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Liouville theorems for the Navier–Stokes equations and applications

by

Gabriel Koch
University of Chicago
Chicago, IL, U.S.A.

Nikolai Nadirashvili
CNRS, Université de Provence
Marseille, France

Gregory A. Seregin
Oxford University
Oxford, U.K.

Vladimir Šverák
University of Minnesota
Minneapolis, MN, U.S.A.

1. Introduction

It is a well-known principle in the regularity theory of partial differential equations that rescaling procedures are very useful in studying potential singularities. For example, for a minimal surface $\Sigma\subset\mathbf{R}^n$ for which $0\in\Sigma$ is a singular point, one should look at the surfaces $\lambda\Sigma$ in the limit $\lambda\to\infty$; see for example [15]. This "blow-up" procedure, probably first introduced by De Giorgi in his study of minimal surfaces, has become indispensable in the study of singularities of various geometric equations (see for example [16], [27] and [30]). Analogous ideas were introduced in the study of many other classes of equations, such as semilinear elliptic and parabolic equations [10], [12], [13], [25], [26], [29], [30], the Navier–Stokes equations [4], [7] and dispersive equations [18], [31], to name a few. The blow-up procedure can be compared to infinite magnification and therefore typically produces solutions of the original equation which are in some sense global. The study of such global solutions is often a valuable stepping stone towards understanding the structure of potential singularities (or the absence of singularities). In this paper we address some of these issues in the context of the Navier–Stokes equations

$$\begin{aligned} u_t+u\nabla u+\nabla p-\Delta u&=0,\\ \operatorname{div} u&=0. \end{aligned} \tag{1.1}$$

The scaling symmetry of the equations is $u(x,t)\mapsto\lambda u(\lambda x,\lambda^2 t)$, $p(x,t)\mapsto\lambda^2 p(\lambda x,\lambda^2 t)$ and can be used to "zoom in" on a solution near a potential singularity. There are some

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free parameters in this process, as we can choose where exactly (in space and time) we magnify (it does not have to be exactly at a singularity, it can for example be just before the singularity occurs), and which properties of the rescaled solutions we wish to control. In this paper we study the situation in which we choose the L^∞-norm of the rescaled velocity on a certain time interval as the parameter we wish to control. The pressure will play no explicit role in the process. As we will see in §5, this leads naturally to the following global problem:

Characterize solutions of (1.1) *in* $\mathbf{R}^n \times (-\infty, 0)$ *with* (*globally*) *bounded velocity* u.

Following [16], we will call solutions defined in $\mathbf{R}^n \times (-\infty, 0)$ *ancient solutions*. Stated in this terminology, we are interested in ancient solutions of (1.1) with bounded velocity. A first guess might be that such solutions should be constant. To make this a plausible conjecture, one must be slightly more precise. Equation (1.1) has trivial non-constant solutions of the form $u(x,t)=b(t)$, $p(x,t)=-b'(t)x$ and so we need a definition of solutions which would eliminate these "parasitic solutions". The right definition seems to be that of a *mild solution* (see §3), which was probably introduced in [17]. (Implicitly it is already used in Leray's paper [22].) Another natural definition often used in the literature is that of a *weak solution*, also essentially introduced in Leray's paper [22], which is defined using divergence-free test functions; see §3. This notion of solution does allow the parasitic solutions above. In these settings, the best possible result one can hope for which is consistent with what is known about the equations would be that any ancient mild solution with bounded velocity is constant and any ancient weak solution with bounded velocity is of the form $u(x,t)=b(t)$. We will prove that this is indeed the case in dimension 2 and also in the case of axi-symmetric fields in dimension 3, if some additional conditions are satisfied (see §5). The case of general 3-dimensional fields is, as far as we know, completely open. In fact, it is open even in the steady-state case (u independent of t).

The methods we use in the proofs of these results are elementary. The key component of the proof in dimension 2 is the use of the vorticity equation:

$$\omega_t + u\nabla\omega = \Delta\omega. \tag{1.2}$$

This is a scalar equation and ω satisfies the Harnack inequality (see e.g. [8]), which can be used to show that if $\omega \neq 0$, then in large areas of space-time ω has to be almost equal to its maximum/minimum. (In fact, the strong maximum principle, together with standard compactness results, is sufficient to prove this.) This turns out to be incompatible with the boundedness of u. (One might speculate that with the condition $\operatorname{div} u=0$, a Liouville theorem might be true for (1.2) at a linear level, without using the relation between u

and ω. This, however, appears to be false; see [32].) The ideas behind the proofs of the results for axi-symmetric fields in dimension 3 are similar. In each case, there is a scalar quantity satisfying a maximum principle which is used in a way similar to the 2-dimensional case. The quantities we use and the corresponding maximum principles are all classical.

There is a technical component in the proofs, since one needs to establish that the solutions we work with have sufficient regularity. This part is more or less standard, and we use elementary techniques based on explicit representation formulae to establish the required properties.

In the last section we use the Liouville theorems of §5 to obtain results limiting the types of singularities which may occur in axi-symmetric solutions of the Navier–Stokes equations. These results are inspired by the recent paper [4], where a significant progress in the study of the axi-symmetric case was made using methods quite different from those presented here. Our results on axi-symmetric singularities address some questions which were left open in [4]. Very recently we learned that the authors of [4] have independently proved results similar to those in §6 using their own methods; see [5].

It is known that axi-symmetric solutions with no swirl have to be regular; see [20] and [34]. (We recall that the "no swirl" condition means that in cylindrical coordinates (r,θ,z)—see (5.5)—the u_θ-component of the velocity vanishes.) However, the case of non-zero swirl is open at the time of this writing. We will prove that, under natural assumptions, every potential singularity of an axi-symmetric solution has to be of type II, in the sense of [16]. We recall that a singularity of a Navier–Stokes solution u at time T is called type I if

$$\sup_x |u(x,t)| \leqslant \frac{C}{\sqrt{T-t}}$$

for some $C>0$. Any singularity which is not of type I is called type II. A blow up of u by a type-II singularity is sometimes called *slow blow-up*, see e.g. [16]. Therefore we can rephrase our result by saying that if an axi-symmetric solution develops a singularity, it can only be through slow blow-up. We remark that Leray proved in [22] that if u develops a singularity at T, then

$$\sup_x |u(x,t)| \geqslant \frac{\varepsilon_1}{\sqrt{T-t}}$$

for some $\varepsilon_1>0$. Also, the rate $1/\sqrt{T-t}$ would be the blow-up rate of a self-similar singularity. (It is known that these do not exist; see [23] and [33].)

It is worth mentioning that although our results are obtained by methods which are more or less elementary, it seems that some of them are out of reach for the usual methods used in the theory of Navier–Stokes equations, such as energy methods or perturbation analyses in various function spaces. This is because some special properties of solutions of scalar equations, although simple, cannot be detected at the broad level at which the usual methods used for the Navier–Stokes equations are applied. A similar situation appears in the proof that Leray's self-similar singularities do not exist, see [23] and [33], where a (non-classical) scalar quantity satisfying an elliptic equation is used. At the time of this writing, there is no known similar quantity for the general 3-dimensional problem.

2. Preliminaries

Let $\Omega\subset\mathbf{R}^n$ be a bounded domain and let $T>0$. We consider the parabolic equation in $\Omega\times(0,T)$ of the form

$$u_t+a(x,t)\nabla u-\Delta u=0, \tag{2.1}$$

with $a\in L^\infty_{x,t}(\Omega\times(0,T))$. A suitable notion of solution is for example a weak solution. By definition, u is a *weak solution* of (2.1) if u and $\nabla_x u$ (the distributional derivative) belong to $(L^2_{x,t})_{\text{loc}}(\Omega\times(0,T))$ and the equation is satisfied in the sense of distributions. It then follows from standard regularity that in fact u_t and $\nabla^2_x u$ belong to $(L^p_{x,t})_{\text{loc}}(\Omega\times(0,T))$ for every $p\in(0,\infty)$, and the equation is satisfied pointwise almost everywhere in $\Omega\times(0,T)$. Moreover, by the local energy estimate, the local L^2-norm of ∇u is controlled by the local L^2-norm of u; see for example [21]. Therefore, there is no difference between weak solutions and strong solutions, and we can just use the term "solution" in the context of (2.1). We recall that the "parabolic boundary" of $\Omega\times(0,T)$ is

$$\partial_{\text{par}}(\Omega\times(0,T))=(\bar{\Omega}\times\{0\})\cup(\partial\Omega\times[0,T]).$$

When $x\in\Omega$, the space-time points (x,T) belong to the "parabolic interior" of $\Omega\times(0,T)$ and $u(x,T)$ is well defined. We recall that the solutions of (2.1) satisfy the *strong maximum principle*: If u is a bounded solution in $\Omega\times(0,T)$ such that $u(\bar{x},T)=\sup_{\Omega\times(0,T)} u$ for some $\bar{x}\in\Omega$, then u is constant in $\Omega\times(0,T)$. In fact, a much stronger statement is true: non-negative solutions of (2.1) satisfy the parabolic Harnack inequality; see e.g. [8]. The Harnack inequality immediately implies the strong maximum principle. For our purposes in this paper the strong maximum principle is sufficient—we will not need the full strength of the Harnack inequality. Our key tool will be the following lemma which essentially says that the statement of the strong maximum principle is in some sense stable under perturbations. (This stability can be made much more precise with the Harnack inequality.) The lemma is certainly known in one form or another, but we were unable to locate in the literature the precise statement we need.

