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# On the regularization of conservative maps

by

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## 1. Introduction

Let $M$ and $N$ be $C^\infty$ manifolds(1) and let $C^r(M,N)$, $r\in\mathbb{N}\cup\{\infty\}$, be the space of $C^r$ maps from $M$ to $N$, endowed with the Whitney topology. It is a well-known fact that $C^\infty$ maps are dense in $C^r(M,N)$. Such a result is very useful in differentiable topology and in dynamical systems (as we will discuss in more detail). On the other hand, in closely related contexts, it is the non-existence of a regularization theorem that turns out to be remarkable: if homeomorphisms could always be approximated by diffeomorphisms then the whole theory of exotic structures would not exist.

Palis and Pugh [20] seem to have been the first to ask about the corresponding regularization results in the case of conservative and symplectic maps. Here one fixes $C^\infty$ volume forms(2) (in the conservative case) or symplectic structures (symplectic case), and asks whether smoother maps in the corresponding class are dense with respect to the induced Whitney topology. The first result in this direction was due to Zehnder [28], who provided regularization theorems for symplectic maps, based on the use of generating functions. He also provided a regularization theorem for conservative maps, but only when $r>1$ (he did manage to treat also non-integer $r$). The case $r=1$ however has remained open since then (due in large part to intrinsic difficulties relating to the partial differential equations (PDEs) involved in Zehnder's approach, which we will discuss below), except in dimension 2, where it is equivalent to the symplectic case. This is the problem we address in this paper. Let $C^r_{\mathrm{vol}}(M,N)\subset C^r(M,N)$ be the subset of maps that preserve the fixed smooth volume forms.

(1) All manifolds will be assumed to be Hausdorff, paracompact and without boundary, but possibly not compact.

(2) For non-orientable manifolds, a volume form should be understood up to sign.

THEOREM 1. *$C^\infty$ maps are dense in $C^1_{\mathrm{vol}}(M,N)$.*

Let us point out that the corresponding regularization theorem for conservative flows was obtained much earlier by Zuppa [29] in 1979. In fact, in a more recent approach of Arbieto–Matheus [1], it is shown that a result of Dacorogna–Moser [13] allows one to reduce to a local situation where the regularization of vector fields which are divergence free can be treated by convolutions. However, attempts to reduce the case of maps to the case of flows through a suspension construction have not been successful.

Let us discuss a bit an approach to this problem which is successful in higher regularity, and the difficulties that appear when considering $C^1$-conservative maps. Let us assume for simplicity that $M$ and $N$ are compact, as all difficulties are already present in this case. Let $f\in C^r_{\mathrm{vol}}(M,N)$, and let $\omega_M$ and $\omega_N$ be the smooth volume forms. Approximate $f$ by a smooth non-conservative map $\tilde f$. Then $\tilde f^*\omega_N$ is $C^{r-1}$ close to $\omega_M$. If we can solve the equation $h^*\tilde f^*\omega_N=\omega_M$, where $h$ is $C^r$ close to the identity, then the desired approximation could be obtained by taking $\tilde f\circ h$. Looking at the local problem one must solve to get $h$, it is natural to turn our attention to the $C^r$ solutions of the equation $\det Dh=\phi$, where $\phi:\mathbb{R}^n\to\mathbb{R}$ is smooth and close to 1.

Unfortunately, though $\phi$ is smooth, we only know a priori that the $C^{r-1}$ norm of $\phi$ is small. This turns out to be quite sufficient to get control on $h$ if $r\geqslant 2$, according to the Dacorogna–Moser technique. But when $r=1$, the analysis of the equation is different, as was shown by Burago–Kleiner [11] and McMullen [18]. This is well expressed in the following result, Theorem 1.2 of [11]: *Given $c>0$ there exists a continuous function $\phi:[0,1]^2\to[1,1+c]$ such that there is no bi-Lipschitz map $h:[0,1]^2\to\mathbb{R}^2$ with $\det Dh=\phi$.*

This implies that continuous volume forms on a $C^\infty$ manifold need not be $C^1$ equivalent to smooth volume forms. This is in contrast with the fact that all smooth volume forms are $C^\infty$ equivalent up to scaling [19], and the differential topology fact that all $C^1$ structures on a $C^\infty$ manifold are $C^1$ equivalent.

*Remark* 2. One can define a $C^r_{\mathrm{vol}}$ structure on a manifold as a maximal atlas whose chart transitions are $C^r$ maps preserving the usual volume of $\mathbb{R}^n$ (see [26, Example 3.1.12]). Then Theorem 1 (and its equivalent result for higher differentiability [28]) can be used to conclude that any $C^r_{\mathrm{vol}}$ structure is compatible with a $C^\infty_{\mathrm{vol}}$ structure (unique up to $C^\infty_{\mathrm{vol}}$ diffeomorphisms by [19]), by following the proof of the corresponding statement for $C^r$ structures (see [17, Theorem 2.9]). For $r\geqslant 2$, a $C^r_{\mathrm{vol}}$ structure is the same as a $C^r$ structure together with a $C^{r-1}$ volume form by [13], but not all continuous volume forms on a $C^\infty$ manifold arise from a $C^1_{\mathrm{vol}}$ structure, by [11, Theorem 1.2] quoted above.

