$ is closed in $\mathbb{K}^{(2)}$. Hence, using the previous identity in distribution, we see that property (i) reduces to the fact that $\#\,\mathrm{Geo}(\varrho \to \mathbf{p}(U)) = 1$ a.s. But this follows from Corollary 7.5.

The arguments for (ii) and (iii) are similar. We use the identity in distribution

$$(\mathbf{m}_\infty, D, \varrho) \overset{(\mathrm{d})}{=} (\mathbf{m}_\infty, D, \mathbf{p}(U)),$$

and the fact that, for every integer $k \geqslant 2$, the set of all pointed spaces (E, d, x) such that there exist a point $x' \in E$ and at least k distinct geodesics from x to x' is a measurable subset of $\mathbb{K}^{(1)}$. The details are left to the reader. □

9. Asymptotics for large planar maps

In this section we discuss the applications of our results to large planar maps, and in particular we prove Propositions 1.1, 1.2 and 1.3 which were stated in §1. Note that, in contrast with the previous sections, we do not limit ourselves to values of n belonging to a suitable sequence converging to ∞.

Proof of Proposition 1.1. We prove the stronger form of Proposition 1.1, where $\mathrm{Geo}_n(\partial_n, a)$ is replaced by $\overline{\mathrm{Geo}}_n(\partial_n, a)$. Let us fix $\delta>0$ and, for every $n \geqslant 1$, let V_n be a vertex that is uniformly distributed over $\mathbf{m}_n$. The desired result is equivalent to saying that

$$\lim_{n\to\infty} P(\text{there exist } \gamma, \gamma' \in \overline{\mathrm{Geo}}_n(\partial_n, V_n) \text{ such that } d(\gamma, \gamma') \geqslant \delta n^{1/4}) = 0.$$

We argue by contradiction and suppose that there exist $\eta>0$ and a subsequence $\bar{n}_k \uparrow \infty$ such that, for every k,

$$P(\text{there exist } \gamma, \gamma' \in \overline{\mathrm{Geo}}_{\bar{n}_k}(\partial_{\bar{n}_k}, V_{\bar{n}_k}) \text{ such that } d(\gamma, \gamma') \geqslant \delta \bar{n}_k^{1/4}) \geqslant \eta.$$

From this sequence $\{\bar{n}_k\}_{k\geqslant 1}$ we can now extract another subsequence $\{n_k\}_{k\geqslant 1}$ such that the properties stated in Theorem 2.2 hold. By Proposition 8.2, we may assume that the random vertices V_{n_k} are constructed in such a way that

$$(\mathbf{m}_{n_k}, \varkappa_p n_k^{-1/4} d_{n_k}, (\partial_{n_k}, V_{n_k})) \xrightarrow{\text{a.s.}} (\mathbf{m}_\infty, D, (\varrho, \mathbf{p}(U))), \quad \text{as } k \to \infty, \tag{38}$$

where U is uniformly distributed over $[0,1]$ and independent of the sequence $\{M_n\}_{n\geqslant 1}$. Consider then the event

$$F=\limsup_{k\to\infty}\{\text{there exist } \gamma,\gamma'\in\overline{\mathrm{Geo}}_{n_k}(\partial_{n_k},V_{n_k}) \text{ such that } d(\gamma,\gamma')\geqslant\delta n_k^{1/4}\}.$$

By our assumptions, $P(F)\geqslant\eta>0$.

From now on, we argue on the event F. We can then find a (random) subsequence $\{m_k\}_{k\geqslant 1}$ of $\{n_k\}_{k\geqslant 1}$ such that, for every k, there exist $\gamma_k,\gamma'_k\in\overline{\mathrm{Geo}}_{m_k}(\partial_{m_k},V_{m_k})$ such that $d(\gamma_k,\gamma'_k)\geqslant\delta m_k^{1/4}$. On the other hand, by (38), we can find, for every k, a correspondence $\mathcal{C}_k$ between the 2-pointed spaces $(\mathbf{m}_{m_k},\varkappa_p m_k^{-1/4}d_{m_k},(\partial_{m_k},V_{m_k}))$ and $(\mathbf{m}_\infty,D,(\varrho,\mathbf{p}(U)))$, in such a way that $\mathrm{dist}(\mathcal{C}_k)$ tends to 0 as $k\to\infty$. This implies in particular that $\varkappa_p m_k^{-1/4}d_{m_k}(\partial_{m_k},V_{m_k})$ converges to $D(\varrho,\mathbf{p}(U))=\overline{Z}_U$. For every rational $r\in[0,\overline{Z}_U[$, and for k sufficiently large, we can find $x_k(r),x'_k(r)\in\mathbf{m}_\infty$ such that $(\gamma_k(\lfloor\varkappa_p^{-1}m_k^{1/4}r\rfloor),x_k(r))\in\mathcal{C}_k$ and $(\gamma'_k(\lfloor\varkappa_p^{-1}m_k^{1/4}r\rfloor),x'_k(r))\in\mathcal{C}_k$. Via a compactness argument and extracting a diagonal subsequence, we may assume that $x_k(r)$ (resp. $x'_k(r)$) converges as $k\to\infty$, for every rational $r\in[0,\overline{Z}_U[$, and we denote the limit by $\gamma_\infty(r)$ (resp. $\gamma'_\infty(r)$). The fact that each γ_k is an approximate geodesic, together with the property $\mathrm{dist}(\mathcal{C}_k)\to 0$, implies that $D(\gamma_\infty(r),\gamma_\infty(r'))=|r-r'|$ for all rationals $r,r'\in[0,\overline{Z}_U[$. It easily follows that γ_∞ can be extended to a geodesic from ϱ to $\mathbf{p}(U)$ in $\mathbf{m}_\infty$. Similarly, we can extend γ'_∞ to a geodesic from ϱ to $\mathbf{p}(U)$ in $\mathbf{m}_\infty$. However,

$$\max_{0\leqslant r\leqslant\overline{Z}_U} D(\gamma_\infty(r),\gamma'_\infty(r))\geqslant\liminf_{k\to\infty}\varkappa_p m_k^{-1/4}d(\gamma_k,\gamma'_k)\geqslant\varkappa_p\delta>0,$$

and so we find that there are two distinct geodesics from ϱ to $\mathbf{p}(U)$ in $\mathbf{m}_\infty$. This holds on the event F, which has positive probability, thus contradicting Corollary 7.5. □