LEMMA 2.1. *Let us consider equation* (2.1) *with bounded measurable coefficient a in* $\Omega\times(0,T)$. *Let* K *be a compact subset of* Ω, $\Omega'\subset\bar{\Omega}'\subset\Omega$, *and* $\tau>0$. *Then, for each* $\varepsilon>0$, *there exists* $\delta=\delta(\Omega,\Omega',K,T,\|a\|_{L^\infty_{x,t}},\tau,\varepsilon)>0$ *such that if u is a bounded solution of* (2.1) *with* $\sup_{\Omega\times(0,T)}|u|=M$ *and* $\sup_{x\in K}u(x,T)\geqslant M(1-\delta)$, *then* $u(x,t)\geqslant M(1-\varepsilon)$ *in* $\Omega'\times(\tau,T)$.

Proof. We can take $M=1$ without loss of generality. Assuming the statement fails for some $\varepsilon>0$, there must exist a sequence of coefficients $a^{(k)}$, solutions $u^{(k)}$ of (2.1) with $a=a^{(k)}$, and points $x_k\in K$ and $(y_k,t_k)\in\Omega'\times(\tau,T)$ such that $|a^{(k)}|\leqslant C$, $|u^{(k)}|\leqslant 1$, $u^{(k)}(x_k,T)\to 1$ and $u^{(k)}(y_k,t_k)\leqslant 1-\varepsilon$. We can assume, after passing to a subsequence, that $a^{(k)}$ converge weakly* in $L^\infty_{x,t}$ to $\bar{a}$, $u^{(k)}$ converge locally uniformly in $\Omega\times(0,T)$ to $\bar{u}$, $x_k\to\bar{x}\in K$ and $(y_k,t_k)\to(\bar{y},\bar{t})\in\bar{\Omega}'\times[\tau,T]$. The regularity properties of solutions of (2.1) discussed above imply that $\bar{u}$ solves (2.1) with $a=\bar{a}$, $|\bar{u}|\leqslant 1$ in $\Omega\times(0,T)$, $\bar{u}(\bar{x},T)=1$ and $\bar{u}(\bar{y},\bar{t})\leqslant 1-\varepsilon$. This, however, is impossible due to the strong maximum principle. □

3. Bounded solutions of the linear Stokes problem

Let us first recall some basic facts about the Cauchy problem for the linear Stokes system, with $u=(u_1,...,u_n)\colon\mathbf{R}^n\times(0,\infty)\to\mathbf{R}^n$ and the right-hand side in divergence form:

$$\left.\begin{aligned} u_t+\nabla p-\Delta u &= \frac{\partial}{\partial x_k}f_k \\ \operatorname{div} u &= 0 \end{aligned}\right\} \quad \text{in } \mathbf{R}^n\times(0,\infty), \tag{3.1}$$

$$u(\,\cdot\,,0)=u_0 \qquad \text{in } \mathbf{R}^n. \tag{3.2}$$

Here $f_k=(f_{1k},...,f_{nk})$ for $k=1,...,n$. Denoting the Helmholtz projection of vector fields on div-free fields by P, and the solution operator of the heat equation by S, we have the well-known representation formula

$$u(t)=S(t)u_0+\int_0^t S(t-s)P\frac{\partial}{\partial x_k}f_k(s)\,ds, \tag{3.3}$$

where, as usual, $u(t)$ denotes the function $u(\,\cdot\,,t)$, etc.

This can be written more concretely in terms of the kernel

$$K_{ij}(x,t)=\left(-\delta_{ij}\Delta+\frac{\partial^2}{\partial x_i\partial x_j}\right)\Phi(x,t),$$

where the "generating function" Φ is defined in terms of the fundamental solution of the Laplace operator G and the heat kernel Γ by

$$\Phi(x,t)=\int_{\mathbf{R}^n}G(y)\Gamma(x-y,t)\,dy, \tag{3.4}$$

which is the same as

$$\Phi(\,\cdot\,,t)=S(t)G.$$

See for example [24]. Letting

$$K_{ijk}=\frac{\partial}{\partial x_k}K_{ij}\,,$$

we can rewrite (3.3) as

$$u_i(x,t)=\int_{\mathbf{R}^n}\Gamma(x-y,t)u_{0i}(y)\,dy+\int_0^t\int_{\mathbf{R}^n}K_{ijk}(x-y,t-s)f_{jk}(y,s)\,dy\,ds. \tag{3.5}$$

Note also the obvious estimates

$$|K_{ij}(x,t)|\leqslant\frac{C}{(|x|^2+t)^{n/2}} \tag{3.6}$$

and

$$|K_{ijk}(x,t)|\leqslant\frac{C}{(|x|^2+t)^{(n+1)/2}}. \tag{3.7}$$

As a consequence of (3.7), the expression (3.5) is well defined for $f\in L^\infty_{x,t}$. We remark that, in contrast, solutions of

$$\left.\begin{aligned} u_t+\nabla p-\Delta u=f\\ \operatorname{div}u=0\end{aligned}\right\}\quad\text{in } \mathbf{R}^n\times(0,\infty), \tag{3.8}$$

$$u(\,\cdot\,,0)=u_0\qquad\text{in } \mathbf{R}^n \tag{3.9}$$

are not well defined for $f\in L^\infty_{x,t}$, although the ambiguity is small. This can also be seen without using the explicit form of the kernel, in the following way: One can write, for each t, the Helmholtz decomposition of $f(x,t)$ as $f(x,t)=Pf(x,t)+\nabla_x\phi(x,t)$. The projection P can be naturally defined on $L^\infty(\mathbf{R}^n)$ (which is mapped by P into $\mathrm{BMO}(\mathbf{R}^n)$) only modulo constants, which creates an ambiguity. However, if the right-hand side is in divergence form, this ambiguity is cancelled by the extra derivative.

By definition, a *mild solution* of the Cauchy problem (3.1) and (3.2) is a function u defined by the formula (3.5). We note that this definition does not involve the pressure. One can obtain (formally) an explicit formula for the pressure, but, unlike the formula for the velocity field u, it defines p only modulo a function of t (constant in x for each t) when f_k is in $L^\infty_{x,t}$.

The definition of mild solutions immediately implies their uniqueness. Also, we have standard estimates for u in terms of $f=(f_1,...,f_n)=(f_{ij})^n_{i,j=1}$. In particular, for $u_0=0$, we have the estimates

$$\|u\|_{C^\alpha_{\mathrm{par}}(Q(z_0,R))}\leqslant C(\alpha,R)\|f\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}, \tag{3.10}$$

$$\|\nabla_x u\|_{L^p_{x,t}(Q(z_0,R))}\leqslant C(p,R)\|f\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))} \tag{3.11}$$

for any $\alpha\in(0,1)$ and $p\in(1,\infty)$, where $Q(z_0,R)=Q((x_0,t_0),R)=B(x_0,R)\times(t_0-R^2,t_0)$ is any parabolic ball contained in $\mathbf{R}^n\times(0,T)$. The space C^α_{par} is defined by means of the parabolic distance $\sqrt{|x-x'|^2+|t-t'|}$.

