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Contour lines of the two-dimensional discrete Gaussian free field

by

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During the revision process of this article, Oded Schramm unexpectedly died. I am deeply indebted for all I learned working with him, for his profound personal warmth, for his legendary vision and skill. There was never a better colleague, never a better friend. He will be dearly missed. (Scott Sheffield)

1. Introduction

1.1. Main result

The 2-dimensional massless Gaussian free field (GFF) is a 2-dimensional-time analog of Brownian motion. Just as Brownian motion is a scaling limit of simple random walks and various other 1-dimensional systems, the GFF is a scaling limit of several discrete models for random surfaces. Among these is the discrete Gaussian free field (DGFF), also called the harmonic crystal. We presently discuss the basic definitions and describe the main results of the current work, postponing an overview of the history and general context to §1.3.

Let $G=(V,E)$ be a finite graph and let $V_\partial \subset V$ be some non-empty set of vertices. Let Ω be the set of functions $h\colon V\to\mathbb{R}$ that are zero on V_∂. Clearly, Ω may be identified with $\mathbb{R}^{V\setminus V_\partial}$. The DGFF on G with zero boundary values on V_∂ is the probability measure on Ω whose density with respect to the Lebesgue measure on $\mathbb{R}^{V\setminus V_\partial}$ is proportional to

$$\exp\biggl(\sum_{\{u,v\}\in E} -\frac{1}{2}(h(v)-h(u))^2\biggr). \tag{1.1}$$

Note that under the DGFF measure, h is a multi-dimensional Gaussian random variable. Moreover, the DGFF is a rather natural discrete model for a random field: the term $-\frac{1}{2}(h(v)-h(u))^2$ corresponding to each edge $\{u,v\}$ penalizes functions h which have a large gradient along the edge.

Now fix some function $h_\partial\colon V_\partial\to\mathbb{R}$, and let Ω_{h_∂} denote the set of functions $h\colon V\to\mathbb{R}$ that agree with h_∂ on V_∂. The probability measure on Ω_{h_∂} whose density with respect to the Lebesgue measure on $\mathbb{R}^{V\setminus V_\partial}$ is proportional to (1.1) is the DGFF with boundary values given by h_∂.

Let TG be the usual triangular grid in the complex plane, i.e., the graph whose vertex set is the integer span of 1 and $e^{i\pi/3}=\frac{1}{2}(1+i\sqrt{3})$, with straight edges joining v and w whenever $|v-w|=1$. A TG-*domain* $D\subset\mathbb{R}^2\cong\mathbb{C}$ is a domain whose boundary is a simple closed curve comprised of edges and vertices in TG. Let $V=V_{\overline{D}}$ be the set of TG-vertices in the closure of D, let $G=G_D$ be the induced subgraph of TG with vertex set $V_{\overline{D}}$, and write $V_\partial=\partial D\cap V_{\overline{D}}$. While introducing our main results, we will focus on graphs G_D and boundary sets V_∂ of this form (though analogous results hold if we replace TG with another doubly periodic planar graph; see §1.5).

We may assume that any function $f\colon V\to\mathbb{R}$ is interpolated to a continuous function on the closure of D which is affine on each triangle of TG. We often interpret f as a surface embedded in 3 dimensions and refer to $f(v)$ as the *height* of the surface at v.

Let $\partial D=\partial_+\cup\partial_-$ be a partition of the boundary of a TG-domain D into two disjoint arcs whose endpoints are midpoints of two distinct TG-edges in ∂D. Fix two constants $a,b>0$. Let h be an instance of the DGFF on (G_D,V_∂), with boundary function h_∂ equal to $-a$ on the vertices in ∂_- and equal to b on the vertices in ∂_+. Then h (linearly interpolated on triangles) almost surely assumes the value zero on a unique piecewise linear path γ_h connecting the two boundary edges containing endpoints of ∂_+.

In §1.4, we will briefly review the definition of SLE(4) (a particular type of random chordal path connecting a pair of boundary points of D whose randomness comes from a 1-dimensional Brownian motion), along with the variants of SLE(4) denoted $\mathrm{SLE}(4;\varrho_1,\varrho_2)$. Our main result, roughly stated, is the following.

THEOREM 1.1. *Let D be a* TG*-domain, $\partial D=\partial_+\cup\partial_-$ and let h and γ_h be as above. There is a constant $\lambda>0$ such that if $a=b=\lambda$, then as the triangular mesh gets finer, the random path γ_h converges in distribution to* SLE(4)*. If $a,b\geqslant\lambda$ are not assumed to equal λ, then the convergence is to* $\mathrm{SLE}(4;a/\lambda-1,b/\lambda-1)$.

See §1.5 for a more precise version, which describes the topology under which the convergence is attained. As explained there, we can also prove convergence in a weaker form when the conditions $a,b\geqslant\lambda$ are relaxed.

We will elaborate on the role of the constant λ in §1.2. This constant depends only on the lattice used. Although we do not prove it in this paper, for the triangular grid the value of λ is $\lambda_{\mathrm{TG}}:=3^{-1/4}\sqrt{\frac{1}{8}\pi}$ (see §1.7).

Figure 1.1 illustrates a dual perspective on an instance of γ_h. Here, each vertex in the closure of a rhombus-shaped TG-domain D is replaced with a hexagon in the honeycomb lattice. Call hexagons *positive* or *negative* according to the sign of h. Then there is a cluster of positive hexagons that includes the positive boundary hexagons, a similar cluster of negative hexagons, and a path γ forming the boundary between these two clusters. Figure 1.1 depicts a computer generated instance of the DGFF—with $\pm\lambda$ boundary conditions—and the corresponding γ. Followed from bottom to top, the interface γ turns right when it hits a negative hexagon, left when it hits a positive hexagon. It closely tracks the boundary-hitting zero contour line γ_h in the following sense: the edges in γ are the duals of the edges of TG that are crossed by γ_h. This is because h is almost surely non-zero at each vertex in V, so whenever a zero contour line contains a point on an edge of TG, h must be positive on one endpoint of that edge and negative on the other; hence the dual of that edge separates a positive hexagon from a

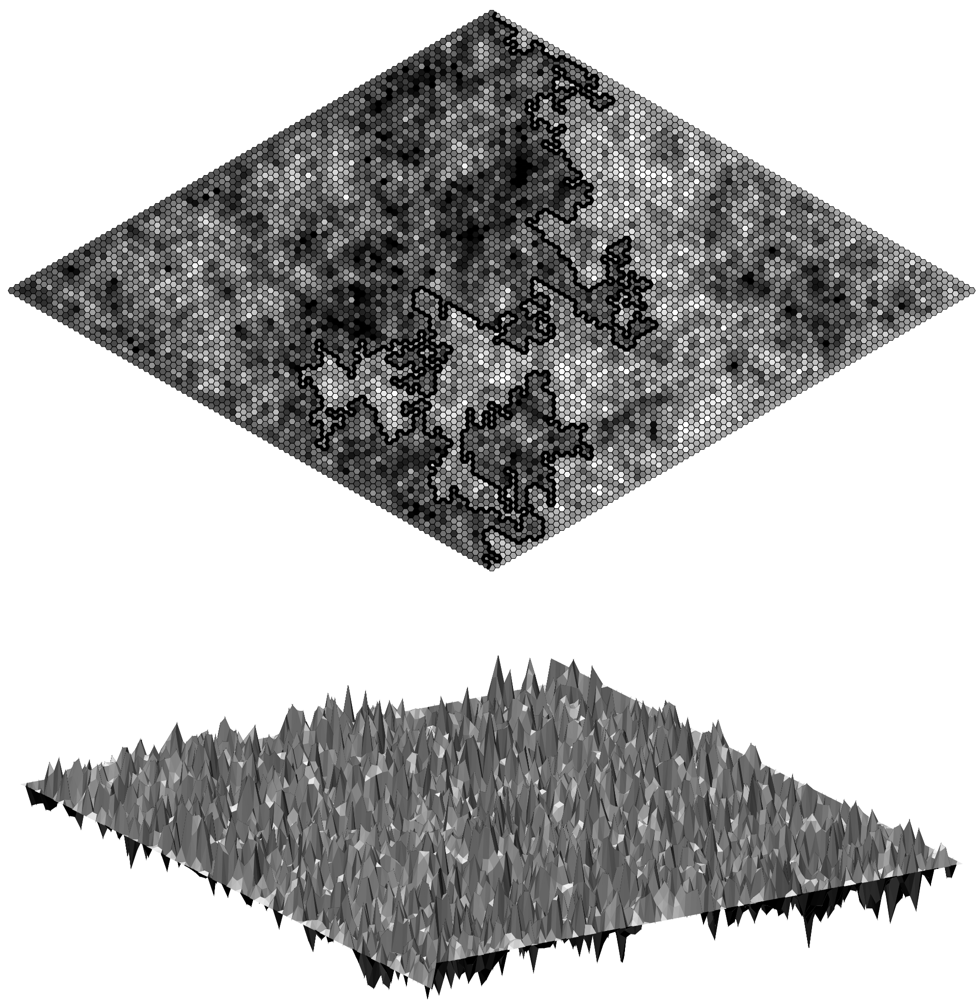

Figure 1.1. (a) DGFF on a 90×90 hexagon array with boundary values λ on the right and $-\lambda$ on the left; faces shaded by height. (b) Surface plot of DGFF.

negative hexagon.

In the fine mesh limit, there will be no difference between γ and γ_h. Thus (by Theorem 1.1) the path in Figures 1.1 (a) and 1.2 (a) approximates SLE(4), while the path of Figure 1.3 (a) approximates SLE(4; 2, 2). We will state and prove most of our results in terms of the dual perspective displayed in the figures.

1.2. Conditional expectation and the height gap

We derive the following well-known facts as a warm-up in §2.1 (see also, e.g., [Gi]).

Boundary influence: The law of the DGFF with boundary conditions $h_\partial\colon V_\partial\to\mathbb{R}$ is the same as that of the DGFF with boundary conditions 0 *plus* a deterministic function $\tilde{h}_\partial\colon V\to\mathbb{R}$ which is the unique discrete-harmonic interpolation of h_∂ to V. (By *discrete-harmonic* we mean that for each $v\in V\setminus V_\partial$, the value $h(v)$ is equal to the average value of $h(w)$ over w adjacent to v.) In particular, the expected value of $h(v)$ is discrete-harmonic in $V\setminus V_\partial$.

Markov property: Let $h\colon V\to\mathbb{R}$ be a random function whose law is the DGFF on G with some boundary values h_∂ on V_∂. Then, given the values of h on a superset $V_0\supset V_\partial$, the conditional law of h is that of a DGFF on G with boundary set V_0 and with boundary values equal to the given values.

From these facts it follows that conditioned on the path γ described in the previous section and on the values of h on the hexagons adjacent to γ, the expected value of h is discrete-harmonic in the remainder of G_D. Figures 1.2 and 1.3 illustrate the expected value of h conditioned on the values of h on the hexagons adjacent to γ.

The reader may observe in Figure 1.2 that although the expected value of h given the values along γ varies a great deal among hexagons close to γ, the expected value at five or ten lattice spacings away from γ appears to be roughly constant along either side of γ. On the other hand, in Figure 1.3, away from γ, the expected height appears to be a smooth but non-constant function. In a sense we make precise in §3 (see Theorem 3.28), the values $-\lambda$ and λ describe the expected value of h, conditioned on γ, at the vertices near (but not microscopically near) the left and right sides of γ; in the fine mesh limit there is thus an "expected height gap" of 2λ between the two sides of γ. In Figure 1.2 the height expectation appears constant away from γ, because the boundary values of $\pm\lambda$ are the same as the expected values near (but not microscopically near) γ.

Once we have established the height gap result, the proof of Theorem 1.1 (at least for the simplest case that the boundary conditions are $-\lambda$ and λ) is similar to the proof that the harmonic explorer converges to SLE(4), as given by the present authors in [SS],

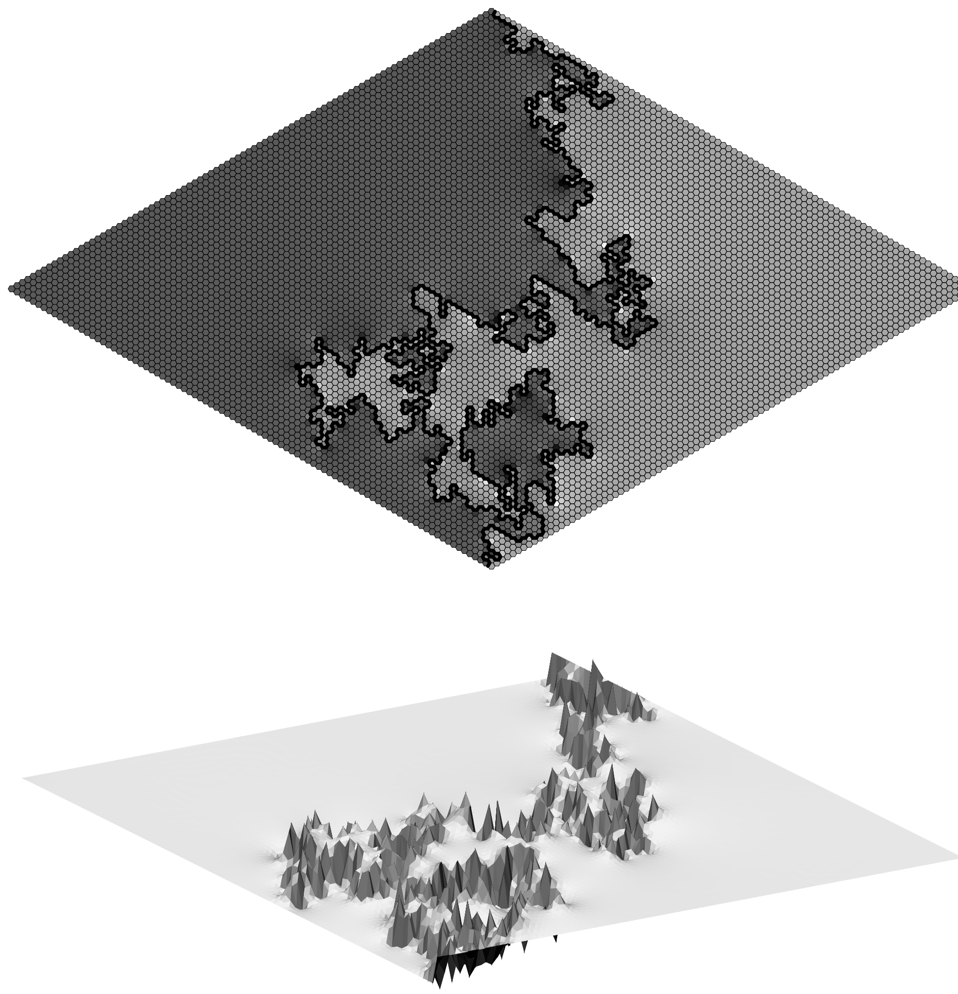

Figure 1.2. (a) Expectation of DGFF with boundary values $\pm\lambda$ given its values at hexagons bordering the interface. (b) Surface plot of the above.

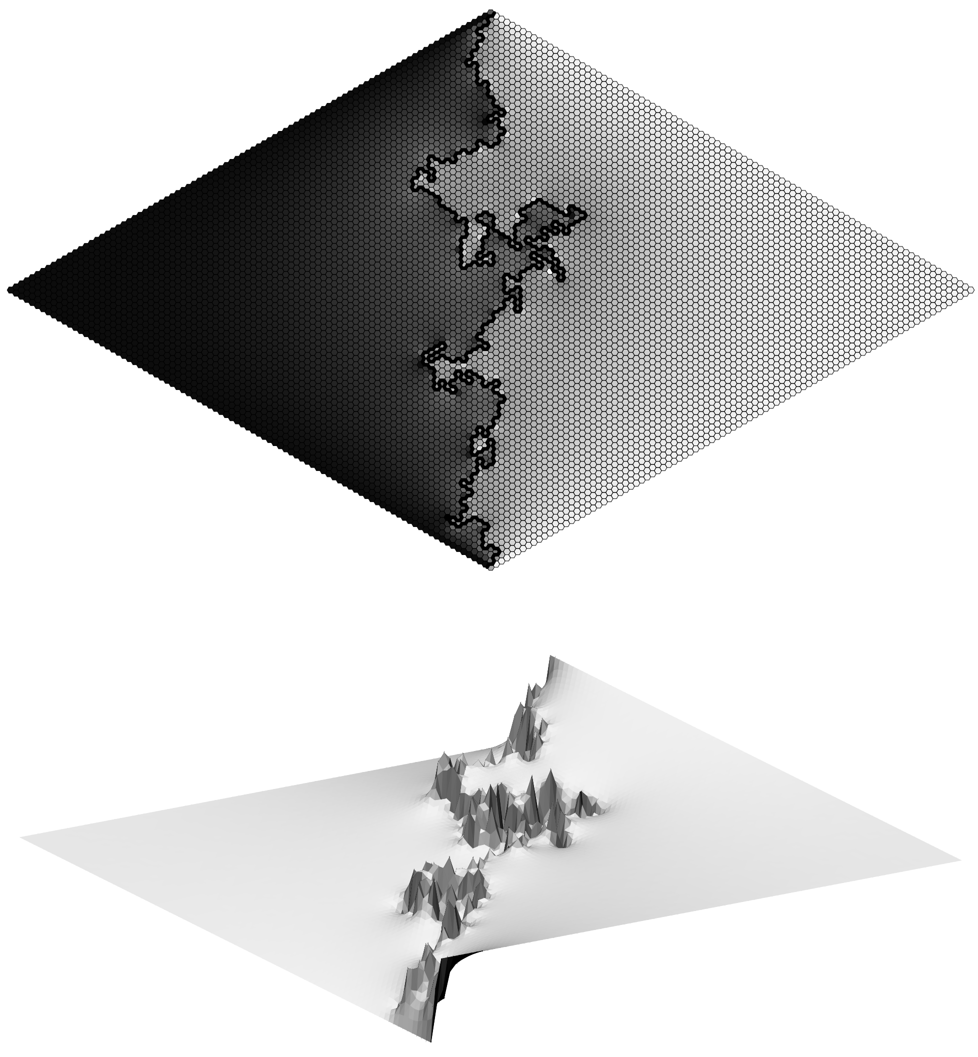

Figure 1.3. (a) Expectation of DGFF given its values at hexagons bordering the interface; exterior boundary values are -3λ on left, and 3λ on right. (b) Surface plot of the above.

which, in turn, follows the same strategy as the proof of convergence of the loop-erased random walk to SLE(2) and the uniform spanning tree Peano curve to SLE(8) in [LSW4].

We will now briefly describe some of the key ideas in the proof of the height gap result. The main step is to show that if one samples a vertex z on γ according to discrete-harmonic measure viewed from a typical point far away from γ, then the absolute value of $h(z)$ is close to independent of the values of h (and the geometry of γ) at points that are *not* microscopically close to z. In other words, if we start a random walk S at a typical point in the interior of D and stop the first time it hits a vertex z which either belongs to V_∂ or corresponds to a hexagon incident to γ, then (conditioned on $z \notin V_\partial$) the random variable $|h(z)|$ (and in particular its conditional expectation) is close to independent of the behavior of γ and h at vertices far away from z.

To prove this, we will actually prove something stronger, namely that (up to multiplication by -1) the collection of all of the values of h (and the geometry of γ) in a microscopic neighborhood of z is essentially independent of the values of h (and the geometry of γ) at points that are not microscopically close to z. One consequence of our analysis is Theorem 3.21, which states that if one takes z to be the origin of a new coordinate system and conditions on the behavior of γ and S outside of a ball of radius R centered at z and S starts outside that ball, then as R tends to infinity the conditional law of the interface γ has a weak limit (which is independent of the sequence of boundary conditions chosen), which is the law of a random infinite path γ on the honeycomb grid TG^* (almost surely containing an edge adjacent to the hexagon centered at the origin $z=0$). We will define a function of such infinite paths γ which (in a certain precise sense) describes the expected value of $|h(z)|$ conditioned on γ; the value λ is the expectation of this function when γ is chosen according to the limiting measure described above.

We remark that many important problems in statistical physics involve classifying the measures that can arise as weak limits of Gibbs measures on finite systems. In such problems, showing the uniqueness of the limiting measure often involves proving that properties of a random system near the origin are approximately independent of the properties of the system far away from the origin. In our case, we need to prove that in some sense the behavior of the triple (h, γ, S) near the the origin (i.e., the first point S hits γ) is close to independent of the behavior of (h, γ, S) far from the origin.

Very roughly speaking, our strategy will be to describe the joint law of (h, γ, S) near the origin and (h, γ, S) far from the origin by considering a different measure in which the two are independent and weighting it by the probability that the inside and outside configurations properly "hook up" with one another. To get a handle on these "hook up" probabilities, we will need to develop various techniques to control the probabilities (conditioned on the values of h on certain sets) that certain zero-height level lines hook up with one another, as well as the probabilities that these level lines avoid certain regions. We will also need bounds on the probability that there exist clusters of positive or negative hexagons crossing certain regions; these are roughly in the spirit of the Russo–Seymour–Welsh theorems for percolation, but the proofs are entirely different. All of the height gap related results are proved in §3.

1.3. GFF definition and background

To help put our DGFF theorems in context and provide further intuition, we now briefly recall the definition of the (continuum) GFF and mention some basic facts described, e.g., in [Sh]. Let $H_s(D)$ be the set of smooth functions supported on compact subsets of a planar domain D, and let $H(D)$ be its Hilbert space completion under the *Dirichlet inner product* $(f,g)_\nabla = \int_D \nabla f \cdot \nabla g \, dx$, where dx refers to area measure. We define an instance of the Gaussian free field to be the formal sum

$$h = \sum_{j=1}^{\infty} \alpha_j f_j,$$

where the α_j's are independent identically distributed 1-dimensional standard (unit variance, zero mean) Gaussians and the f_j's are an orthonormal basis for $H(D)$. Although the sum does not converge pointwise or in $H(D)$, it does converge in the space of distributions [Sh]. In particular, the sum

$$(h,g)_\nabla := \sum_{j=1}^{\infty} \alpha_j (f_j, g)_\nabla$$

is almost surely convergent for every $g \in H_s(D)$.

It is worthwhile to take a moment to compare with the situation where D is 1-dimensional. If D were a bounded open interval in $\mathbb{R}$, then the partial sums of

$$h = \sum_{j=1}^{\infty} \alpha_j f_j$$

would almost surely converge uniformly to a limit, whose law is that of the Brownian bridge, having the value zero at the interval's endpoints. If D were the interval $(0,\infty)$, then the partial sums would converge (uniformly on compact sets) to a function whose law is that of ordinary Brownian motion B_t, indexed by $t \in [0,\infty)$, with $B_0 = 0$ [Sh].

Let g be a *conformal* (i.e., bijective analytic) map from D to another planar domain D'. When g is a rotation, dilation, or translation, it is obvious that

$$\int_{D'} \nabla(f_1 \circ g^{-1}) \cdot \nabla(f_2 \circ g^{-1}) \, dx = \int_D (\nabla f_1 \cdot \nabla f_2) \, dx$$

for any $f_1, f_2 \in H_s(D)$, and an elementary change of variables calculation gives this equality for any conformal g. Taking the completion to $H(D)$, we see that the Dirichlet inner product—and hence the 2-dimensional GFF—is invariant under conformal transformations of D.

Up to a constant, the DGFF on a TG-domain D can be realized as a projection of the GFF on D onto the subspace of $H(D)$ consisting of functions which are continuous and are affine on each triangle of D [Sh]. Note that if f is such a function, then

$$(f,f)_\nabla = \frac{\sqrt{3}}{6} \sum (|f(k)-f(j)|^2 + |f(l)-f(j)|^2 + |f(l)-f(k)|^2),$$

where the sum is over all triangles (j,k,l) in $V_{\overline{D}}$. This is because the area of each triangle is $\frac{1}{4}\sqrt{3}$ and the norm of the gradient squared in the triangle is

$$\tfrac{2}{3}(|f(k)-f(j)|^2 + |f(l)-f(j)|^2 + |f(l)-f(k)|^2).$$

Since each interior edge of D is contained in two triangles, for such f,

$$\|f\|_\nabla^2 = \frac{1}{\sqrt{3}} \sum_{\{j,k\}\in E_I} |f(k)-f(j)|^2 + \frac{1}{2\sqrt{3}} \sum_{\{j,k\}\in E_\partial} |f(k)-f(j)|^2, \tag{1.2}$$

where E_I and E_∂ are the interior and boundary (undirected) edges of TG in $\overline{D}$. We will refer to the sum $\sum_{E_I} |f(k)-f(j)|^2$ as the *discrete Dirichlet energy* of f. It is equivalent—up to the constant factor $3^{-1/2}$ and an additive term depending only on the boundary values of f—to the *Dirichlet energy* $(f,f)_\nabla$ of the piecewise affine interpolation of f to D.

The above analysis suggests a natural coupling between the GFF and a sequence of DGFF approximations to the GFF (obtained by taking finer mesh approximations of the same domain). The GFF can also be obtained as a scaling limit of other discrete random surface models (e.g., solid-on-solid, dimer-height-function, and $\nabla\phi$-interface models) [Ke] [NS], [Sp]. Its Laplacian is a scaling limit of some Coulomb gas models, which describe random electrostatic charge densities in 2-dimensional domains [F], [FS], [Ko], [KT], [Sp]. Physicists often use heuristic connections to the GFF to predict properties of 2-dimensional statistical physics models that are not obviously random surfaces or Coulomb gases (e.g., Ising and Potts models, $O(n)$ loop models) [dN], [DMS], [D], [Ka], [KN], [N1], [N2]. As a model for the field theory of non-interacting massless bosons, the GFF is a starting point for many constructions in quantum field theory, conformal field theory, and string theory [BPZ], [DMS], [Ga], [GJ].