Proof of Proposition 1.2. This is very similar to the proof of Proposition 1.1. We argue by contradiction and assume that there is an increasing sequence $\{\bar{n}_k\}_{k\geqslant 1}$ such that, for every $k\geqslant 1$,

$$P(\text{there exists } a\in\mathbf{m}_{\bar{n}_k} \text{ such that } \overline{\mathrm{Mult}}_{\bar{n}_k,\delta}(\partial_{\bar{n}_k},a)\geqslant 4)\geqslant\eta$$

for some $\eta>0$. We can then extract a subsequence $\{n_k\}_{k\geqslant 1}$ and construct the random maps M_{n_k} in such a way that we have the almost sure convergence

$$(\mathbf{m}_{n_k},\varkappa_p n_k^{-1/4}d_{n_k},\partial_{n_k})\xrightarrow{\text{a.s.}}(\mathbf{m}_\infty,D,\varrho),\quad\text{as } k\to\infty,\tag{39}$$

in $\mathbb{K}^{(1)}$. The event

$$F=\limsup_{k\to\infty}\{\text{there exists } a\in\mathbf{m}_{n_k} \text{ such that } \overline{\mathrm{Mult}}_{n_k,\delta}(\partial_{n_k},a)\geqslant 4\}$$

has probability bounded below by η. Furthermore, on the event F, we can use the convergence (39) and compactness arguments to construct a point x of $\mathbf{m}_\infty$ which is connected to ϱ by at least four distinct geodesics, thus contradicting Theorem 1.4. We leave the details to the reader. □

Remark. As we already mentioned in the introduction, the statements of Propositions 1.1 and 1.2 remain valid if ∂_n is replaced by a vertex chosen uniformly at random in $\mathbf{m}_n$. Let us outline the argument in the case of Proposition 1.1. We consider two random vertices V_n and V_n' that are distributed independently uniformly over the vertex set of $\mathbf{m}_n$. From the argument in the proof of Proposition 8.2, we can construct V_n and V_n', and the random maps M_n, in such a way that along a suitable sequence $\{n_k\}_{k\geqslant 1}$ we have the almost sure convergence

$$(\mathbf{m}_n, \varkappa_p n^{-1/4} d_n, (V_n, V_n')) \xrightarrow{\text{a.s.}} (\mathbf{m}_\infty, D, (\mathbf{p}(U), \mathbf{p}(U'))), \quad \text{as } n\to\infty, \tag{40}$$

where U and U' are independent and uniformly distributed over $[0,1]$, and (U,U') is independent of the sequence $\{M_n\}_{n\geqslant 1}$. Then a simple adaptation of the proof of Proposition 1.1, using Corollary 8.3 (i), shows that $P(\overline{\mathrm{Mult}}_{n,\delta}(V_n, V_n')\geqslant 2)$ tends to 0, as $n\to\infty$, along the sequence $\{n_k\}_{k\geqslant 1}$. Since from any sequence $\{n_k'\}_{k\geqslant 1}$ we can extract a subsequence $\{n_k\}_{k\geqslant 1}$ such that the previous conclusion holds, the desired result follows.

Proof of Proposition 1.3. Recall our notation $v_0^n, v_1^n, ..., v_{pn}^n$ for the contour sequence of τ_n°. Let $\alpha>0$. We denote by $\mathcal{H}_n(\alpha)$ the set of all triplets (j_1,j_2,j_3) of integers such that $0\leqslant j_1\leqslant j_2\leqslant j_3\leqslant pn$ and the following properties hold:

- $v_{j_1}^n = v_{j_2}^n = v_{j_3}^n$;
- $\max\{\min_{j_1\leqslant j\leqslant j_2}\Lambda_j^n, \min_{j_2\leqslant j\leqslant j_3}\Lambda_j^n\}\leqslant \Lambda_{j_1}^n - \alpha n^{1/4}$.

Suppose that $(j_1,j_2,j_3)\in\mathcal{H}_n(\alpha)$. Then we have $C_{j_1}^n = C_{j_2}^n = C_{j_3}^n$, $\Lambda_{j_1}^n = \Lambda_{j_2}^n = \Lambda_{j_3}^n$ and $C_j^n \geqslant C_{j_1}^n$ if $j_1\leqslant j\leqslant j_3$. From the properties of the spatial contour function, it is also clear that Λ^n takes all integer values between $\Lambda_{j_1}^n$ and $\Lambda_{j_1}^n - \alpha n^{1/4}$ on each of the intervals $[j_1,j_2]$, $[j_2,j_3]$ and $[j_3,pn]$.

We denote the event $\{\mathcal{H}_n(\alpha)\neq\varnothing\}$ by $\Delta_n(\alpha)$.

Lemma 9.1. *We have*

$$\lim_{\alpha\to 0}\liminf_{n\to\infty} P(\Delta_n(\alpha)) = 1.$$

We postpone the proof of Lemma 9.1 to the appendix.

Let $\delta>0$. For $i=1,2$, we say that the event $\Delta_n^{(i)}(\alpha,\delta)$ holds if, for every triplet $(j_1,j_2,j_3)\in\mathcal{H}_n(\alpha)$ and for every $j\in[j_i,j_{i+1}]$ such that

$$\Lambda_j^n = \min_{j_i\leqslant \ell\leqslant j}\Lambda_\ell^n \quad \text{and} \quad \Lambda_{j_1}^n - \tfrac{2}{3}\alpha n^{1/4} \leqslant \Lambda_j^n \leqslant \Lambda_{j_1}^n - \tfrac{1}{3}\alpha n^{1/4}$$

we have

$$d_n(j,j') \geqslant \delta n^{1/4} \quad \text{for every } j' \in \{j_{i+1},...,pn\}. \tag{41}$$

We claim that, for every $\alpha>0$,

$$\sup_{n\geqslant 1} P(\Delta_n(\alpha)\setminus(\Delta_n^{(1)}(\alpha,\delta)\cap\Delta_n^{(2)}(\alpha,\delta))) \to 0, \quad \text{as } \delta\to 0. \tag{42}$$

Assuming the claim, it is easy to complete the proof of Proposition 1.3. Let $\eta>0$. Using Lemma 9.1 and (42), we can choose first $\alpha>0$, and then $\delta>0$ in such a way that

$$\liminf_{n\to\infty} P(\Delta_n(\alpha)\cap(\Delta_n^{(1)}(\alpha,\delta)\cap\Delta_n^{(2)}(\alpha,\delta))) > 1-\eta.$$

However, on the event $\Delta_n(\alpha)\cap(\Delta_n^{(1)}(\alpha,\delta)\cap\Delta_n^{(2)}(\alpha,\delta))$, there is at least one point $a\in\mathbf{m}_n$ such that $\mathrm{Mult}_{n,\delta}(\partial_n,a)\geqslant 3$. Indeed we pick a triplet $(j_1,j_2,j_3)\in\mathcal{H}_n(\alpha)$ and set

$$a = v^n_{j_1} = v^n_{j_2} = v^n_{j_3}.$$

We consider the "simple geodesics" γ_1, γ_2 and γ_3 defined by

$$\begin{aligned}\gamma_i(\ell) &= v^n_{m_i(\ell)}, \quad \text{where } m_i(\ell) = \inf\{j\geqslant j_i : \Lambda^n_j = \Lambda^n_{j_1} - \ell\} \text{ for } 0\leqslant\ell\leqslant\Lambda^n_{j_1}-1,\\ \gamma_i(\Lambda^n_{j_1}) &= \partial_n,\end{aligned}$$

for $i=1,2,3$. It is then easy to verify that γ_1, γ_2 and γ_3 are three geodesics from a to ∂_n in the map M_n, and the property (41) guarantees that $d(\gamma_i,\gamma_{i'})\geqslant\delta n^{1/4}$ for every $i,i'\in\{1,2,3\}$ with $i\neq i'$.