Taking difference quotients, we see that, for $u_0=0$, we have similar estimates for spatial derivatives:

$$\|\nabla_x^k u\|_{C^\alpha_{\mathrm{par}}(Q(z_0,R))}\leqslant C(\alpha,R)\|\nabla_x^k f\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))},\tag{3.12}$$

$$\|\nabla_x^{k+1} u\|_{L^p_{x,t}(Q(z_0,R))}\leqslant C(p,R)\|\nabla_x^k f\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}.\tag{3.13}$$

Moreover, a routine inspection of representation formula (3.5) shows that, when $u_0=0$, the time derivative satisfies, for $k=0,1,\ldots$,

$$\|\nabla_x^k u_t\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}\leqslant C(T,k)\|\nabla_x^{k+2} f\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}.\tag{3.14}$$

We sketch the calculation leading to the last estimate in the case $k=0$ for the convenience of the reader: Clearly it is enough to estimate $|u_t(0,t)|$. Let Φ be the generating function defined in (3.4), which will be considered as a function of $\mathbf{R}^n\times\mathbf{R}$, with $\Phi=0$ for negative values of t. We can write

$$u_i=(L_{ijk}\Phi)*f_{jk},\tag{3.15}$$

where L_{ijk} is a homogeneous constant coefficient operator in x of order 3 and $*$ denotes space-time convolution. Applying the heat operator to (3.15), we can write, with a slight abuse of notation,

$$(\partial_t-\Delta)u_i=(L_{ijk}(\partial_t-\Delta)\Phi)*f_{jk}=(L_{ijk}G(x)\delta(t))*f_{jk},\tag{3.16}$$

where G is the fundamental solution of the Laplacian and $\delta(t)$ is the Dirac distribution in t. We consider a smooth cut-off function $\eta=\eta(x)$ on $\mathbf{R}^n$ with $\eta=1$ in the unit ball $B(0,1)$ and $\eta=0$ outside of $B(0,2)$, and set $f'=\eta f$ and $f''=(1-\eta)f$. Let us first look at u_i', the contribution to u_i in (3.16) coming from f'. We can move two derivatives from L_{ijk} to f'_{jk} to obtain an estimate of $(\partial_t-\Delta)u_i'(0,t)$ in terms of the $L^\infty_{x,t}$-norm of the second derivatives of f'_{jk}. The estimate of $(\partial_t-\Delta)u_i''(0,t)$ (with the obvious meaning of u_i'') is even simpler, since $L_{ijk}G$ is integrable in $\mathbf{R}^n\setminus B(0,1)$ and therefore $(\partial_t-\Delta)u_i''(0,t)$ can be estimated in terms of the $L^\infty_{x,t}$-norm of f''_{jk}. Once we have the estimate for $(\partial_t-\Delta)u$, the estimate for u_t follows from (3.12).

To define the notion of weak solution of equation (3.1), we follow the standard procedures and introduce the space $\mathcal{V}_T$ of smooth compactly supported div-free vector fields $\varphi:\mathbf{R}^n\times(0,T)\to\mathbf{R}^n$. We then say that a bounded measurable vector field $u:\mathbf{R}^n\times(0,T)\to\mathbf{R}^n$ is a *weak solution* of (3.1) if $\operatorname{div}u=0$ in $\mathbf{R}^n\times(0,T)$ (in the sense of distributions) and, for each $\varphi\in\mathcal{V}_T$,

$$\int_0^T\int_{\mathbf{R}^n}u(\varphi_t+\Delta\varphi)\,dx\,dt=\int_0^T\int_{\mathbf{R}^n}f_k\frac{\partial}{\partial x_k}\varphi\,dx\,dt.$$

LEMMA 3.1. *For a fixed* $f\in L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$ *let* $u\in L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$ *be any weak solution of* (3.1) *in* $\mathbf{R}^n\times(0,T)$, *and denote by* v *the mild solution of the Cauchy problem* (3.1) *and* (3.2) *with* $u_0=0$. *Then* $u(x,t)=v(x,t)+w(x,t)+b(t)$, *where* w *satisfies the heat equation* $w_t-\Delta w=0$ *in* $\mathbf{R}^n\times(0,T)$ *and* b *is a bounded measurable* $\mathbf{R}^n$*-valued function on* $(0,T)$. *Moreover, we have the estimates*

$$\|w\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}\leqslant C(T)\|u\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}, \tag{3.17}$$

$$\|b\|_{L^\infty(0,T)}\leqslant C(T)\|u\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}. \tag{3.18}$$

Proof. In view of the estimates (3.10), it is enough to consider only the case $f=0$. Let $\phi:\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}$ be a mollifier compactly supported in $\mathbf{R}^n\times(-1,0)$. For any $\varepsilon>0$ let $\phi_\varepsilon(x,t)=\varepsilon^{-(n+1)}\phi(x/\varepsilon,t/\varepsilon)$ and let $u_\varepsilon:\mathbf{R}^n\times(0,T-\varepsilon)$ be defined by $u_\varepsilon=\phi_\varepsilon*u$ (space-time convolution). Let w_ε be the solution of the heat equation in $\mathbf{R}^n\times(0,T)$ with initial datum $w_\varepsilon(x,0)=u_\varepsilon(x,0)$. The (smooth and bounded) function $h_\varepsilon=\operatorname{curl}(u_\varepsilon-w_\varepsilon)$ satisfies the heat equation in $\mathbf{R}^n\times(0,T-\varepsilon)$ with initial datum $h_\varepsilon(x,0)=0$ and therefore it must vanish. Since bounded solutions of the system $\operatorname{curl} z=0$ and $\operatorname{div} z=0$ in $\mathbf{R}^n$ are constant by Liouville's theorem, we see that $u_\varepsilon(x,t)-w_\varepsilon(x,t)=b_\varepsilon(t)$ for a suitable $b_\varepsilon:(0,T-\varepsilon)\to\mathbf{R}^n$. By compactness properties of families of bounded solutions of the heat equation, we see that if $\varepsilon\to0$ along a suitable sequence, the functions b_ε converge a.e. to an L^∞ function $b:(0,T)\to\mathbf{R}^n$. The estimates follow from the constructions. □

Remark 3.1. In the above decomposition, the function v is of course uniquely determined by f, whereas the functions w and b are determined up to a constant (independent of time). In other words, the (distributional) derivative $b'(t)$ is uniquely determined by u and f.

4. Bounded solutions of Navier–Stokes

Let us now consider the Cauchy problem for the Navier–Stokes equations:

$$\left.\begin{aligned} u_t+u\nabla u+\nabla p-\Delta u&=0\\ \operatorname{div} u&=0\end{aligned}\right\}\quad \text{in } \mathbf{R}^n\times(0,\infty), \tag{4.1}$$

$$u(\,\cdot\,,0)=u_0\quad \text{in } \mathbf{R}^n. \tag{4.2}$$

The considerations of the previous section can be repeated with $f_k=-u_ku$. In particular, a function $u\in L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$ is defined to be (i) a *mild solution* of the Cauchy problem (4.1) and (4.2) if (3.5) is valid with $f_k=-u_ku$ and (ii) a *weak solution* of equation (4.1) in $\mathbf{R}^n\times(0,T)$ if $\operatorname{div} u=0$ in $\mathbf{R}^n\times(0,T)$ (in the sense of distributions) and, for each $\varphi\in\mathcal{V}_T$,

$$\int_0^T\int_{\mathbf{R}^n} u(\varphi_t+\Delta\varphi)\,dx\,dt=\int_0^T\int_{\mathbf{R}^n}\left(-u_ku\frac{\partial}{\partial x_k}\varphi\right)dx\,dt.$$

Remark 4.1. It is obvious that the notions of weak solution and mild solution are also well defined under the assumption that $u\in L^\infty_{x,t}(\mathbf{R}^n\times(0,T'))$ for each $T'<T$ (with the possibility that $\|u\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T'))}\to\infty$ as $T'\nearrow T$). This is a natural setting in which potential singularities of solutions of the Cauchy problem can be studied. Even if one considers the Cauchy problem for u_0 in spaces other than $L^\infty(\mathbf{R}^n)$, such as $L^n(\mathbf{R}^n)$ ([17]) or $\mathrm{BMO}^{-1}(\mathbf{R}^n)$ ([19]), the local-in-time solution $u:\mathbf{R}^n\times(0,T)\to\mathbf{R}^n$ which is constructed for u_0 in these spaces typically belongs to $L^\infty_{x,t}(\mathbf{R}^n\times(\tau,T-\tau))$ for any $\tau>0$.