We notice also the following amusing consequence of [11, Theorem 1.2], which we leave as an exercise: *A generic continuous volume form on a $C^1$ surface has no non-trivial symmetries*, *that is*, *the identity is the only diffeomorphism of the surface preserving the volume form.* This highlights that the correct framework to do $C^1$-conservative dynamics is the $C^1_{\mathrm{vol}}$ category (and not $C^1$ plus continuous volume form category).

The equation $\det Dh = \phi$ has been studied also in other regularity classes (such as Sobolev classes) by Ye [27] and Rivière–Ye [22], but this has not helped with the regularization theorem in the $C^1$ case.

The approach taken in this paper is very simple, ultimately constructing a smooth approximation by taking independent linear approximations (derivative) in a very dense set, and carefully modifying and gluing them into a global map (with a mixture of bare-hand techniques and some results from the PDE approach *in high regularity*). A key point is to enforce that the choices involved in the construction are made through a *local* decision process. This is useful to avoid long-range effects, which if left out would lead us to a discretized version of the PDE approach in low regularity, with the associated difficulties. To ensure locality, we use the original unregularized map $f$ as *background data* for making the decisions. The actual details of the procedure are best understood by going through the proof, since the difficulties of this problem lie in the details.

### 1.1. Dynamical motivation

In the discussion below, we restrict ourselves to diffeomorphisms of compact manifolds for definiteness.

There is a good reason why the regularization problem for conservative maps has first been introduced in a dynamical context. In dynamics, low regularity is often used in order to be able to have the strongest perturbation results available, such as the closing lemma [21], the connecting lemma [16] and the simple but widely used Franks' lemma [15]. Currently such results are only proved precisely for the $C^1$ topology (even getting to $C^{1+\alpha}$ would be an amazing progress), except when considering 1-dimensional dynamics. On the other hand, higher regularity plays a fundamental role when distortion needs to be controlled, which is the case for instance when the ergodic theory of the maps is the focus ($C^{1+\alpha}$ is a basic hypothesis of Pesin theory, and for most results on stable ergodicity such as [12], though more regularity is necessary if KAM (Kolmogorov–Arnold–Moser) methods are involved [23]). While dynamics in the smooth and the low regularity worlds may often seem to be different altogether (compare [10] and [5]), it turns out that their characteristics can often be combined (both in the conservative and the dissipative setting), yielding for instance great flexibility in obtaining interesting examples: see the construction of non-uniformly hyperbolic Bernoulli maps [14] which uses $C^1$-perturbation techniques of [3].

In the dissipative and symplectic settings, regularization theorems have been an important tool in the analysis of $C^1$-generic dynamics: for instance, Zehnder's theorem is used in the proof of [2] that ergodicity is $C^1$ generic for partially hyperbolic symplectic diffeomorphisms.([3]) Thus it is natural to expect that Theorem 1 will lead to several applications on $C^1$-generic conservative dynamics. Indeed many recent results have been stated about certain properties of $C^2$ maps being dense in the $C^1$ topology, without being able to conclude anything about $C^1$ maps only due to the non-availability of Theorem 1. Thus it had been understood for some time that proving Theorem 1 would have many immediate applications. Just staying with examples in the line of [2], we point out that [8] now implies that ergodicity is $C^1$ generic for partially hyperbolic maps with 1-dimensional center (see [8, §4]), and the same applies to the case of 2-dimensional center, in view of the recent work [24].

Though we do not aim to be exhaustive in the discussion of applications here, we give a few other examples which were pointed out to us by Bochi and Viana:

(1) Any $C^1_{\mathrm{vol}}$-robustly transitive diffeomorphism admits a dominated splitting (conjectured, e.g., in [6, p. 365]), a result obtained for $C^{1+\alpha}$ diffeomorphisms in [1] using a pasting lemma. (We note that this work also allows one to extend the pasting lemma of [1] itself, and hence its other consequences, to the $C^1$ case.)

(2) A $C^1$-generic conservative non-Anosov diffeomorphism has only hyperbolic sets of zero Lebesgue measure. Zehnder's theorem has been used in [3] and [5] to achieve this conclusion in the symplectic case, and such a result is necessary for the conclusion of the central dichotomy of [3]. It is based on a statement about $C^2$-conservative maps obtained in [4], so the conclusion for conservative maps now follows directly from Theorem 1. We hope that results in this direction will play a role in further strengthenings of [5].

(3) The existence of locally generic non-uniformly hyperbolic ergodic conservative diffeomorphisms with non-simple Lyapunov spectrum [9], [25] (the proof, conditional to the existence of regularization, is nicely sketched in [7, p. 260]).

### 1.2. Outline of the proof

Our basic idea is to construct the approximation of a diffeomorphism "from inside", growing it up through a growing frame while paying attention to compatibilities.

Let us think first of the case where we have a $C^1_{\mathrm{vol}}$ map $f:\mathbb{R}^n \to \mathbb{R}^n$, whose derivative is bounded and uniformly continuous. We wish to approximate $f$ by a $C^\infty_{\mathrm{vol}}$ map, $C^1$

([3]) It was this fact indeed that convinced the author to work on Theorem 1.

uniformly. We break $\mathbb{R}^n$ into small cubes with vertices in a multiple of the lattice $\mathbb{Z}^n$. In each cube, the derivative of $f$ varies little. Thus $f$ restricted to each cube certainly admits a nice $C^\infty$ approximation: in fact, we can just approximate it by an affine map. Annoyingly, those approximations do not match.