We still have to prove our claim (42). We verify that

$$\sup_{n\geqslant 1} P(\Delta_n(\alpha)\setminus\Delta_n^{(1)}(\alpha,\delta)) \to 0, \quad \text{as } \delta\to 0, \tag{43}$$

and the same argument applies if $\Delta_n^{(1)}(\alpha,\delta)$ is replaced by $\Delta_n^{(2)}(\alpha,\delta)$. We argue by contradiction and assume that (43) does not hold. Then we can find $\varepsilon>0$ such that, for every $k\geqslant 1$,

$$\sup_{n\geqslant 1} P\left(\Delta_n(\alpha)\setminus\Delta_n^{(1)}\left(\alpha,\frac{1}{k}\right)\right) > \varepsilon.$$

Hence, for every $k\geqslant 1$, we can find a positive integer $\bar{n}_k$ such that

$$P\left(\Delta_{\bar{n}_k}(\alpha)\setminus\Delta_{\bar{n}_k}^{(1)}\left(\alpha,\frac{1}{k}\right)\right) > \varepsilon.$$

The sequence $\{\bar{n}_k\}_{k\geqslant 1}$ must converge to ∞ (for any fixed n, the condition (41) holds automatically if δ is small enough). So from this sequence, we can extract a monotone

increasing subsequence $\{n_k\}_{k\geqslant 1}$ along which the almost sure convergence (3) holds for a suitable choice of the random maps M_{n_k}. Then we have

$$P\left(\limsup_{k\to\infty}\left(\Delta_{n_k}(\alpha)\setminus\Delta_{n_k}^{(1)}\left(\alpha,\frac{1}{k}\right)\right)\right)\geqslant\varepsilon.$$

Furthermore, on the event

$$\limsup_{k\to\infty}\left(\Delta_{n_k}(\alpha)\setminus\Delta_{n_k}^{(1)}\left(\alpha,\frac{1}{k}\right)\right),$$

we can find a (random) sequence of values of k along which the following holds: There exists a triplet $(j_1^k,j_2^k,j_3^k)\in\mathcal{H}_{n_k}(\alpha)$ and two integers $m_k\in[j_1^k,j_2^k]$ and $m_k'\in[j_2^k,pn_k]$ such that

$$\Lambda_{m_k}^{n_k}=\min_{j_1^k\leqslant\ell\leqslant m_k}\Lambda_\ell^{n_k},\quad \Lambda_{j_1^k}^{n_k}-\tfrac{2}{3}\alpha n_k^{1/4}\leqslant\Lambda_{m_k}^{n_k}\leqslant\Lambda_{j_1^k}^{n_k}-\tfrac{1}{3}\alpha n_k^{1/4}$$

and

$$d_{n_k}(m_k,m_k')\leqslant\frac{1}{k}n_k^{1/4}. \tag{44}$$

By a compactness argument, we may assume that

$$\frac{j_1^k}{pn_k}\to s,\quad \frac{j_2^k}{pn_k}\to t,\quad \frac{m_k}{pn_k}\to r\quad\text{and}\quad \frac{m_k'}{pn_k}\to r',$$

as $k\to\infty$. We have then $s\leqslant r\leqslant t\leqslant r'$, $s\sim t$ (and so in particular $\overline{Z}_s=\overline{Z}_t$) and $D(r,r')=0$ from (3) and (44). Furthermore, the convergence (3) also implies that

$$\inf_{s\leqslant u\leqslant t}\overline{Z}_u\leqslant\overline{Z}_s-\alpha\varkappa_p,$$

and that

$$\overline{Z}_r=\inf_{s\leqslant u\leqslant r}\overline{Z}_u\quad\text{and}\quad \overline{Z}_s-\tfrac{2}{3}\alpha\varkappa_p\leqslant\overline{Z}_r\leqslant\overline{Z}_s-\tfrac{1}{3}\alpha\varkappa_p.$$

From these properties, it follows that neither of the conditions $r\sim r'$ or $r\approx r'$ can hold. This contradicts the equality $D(r,r')=0$, and this contradiction completes the proof. □

Remark. In the same way as Propositions 1.1 and 1.2, Proposition 1.3 still holds if the root vertex ∂_n is replaced by a vertex chosen uniformly at random in $\mathbf{m}_n$. A way to obtain this result is to use the variant of the Bouttier–Di Francesco–Guitter bijection that is presented in [27, §2.3]. We omit the details of the argument.

We conclude this section with a discrete version of Corollary 7.7.

PROPOSITION 9.2. *Let $\chi>0$ and $\varepsilon>0$, and let Geo_n^χ denote the set of all discrete geodesics starting from the root ∂_n and with length greater than $\chi n^{1/4}$ in the map M_n. Then we can choose a constant $\beta\in]0,\chi[$ so that, for every $\delta>0$,*

$$\liminf_{n\to\infty} P\Big(\sup_{\gamma,\gamma'\in\mathrm{Geo}_n^\chi}\ \sup_{0\leqslant i\leqslant\lfloor\beta n^{1/4}\rfloor} d_n(\gamma(i),\gamma'(i))\leqslant\delta n^{1/4}\Big)\geqslant 1-\varepsilon.$$

Proposition 9.2 is derived from Corollary 7.7 and the Gromov–Hausdorff convergence in Theorem 2.2 in a way very similar to the above proof of Proposition 1.1. To be specific, if $\alpha>0$ is the random variable appearing in the statement of Corollary 7.7 when $\eta=\varkappa_p\chi$, we choose the constant $\beta>0$ so that

$$P(\varkappa_p^{-1}\alpha>\beta)>1-\varepsilon.$$

From the proof of Corollary 7.7, we see that the choice of β is determined from the values of χ and ε and the distribution of the process $\overline{Z}$, so that it does not depend on the particular subsequence used in Theorem 2.2. The remaining part of the argument is straightforward.

The recent article [9] gives explicit calculations related to the confluence property of discrete geodesics, in the particular case of quadrangulations.