Because of the conformal invariance of the GFF, physicists and mathematicians have hypothesized that discrete random surface models that are believed or known to converge to the Gaussian free field (e.g., the discrete Gaussian free field, the height function of the oriented $O(n)$ loop model with $n=2$, height functions for domino and lozenge tilings) have level sets with conformally invariant scaling limits [Co], [DS1], [DS2], [SD], [KDH], [HK], [Ke], [KH], [KHS], [N1]. Our results confirm this hypothesis for the discrete Gaussian free field.

Various properties of the DGFF contour lines (such as winding exponents and the fact that the fractal dimension is $\frac{3}{2}$) have been predicted correctly in the physics literature [Co], [DS2], [DS1], [SD], [HK], [KDH], [KH], [KHS], [N1]. The techniques used to make these predictions are also described in detail in the survey papers [D], [KN], [N2]. Analogous results about winding exponents and fractal dimension have now been proved rigorously for SLE [Sch], [RS], [B].

The study of level lines of the DGFF and related random surfaces is also related to the study of equipotential lines of random charge distributions in statistical physics. The so-called 2-*dimensional Coulomb gas* is a model for electrostatics in which the force between charged particles is inversely proportional to the distance between them. In this model, a continuous function $f\in H_s(D)$ is the *Coulomb gas electrostatic potential function* ("grounded" at the boundary of D) of $-\Delta f$, when Δf is interpreted as a charge density function. The value $(f,f)_\nabla$ is then the total potential energy—also called the *energy of assembly* of the charge distribution $-\Delta f$. In the Coulomb gas model, this is the amount of energy required to move from a configuration in which the charge density is zero throughout D to a configuration in which the charge density is given by $-\Delta f$.

In statistical physics, it is often natural to consider a probability distribution on configurations in which the probability of a configuration with potential energy H is proportional to e^{-H}. If ϱ is a smooth charge distribution, then its energy of assembly is given by $(-\Delta^{-1}\varrho, -\Delta^{-1}\varrho)_\nabla=(\varrho, -\Delta^{-1}\varrho)$; if we define ϱ to be the standard Gaussian in $\Delta H(D)$ determined by this quadratic form, then ϱ is the Laplacian of the Gaussian free field (which, like the GFF itself, is well defined as a random distribution but not as a function). In other words, the Laplacian of a Gaussian free field is a random distribution that we may interpret as a model for random charge density in a statistical physical Coulomb gas.

However, we stress that when physicists refer to the Coulomb gas method for $O(n)$ model computations, they typically have in mind a more complicated Coulomb gas model in which the charges are required to be discrete (i.e., ϱ is required to be a sum of unit positive and negative point masses) and hard core constraints may be enforced.

The surveys [BEF], [Gi], [Sp] contain additional references on lattice spin models that have the GFF as a scaling limit and Coulomb gas models that have its Laplacian as a scaling limit—for example, the *harmonic crystal* (also known as the *discrete Gaussian free* *field*) with quadratic nearest-neighbor potential, the more general *anharmonic crystal*, the *discrete-height Gaussian* (where h is a function on a lattice, with values restricted to integers), the *Villain gas* (where h is a function on a lattice and the values of its discrete Laplacian $\mathfrak{p}=-\Delta h$ are restricted to integers), and the *hard core Coulomb gas* (where h is a function on a lattice and its discrete Laplacian $\mathfrak{p}=-\Delta h$ is ± 1 valued).

The physics literature on applications of the GFF to field theory and statistical physics is large, and the authors themselves are only familiar with parts of it. Outside of these areas, there is a body of experimental and computational research on contour lines of random topographical surfaces, such as the surface of the earth. Mandelbrot's famous *How Long Is the Coast of Britain?* [M], which prefigured the notion of "fractal" introduced by Mandelbrot years later, is an early example. The results about contour lines in these studies (including fractal dimension computations) are less detailed than the ones provided here and are not all mathematically rigorous. However, some of the models are similar in spirit to the GFF, involving functions whose Fourier coefficients are independent Gaussians. An eclectic overview of this literature appears in [I].

1.4. SLE background and prior convergence results

We now give a brief definition of (chordal) SLE($\varkappa$) for $\varkappa>0$. See also the surveys [W], [KN], [L4], [Ca] or [L3]. The discussion below along with further discussion of the special properties of SLE(4) appears in another paper by the current authors [SS]. That paper shows that SLE(4) is the scaling limit of a random interface called the harmonic explorer (designed in part to be a toy model for the DGFF contour line addressed here).

Let $T>0$. Suppose that $\gamma\colon[0,T]\to\overline{\mathbb{H}}$ is a continuous simple path in the closed upper half-plane $\overline{\mathbb{H}}$ which satisfies $\gamma[0,T]\cap\mathbb{R}=\{\gamma(0)\}=\{0\}$. For every $t\in[0,T]$, there is a unique conformal homeomorphism $g_t\colon\mathbb{H}\setminus\gamma[0,t]$ which satisfies the so-called *hydrodynamic* normalization at infinity

$$\lim_{z\to\infty}(g_t(z)-z)=0.$$

The limit

$$\operatorname{cap}_\infty(\gamma[0,t]):=\lim_{z\to\infty}\frac{z(g_t(z)-z)}{2}$$

is real and monotone increasing in t. It is called the (half-plane) *capacity* of $\gamma[0,t]$ from ∞, or just *capacity*, for short. Since $\operatorname{cap}_\infty(\gamma[0,t])$ is also continuous in t, it is natural to reparameterize γ so that $\operatorname{cap}_\infty(\gamma[0,t])=t$. Loewner's theorem states that in this case the maps g_t satisfy his differential equation

$$\partial_t g_t(z)=\frac{2}{g_t(z)-W_t},\quad g_0(z)=z, \tag{1.3}$$

where $W_t = g_t(\gamma(t))$. (Since $\gamma(t)$ is not in the domain of definition of g_t, the expression $g_t(\gamma(t))$ should be interpreted as a limit of $g_t(z)$ as $z \to \gamma(t)$ inside $\mathbb{H} \setminus \gamma[0,t]$. This limit does exist.) The function $t \mapsto W_t$ is continuous in t, and is called the *driving parameter* for γ.

One may also try to reverse the above procedure. Consider the Loewner evolution defined by the ordinary differential equation (ODE) (1.3), where W_t is a continuous, real-valued function. For a fixed z, the evolution defines $g_t(z)$ as long as $|g_t(z) - W_t|$ is bounded away from zero. For $z \in \overline{\mathbb{H}}$ let τ_z be the first time $t \geqslant 0$ in which $g_t(z)$ and W_t collide, or set $\tau_z = \infty$ if they never collide. Then $g_t(z)$ is well defined on $\{z \in \overline{\mathbb{H}} : \tau_z \geqslant t\}$. The set $K_t := \{z \in \overline{\mathbb{H}} : \tau_z \leqslant t\}$ is sometimes called the *evolving hull* of the evolution. In the case discussed above where the evolution is generated by a simple path γ parameterized by capacity and satisfying $\gamma(t) \in \mathbb{H}$ for $t > 0$, we have $K_t = \gamma[0,t]$.

The path of the evolution is defined as $\gamma(t) = \lim_{z \to W_t} g_t^{-1}(z)$, where z tends to W_t from within the upper half-plane $\mathbb{H}$, provided that the limit exists and is continuous. However, this is not always the case. The process (chordal) SLE($\varkappa$) in the upper half-plane, beginning at 0 and ending at ∞, is the path $\gamma(t)$ when W_t is $B_t \sqrt{\varkappa}$, where $B_t = B(t)$ is a standard 1-dimensional Brownian motion. (*Standard* means $B(0) = 0$ and $\mathbf{E}[B(t)^2] = t$, $t \geqslant 0$. Since $(B_t \sqrt{k} : t \geqslant 0)$ has the same distribution as $(B_{\varkappa t} : t \geqslant 0)$, taking $W_t = B_{\varkappa t}$ is equivalent.) In this case, almost surely $\gamma(t)$ does exist and is a continuous path. See [RS] ($\varkappa \neq 8$) and [LSW4] ($\varkappa = 8$).

We now define the processes SLE($\varkappa; \varrho_1, \varrho_2$). Given a Loewner evolution defined by a continuous W_t, we let x_t and y_t be defined by $x_t := \sup\{g_t(x) : x < 0 \text{ and } x \notin K_t\}$ and $y_t := \inf\{g_t(x) : x > 0 \text{ and } x \notin K_t\}$. When the Loewner evolution is generated by a simple path $\gamma(t)$ satisfying $\gamma(t) \in \mathbb{H}$ for $t > 0$, these points x_t and y_t can be thought of as the two images of 0 under g_t. Note that, by (1.3),

$$\partial_t x_t = \frac{2}{x_t - W_t} \quad \text{and} \quad \partial_t y_t = \frac{2}{y_t - W_t} \tag{1.4}$$

for all t such that $x_t < W_t < y_t$. Beginning from an initial time r for which $x_r < W_r < y_r$, we define SLE($\varkappa; \varrho_1, \varrho_2$) to be the evolution that makes (x_t, W_t, y_t) a solution to the stochastic differential equation (SDE) system

$$dW_t = \sqrt{\varkappa}\, dB_t + \frac{\varrho_1\, dt}{W_t - x_t} + \frac{\varrho_2\, dt}{W_t - y_t}, \quad dx_t = \frac{2\, dt}{x_t - W_t}, \quad dy_t = \frac{2\, dt}{y_t - W_t}, \tag{1.5}$$

noting that existence and uniqueness of solutions to this SDE (at least from the initial time r until the first $s > r$ for which either $x_s = W_s$ or $W_s = y_s$) follow easily from standard results in [RY]. (The x_t and y_t are called *force points* because they apply a “force” affecting the drift of the process W_t by an amount inversely proportional to their distance from W_t.)

Some subtlety is involved in extending the definition of $\mathrm{SLE}(\varkappa;\varrho_1,\varrho_2)$ beyond times when W_t hits the force points, and in starting the process from the natural initial values $x_0=W_0=y_0=0$. This is closely related to the issues which come up when defining the Bessel processes of dimension less than 2 and will be discussed in more detail in §4.

Although many random self-avoiding lattice paths from the statistical physics literature are conjectured to have forms of SLE as scaling limits, rigorous proofs have thus far appeared only for a few cases: site percolation cluster boundaries on the hexagonal lattice (SLE(6), [Sm]; see also [CN]), branches (loop-erased random walk) and outer boundaries (random Peano curves) of uniform spanning trees (forms of SLE(2) and SLE(8), respectively, [LSW4]), the harmonic explorer (SLE(4), [SS]), and boundaries of simple random walks (forms of $\mathrm{SLE}\big(\frac{8}{3}\big)$, [LSW3]).

In the latter case, conformal invariance properties follow almost immediately from the conformal invariance of 2-dimensional Brownian motion. In each of the other cases listed above, the initial step of the proof is to show that a certain function of the partially generated paths $\gamma([0,t])$, which is a martingale in t when γ is $\mathrm{SLE}(\varkappa)$ for the appropriate $\varkappa$, has a discrete analog which is (approximately or exactly) a martingale for the discrete paths and is approximately equivalent to the continuous version in the fine mesh limit. For loop-erased random walk, harmonic explorer, and uniform spanning tree Peano curves, this initial step is the easy part of the argument; it follows almost immediately from the fact that simple random walk converges to Brownian motion. The analogous step for site percolation on the hexagonal lattice, as given by [Sm], is an ingenious but nonetheless short and simple argument.

By contrast, the analogous step in this paper (which requires the proof of the height gap lemma, as given in §3) is quite involved; it is the most technically challenging part of the current work and includes many new techniques and lemmas about the geometry of DGFF contours that we hope are interesting for their own sake.

Another way in which the DGFF differs from percolation, the harmonic explorer, and the uniform spanning tree is that it has a natural continuum analog (the GFF) which can be easily rigorously constructed without any reference to SLE, and which is itself (like Brownian motion) an object of great significance. It becomes natural to ask whether the DGFF results enable us to define the "contour lines" of the continuum GFF in a canonical way; we plan to answer this question (affirmatively) in a subsequent work (see §1.7).

A final difference is that, for the DGFF, there is a continuum of choices for left and right boundary conditions (a and b) which are equally natural a priori, so we are led to consider a family of paths $\mathrm{SLE}(4;a/\lambda-1,b/\lambda-1)$ instead of simply SLE(4). (The case $a=b=0$ is particularly natural; see Figure 1.4.) In these processes, the driving parameters W_t are generally no longer Brownian motions (rather, they are continuous semimartingales with constant quadratic variation and a drift term that can become singular on a fractal set). Proving driving parameter convergence to these processes requires some rather general convergence infrastructure (§4.4), which we hope will be useful in other settings as well.

1.5. Precise statement of main result

Let $\mathbb{H}$ be the upper half-plane. Let D be any TG-domain and let $\partial D=\partial_+\cup\partial_-$ be a partition of the boundary of D into two disjoint arcs whose endpoints are midpoints of two TG-edges contained in ∂D. As before, let V denote the vertices of TG in $\overline{D}$. Let $h_\partial=-a$ on $\partial_-\cap V$ and $h_\partial=b$ on $\partial_+\cap V$, where a and b are positive constants.

Let $h: V\to\mathbb{R}$ be an instance of the DGFF with boundary conditions h_∂. Let ϕ_D be any conformal map from D to $\mathbb{H}$ that maps ∂_+ bijectively onto the positive real ray $(0,\infty)$. (Note that ϕ_D is unique up to positive scaling.)

There is almost surely a unique interface $\gamma\subset\overline{D}$ between hexagons in the dual grid TG^* containing TG-vertices where h is positive and such hexagons where h is negative, such that the endpoints of γ are on ∂D. In fact, the endpoints of γ are the same as the endpoints of ∂_+. (As mentioned above, this interface γ stays within a bounded distance from the zero-height contour line γ_D of the affine interpolation of h.) Now, $\phi_D\circ\gamma$ is a random path on $\mathbb{H}$ connecting 0 to ∞. We will show that this path converges to a form of SLE(4). Rather than considering a fixed domain $\widehat{D}$ and a sequence of discrete domains D_n approximating $\widehat{D}$, with the mesh tending to 0, we will employ a setup that is more general in which the mesh is fixed (the triangular lattice will not be rescaled), and we consider domains D that become "larger". The correct sense of "large" is measured by

$$r_D=r_{D,\phi}:=\operatorname*{rad}_{\phi_D^{-1}(i)}(D),$$

where $\mathrm{rad}_x(D)$ denotes the radius of D viewed from x, i.e., $\inf_{y\notin D}|x-y|$. Of course, if $\phi_D^{-1}(i)$ is at a bounded distance from ∂D, then the image of the triangular grid under ϕ_D is not fine near i, and there is no hope for approximating SLE by $\phi_D\circ\gamma$.

We have chosen to use $\mathbb{H}$ as our canonical domain (mapping all other paths into $\mathbb{H}$), because it is the most convenient domain in which to define chordal SLE. However, to make the completion of $\mathbb{H}$ a compact metric space, we will endow $\mathbb{H}$ with the metric it inherits from its conformal map onto the unit disk $\mathbb{U}$. Namely, we let $d_*(\cdot\,,\cdot)$ be the metric on $\overline{\mathbb{H}}\cup\{\infty\}$ given by $d_*(z,w)=|\Psi(z)-\Psi(w)|$, where $\Psi(z):=(z-i)/(z+i)$ maps $\overline{\mathbb{H}}\cup\{\infty\}$ onto $\overline{\mathbb{U}}$. If $z\in\overline{\mathbb{H}}$, then $d_*(z_n,z)\to0$ is equivalent to $|z_n-z|\to0$, and $d_*(z_n,\infty)\to0$ is equivalent to $|z_n|\to\infty$.

If γ_1 and γ_2 are distinct unparameterized simple paths in $\overline{\mathbb{H}}$, then we define $d_{\mathcal{U}}(\gamma_1,\gamma_2)$ to be the infimum over all pairs (η_1,η_2) of parameterizations of γ_1 and γ_2 in $[0,1]$ (i.e., $\eta_j:[0,1]\to\overline{\mathbb{H}}$ is a simple path satisfying $\eta_j([0,1])=\gamma_j$ for $j=1,2$) of the uniform distance $\sup\{d_*(\eta_1(t),\eta_2(t)):t\in[0,1]\}$ with respect to the metric d_*.

Our strongest result is in the case where $a,b\geqslant\lambda$. We prove the following result.

THEOREM 1.2. *There is a constant $\lambda>0$ such that if $a,b\geqslant\lambda$, then, as $r_D\to\infty$, the random paths $\phi_D\circ\gamma$ described above converge in distribution to* $\mathrm{SLE}(4;a/\lambda-1,b/\lambda-1)$ *with respect to the metric $d_{\mathcal{U}}$.*

In other words, for every $\varepsilon>0$ there is some $R=R(\varepsilon)$ such that if $r_D>R$, then there is a coupling of $\phi_D\circ\gamma$ and a path γ_{SLE} whose law is that of $\mathrm{SLE}(4;a/\lambda-1,b/\lambda-1)$ *such that*

$$\mathbf{P}[d_{\mathcal{U}}(\phi_D\circ\gamma,\gamma_{\mathrm{SLE}})>\varepsilon]<\varepsilon.$$

We first comment that it follows that when r_D is large, γ is "close" to $\phi_D^{-1}\circ\gamma_{\mathrm{SLE}}$. For example, if $r_D\to\infty$ and $r_D^{-1}D$ tends to a bounded domain $\widehat{D}$ whose boundary is a simple closed path in such a way that the boundaries of the domains may be parameterized to give uniform convergence of parameterized paths and if $r_D^{-1}\partial_+$ converges, then $r_D^{-1}\gamma$ converges in law to the corresponding SLE in $\widehat{D}$. To prove this from Theorem 1.2, we only need to note that in this case the maps $r_D^{-1}\phi_D^{-1}$ converge uniformly in $\overline{\mathbb{H}}\cup\{\infty\}$ (see, e.g., [P, Proposition 2.3]).

When we relax the assumption $a,b\geqslant\lambda$ to $a,b>0$, we still prove some sort of convergence to $\mathrm{SLE}(4;a/\lambda-1,b/\lambda-1)$, but with respect to a weaker topology. In fact, we can allow a and b to be zero or even slightly negative, but in this case we need to appropriately adjust the above definition of the interface γ. Say that a hexagon in the hexagonal grid TG^* dual to TG is positive if either the center v of the hexagon is in D and $h(v)>0$, or $v\in\partial_+$. Likewise, say that the hexagon is negative if $v\in D$ and $h(v)<0$, or $v\in\partial_-$. Let γ be the unique oriented path in TG^* that joins the two endpoints of ∂_+, has only positive hexagons adjacent to its right-hand side and only negative hexagons adjacent to its left-hand side. (If $a,b>0$, this definition clearly agrees with the previous definition of γ.) We prove the following.

THEOREM 1.3. *For every constant $\bar{\Lambda}>0$ there is a constant $\Lambda_0=\Lambda_0(\bar{\Lambda})>0$ such that if $a,b\in[-\Lambda_0,\bar{\Lambda}]$ and γ is the* DGFF *interface defined above, then as $r_D\to\infty$ the Loewner driving term $\widehat{W}_t$ of $\phi_D\circ\gamma$, parameterized by capacity from ∞, converges in law to the driving term W_t of* $\mathrm{SLE}(4;a/\lambda-1,b/\lambda-1)$ *with respect to the topology of locally uniform convergence. That is, for every $T,\varepsilon>0$ there is some $R>0$ such that if $r_D>R$, then γ and* $\mathrm{SLE}(4;a/\lambda-1,b/\lambda-1)$ *may be coupled so that with probability at least $1-\varepsilon$,*

$$\sup\{|\widehat{W}_t-W_t|:t\in[0,T]\}<\varepsilon.$$

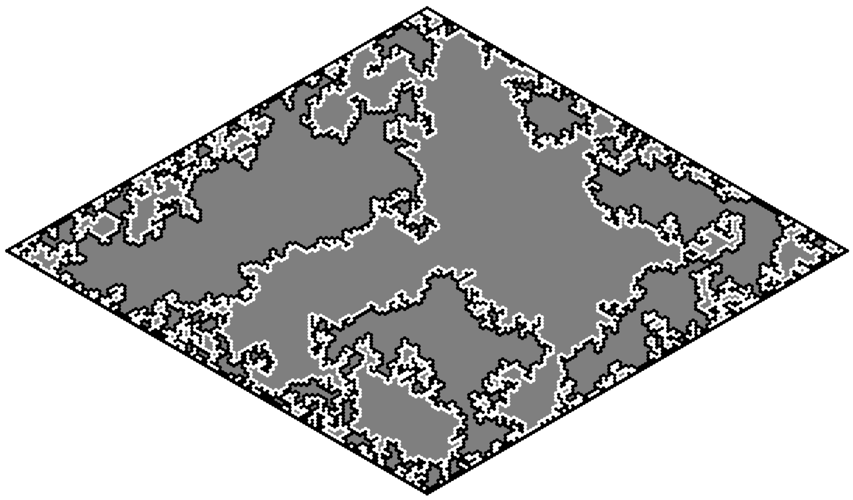

Figure 1.4. Zero-height interfaces starting and ending on the boundary, shown for the discrete GFF on a 150×150 hexagonal array with zero boundary. Interior white hexagons have height greater than zero; interior black hexagons have height less than zero; boundary hexagons (of height zero) are black. Hexagons that are not incident to a zero-height interface that reaches the boundary are grey.

Some (essentially well-known) geometric consequences of this kind of convergence are proved in §4.7.

A particularly interesting case of Theorem 1.3 is the case $a=b=0$, corresponding to the DGFF with zero boundary values. In this case, when h is interpolated linearly to triangles, its zero level set will almost surely include a finite number of piecewise linear arcs in D whose endpoints on ∂D are vertices of TG. A dual representation of this set of arcs is shown in Figure 1.4. For any fixed choice of endpoints on the boundary, the interface connecting those endpoints will converge to SLE$(4;-1,-1)$. The limit of the complete set of arcs in Figure 1.4 is in some sense a coupling of SLE$(4;-1,-1)$ processes, one for each pair of boundary points.

Finally, we discuss the generalizations: replace TG with an arbitrary weighted doubly periodic planar lattice—i.e., a connected planar graph $\mathfrak{G} \subset \mathbb{R}^2$ invariant under two linearly independent translations, T_1 and T_2, such that every compact subset of $\mathbb{R}^2$ meets only finitely many vertices and edges, together with a map w from the edges of $\mathfrak{G}$ to the positive reals, which is invariant under T_1 and T_2.

A $\mathfrak{G}$*-domain* $D \subset \mathbb{R}^2$ is a domain whose boundary is a simple closed curve comprised of edges and vertices in $\mathfrak{G}$. Let $V = V_{\overline{D}}$ be the set of $\mathfrak{G}$-vertices in the closure of D, let $G = G_D = (V, E)$ be the induced subgraph of $\mathfrak{G}$ with vertex set $V_{\overline{D}}$, and write $V_\partial = \partial D \cap V_{\overline{D}}$. Given a boundary value function $h_\partial : V_\partial \to \mathbb{R}$, the edge weighted DGFF on G has a density with respect to the Lebesgue measure on $\mathbb{R}^{V\setminus V_\partial}$ which is proportional to

$$\exp\biggl(\sum_{\{u,v\}\in E} -\frac{1}{2}w(\{u,v\})(h(v)-h(u))^2\biggr).$$

If every face of $\mathfrak{G}$ has three edges, then every vertex in the dual graph is an endpoint of exactly three edges, and the boundary between positive and negative faces can be defined as a simple path in this dual lattice, similar to the one shown in Figure 1.1 (a). If not every face of $\mathfrak{G}$ has three edges, then we may "triangulate" $\mathfrak{G}$ by adding additional edges to $\mathfrak{G}$, while maintaining the invariance under T_1 and T_2, to make this the case (and set w to zero on these edges so that their presence does not affect the law of the DGFF).

We define the weighted random walk on $\mathfrak{G}$ to be the Markov chain with transition probability $w(\{u,v\})/\sum_{v'} w(\{u,v'\})$ from u to v, where we take $w(\{u,v\})=0$ unless u and v are neighbors in $\mathfrak{G}$. It is well known and easy to prove that such a walk on the rescaled lattice $\varepsilon\mathfrak{G}$ converges to a linear transformation of time-scaled 2-dimensional Brownian motion when ε tends to zero (but since we could not find a reference, we very briefly explain this in §5). It is convenient to replace the embedding of $\mathfrak{G}$ into $\mathbb{R}^2$ described above with a linear transformation of that embedding that causes this limit to be standard Brownian motion.

THEOREM 1.4. *Both Theorems* 1.2 *and* 1.3 *continue to hold if* TG *is replaced by a general weighted doubly periodic planar lattice* $\mathfrak{G}$, *as described above*, *provided that* $\mathfrak{G}$ *is embedded in* $\mathbb{R}^2$ *in such a way that the weighted random walk converges to Brownian motion.*

If $\mathfrak{G}$ is the grid $\mathbb{Z}^2$, then one natural way to triangulate $\mathfrak{G}$ is to add all the edges of the form $\{(x,y),(x+1,y+1)\}$. Another would be to add the edges $\{(x,y),(x+1,y-1)\}$. The above theorem implies, perhaps surprisingly, that the limiting law of the zero-height interface is the same in either case, with no need for a linear change of coordinates.