Our next attempt is to build the approximation more slowly. First we will construct an approximation in a neighborhood of the set of vertices of the cubes, then extend it to an approximation near the set of edges, etc. Progressing through the $k$-faces of the cubes, $0 \leqslant k \leqslant n$, we will eventually get a map defined everywhere.

The first step is easy: consider a small $\varepsilon$-neighborhood $V_0$ of the set of vertices of all the cubes. In this set, we can define an approximation $f_0$ of $f$ which is just affine in each connected component. Next, consider an $\varepsilon^2$-neighborhood $V_1$ of the set of edges. The connected components of $V_1 \setminus V_0$ intersect, each, a single edge. We can extend $f_0|_{V_0 \cap V_1}$ to a map $f_1$ defined in $V_1$: the extension argument follows [13], and is based on the fact that $f_0$ admits a nice $C^\infty$ (a priori not volume-preserving) extension. This extension, behaves well at the scale of the cubes (after rescaling to unit size, the extension is $C^\infty$ close to affine), which yields the estimates necessary to apply the (high regularity) Dacorogna–Moser argument.

We repeat this process until getting a map $f_{n-1}$ defined in a neighborhood $V_{n-1}$ of the faces of the cubes. It is important to emphasize that, along this process, all decisions taken are local: for instance, to know what to do near an edge we only need to look at what we have done near the vertices of this edge. This eliminates long range effects in the process.

At the last moment however, we face a new difficulty: there is an obstruction to the extension of $f_{n-1}$ to a volume-preserving map. In fact, $\mathbb{R}^n \setminus V_{n-1}$ is disconnected, and for an extension (close to $f$) to exist, the boundary $P$ (a topological sphere) of each hole must be mapped, under $f_{n-1}$ to a topological sphere $P'$ such that the bounded components of $\mathbb{R}^n \setminus P$ and $\mathbb{R}^n \setminus P'$ have the same volume.

To account for this, one could try to modify the map $f_{n-1}$, so that the volume of the "holes in the image" is the same as the volume of the "holes in the domain". In fact, if we have a volume-preserving map such as $f_{n-1}$, defined in a neighborhood of the boundaries of the cubes, it is easy to modify it to "shift mass" between adjacent "holes in the image". We could try to correct an increasing family of holes: choose one hole and an adjacent one, move mass so that the first one becomes fine, then choose another adjacent hole to the second one, move mass, etc. But this introduces possible long range effects: the decisions taken early on, in some specific place, affect what we have to do much later, and far away. Thus it is better to try to do it simultaneously. How to prescribe how much mass should be moved from which hole to which hole? Trying to solve this takes us to some difference equation: we are given a function $d$ from the set of cubes to $\mathbb{R}$ (measuring the excess or deficit of volume of the "hole in the image"), and we want to find some function $s$ from the set of faces to $\mathbb{R}$ such that the sum of $s$ over all faces of each cube equals $d$. This is just some discretized form of the divergence equation, and we do not want to follow this path, since, as described before, the divergence equation is hard to solve in the regularity we are dealing with.

We will instead proceed differently, being careful to make the constructions of $f_0,\dots,f_{n-1}$, so that the problem will not show up at the last step: we make the corrections along the way, which breaks the problem into simple ones (we want to be able to make local decisions). To make the decisions, we use an important guide: the "background" map $f$, which is known to be volume-preserving. When constructing $f_0$, we make sure that $f_0$, near each vertex $p$, is fair to all cubes $C$ that have $p$ as a vertex: thus if $B$ is the connected component of $V_0$ containing $p$, we want that $f_0(B)\cap f(C)$ and $B\cap C$ have the same volume. This can be done, starting with a careless attempt at defining $f_0$, such as the one considered before, by a "moving mass" argument, which this time has no longer range effects. Later, when defining $f_1$ near an edge $q$, the fairness property of $f_0$ will allow us to be fair to all cubes that have $q$ as an edge. This goes on until $f_{n-1}$, when we find out that the fairness condition implies that there is no problem with the holes any more. We can then extend $f_{n-1}$ to the desired approximation $f_n$ of $f$.

This concludes the argument in this case. We can adapt this argument to deal with, instead of the entire $\mathbb{R}^n$, some domain in $\mathbb{R}^n$. We just need to consider a suitable decomposition into cubes which has locally bounded geometry, and the Whitney decomposition will do. In fact, we can prove a more detailed result about domains, with "matching conditions" (thus, if $f$ is already smooth somewhere, we do not need to modify $f$ there along the approximation(4)). Once the case of domains in $\mathbb{R}^n$ is taken care of, we can deal with the case of manifolds as well by a triangulation argument, building the approximation through vertices, edges, etc., but with a much easier argument (since we can prescribe matching conditions).

This paper is organized as follows. We first describe the kind of extension result we will repeatedly make use of, obtained using the Dacorogna–Moser technique. Then we show how to move mass between cubes, to achieve fairness. Next, we formulate and prove a version of the approximation theorem with matching conditions. We conclude with the application of this result to the case of manifolds.

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(4) We note that this kind of result is more relevant for "pasting lemma" applications [1] than Theorem 1 itself.