Appendix

In this appendix, we prove Lemmas 5.5 and 9.1. As previously, the integer $p\geqslant 2$ is fixed, and all constants that will appear depend on p, even though this will not be mentioned explicitly.

Proof of Lemma 5.5. The proof depends on an intermediate estimate on the distribution of labels in a labeled p-tree. We say that a pair $(\tau,(\ell_v)_{v\in\tau^\circ})$ consisting of a p-tree τ and a collection of integer labels assigned to the white vertices of τ is a *labeled p-tree* if it satisfies property (b) in the definition of a p-mobile (cf. §2.6). We say that it is a *standard labeled p-tree* if in addition $\ell_\varnothing=0$. In a labeled p-tree, labels may take negative values.

Let $(\tau,(\ell_v)_{v\in\tau^\circ})$ be a standard labeled p-tree, and let $v\in\tau^\bullet$. Let $v_{(0)},v_{(1)},\dots,v_{(p-1)}$ be the neighboring vertices of v listed as in property (b) of the definition of a p-mobile. Simple combinatorics shows that given $\ell_{v_{(0)}}$ there are $\binom{2p-1}{p-1}$ possible choices for $\ell_{v_{(1)}},\dots,\ell_{v_{(p-1)}}$. As an immediate consequence, given a p-tree τ with n black vertices, there are $\binom{2p-1}{p-1}^n$ choices for the labels ℓ_v, $v\in\tau^\circ$, that produce a standard labeled p-tree.

We denote by Π_n^p the uniform distribution over the set of all standard labeled p-trees with n black vertices.

LEMMA A.1. *For every integer $k\geqslant 1$, there exists a constant C_k such that, for every $n\geqslant 1$ and every $x>0$,*

$$\Pi_n^p\Big(\max_{v\in\tau^\circ}|\ell_v|\geqslant xn^{1/4}\Big)\leqslant\frac{C_k}{x^k}. \tag{45}$$

Consequently, for every $\varepsilon>0$, there exists a constant $C_{(\varepsilon)}$ such that, for every $n\geqslant 1$ and every $x\geqslant\varepsilon$,

$$\Pi_n^p\Big(\min_{v\in\tau^\circ}\ell_v\leqslant -xn^{1/4}\Big)\leqslant 1-\exp(-C_{(\varepsilon)}x^{-4}).$$

Remark. When $p=2$, the bound (45) follows from the much stronger exponential estimate in [13, Proposition 4]. It is very plausible that such an exponential estimate holds for general p, but we content ourselves with the weaker bound (45). This bound is sufficient to get the second assertion of the lemma, which is what we really need.

Proof. We write $(\tilde\tau_n,(\tilde\ell_v^n)_{v\in\tilde\tau_n^\circ})$ for a random labeled p-tree which is uniformly distributed over the set of all standard labeled p-trees with n black vertices. From the remarks preceding the lemma, it is clear that $\tilde\tau_n$ is uniformly distributed over the set of all p-trees with n black vertices. For our purposes, it will be convenient to view $\tilde\tau_n$ as a conditioned multitype Galton–Watson tree. To this end, consider the random p-tree τ_* whose distribution is informally specified as follows:

(i) for every $v\in\tau_*^\bullet$, $k_v(\tau_*)=p-1$;

(ii) for every $v\in\tau_*^\circ$, $k_v(\tau_*)$ is geometrically distributed with parameter p^{-1}, that is

$$P(k_v(\tau_*)=k\,|\,v\in\tau_*)=(1-p^{-1})p^{-k},\quad k\geqslant 0.$$

Then $\tilde\tau_n$ has the same distribution as τ_* conditioned on the event $\#\tau_*^\bullet=n$, or equivalently on the event $\#\tau_*^\circ=(p-1)n+1$.

Now note that τ_*° can be viewed as a random plane tree, by declaring that two vertices v and v' of τ_*° are linked by an edge if v is the grandparent of v' or conversely (and of course keeping the root and the order structure). With this interpretation, τ_*° is a Galton–Watson tree, whose offspring distribution is the law of $p-1$ times a geometric random variable with parameter p^{-1}.

Recall that $\tilde\tau_n^\circ$ has the same distribution as τ_*° conditioned on the event

$$\#\tau_*^\circ=(p-1)n+1.$$

We denote by $\tilde v_0^n,\dots,\tilde v_{2(p-1)n}^n$ the depth-first search sequence of $\tilde\tau_n^\circ$ viewed as a plane tree, and we set

$$\mathcal{C}_i^n=\tfrac{1}{2}|\tilde v_i^n|$$

for every $i\in\{0,1,...,2(p-1)n\}$. An estimate of Gittenberger [17] gives two positive constants C_0 and C_1, which do not depend on n, such that, for every $i,j\in\{0,1,...,2(p-1)n\}$ and every $x>0$,

$$P\Big(\mathcal{C}_i^n+\mathcal{C}_j^n-2\min_{i\wedge j\leqslant q\leqslant i\vee j}\mathcal{C}_q^n\geqslant x\Big)\leqslant\frac{C_0n}{|i-j|}\exp\Big(-\frac{C_1x}{\sqrt{|i-j|}}\Big). \tag{46}$$

Let $N>4$ be an integer. As an immediate consequence of the bound (46), we get

$$E\Big[\Big(n^{-1/2}\Big(\mathcal{C}_i^n+\mathcal{C}_j^n-2\min_{i\wedge j\leqslant q\leqslant i\vee j}\mathcal{C}_q^n\Big)\Big)^N\Big]\leqslant C'_N\Big(\frac{|i-j|}{n}\Big)^{N/2-1}, \tag{47}$$

with a constant C'_N that does not depend on n.

Fix $i,j\in\{0,1,...,2(p-1)n\}$ such that $i<j$, and let $m\in\{i,i+1,...,j\}$ be such that $\tilde v_m^n$ is the most recent common ancestor to $\tilde v_i^n$ and $\tilde v_j^n$ in the tree $\tilde\tau_n^\circ$. Then, conditionally on $\tilde\tau_n$, $\tilde\ell^n_{\tilde v_i^n}-\tilde\ell^n_{\tilde v_m^n}$ is distributed as the sum of $\frac12(|\tilde v_i^n|-|\tilde v_m^n|)$ integer-valued independent random variables, which are centered and bounded by p in absolute value. A similar property holds for $\tilde\ell^n_{\tilde v_j^n}-\tilde\ell^n_{\tilde v_m^n}$. Noting that

$$\tfrac12(|\tilde v_i^n|+|\tilde v_j^n|-2|\tilde v_m^n|)=\mathcal{C}_i^n+\mathcal{C}_j^n-2\min_{i\wedge j\leqslant q\leqslant i\vee j}\mathcal{C}_q^n,$$

we can use standard arguments to derive the bound

$$E[|\tilde\ell^n_{\tilde v_i^n}-\tilde\ell^n_{\tilde v_j^n}|^{2N}\,|\,\tilde\tau_n]\leqslant C''_N\Big(\mathcal{C}_i^n+\mathcal{C}_j^n-2\min_{i\wedge j\leqslant q\leqslant i\vee j}\mathcal{C}_q^n\Big)^N, \tag{48}$$

where the constant C''_N only depends on N.