1.6. Outline

In §2 we introduce the basic notation and assumptions that are necessary for the height gap results proved in §3. In §§3.1–3.4 we develop bounds and estimates related to the geometry of zero-height interfaces. The random walk S comes into the picture in §3.5, where we develop results about the near-independence of the triple (h,γ,S) on microscopic and macroscopic scales. In §3.6 we apply these results to prove uniqueness of the limiting measure, namely Theorem 3.21, and in §3.7 we apply this to prove our main height gap result, Theorem 3.28.

Based on Theorem 3.28, the convergence in the case where the boundary values are $\pm\lambda$ is not too hard, using the method from [LSW4]. However, to prove the convergence to $\mathrm{SLE}(4; a/\lambda-1, b/\lambda-1)$ in §4, we need to contend with a few other issues which stem from the fact that the driving parameter of $\mathrm{SLE}(4; a/\lambda-1, b/\lambda-1)$ is the solution to an SDE with a drift term that blows up to infinity on a fractal set of times. To overcome these difficulties, we change coordinates to a coordinate system in which the drift terms stay bounded. In §4.4 we define and study *approximate diffusions*. These are random processes that are not necessarily Markov, but satisfy an approximate discrete version of an SDE. The Loewner driving term of the DGFF interface (before going to the scaling limit) is the approximate diffusion we are interested in. The main point is that an approximate diffusion of an SDE is shown to be close to the corresponding true diffusion satisfying the same SDE, under appropriate regularity conditions. This is how the convergence of the driving term of the DGFF interface to the driving term of the corresponding $\mathrm{SLE}(\varkappa; \varrho_1, \varrho_2)$ is established. In §4.7 and §4.8 more geometric convergence results are deduced from the convergence of the Loewner driving term of the interface.

Finally, the rather brief §5 then describes the (very minor) modifications required for the generalization to other lattices, Theorem 1.4.

1.7. Sequel

This paper is actually the first of two papers the current authors are writing about this subject. In the second paper we will make sense of the "contour lines" of the continuum GFF. An instance h of the continuum GFF is a random distribution, not a random function; however, given an instance of the GFF on a domain D, we can project h onto the space of functions which are piecewise linear on a triangulation of D to yield an instance of the DGFF which is, in some sense, a piecewise linear approximation to h. We can then define the level lines of the GFF to be the limits of the level lines of its piecewise linear approximations (after proving that these limits exist). We will also characterize these random paths directly—without reference to discrete approximations—by showing that they are the unique path-valued functions of h which satisfy a simple Markov property. Similar techniques allow us to describe the contour lines of h that form loops (instead of starting and ending at points on the boundary of D).

The determination of the value of λ for a given lattice is not too hard, but fits better with the general spirit of our next paper on the subject, in which we will prove, in particular, that $\lambda_{\mathrm{TG}}=3^{-1/4}\sqrt{\frac{1}{8}\pi}$. If the DGFF is scaled so that its fine mesh limit is the ordinary GFF, we have $\lambda=\sqrt{\frac{1}{8}\pi}$.

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2. Preliminaries

2.1. A few general properties of the DGFF

In this subsection, we recall a few well-known properties of the DGFF that are valid on any finite graph. Let (V,E) be a finite graph, and let $V_\partial \subset V$ be a non-empty set of vertices. Let h denote a sample from the DGFF with boundary values given by some function $h_\partial: V_\partial \to \mathbb{R}$.

When f is a function on V, the discrete gradient ∇f is the function on the set of ordered pairs (v,u) such that $\{v,u\}\in E$ defined by $\nabla f((u,v))=f(v)-f(u)$. When defining the norm of the gradient we sum over undirected edges, i.e., we write

$$\|\nabla f(v)\|^2 = \sum_{\{u,v\}\in E} (f(v)-f(u))^2. \tag{2.1}$$

Thus, the probability density of h is proportional to $e^{-\|\nabla h(v)\|^2/2}$. Therefore, when $h_\partial=0$ on V_∂, h is a standard Gaussian with respect to the norm $\|\nabla h\|$ on Ω. The (discrete) Dirichlet inner product that defines this norm can be written

$$(f,g)_\nabla = \sum_{\{u,v\}\in E} (f(v)-f(u))(g(v)-g(u)). \tag{2.2}$$

Now write $\Delta f(v)=\sum_{u\sim v}(f(u)-f(v))$, where the sum is over all neighbors u of v. By expanding and rearranging the summands in (2.2), we find

$$(f,g)_\nabla = -(\Delta f, g). \tag{2.3}$$

Let $V_0\subset V_D$. We claim that the vector space of functions $f: V\to\mathbb{R}$ that are zero on $V\setminus V_0$, and the vector space of functions $f: V\to\mathbb{R}$ that are discrete-harmonic on V_0 (i.e., $\Delta f=0$ on V_0) are orthogonal to each other with respect to the inner product $(\cdot,\cdot)_\nabla$, and together they span $\mathbb{R}^V$. This basic observation will be used frequently below. Indeed, that they are orthogonal follows immediately from (2.3), and a dimension count now shows that the two spaces together span $\mathbb{R}^V$.

The following consequence of this orthogonality property will be used below. Let $V_0\subset V$ satisfy $V_0\supset V_\partial$ and let h_0 denote the function that is discrete-harmonic in $V\setminus V_0$

and equal to h in V_0. Then $h-h_0$ and h_0 are independent random variables, because $h=h_0+(h-h_0)$ is the corresponding orthogonal decomposition of h. It also follows that

$$h-h_0 \text{ is the DGFF in } V\setminus V_0 \text{ with zero boundary values on } V_0. \tag{2.4}$$

Observe that the Markov property and the effect of boundary conditions that were mentioned in §1.2 are immediate consequences of (2.4). An additional useful consequence is that the law of h_0 is proportional to $e^{-(h_0,h_0)_\nabla^2/2}$ times the Lebesgue measure on $\mathbb{R}^{V_0}$.

We now derive a useful well-known expression for the expectation of $h(v)h(u)$:

$$\mathbf{E}[h(v)h(u)]-\mathbf{E}[h(v)]\mathbf{E}[h(u)] = \frac{G(u,v)}{\deg(v)}, \tag{2.5}$$

where $G(u,v)$ is the expected number of visits to v by a simple random walk started at u before it hits V_∂ and $\deg(v)$ is the degree of v; that is, the number of edges incident with it. (The function G is known as the Green function.) As we have noted in the introduction, h is the sum of the discrete-harmonic extension of h_∂ and a DGFF with zero boundary values. It therefore suffices to prove (2.5) in the case where $h_\partial=0$ on V_∂. In this case, $\mathbf{E}[h(v)]=0=\mathbf{E}[h(u)]$. Setting $G_v(u)=G(u,v)$, we observe (or recall) that $\Delta G_v(u)=-\deg(v)1_v(u)$. Thus,

$$h(v)=(h,1_v)=-\frac{(h,\Delta G_v)}{\deg(v)} \overset{(2.3)}{=} \frac{(h,G_v)_\nabla}{\deg(v)}.$$

If X is a standard Gaussian in $\mathbb{R}^n$, and $x,y\in\mathbb{R}^n$, then $\mathbf{E}[(X\cdot x)(X\cdot y)]=x\cdot y$. Consequently, when $h_\partial=0$, we have

$$\begin{aligned}\mathbf{E}[h(v)h(u)] &= \frac{\mathbf{E}[(h,G_v)_\nabla(h,G_u)_\nabla]}{\deg(v)\deg(u)} = \frac{(G_v,G_u)_\nabla}{\deg(v)\deg(u)} \\ &= \frac{-(G_v,\Delta G_u)}{\deg(v)\deg(u)} = \frac{(G_v,1_u)}{\deg(v)} = \frac{G(u,v)}{\deg(v)}.\end{aligned}$$

This proves (2.5).

2.2. Some assumptions and notation

We will make frequent use of the following notation and assumptions:

(h) A bounded domain (non-empty, open, connected set) $D\subset\mathbb{R}^2$ whose boundary ∂D is a subgraph of TG is fixed. The set of vertices of TG in D is denoted by V_D and V_∂ denotes the set of vertices in ∂D. A constant $\bar\Lambda>0$ is fixed, as well as a function $h_\partial\colon V_\partial\to\mathbb{R}$ satisfying $\|h_\partial\|_\infty\leqslant\bar\Lambda$. The DGFF on D with boundary values given by h_∂ is denoted by h. Also set $V=V_{\overline{D}}=V_D\cup V_\partial$.

We denote by TG^* the hexagonal grid which is dual to the triangular lattice TG—so that each hexagonal face of TG^* is centered at a vertex of TG. Generally, a TG^*-hexagon will mean a closed hexagonal face of TG^*. Denote by $\mathfrak{B}_R$ the union of all TG^*-hexagons that intersect the ball $B(0,R)$.

Sometimes, in addition to (h) we will need to assume:

(D) The domain D is simply connected, and (to avoid minor but annoying trivialities) ∂D is a simple closed curve. We fix two distinct midpoints of TG-edges x_∂ and y_∂ on ∂D. Let the counterclockwise (respectively, clockwise) arc of ∂D from x_∂ to y_∂ be denoted by ∂_+ (respectively, ∂_-).

If $H\subset D$ is a TG^*-hexagon, we write $h(H)$ as a shorthand for the value of h on the center of H (which is a vertex of TG). Assuming (h) and (D), let $\mathfrak{D}^+$ denote the union of all TG^*-hexagons contained in D where h is positive together with the intersection of $\overline{D}$ with TG^*-hexagons centered at vertices in ∂_+. Let $\mathfrak{D}^-$ be the closure of $\overline{D}\setminus\mathfrak{D}^+$ (which almost surely consists of TG^*-hexagons in D where $h<0$ and the intersection of $\overline{D}$ with TG^*-hexagons whose center is in ∂_-). Then $\partial\mathfrak{D}^-\cap\partial\mathfrak{D}^+$ necessarily consists of the interface we previously called γ, and a collection of disjoint simple closed paths. We use the term *interface* (or *zero-height interface*) to describe a simple (or simple closed) path in $\partial\mathfrak{D}^-\cap\partial\mathfrak{D}^+$ oriented so that $\mathfrak{D}^+$ is on its right (that is, oriented clockwise around $\mathfrak{D}^+$).

Throughout, the notation $O(s)$ represents any quantity f such that $|f|\leqslant Cs$ for some absolute constant C. We use the notation $O_{\bar\Lambda}(s)$ if the constant also depends on $\bar\Lambda$. When introducing a constant c, we often write $c=c(a,b)$ as shorthand to indicate that c may depend on a and b.

2.3. Simple random walk background

We need to recall a very useful property of the discrete-harmonic measure of simple random walk.

LEMMA 2.1. (Hit near) *Let v be a vertex of the grid* TG, *and let H be a connected subgraph of* TG. *Set $d=\operatorname{diam} H$. The probability that a simple random walk on* TG *started from v exits the ball $B(v,d)$ before hitting H is at most $c(\operatorname{dist}(v,H)/d)^{\zeta_1}$, where c and $\zeta_1\in(0,1)$ are absolute constants.*

Likewise, the same bound applies to the probability that a simple random walk started at some vertex outside $B(v,d)$ will hit $B(v,\operatorname{dist}(v,H))$ before H.

In fact, we may take $\zeta_1=\frac{1}{2}$. The continuous version of this statement is known as the Beurling projection theorem (the extremal case is when H is a line segment). The above statement can probably be deduced from the discrete Beurling theorem as given in [L1, Theorem 2.5.2], though the setting there is slightly different. In any case, since we do not require any particular value for ζ_1, the lemma is rather easily proved directly (see [Sch, Lemma 2.1]).

We will also use the (well-known) discrete Harnack principle, in the following form.

LEMMA 2.2. (Harnack principle) *Let D, V, V_∂ and V_D be as in* (h), *let $v, u \in V_D$, and let $f: V \to \mathbb{R}$ be discrete-harmonic in V_D.*

(1) *If v and u are neighbors, then*

$$|f(v)-f(u)| \leqslant O(1)\frac{\|f\|_\infty}{\operatorname{dist}(v,\partial D)}.$$

(2) *If we assume that $f \geqslant 0$ and that there is a path of length ℓ from v to u whose minimal distance to ∂D is ϱ, then*

$$f(u) \geqslant f(v)\exp\left(-\frac{O(\ell+\varrho)}{\varrho}\right).$$

The proof of statement (1) can be obtained by noting that $f(v)$ is the expected value of f evaluated at the first hitting point on V_∂ of a random walk started at v, and observing that a random walk started at v and a random walk started at u may be coupled so that they meet before reaching a distance of $r:=\operatorname{dist}(v,\partial D)$ with probability $1-O(1/r)$ and walk together after they meet. (See also the more general [LSW4, Lemma 6.2], for example.)

To prove statement (2), let $W:=\{w\in V: f(w)\geqslant f(v)\}$, and observe that the maximum principle implies that W contains a path from v to ∂D. If we assume that $\ell \leqslant \frac{1}{2}\varrho$, then a random walk started at u has probability bounded away from zero to hit W before ∂D, which by the optional sampling theorem implies (2) in this case. The case $\ell > \frac{1}{2}\varrho$ now follows by induction on $\lceil 2\ell/\varrho \rceil$.

We also need the following well-known estimate on the Green function $G(u,v)$.

LEMMA 2.3. *Let D, V_D and V be as in* (h), *let $u, v \in V_D$, and let $\varepsilon > 0$. Set $r :=$* $\operatorname{dist}(u,\partial D)$, *and suppose that within distance r/ε from u there is a connected component of ∂D of diameter at least εr. If $|u-v| \leqslant \frac{2}{3} r$, say, then $G_D(v,u)$ (the expected number of visits to u by a simple random walk starting at v before hitting ∂D) satisfies*

$$G_D(v,u) = \exp(O_\varepsilon(1)) \log\frac{r+1}{|u-v|+1}.$$

The probability $H_D(v,u)$ that a random walk started from v hits u before ∂D can be expressed as $G_D(v,u)/G_D(u,u)$ and hence by the lemma

$$H_D(v,u) = \exp(O_\varepsilon(1))\left(1-\frac{\log(|u-v|+1)}{\log(r+1)}\right). \tag{2.6}$$

Let p_R denote the probability that a simple random walk started at 0 does not return to 0 before exiting the ball $B(0,R)$. We now show that the lemma follows from the well-known estimate

$$p_R = \frac{\exp(O(1))}{\log(R+2)}. \tag{2.7}$$

In the setting of the lemma, consider some vertex $w \in V_D$ such that $|w-u|>r$. It is easy to see that with probability bounded away from zero (by a function of ε), a simple random walk started from w will hit ∂D before u. Hence, $q:=\min\{1-H_D(w,u): w\in V_D \setminus B(u,r)\}$ is bounded away from 0 by a positive function of ε. Clearly,

$$\frac{1}{qp_r} \geqslant G_D(u,u) \geqslant \frac{1}{p_r}.$$

This proves the case $u=v$ of the lemma from (2.7). Now start a simple random walk at a vertex $v\neq u$ satisfying $|v-u| \leqslant \frac{2}{3}r$, and let the walk stop when it hits ∂D. It is easy to see that the probability that the walk visits v after exiting the ball $B\big(v, \frac{1}{4}|v-u|\big)$, say, is within a bounded factor from $H_D(v,u)$. Therefore, the expected number of visits to v after exiting $B\big(v, \frac{1}{4}|v-u|\big)$ is within a bounded factor of $G_D(v,u)$. But the former is the same as $G_D(v,v) - G_{B(v,|v-u|/4)}(v,v)$. The lemma follows.

We have not found a reference proving (2.7) in a way that generalizes to the setting of Theorem 1.4, though the result is well known. In fact, it easily follows from Rayleigh's method, as explained in [DoSn, §2.2]. In that book, the goal is to show that $p_R \to 0$, as $R\to\infty$, for lattices in the plane but not in $\mathbb{R}^3$. However, the method easily yields the quantitative bounds (2.7).

3. The height gap in the discrete setting

3.1. A priori estimates

This subsection contains some technical (and uninspiring) estimates that are necessary to carry out the technical (but hopefully interesting) coupling argument of the later parts of the section.

Suppose that β is some path in the hexagonal grid TG^*, which has a positive probability to be a subset of a zero-height interface h. We will need to understand rather well the behavior of h conditioned on β being a subset of a contour line. This conditioning amounts to conditioning h to be positive on vertices adjacent to β on one side and negative on vertices adjacent to β on the other side. (Here and in the following, a vertex v of TG is *adjacent* to β if β intersects the interior of one of the six boundary edges of the TG^*-hexagon centered at v.) Thus, the following lemma will be rather useful.

LEMMA 3.1. (Expectation bounds) *There is a finite* $c=c(\bar{\Lambda})>0$ *such that the following holds. Assume* (h). *Let* V_+ *and* V_- *be non-empty disjoint subsets of* V_D, *and set* $U:=V_\partial\cup V_+\cup V_-$. *Suppose that every vertex in* V_+ *has a neighbor in* $V_-\cup V_\partial$ *and every vertex in* V_- *has a neighbor in* $V_+\cup V_\partial$. *Let* $\mathcal{K}$ *be the event that* $h>0$ *on* V_+ *and* $h<0$ *on* V_-. *Then for every* $v\in V_+\cup V_-$,

$$\mathbf{E}[e^{|h(v)|}\,|\,\mathcal{K}]<c, \tag{3.1}$$

and

$$\frac{1}{c}<\mathbf{E}[|h(v)|\,|\,\mathcal{K}]<c. \tag{3.2}$$

Moreover,

$$\mathbf{E}[|h(v)|^{-1/2}\,|\,\mathcal{K}]<c. \tag{3.3}$$

Let $B\subset D$ *be a disk whose radius* r *is smaller than its distance to* U *and assume that* $B\cap V_D\neq\varnothing$. *Then*

$$\mathbf{E}[\max\{|\bar{h}(v)|:v\in V_D\cap B\}\,|\,\mathcal{K}]<c, \tag{3.4}$$

where $\bar{h}$ *denotes the discrete-harmonic extension of the restriction of* h *to* U (*which is also the conditional expectation of* h *given its restriction to* U). *Moreover*, *if* $\varepsilon>0$ *and* U *has a connected component whose distance from* B *is* R *and whose diameter is at least* εR, *then*

$$E\left[\left(\frac{1}{|V_D\cap B|}\sum_{v\in V_D\cap B}h(v)\right)^2\,\middle|\,\mathcal{K}\right]\leqslant c+c'\log\frac{R}{r}, \tag{3.5}$$

where $c'=c'(\varepsilon)$.

Proof. The proof of (3.1) is a maximum principle type argument. Suppose that v_1 maximizes $\mathbf{E}[e^{|h(v)|}|\mathcal{K}]$ among $v\in U$, and let $M:=\mathbf{E}[e^{|h(v_1)|}|\mathcal{K}]$. Clearly, $M<\infty$. Assume first that $v_1\in V_+$. Set $\widetilde{V}:=U\setminus\{v_1\}$, and for $v\in\widetilde{V}$ let p_v be the probability that the simple random walk starting at v_1 first hits $\widetilde{V}$ in v. We may write

$$h(v_1)=X+Y,$$

where

$$Y:=\sum_{v\in\widetilde{V}}p_v h(v) \tag{3.6}$$

and X is a centered Gaussian independent of $(h(v):v\in\widetilde{V})$. Moreover, by (2.5), $\mathbf{E}[X^2]$ is $\frac{1}{6}$ times the expected number of visits to v_1 by a simple random walk starting from v_1 until it first hits $\widetilde{V}$. Set $a:=\mathbf{E}[X^2]$. Since v_1 has a neighbor in $\widetilde{V}$, it follows that $a=O(1)$.

Also, clearly, a is bounded away from zero, since this random walk has at least one visit to v_1.

For every $y\in\mathbb{R}$, we have that

$$\mathbf{E}[e^{h(v_1)}\mid Y=y,\mathcal{K}]=\mathbf{E}[e^{X+y}\mid Y=y,\mathcal{K}]=e^y\mathbf{E}[e^X\mid X+y>0]=e^y\frac{\int_{-y}^{\infty}\exp(x-x^2/2a)\,dx}{\int_{-y}^{\infty}\exp(-x^2/2a)\,dx}.$$

If $y>0$, the right-hand side is bounded by $O(e^y)$. If $y\leqslant 0$, then both integrals on the right are comparable to their value when the upper integration limit is reduced to $-y+1$. But then the ratio between the corresponding integrands is bounded by e^{1-y}, which implies the same for the ratio of the integrals. Consequently, the right-hand side is $O(1)$ when $y\leqslant 0$. We take expectations conditioned on $\mathcal{K}$ and get

$$M=O(1)\mathbf{E}[e^Y\mid\mathcal{K}]+O(1). \tag{3.7}$$

Repeated use of Hölder's inequality shows that for non-negative random variables $x_1,...,x_n$ and non-negative constants $p_1,...,p_n$ such that $\sum_{j=1}^n p_j\leqslant 1$ we have

$$\mathbf{E}\biggl[\prod_{j=1}^n x_j^{p_j}\biggr]\leqslant\prod_{j=1}^n\mathbf{E}[x_j]^{p_j}.$$

Thus, (3.6) and (3.7) give

$$M\leqslant O(1)\prod_{v\in\widetilde{V}}\mathbf{E}[e^{h(v)}\mid\mathcal{K}]^{p_v}+O(1).$$

Clearly,

$$\mathbf{E}[e^{h(v)}\mid\mathcal{K}]\leqslant\begin{cases}1, & \text{if } v\in V_-,\\ M, & \text{if } v\in V_+,\\ e^{\bar\Lambda}, & \text{if } v\in V_\partial.\end{cases}$$

Setting $p_+:=\sum_{v\in\widetilde{V}\cap V_+}p_v$, we therefore obtain

$$M\leqslant O(1)M^{p_+}e^{(1-p_+)\bar\Lambda}+O(1). \tag{3.8}$$

Since v_1 has a neighbor in $U\setminus V_+$, we have $p_+\leqslant\frac{5}{6}$ and $M\leqslant O(1)e^{\bar\Lambda}$ follows. A symmetric argument applies if $v_1\in V_-$. If $v_1\in V_\partial$, then obviously $M\leqslant e^{\bar\Lambda}$. Thus (3.1) holds with $c=O(e^{\bar\Lambda})$. The right-hand inequality in (3.2) is an immediate consequence of (3.1) (possibly with a different c).

We now prove (3.4). Given $v\in V_D$ and $u\in U$, let $H(v,u)$ denote the probability that a simple random walk started at v first hits U at u. Suppose that $v,v'\in B\cap V_D$ and $u\in U$. The discrete Harnack principle (Lemma 2.2) then gives $H(v',u)\leqslant H(v,u)O(1)$. Thus,

$$|\bar h(v')|=\biggl|\sum_{u\in U}H(v',u)h(u)\biggr|\leqslant\sum_{u\in U}O(1)H(v,u)|h(u)|.$$

The right-hand side therefore bounds $\max\{|\bar{h}(v')|:v'\in V_D\cap B\}$. Now, (3.4) follows from the right-hand inequality of (3.2) and $\sum_{u\in U} H(v,u)=1$.

We now prove (3.3). Consider some $v\in V_+$. We first claim that if w neighbors with v and $w\notin U$, then $\mathbf{E}[h(w)^2\,|\,\mathcal{K}]=O_{\bar{\Lambda}}(1)$. To see this, first note that $\bar{h}(w)$ (where as above $\bar{h}$ is the discrete-harmonic extension of the values of h on U) is a linear combination of the values $\bar{h}(v)$ for $v\in U$ (which are the same as the values of h there); since each of these values has mean and variance which are $O_{\bar{\Lambda}}(1)$ (by (3.1)), the mean and variance of $\bar{h}(w)$ are also $O_{\bar{\Lambda}}(1)$. By (2.5), conditioned on $\bar{h}$, the value $h(w)$ is Gaussian with variance $\frac{1}{6}$ times the expected number of times a random walk started at w visits w before hitting U, which is $O(1)$ because U contains a neighbor of w.

Let Z denote the average of h on the neighbors of v, and let $Z':=h(v)-Z$. The above implies that $\mathbf{E}[Z^2\,|\,\mathcal{K}]=O_{\bar{\Lambda}}(1)$. Since Z' is a centered Gaussian with variance $\frac{1}{6}$, for every $z\in\mathbb{R}$,

$$\mathbf{E}[h(v)^{-1/2}\,|\,\mathcal{K},Z=z]=\mathbf{E}[(Z'+z)^{-1/2}\,|\,Z'+z>0]=\frac{\int_0^\infty x^{-1/2}e^{-6(x-z)^2/2}\,dx}{\int_0^\infty e^{-6(x-z)^2/2}\,dx}.$$

If $z\geqslant -2$, then this is clearly bounded. Assume therefore that $z<-2$. It is easy to verify that the integrals in the numerator and denominator are comparable to the same integrals restricted to the range $x\in[0,|z|^{-1}]$. But in this range, the maximum value of

$$\exp\bigl(-\tfrac{6}{2}(x-z)^2\bigr)$$

is comparable to the minimum value of the same quantity. Consequently, when $z<-2$,

$$\begin{aligned}\mathbf{E}[h(v)^{-1/2}\,|\,\mathcal{K},Z=z]&=O(1)\frac{\int_0^{|z|^{-1}} x^{-1/2}e^{-6(x-z)^2/2}\,dx}{\int_0^{|z|^{-1}} e^{-6(x-z)^2/2}\,dx}\\&=O(1)\frac{\int_0^{|z|^{-1}} x^{-1/2}\,dx}{\int_0^{|z|^{-1}} dx}=O(1)|z|^{1/2},\end{aligned}$$

which gives

$$\mathbf{E}[h(v)^{-1/2}\,|\,\mathcal{K}]=O(1)+O(1)\mathbf{E}[|Z|^{1/2}\,|\,\mathcal{K}].$$

Since $\mathbf{E}[Z^2|\mathcal{K}]=O_{\bar{\Lambda}}(1)$, we certainly have $\mathbf{E}[|Z|^{1/2}|\mathcal{K}]=O_{\bar{\Lambda}}(1)$. This proves (3.3). Now the left-hand inequality in (3.2) is an immediate consequence.