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## 2. Extending conservative maps

Fix two connected open sets with smooth boundary $B_1, B_2 \subset \mathbb{R}^n$ with $\overline{B}_1 \subset B_2$ and $\overline{B}_2 \setminus B_1$ smoothly diffeomorphic to $\partial B_1 \times [0,1]$. For the proof of Theorem 1, we will need the following slight variations of Theorems 2 and 1 of Dacorogna–Moser [13].

THEOREM 3. *Let $\phi: \mathbb{R}^n \to \mathbb{R}$ be a $C^\infty$ function with $\int_{\mathbb{R}^n} \phi(z)\, dz = 0$ supported inside $B_1$. Then there exists $v \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ supported inside $B_2$ with* $\operatorname{div} v = \phi$. *Moreover, if $\phi$ is $C^\infty$ small, then $v$ is $C^\infty$ small.*

*Proof.* Theorem 2 of [13] states in a more general context, that there exists a map $w: \overline{B}_2 \to \mathbb{R}^n$ with $\operatorname{div} w = \phi$ and $w|_{\partial B_2} = 0$, and if $\phi$ is $C^\infty$ small then $w$ is also $C^\infty$ small. It is thus enough to find some $C^\infty$ map $u: \overline{B}_2 \to \mathbb{R}^n$ (small if $\phi$ is small) with $\operatorname{div} u = 0$ and $u|_{\overline{B}_2 \setminus B_1} = w$, and let $v|_{\overline{B}_2} = w - u$ and $v|_{\mathbb{R}^n \setminus \overline{B}_2} = 0$. This procedure is the standard one already used in [13].

There is a duality between smooth vector fields $u$ and smooth $(n-1)$-forms $u^*$, given by $u^*(x)(y_1, ..., y_{n-1}) = \det(u(x), y_1, ..., y_{n-1})$. The duality transforms the equation $\operatorname{div} u = 0$ into $du^* = 0$. The form $w^*$ is thus closed in $\overline{B}_2 \setminus B_1$, and the boundary condition $w|_{\partial B_2} = 0$ implies that it is exact in $\overline{B}_2 \setminus B_1$. Solve the equation $d\alpha = w^*$ in $\overline{B}_2 \setminus B_1$ and extend $\alpha$ smoothly to $\overline{B}_2$ (notice that $\alpha$ can be required to be small if $w$ is small). Let $u$ be a vector field on $\overline{B}_2$ given by $d\alpha = u^*$. Then $u|_{\overline{B}_2 \setminus B_1} = w$, and since $du^* = 0$ in $\overline{B}_2$, we have $\operatorname{div} u|_{\overline{B}_2} = 0$. □

THEOREM 4. *Let $\phi: \mathbb{R}^n \to (-1, \infty)$ be a $C^\infty$ function with $\int_{\mathbb{R}^n} \phi(z)\, dz = 0$ supported inside $B_1$. Then there exists $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, with $\psi - \mathrm{id}$ supported inside $B_1$, such that* $\det \psi = 1 + \phi$. *Moreover, if $\phi$ is $C^\infty$ small, then $\psi - \mathrm{id}$ is $C^\infty$ small.*

*Proof.* As in [13], the solution is given explicitly as $\psi = \psi_1$, where $\psi_t(x)$ is the solution of the differential equation

$$\frac{d\psi_t(x)}{dt} = \frac{v(\phi_t(x))}{t + (1-t)(1 + \phi(\psi_t(x)))}$$

with $\psi_0(x) = x$ and $v$ comes from the previous theorem. □

Corollary 5. *Let $K$ be a compact set, $U$ be a neighborhood of $K$ and let $f\in C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ be $C^\infty$ close to the identity and such that $f|_U$ is volume-preserving. Assume that for every bounded connected component $W$ of $\mathbb{R}^n\setminus K$, $W$ and $f(W)$ have the same volume. Then there exists a $C^\infty$-conservative map close to the identity such that $\tilde f=f$ on $K$.*

*Proof.* We may modify $f$ away from $K$ so that $f-\mathrm{id}$ is compactly supported and $\det f-1$ is supported inside some open set $\widetilde B_1$, which can be assumed to have smooth boundary, disjoint from some neighborhood of $K$. Let $m$ be the number of connected components of $\widetilde B_1$. We can assume that each connected component of $\mathbb{R}^n\setminus K$ contains at most one connected component of $\widetilde B_1$ (otherwise we just enlarge $\widetilde B_1$ suitably). For each connected component $B_1^i$ of $\widetilde B_1$, select a small $\varepsilon$-neighborhood $B_i^2$ of $\overline B{}_1^i$. Let $\phi_i$ be given by $\phi_i|_{B_1^i}=\det f-1$ and $\phi_i|_{\mathbb{R}^n\setminus B_1^i}=0$. Then $\int_{\mathbb{R}^n}\phi_i(z)\,dz=0$. Indeed, if $B_1^i$ is contained in a bounded connected component of $\mathbb{R}^n\setminus K$, this follows immediately from $f$ preserving the volumes of such sets, and if $B_1^i$ is contained in the unbounded component $W$ of $\mathbb{R}^n\setminus K$, one uses that $f$ preserves the volume of $W\cap B$ for all sufficiently large balls $B$ (to see this one uses that $f-\mathrm{id}$ is compactly supported). Applying the previous theorem, one gets maps $\psi_i$ with $\psi_i-\mathrm{id}$ supported inside $B_1^i$. We then take $\tilde f=f\circ\psi_1^{-1}\circ\ldots\circ\psi_m^{-1}$. □

## 3. Moving mass

In this section we will consider the $L^\infty$ norm in $\mathbb{R}^n$. The closed ball of radius $r>0$ around $p\in\mathbb{R}^n$ will be denoted by $B(p,r)$ (this ball is actually a cube). The canonical basis of $\mathbb{R}^n$ will be denoted by $e_1,\ldots,e_n$.