Set $\mathcal{V}_i^n=\ell^n_{\tilde v_i^n}$ for $i\in\{0,1,...,2(p-1)n\}$ and extend $\mathcal{V}^n$ to $[0,2(p-1)n]$ by interpolating linearly between integers. From (47) and (48), we have, for $s,t\in\{0,1,...,2(p-1)n\}$,

$$E\Big[\Big|\frac{\mathcal{V}_s^n-\mathcal{V}_t^n}{n^{1/4}}\Big|^{2N}\Big]\leqslant\overline{C}_N\Big(\frac{|s-t|}{n}\Big)^{N/2-1}, \tag{49}$$

where $\overline{C}_N=C'_NC''_N$. This bound remains valid for all reals $s,t\in[0,2(p-1)n]$, with a possibly different constant $\overline{C}_N$ that still depends only on N.

Set $\widehat{\mathcal{V}}_s^n=n^{-1/4}\mathcal{V}_{ns}^n$ for every $s\in[0,2(p-1)]$. From (49), for all $s,t\in[0,2(p-1)]$ we have

$$E[|\widehat{\mathcal{V}}_s^n-\widehat{\mathcal{V}}_t^n|^{2N}]\leqslant\overline{C}_N|s-t|^{N/2-1}.$$

Since $\frac12N-1>1$, an application of the Kolmogorov criterion (see e.g. [35, Theorem I.2.1]) gives a constant $\widetilde{C}_N$, which does not depend on n, such that

$$E\Big[\sup_{0\leqslant s\leqslant2(p-1)}|\widehat{\mathcal{V}}_s^n|^{2N}\Big]\leqslant\widetilde{C}_N.$$

The bound (45) follows.

To get the second assertion, first note that for every $\varepsilon>0$ there exists a constant $C'_{(\varepsilon)}>0$ such that, for every integer $n\geqslant 1$,

$$1-\Pi_n^p\Big(\min_{v\in\tau^\circ}\ell_v\leqslant-\varepsilon n^{1/4}\Big)\geqslant C'_{(\varepsilon)}. \tag{50}$$

Indeed, it follows from [27, Theorem 11] that

$$\limsup_{n\to\infty}\Pi_n^p\Big(\min_{v\in\tau^\circ}\ell_v\leqslant-\varepsilon n^{1/4}\Big)\leqslant P\Big(\min_{0\leqslant s\leqslant 1}Z^{\mathbf{e}}_s\leqslant-\varkappa_p\varepsilon\Big),$$

and the limit is strictly less than 1, as can be seen for instance from the invariance of the continuum random tree under uniform re-rooting. Using (50) and the first assertion of the lemma with $k=4$, we get for every $x\geqslant\varepsilon$,

$$\Pi_n^p\Big(\min_{v\in\tau^\circ}\ell_v\leqslant-xn^{1/4}\Big)\leqslant(1-C'_{(\varepsilon)})\wedge C_4x^{-4}\leqslant 1-\exp(-C_{(\varepsilon)}x^{-4})$$

for some other constant $C_{(\varepsilon)}$. This completes the proof of the lemma. □

We now turn to the proof of Lemma 5.5. We will in fact prove a dual version of this lemma. Precisely, we will show that, for every choice of $U\in]0,1[$, $u_0\in]U,1[$, $\delta\in\big]0,\frac{1}{2}(1-u_0)[$ and $\eta,\eta'>0$, there exists a constant $K=K(U,u_0,\eta,\eta',\delta)$ such that the following holds. Whenever $u\in]U,u_0[$ and $\alpha>0$, we have, for all sufficiently large n,

$$P\Big(\inf_{\lfloor pnu\rfloor\leqslant i\leqslant\lfloor pn(u+\delta)\rfloor}C_i^n\geqslant C^n_{\lfloor pnu\rfloor}-\alpha n^{1/2}\,\Big|\,\mathcal{G}^n(0,\lfloor pnu\rfloor)\Big)\leqslant K\alpha \tag{51}$$

on the event

$$\Big\{C^n_{\lfloor pnu\rfloor}\geqslant\eta n^{1/2},\inf_{\eta n^{1/2}/2\leqslant j\leqslant C^n_{\lfloor pnu\rfloor}}W^n_{\lfloor pnu\rfloor}(j)\geqslant\eta' n^{1/4}\Big\}. \tag{52}$$

Strictly speaking, this dual version is not equivalent to Lemma 5.5, because of the lack of symmetry in property (b) of the definition of a p-mobile. However, the reader will easily check that the same arguments we will use to prove the dual version also apply to the statement of Lemma 5.5, at the cost of a somewhat heavier notation.

In proving the bound (51), we may and will assume that $\alpha\leqslant\eta/4p$. Let us fix $u\in]U,u_0[$, and to simplify notation set $N=C^n_{\lfloor pnu\rfloor}$. Recall our notation $v_0^n,v_1^n,\dots,v_{pn}^n$ for the contour sequence of τ_n°, and let

$$w_0^n=\varnothing,\quad w_1^n,\quad w_2^n,\quad\dots,\quad w_N^n=v^n_{\lfloor pnu\rfloor}$$

be the white vertices of τ_n that are ancestors of $v^n_{\lfloor pnu\rfloor}$, listed in such a way that $|w_i^n|=2i$ for $0\leqslant i\leqslant N$. Also let $\widetilde{w}_1^n,\widetilde{w}_2^n,\dots,\widetilde{w}_N^n$ be the black vertices that are ancestors of $v^n_{\lfloor pnu\rfloor}$, listed in such a way that $|\widetilde{w}_i^n|=2i-1$ for $1\leqslant i\leqslant N$.

Consider now all vertices $x\in\tau_n^\circ$ that are of one of the following two types:

- either $x\in\{w_0^n, w_1^n, ..., w_N^n\}$, or
- there exists $i\in\{1, ..., N\}$ such that x is a child of $\widetilde{w}_i^n$ that has not yet appeared in the sequence $v_0^n, v_1^n, ..., v_{\lfloor pnu\rfloor}^n$.

Let $x_0^n, x_1^n, ..., x_M^n$ be the sequence consisting of all such vertices x listed in such a way that $|x_i^n|\leqslant|x_{i'}^n|$ if $i\leqslant i'$, and in lexicographical order for vertices of the same generation. Notice that $N\leqslant M\leqslant(p-1)N$.