We now prove (3.5). Consider two vertices $v,u\in B\cap V_D$. Assume that within distance R from B there is a connected component of U of diameter at least εR. Since

$$\mathbf{E}[h(v)\,|\,\mathcal{K},\bar{h}]=\bar{h}(v),$$

we have that

$$\mathbf{E}[h(v)h(u)\,|\,\mathcal{K}]=\mathbf{E}[(h(v)-\bar{h}(v))(h(u)-\bar{h}(u))\,|\,\mathcal{K}]+\mathbf{E}[\bar{h}(v)\bar{h}(u)\,|\,\mathcal{K}]. \tag{3.9}$$

Now, $\bar{h}(v)$ is just a weighted average of $h(w)$ with $w\in U$ (according to the discrete-harmonic measure from v). Consequently,

$$\mathbf{E}[\bar{h}(v)^2\,|\,\mathcal{K}]\leqslant\max_{w\in U}\mathbf{E}[h(w)^2\,|\,\mathcal{K}]\leqslant\max_{w\in U}2\mathbf{E}[e^{|h(w)|}\,|\,\mathcal{K}]\leqslant 2c$$

and the Cauchy–Schwarz inequality implies that the last summand in (3.9) is also bounded by $2c$. Since $h-\bar{h}$ is the Gaussian free field with zero boundary values on U and is independent of the restriction of h to U, by (2.5) we have that

$$\mathbf{E}[(h(v)-\bar{h}(v))(h(u)-\bar{h}(u))\,|\,\mathcal{K}]$$

is $\frac{1}{6}$ times the expected number of visits to u by a random walker started at v which stops when it hits U. Thus

$$\sum_{u\in B\cap V_D}\mathbf{E}[(h(v)-\bar{h}(v))(h(u)-\bar{h}(u))\,|\,\mathcal{K}]$$

is $\frac{1}{6}$ times the expected number of steps that the walker spends in B, which is

$$O_\varepsilon(1)|B\cap V_D|\log\frac{R}{r},$$

by Lemma 2.3. The estimate (3.5) is now obtained by averaging (3.9) over all $v,u\in B\cap V_D$. □

The following lemma provides a variant of the right-hand bound in (3.2) in the case where instead of looking for a zero-height interface of h, we consider instead a zero-height interface of $h-g$, for some fixed function g.

LEMMA 3.2. (Further expectation bounds) *In the setting of Lemma* 3.1, *suppose that* $g:\overline{D}\to\mathbb{R}$ *is zero on* V_∂ *and Lipschitz in* $\overline{D}$. *Let* $\mathcal{K}_g$ *be the event that* $h>g$ *on* V_+ *and* $h<g$ *on* V_-. *Then for every* $v\in V_+\cup V_-$,

$$\mathbf{E}[|h(v)-g(v)|\,|\,\mathcal{K}_g]\leqslant c(1+\bar{\Lambda})+c\frac{\|g\|_\infty}{\log(q+2)}, \tag{3.10}$$

where $c>0$ *is a universal constant and* $q=\|g\|_\infty/\|\nabla g\|_\infty$.

Proof. The proof is similar to the proof of Lemma 3.1. Suppose that v_1 maximizes $\mathbf{E}[h(v)-g(v)|\mathcal{K}_g]$ among $v\in V_+$, and let $M:=\mathbf{E}[h(v_1)-g(v_1)|\mathcal{K}_g]$. By symmetry, it is enough to get a bound on M. We define $\widetilde{V}$, X, Y and p_v as in the proof of Lemma 3.1.

For every $y\in\mathbb{R}$ we have

$$\mathbf{E}[h(v_1)-g(v_1)\,|\,Y=y,\mathcal{K}_g]=\mathbf{E}[X+y-g(v_1)\,|\,X+y-g(v_1)>0]\leqslant O(1)+(y-g(v_1))_+,$$

where $(x)_+:=\max\{0,x\}$. Consequently,

$$\begin{aligned}M&\leqslant O(1)+\mathbf{E}[(Y-g(v_1))_+\,|\,\mathcal{K}_g]\\&\leqslant O(1)+\sum_{v\in\widetilde{V}}p_v\mathbf{E}[(h(v)-g(v_1))_+\,|\,\mathcal{K}_g]\\&\leqslant O(1)+\sum_{v\in\widetilde{V}}p_v\mathbf{E}[(h(v)-g(v))_+\,|\,\mathcal{K}_g]+\sum_{v\in\widetilde{V}}p_v|g(v)-g(v_1)|\\&\leqslant O(1)+\bar{\Lambda}+M\sum_{v\in V_+}p_v+\sum_{v\in\widetilde{V}}p_v|g(v)-g(v_1)|.\end{aligned}$$

Consider a simple random walk started at v_1 and stopped when it first hits $\widetilde{V}$. Denote the vertex where it first hits $\widetilde{V}$ by w. (Then, $\mathbf{P}[v=w]=p_v$.) Set $r:=q/\log(q+2)$. Since v_1 has a neighbor in $\widetilde{V}$, we have $\mathbf{P}[|v-v_1|>r]\leqslant O(1)/\log(r+2)$, by standard random walk estimates. When $|v-v_1|\leqslant r$, we have $|g(v)-g(v_1)|\leqslant O(1)r\|\nabla g\|_\infty$. Therefore,

$$\sum_{v\in\widetilde{V}}p_v|g(v)-g(v_1)|\leqslant O(1)r\|\nabla g\|_\infty+O(1)\frac{\|g\|_\infty}{\log(r+2)}\leqslant O(1)\frac{\|g\|_\infty}{\log(q+2)}.$$

This gives

$$M\sum_{v\in\widetilde{V}\setminus V_+}p_v\leqslant O(1)+\bar{\Lambda}+O(1)\frac{\|g\|_\infty}{\log(q+2)}.$$

Because $\sum_{v\in\widetilde{V}\setminus V_+}p_v$ is bounded away from zero, the proof is now complete. □

Our next result establishes the continuity of the conditional distribution of h in the specified data. More precisely, the following proposition holds.

PROPOSITION 3.3. (Heights interface continuity) *For every $\varepsilon>0$ there is some $R=R(\varepsilon,\bar{\Lambda})>1/\varepsilon$ such that the following holds. Let D, V_D, V_∂, h_∂, V_+, V_-, U and $\mathcal{K}$ be as in Lemma* 3.1, *and let $\widehat{D}$, $\widehat{V}_D$, $\widehat{V}_\partial$, $\hat{h}_\partial$, $\widehat{V}_+$, $\widehat{V}_-$, $\widehat{U}$ and $\widehat{\mathcal{K}}$ be another such system, which is also assumed to satisfy $\|\hat{h}_\partial\|_\infty\leqslant\bar{\Lambda}$. Let $h_{\mathcal{K}}$ be a DGFF in D with boundary values given by h_∂ conditioned on $\mathcal{K}$, and let $h_{\widehat{\mathcal{K}}}$ be a DGFF in $\widehat{D}$ with boundary values given by $\hat{h}_\partial$ conditioned on $\widehat{\mathcal{K}}$. Suppose that within $\mathfrak{B}_R$, the two systems are the same*; *that is, $D\cap\mathfrak{B}_R=\widehat{D}\cap\mathfrak{B}_R$, $h_\partial|_{\mathfrak{B}_R}=\hat{h}_\partial|_{\mathfrak{B}_R}$, $V_+\cap\mathfrak{B}_R=\widehat{V}_+\cap\mathfrak{B}_R$ and $V_-\cap\mathfrak{B}_R=\widehat{V}_-\cap\mathfrak{B}_R$. Further suppose that $0\in U$. Then there is a coupling of $h_{\mathcal{K}}$ and $h_{\widehat{\mathcal{K}}}$ such that for every vertex $v\in\mathfrak{B}_{1/\varepsilon}$ we have $\mathbf{E}[|h_{\mathcal{K}}(v)-h_{\widehat{\mathcal{K}}}(v)|]<\varepsilon$.*

The following lemma will be needed in the proof.

LEMMA 3.4. *Let X be a 1-dimensional Gaussian of zero mean and unit variance. Let $x,\hat{x}\in\mathbb{R}$, let Z be a random variable whose distribution is the same as that of $X+x$ conditioned on $X+x>0$, and let $\widehat{Z}$ be a random variable whose distribution is the same as that of $X+\hat{x}$ conditioned on $X+\hat{x}>0$. Then there is a coupling of Z and $\widehat{Z}$ such that $|Z-\widehat{Z}|<|x-\hat{x}|$ almost surely if $x\neq\hat{x}$. Moreover, there is a continuous function $\delta(x,\hat{x})$ satisfying $\delta(x,\hat{x})<1$ such that $\mathbf{E}[|Z-\widehat{Z}|]\leqslant\delta(x,\hat{x})|x-\hat{x}|$ under this coupling.*

The coupling that we use is what is known as the *quantile coupling* of Z and $\widehat{Z}$.

Proof. Let $F(s)=\mathbf{P}[X<s]$, and let $G=F^{-1}$. Set $t:=F(-x)$ and $\hat{t}:=F(-\hat{x})$. Let p be a random variable uniformly distributed in $[0,1]$. Then

$$Z(t):=x+G(t+p(1-t))=G(t+p-tp)-G(t)$$

has the same distribution as Z. Therefore, $(Z(t),Z(\hat{t}))$ is a coupling of Z and $\widehat{Z}$. Consequently, to verify the first claim it is sufficient to show that $|\partial_t Z(t)|<\partial_t G(t)$. In fact, we will prove the stronger statement, $-\partial_t G(t)<\partial_t Z(t)<0$ for all $p\in(0,1)$, which is equivalent to

$$0<\partial_t G(t+p-tp)<\partial_t G(t).$$

The left-hand inequality is immediate, because $G'>0$ on $(0,1)$. The right-hand inequality translates to $(1-p)G'(t+p-tp)<G'(t)$, which we rewrite as

$$(1-(t+p-tp))G'(t+p-tp)<(1-t)G'(t).$$

This is equivalent to $(1-t_p)/F'(G(t_p))<(1-t)/F'(G(t))$, where $t_p:=t+p-tp>t$. Now, note that

$$\frac{1-t}{F'(G(t))}=\frac{\int_{-x}^{\infty}\exp(-s^2/2)\,ds}{\exp(-x^2/2)}=\int_{-x}^{\infty}e^{(x^2-s^2)/2}\,ds=\int_0^{\infty}e^{xs-s^2/2}\,ds$$

is strictly decreasing in t, because x is strictly decreasing in t. This proves the first claim. The second claim follows with

$$\delta(x,\hat{x}):=\int_0^1\frac{Z(t)-Z(\hat{t})}{x-\hat{x}}\,dp\quad\text{when } x\neq\hat{x}$$

and

$$\delta(x,x):=\int_0^1\frac{Z'(t)}{-G'(t)}\,dp. \qquad \square$$

Proof of Proposition 3.3. Fix a coupling of $h_{\mathcal{K}}$ and $h_{\widehat{\mathcal{K}}}$ that minimizes

$$\sum_{v\in V_D\cap\widehat{V}_D}\mathbf{E}[|h_{\mathcal{K}}(v)-h_{\widehat{\mathcal{K}}}(v)|].$$

Standard continuity and compactness arguments show that there is such a coupling. Set $f(v):=\mathbf{E}[|h_{\mathcal{K}}(v)-h_{\widehat{\mathcal{K}}}(v)|]$ for vertices $v\in(V_D\cup V_\partial)\cap(\widehat{V}_D\cup\widehat{V}_\partial)$.

First, we claim that f is discrete-subharmonic on vertices in $\mathfrak{B}_R\setminus U$. Indeed, fix a vertex $w\in\mathfrak{B}_R\setminus U$. The conditional distribution of $h_{\mathcal{K}}(w)$ given the value of $h_{\mathcal{K}}$ at every vertex but w is that of $x+AX$, where x is the average of $h_{\mathcal{K}}$ on the neighbors of w, X is a standard Gaussian, and A is the lattice-dependent constant $1/\sqrt{6}$ (since each vertex has six neighbors in TG). Similarly, $h_{\widehat{\mathcal{K}}}(w)=\hat{x}+A\widehat{X}$. By the choice of coupling, when we fix the values of $h_{\mathcal{K}}$ and $h_{\widehat{\mathcal{K}}}$ off of w, the corresponding conditioned coupling of $h_{\mathcal{K}}(w)$ and $h_{\widehat{\mathcal{K}}}(w)$ minimizes the conditioned expectation of $|h_{\mathcal{K}}(w)-h_{\widehat{\mathcal{K}}}(w)|$. But one such conditioned coupling is obtained by taking $X=\widehat{X}$. Thus, $f(w)\leqslant\mathbf{E}[|x-\hat{x}|]$, which implies that f is discrete-subharmonic at w, since

$$\sum_{u\sim w}|h_{\mathcal{K}}(u)-h_{\widehat{\mathcal{K}}}(u)|\geqslant\biggl|\sum_{u\sim w}h_{\mathcal{K}}(u)-\sum_{u\sim w}h_{\widehat{\mathcal{K}}}(u)\biggr|, \tag{3.11}$$

where the sums are over the neighbors of w.

Next, consider any vertex $v\in V_+\cap\mathfrak{B}_R$. As before, we may write $h_{\mathcal{K}}(v)=x+AX$, where x is the average of $h_{\mathcal{K}}$ on the neighbors of v and X is a random variable whose conditional law given x is that of a standard Gaussian conditional on $x+AX>0$. Lemma 3.4 applied with x/A instead of x and $\hat{x}/A$ instead of $\hat{x}$ implies that f is also discrete-subharmonic at v. We claim that there is a constant $b=b(\bar{\Lambda},\varepsilon)>0$ such that

$$\Delta f(v)\geqslant b,\quad \text{if } f(v)\geqslant\tfrac{1}{2}\varepsilon, \tag{3.12}$$

where Δ denotes the discrete Laplacian on TG. Indeed, the optimality of the coupling gives

$$\mathbf{E}[|h_{\mathcal{K}}(v)-h_{\widehat{\mathcal{K}}}(v)|\,|\,x,\hat{x}]\leqslant\delta\Big(\frac{x}{A},\frac{\hat{x}}{A}\Big)|x-\hat{x}|,$$

where $\delta(\cdot\,,\cdot)<1$ is as in the lemma. Thus,

$$f(v)\leqslant\mathbf{E}\Big[\delta\Big(\frac{x}{A},\frac{\hat{x}}{A}\Big)|x-\hat{x}|\Big].$$

By (3.11) with v in place of w, we have

$$\Delta f(v)\geqslant\mathbf{E}[|x-\hat{x}|]-f(v)\geqslant\mathbf{E}\Big[\Big(1-\delta\Big(\frac{x}{A},\frac{\hat{x}}{A}\Big)\Big)|x-\hat{x}|\Big]. \tag{3.13}$$

For every neighbor u of v we have, by (3.5), that $\mathbf{E}[h_{\mathcal{K}}(u)^2] < O_{\bar\Lambda}(1)$. It easily follows that $\mathbf{E}[x^2] < O_{\bar\Lambda}(1)$. Hence, the Cauchy–Schwarz inequality implies that there is a constant $b_0(\varepsilon, \bar\Lambda) > 0$ such that $\mathbf{E}[|x| 1_{\mathcal{A}}] < \frac{1}{8}\varepsilon$ for every event $\mathcal{A}$ satisfying $\mathbf{P}[\mathcal{A}] < b_0$. There is a constant $b_1 = b_1(\varepsilon, \bar\Lambda)$ such that $\mathbf{P}[|x| > b_1] < \frac{1}{2} b_0$. The same inequalities will hold with $\hat x$ and $h_{\widehat{\mathcal{K}}}$ in place of x and $h_{\mathcal{K}}$. Therefore,

$$\begin{aligned}\mathbf{E}[|x-\hat x|] &= \mathbf{E}[|x-\hat x| 1_{|x|\vee|\hat x| \leqslant b_1}] + \mathbf{E}[|x-\hat x| 1_{|x|\vee|\hat x| > b_1}] \\ &\leqslant \mathbf{E}[|x-\hat x| 1_{|x|\vee|\hat x| \leqslant b_1}] + \mathbf{E}[(|x|+|\hat x|) 1_{|x|\vee|\hat x| > b_1}] \\ &\leqslant \mathbf{E}[|x-\hat x| 1_{|x|\vee|\hat x| \leqslant b_1}] + \tfrac{1}{4}\varepsilon.\end{aligned}$$

Thus, if we assume that $f(v) \geqslant \frac{1}{2}\varepsilon$, then also $\mathbf{E}[|x-\hat x|] \geqslant \frac{1}{2}\varepsilon$, and therefore

$$\mathbf{E}[|x-\hat x| 1_{|x|\vee|\hat x| \leqslant b_1}] \geqslant \tfrac{1}{4}\varepsilon.$$

Therefore, (3.13) gives (3.12) with

$$b := \frac{\varepsilon}{4} \min\Big\{ 1 - \delta\Big(\frac{x}{A}, \frac{\hat x}{A}\Big) : x, \hat x \in [-b_1, b_1] \Big\}.$$

Clearly, f is also discrete-subharmonic on $V_- \cap \mathfrak{B}_R$ and (3.12) also holds for $v \in V_- \cap \mathfrak{B}_R$ and (trivially) for $v \in V_\partial \cap \mathfrak{B}_R$.

Next, we prove that for all vertices $w \in \mathfrak{B}_R$ we have

$$f(w) \leqslant O_{\bar\Lambda}(1) \sqrt{\log R}. \tag{3.14}$$

Fix such a w, and assume that $w \notin U$. We may decompose $h_{\mathcal{K}}(w)$ as a sum $h_{\mathcal{K}}(w) = y + Y$, where y is the value at w of the discrete-harmonic extension of the restriction of $h_{\mathcal{K}}$ to U, and Y is a centered Gaussian whose variance is $\frac{1}{6}$ times the expected number of visits to w by a simple random walk started at w that is stopped when it hits U. A simple random walk on TG started at w has probability at least a positive constant times $1/\log R$ to reach distance R from w before returning to w, and once it does reach this distance, it has probability bounded away from zero to hit 0 before returning to w. Since $0 \in U$, it follows that $\mathbf{E}[Y^2] = O(\log R)$. Thus, $\mathbf{E}[|Y|] = O(\sqrt{\log R}\,)$. As y is the average of the value of $h_{\mathcal{K}}$ on U with respect to the discrete-harmonic measure from w, it follows from (3.2) that $\mathbf{E}[|y|] = O_{\bar\Lambda}(1)$. Thus, we have $\mathbf{E}[|h_{\mathcal{K}}(w)|] \leqslant O_{\bar\Lambda}(\sqrt{\log R}\,)$. This certainly also holds if $w \in U$, and a similar estimate holds for $h_{\widehat{\mathcal{K}}}(w)$. Now (3.14) follows, since $f(w) \leqslant \mathbf{E}[|h_{\mathcal{K}}(w)|] + \mathbf{E}[|h_{\widehat{\mathcal{K}}}(w)|]$.

We now show that the established properties of f imply that $f \leqslant \varepsilon$ on $\mathfrak{B}_{1/\varepsilon}$ if R is sufficiently large. Fix some vertex $w \in \mathfrak{B}_{1/\varepsilon}$, and let S_t be a simple random walk on TG started at w. Let t_1 be the first time t such that $|S_t|>\frac{1}{2}R$ or $S_t=0$. Since f is discrete-subharmonic on $V_D\cap\mathfrak{B}_R$, we have that $t\mapsto f(S_{t\wedge t_1})$ is a submartingale. The optional sampling theorem implies that $f(w)\leqslant\mathbf{E}[f(S_{t_1})]$. By standard random walk estimates, $\mathbf{P}\big[|S_{t_1}|>\frac{1}{2}R\big]\leqslant O(1)|\log\varepsilon|/\log R$. (We assume, with no loss of generality, that $\varepsilon<\frac{1}{2}$, say.) Consequently,

$$f(w)\leqslant\mathbf{E}[f(S_{t_1})]\leqslant f(0)+O(1)\frac{|\log\varepsilon|}{\log R}\max\{f(u):u\in V_D\cap\mathfrak{B}_R\}\leqslant f(0)+O_{\bar\Lambda}(1)\frac{|\log\varepsilon|}{\sqrt{\log R}}.$$

This proves that $f(w)\leqslant\varepsilon$ if $f(0)\leqslant\frac{1}{2}\varepsilon$ and R is sufficiently large.

Now assume that $f(0)\geqslant\frac{1}{2}\varepsilon$. Let $\widetilde{S}_t$ be a simple random walk starting at 0. Let

$$t_*:=\min\big\{t:|\widetilde{S}_t|>\tfrac{1}{2}R\big\}$$

and let n_s be the number of $t\in\{0,...,s-1\}$ such that $\widetilde{S}_t=0$. By (3.12) and our assumption that $f(0)\geqslant\frac{1}{2}\varepsilon$, we have that $t\mapsto f(\widetilde{S}_{t\wedge t_*})-bn_{t\wedge t_*}$ is a submartingale. Thus,

$$\begin{aligned}0\leqslant f(0)\leqslant\mathbf{E}[f(\widetilde{S}_{t_*})]-b\mathbf{E}[n_{t_*}]&\leqslant\max\{f(u):u\in V_D\cap\mathfrak{B}_R\}-b\mathbf{E}[n_{t_*}]\\&\leqslant O_{\bar\Lambda}(1)\sqrt{\log R}-b\mathbf{E}[n_{t_*}].\end{aligned}$$

Now note that as $R\to\infty$, while ε is fixed, $\mathbf{E}[n_{t_*}]$ grows at least as fast as a positive constant times $\log R$, because the probability for $\widetilde{S}_t$ not to return to 0 after any specific visit to 0 is bounded by $O(1/\log R)$. Thus, the above rules out the possibility that $f(0)\geqslant\frac{1}{2}\varepsilon$ if R is sufficiently large. This completes the proof. □

As a corollary of the proposition, we now show that the correlation in the values of h at two vertices in U decays when the distance between them tends to infinity.

Corollary 3.5. (Correlation decay) *For every $\varepsilon>0$ there is some $R=R(\varepsilon,\bar\Lambda)$ such that the following holds. Let D, V_D, V_∂, h_∂, V_+, V_-, U and $\mathcal{K}$ be as in Lemma* 3.1, *and let $v_1,v_2\in U$ satisfy $|v_1-v_2|>R$. Then*

$$\big|\mathbf{E}[h(v_1)h(v_2)\,|\,\mathcal{K}]-\mathbf{E}[h(v_1)\,|\,\mathcal{K}]\mathbf{E}[h(v_2)\,|\,\mathcal{K}]\big|<\varepsilon.$$

Proof. Suppose, without loss of generality, that $v_2\in V_+$. Fix some $a>0$ and let $X:=1_{\{0<h(v_2)\leqslant a\}}$. We may apply Proposition 3.3 to our present setup and to the setup where the value of $h(v_2)$ is fixed at some constant $y\in(0,a]$ and $v_2\in\partial D$. Thus, the proposition would apply, provided that $\bar\Lambda$ is replaced by $\bar\Lambda\vee a$. Consequently, we find that there is an $R'=R'(\varepsilon,\bar\Lambda,a)$ such that if $|v_1-v_2|>R'$, then

$$\big|\mathbf{E}[h(v_1)\,|\,h(v_2),\mathcal{K}]-\mathbf{E}[h(v_1)\,|\,\mathcal{K}]\big|X\leqslant\frac{\varepsilon}{2a}.$$

Since $h(v_2)X\leqslant a$, $X^2=X$ and $h(v_2)X$ is $h(v_2)$-measurable, this gives

$$\begin{aligned}\tfrac{1}{2}\varepsilon &\geqslant \bigl|\mathbf{E}[h(v_1)\,|\,h(v_2),\mathcal{K}]h(v_2)X-\mathbf{E}[h(v_1)\,|\,\mathcal{K}]h(v_2)X\bigr|\\ &= \bigl|\mathbf{E}[h(v_1)h(v_2)X\,|\,h(v_2),\mathcal{K}]-\mathbf{E}[h(v_1)\,|\,\mathcal{K}]h(v_2)X\bigr|.\end{aligned}$$

Taking expectations conditioned on $\mathcal{K}$ now gives

$$\bigl|\mathbf{E}[h(v_1)h(v_2)X\,|\,\mathcal{K}]-\mathbf{E}[h(v_1)\,|\,\mathcal{K}]\mathbf{E}[h(v_2)X\,|\,\mathcal{K}]\bigr|\leqslant \tfrac{1}{2}\varepsilon. \tag{3.15}$$

Since

$$(1-X)h(v_2)^2\leqslant \frac{(1-X)|h(v_2)|^3}{a}\leqslant \frac{6e^{|h(v_2)|}}{a}$$

and $h(v_1)^2\leqslant 2e^{|h(v_1)|}$, if c denotes the constant satisfying (3.1), then the Cauchy–Schwarz inequality gives

$$\mathbf{E}[h(v_1)h(v_2)(1-X)\,|\,\mathcal{K}]^2\leqslant \mathbf{E}[h(v_1)^2\,|\,\mathcal{K}]\mathbf{E}[h(v_2)^2(1-X)\,|\,\mathcal{K}]\leqslant 2c\frac{6c}{a}.$$

Similarly,

$$0\leqslant \mathbf{E}[|h(v_2)|(1-X)\,|\,\mathcal{K}]\leqslant E\left[\frac{2e^{|h(v_2)|}}{a}\,\middle|\,\mathcal{K}\right]\leqslant \frac{2c}{a}.$$

Consequently, if a is chosen sufficiently large then

$$\bigl|\mathbf{E}[h(v_1)h(v_2)X\,|\,\mathcal{K}]-\mathbf{E}[h(v_1)h(v_2)\,|\,\mathcal{K}]\bigr|=\bigl|\mathbf{E}[h(v_1)h(v_2)(1-X)\,|\,\mathcal{K}]\bigr|<\tfrac{1}{4}\varepsilon$$

and

$$\bigl|\mathbf{E}[h(v_1)\,|\,\mathcal{K}]\mathbf{E}[h(v_2)X\,|\,\mathcal{K}]-\mathbf{E}[h(v_1)\,|\,\mathcal{K}]\mathbf{E}[h(v_2)\,|\,\mathcal{K}]\bigr|<\tfrac{1}{4}\varepsilon.$$

The corollary now follows from (3.15). □

Next, we provide a simple lemma which bounds the amount in which adding a function to h affects its distribution.