The existence and uniqueness of *local-in-time* mild solutions of the Cauchy problem (4.1) and (4.2) with $u_0\in L^\infty$ was addressed in [11]. We briefly outline a slightly modified approach using standard perturbation theory. We define the bilinear form

$$B: L^\infty_{x,t}(\mathbf{R}^n\times(0,T))\times L^\infty_{x,t}(\mathbf{R}^n\times(0,T))\longrightarrow L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$$

by

$$B(u,v)_i(x,t)=\int_0^t\int_{\mathbf{R}^n}(-K_{ijk}(x-y,t-s)u_k(y,s)v_j(y,s))\,dy\,ds, \tag{4.3}$$

and we denote by U the heat extension of the initial datum u_0. The equation for u then becomes

$$u=U+B(u,u), \tag{4.4}$$

and can be solved in $L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$ for sufficiently small T by a fixed point argument, since estimate (3.7) easily implies that

$$\|B(u,v)\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}\leqslant C\sqrt{T}\,\|u\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}\|v\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}. \tag{4.5}$$

We remark that (3.10) implies that the solutions of (4.4) have enough regularity to allow us to treat (4.4) as an ordinary differential equation in t, without making assumptions about u other than $u\in L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$.

We recall now the regularity properties of mild solutions in $L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$. The following (optimal) result will not be needed here in its full generality, but we feel it is still worth mentioning.

Proposition 4.1. *Let $u\in L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$ be a mild solution of* (4.1) *and* (4.2) *with $u_0\in L^\infty$. Then, for $k,l=0,1,\dots$, the functions $t^{k/2+l}\nabla^k_x\partial^l_t u$ are bounded and, for $T'=\varepsilon(k,l)\|u_0\|^{-2}_{L^\infty(\mathbf{R}^n)}$ (where $\varepsilon(k,l)>0$ is a small constant), we have*

$$\|t^{k/2+l}\nabla^k_x\partial^l_t u\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T'))}\leqslant C(k,l)\|u_0\|_{L^\infty(\mathbf{R}^n)}. \tag{4.6}$$

Proof. This can be proved in the same way as the corresponding results in [14], [6] and [9], where the authors work in function spaces other than $L^\infty_{x,t}$. The key is an estimate of B with the same form as (4.5), but in spaces with norms given by the expression on the left-hand side of (4.6). In the context of the $L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$-based norms we use here, the proof is in fact much simpler than in that of the spaces used in the above papers, due to the elementary nature of estimate (4.5). □

Remark 4.2. Estimate (4.6) says that the local-in-time smoothing properties of the Navier–Stokes equations for $u_0\in L^\infty$ are the same as those of the heat equation. Since the solution u is constructed essentially as a power series perturbation around the heat extension U of u_0, this may not be surprising.

LEMMA 4.1. *Let $u^{(k)}\in L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$ be a sequence of mild solutions of (4.1) and (4.2) with initial conditions $u_0^{(k)}$. Assume that $\|u^{(k)}\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}\leqslant C$, with C independent of k. Then, a subsequence of the sequence $u^{(k)}$ converges locally uniformly in $\mathbf{R}^n\times(0,T)$ to a mild solution $u\in L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$ with initial datum $u(x,0)$ given by the weak* limit of a suitable subsequence of the sequence $u_0^{(k)}$.*

Proof. This is a routine consequence of (4.6), and the decay estimate (3.7) for the kernel K_{ijk}. □

We now turn to regularity properties of bounded weak solutions. Let

$$u\in L^\infty_{x,t}(\mathbf{R}^n\times(0,T))$$

be a weak solution of (4.1) in $\mathbf{R}^n\times(0,T)$, and let $M=\|u\|_{L^\infty_{x,t}(\mathbf{R}^n\times(0,T))}$. Let v be the mild solution of the linear Cauchy problem (3.1) and (3.2) with $f_k=-u_k u$ and $u_0=0$. By Lemma 3.1, we can write $u=v+w+b$ with the L^∞-norms of v, w and b bounded by $N=C_1(T)M^2+C_2(T)M$, $w_t-\Delta w=0$ and b is a function of t only. Hence for $k=0,1,2,...$ and $\delta>0$ the derivatives $\nabla_x^k(w+b)$ are bounded by $C(k,\delta)N$ in $\mathbf{R}^n\times(\delta,T)$ by estimates for the heat equation. Moreover, we have the L^p-estimate (3.11) for $\nabla_x v$. Therefore $\omega=\operatorname{curl} u$ belongs to $L^p_{x,t}(Q(z_0,R))$ for any $p\in(1,\infty)$ and any $Q(z_0,R)\subset\mathbf{R}^n\times(\delta,T)$, with

$$\|\omega\|_{L^p_{x,t}(Q(z_0,R))}\leqslant C(p,\delta,R,M). \tag{4.7}$$

Following [28], we can now use the equation for ω to obtain estimates for higher derivatives $\nabla_x^k u$. For $n=3$, the equation for ω is

$$(\omega_i)_t-\Delta\omega_i=\frac{\partial}{\partial x_j}(\omega_j u_i-\omega_i u_j), \tag{4.8}$$

and it is easy to check that in our situation this equation is satisfied in the sense of distributions. Equation (4.8) gains ω one spatial derivative in $L^p_{x,t}$. The standard bootstrapping arguments and regularity estimates for harmonic functions now give

$$\|\nabla^k_x u\|_{L^p_{x,t}(Q(z_0,R))} \leqslant C(k,\delta,R,M) \tag{4.9}$$

for each $Q_(z_0,R)\subset \mathbf{R}^n\times(\delta,T)$. Therefore, using standard imbeddings, for $k=0,1,2\ldots$ we have

$$\|\nabla^k_x u\|_{L^\infty_{x,t}(\mathbf{R}^n\times(\delta,T))} \leqslant C(k,\delta,T,M). \tag{4.10}$$

Finally, using (3.14) we also obtain, for $k=0,1,2\ldots$,

$$\|\nabla^k_x \partial_t(u-b)\|_{L^\infty_{x,t}(\mathbf{R}^n\times(\delta,T))} \leqslant C(k,\delta,R,M). \tag{4.11}$$

(We adopt the usual convention that the value of C can change from line to line.)

5. Liouville theorems

Let us first consider the Navier–Stokes equations in two space dimensions.

THEOREM 5.1. *Let u be a bounded weak solution of the Navier–Stokes equations in $\mathbf{R}^2\times(-\infty,0)$. Then $u(x,t)=b(t)$ for a suitable bounded measurable $b\colon(-\infty,0)\to\mathbf{R}^2$.*

Proof. In two space dimensions the vorticity is a scalar quantity defined by

$$\omega = u_{2,1}-u_{1,2}, \tag{5.1}$$

where the indices after comma mean derivatives, i.e. $u_{2,1}=\partial u_2/\partial x_1$, etc. By the results of §4, the function ω is uniformly bounded together with its spatial derivatives. Moreover, its time derivative is also uniformly bounded. The vorticity equation in dimension 2 is

$$\omega_t + u\nabla\omega - \Delta\omega = 0. \tag{5.2}$$

Let

$$M_1 = \sup_{\mathbf{R}^2\times(-\infty,0)} \omega \quad \text{and} \quad M_2 = \inf_{\mathbf{R}^2\times(-\infty,0)} \omega,$$

and assume that $M_1>0$. Applying Lemma 2.1 to $\omega-\frac12(M_1+M_2)$, we see that there exist arbitrarily large parabolic balls $Q_R=Q((\bar x,\bar t),R)=B(\bar x,R)\times(\bar t-R^2,\bar t)$ such that $\omega\geqslant\frac12 M_1$ in $Q((\bar x,\bar t),R)$. For such parabolic balls, we have

$$\int_{Q_R} \omega\,dx\,dt \geqslant \frac{1}{2}\pi M_1 R^4. \tag{5.3}$$

On the other hand, denoting by n the normal to the boundary of $B(\bar{x}, R)$, we can also write

$$\int_{Q_R} \omega \, dx \, dt = \int_{Q_R} (u_{2,1} - u_{1,2}) \, dx \, dt = \int_{\partial B(\bar{x}, R) \times (\bar{t} - R^2, \bar{t})} (u_2 n_1 - u_1 n_2) \, ds \, dt \leqslant CR^3. \quad (5.4)$$