Lemma 6. *Fix $0<\delta<\frac{1}{10}$. Let $S\subset\{1,\ldots,n\}$ be a subset with $0\leqslant k\leqslant n-1$ elements. Let $P\subset\mathbb{R}^n$ be the (finite) set of all $p$ of the form $\sum_{i\notin S}u_ie_i$ with $u_i=\pm1$. Let*

$$B=\bigcup_{p\in P}B(p,1),$$

*and let $B'$ be the open $\delta$-neighborhood of $B$. Let $W$ be a Borel set whose $\delta$-neighborhood is contained in $B$ and which contains $B(0,\delta)$. If $F\in C^1_{\mathrm{vol}}(B',\mathbb{R}^n)$ is $C^1$ close to the identity, then there exists $s\in C^\infty_{\mathrm{vol}}(\mathbb{R}^n,\mathbb{R}^n)$ such that*

(1) *$s|_{\mathrm{int}\,B(0,10)}$ is $C^\infty$ close to the identity;*
(2) *$\mathrm{vol}(F(s(W))\cap B(p,1))=\mathrm{vol}(W\cap B(p,1))$ for $p\in P$;*
(3) *$s$ is the identity outside the $\delta$-neighborhood of the subspace generated by $\{e_i\}_{i\notin S}$.*

*Proof.* Notice that there are $2^{n-k}$ elements in $P$. Call two elements $p,p'\in P$ *adjacent* if $p-p'=\pm2e_l$ for some $1\leqslant l\leqslant n$.

Let $p$ and $p'$ be adjacent. Let $q=q(p,p')=\frac{1}{4}\delta(p'+p)$, and let $C=C(p,p')$ be the cylinder consisting of all $z\in\mathbb{R}^n$ of the form $z+t(p'-p)$, where $t\in\mathbb{R}$ and $z\in B\big(q,\frac{1}{4}\delta\big)$. Let $\phi=\phi^{p,p'}:\mathbb{R}^n\to[0,1]$ be a $C^\infty$ function such that $\phi(q)=1$, $\phi|_{\mathbb{R}^n\setminus C}=0$ and $\phi(z+t(p-p'))=\phi(z)$ for $t\in\mathbb{R}$. For $t\in\mathbb{R}$, let $s_t=s_t^{p,p'}\in C^\infty_{\mathrm{vol}}(\mathbb{R}^n,\mathbb{R}^n)$ be given by $s_t(z)=z+t\phi(z)(p-p')$.

Let us show that for $|t|<\frac{1}{100}\delta$, we have

$$\begin{aligned}\mathrm{vol}(s_t(W)\cap B(p,1))-\mathrm{vol}(W\cap B(p,1)) &= \mathrm{vol}(s_t(C\cap B(0,\delta))\cap B(p,1)) \\ &\quad -\mathrm{vol}(C\cap B(0,\delta)\cap B(p,1)) \\ &= t\int_{B(0,1/2)}\phi(z)\,dz.\end{aligned} \tag{3.1}$$

Indeed, since the $\delta$-neighborhood of $W$ is contained in $B$, and $s_t$ is the identity outside $C$, if $z\in W\cap B(p,1)$ belongs (respectively, does not belong) to $C\cap B(0,\delta)$, then $s_t(z)$ belongs (respectively, does not belong) to $B(p,1)$ as well. Since $C\cap B(0,\delta)\subset W$, this justifies the first equality. The second equality is a straightforward computation.

Let $B''$ be the open $\frac{1}{2}\delta$-neighborhood of $B$. It is easy to see that if $\widetilde{F}\in C^1_{\mathrm{vol}}(B'',\mathbb{R}^n)$ is $C^1$ close to the identity then $\mathrm{vol}(\widetilde{F}(s_t(W))\cap B(\tilde{p},1))=\mathrm{vol}(\widetilde{F}(W)\cap B(\tilde{p},1))$ for every $\tilde{p}\in P\setminus\{p,p'\}$, since in this case we actually have $\widetilde{F}(s_t(W))\cap B(\tilde{p},1)=\widetilde{F}(W)\cap B(\tilde{p},1)$.