LEMMA 3.6. (DGFF distortion) *Assume* (h). *Let* $f\colon V_D\cup V_\partial\to\mathbb{R}$ *satisfy* $f=0$ *on* V_∂. *Let* μ *be the law of* h, *and let* μ_f *be the law of* $\tilde h:=h+f$. *Then, for every event* $\mathcal{A}$,

$$\mu_f[\mathcal{A}]\leqslant \exp\biggl(\frac{\|\nabla f\|^2}{2}\biggr)\mu[\mathcal{A}]^{1/2}.$$

Proof. Suppose that X is a standard Gaussian in $\mathbb{R}^n$, $y\in\mathbb{R}^n$ is some fixed vector, and $A\subset\mathbb{R}^n$ is measurable. Then

$$\begin{aligned}\mathbf{P}[X+y\in A] &= c_n\int_{\mathbb{R}^n} 1_A \exp\biggl(-\frac{\|x-y\|^2}{2}\biggr)\,dx\\ &= c_n\int_{\mathbb{R}^n} 1_A \exp\biggl(x\cdot y-\frac{\|y\|^2}{2}\biggr)\exp\biggl(-\frac{\|x\|^2}{2}\biggr)\,dx,\end{aligned}$$

where $c_n^{-1}=\int_{\mathbb{R}^n}\exp(-\|x\|^2/2)\,dx$ and the integrals are with respect to the Lebesgue measure in $\mathbb{R}^n$. (This is the Cameron–Martin formula.) We may think of the right-hand side as the inner product of 1_A and $\exp(x\cdot y-\|y\|^2/2)$ with respect to the Gaussian measure. Hence, the Cauchy–Schwarz inequality gives

$$\mathbf{P}[X+y\in A]\leqslant \mathbf{P}[X\in A]^{1/2}\mathbf{E}[\exp(2X\cdot y-\|y\|^2)]^{1/2}=\mathbf{P}[X\in A]^{1/2}\exp\left(\frac{\|y\|^2}{2}\right).$$

Let $\bar{h}$ denote the discrete-harmonic extension of h_∂. Then $h-\bar{h}$ is the DGFF with zero boundary values, and hence is a standard Gaussian on $\mathbb{R}^{V_D}$ with respect to the norm $g\mapsto\|\nabla g\|_2$. The lemma follows. □

3.2. Near independence

In this subsection we build on the infrastructure developed above to prove that under appropriate assumptions the shape of an interface inside a ball does not depend too strongly on the shape of an interface outside a slightly larger ball. More precisely, we have the following result.

PROPOSITION 3.7. (Near independence) *Let $C>1$ and let $R>10^3C$. Assume* (h) *and $\mathfrak{B}_{5R}\subset D$. Let $R_1,R_2,R_3\in[R,5R]$ satisfy*

$$R_1+\frac{R}{C}<R_2,\quad R_2+\frac{R}{C}<R_3\quad\textit{and}\quad R_3+\frac{R}{C}<5R.$$

Let V_+^3 and V_-^3 be disjoint sets of vertices in $D\setminus\mathfrak{B}_{R_3}$ and let V_+^1 and V_-^1 be disjoint sets of vertices in $\mathfrak{B}_{R_1}$. Suppose that every vertex of V_+^1 neighbors with a vertex in V_-^1, every vertex of V_-^1 neighbors with a vertex in V_+^1, and similarly for V_-^3 and V_+^3. Also suppose that a random walk started at 0 has probability at least $1/C$ to hit $V_-^3\cup V_+^3$ before exiting $\mathfrak{B}_{5R}$. Let $\mathcal{K}_1$ be the event that $h>0$ on V_+^1 and that $h<0$ on V_-^1, and let $\mathcal{K}_3$ be the corresponding event for V_-^3 and V_+^3. Let $a(V_+^1,V_-^1)$ be the probability of $\mathcal{K}_1$ for the DGFF on $\mathfrak{B}_{R_2}$ with zero boundary values outside $\mathfrak{B}_{R_2}$. Then there is a constant $c=c(\bar{\Lambda},C)>0$ such that

$$\frac{1}{c}a(V_+^1,V_-^1)\leqslant\mathbf{P}[\mathcal{K}_1\,|\,\mathcal{K}_3]\leqslant ca(V_+^1,V_-^1).$$

Proof. For $j=0,...,8$ set $r_j=R_1+\frac{1}{8}j(R_2-R_1)$. Then $r_8=R_2$ and $r_{j+1}\geqslant r_j+\frac{1}{8}C^{-1}R$. We set $W^j:=(V_D\cup V_\partial)\cap\mathfrak{B}_{r_j}=V_D\cap\mathfrak{B}_{r_j}$, $W_j:=(V_D\cup V_\partial)\setminus W^j$ and $W_j^k:=W_j\cap W^k$. Let $\tilde{h}$ denote the discrete-harmonic extension of the restriction of h to $W^1\cup W_7$. We may identify $\tilde{h}$ with a point in $\mathbb{R}^{W^1\cup W_7}$; namely, its restriction to $W^1\cup W_7$. As we have noted after (2.4), the probability density of $\tilde{h}$ with respect to the Lebesgue measure on $\mathbb{R}^{W^1 \cup W_7}$ is proportional to $\exp(-\|\nabla \tilde{h}\|^2/2)$. Hence,

$$\mathbf{P}[\mathcal{K}_1 \,|\, \mathcal{K}_3] = \frac{\int_{\mathcal{K}_1 \cap \mathcal{K}_3} \exp(-\|\nabla \tilde{h}\|^2/2)\, dx}{\int_{\mathcal{K}_3} \exp(-\|\nabla \tilde{h}\|^2/2)\, dx}, \tag{3.16}$$

where the integrals are with respect to the Lebesgue measure on $\mathbb{R}^{W^1 \cup W_7}$. Let h_1 be the function that agrees with h on W^1, is discrete-harmonic on W_1^7, and is zero in W_7. Let h_3 be the function that agrees with h on W_7, is discrete-harmonic on W_1^7, and is zero in W^1. Clearly, $\tilde{h} = h_1 + h_3$.

We claim that

$$\int_{\mathcal{K}_1 \cap \mathcal{K}_3} \exp\left(-\frac{\|\nabla \tilde{h}\|^2}{2}\right) dx \asymp \int_{\mathcal{K}_1 \cap \mathcal{K}_3} \exp\left(-\frac{\|\nabla h_1\|^2}{2}\right) \exp\left(-\frac{\|\nabla h_3\|^2}{2}\right) dx, \tag{3.17}$$

where $\asymp$ means equivalence up to multiplicative constants depending on $\bar{\Lambda}$ and C. Let $V_+ := V_+^1 \cup V_+^3$ and $V_- := V_-^1 \cup V_-^3$. Fix some $v \in W_2^6$. Let $\bar{h}$ denote the discrete-harmonic extension of the restriction of h to $V_+ \cup V_-$. Then $h(v) - \tilde{h}(v)$ and $\tilde{h}(v) - \bar{h}(v)$ are independent Gaussian random variables, and both are independent of $\bar{h}$ (by the orthogonality property noted in §2.1). By (2.5), the variance of $h(v) - \tilde{h}(v)$ is $\frac{1}{6}$ times the expected number of visits to v by a random walk started at v, which is stopped when it hits $W^1 \cup W_7$, and the variance of $h(v) - \bar{h}(v)$ is $\frac{1}{6}$ times the expected number of visits to v by the same random walk stopped when it hits $V_+ \cup V_-$. Consequently, the variance of $\tilde{h}(v) - \bar{h}(v)$ is $\frac{1}{6}$ times the expected number of visits to v after the first hit of $W^1 \cup W_7$ and before the first hit of $V_+ \cup V_-$. Note that the probability to hit v by a random walk started in $W^1 \cup W_7$ before exiting $\mathfrak{B}_{5R}$ is $O(1/\log R)$ and conditioned on hitting v before exiting $\mathfrak{B}_{5R}$ the number of visits to v prior to exiting $\mathfrak{B}_{5R}$ is $O(\log R)$. Our assumption on the probability to hit $V_-^3 \cup V_+^3$ therefore easily implies that $\mathbf{E}[(\tilde{h}(v) - \bar{h}(v))^2] = O_C(1)$ and hence $\mathbf{E}[|\tilde{h}(v) - \bar{h}(v)|] = O_C(1)$. Since $\mathcal{K}_1 \cap \mathcal{K}_3$ is determined by $\bar{h}$, it is independent of $\tilde{h}(v) - \bar{h}(v)$ and, consequently, $\mathbf{E}[|\tilde{h}(v) - \bar{h}(v)| \,|\, \mathcal{K}_1, \mathcal{K}_3] = O_C(1)$. Now, by (3.4), we have $\mathbf{E}[|\bar{h}(v)| \,|\, \mathcal{K}_1, \mathcal{K}_3] = O_{\bar{\Lambda}}(1)$. Combining these estimates, we get $\mathbf{E}[|\tilde{h}(v)| \,|\, \mathcal{K}_1, \mathcal{K}_3] = O_{C,\bar{\Lambda}}(1)$.

We will now apply the argument used to prove (3.4) in order to establish

$$\mathbf{E}[\max\{|\tilde{h}(v)| : v \in W_3^5\} \,|\, \mathcal{K}_1, \mathcal{K}_3] = O_{C,\bar{\Lambda}}(1). \tag{3.18}$$

Indeed, let A denote the set of vertices in W_2^6 neighboring with some vertex outside W_2^6, and let $H(v,u)$ denote the probability that a simple random walk started at v first hits A in u. As in the proof of (3.4), $H(v,u) \leqslant O_C(1) H(v',u)$ for $v, v' \in W_3^5$. Now (3.18) follows as in the proof of (3.4).

Next, we want to show that (3.18) holds with h_1 and h_3 replacing $\tilde{h}$; that is,

$$\mathbf{E}[M \mid \mathcal{K}_1, \mathcal{K}_3] = O_{C,\bar{\Lambda}}(1), \tag{3.19}$$

where $M := \max\{|h_j(v)| : v \in W_3^5, j = 1, 3\}$. Let v_1 be the vertex $v \in W_3^5$ where $|h_1(v)|$ is maximized, and let v_3 be the vertex $v \in W_3^5$ where $|h_3(v)|$ is maximized. Then $M = \max\{|h_1(v_1)|, |h_3(v_3)|\}$. Assume, for now, that $M = |h_3(v_3)|$. The maximum principle for discrete-harmonic functions implies that v_3 neighbors with a vertex outside $\mathfrak{B}_{r_5}$ and v_1 neighbors with a vertex in $\mathfrak{B}_{r_3}$. Let p be the probability that a simple random walk started at v_3 exits $\mathfrak{B}_{R_3}$ before hitting a vertex neighboring with a vertex in $\mathfrak{B}_{r_3}$. Then p is bounded away from 0 by a function of C. Since h_1 composed with a simple random walk is a martingale while the walk stays in W_1^7, we get $|h_1(v_3)| \leqslant (1-p)|h_1(v_1)| \leqslant (1-p)M$. As $\tilde{h} = h_1 + h_3$, we get $|\tilde{h}(v_3)| \geqslant |h_3(v_3)| - |h_1(v_3)| = M - |h_1(v_3)| \geqslant pM$. The case $M = |h_1(v_1)|$ is similarly treated. Using (3.18), we then get (3.19).

Next, we want to prove that

$$|\nabla h_1 \cdot \nabla h_3| = O(M^2). \tag{3.20}$$

Since h_1 is discrete-harmonic in W_1^7, if $v \in W_1^7$ we have

$$\sum_{u \sim v} (h_1(v) - h_1(u)) h_3(v) = 0,$$

where the sum is over the neighbors of v. This is also true for $v \in W^1$, since h_3 is zero there. Consequently,

$$\sum_{v \in W^4} \sum_{u \sim v} (h_1(v) - h_1(u)) h_3(v) = 0. \tag{3.21}$$

Similarly, we find that

$$\sum_{u \in W_4} \sum_{v \sim u} (h_3(u) - h_3(v)) h_1(u) = 0. \tag{3.22}$$

Set $\partial W^4 := \{(v, u) \in W^4 \times W_4 : u \sim v\}$. By considering the contribution of each edge $[v, u]$ to $\nabla h_1 \cdot \nabla h_3$, we compare the sum of the left-hand sides of (3.21) and (3.22) to $\nabla h_1 \cdot \nabla h_3$ and conclude that

$$\begin{aligned} \nabla h_1 \cdot \nabla h_3 &= \sum_{(v,u) \in \partial W^4} (h_3(v) h_1(u) - h_1(v) h_3(u)) \\ &= \sum_{(v,u) \in \partial W^4} ((h_1(u) - h_1(v)) h_3(u) + (h_3(v) - h_3(u)) h_1(u)). \end{aligned} \tag{3.23}$$

The number of summands is clearly $O(R)$. Note that for every $v \in W^4$ neighboring with a vertex $u \in W_4$, there is a disk of radius proportional to R/C such that all the vertices in that disk are in W_3^5. Consequently, the discrete Harnack principle (Lemma 2.2) gives $|h_1(u)-h_1(v)|=O(M/R)$ and $|h_3(u)-h_3(v)|=O(M/R)$. Hence, (3.23) gives (3.20).

Now, (3.19) implies that the expectation of $|\nabla h_1\cdot\nabla h_3|^{1/2}$ conditioned on $\mathcal{K}_1\cap\mathcal{K}_3$ is bounded by a function of C and $\bar\Lambda$. In particular, there is a constant $c_1=c_1(\bar\Lambda,C)>0$ such that

$$\mathbf{P}[|\nabla h_1\cdot\nabla h_3|<c_1\,|\,\mathcal{K}_1,\mathcal{K}_3]>\frac{1}{c_1}.$$

In terms of Lebesgue measure, this may be written as

$$\int_{\mathcal{K}_1\cap\mathcal{K}_3}\exp\biggl(-\frac{\|\nabla\tilde h\|^2}{2}\biggr)\,dx<c_1\int_{\mathcal{K}_1\cap\mathcal{K}_3}\exp\biggl(-\frac{\|\nabla\tilde h\|^2}{2}\biggr)1_{\{|\nabla h_1\cdot\nabla h_3|<c_1\}}\,dx.$$

Since $\tilde h=h_1+h_3$, this implies that

$$\int_{\mathcal{K}_1\cap\mathcal{K}_3}\exp\biggl(-\frac{\|\nabla\tilde h\|^2}{2}\biggr)\,dx<c_1e^{c_1}\int_{\mathcal{K}_1\cap\mathcal{K}_3}\exp\biggl(-\frac{\|\nabla h_1\|^2+\|\nabla h_3\|^2}{2}\biggr)\,dx,$$

which gives one side of (3.17).

The other direction is proved in essentially the same way. Under the probability measure weighted by $\exp\bigl(-\frac{1}{2}(\|\nabla h_1\|^2+\|\nabla h_3\|^2)\bigr)$ (with respect to the Lebesgue measure on $\mathbb{R}^{W^1\cup W_7}$), h_1 restricted to W^1 has the law of the DGFF with zero boundary values on W_7 restricted to W^1. Similarly, with this weighting, h_3 restricted to W_7 has the law of the DGFF with zero boundary values on W^1 and with boundary values given by h_∂ on ∂D, restricted to W_7. Moreover, under this measure, h_1 and h_3 are clearly independent. The above arguments show that under this measure (3.19) holds (where $\mathcal{K}_1$ refers to h_1 while $\mathcal{K}_3$ refers to h_3). Since (3.20) is still valid, the opposite inequality in (3.17) is then easily established.

We may also apply (3.17) in the case where $V_+^1=V_-^1=\varnothing$, and hence $\mathcal{K}_1$ has full measure. Since

$$\begin{aligned}\int_{\mathcal{K}_1\cap\mathcal{K}_3}&\exp\biggl(-\frac{\|\nabla h_1\|^2}{2}\biggr)\exp\biggl(-\frac{\|\nabla h_3\|^2}{2}\biggr)\,dx\\&=\int_{\mathbb{R}^{W^1}}1_{\mathcal{K}_1}\exp\biggl(-\frac{\|\nabla h_1\|^2}{2}\biggr)\,dx\int_{\mathbb{R}^{W_7}}1_{\mathcal{K}_3}\exp\biggl(-\frac{\|\nabla h_3\|^2}{2}\biggr)\,dx,\end{aligned}$$

from (3.16) we get

$$\mathbf{P}[\mathcal{K}_1\,|\,\mathcal{K}_3]\asymp\frac{\int_{\mathcal{K}_1}\exp(-\|\nabla h_1\|^2/2)\,dx}{\int_{\mathbb{R}^{W_1}}\exp(-\|\nabla h_1\|^2/2)\,dx},$$

which completes the proof. □

3.3. Narrows and obstacles

The present subsection and the next will use the infrastructure developed in §3.1 to prove some bounds on the probabilities that contour lines cross certain regions in specified ways. This is roughly in the spirit of the Russo–Seymour–Welsh theorem for percolation, though the proofs are entirely different.

The following lemma is an estimate for having a crossing by TG^*-hexagons where h is negative between two arcs on the boundary of some subset of the domain, conditioned on some zero-height interface paths. The statement below is slightly complicated, because we need to keep the geometric assumptions quite general. In percolation, boundary values do not play a role, of course. But in our case we need the crossing estimate in the case where one boundary arc of the domain is conditioned to be an interface.

LEMMA 3.8. (Narrows) *For every $\varepsilon>0$ there is a $\delta=\delta(\bar{\Lambda},\varepsilon)>0$ such that the following crossing estimate holds. Assume* (h) *and* (D). *Let $\mathcal{K}$ be the event that a fixed collection $\{\gamma_1,...,\gamma_k\}$ of oriented paths in* TG^* *are contained in oriented zero-height interfaces of h, and suppose that $\mathbf{P}[\mathcal{K}]>0$. Let $\alpha\subset D\setminus(\gamma_1\cup...\cup\gamma_k)$ be a simple path that has both its endpoints on the right-hand side* (*positive side*) *of γ_1. Let A be the domain bounded by α and a subarc of γ_1, and assume that A does not meet the left side of γ_1 and that $\bar{A}\cap(\gamma_2\cup\gamma_3\cup...\cup\gamma_k\cup\partial D)=\varnothing$. Let α_1, α_2 and α' be three disjoint subarcs of α, where α_1 contains one endpoint of α, and α_2 contains the other endpoint of α.* (*See Figure* 3.1.) *Suppose that each point in α' is contained in a* TG^**-hexagon whose center is outside A. Set $d_1:=\sup_{z\in\alpha'}\operatorname{dist}(z,\gamma_1)$. Let $\mathcal{C}$ be the event that there is a path crossing from α_1 to α_2 in A inside hexagons where h is negative. Let d^* be the infimum diameter of any path connecting α' to $\gamma_1\cup...\gamma_k\cup\partial D$ which does not contain a subpath connecting $\alpha\setminus(\alpha_1\cup\alpha_2)$ to γ_1 in A. If*

$$d_1+1\leqslant\delta\min\{d^*,\operatorname{dist}(\alpha_1,\alpha'),\operatorname{dist}(\alpha_2,\alpha'),\operatorname{diam}\alpha'\},\tag{3.24}$$

then

$$\mathbf{P}[\mathcal{C}\mid\mathcal{K}]<\varepsilon.$$

The idea of the proof is to observe the effect that such a crossing would have on certain averages of heights of vertices, and thereby conclude that it is unlikely. The challenge in the implementation of this strategy is to condition on a crossing in such a way that the expected heights are easy to estimate.

Proof. Set $N:=\lfloor\frac{1}{2}\operatorname{diam}\alpha'\rfloor$. Assume that (3.24) holds. Note that

$$\operatorname{diam}\gamma_1\geqslant\operatorname{diam}\alpha'-2d_1.$$

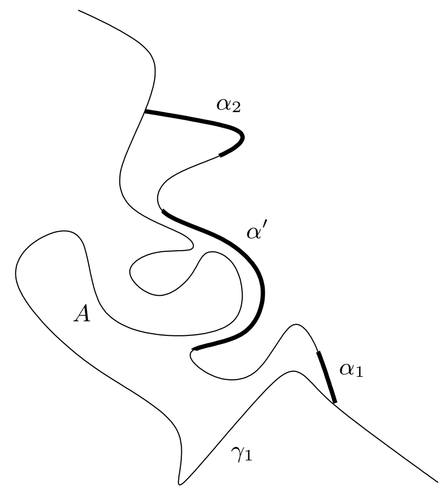

Figure 3.1. Setup in the narrows lemma.

Choose some point $z_0 \in \alpha'$. For $j=1, ..., N$ let z_j be a point in α' at distance j from z_0. Now for each z_j we let s_j be some center of a hexagon that contains z_j satisfying $s_j \notin A$. Then $|s_j - s_{j'}| \geqslant |j-j'| - O(1)$. Let U be the union of V_∂ and the set of vertices adjacent to any one of the paths $\gamma_1, ..., \gamma_k$. (These are precisely the vertices v where h takes boundary values, or the sign of $h(v)$ is determined by $\mathcal{K}$.) Set

$$X := \frac{1}{N} \sum_{j=1}^{N} h(s_j),$$

$b := \mathbf{E}[X \,|\, \mathcal{K}]$ and fix some $\varepsilon' > 0$. We first claim that

$$\mathbf{E}[(X-b)^2 \,|\, \mathcal{K}] < \varepsilon' \tag{3.25}$$

if $\delta = \delta(\varepsilon', \bar{\Lambda}) > 0$ is sufficiently small. Let h_U denote the discrete-harmonic extension of the restriction of h to U and set $X_U := N^{-1} \sum_{j=1}^{N} h_U(s_j)$. Note that $\mathbf{E}[X - X_U \,|\, \mathcal{K}, h_U] = 0$, and hence $\mathbf{E}[X_U \,|\, \mathcal{K}] = b$. For each $u \in U$ and $j \in \{1, ..., N\}$ let $p(j,u)$ denote the probability that a simple random walk started from s_j first hits U at u. Also set $p(u) := N^{-1} \sum_{j=1}^{N} p(j,u)$. Then $X_U = \sum_{u \in U} p(u) h(u)$. Consequently,

$$\mathbf{E}[(X_U - b)^2 \,|\, \mathcal{K}] = \sum_{u,u' \in U} p(u)p(u')(\mathbf{E}[h(u)h(u') \,|\, \mathcal{K}] - \mathbf{E}[h(u) \,|\, \mathcal{K}]\mathbf{E}[h(u') \,|\, \mathcal{K}]).$$

Let $Z(u,u')$ denote the term in parentheses corresponding to the summand involving u and u'. Then $\mathbf{E}[(X_U - b)^2 | \mathcal{K}]$ is just the average of the conditioned covariances $Z(u,u')$

weighted by $p(u)p(u')$. We know from (3.1) that $\mathbf{E}[h(u)^2\,|\,\mathcal{K}]$ is bounded by a constant depending only on $\bar{\Lambda}$. It then follows by the Cauchy–Schwarz inequality that the same is true for $Z(u,u')$. Consequently, to prove that $\mathbf{E}[(X_U-b)^2\,|\,\mathcal{K}]$ is small, it suffices to show that when (u,u') is chosen with probability $p(u)p(u')$ it is very likely that $|Z(u,u')|$ is small. Suppose that we select j from $\{1,...,N\}$ uniformly at random and given j select $u\in U$ with probability $p(j,u)$. Independently, we also select (j',u') with the same distribution. It suffices to show that $|Z(u,u')|$ is likely to be small, and by Corollary 3.5 it suffices to show that the distance between u and u' is likely to be large. Since $|s_j-s_{j'}|=|j-j'|+O(1)$ is unlikely to be much smaller than $\operatorname{diam}\alpha'$, which is larger than $\delta^{-1}(d_1+1)$, it follows from Lemma 2.1 (hit near) that for any fixed R the probability that $|u-u'|<R$ tends to zero as $\delta\to 0$. Consequently,

$$\mathbf{E}[(X_U-b)^2\,|\,\mathcal{K}]<\tfrac{1}{2}\varepsilon',$$

provided that δ is sufficiently small.