Clearly (5.3) is not compatible with (5.4), unless $M_1 \leqslant 0$. In the same way, we conclude that $M_2 \geqslant 0$ and therefore ω must vanish identically. Hence $\operatorname{curl} u = 0$ in $\mathbf{R}^2 \times (-\infty, 0)$ which, together with $\operatorname{div} u = 0$ and the boundedness of u, implies (by the classical Liouville theorem for harmonic functions) that u is constant in x for each t. □

It is not known if a result similar to Theorem 5.1 remains true in three spatial dimensions. In fact, the problem is open even in the steady-state case. However, under the additional assumption that the solutions are axi-symmetric, one can obtain some results which seem to be of interest. We recall that a vector field u in $\mathbf{R}^3$ is axi-symmetric if it is invariant under rotations about a suitable axis, which is often identified with the x_3-coordinate axis. In other words, a field u is axi-symmetric if $u(Rx) = Ru(x)$ for every rotation R of the form

$$R = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In cylindrical coordinates (r, θ, z) given by

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta \quad \text{and} \quad x_3 = z, \quad (5.5)$$

the axi-symmetric fields are given by

$$u = u_r \frac{\partial}{\partial r} + u_\theta \frac{\partial}{r\partial\theta} + u_z \frac{\partial}{\partial z},$$

where the coordinate functions u_r, u_θ and u_z depend only on r and z. In these coordinates, the Navier–Stokes equations become

$$(u_r)_t + u_r u_{r,r} + u_z u_{r,z} - \frac{{u_\theta}^2}{r} + p_{,r} = \Delta u_r - \frac{u_r}{r^2}, \quad (5.6)$$

$$(u_\theta)_t + u_r u_{\theta,r} + u_z u_{\theta,z} + \frac{u_r u_\theta}{r} = \Delta u_\theta - \frac{u_\theta}{r^2}, \quad (5.7)$$

$$(u_z)_t + u_r u_{z,r} + u_z u_{z,z} + p_{,z} = \Delta u_z, \quad (5.8)$$

$$\frac{(r u_r)_{,r}}{r} + u_{z,z} = 0, \quad (5.9)$$

where Δ is the scalar Laplacian (expressed in the coordinates (r, θ, z)), $u_{r,z}$ denotes the partial derivative $\partial u_r / \partial z$, etc. The equation for u_θ is of special interest, as it is decoupled from the pressure. The role of the non-linear terms in this equation can be seen by considering the inviscid case (Euler's equations), wherein equation (5.7) is replaced by

$$(u_\theta)_t + u_r u_{\theta,r} + u_z u_{\theta,z} + \frac{u_r u_\theta}{r} = 0, \tag{5.10}$$

which is the same as

$$(ru_\theta)_t + u_r(ru_\theta)_{,r} + u_z(ru_\theta)_{,z} = 0. \tag{5.11}$$

Equation (5.11) says that the quantity ru_θ "moves with the flow". This is a special case of Kelvin's law that the integral of $u_i\,dx_i$ along curves moving with the flow is constant. In the situation considered here, the curves are circles centered at the x_3-axis and lying in planes perpendicular to it.

In view of (5.11), it is natural to rewrite (5.7) as an equation for ru_θ:

$$(ru_\theta)_t + u_r(ru_\theta)_{,r} + u_z(ru_\theta)_{,z} = \Delta(ru_\theta) - \frac{2}{r}(ru_\theta)_{,r}. \tag{5.12}$$

The infinitesimal version of Kelvin's law, which is Helmholtz's law that vorticity "moves with the flow" (for inviscid flows), gives in the case of axi-symmetric flows without swirl ($u_\theta = 0$) another quantity which moves with the flow, namely ω_θ/r. Here $\omega = \operatorname{curl} u$, as usual, and in cylindrical coordinates we write

$$\omega = \omega_r \frac{\partial}{\partial r} + \omega_\theta \frac{\partial}{r\partial\theta} + \omega_z \frac{\partial}{\partial z}.$$

(For axi-symmetric flows without swirl we have $\omega_r = \omega_z = 0$ and we can write

$$\omega = \omega_\theta \frac{\partial}{r\partial\theta}.$$

Therefore the situation is similar to 2-dimensional flows.)

Hence, for axi-symmetric solutions of Euler's equations without swirl, we have

$$\left(\frac{\omega_\theta}{r}\right)_t + u_r\left(\frac{\omega_\theta}{r}\right)_{,r} + u_z\left(\frac{\omega_\theta}{r}\right)_{,z} = 0. \tag{5.13}$$

This is nothing but the θ-component of the equation for ω, and can of course be obtained by simple calculation, without any consideration of the Helmholtz law. For axi-symmetric solutions of the Navier–Stokes equations without swirl, the last equation becomes

$$\left(\frac{\omega_\theta}{r}\right)_t + u_r\left(\frac{\omega_\theta}{r}\right)_{,r} + u_z\left(\frac{\omega_\theta}{r}\right)_{,z} = \Delta\left(\frac{\omega_\theta}{r}\right) + \frac{2}{r}\left(\frac{\omega_\theta}{r}\right)_{,r}. \tag{5.14}$$

Remark 5.1. For a smooth vector field u, the apparent singularity of ω_θ/r is only an artifact of the coordinate choice. The quantity ω_θ/r is actually a smooth function, even across the x_3-axis, as long as u is smooth.

The diffusion term on the right-hand side of equation (5.14) can be interpreted as the 5-dimensional Laplacian acting on SO(4)-invariant functions in $\mathbf{R}^5$. We write $r=\sqrt{y_1^2+y_2^2+y_3^2+y_4^2}$ and $y_5=z$, and we note that for $\tilde{f}(y_1, ..., y_5)=f(r,z)$, we have

$$\Delta_y \tilde{f}(y_1, ..., y_5) = \left(\frac{\partial^2 f}{\partial r^2} + \frac{3\partial f}{r\partial r} + \frac{\partial^2 f}{\partial z^2}\right)(r,z). \tag{5.15}$$

Therefore, with a slight abuse of notation, we can write the equation (5.14) as

$$\left(\frac{\omega_\theta}{r}\right)_t + u_r\left(\frac{\omega_\theta}{r}\right)_{,r} + u_z\left(\frac{\omega_\theta}{r}\right)_{,z} = \Delta_5\left(\frac{\omega_\theta}{r}\right). \tag{5.16}$$

THEOREM 5.2. *Let u be a bounded weak solution of the Navier–Stokes equations in $\mathbf{R}^3\times(-\infty,0)$. Assume that u is axi-symmetric with no swirl. Then, $u(x,t)=(0,0,b_3(t))$ for some bounded measurable function $b_3\colon(-\infty,0)\to\mathbf{R}$.*

Proof. The idea of the proof is the same as in the 2-dimensional case. By the results of §4, we have $|\nabla_x^k u|\leqslant C_k$ in $\mathbf{R}^3\times(-\infty,0)$, and this implies that ω_θ/r is bounded in $\mathbf{R}^3\times(-\infty,0)$. Let

$$M_1 = \sup_{\mathbf{R}^3\times(-\infty,0)} \frac{\omega_\theta}{r} \quad \text{and} \quad M_2 = \inf_{\mathbf{R}^3\times(-\infty,0)} \frac{\omega_\theta}{r},$$

and assume that $M_1>0$. Applying Lemma 2.1 to the solution $\omega_\theta/r-\frac{1}{2}(M_1+M_2)$ of equation (5.16), considered as an equation in $\mathbf{R}^5\times(-\infty,0)$, we see that $\omega_\theta/r\geqslant\frac{1}{2}M_1$ in arbitrarily large parabolic balls (with suitably chosen centers). However, this would mean that ω_θ is unbounded, a contradiction. Therefore $M_1\leqslant 0$. In the same way we show that $M_2\geqslant 0$, and hence ω_θ vanishes identically. For axi-symmetric vector fields with no swirl, this means that $\omega=0$ and the proof is finished by again applying the Liouville theorem to the system $\operatorname{curl} u=0$, $\operatorname{div} u=0$. □

The validity of Theorem 5.2 in the absence of the "no swirl" assumption is still an open problem. The following theorem, however, is a partial result in that direction.

THEOREM 5.3. *Let u be a bounded weak solution of the Navier–Stokes equations in $\mathbf{R}^3\times(-\infty,0)$. Assume that u is axi-symmetric and, in addition, satisfies*

$$|u(x,t)| \leqslant \frac{C}{\sqrt{x_1^2+x_2^2}} \quad \text{in } \mathbf{R}^3\times(-\infty,0). \tag{5.17}$$

Then, $u=0$ in $\mathbf{R}^3\times(-\infty,0)$.