We claim that there exists $t\in\mathbb{R}$ small such that

$$\mathrm{vol}(\widetilde{F}(s_t(W))\cap B(p,1))=\mathrm{vol}(W\cap B(p,1)).$$

Indeed, for $|t|<\frac{1}{100}\delta$, $\mathrm{vol}(\widetilde{F}(s_t(W))\cap B(p,1))-\mathrm{vol}(s_t(W)\cap B(p,1))$ is small if $\widetilde{F}$ is close to the identity. The claim follows from (3.1) and the obvious continuity of

$$t\longmapsto \mathrm{vol}(\widetilde{F}(s_t(W))\cap B(p,1)).$$

As a graph, $P$ is just a hypercube, so there exists an ordering $p_1,...,p_{2^{n-k}}$ of the elements of $P$ such that for $1\leqslant i\leqslant 2^{n-k}-1$, $p_i$ and $p_{i+1}$ are adjacent. Given $F$, we define sequences $F_{(l)}\in C^1_{\mathrm{vol}}(B'',\mathbb{R}^n)$ and $s_{(l)}\in C^\infty_{\mathrm{vol}}(\mathbb{R}^n,\mathbb{R}^n)$, $0\leqslant l\leqslant 2^{n-k}-1$, by induction as follows. We let $s_{(0)}=\mathrm{id}$, $F_{(l)}=F\circ s_{(l)}$ for $0\leqslant l\leqslant 2^{n-k}-1$, and for $1\leqslant l\leqslant 2^{n-k}-1$ we let $s_{(l)}=s_t^{p,p'}\circ s_{(l-1)}$, where $p=p_i$, $p'=p_{i+1}$ and $t$ is given by the claim applied with $\widetilde{F}=F_{(l-1)}$. As long as $F$ is sufficiently close to the identity, we get inductively that $F_{(l)}$ is close to the identity, so this construction can indeed be carried out.

Let us show that $s=s_{(2^{n-k}-1)}$ has all the required properties. Properties (1) and (3) are rather clear. By construction, we get inductively that

$$\mathrm{vol}(F_{(l)}(W)\cap B(p,1))=\mathrm{vol}(W\cap B(p,1))$$

for $p \in \{p_1, ..., p_l\}$, so it is clear that $\mathrm{vol}(F(s(W)) \cap B(p,1)) = \mathrm{vol}(W \cap B(p,1))$, except possibly for $p = p_{2^{n-k}}$. But

$$\sum_{p \in P} \mathrm{vol}(F(s(W)) \cap B(p,1)) = \mathrm{vol}(F(s(W)) \cap B) = \mathrm{vol}(W \cap B) = \sum_{p \in P} \mathrm{vol}(W \cap B(p,1)),$$

so we must have $\mathrm{vol}(F(s(W)) \cap B(p,1)) = \mathrm{vol}(W \cap B(p,1))$ also for $p = p_{2^{n-k}}$, and property (2) follows. □

## 4. Proof of Theorem 1

### 4.1. Charts

If $U \subset \mathbb{R}^n$ is open and $f: U \to \mathbb{R}^n$ is a bounded $C^r$ map with bounded derivatives up to order $r$, we let $\|f\|_{C^r}$ be the natural $C^r$ norm.

THEOREM 7. *Let $W$ be an open subset of $\mathbb{R}^n$ and let $f \in C^1_{\mathrm{vol}}(W, \mathbb{R}^n)$ be a map with bounded uniformly continuous derivative. Let $K_0 \subset W$ be a compact set such that $f$ is $C^\infty$ in a neighborhood of $K_0$. Let $U \subset W$ be open. Then for every $\varepsilon > 0$ there exists $\tilde{f} \in C^1_{\mathrm{vol}}(W, \mathbb{R}^n)$ such that $\tilde{f}|_U$ is $C^\infty$, $\tilde{f}$ coincides with $f$ in $W \setminus U$ and in a neighborhood of $K_0$, and $\|f - \tilde{f}\|_{C^1} < \varepsilon$.*

*Proof.* We consider the $L^\infty$ metric in $\mathbb{R}^n$. Let $\theta > 0$ be such that the $\theta$-neighborhood of $K_0$ is contained in $W$ and $f$ is $C^\infty$ in it. We will now introduce a Whitney decomposition of $U$.

If $0 \leqslant m \leqslant n$, an *$m$-cell* $x$ is some set of the form $\prod_{k=1}^n [2^{-t} a_k, 2^{-t}(a_k + b_k)]$, where $t \in \mathbb{Z}$, $a_k \in \mathbb{Z}$ and $b_k \in \{0,1\}$ with $\#\{k : b_k = 1\} = m$. For $m \geqslant 1$, we let its interior $\mathrm{int}\, x$ be $\prod_{k=1}^n (2^{-t} a_k, 2^{-t}(a_k + b_k))$, with the convention that $(a,a) = \{a\}$ for $a \in \mathbb{R}$. Let $\partial x$ be $x \setminus \mathrm{int}\, x$.

We say that an $n$-cell $x$ is *$\varepsilon$-small* if its diameter is at most $\varepsilon$, and every $n$-cell of the same diameter as $x$ which intersects $x$ is contained in $U$. We say that a dyadic $n$-cell is *$\varepsilon$-good* if it is a maximal (with respect to inclusion) $\varepsilon$-small $n$-cell. We say that a dyadic $m$-cell, $0 \leqslant m \leqslant n-1$, is *$\varepsilon$-good* if it is the intersection of all $\varepsilon$-good $n$-cells that intersect its interior.