It is easily verified that the quantities N, M and the random vertices $x_0^n, x_1^n, ..., x_M^n$ are measurable with respect to the σ-field $\mathcal{G}^n(0, \lfloor pnu\rfloor)$. This measurability property does not hold for the labels $\ell_{x_i^n}^n$, $0\leqslant i\leqslant M$ (unless $p=2$). So we let $\widehat{\mathcal{G}}^n(0, \lfloor pnu\rfloor)$ be the σ-field generated by $\mathcal{G}^n(0, \lfloor pnu\rfloor)$ and the labels $\ell_{x_i^n}^n$, $0\leqslant i\leqslant M$, and we will prove the stronger form of (51) where the σ-field $\mathcal{G}^n(0, \lfloor pnu\rfloor)$ is replaced by $\widehat{\mathcal{G}}^n(0, \lfloor pnu\rfloor)$.

Let k and m be two positive integers, with $k\leqslant m\leqslant(p-1)k$, and let $\beta_0, \beta_1, ..., \beta_{\lfloor pnu\rfloor}$, $\nu_0, \nu_1, ..., \nu_{\lfloor pnu\rfloor}$ and $l_0, l_1, ..., l_m$ be non-negative integers such that

$$P(N=k \text{ and } M=m; C_j^n=\beta_j \text{ and } \Lambda_j^n=\nu_j \text{ for all } j\in\{0, ..., \lfloor pnu\rfloor\}; \ell_{x_i^n}^n=l_i \text{ for all } i\in\{0, ..., m\})>0.$$

Recalling the event considered in (52), we also assume that

$$k\geqslant\eta n^{1/2} \quad \text{and} \quad l_i\geqslant\tfrac{1}{2}\eta' n^{1/4} \text{ for } i\in\big\{m-\big\lfloor\tfrac{1}{2}\eta n^{1/2}\big\rfloor, ..., m\big\}.$$

(To justify this last assumption, note that the label of a child of $\widetilde{w}_i^n$ can differ by at most p from the label of $\widetilde{w}_i^n$, for any $i\in\{1, ..., N\}$.) Consider then the conditional probability

$$P^{*,n}=P(\cdot\,|\,N=k \text{ and } M=m; C_j^n=\beta_j \text{ and } \Lambda_j^n=\nu_j \text{ for all } j\in\{0, ..., \lfloor pnu\rfloor\}; \ell_{x_i^n}^n=l_i \text{ for all } i\in\{0, ..., m\}).$$

In order to get (51), we need to verify the bound

$$P^{*,n}(B_n(u, \alpha, \delta))\leqslant K\alpha, \tag{53}$$

where

$$B_n(u, \alpha, \delta)=\Big\{\inf_{\lfloor pnu\rfloor\leqslant i\leqslant\lfloor pn(u+\delta)\rfloor} C_i^n\geqslant C_{\lfloor pnu\rfloor}^n-\alpha n^{1/2}\Big\},$$

and the constant K depends only on U, u_0, η, η' and δ.

From now on, we argue under the conditional probability $P^{*,n}$. For $i\in\{0, 1, ..., m\}$, we define a labeled p-tree $(\tau_{n,(i)}, (\ell_v^{n,(i)})_{v\in\tau_{n,(i)}^\circ})$ in the following way:

- if $x_i^n\notin\{w_0^n, w_1^n, ..., w_k^n\}$, then $\tau_{n,(i)}$ is just the subtree of descendants of x_i^n, i.e.

$$\tau_{n,(i)}=\{v\in\mathcal{U}: x_i^n v\in\tau_n\},$$

where $x_i^n v$ stands for the concatenation of the words x_i^n and v;

- if $x_i^n = w_j^n$ for some $j \in \{0,1,...,k\}$, then $\tau_{n,(i)}$ consists of $\varnothing$ and all $v \in \mathcal{U}$ such that $x_i^n v$ is a vertex of τ_n which has not yet appeared in $\{v_0^n, v_1^n, ..., v_{\lfloor pnu \rfloor}^n\}$;
- $\ell_v^{n,(i)} = \ell_{x_i^n v}^n$ for every $i \in \{0,1,...,m\}$ and $v \in \tau_{n,(i)}^\circ$.

We need to describe the distribution of the collection $(\tau_{n,(i)}, (\ell_v^{n,(i)})_{v \in \tau_{n,(i)}^\circ})$, $0 \leqslant i \leqslant m$, under $P^{*,n}$. To this end, introduce another collection $(T_i, (L_v^i)_{v \in T_i^\circ})$, $0 \leqslant i \leqslant m$, of independent random labeled p-trees defined under a probability measure $\mathbf{P}$ and such that, for $i \in \{0,1,...,m\}$,

- T_i is distributed as the multitype Galton–Watson tree τ_* considered in the proof of the lemma above;
- given T_i, the labels L_v^i, $v \in T_i^\circ$, are distributed uniformly at random among admissible choices making $(T_i, (L_v^i)_{v \in T_i^\circ})$ a p-labeled tree with $L_\varnothing^i = l_i$.

Consider also the two events

$$A_1 = \{L_v^i > 0 \text{ for every } i \in \{0,1,...,m\} \text{ and } v \in T_i^\circ\},$$
$$A_2 = \left\{ p \sum_{i=0}^m \#T_i^\bullet + m = pn - \lfloor pnu \rfloor \right\}.$$

Then the distribution of $(\tau_{n,(i)}, (\ell_v^{n,(i)})_{v \in \tau_{n,(i)}^\circ})_{0 \leqslant i \leqslant m}$ under $P^{*,n}$ coincides with the distribution of $(T_i, (L_v^i)_{v \in T_i^\circ})_{0 \leqslant i \leqslant m}$ under $\mathbf{P}(\cdot \,|\, A_1 \cap A_2)$. This follows from the fact that, given the information provided by the conditioning event in $P^{*,n}$, the distribution of $(\tau_n, (\ell_v^n)_{v \in \tau_n})$ is uniform over the set of all p-mobiles that are compatible with this information.