Set $X_j:=h(s_j)-h_U(s_j)$. Recall from §2.1 that, given the restriction of h to U, the function $h-h_U$ is the DGFF on $V_D\setminus U$ with zero boundary values on U. Therefore, by (2.5), $\mathbf{E}[X_iX_j\,|\,\mathcal{K},h_U]=\frac{1}{6}G(s_i,s_j)$, where $G(v,u)$ is the expected number of visits to u by a random walker started at v and stopped when it hits U. From Lemmas 2.1 and 2.3,

$$G(s_j,s_{j'})\leqslant\begin{cases} O(1)\log(d_1/(|s_j-s_{j'}|\vee 1)), & \text{if } |s_j-s_{j'}|<\frac{1}{2}d_1,\\ O(1)(d_1/(|s_j-s_{j'}|\vee 1))^{\zeta_1}, & \text{if } |s_j-s_{j'}|\geqslant\frac{1}{2}d_1.\end{cases}$$

Since $|s_j-s_{j'}|\geqslant|j-j'|-O(1)$ and $\zeta_1\in(0,1)$, these estimates give

$$\mathbf{E}[(X-X_U)^2\,|\,\mathcal{K}]=\frac{1}{N^2}\sum_{j,j'=1}^{N}\frac{G(s_j,s_{j'})}{6}=O(1)\left(\frac{d_1}{N}\right)^{\zeta_1}.$$

Now (3.25) follows for sufficiently small $\delta=\delta(\varepsilon',\bar{\Lambda})>0$, since $X-X_U$ is independent of X_U and they are also independent given $\mathcal{K}$.

We now claim that $b\geqslant 0$ if δ is sufficiently small. Let c be the constant given by Lemma 3.1. If u is a fixed vertex adjacent to γ_1 on the right, then $\mathbf{E}[h(u)\,|\,\mathcal{K}]>1/c$, by (3.2). On the other hand, $\mathbf{E}[|h(u)|\,|\,\mathcal{K}]<c$ for every $u\in U$. By (3.24) and Lemma 2.1, it follows that when δ is small with high probability a random walk starting at any s_j is likely to first hit U at a vertex adjacent to the right-hand side of γ_1. Thus, when δ is small, we have $b=\mathbf{E}[X_U\,|\,\mathcal{K}]>0$. Also, clearly, $b\leqslant c$.

Now set $a=b+1$. Let Q denote the union of the closed hexagons in TG^* for which $h(H)\in[0,a]$ and let Q' denote the union of the edges in TG^* that are on the common boundary of two hexagons H_1 and H_2 satisfying $h(H_1)<0$ and $a<h(H_2)$. Let Q_0 denote

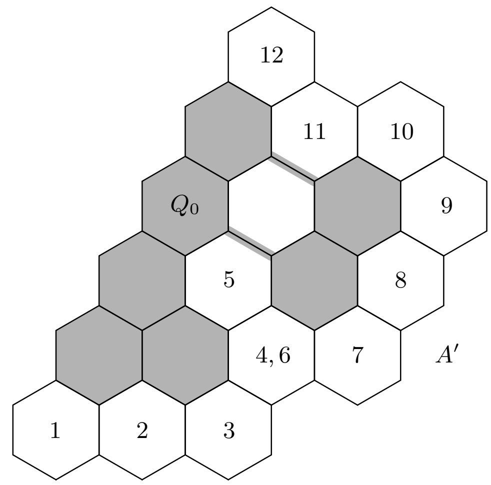

Figure 3.2. A portion of the sequence of hexagons adjacent to $\partial A' \setminus \partial A$.

the connected component of $(Q \cup Q') \cap \bar{A}$ that contains $\gamma_1 \cap \partial A$. Let $\mathcal{Q}$ denote the event $Q_0 \cap (\alpha \setminus (\alpha_1 \cup \alpha_2)) = \varnothing$. If $\mathcal{C}$ holds, then the corresponding crossing by hexagons where h is negative separates $\gamma_1 \cap \partial A$ from $\alpha \setminus (\alpha_1 \cup \alpha_2)$ in A, and hence $\mathcal{Q}$ holds as well. Thus, $\mathcal{C} \subset \mathcal{Q}$.

On the event $\mathcal{Q}$, let A' be the connected component of $\bar{A} \setminus Q_0$ that contains α', and let U_Q denote the set of centers of hexagons H such that $\overline{H \cap A'} \cap Q_0 \neq \varnothing$. Clearly, $h(v) < 0$ or $h(v) > a$ for each $v \in U_Q$. Since A' and Q_0 are connected, it is immediate to verify (using the Jordan planar curve theorem) that A' is simply connected and $\partial A' \setminus \partial A \subset Q_0$ is connected. The closed hexagons of TG^* with centers in U_Q form a sequence (possibly with repetitions) with each pair H, H' of consecutive hexagons along the sequence satisfying $H \cap H' \setminus Q_0 \neq \varnothing$ and $H \cap H' \cap Q_0 \neq \varnothing$. (See Figure 3.2.) If $v, u \in U_Q$ are centers of consecutive hexagons in this sequence, then it is impossible that $h(v) < 0$ and $h(u) > a$ (otherwise, the boundary between the hexagons would be in Q'). Thus, either $h(U_Q) \subset (a, \infty)$, or $h(U_Q) \subset (-\infty, 0)$. Let $\mathcal{Q}_+$ be the event that $\mathcal{Q}$ occurs and $\min h(U_Q) > a$, and let $\mathcal{Q}_-$ be the event that $\mathcal{Q}$ occurs and $\max h(U_Q) < 0$.

We now want to estimate $\mathbf{E}[X \mid \mathcal{K}, \mathcal{Q}_-]$ and $\mathbf{E}[X \mid \mathcal{K}, \mathcal{Q}_+]$. Let U' be the set of vertices that are either in hexagons adjacent to Q_0 or in U. Since $a = O_{\bar{\Lambda}}(1)$, it is clear that the proof of (3.2) gives $\mathbf{E}[|h(u)| \mid \mathcal{K}, U', \mathcal{Q}_\pm] = O_{\bar{\Lambda}}(1)$ for $u \in U'$. On the other hand, Lemma 2.1 shows that at least $1 - O(\delta^{\zeta_1})$ of the discrete-harmonic measure on U' starting from every s_j is in U_Q. If $\mathcal{Q}_-$ holds and $u \in U_Q$, then $\mathbf{E}[h(u) \mid \mathcal{K}, \mathcal{Q}_-, U_Q]$ is negative and bounded away from zero, by the corresponding analog of the left-hand side of (3.2). Thus, we find that $\mathbf{E}[X \mid \mathcal{K}, \mathcal{Q}_-]$ is negative and bounded away from zero when $\delta = \delta(\varepsilon', \bar{\Lambda}) > 0$ is small.

Since $\mathbf{E}[(X-b)^2|\mathcal{K}]<\varepsilon'$ and $b\geqslant 0$, we conclude that by choosing $\varepsilon'>0$ small it can be guaranteed that $\mathbf{P}[\mathcal{Q}_-|\mathcal{K}]<\frac{1}{2}\varepsilon$.

On $\mathcal{Q}_+$, we clearly have $h(u)>a\geqslant b+1$ on every $u\in U_Q$. Thus, as above, it follows that when δ is small $\mathbf{E}[X|\mathcal{K},\mathcal{Q}_+]>b+\frac{1}{2}$. This again implies that $\mathbf{P}[\mathcal{Q}_+|\mathcal{K}]$ can be made smaller than $\frac{1}{2}\varepsilon$. Since $\mathcal{C}\subset\mathcal{Q}_+\cup\mathcal{Q}_-$, this completes the proof. □

Next, we formulate an analogous lemma for crossings near the boundary of the domain.

LEMMA 3.9. (Domain boundary narrows) *There is a constant* $\Lambda_0=\Lambda_0(\bar\Lambda)>0$ *such that for every* $\varepsilon>0$ *there is a* $\delta=\delta(\bar\Lambda,\varepsilon)>0$ *such that the following crossing estimate holds. Assume* (h) *and* (D), *assume that* ∂_+ *is a simple path contained in* ∂D *and that* $h_\partial\geqslant-\Lambda_0$ *on* $\partial_+\cap V_\partial$. *Set* $\partial_-:=\partial D\setminus\partial_+$. *Let* $\mathcal{K}$ *be the event that a fixed collection* $\{\gamma_1,...,\gamma_k\}$ *of oriented paths in* TG* *are contained in oriented zero-height interfaces of* h, *and suppose that* $\mathbf{P}[\mathcal{K}]>0$. *Let* $\alpha\subset D\setminus(\gamma_1\cup...\cup\gamma_k)$ *be a simple path that has both its endpoints on* ∂_+. *Let* A *be the domain bounded by* α *and a subarc of* ∂_+, *and assume that* $\bar A\cap(\gamma_1\cup\gamma_2\cup...\cup\gamma_k\cup\partial_-)=\varnothing$. *Let* $\alpha'\subset\alpha$ *be a subarc. Suppose that each point in* α' *is contained in a hexagon whose center is outside* A. *Set* $d_1:=\sup_{z\in\alpha'}\operatorname{dist}(z,\partial_+)$. *Let* α_1 *be a subarc of* α *that contains one of the endpoints of* α, *and let* α_2 *be a subarc of* α *that contains the other endpoint of* α. *Let* $\mathcal{C}$ *be the event that there is a path crossing from* α_1 *to* α_2 *in* A *inside hexagons where* h *is negative. Let* d^* *be the infimum diameter of any path connecting* α' *to* $\gamma_1\cup...\gamma_k\cup\partial D$ *which does not contain a subpath connecting* $\alpha\setminus(\alpha_1\cup\alpha_2)$ *to* ∂_+ *in* A. *If*

$$d_1+1\leqslant\delta\min\bigl\{d^*,\operatorname{dist}(\alpha_1,\alpha'),\operatorname{dist}(\alpha_2,\alpha'),\operatorname{diam}\alpha'\bigr\},\tag{3.26}$$

then

$$\mathbf{P}[\mathcal{C}\,|\,\mathcal{K}]<\varepsilon.$$

Proof. The proof is slightly simpler but essentially the same as that of Lemma 3.8 (narrows). We use the same notation as in that lemma, and only indicate the few differences in the proof. In the present setting $b=\mathbf{E}[X|\mathcal{K}]$ can be made larger than $-2\Lambda_0$ by taking $\delta>0$ small. Here, we define Q_0 as the connected component of $(Q\cup Q'\cup\partial_+)\cap\bar A$ that contains $\partial_+\cap\partial A$. Observe that $U_Q\cap\partial D=\varnothing$ on $\mathcal{Q}_-\cap\mathcal{K}$. It follows that $\mathbf{E}[X|\mathcal{K},\mathcal{Q}_-]$ is negative and bounded away from zero (by a function of $\bar\Lambda$) when $\delta>0$ is small. By taking $\Lambda_0>0$ sufficiently small, we can make sure that $\mathbf{E}[X|\mathcal{K},\mathcal{Q}_-]<-3\Lambda_0$. But since $\mathbf{E}[(X-b)^2|\mathcal{K}]$ is arbitrarily small and $b\geqslant-2\Lambda_0$, this makes $\mathbf{P}[\mathcal{Q}_-|\mathcal{K}]$ small. The rest of the argument is essentially the same. □

The previous lemmas will help us control the behavior of the continuation of contours near existing contours or the boundary of the domain. The next lemma will help us control the behavior in the interior away from existing contours.

Lemma 3.10. (Obstacle) *For every $\varepsilon>0$ there is some constant $c=c(\varepsilon,\bar\Lambda)>0$ such that the following estimate holds. Assume* (h) *and* (D). *Let $\mathcal{K}$ be the event that a fixed collection $\{\gamma_1,...,\gamma_k\}$ of oriented paths in* TG^* *are contained in oriented zero-height interfaces of h, and suppose that $\mathbf{P}[\mathcal{K}]>0$. Let U be the union of V_∂ and the vertices of* TG *adjacent to $\bar\gamma:=\gamma_1\cup...\cup\gamma_k$. Let g be a function defined on the vertices of* TG *that is* 0 *on U. Let $\widehat\gamma_g$ denote the union of the interfaces of $h+g$ that contain any one of the paths $\gamma_1,...,\gamma_k$. Let $B(z_0,r)$ be a disk of radius r that is centered at some vertex z_0 satisfying $|g(z_0)|\geqslant\frac{1}{2}\|g\|_\infty$. Let $d>0$ and suppose that at distance at most $\varepsilon^{-1}d$ from z_0 there is a connected component of $\bar\gamma\cup\partial D$ whose diameter is at least εd. Also assume that $\|g\|_\infty/\|\nabla g\|_\infty>cr>c$. Then*

$$\mathbf{P}[\widehat\gamma_g\cap B(z_0,r)\neq\varnothing\mid\mathcal{K}]\leqslant c\|g\|_\infty^{-2}\log\frac{d}{r}.$$

Proof. With no loss of generality, we assume that $g(z_0)>0$. Set $q:=\|g\|_\infty/\|\nabla g\|_\infty$ and $r_1:=\frac{1}{10}q$. Since between any two vertices z and z' in TG there is a path in TG whose length is at most $2|z-z'|$,

$$\min\{g(z):z\in B(z_0,r_1)\}\geqslant g(z_0)-2r_1\|\nabla g\|_\infty=g(z_0)-\tfrac{1}{5}\|g\|_\infty\geqslant\tfrac{1}{4}\|g\|_\infty.$$

As $g=0$ on U, it follows that $\varepsilon^{-1}d\geqslant r_1$. Since we are assuming that $q>cr$, and we may assume that c is a large constant which may depend on ε, it follows that $d/r>100$, say. Thus, we also assume, with no loss of generality, that $\|g\|_\infty\geqslant\sqrt{c}$, since the required inequality is trivial otherwise.

Let X denote the average value of h on the vertices in $B(z_0,r)$. The inequality (3.5) and $d/r>100$ give

$$\mathbf{E}[X^2\mid\mathcal{K}]\leqslant O_{\varepsilon,\bar\Lambda}(1)\log\frac{d}{r}. \tag{3.27}$$

If γ_1 is not a closed path, we start exploring the interface of $h+g$ containing γ_1 starting from one of the endpoints of γ_1 until that interface is completed or $B(z_0,r)$ is hit, whichever occurs first. (This may entail going through several of the interfaces γ_j, $j>1$.) If that interface is completed before we hit $B(z_0,r)$, we continue and explore the interface of $h+g$ containing γ_2, and so forth, until finally either all of $\widehat\gamma_g$ is explored or $B(z_0,r)$ is hit. Let $\mathcal{Q}$ denote the event that $B(z_0,r)$ is hit, and let β be the interfaces explored up to the time when the exploration terminates.

Let U' be the union of U with the vertices adjacent to β. Since we are assuming that $q>cr$, $r>1$ and $\|g\|_\infty\geqslant\sqrt{c}$, and since c may be chosen arbitrarily large, Lemma 3.2 shows that for every vertex $v\in U'$, we have that

$$\mathbf{E}[|h(v)+g(v)|\mid\mathcal{K},\mathcal{Q},\beta]\leqslant\frac{\|g\|_\infty}{100}. \tag{3.28}$$

(Note that conditioning on $\mathcal{Q}$ and β amounts to conditioning that $h+g>0$ on vertices adjacent to the right-hand side of β and $h+g<0$ on vertices adjacent to the left-hand side. Consequently, the lemma applies.)

Now let x be any vertex in $B(z_0, r)$. For each $u\in U'$, let p_u denote the probability that a simple random walk started at x will first hit U' in u; that is, the discrete-harmonic measure from x. Then (3.28) gives

$$\mathbf{E}[h(x) \mid \mathcal{K}, \mathcal{Q}, \beta] \leqslant \frac{\|g\|_\infty}{100} - \sum_{u\in U'} p_u g(u). \tag{3.29}$$

Since $\beta\cap B(z_0, r)\neq\varnothing$ and β intersects the complement of $B(z_0, r_1)$, Lemma 2.1 gives

$$\sum_{u\in U'\setminus B(z_0,r_1)} p_u \leqslant O(1)\Big(\frac{r}{r_1}\Big)^{\zeta_1}.$$

Since we are assuming that $q\geqslant cr$, we may assume that the right-hand side is less than $\frac{1}{10}$. Recall that $g\geqslant\frac{1}{4}\|g\|_\infty$ inside $B(z_0, r_1)$. Outside $B(z_0, r_1)$, the trivial estimate $g\geqslant-\|g\|_\infty$ applies. When these estimates are applied to (3.29), one gets

$$\begin{aligned}\mathbf{E}[h(x) \mid \mathcal{K}, \mathcal{Q}, \beta] - \frac{\|g\|_\infty}{100} &\leqslant -\sum_{u\in U'\cap B(z_0,r_1)} p_u g(u) - \sum_{u\in U'\setminus B(z_0,r_1)} p_u g(u) \\ &\leqslant -\Big(\frac{\|g\|_\infty}{4}\Big)\Big(1-\frac{1}{10}\Big) + \frac{\|g\|_\infty}{10} = -\frac{\|g\|_\infty}{8}.\end{aligned}$$

We may take expectation with respect to β and average with respect to x to conclude that $\mathbf{E}[X \mid \mathcal{K}, \mathcal{Q}]\leqslant-\frac{1}{9}\|g\|_\infty$, which implies, by Jensen's inequality, that

$$\mathbf{E}[X^2 \mid \mathcal{K}, \mathcal{Q}] \geqslant \frac{\|g\|_\infty^2}{81}.$$

Since

$$\mathbf{P}[\mathcal{Q} \mid \mathcal{K}] \leqslant \frac{\mathbf{E}[X^2 \mid \mathcal{K}]}{\mathbf{E}[X^2 \mid \mathcal{Q}, \mathcal{K}]},$$

the lemma now follows from the above and (3.27). □

3.4. Barriers

In this subsection we apply Lemmas 3.8, 3.10 and 3.6 to get a flexible (though slightly complicated) criterion giving lower bounds for the probability that contours avoid certain sets. The complications arise from the need to handle pre-existing contours that are highly non-smooth on large scales.

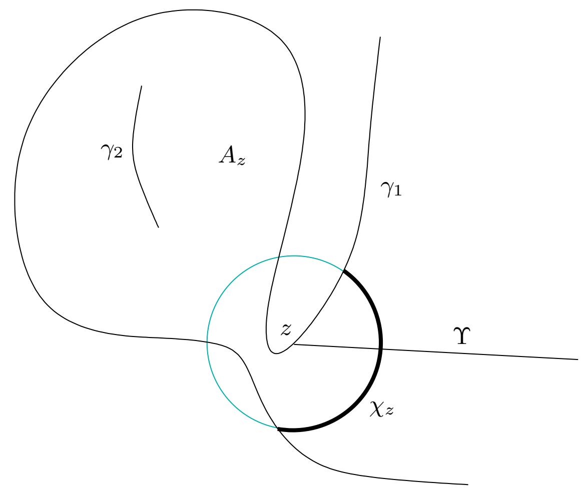

Figure 3.3. The domain A_z in a situation where condition (3) fails. (The figure does not show detail on the scale of the lattice; that is, $D(\bar\gamma)$ appears as $D\setminus\bar\gamma$.)

The following relative notion of distance will sometimes be used below:

$$\operatorname{dist}(A,B;X):=\inf\{\operatorname{diam}\alpha:\alpha \text{ is a path in } \overline{X} \text{ connecting } A \text{ and } B\}. \qquad (3.30)$$

If $\bar\gamma$ is a collection of paths in $\overline{D}$, let $D(\bar\gamma)$ denote the complement in D of the union of the closed triangles of TG meeting $\bar\gamma$.

We now define the notion of barrier. Assume (h). Let $\gamma_1,\gamma_2,...,\gamma_k$ be disjoint simple paths or simple closed paths in $\overline{D}$. Set $\bar\gamma:=\gamma_1\cup...\cup\gamma_k$. Let $\Upsilon\subset\overline{D(\bar\gamma)}$ be a path, which is contained in $D(\bar\gamma)$, except possibly for its two endpoints. Fix some $R>0$ and $\varepsilon>0$. We call Υ an (ε,R)-*barrier* for the configuration $(D,\bar\gamma)$ if the following conditions hold:

(1) $\varepsilon R<\operatorname{diam}\Upsilon\leqslant R$;

(2) within distance $\varepsilon^{-1}R$ from Υ there is a connected component of $\bar\gamma\cup\partial D$ whose diameter is at least εR;

(3) if $z\in\partial D(\bar\gamma)$ is an endpoint of Υ and χ_z is the connected component of $\partial B(z,\varepsilon R)\cap D(\bar\gamma)$ first encountered when traversing Υ from z (which exists by (1)), then the connected component A_z of $D(\bar\gamma)\setminus\chi_z$ that contains points of Υ arbitrarily close to z satisfies (a) ∂A_z consists of χ_z and a simple path contained in $\partial D(\bar\gamma)$ and (b) $\Upsilon\cap A_z\cap\partial B(z,r)$ consists of a single point for every $r\in(0,\varepsilon R)$;

(4) for every point $w\in\Upsilon$ such that $\operatorname{dist}(w,\partial D(\bar\gamma))\leqslant\frac{1}{5}\varepsilon R$ there is an endpoint $z\in\Upsilon\cap\partial D(\bar\gamma)$ such that $w\in A_z$ and $|w-z|<\frac{1}{2}\varepsilon R$ (roughly, Υ does not get close to $\partial D(\bar\gamma)$ except near its endpoints on $\partial D(\bar\gamma)$).

One example where condition (3) fails is given in Figure 3.3. If we remove the strand of γ_2 from that figure, we get an example that illustrates that $A_z\subset\overline{B(z,\varepsilon R)}$ does not follow from the above conditions. Note that it may happen that $\partial A_z \setminus \chi_z$, which is a simple path, by (3), consists of an arc in ∂D together with one or two arcs in $\bar{\gamma}$ that have endpoints in ∂D.

We now use barriers to "manipulate" contours of the DGFF.

THEOREM 3.11. (Barriers) *For every* $\varepsilon > 0$, $m \in \mathbb{N}_+$ *and* $\bar{\Lambda} > 0$ *there is a*

$$p = p(\varepsilon, \bar{\Lambda}, m) > 0$$

such that the following estimate holds. Assume (h) *and* (D). *Let* $\mathcal{K}$ *be the event that a fixed collection* $\{\gamma_1, ..., \gamma_k\}$ *of oriented paths in* TG^* *is contained in the oriented zero-height interfaces of* h, *and suppose that* $\mathbf{P}[\mathcal{K}] > 0$. *Set*

$$\bar{\gamma} := \bigcup_{j=1}^{k} \gamma_j.$$

Let V_+ (*respectively*, V_-) *be the set of vertices adjacent to* $\bar{\gamma}$ *on the right-hand* (*respectively*, *left-hand*) *side. Let* $R > 0$ *and let* $Y = Y_+ \cup Y_-$ *be a collection of* m (ε, R)*-barriers for the configuration* $(D, \bar{\gamma})$. *Assume that the endpoints of these barriers are not on* ∂D *and that for every* $\Upsilon \in Y_+$ (*respectively*, $\Upsilon \in Y_-$) *and every endpoint* $z \in \Upsilon \cap \partial D(\bar{\gamma})$, *the vertices in* $\overline{A_z(\Upsilon)} \cap \partial D(\bar{\gamma})$ *are in* V_+ (*respectively*, V_-), *where* $A_z(\Upsilon)$ *is as in condition* (3). *Also assume that* $\operatorname{dist}(\bigcup Y_+, \bigcup Y_-; D(\bar{\gamma})) \geqslant 2\varepsilon R$. *In the situation where* $\varepsilon R = O(1)$, *we also need to assume that there is no hexagon in* TG^* *meeting both* $V_+ \cup (\bigcup Y_+)$ *and* $V_- \cup (\bigcup Y_-)$ *and there is no hexagon meeting* $\bigcup Y$ *and* ∂D. *Let* $\widehat{\gamma}$ *denote the union of the zero-height interfaces of* h *which contain any one of the arcs* $\gamma_1, ..., \gamma_k$. *Then*

$$\mathbf{P}[(\widehat{\gamma} \setminus \bar{\gamma}) \cap (\bigcup Y) = \varnothing \mid \mathcal{K}] > p.$$

The basic idea of the proof is as follows. We define a function g that is large (positive) near $\bigcup Y_+$ away from $\partial D(\bar{\gamma})$ and is negative and large in absolute value near $\bigcup Y_-$ away from $\partial D(\bar{\gamma})$. The obstacle lemma (Lemma 3.10) will then imply that $\widehat{\gamma}_g$, as defined there, is unlikely to hit Υ, except near endpoints of barriers. The narrows lemma (Lemma 3.8) will be used to show that $\widehat{\gamma}_g$ is also unlikely to hit $\bigcup Y$ near endpoints. Finally, the distortion lemma (Lemma 3.6) will be used to conclude that with probability bounded away from zero, $\widehat{\gamma}$ will not hit Υ.