Given $\varepsilon > 0$, we say that an $\varepsilon$-good $m$-cell $x$ has *rank* $t = t(x)$ if the minimal diameter of the $\varepsilon$-good $n$-cells containing it is $2^{-t}$ (if $m > 0$, $2^{-t}$ is just the diameter of $x$). The rank is designed to give a measure of the intrinsic scale of the $\varepsilon$-good cells near $x$, so the more cumbersome definition is needed to be meaningful for $m = 0$. Notice that if $x$ and $y$ are $\varepsilon$-good cells and $x \cap y \neq \varnothing$ then $|t(x) - t(y)| \leqslant 1$ (otherwise either $x$ or $y$ would not satisfy the maximality requirement of an $\varepsilon$-good $m$-cell). Each $\varepsilon$-good $m$-cell $x$ is contained in $2^{n-m}$ $n$-cells of diameter $2^{-t(x)}$, called neighbors of $x$ (which are not necessarily $\varepsilon$-good).

Fix some small $\varepsilon>0$. From now on, by $m$-cell we will understand an $\varepsilon$-good $m$-cell. Let $N_m$ be the set of $m$-cells. By construction, the interiors of distinct cells are always disjoint, and their union covers $U$. This is what we meant by a Whitney decomposition of $U$. The local geometry of the Whitney decomposition has some bounded complexity (depending on the dimension): there exists $C_0=C_0(n)$ such that each $m$-cell contains at most $C_0$ $k$-cells, $0\leqslant k\leqslant m$. Moreover, each $x\in N_m$ is the union of the interior of the $k$-cells, $0\leqslant k\leqslant m$, contained in $x$.

For $x\in N_m$, let $D(x)$ be the $2^{-10(m+1)}2^{-t(x)}$-neighborhood of $x$, and let

$$I(x)=\bigcup_y D(y),$$

where the union is taken over all proper subcells $y\subset x$. Thus $I(x)$ is a neighborhood of the boundary of $x$. Let $B(x)=D(x)\cup I(x)$ (a somewhat larger neighborhood of $x$) and $J(x)=D(x)\setminus I(x)$ (and thus $J(x)$ is obtained by truncating a neighborhood of $x$ near the boundary of $x$). Notice that if $x$ and $y$ are distinct cells, then $J(x)$ and $J(y)$ are disjoint. Let $R(x)$ be the interior of the union of all $n$-cells intersecting $x$. Thus $R(x)$ is again a neighborhood of $x$, larger than $B(x)$ and $D(x)$. Notice that the $2^{-100(m+1)}2^{-t(x)}$-neighborhood of $J(x)$ is contained in the interior of the union of the neighbors of $x$.

For a cell $x$, let $b$ be its barycenter, let $\lambda_x:\mathbb{R}^n\to\mathbb{R}^n$ be given by $\lambda_x(z)=b+2^{-t(x)+1}z$ and let $H_x(z)=f(b)+Df(b)(z-b)$. We say that $h\in C^\infty_{\mathrm{vol}}(R(x),\mathbb{R}^n)$ is *$x$-nice* if $h-f$ is $C^1$ small, $(\lambda_x^{-1}\circ h\circ\lambda_x)-(\lambda_x^{-1}\circ H_x\circ\lambda_x)$ is $C^\infty$ small, and for every neighbor $y$ of $x$, $\mathrm{vol}(f^{-1}(h(J(x)))\cap y)=\mathrm{vol}(J(x)\cap y)$. Notice that this last condition implies that for every $y\in N_n$ containing $x$, $\mathrm{vol}(f^{-1}(h(J(x)))\cap y)=\mathrm{vol}(J(x)\cap y)$.

A family $\{h_x\}_{x\in N_m}$ is said to be *nice* if each $h_x$ is $x$-nice and $\|h_x-f\|_{C^1}\to0$ uniformly as $\mathrm{rank}(x)\to\infty$. Let $\varrho=2^{-100n}$. We will now inductively construct nice families $\{\tilde h_x\}_{x\in N_m}$, $0\leqslant m\leqslant n$, such that $\tilde h_x=\tilde h_y$ in a $2^{-t(x)}\varrho^{m+1}$-neighborhood of $B(y)$ whenever $y$ is a subcell of $x$, and such that if $x$ is $2^{-(m+1)}\theta$-close to $K_0$ then $\tilde h_x=f|_{R(x)}$.

Let $x\in N_0$. If $\varepsilon$ is small and $x$ is $\frac12\theta$-close to $K_0$, then $\tilde h_x=f|_{R(x)}$ is $x$-nice. Otherwise, if $\varepsilon$ is sufficiently small, then by a $C^1$ small modification of $H_x$ we obtain a map $\tilde h_x$ which is $x$-nice. The easiest way to see this is to first conjugate by $\lambda_x$, bringing things to unit scale. More precisely, we get into the setting of Lemma 6 (with $k=0$, and hence $S=\varnothing$) by putting $F=\lambda_x^{-1}\circ f^{-1}\circ H_x\circ\lambda_x$ and $W=\lambda_x^{-1}(J(x))$. Let $s$ be the map given by Lemma 6. Then $\tilde h_x=H_x\circ\lambda_x\circ s\circ\lambda_x^{-1}$ is $x$-nice. Moreover, $\{\tilde h_x\}_{x\in N_0}$ is a nice family since the estimates improve as the rank grows (indeed, as the rank grows, one looks at smaller and smaller scales, and the derivative varies less and less).

Let now $1 \leqslant m \leqslant n-1$ and assume that for every $k \leqslant m-1$ we have defined a nice family $\{\tilde{h}_x\}_{x \in N_k}$ with the required compatibilities.