Proof. We will use the cylindrical coordinates (r,θ,z) given by (5.5). We set $f=ru_\theta$ and recall that

$$f_t + u_r f_{,r} + u_z f_{,z} = \Delta f - \frac{2}{r} f_{,r}. \tag{5.18}$$

For $\lambda>0$ we let $f^\lambda(x,t)=f(\lambda x,\lambda^2 t)$ and $u^\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t)$. We note that f^λ again satisfies (5.18) with u replaced by u^λ, a consequence of the fact that u^λ satisfies the Navier–Stokes equations. Under our assumptions we have

$$|f^\lambda|\leqslant C \quad \text{in } \mathbf{R}^3\times(-\infty,0) \text{ uniformly in } \lambda>0, \tag{5.19}$$

$$|u^\lambda|\leqslant \frac{C}{r} \quad \text{in } \mathbf{R}^3\times(-\infty,0) \text{ uniformly in } \lambda>0. \tag{5.20}$$

Let $M=\sup_{\mathbf{R}^3\times(-\infty,0)} f$ and $m=\inf_{\mathbf{R}^3\times(-\infty,0)} f$. We will show that $M\leqslant 0$ and $m\geqslant 0$. Arguing by contradiction, let us first assume that $M>0$. By rescaling $f\mapsto f^\lambda$, we can move points where f^λ is "almost equal to M" close to the x_3-axis. Using this and applying Lemma 2.1 to $f^\lambda-\frac{1}{2}(M+m)$, we see that for any (large) $T_1>0$ and $L>0$ and any (small) $\varepsilon>0$ we can find $\lambda>0$ such that $f^\lambda\geqslant M-\varepsilon$ in a space-time region $\mathcal{R}_1$ of the form

$$\mathcal{R}_1=\bigl\{x\in\mathbf{R}^3 : 1\leqslant \sqrt{x_1^2+x_2^2}\leqslant 2 \text{ and } -L+\bar{x}_3\leqslant x_3\leqslant L+\bar{x}_3\bigr\}\times(\bar{t}-T_1,\bar{t}\,). \tag{5.21}$$

Consider a smooth axi-symmetric cut-off function φ on $\mathbf{R}^3\times(-\infty,0)$ of the form

$$\varphi(r,z,t)=\xi(r)\eta(z-\bar{x}_3)\zeta(t),$$

where $\xi\colon[0,\infty)\to[0,1]$ is a smooth function supported in $[0,2)$ with $\xi=1$ in $[0,1]$ and $|\xi'|+|\xi''|\leqslant 4$ in $[0,\infty)$, $\eta\colon(-\infty,\infty)\to[0,1]$ is a smooth function supported in $(-L,L)$ with $\eta=1$ in $(-L+1,L-1)$ and $|\eta'|+|\eta''|\leqslant 4$ in $(-\infty,\infty)$, and $\zeta\colon(-\infty,0)\to[0,1]$ is a smooth function supported in $(\bar{t}-T_1,\bar{t})$ with $\zeta=1$ in $(\bar{t}-T_1+1,\bar{t}-1)$ and $|\zeta'|\leqslant 2$ in $(-\infty,0)$. Multiplying the equation for f^λ by φ and integrating over space-time, we obtain

$$\int_{-\infty}^0\int_{\mathbf{R}^3}(f_t^\lambda+u_r^\lambda f_{,r}^\lambda+u_z^\lambda f_{,z}^\lambda-\Delta f^\lambda)\varphi\,dx\,dt=\int_{-\infty}^0\int_{\mathbf{R}^3}\left(-\frac{2}{r}f_{,r}^\lambda\varphi\right)dx\,dt. \tag{5.22}$$

This equality will be shown to be impossible when $M>0$. In the integral on the left-hand side of (5.22) one can change f^λ to $f^\lambda-M$ and integrate by parts, to obtain

$$\int_{-\infty}^0\int_{\mathbf{R}^3}(f^\lambda-M)(-\varphi_t-u^\lambda\nabla\varphi-\Delta\varphi)\,dx\,dt=\mathrm{I}+\mathrm{II}+\mathrm{III}. \tag{5.23}$$

We have $|f^\lambda-M|\leqslant\varepsilon$ in $\mathcal{R}_1$. Using (5.19) and (5.20), it is not hard to see that for a suitable constant $\gamma>0$ which is independent of the parameters ε, L and T_1, we have

$$|\mathrm{I}|\leqslant\gamma(CL+\varepsilon L),\quad |\mathrm{II}|\leqslant\gamma(C^2T_1+\varepsilon CLT_1)\quad\text{and}\quad |\mathrm{III}|\leqslant\gamma(CT_1+\varepsilon LT_1). \tag{5.24}$$

On the other hand, the right-hand side of (5.22) can be written as follows:

$$\int_{-\infty}^0\int_{\mathbf{R}^3}\left(-\frac{2}{r}f_{,r}^\lambda\varphi\right)dx\,dt=4\pi\int_{-\infty}^0\int_{-\infty}^\infty\int_0^\infty f^\lambda\varphi_{,r}\,dr\,dz\,dt. \tag{5.25}$$

The key point here is that f^λ vanishes at the x_3-axis and equals $M+O(\varepsilon)$ on most of the support of $(\varphi)_r$. It is easy to check that the last integral in (5.25) can be written as

$$-4\pi M\int_{-\infty}^0\int_{-\infty}^\infty \varphi(0,z,t)\,dz\,dt+\int_{-\infty}^0\int_{-\infty}^\infty\int_0^\infty 4\pi(f^\lambda-M)\xi'(r)\eta(z-\bar{x}_3)\zeta(t)\,dr\,dz\,dt. \tag{5.26}$$

This expression can be estimated from above by $-8\pi M(L-1)(T_1-2)+\gamma\varepsilon LT_1$, where γ is a constant independent of ε, L and T_1. This leads to a contradiction to (5.24) and (5.22) if L and T_1 are sufficiently large and ε is sufficiently small. We have proved that $\sup f\leqslant 0$. It follows in a similar way that $\inf f\geqslant 0$ and therefore f must vanish. This means that the solution u is swirl-free and we can apply Theorem 5.2 to conclude that $u=0$ in $\mathbf{R}^3\times(-\infty,0)$. □

6. Singularities and ancient solutions

We will now consider the consequences of an assumption that a singularity exists in a solution of the Cauchy problem for the Navier–Stokes equations (4.1) and (4.2). We aim to show that singularities generate bounded *ancient solutions*, which are solutions defined in $\mathbf{R}^n\times(-\infty,0)$. More precisely, an *ancient weak solution* of the Navier–Stokes equations is a weak solution defined in $\mathbf{R}^n\times(-\infty,0)$, and u is an *ancient mild solution* if there is a sequence $T_l\to-\infty$ such that $u(\cdot,T_l)$ is well defined and u is a mild solution of the Cauchy problem in $\mathbf{R}^n\times(T_l,0)$ with initial datum $u(\cdot,T_l)$. (We remark that even if u is a bounded weak solution of Navier–Stokes in $\mathbf{R}^n\times(-\infty,0)$, the function $u(\cdot,t)$ may not be well defined for each t; see §4. On the other hand, $u(\cdot,t)$ is well defined for almost every t for any $u\in L^\infty_{x,t}(\mathbf{R}^n\times(-\infty,0))$.)

LEMMA 6.1. *Let u_l be a sequence of bounded mild solutions of the Navier–Stokes equations defined in $\mathbf{R}^n\times(T_l,0)$ (for some initial data) with a uniform bound $|u_l|\leqslant C$, and $T_l\searrow-\infty$. Then, we can choose a subsequence along which u_l converges locally uniformly in $\mathbf{R}^n\times(-\infty,0)$ to an ancient mild solution u satisfying $|u|\leqslant C$ in $\mathbf{R}^n\times(-\infty,0)$.*

Proof. This is an easy consequence of the results in §4. □

Remark 6.1. Another easy result, which is nevertheless a useful addendum to the Liouville theorems of §5 is the following: A bounded ancient mild solution $u(x,t)$ of the Navier–Stokes equations which is of the form $u(x,t)=b(t)$ is constant (independent of t).

We leave the proof of the last statement to the reader as a simple exercise.