Let us set $m' = m - \lfloor \frac{1}{2} \eta n^{1/2} \rfloor$ to simplify notation. Consider the event

$$A_1' = \{L_v^i > 0 \text{ for every } i \in \{0,1,...,m'\} \text{ and } v \in T_i^\circ\}.$$

Claim. *There exists a constant $\overline{K}$, depending only on η', such that for all sufficiently large n,*

$$\mathbf{P}(A_1' \cap A_2) \leqslant \overline{K} \mathbf{P}(A_1 \cap A_2).$$

Let us verify the claim. To simplify notation, set

$$L(T_i) = \min\{L_v^i : v \in T_i\}$$

for every $i \in \{0,1,...,m\}$. Then it suffices to prove that, for every choice of the integers $\sigma_0, \sigma_1, ..., \sigma_m \geqslant 0$ such that $p \sum_{i=0}^m \sigma_i + m \leqslant pn$,

$$\mathbf{P}(L(T_i) > 0 \text{ for all } i \in \{m'+1,...,m\} \,|\, \#T_i^\bullet = \sigma_i \text{ for all } i \in \{0,1,...,m\}) \geqslant \overline{K}^{-1},$$

with a constant $\overline{K}$ depending only on η'. By independence, this conditional probability is equal to

$$\prod_{i=m'+1}^{m} \mathbf{P}(L(T_i)>0 \,|\, \#T_i^\bullet=\sigma_i) = \prod_{i=m'+1}^{m} \Big(1-\Pi_{\sigma_i}^p\Big(\min_{v\in\tau^\circ} \ell_v \leqslant -l_i\Big)\Big),$$

with the notation of the lemma above. Now recall that $l_i \geqslant \frac{1}{2}\eta' n^{1/4}$ for $i\in\{m'+1,...,m\}$. Using the second assertion of the lemma, we get, for $i\in\{m'+1,...,m\}$,

$$\Pi_{\sigma_i}^p\Big(\min_{v\in\tau^\circ} \ell_v \leqslant -l_i\Big) \leqslant 1-\exp\Big(-\widetilde{K}\frac{\sigma_i}{l_i^4}\Big),$$

where $\widetilde{K}$ only depends on η'. Hence,

$$\begin{aligned}\prod_{i=m'+1}^{m} \Big(1-\Pi_{\sigma_i}^p\Big(\min_{v\in\tau^\circ} \ell_v \leqslant -l_i\Big)\Big) &\geqslant \prod_{i=m'+1}^{m} \exp\Big(-\widetilde{K}\frac{\sigma_i}{l_i^4}\Big)\\ &\geqslant \exp\Big(-\frac{16\widetilde{K}}{(\eta')^4 n}\sum_{i=m'+1}^{m}\sigma_i\Big) \geqslant \exp\Big(-\frac{16\widetilde{K}}{(\eta')^4}\Big),\end{aligned}$$

since $\sum_{i=0}^m \sigma_i \leqslant n$. The claim now follows.

From the claim and the obvious property $A_1\subset A_1'$, we get

$$\mathbf{P}(\cdot \,|\, A_1\cap A_2) \leqslant \overline{K}\mathbf{P}(\cdot \,|\, A_1'\cap A_2). \tag{54}$$

Under the probability measure $P^{*,n}$, if the event $B_n(u,\alpha,\delta)$ holds, then we must have

$$\sum_{i=m-p(\lfloor \alpha n^{1/2}\rfloor+1)+1}^{m} (1+p\#\tau_{n,(i)}^\bullet) \geqslant \lfloor pn(u+\delta)\rfloor - \lfloor pnu\rfloor,$$

because, from the properties of the contour sequence and our definitions, the ancestor of $v_{\lfloor pnu\rfloor}^n$ at generation $2(C_{\lfloor pnu\rfloor}^n - \lfloor \alpha n^{1/2}\rfloor - 1)$ is visited by the contour sequence after time $\lfloor pnu\rfloor$ and before time

$$\lfloor pnu\rfloor + \sum_{i=m-p(\lfloor \alpha n^{1/2}\rfloor+1)+1}^{m} (1+p\#\tau_{n,(i)}^\bullet).$$

From this observation and (54), we get the bound

$$\begin{aligned}&P^{*,n}(B_n(u,\alpha,\delta))\\ &\quad\leqslant \overline{K}\mathbf{P}\Big(\sum_{i=m-p(\lfloor \alpha n^{1/2}\rfloor+1)+1}^{m} (1+p\#T_i^\bullet) \geqslant \lfloor pn(u+\delta)\rfloor - \lfloor pnu\rfloor \,\Big|\, A_1'\cap A_2\Big).\end{aligned} \tag{55}$$

Notice that the event A_1' does not involve the labeled trees $(T_i,(L_v^i)_{v\in T_i^\circ})$ for $m'<i\leqslant m$. Hence the distribution of the collection $(\#T_i^\bullet, m'<i\leqslant m)$ under $\mathbf{P}(\,\cdot\,|A_1'\cap A_2)$ is exchangeable.

On the other hand, put $q=\lfloor \eta/2p\alpha\rfloor -1\geqslant 1$ (recall that we assumed $\alpha\leqslant\eta/4p$). Then $qp(\lfloor \alpha n^{1/2}\rfloor+1)<m-m'$ when n is large. On the event A_2, we have the trivial estimate

$$\#\biggl\{j\in\{1,\dots,q\}:\sum_{m-jp(\lfloor \alpha n^{1/2}\rfloor+1)<i\leqslant m-(j-1)p(\lfloor \alpha n^{1/2}\rfloor+1)}(1+p\#T_i^\bullet)\geqslant \lfloor pn(u+\delta)\rfloor-\lfloor pnu\rfloor\biggr\}\\ \leqslant \frac{pn-\lfloor pnu\rfloor}{\lfloor pn(u+\delta)\rfloor-\lfloor pnu\rfloor}\leqslant\frac{2(1-u)}{\delta},$$

when n is large. By exchangeability, the expected value of the left-hand side under the conditional probability $\mathbf{P}(\,\cdot\,|A_1'\cap A_2)$ equals

$$q\mathbf{P}\biggl(\sum_{i=m-p(\lfloor \alpha n^{1/2}\rfloor+1)+1}^{m}(1+p\#T_i^\bullet)\geqslant \lfloor pn(u+\delta)\rfloor-\lfloor pnu\rfloor\,\biggr|\,A_1'\cap A_2\biggr)$$

and, using (55), we finally obtain that

$$P^{*,n}(B_n(u,\alpha,\delta))\leqslant \overline{K}\,\frac{2(1-u)}{\delta}\biggl(\biggl\lfloor\frac{\eta}{2p\alpha}\biggr\rfloor-1\biggr)^{-1}.$$

The estimate (53) follows, and this completes the proof. □

Proof of Lemma 9.1. As we already observed after the statement of Theorem 2.2, we may choose the uniformly distributed p-mobiles $(\tau_n,(\ell_v^n)_{v\in\tau_n^\circ})$ in such a way that the convergence

$$(\lambda_p n^{-1/2}C^n_{\lfloor pnt\rfloor},\varkappa_p n^{-1/4}\Lambda^n_{\lfloor pnt\rfloor})_{0\leqslant t\leqslant 1}\xrightarrow{\text{a.s.}}(\bar{\mathbf{e}}_t,\bar{Z}_t)_{0\leqslant t\leqslant 1}\tag{56}$$

holds as $n\to\infty$ and not only along a subsequence $\{n_k\}_{k\geqslant 1}$ as in (3).