If $x \in N_m$ intersects a $2^{-(m+1)}\theta$-neighborhood of $K_0$ just take $\tilde{h}_x = f|_{R(x)}$ as definition and it will satisfy the other compatibility by hypothesis. Otherwise, let $Q$ be the open $\varrho^m$-neighborhood of $B(x)$ and define a map $h_x \in C^\infty_{\mathrm{vol}}(Q, \mathbb{R}^n)$ such that $h_x = \tilde{h}_y$ in the $\varrho^m$-neighborhood of $B(y)$ for every subcell $y \subset x$. Restricting $h_x$ to the $\frac{1}{2}\varrho^m$-neighborhood of $I(x)$, which is a full compact set (that is, it does not disconnect $\mathbb{R}^n$), since $m \leqslant n-1$, and extending it to $R(x)$ using Corollary 5, we get that $h_x^{(1)} \in C^\infty_{\mathrm{vol}}(R(x), \mathbb{R}^n)$ which is $C^\infty$ close to $H_x$ after rescaling by $\lambda_x$. By a $C^1$ small modification of $h^{(1)}(x)$ outside the $\varrho$-neighborhood of $I(x)$, we can obtain a nice family $\{\tilde{h}_x\}_{x \in N_m}$. This is an application of Lemma 6 (with $k=m$) analogous to the one described before. This time, we let $F = \lambda_x^{-1} \circ f^{-1} \circ h_x^{(1)} \circ \lambda_x$ and $W = \lambda_x^{-1}(J(x))$. We choose $S$ as the subset of the canonical basis of $\mathbb{R}^n$ which spans the tangent space to $x$ at some (any) interior point. Letting $s$ be the map given by Lemma 6, the desired maps are given by $\tilde{h}_x = h_x^{(1)} \circ \lambda_x \circ s \circ \lambda_x^{-1}$.

By induction, we can construct the nice families as above for $0 \leqslant m \leqslant n-1$. Let now $x \in N_n$. As before, when $x$ is close to $K_0$ the definition is forced and there is no problem of compatibility by hypothesis. Otherwise, let $Q$ be the open $\varrho^n$-neighborhood of $I(x)$. As before, define a map $h_x : Q \to \mathbb{R}^n$ by gluing the definitions of $\tilde{h}_y$ for subcells of $x$. Notice that $\mathbb{R}^n \setminus I(x)$ has two connected components, and the bounded one is contained in $x$. By construction, $I(x)$ is the disjoint union of the $J(y)$ contained in it. Thus the volumes of $h_x(I(x)) \cap f(x)$ and $I(x) \cap x$ are equal. This implies that the bounded component of $\mathbb{R}^n \setminus h_x(I(x))$ has the same volume as the bounded component of $\mathbb{R}^n \setminus I(x)$. We can restrict $h(x)$ to the $\frac{1}{2}\varrho^n$-neighborhood of $I(x)$ and extend it to a map $\tilde{h}_x \in C^\infty_{\mathrm{vol}}(R(x), \mathbb{R}^n)$ which is $x$-nice using Corollary 5 (after rescaling by $\lambda_x$ and then rescaling back). Thus we obtain a nice family $\{\tilde{h}_x\}_{x \in N_n}$ with all the compatibilities.

The nice family $\{\tilde{h}_x\}_{x \in N_n}$ is such that whenever two $n$-cells $x$ and $y$ intersect, we have $\tilde{h}_x = \tilde{h}_y$ in a neighborhood of the intersection. Let $\tilde{f} : W \to \mathbb{R}^n$ be defined by $\tilde{f}(z) = f(z)$, for $z \notin U$, and $\tilde{f}(z) = \tilde{h}_x(z)$ for every $z \in x$, $x \in N_n$. Then $\tilde{f} \in C^1_{\mathrm{vol}}(W, \mathbb{R}^n)$, since near $\partial U \cap W$ the rank of an $n$-cell $x$ is big and hence $\|\tilde{h}_x - f|_{R(x)}\|_{C^1}$ is small. Moreover $\|\tilde{f} - f\|_{C^1}$ is small everywhere, and $\tilde{f} = f$ in a neighborhood of $K_0$ by construction. □

### 4.2. Manifolds

We now conclude the proof of Theorem 1 by a triangulation argument. Triangulate $M$ so that for every simplex $D$ there are smooth charts $g_i : W_i \to \mathbb{R}^n$ and $\tilde{g}_i : \widetilde{W}_i \to \mathbb{R}^n$ such that $f(W_i) \subset \widetilde{W}_i$ and $D$ is precompact in $W_i$. Such charts may be assumed to be volume-preserving by [19].

Enumerate the vertices. Apply Theorem 7 in charts to smooth $f$ in a neighborhood of the first vertex without changing $f$ in a neighborhood of simplices that do not contain this vertex. Repeat with the subsequent vertices. Now suppose we have already smoothed $f$ in a neighborhood of $m$-simplices, for some $0 \leqslant m \leqslant n-1$. Enumerate the $(m+1)$-simplices and apply Theorem 7 in charts to smooth it in a neighborhood of the first $(m+1)$-simplex, without changing it in a neighborhood of simplices that do not contain it (in particular we do not change it near its boundary). Repeat with the subsequent $(m+1)$-simplices. After $n$ steps we will have smoothed $f$ on the whole $M$.

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