Recall from §4 that for any $u_0\in L^\infty(\mathbf{R}^n)$ the Cauchy problem (4.1), (4.2) has a unique local-in-time mild solution u. Assume now that the mild solution develops a singularity in finite time, and that $(0,T)$ is its maximal time interval of existence. Let $h(t)=\sup_{x\in\mathbf{R}^n}|u(x,t)|$. By a classical result of Leray ([22]), we have

$$h(t)\geqslant\frac{\varepsilon_1}{\sqrt{T-t}}\tag{6.1}$$

for some $\varepsilon_1>0$. Let $H(t)=\sup_{0\leqslant s\leqslant t}h(s)$. It is easy to see that there exists a sequence $t_k\nearrow T$ such that $h(t_k)=H(t_k)$. Let us choose a sequence of numbers $\gamma_k\searrow 1$. For all k, let $N_k=H(t_k)$ and choose $x_k\in\mathbf{R}^n$ such that $M_k=|u(x_k,t_k)|\geqslant N_k/\gamma_k$. Let us set

$$v^{(k)}(y,s)=\frac{1}{M_k}u\biggl(x_k+\frac{y}{M_k},t_k+\frac{s}{M_k^2}\biggr).\tag{6.2}$$

The functions $v^{(k)}$ are defined in $\mathbf{R}^n\times(A_k,B_k)$, with

$$A_k=-M_k^2t_k\quad\text{and}\quad B_k=M_k^2(T-t_k)\geqslant\frac{\varepsilon_1^2}{\gamma_k^2},$$

and satisfy

$$|v^{(k)}|\leqslant\gamma_k\text{ in }\mathbf{R}^n\times(A_k,0)\quad\text{and}\quad|v^{(k)}(0,0)|=1.\tag{6.3}$$

Also, $v^{(k)}$ are mild solutions of the Navier–Stokes equations in $\mathbf{R}^n\times(A_k,0)$ with initial data $v_0^{(k)}(y)=(1/M_k)u_0(x_k+y/M_k)$. By Lemma 6.1, there is a subsequence of $v^{(k)}$ converging to an ancient mild solution v of the Navier–Stokes equations. By our construction, we have $|v|\leqslant 1$ in $\mathbf{R}^n\times(-\infty,0)$ and $|v(0,0)|=1$.

We have proved the following statement.

PROPOSITION 6.1. *A finite-time singularity arising from a mild solution generates a bounded ancient mild solution which is not identically zero.*

Without further information about the situation at hand, the proposition may not be very useful. By itself, the existence of non-zero bounded ancient solutions is not surprising. (Consider constants, for example.) However, if (non-zero) constant solutions can be excluded (for example by a scale-invariant estimate) and a Liouville-type theorem for ancient solutions is available, then finite-time singularities can be ruled out. In fact, in certain situations one does not need the full Liouville theorem, and in the presence of suitable scale-invariant estimates the Hölder estimate (3.10) for the rescaled solutions is sufficient to rule out singularities. To illustrate this with a simple example, we give a proof of a known result, the Ladyzhenskaya–Prodi–Serrin regularity criterion, by the above technique. Assume that a finite $T>0$ is the maximal time of existence of a mild solution (with a suitable initial condition). Let $p,q>1$ with $n/p+2/q=1$, $q<\infty$. We will show that $\|u\|_{L_t^qL_x^p(\mathbf{R}^n\times(0,T))}=\infty$. To see this, it is enough to note that if the $L_t^qL_x^p$-norm of u was finite, the function v constructed by the above procedure would have to vanish identically a.e., due to the invariance of the $L_t^q L_x^p$-norm under the scaling used in the procedure, along with the fact that the finiteness of the $L_t^q L_x^p$-norm implies its "local smallness". On the other hand, v has to be smooth (by the results of §4) and $|v(0,0)|=1$. This leads to a contradiction, and hence $\|u\|_{L_t^q L_x^p(\mathbf{R}^n\times(0,T))}=\infty$ as claimed.

The above reasoning also suggests that any mild ancient solution u of the Navier–Stokes equations, with $\|u\|_{L_t^q L_x^p(\mathbf{R}^n\times(-\infty,0))}<\infty$ for some p and q as above, must vanish. This can indeed be proved. We will not need this result in what follows, and therefore we omit the proof, leaving it to the interested reader as an exercise. We emphasize that we assume that $q<\infty$ (and hence $p>n$). We believe that the statement is also true in the case $p=n$ and $q=\infty$, but we do not have a proof for this case.

A more interesting application of the above "blow-up procedure" gives Theorems 6.1 and 6.2 below, which can be thought of as generalizations of recent results in [4].

THEOREM 6.1. *Let u be an axi-symmetric vector field in $\mathbf{R}^3\times(0,T)$ which belongs to $L^\infty_{x,t}(\mathbf{R}^3\times(0,T'))$ for each $T'<T$. Assume that u is a weak solution of the Navier–Stokes equations in $\mathbf{R}^3\times(0,T)$ and that*

$$|u(x,t)|\leqslant\frac{C}{\sqrt{x_1^2+x_2^2}}\quad \text{in } \mathbf{R}^3\times(0,T). \tag{6.4}$$

Then, $|u|\leqslant M=M(C)$ in $\mathbf{R}^3\times(0,T)$. Moreover, u is a mild solution of the Navier–Stokes equations (for a suitable initial datum).

Remark 6.2. By the results of §4 regarding mild solutions, we see that u is in fact smooth in $\mathbf{R}^3\times(0,T)$ with pointwise bounds on all derivatives in $\mathbf{R}^3\times(\tau,T)$ for any fixed $\tau>0$.

Proof. We first prove the statement assuming that u is a mild solution (for a suitable initial datum). This situation is in fact the main point of the theorem. The fact that we can weaken the assumptions from mild solutions to weak solutions in the formulation of the theorem (while keeping the other assumptions the same) is only of marginal interest.

Arguing by contradiction, let us assume that u is a mild solution which is bounded in $\mathbf{R}^3\times(0,T')$ for each $T'<T$ and develops a singularity at time T. We now use the rescaling procedure described in the paragraph preceding Proposition 6.1 to construct a bounded ancient mild solution v. Let x_k and M_k be as in the construction. We will write $x_k=(x_k',x_{3k})$, with $x_k'=(x_{1k},x_{2k})$. An obvious consequence of assumption (6.4) is that $|x_k'|\leqslant C/M_k$. This implies that the functions $v^{(k)}(y,s)$ are axi-symmetric with respect to an axis parallel to the y_3-axis and at distance at most C from it. Therefore we can assume (by passing to a suitable subsequence first) that the limit function v is axi-symmetric

with respect to a suitable axis. Moreover, since assumption (6.4) is scale-invariant, it will again be satisfied (in suitable coordinates) by v. Applying Theorem 5.3 and using (6.4), we see that $v=0$. On the other hand, $|v(0,0)|=1$, a contradiction. This finishes the main part of the proof.

It remains to show that, under the assumptions of the theorem, u is a mild solution. To do this, we inspect the decomposition of u constructed in Lemma 3.1 with $f_k=-u_k u$. Using the decay of the kernel (3.7) and of the heat kernel, it is easy to check that, under the assumption (6.4), all the terms in the decomposition $u=v+w+b$ will again satisfy (6.4). It follows easily that b must vanish, and therefore u is a mild solution. □

Theorem 6.1 can be used to prove the following result.

THEOREM 6.2. *Let u be an axi-symmetric vector field in $\mathbf{R}^3\times(0,T)$ which belongs to $L^\infty_{x,t}(\mathbf{R}^3\times(0,T'))$ for each $T'<T$. Assume that u is a weak solution of the Navier–Stokes equations in $\mathbf{R}^3\times(0,T)$ satisfying*

$$|u|\leqslant\frac{C}{\sqrt{T-t}}\quad \text{in } \mathbf{R}^3\times(0,T). \tag{6.5}$$

In addition, assume that there exists some $R_0>0$ such that

$$|u(x,t)|\leqslant\frac{C}{\sqrt{x_1^2+x_2^2}}\quad \text{for } \sqrt{x_1^2+x_2^2}\geqslant R_0 \text{ and } 0<t<T, \tag{6.6}$$

as is for example the case when u is a mild solution with initial datum u_0 decaying sufficiently fast at ∞.

Then, $|u|\leqslant M=M(C)$ in $\mathbf{R}^3\times(0,T)$. Moreover, u is a mild solution of the Navier–Stokes equations (for a suitable initial datum).