Proof. Let

$$\begin{aligned} \hat{A}_+^1 &:= \bigcup \{A_z(\Upsilon) \cap B(z, \varepsilon R) : \Upsilon \in Y_+ \text{ and } z \in \Upsilon \cap \bar{\gamma}\}, \\ \hat{A}_+^2 &:= \big\{z \in D(\bar{\gamma}) : \operatorname{dist}(z, \textstyle\bigcup Y_+; D(\bar{\gamma})) \leqslant \tfrac{1}{10}\varepsilon R\big\}, \end{aligned}$$

and $\hat{A}_+ := \hat{A}_+^1 \cup \hat{A}_+^2$. Similarly, define $\hat{A}_-$, with Y_- replacing Y_+. We fix constants $c_0 > 0$, large, and $\delta > 0$ much smaller than ε, and set

$$g(z) := \begin{cases} c_0 \delta^{-1} R^{-1} \min\{\delta R, \operatorname{dist}(z, \mathbb{R}^2 \setminus \hat{A}_+)\}, & \text{if } z \in \hat{A}_+, \\ -c_0 \delta^{-1} R^{-1} \min\{\delta R, \operatorname{dist}(z, \mathbb{R}^2 \setminus \hat{A}_-)\}, & \text{if } z \in \hat{A}_-, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\hat{A}_+ \cap \hat{A}_- = \varnothing$ and $\|g\|_\infty = c_0$.

Let $Z := \{z \in \mathbb{R}^2 : |g(z)| = c_0\}$. Let $c_1 > 1$ be some large constant and set $r := \delta R / c_1$. Let W be a maximal collection of vertices in $\{v : |g(v)| \geqslant \frac{1}{2} c_0\}$ such that the distance between any two distinct vertices in W is at least $\frac{1}{3} r$, and for $a \in W$ let B_a denote the disk of radius r centered at a. Then the disks B_a, $a \in W$, cover Z, assuming that $r > 1$ and $c_1 > 100$, say. Note that

$$|W| = O(m) \left(\frac{R}{r}\right)^2 = O(m) \left(\frac{c_1}{\delta}\right)^2.$$

Fix some $a \in W$. We wish to invoke Lemma 3.10 to get a good upper bound on $\mathbf{P}[\widehat{\gamma}_g \cap B_a \neq \varnothing \,|\, \mathcal{K}]$. We now verify the assumptions of the lemma. We note that in our case $\|g\|_\infty = c_0$ and $\|\nabla g\|_\infty = c_0 / \delta R$. Thus, $q := \|g\|_\infty / \|\nabla g\|_\infty = \delta R$. Consequently, we set c_1 to be the maximum of 100 and twice the constant c in the lemma, and the assumption $q > cr$ is satisfied. The assumption $r > 1$ will hold once R is large enough, which we assume for now (we promise that δ will be a constant depending only on ε, m and $\bar{\Lambda}$). For the d in the lemma we may take $d = R$. Thus, $\log(d/r) = \log(c_1/\delta)$, and the lemma gives

$$\mathbf{P}[\widehat{\gamma}_g \cap B_a \neq \varnothing \,|\, \mathcal{K}] \leqslant O_{\varepsilon, \bar{\Lambda}}(1) \frac{1}{c_0^2} \log \frac{c_1}{\delta}.$$

Since $\bigcup_{a \in W} B_a \supset Z$ and $|W| = O(m)(c_1/\delta)^2$, we conclude that

$$\mathbf{P}[\widehat{\gamma}_g \cap Z \neq \varnothing \,|\, \mathcal{K}] \leqslant O_{\varepsilon, \bar{\Lambda}}(m) \frac{1}{c_0^2} \left(\frac{c_1}{\delta}\right)^2 \log \frac{c_1}{\delta}.$$

Although we have not specified δ yet, we choose $c_0 = c_0(c_1, \delta, \varepsilon, m, \bar{\Lambda})$ so that

$$\mathbf{P}[\widehat{\gamma}_g \cap Z \neq \varnothing \,|\, \mathcal{K}] \leqslant \tfrac{1}{10}. \tag{3.31}$$

Now that we have established that it is unlikely that $\widehat{\gamma}_g$ intersects Z, we need to worry about the case in which $\widehat{\gamma}_g$ circumvents Z but hits $\bigcup Y \setminus Z$. This can only happen near endpoints of barriers. Let us fix some $\Upsilon \in Y_+$ that has an endpoint, say z_1, on $\partial D(\bar{\gamma})$. Let $\widetilde{\chi}_{z_1}$ be the arc $\partial A_{z_1}(\Upsilon) \setminus \chi_{z_1}$. (It is an arc, by condition (3) of the definition of barrier.) We will now prepare the geometric setup that will enable the use of Lemma 3.8 to prove that $\mathbf{P}[\widehat{\gamma}_g \cap A_{z_1}(\Upsilon) \cap \Upsilon \setminus Z \neq \varnothing \,|\, \mathcal{K}]$ is small.

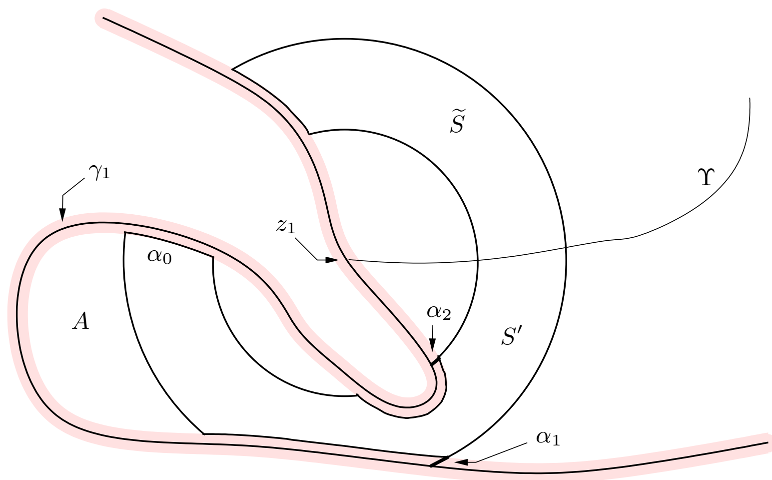

Figure 3.4. The set S, and the paths α_0 and $\widetilde{\alpha}_0$. The shaded region is Q.

Let Q be the set of all hexagons of the grid TG^* whose distance from $\partial D(\bar{\gamma})$ is at most $2\delta R$. Let S be the connected component of

$$A_{z_1} \cap B\big(z_1, \tfrac{7}{8}\varepsilon R\big) \setminus \big(Q \cup B\big(z_1, \tfrac{5}{8}\varepsilon R\big)\big)$$

that intersects Υ. (See Figure 3.4.) Condition (4) in the definition of barriers and our assumption that $\delta \ll \varepsilon$ guarantees that there is a unique such component S. We have $S \subset Z$, and so $\widehat{\gamma}_g$ is unlikely to hit S.

Let S' and $\widetilde{S}$ be the two connected components of $S \setminus \Upsilon$. Consider the connected component M_1 of $\partial B\big(\frac{7}{8} z_1 \varepsilon R\big) \setminus \widetilde{\chi}_{z_1}$ that intersects Υ. Let α_1 be the arc in $M_1 \setminus \bar{S}$ that has one endpoint in $\bar{S}'$ and the other in $\widetilde{\chi}_{z_1}$. Likewise, let M_2 be the connected component of $\partial B\big(\frac{5}{8} z_1 \varepsilon R\big) \setminus \widetilde{\chi}_{z_1}$ that intersects Υ, and let α_2 be the arc in $M_2 \setminus \bar{S}$ that has one endpoint in $\bar{S}'$ and the other in $\widetilde{\chi}_{z_1}$. Let α denote the union of $\alpha_1 \cup \alpha_2$ with the arc of $\partial S' \setminus \Upsilon$ that connects the endpoint of α_1 with the endpoint of α_2. Let $A \subset A_{z_1}(\Upsilon)$ be the domain whose boundary consists of $\alpha_1 \cup \alpha_2$, an arc of $\partial S'$ connecting α_1 and α_2 and an arc of $\widetilde{\chi}_{z_1}$ connecting α_1 and α_2. Let α_0 be the unique arc of $\partial A \setminus \big(\partial B\big(\frac{5}{8} z_1 \varepsilon R\big) \cup \partial B\big(\frac{7}{8} z_1 \varepsilon R\big)\big)$ connecting $\partial B\big(\frac{5}{8} z_1 \varepsilon R\big)$ with $\partial B\big(\frac{7}{8} z_1 \varepsilon R\big)$. Finally, let α' be any subarc of α_0 whose diameter is $\frac{1}{16}\varepsilon R$ that intersects the circle $\partial B\big(z_1, \frac{3}{4}\varepsilon R\big)$. We use A, α_1, α_2 and α' in Lemma 3.8. In the present situation, the value of d_1 of that lemma is $d_1 = 2\delta R + O(1)$. If we have a path connecting α' to $\partial D \cup \bar{\gamma}$, it must either exit $B\big(z_1, \frac{7}{8}\varepsilon R\big) \setminus B\big(z_1, \frac{5}{8}\varepsilon R\big)$, hit $\Upsilon \setminus B\big(z_1, \frac{5}{8}\varepsilon R\big)$ (whose distance from $\bar{\gamma}$ is at least $\frac{1}{5}\varepsilon R$) or connect α_0 to $\widetilde{\chi}_{z_1}$ inside A. Consequently, the minimum on the right-hand side in (3.24) is presently at least $\frac{1}{16}\varepsilon R$. The lemma now implies that if we choose our current $\delta = \delta(\varepsilon, m, \bar{\Lambda}) > 0$ sufficiently small

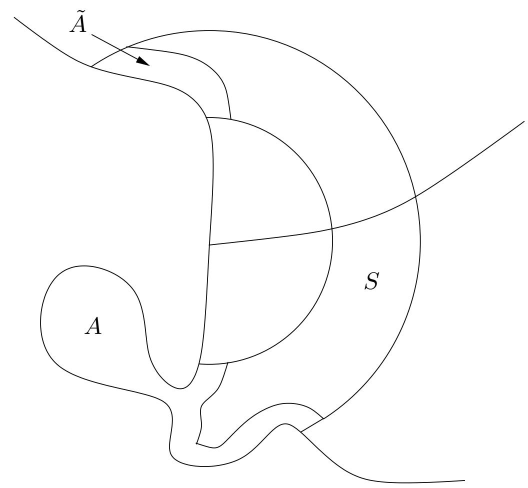

Figure 3.5. No way to penetrate.

and make sure that $R>1/\delta$, then

$$\mathbf{P}[\mathcal{C}\,|\,\mathcal{K}]\leqslant\frac{1}{10m},$$

where $\mathcal{C}$ is the event that there is a crossing of hexagons satisfying $h<0$ between α_1 and α_2 inside A. If there is no such crossing, then also $\widehat{\gamma}_g$ does not make such a crossing, because $g\geqslant 0$ in A.

Likewise, we may define $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, $\widetilde{\alpha}'$, $\tilde{A}$ and $\widetilde{\mathcal{C}}$, when we replace S' by $\widetilde{S}$ in the above paragraph. The same argument shows that $\mathbf{P}[\widetilde{\mathcal{C}}|\mathcal{K}]\leqslant\frac{1}{10}m^{-1}$.

Condition (3) of the definition of a barrier implies that $\overline{S}\cup A\cup\tilde{A}\cup\chi_{z_1}$ separates $\Upsilon\cap A_{z_1}(\Upsilon)$ from all the endpoints of the strands $\gamma_1,...,\gamma_k$. Consequently, in order for $\widehat{\gamma}_g$ to hit $\Upsilon\cap A_{z_1}(\Upsilon)$, we must have $\widehat{\gamma}_g\cap S\neq\varnothing$ or $\mathcal{C}\cup\widetilde{\mathcal{C}}$. See Figure 3.5 (also compare Figure 3.4). A similar argument applies for every other endpoint of a barrier. Since $S\subset Z$, we conclude that

$$\mathbf{P}[\widehat{\gamma}_g\cap(\cup Y)\neq\varnothing\,|\,\mathcal{K}]\leqslant\tfrac{3}{10}. \tag{3.32}$$

We are really more interested in $\widehat{\gamma}$ than in $\widehat{\gamma}_g$. To do the translation, we will appeal to Lemma 3.6. For this purpose, note that

$$\|\nabla g\|_\infty=O(1)\frac{c_0}{\delta R}=O_{\varepsilon,\bar{\Lambda},m}\left(\frac{1}{R}\right)$$

and that g is supported in a union of m sets of diameter $O(R)$. Consequently, we have $\|\nabla g\|_2=O_{\varepsilon,\bar{\Lambda},m}(1)$. Let U be the set of vertices in $\partial D(\bar{\gamma})$, and let h_U denote the restriction of h to U. By (3.32), even if we further condition on h_U, the left-hand side stays bounded away from one on an event whose probability is bounded away from zero, namely,

$$\mathbf{P}\big[\mathbf{P}[\widehat{\gamma}_g\cap(\bigcup Y)=\varnothing\,|\,\mathcal{K},h_U]>\tfrac{1}{10}\,\big|\,\mathcal{K}\big]>\tfrac{1}{10}.$$

Because $g=0$ on U, we may apply Lemma 3.6 and conclude that for h_U such that the inner inequality above holds,

$$O_{\varepsilon,\bar{\Lambda},m}(1)\mathbf{P}[\widehat{\gamma}\cap(\bigcup Y)=\varnothing\,|\,\mathcal{K},h_U]\geqslant 1.$$

Since this set of h_U has conditioned probability at least $\frac{1}{10}$, the theorem follows.

It remains to remove the assumption that R is larger than some fixed constant $R_0=R_0(\varepsilon,m,\bar{\Lambda})$. Assume now that R is bounded. It is not too hard to see that the event $\mathcal{H}$ that $h(H)\in(0,1)$ for every hexagon meeting $\bigcup Y_+$ and $h\in(-1,0)$ for every hexagon meeting $\bigcup Y_-$ has probability bounded below by (a rather small) positive constant. This is proved by considering these $O(mR^2)$ hexagons one by one. On the event $\mathcal{H}$, we have $\widehat{\gamma}\cap(\bigcup Y)=\varnothing$. This completes the proof. □

Remark 3.12. There is a corresponding analog of Theorem 3.11 in the case where the endpoints of the barriers are permitted to land on ∂D. In that case, it is necessary to assume that $h_\partial\geqslant-\Lambda_0$ (respectively, $h_\partial\leqslant\Lambda_0$) on $\partial A_z(\Upsilon)\cap V_\partial$ if $\Upsilon\in Y_+$ (respectively, Y_-) and $z\in\partial D\cap\Upsilon$, where $\Lambda_0=\Lambda_0(\bar{\Lambda})>0$ is the constant given by Lemma 3.9. We refrain from stating a complete formulation of this variant, though it will be useful. The proof is the same, except that Lemma 3.9 is used to deal with the narrows near ∂D, instead of the narrows lemma (Lemma 3.8).

3.5. Meeting of random walk and interface

We now need to further develop the basic setup and introduce some more notation. If α is a path in the hexagonal grid TG*, we let $V(\alpha)$ denote the set of TG-vertices adjacent to it. If α is an arc of an oriented zero-height interface of h, let $V_+(\alpha)$ denote the vertices adjacent to it on its right-hand side, and let $V_-(\alpha)$ denote the vertices adjacent to it on its left-hand side.

In addition to our previous assumptions (h) and (D), we will use the following setup:

(S) Let γ denote the interface of h from x_∂ to y_∂. Let v_0 be some vertex of TG in D, and let S be a simple random walk on the vertices of TG started at v_0 that is independent of h. Let τ be the first time t such that $S_t\in\partial D(\gamma)$.

The point S_τ will play a special role. Essentially, we will be interested in the configuration "as viewed from S_τ"; that is, in the coordinate system where S_τ is translated to 0.

In order to eliminate too much additional notation, it will be convenient to consider the event $S_\tau=0$ instead. Let τ_0 be the first t such that $S_t=0$, and let $\breve{S}$ denote the reversed walk $\breve{S}_t:=S_{\tau_0-t}$, $t=0,1,...,\tau_0$.

For $\sigma=0,1,...,5$, let e_σ denote the edge $\big[0,\exp\big(\frac{1}{3}\pi i\sigma\big)\big]$ of the triangular grid TG, and let e_σ^* denote the dual edge in TG*. Let $\mathcal{Z}_0^\sigma$ denote the event $\mathcal{Z}_0^\sigma:=\{S_\tau=0\}\cap\{e_\sigma^*\subset\gamma\}$.

Fix some large R. Suppose that $\mathfrak{B}_R\subset D$ and $v_0\notin\mathfrak{B}_R$. Let $\mathrm{ext}_R\gamma$ denote the union of the components of $\gamma\setminus\mathfrak{B}_R$ containing x_∂ and y_∂. (If $\gamma\cap\mathfrak{B}_R=\varnothing$, then $\mathrm{ext}_R\gamma=\gamma$.) If there is an interface of h containing e_σ^*, denote it by $\hat\beta=\hat\beta^\sigma$, and let $\beta=\beta_R^\sigma$ be the connected component of $\hat\beta\cap\mathfrak{B}_R$ that contains e_σ^*. Otherwise, set $\beta=\hat\beta=\varnothing$. Let $\mathrm{int}_R\breve{S}$ denote the part of $\breve{S}$ up to the first exit of $\mathfrak{B}_R$, and let $\mathrm{ext}_R S$ denote the part of S up to the first entry to $\mathfrak{B}_R$.

Set $\Phi_R:=(D,\partial_+,h_\partial,v_0,\mathrm{ext}_R\gamma,\mathrm{ext}_R S)$ and $\Theta_R=\Theta_R(\sigma):=(\beta_R^\sigma,\mathrm{int}_R\breve{S})$. Our goal is to show that conditioned on $\mathcal{Z}_0^\sigma$, the distribution of β does not depend strongly on Φ_{4R}. (A precise version of this statement is given in Corollary 3.16 below.) To this end we will use something like

$$\mathbf{P}[\Theta_R=\vartheta\mid\Phi_{3R},\mathcal{Z}_0^\sigma]=\frac{\mathbf{P}[\Theta_R=\vartheta\mid\Phi_{3R}]\mathbf{P}[\mathcal{Z}_0^\sigma\mid\Theta_R=\vartheta,\Phi_{3R}]}{\mathbf{P}[\mathcal{Z}_0^\sigma\mid\Phi_{3R}]}. \tag{3.33}$$

This equality is obtained by applying Bayes' formula to the measure $\mathbf{P}[\,\cdot\,|\Phi_{3R}]$. The following lemma takes care of the first factor in the numerator on the right-hand side.

LEMMA 3.13. *Assume* (h), (D) *and* (S). *There exists a constant $c=c(\bar\Lambda)>0$ and a function $p_R(\cdot)$ such that if $R>50$, $R'\in\big[\frac{5}{4}R,3R\big]$, $D\supset\mathfrak{B}_{4R}$ and $v_0\notin\mathfrak{B}_{4R}$, then for all $\vartheta=(\tilde\beta,\widetilde{S})$ such that $\tilde\beta\neq\varnothing$ one has*

$$\frac{1}{c}p_R(\vartheta)\leqslant\mathbf{P}[\Theta_R=\vartheta\mid\gamma\cap\mathfrak{B}_{4R}\neq\varnothing,\Phi_{R'}]\leqslant c\,p_R(\vartheta).$$

The function p_R may depend on R and ϑ, but not on anything else (*in particular*, *not on D, v_0, $\Phi_{R'}$ or h_∂*).

Proof. The corresponding statement with Θ_R replaced by β, the first coordinate of Θ_R, is an immediate consequence of Proposition 3.7.

We assume that $v_0\notin\mathfrak{B}_{4R}$. The configuration $\Phi_{R'}$ determines the first vertex, say q, inside $\mathfrak{B}_{R'}$ visited by the random walk S. The continuation of the walk is just a simple random walk starting at q. Suppose that we had another such walk starting at a vertex $q'\in\mathfrak{B}_{R'}$. It is easy to see that with probability bounded away from zero the walk starting at q visits q' before 0. If that happens, we couple the continuation of the walk to be the same as the walk which starts at q' (otherwise, we let them be independent). On the event that the walk started at q hits q' before 0, the corresponding $\mathrm{int}_R\breve{S}$ for both walks will be the same. This proves the corresponding statement about the second coordinate of Θ_R. Since the two coordinates are independent given $\Phi_{R'}$, the lemma follows. $\square$

Proving an analogous result for the second factor in the numerator of the right-hand side of (3.33) will be considerably more difficult. To this end, we now define a measure of the *quality* $Q=Q_R$ of the configurations Φ_R and Θ_R.

If $\gamma\cap\mathfrak{B}_R\neq\varnothing$, let x^R (respectively, y^R) denote the endpoint in $\partial\mathfrak{B}_R$ of the component of $\operatorname{ext}_R\gamma$ containing x_∂ (respectively, y_∂). When $v_0\notin\mathfrak{B}_R$, let q^R denote the vertex in $\mathfrak{B}_R$ first visited by S. If $\gamma\cap\mathfrak{B}_R=\varnothing$ or $\operatorname{ext}_R S$ visits $\partial D(\operatorname{ext}_R\gamma)$, then set $Q(\Phi_R)=0$. Otherwise, define

$$Q(\Phi_R):=\frac{\operatorname{dist}(x,\operatorname{ext}_R S)\wedge\operatorname{dist}(y,\operatorname{ext}_R S)\wedge\operatorname{dist}(q,\operatorname{ext}_R\gamma)\wedge|x-y|}{R}\wedge\frac{1}{100},$$

where $x=x^R$, $y=y^R$ and $q=q^R$. This is a measure of the separation between the strands comprising Φ_R. Similarly, define $Q(\Theta_R)$, as follows. Suppose that $v_0\notin\mathfrak{B}_R\subset D$ and let $\hat{q}^R$ be the first vertex outside of $\mathfrak{B}_R$ visited by $\breve{S}$. Fix an orientation of e^*_σ. If $\hat{\beta}\not\subset\mathfrak{B}_R$, let $\hat{x}^R$ and $\hat{y}^R$ be the two endpoints of the component of $\beta=\beta_R$ containing e^*_σ, chosen so that the orientation of the arc of β from $\hat{x}^R$ to $\hat{y}^R$ agrees with that of e^*_σ. If $\hat{\beta}\subset\mathfrak{B}_R$ or if $\breve{S}$ visits any vertex in $\partial D(\beta_R)\setminus\{0\}$, then set $Q(\Theta_R)=0$. Otherwise, set

$$Q(\Theta_R):=\frac{\operatorname{dist}(\hat{x},\operatorname{int}_R\breve{S})\wedge\operatorname{dist}(\hat{y},\operatorname{int}_R\breve{S})\wedge\operatorname{dist}(\hat{q},\beta)\wedge|\hat{x}-\hat{y}|}{R}\wedge\frac{1}{100},$$

where $\hat{x}=\hat{x}^R$, $\hat{y}=\hat{y}^R$ and $\hat{q}=\hat{q}^R$.

LEMMA 3.14. (Compatibility) *Assume* (h), (D) *and* (S). *For every* $\varepsilon>0$ *there is a constant* $c=c(\varepsilon,\bar{\Lambda})>0$ *such that*

$$\frac{1}{c}\,1_{\{Q(\Theta_R)>\varepsilon\}}1_{\{Q(\Phi_{R'})>\varepsilon\}}\leqslant\mathbf{P}[\mathcal{Z}_0^\sigma\mid\Theta_R,\Phi_{R'}]\log R\leqslant c$$

holds whenever $R>c$, $5R>R'>\frac{9}{8}R$ *and* $v_0\notin\mathfrak{B}_{6R}\subset D$.

Proof. We start by proving the lower bound on $\mathbf{P}[\mathcal{Z}_0^\sigma|\Theta_R,\Phi_{R'}]$, which is the harder estimate. Assume that $Q(\Theta_R)\wedge Q(\Phi_{R'})>\varepsilon$, $R>c>10^{10}/\varepsilon$, $5R>R'>\frac{9}{8}R$ and $v_0\notin\mathfrak{B}_{6R}\subset D$. Let $\widehat{D}_S$ denote the connected component of $\mathfrak{B}_R\setminus\beta_R$ that intersects $\operatorname{int}_R\breve{S}$ and let D_S denote the connected component of $D\setminus(\operatorname{ext}_{R'}\gamma\cup\mathfrak{B}_{R'})$ that contains v_0 and therefore $\operatorname{ext}_{R'}S$. Note that the sign of h on vertices in $\widehat{D}_S$ adjacent to β is constant, as is the sign of h on vertices in D_S adjacent to $\operatorname{ext}_{R'}\gamma$. Let $\mathcal{D}$ denote the event that these signs are the same, namely, the sign of h on vertices in $V(\operatorname{ext}_{R'}\gamma)\cap D_S$ is the same as on vertices in $V(\beta_R)\cap\widehat{D}_S$. Using symmetry, Proposition 3.7 immediately implies that $\mathbf{P}[\mathcal{D}|\Phi_{R'},\Theta_R]$ is bounded away from zero. (Although β is determined by Θ_R, its orientation as a subarc of an oriented zero-height interface of h is not determined by Θ_R.)