Let $\beta>0$ and let $F_n(\beta)$ denote the event on which there exist three integers

$$k_1^n\leqslant k_2^n\leqslant k_3^n$$

in $[0,pn]$ such that $v^n_{k_1^n}=v^n_{k_2^n}=v^n_{k_3^n}$, $k_2^n-k_1^n\geqslant\beta n$ and $k_3^n-k_2^n\geqslant\beta n$. Then, it is easy to see that

$$\lim_{\alpha\to 0}\limsup_{n\to\infty}P(F_n(\beta)\setminus\Delta_n(\alpha))=0.\tag{57}$$

Indeed, on the event

$$\limsup_{n\to\infty}(F_n(\beta)\setminus\Delta_n(\alpha))$$

we can use (56) and a compactness argument to find three times $s_1 \leqslant s_2 \leqslant s_3$ in $[0,1]$ such that $s_1 \sim s_2 \sim s_3$, $s_2 - s_1 \geqslant \beta/p$, $s_3 - s_2 \geqslant \beta/p$ and furthermore

$$\min_{r\in[s_1,s_2]} \overline{Z}_r \geqslant \overline{Z}_{s_1} - \varkappa_p \alpha \quad \text{or} \quad \min_{r\in[s_2,s_3]} \overline{Z}_r \geqslant \overline{Z}_{s_1} - \varkappa_p \alpha.$$

Lemma 2.1 now shows that the probability of the existence of such a triplet (s_1, s_2, s_3) tends to 0 as $\alpha \to 0$, which implies (57).

Due to (57), Lemma 9.1 will follow if we can verify that

$$\lim_{\beta\to 0} \liminf_{n\to\infty} P(F_n(\beta)) = 1. \tag{58}$$

In the case $p=2$, the proof of (58) is easy from (56). Indeed (56) implies the existence of vertices of τ_n whose removal produces three components each containing a number of vertices of order n. If $p=2$, these vertices must be white vertices, but in the general case, they could possibly be black vertices, and so we need a different argument.

A proof of (58) can be given along the lines of the proof of Lemma 5.5. The arguments are relatively straightforward but somewhat tedious, and for this reason we will leave certain details to the reader. We use the notation introduced in the proof of Lemma 5.5, with the particular choice $u = \frac{1}{2}$. In particular, we let $N = C^n_{\lfloor pn/2 \rfloor}$ and $M \in \{N, ..., (p-1)N\}$, and the random vertices $x^n_0, ..., x^n_M$ and the labeled trees

$$(\tau_{n,(i)}, (\ell^{n,(i)}_v)_{v\in\tau^\circ_{n,(i)}})$$

for $0 \leqslant i \leqslant M$ be defined as in the previous proof, with $u = \frac{1}{2}$. Given $\eta, \eta' > 0$, we also set $M' = (M - 2\lfloor \eta n^{1/2} \rfloor)^+$ and $M'' = (M - \lfloor \eta n^{1/2} \rfloor)^+$, and we denote by $A^n_{\eta,\eta'}$ the event

$$A^n_{\eta,\eta'} = \Big\{ N > 4\eta n^{1/2}; W^n_{\lfloor pn/2 \rfloor}(j) > \eta' n^{1/4} + p \text{ for all } j \in \{N - 2\lfloor \eta n^{1/2} \rfloor, ..., N\}; \\ \sup_{v\in\tau^\circ_{n,(i)}} |\ell^{n,(i)}_v - \ell^{n,(i)}_\varnothing| \leqslant \eta' n^{1/4} \text{ for all } i \in \{M', ..., M''\} \Big\}.$$

Let $\varepsilon > 0$. We can fix η and η' so that $P(A^n_{\eta,\eta'}) > 1 - \varepsilon$ for every large enough n. Furthermore, due to the convergence (56), we can choose a constant $\beta_1 = \beta_1(\eta) > 0$ small enough so that the probability that the contour sequence of τ°_n visits the vertex $x^n_{M''}$ between times $\lfloor \frac{1}{2} pn \rfloor$ and $\lfloor \frac{1}{2} pn \rfloor + \beta_1 n$ is bounded above by ε when n is large.

By making the distribution of the collection $(\tau_{n,(i)}, (\ell^{n,(i)}_v)_{v\in\tau^\circ_{n,(i)}})_{0\leqslant i\leqslant M}$ explicit as in the preceding proof, one verifies that the distribution of the vector

$$(\#\tau^\circ_{n,(M'+i)})_{0\leqslant i\leqslant \lfloor \eta n^{1/2} \rfloor}$$

under the conditioned measure $P(\cdot\,|\,A^n_{\eta,\eta'})$ is exchangeable. Notice that the proportion of indices $i\in\{M',...,M''\}$ such that x^n_i is an ancestor of $v^n_{\lfloor pn/2\rfloor}$ is at least $1/p$ when n is large.

Furthermore, from the convergence (56) and properties of the limiting random function $\bar{\mathbf{e}}$ (recall from §2.4 that $\bar{\mathbf{e}}$ is defined by a simple transformation of the Brownian excursion $\mathbf{e}$), one gets that, for every integer $K\geqslant 1$, we can choose $\beta>0$ sufficiently small so that

$$\liminf_{n\to\infty} P(A^n_{\eta,\eta'}\cap\{\#\{i\in\{0,1,...,\lfloor\eta n^{1/2}\rfloor\}:\#\tau^\circ_{n,(M'+i)}>\beta n\}\geqslant K\})\geqslant 1-2\varepsilon.$$

Using an exchangeability argument, we can then find a constant $\beta_0\in]0,\beta_1(\eta)]$ such that the following holds: With probability greater than $1-3\varepsilon$ when n is large, the event $A^n_{\eta,\eta'}$ holds and there exists $i_0\in\{M',...,M''\}$ such that $x^n_{i_0}$ is an ancestor of $v^n_{\lfloor pn/2\rfloor}$ and

$$\#\tau^\circ_{n,(i_0)}>\beta_0 n.$$

On the event where i_0 is defined, let k^n_1 stand for the time of the first occurence of $x^n_{i_0}$ in the contour sequence of τ°_n, let k^n_2 be the time of the first occurence of $x^n_{i_0}$ after time $\lfloor\frac{1}{2}pn\rfloor$ and let k^n_3 be the time of the last occurence of $x^n_{i_0}$. Then the triplet (k^n_1,k^n_2,k^n_3) will be defined and will satisfy the property of the definition of $F_n(\beta_0)$, with a probability greater than $1-4\varepsilon$ when n is large. Our claim (58) now follows. □

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