We remark that the statement fails, for trivial reasons, if we drop assumption (6.6). (Consider $u(x,t)=b(t)$.) The fact that (6.6) is satisfied when u_0 decays sufficiently fast at ∞ (e.g. when it is compactly supported) follows for example from [1] and [2].

Proof. We have seen in the proof of Theorem 6.1 that (6.6) implies that u is a mild solution for a suitable initial datum and is therefore smooth in open subsets of $\mathbf{R}^3\times(0,T)$. We define

$$f(x,t)=|x'||u(x,t)|=\sqrt{x_1^2+x_2^2}\,|u(x,t)|, \tag{6.7}$$

where, as above, $x'=(x_1,x_2)$. By Theorem 6.1, it is enough to prove that f is bounded in $\mathbf{R}^3\times(0,T)$. Let $h(t)=\sup_{\mathbf{R}^3}f(x,t)$ and $H(t)=\sup_{0\leqslant\tau\leqslant t}h(\tau)$. Assume that f is not bounded and choose $t_k\nearrow T$ and $x_k\in\mathbf{R}^3$ such that $M_k=f(x_k,t_k)=h(t_k)=H(t_k)\nearrow\infty$. Let $\lambda_k=|x'_k|$ and, for $y\in\mathbf{R}^3$ and $s\in(-T\lambda_k^{-2},0)$, define

$$v^{(k)}(y,s)=v^{(k)}(y',y_3,s)=\lambda_k u(\lambda_k y',\lambda_k y_3+x_{3k},T+\lambda_k^2 s). \tag{6.8}$$

We note that the sequence λ_k is bounded, due to (6.6). Set $s_k = -(T-t_k)\lambda_k^{-2}$. Since (6.5) is invariant under the Navier–Stokes scaling, the functions $v^{(k)}$ satisfy

$$|v^{(k)}| \leqslant \frac{C}{\sqrt{-s}} \quad \text{in } \mathbf{R}^3 \times (-T\lambda_k^{-2}, 0), \tag{6.9}$$

where C is the same as in (6.5).

Moreover, from the construction we have

$$|v^{(k)}(y,s)| \leqslant \frac{M_k}{|y'|} \quad \text{in } \mathbf{R}^3 \times (-T\lambda_k^{-2}, s_k). \tag{6.10}$$

Note also that, by the elementary inequality $\min\{1/a, 1/b\} \leqslant 2/(a+b)$, estimates (6.9) and (6.10) imply that

$$|v^{(k)}(y,s)| \leqslant \frac{2CM_k}{M_k\sqrt{-s}+C|y'|} \quad \text{in } \mathbf{R}^3 \times (-T\lambda_k^{-2}, s_k). \tag{6.11}$$

Let $\gamma \subset \mathbf{R}^3$ be the unit circle $\{y \in \mathbf{R}^3 : |(y_1, y_2)| = 1 \text{ and } y_3 = 0\}$. We have, by construction, $(|v^{(k)}(\cdot, s_k)|)|_\gamma = M_k$ which, together with (6.9) shows that $s_k \geqslant -C^2 M_k^{-2}$.

Therefore, roughly speaking, as $k \to \infty$, the sequence $v^{(k)}$ blows up along γ. If we knew that the $v^{(k)}$ satisfied local energy estimates with bounds independent of k, the blow-up along γ would be in contradiction with the partial regularity theory in [3], since the 1-dimensional Hausdorff measure of the blow-up set must be zero. One can in fact work along these lines and finish the proof, but the procedure is not simple.

One can alternatively finish the proof by another scaling argument (one could do both scalings in one step, but the two-step procedure seems to be more transparent): Denoting by e_1 the vector $(1,0,0)$, for $x \in \mathbf{R}^3$ and $\tau \in (A_k, 0]$, where $A_k = M_k^2(-T\lambda_k^{-2} - s_k)$, we define

$$w^{(k)}(x,\tau) = \frac{1}{M_k} v^{(k)}\left(e_1 + \frac{x}{M_k}, s_k + \frac{\tau}{M_k^2}\right). \tag{6.12}$$

We will consider the cylinders

$$\mathcal{C}_k = \left\{x \in \mathbf{R}^3 : \sqrt{(x_1+M_k)^2 + x_2^2} \leqslant \tfrac{1}{2} M_k\right\}. \tag{6.13}$$

It follows from our definitions that

$$|w^{(k)}(0,0)| = 1 \quad \text{and} \quad |w^{(k)}(x,\tau)| \leqslant 2 \text{ in } (\mathbf{R}^3 \setminus \mathcal{C}_k) \times (A_k, 0). \tag{6.14}$$

Note also that (6.11) implies

$$|w^{(k)}(x,\tau)| \leqslant \frac{2CM_k}{M_k\sqrt{-\tau} + C\sqrt{(x_1+M_k)^2 + x_2^2}} \quad \text{in } \mathcal{C}_k \times (A_k, 0) \tag{6.15}$$

and that (6.9) implies

$$|w^{(k)}(x,\tau)| \leqslant \frac{C}{\sqrt{-\tau}} \quad \text{in } \mathbf{R}^3 \times (A_k, 0). \tag{6.16}$$

Since the functions $w^{(k)}$ are mild solutions of the Navier–Stokes equations in $(A_k, 0)$ (for suitable rescalings of the initial datum u_0), in view of bound (6.16) we can choose a subsequence of the sequence $w^{(k)}$, which we again denote by $w^{(k)}$, such that the $w^{(k)}$ converge uniformly on compact subsets of $\mathbf{R}^3 \times (-\infty, 0)$ to an ancient mild solution w. In view of (6.14), we have $|w| \leqslant 2$ in $\mathbf{R}^3 \times (-\infty, 0)$. Moreover, since the solutions $v^{(k)}$ are axi-symmetric and $M_k \nearrow \infty$, it is easy to see that w is independent of the x_2-variable. Applying Theorem 5.1 and Remark 6.1 to the field (w_1, w_3), we conclude that (w_1, w_3) must vanish identically, and this easily implies that $w=0$ in $\mathbf{R}^3 \times (-\infty, 0)$. This would give a contradiction with $|w^{(k)}(0,0)|=1$, if we could prove that $w^{(k)}(0,0) \to w(0,0)$, which is not immediately obvious since our bound of $\sup_x |w^{(k)}(x,\tau)|$ may not be uniform as $\tau \to 0$. However, by (6.14), the only possible problem may occur due to the contribution from the cylinder $\mathcal{C}_k$. In the cylinder, we can use the bound (6.15) to show that the contribution of the dangerous part of $w^{(k)}$ to the representation formula (3.5) is negligible (in the limit $k \to \infty$). Applying the representation formula (3.5) in $\mathbf{R}^3 \times (-1, 0)$ with $w^{(k)}(x, -1)$ as initial datum and $f_{jl} = -w_l^{(k)} w_j^{(k)}$, and using the bound (6.15) together with the decay of the kernel (3.7), one sees that it is enough to estimate the integral

$$I(M) = \int_{-1}^{0} \int_{-\infty}^{\infty} \int_{|x'| \leqslant M/2} \frac{1}{(\sqrt{-\tau} + |x'|/M)^2} \, \frac{1}{(M^2/4 + x_3^2)^2} \, dx' \, dx_3 \, d\tau. \tag{6.17}$$

An easy calculation shows that $I(M) \to 0$ as $M \to \infty$. This shows that the contribution from the region where $|w^{(k)}| \geqslant 2$ to the representation formula (3.5) (with $f_{jl} = -w_l^{(k)} w_j^{(k)}$) is negligible (in the limit $k \to \infty$) and therefore (by (3.10)) the sequence $w^{(k)}$ converges to w uniformly in $\overline{B}(0,1) \times [-1, 0]$. Therefore $|w(0,0)|=1$, which gives the sought-after contradiction. □

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Gabriel Koch
University of Chicago
Department of Mathematics
5734 S. University Avenue
Chicago, IL 60637
U.S.A.
koch@math.uchicago.edu

Nikolai Nadirashvili
CNRS, Université de Provence
LATP, CMI
39 rue Joliot-Curie
FR-13453 Marseille Cedex 13
France
nicolas@cmi.univ-mrs.fr

Gregory A. Seregin
Oxford University
Mathematical Institute
24–29 St Giles'
Oxford OX1 3LB
U.K.
seregin@pdmi.ras.ru

Vladimir Šverák
University of Minnesota
236 Vincent Hall
206 Church St. SE
Minneapolis, MN 55455
U.S.A.
sverak@math.umn.edu

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