We now construct some barriers, as illustrated in Figure 3.6. Let a be the initial point of the arc $\partial\mathfrak{B}_{R'}\cap\overline{D}_S$, when the arc is oriented counterclockwise around $\mathfrak{B}_{R'}$, and

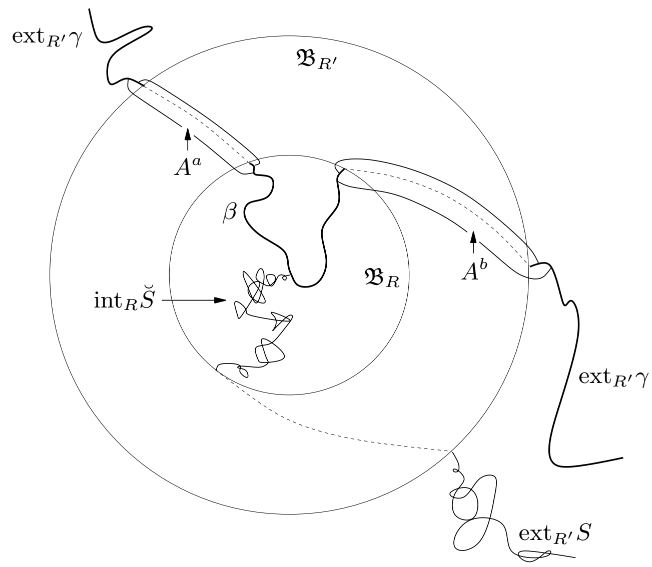

Figure 3.6. The construction of the barriers.

let b be the other endpoint of this arc. Likewise, let $\hat{a}$ be the initial point of the arc $\partial\mathfrak{B}_R\cap\bar{\bar{D}}_S$, when the arc is oriented counterclockwise around $\mathfrak{B}_R$, and let $\hat{b}$ be the other endpoint of this arc. Note that $\{\hat{a},\hat{b}\}=\{\hat{x}^R,\hat{y}^R\}$ and $\{a,b\}=\{x^{R'},y^{R'}\}$.

We now describe a path α_a connecting a to $\hat{a}$ and a path α_b connecting b to $\hat{b}$ and a path α_q connecting $\hat{q}^R$ to $q^{R'}$ such that these paths do not come too close to each other. For example, if the arguments $\arg a$, $\arg b$, $\arg q^{R'}$, $\arg\hat{a}$, $\arg\hat{b}$ and $\arg\hat{q}^R$ are chosen so that $\arg\hat{a}<\arg\hat{q}^R<\arg\hat{b}<\arg\hat{a}+2\pi$, $\arg a<\arg q^R<\arg b<\arg a+2\pi$ and $|\arg a-\arg\hat{a}|\leqslant\pi$, then we may take α_a to be defined in polar coordinates by $\theta=s+tr$, with s and t chosen so that a and $\hat{a}$ are on the path, and similarly for α_b and α_q. It is easy to check that our assumptions guarantee that the distance between any two of these paths is at least $c_1\varepsilon R$ for some constant $c_1>0$. Set $\varepsilon'=\frac{1}{10}c_1\varepsilon$.

Let D_+ be the connected component of $D\setminus(\text{ext}_{R'}\gamma\cup\beta_R\cup\alpha_a\cup\alpha_b)$ that contains $V_+(\text{ext}_{R'}\gamma)$, and let D_- be the other connected component. Let α'_a be the connected component of $\{z\in D_+:\text{dist}(z,\alpha_a)=\varepsilon' R\}$ that meets the circle $\partial B\big(0,\frac{1}{2}(R+R')\big)$. Note that α'_a is a simple path which intersects the circle $\partial B\big(0,\frac{1}{2}(R+R')\big)$ at one point. Let a_0 denote that point. We want to construct a pertubation of α'_a, which will be some $(\varepsilon'',12R)$-barrier, with ε'' not much smaller than ε'. Let a_1 be the closest point to a_0 along α'_a such that the distance from a_1 to $\partial D(\text{ext}_{R'}\gamma)$ is $\frac{1}{10}\varepsilon' R$, and let $\hat{a}_1$ be the closest point to a_0 along α'_a such that the distance from $\hat{a}_1$ to $\partial D(\beta_R)$ is $\frac{1}{10}\varepsilon' R$. Let α^a_+ be the path which is the union of the arc of α'_a connecting a_1 and $\hat{a}_1$ together with a shortest line segment connecting a_1 to $\partial D(\mathrm{ext}_{R'}\gamma)$ and a shortest line segment connecting $\hat{a}_1$ to $\partial D(\beta_R)$.

We claim that α_+^a is an $(\varepsilon'', 12R)$-barrier for the configuration $(D, \mathrm{ext}_{R'}\gamma\cup\beta_R)$ with $\varepsilon''=\frac{1}{1000}\varepsilon'$. Indeed, conditions (1), (2), (4) and (3) (b) in the definition of the barrier clearly hold. To verify condition (3) (a), let z_1 be the endpoint of α_+^a on $\partial D(\mathrm{ext}_{R'}\gamma)$, and let z_1' and z_1'' be the two endpoints of χ_{z_1} on $\partial D(\mathrm{ext}_{R'}\gamma)$. Consider the simple arc $\widetilde{\chi}$ connecting z_1' to z_1'' in $\partial D(\mathrm{ext}_{R'}\gamma)$. By the Jordan curve theorem, $\widetilde{\chi}\cup\chi_{z_1}$ separates the plane into two connected components. Since α_+^a crosses χ_{z_1}, it follows that the part of α_+^a inside $\mathfrak{B}_{R'}$ is outside of A_{z_1}, and thus the endpoints $x^{R'}$, $y^{R'}$ and also β_R are all outside A_{z_1}. It follows that $\partial A_{z_1}=\chi_{z_1}\cup\widetilde{\chi}$, as required. A similar argument applies near the endpoint of α_+^a on $\partial D(\beta_R)$. Thus, α_+^a is indeed an (ε'', R')-barrier. Note also that the above easily implies that $\partial A_{z_1}\subset D_+$. This will be useful below when we apply the barriers theorem (Theorem 3.11).

We similarly construct a path α_-^a in D_- close to α_a. Likewise, we construct barriers α_+^b and α_-^b near the path α_b. The construction is the same, except that we replace α_a by α_b.

On the event $\mathcal{D}$, we may apply Theorem 3.11 with $Y_+=\{\alpha_+^a, \alpha_+^b\}$, $Y_-=\{\alpha_-^a, \alpha_-^b\}$, $Y=Y_+\cup Y_-$ and $\bar{\gamma}=\mathrm{ext}_{R'}\gamma\cup\beta_R$. (Here we use the assumption that $R>c$.) Note that conditioning on Θ_R, $\Phi_{R'}$ and $\mathcal{D}$, amounts to conditioning on $\mathcal{K}$ in the theorem and on the behavior of $\mathrm{int}_R\breve{S}\cup\mathrm{ext}_{R'}S$, which is anyway independent of h. Therefore, there is a $p=p(\varepsilon, \bar{\Lambda})>0$ such that

$$\mathbf{P}[\mathcal{Y}\,|\,\Theta_R, \Phi_{R'}, \mathcal{D}]\geqslant p,$$

where $\mathcal{Y}$ denotes the event

$$\mathcal{Y}:=\{(\gamma\setminus\bar{\gamma})\cap\left(\bigcup Y\right)=\varnothing\}.$$

Let A^a be the connected component of $D(\mathrm{ext}_{R'}\gamma\cup\beta_R)\setminus(\alpha_+^a\cup\alpha_-^a)$ that contains α_a, and let A^b be the connected component of $D(\mathrm{ext}_{R'}\gamma\cup\beta_R)\setminus(\alpha_+^b\cup\alpha_-^b)$ that contains α_b. Again, using the Jordan curve theorem, it is easy to verify that $A^a\cap A^b=\varnothing$. On the event $\mathcal{Y}$, there is no other choice for the strand of γ extending $\mathrm{ext}_{R'}\gamma$ at a, but to be confined to A^a until it hooks up with β at $\hat{a}$, since every other exit from A^a is blocked. Consequently, on $\mathcal{Y}$, we have $\gamma\supset\beta$. A similar argument applies to A^b, and we get

$$\beta\subset\gamma\subset\mathrm{ext}_{R'}\gamma\cup\beta\cup A^a\cup A^b\quad\text{on } \mathcal{Y}.$$

We now turn to the random walk S. For $\mathcal{Z}_0^\sigma$ to hold, we must make sure that $\{S_t:t<\tau_0\}$ does not meet any vertex neighboring with γ. First consider $\mathrm{ext}_{R'}S$. Note that $\mathrm{ext}_{R'}S$ does not intersect ∂A^a, because ∂A^a is contained in $\mathfrak{B}_{R'}\cup B(a, 2\varepsilon' R)$, and we are assuming that $Q(\Phi_{R'})>\varepsilon$. (Recall that $\{a,b\}=\{x^{R'},y^{R'}\}$.) Thus, $\mathrm{ext}_{R'}S\cap A^a=\varnothing$, and we may also conclude that $\mathrm{ext}_{R'}S$ does not visit any vertex adjacent to A^a when R is large. Similar arguments apply to A^b and to $\mathrm{int}_R\breve{S}$. Thus, $\mathrm{int}_R\breve{S}\cup\mathrm{ext}_{R'}S$ does not visit any vertex adjacent to γ on the event $\mathcal{Y}\cap\{Q(\Phi_{R'})>\varepsilon, Q(\Theta_R)>\varepsilon\}$.

Now let S^* be the walk S from the first time it visits $q^{R'}$ until the first time it visits $\hat{q}^R$. Then, conditioned on Θ_R and $\Phi_{R'}$, S^* is just a simple random walk started at $q^{R'}$ conditioned to hit $\hat{q}^R$ before hitting 0. Let A^q denote the $\varepsilon' R$-neighborhood of α_q. Clearly, $\mathrm{dist}(A^q, A^a\cup A^b)\geqslant\varepsilon' R$. The probability that S^* gets within distance $\frac{1}{4}\varepsilon' R$ of $\hat{q}^R$ before exiting A^q is at least some (perhaps small) positive constant depending only on ε' (and hence on ε). Conditional on this event, the probability that S^* visits $\hat{q}^R$ before exiting A^q is within a constant multiple of $1/\log(\varepsilon' R)$, by (2.6). Now let S^{**} be the walk S from the first visit of $\hat{q}^R$ to the last visit of $\hat{q}^R$ before time τ_0. Note that S^{**} and $\mathrm{int}_R\breve{S}$ are independent given $\hat{q}^R$. Thus, given $\Phi_{R'}$, Θ_R, $\mathcal{D}$ and S^*, we may sample S^{**} by starting a random walk from $\hat{q}^R$, stopping when it hits 0, and then removing the part of that walk after the last visit to $\hat{q}^R$. When the latter walk first gets to distance $\frac{1}{2}\varepsilon' R$ from $\hat{q}^R$, it has probability bounded away from zero (by a constant depending only on ε') to hit 0 before $\hat{q}^R$. (This follows, for example, from Lemma 2.2 applied to the function giving for every vertex the probability to hit 0 before $\hat{q}^R$ for a random walk started at that vertex.) Thus, conditioned on $(S^*, \Theta_R, \Phi_{R'})$, with probability bounded away from zero, $S^{**}\subset A^q$. Since

$$\mathcal{Z}_0^\sigma \supset \{S^*\cup S^{**}\subset A^q\}\cap\mathcal{Y},$$

we conclude that

$$O_{\varepsilon,\bar{\Lambda}}(1)\mathbf{P}[\mathcal{Z}_0^\sigma \mid \Phi_{R'}, \Theta_R, \mathcal{D}] \geqslant \frac{1}{\log R}.$$

Above, we have argued that $\mathbf{P}[\mathcal{D}|\Phi_{R'},\Theta_R]$ is bounded away from zero, and so we conclude that the lower bound estimate in the proposition holds.

It remains to prove the upper bound. Conditional on γ, $\mathrm{int}_R\breve{S}$ and $\mathrm{ext}_{R'}S$, the probability that S^* (as defined in the proof of the lower bound) hits $\hat{q}^R$ before hitting γ is clearly $O(1)/\log R$, since the conditional law of S^* is that of a random walk started at $q^{R'}$ and conditioned to hit $\hat{q}^R$ before 0, and the probability that an ordinary random walk started at $q^{R'}$ hits $\hat{q}^R$ before 0 is bounded away from zero. The upper bound now follows, and the proof is complete. □

To make the previous lemma useful, we will need to argue that configurations with quality bigger than ε are not too rare, in an appropriate sense. This is achieved by the following lemma.

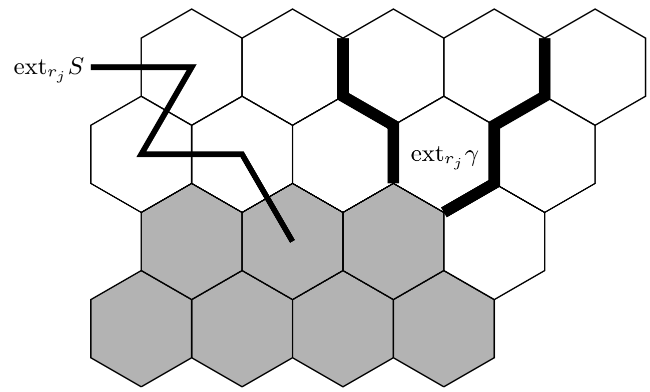

Figure 3.7. An example for $Q_j=0\neq Q(\Phi_{r_j})$. The set $\mathfrak{B}_{r_j}$ is shaded.

LEMMA 3.15. (Separation) *Assume* (h), (D) *and* (S). *Let* $p<1$. *There exists some constant* $c=c(p,\bar{\Lambda})>0$ *such that if* $R>1/c$ *and* $v_0\notin\mathfrak{B}_{6R}\subset D$, *then*

$$\mathbf{P}[Q(\Phi_{3R})\wedge Q(\Theta_{2R})>c\,|\,\Theta_R,\Phi_{4R},\mathcal{Z}_0^\sigma]>p, \tag{3.34}$$

provided that $\mathbf{P}[\mathcal{Z}_0^\sigma\,|\,\Theta_R,\Phi_{4R}]>0$.

The proof of this lemma is modeled after Lawler's separation lemma for Brownian motions from [L2, Lemma 4.2].

Proof. To keep the notation simple, we start by proving a simpler version of the lemma, where we also assume that $Q(\Theta_R)\geqslant\frac{1}{100}$, say (and therefore $Q(\Theta_R)=\frac{1}{100}$), and we prove that

$$\mathbf{P}[Q(\Phi_{3R})>c\,|\,\Theta_R,\Phi_{4R},\mathcal{Z}_0^\sigma]>p. \tag{3.35}$$

We define inductively a random sequence $r_0,r_1,...$ as follows. Set $r_0:=4R$. Suppose that r_j is defined. Set

$$Q_j:=\begin{cases} Q(\Phi_{r_j}), & \text{if } \mathbf{P}[\mathcal{Z}_0^\sigma\,|\,\Phi_{r_j},\Theta_R]>0,\\ 0, & \text{otherwise,}\end{cases}$$

and $r_{j+1}:=(1-10Q_j)r_j$. Note that $Q_j=0$ implies that $\mathbf{P}[\mathcal{Z}_0^\sigma|\Phi_{r_j},\Theta_R]=0$. (An example showing that $Q_j=0\neq Q(\Phi_{r_j})$ is possible is given in Figure 3.7. Such a situation can only occur when $|x^{r_j}-y^{r_j}|=O(1)$.) Also note that $\mathbf{P}[\mathcal{Z}_0^\sigma|\Phi_{r_j},\Theta_R]>0$ if and only if there are paths γ_* and S_* satisfying the following: (1) γ_* is a simple TG*-path in $\overline{D}$ containing β^R and $\mathrm{ext}_{r_j}\gamma$, (2) S_* is a TG-path in D containing $\mathrm{ext}_{r_j}S$ and the reversal of $\mathrm{int}_R\breve{S}$ and (3) S_* does not visit any vertex in $\partial D(\gamma_*)$, except for 0.

We claim that for every $j\in\mathbb{N}$,

$$\mathbf{P}\big[Q_{j+1}\geqslant(2Q_j)\wedge\tfrac{1}{100}\,\big|\,\Phi_{r_j},r_j>2R,\Theta_R\big]>c_0 \tag{3.36}$$

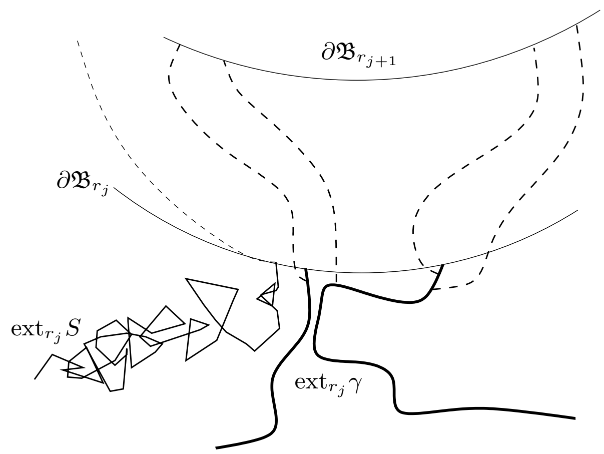

Figure 3.8. The construction of the barriers giving (3.36).

for some constant $c_0=c_0(\bar{\Lambda})>0$. Clearly it suffices to prove this in the case $Q_j>0$. The gap between $\partial\mathfrak{B}_{r_j}$ and $\partial\mathfrak{B}_{r_{j+1}}$ is larger than but comparable to $5r_jQ_j$. Note also that because $\mathfrak{B}_{r_j}$ is a union of TG*-hexagons, $r_jQ_j\geqslant 1/\sqrt{3}$. If $r_jQ_j>20$, say, then, we can easily use barriers as in the proof of Lemma 3.14 (see Figure 3.8) to direct and separate the two strands of $\mathrm{ext}_{r_{j+1}}\gamma\setminus\mathrm{ext}_{r_j}\gamma$ and the walk $\mathrm{ext}_{r_{j+1}}S\setminus\mathrm{ext}_{r_j}S$ so as to obtain (3.36).

Now assume that $r_jQ_j\leqslant 20$. In this case the discrete structure of the lattice is "visible". Let q' be the point on $\partial\mathfrak{B}_{r_j}$ crossed by $\mathrm{ext}_{r_j}S$ in its last step, and let α be a longest arc among the three connected components of $\partial\mathfrak{B}_{r_j}\setminus\{q',x^{r_j},y^{r_j}\}$. Suppose first that q' is not an endpoint of α. Let α be oriented counterclockwise around $\mathfrak{B}_{r_j}$, and let a and b be the initial and terminal points of α, respectively. Let η_a (respectively, η_b) denote the connected component of $\mathrm{ext}_{r_{j+1}}\gamma\setminus\mathrm{ext}_{r_j}\gamma$ that has a (respectively, b) as an endpoint. Assume that $|q'-a|<500$ and $|q'-b|<500$. Consider the event $\mathcal{X}$ that η_a goes as far to the right as possible subject to the conditions that it remains inside $B(q',550)$ and avoids $\mathrm{ext}_{r_j}\gamma$ and that η_b goes as far to the left as possible subject to the conditions that it remains inside $B(q',550)$ and avoids $\mathrm{ext}_{r_j}\gamma$. See Figure 3.9. It is easy to see that $\mathbf{P}[\mathcal{Z}_0^\sigma\,|\,\Phi_{r_j},\Theta_R]>0$ implies that on $\mathcal{X}$ there is a simple TG-path in $B(q',250)$ from q' to $\mathfrak{B}_{r_{j+1}}$ that avoids $\partial D(\mathrm{ext}_{r_{j+1}}\gamma)$. Since the number of edges traversed by these paths is bounded, it is easy to see that the probability that S follows the latter path and $\mathcal{X}$ holds given Φ_{r_j} and Θ_R satisfying the above assumptions is bounded away from zero (for $\mathcal{X}$, note that we can extend the interfaces one step at a time and the probability for every specific step given the previous ones is bounded away from zero). This gives (3.36) in this case. Similar, or simpler, arguments apply if one or more of the assumptions $|q'-a|<500$ and $|q'-b|<500$ do not hold.

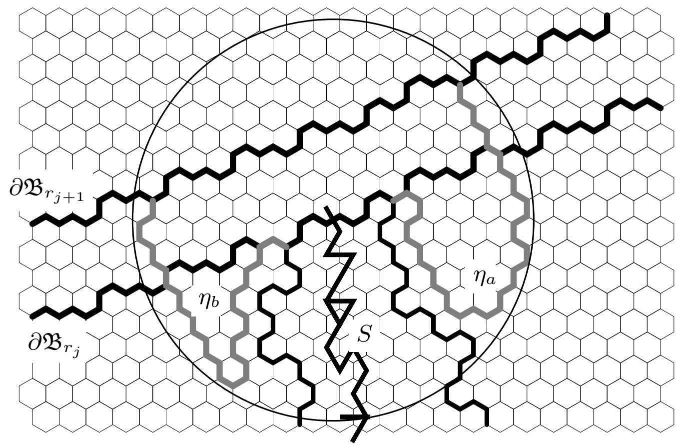

Figure 3.9. The interfaces spreading out.

If q' is an endpoint of α a similar argument may be used. Suppose, for example, that q' is the initial point of α, that y^{r_j} is the other endpoint of α and that $|q'-y^{r_j}|<300$. Then we may consider the possibility that the connected component of $\mathrm{ext}_{r_{j+1}}\gamma\setminus\mathrm{ext}_{r_j}\gamma$ that has y^{r_j} as an endpoint goes as far to the left as possible subject to the requirements that it stays inside $B(y^{r_j},5000)$ and avoids hexagons containing vertices visited by $\mathrm{ext}_{r_j}S$, and that the connected component of $\mathrm{ext}_{r_{j+1}}\gamma\setminus\mathrm{ext}_{r_j}\gamma$ that has x^{r_j} as an endpoint goes as far to the left as possible subject to the requirements that it stays inside $B(x^{r_j},4000)$ and avoids the previous strand extending $\mathrm{ext}_{r_j}\gamma$ at y^{r_j} and finally, the random walk avoids $\partial D(\mathrm{ext}_{r_{j+1}}\gamma)$ and stays in $B(q',300)$ until it hits $\mathfrak{B}_{r_{j+1}}$. A similar argument applies if $|q'-y^{r_j}|\geqslant 300$. This proves (3.36).

We now prove that for every $j\in\mathbb{N}$,

$$\mathbf{P}[Q_{j+2}=0\mid\Phi_{r_j},r_j>2R,\Theta_R,Q_{j+1}<2Q_j]\geqslant c_1, \tag{3.37}$$

for some $c_1=c_1(\bar\Lambda)>0$. Let $x=x^{r_{j+1}}$, $y=y^{r_{j+1}}$ and $q=q^{r_{j+1}}$. Let $S^\dagger$ be the part of the walk S from the first visit to q^{r_j} up to the first visit to q. Note that

$$\mathrm{dist}(x,\mathrm{ext}_{r_j}S)\geqslant 2r_jQ_j>2r_{j+1}Q_j,$$

if $Q_j>0$, and similarly for y. Thus, the event $Q_{j+1}<2Q_j$ is the union of the following five events

$$\mathcal{M}_0:=\{Q_{j+1}=0\},$$
$$\mathcal{M}_1:=\{\mathrm{dist}(S^\dagger,\{x,y\})<(2Q_jr_{j+1})\wedge|x-y|\},$$
$$\mathcal{M}_2:=\{r_{j+1}Q_{j+1}=\mathrm{dist}(q,\mathrm{ext}_{r_{j+1}}\gamma),Q_{j+1}<2Q_j\},$$

$$\mathcal{M}_3 := \{0 < |x-y| = r_{j+1} Q_{j+1}, Q_{j+1} < 2Q_j\},$$
$$\mathcal{M}_4 := \{Q_{j+1} = \tfrac{1}{100}, Q_{j+1} < 2Q_j\}.$$

(In the definition of $\mathcal{M}_1$, $\operatorname{dist}(S^\dagger, \{x,y\})$ means the least distance from a vertex visited by $S^\dagger$ to x or y, of course.) Clearly,

$$O_{\bar\Lambda}(1)\mathbf{P}[Q_{j+2}=0 \mid \Phi_{r_j}, \Theta_R, \mathcal{M}_k] \geqslant 1 \tag{3.38}$$

holds for $k=0$. The same is also true for $k=4$, because $Q_{j+1}=\frac{1}{100}$ implies that the random walk started at q has conditional probability bounded away from zero to hit $\operatorname{ext}_{r_{j+1}}\gamma$ before $\partial\mathfrak{B}_{r_{j+2}}$. A similar argument gives (3.38) when $k=2$.

Now condition on $\mathcal{M}_1$, and let v be the vertex first visited by $S^\dagger$ that is at distance less than $(2Q_j r_{j+1})\wedge|x-y|$ from $\{x,y\}$. Conditioned additionally on Θ_R, $\operatorname{ext}_{r_{j+1}}\gamma$ and the walk $S^\dagger$ until it hits v, there is clearly probability bounded away from zero that $S^\dagger$ hits a vertex adjacent to $\operatorname{ext}_{r_{j+1}}\gamma$ before $\mathfrak{B}_{r_{j+2}}$, and in this case we have $Q_{j+2}=0$. Consequently, (3.38) also holds for $k=1$.

Now condition on $\mathcal{M}_3$, $\Phi_{r_{j+1}}$ and Θ_R. Let z be the midpoint of the segment $[x,y]$, and consider the circle $\partial B(z, 2|x-y|)$. We may build a barrier by using the connected component of $\partial B(z,2|x-y|)\setminus\partial D(\operatorname{ext}_{r_{j+1}}\gamma)$ that intersects $\mathfrak{B}_{r_{j+1}}$ (and possibly perturbing it slightly near its endpoints). If γ does not cross this barrier, then $Q_{j+2}=0$ holds. Thus, we get from Theorem 3.11 that (3.38) also holds for $k=3$. Since $\{Q_{j+1}<2Q_j\}=\bigcup_{k=0}^4 \mathcal{M}_k$, and (3.38) holds for $k=0,1,...,4$, it follows that (3.37) holds as well.

Set $s_n := 2R\prod_{k=0}^{n-1}(1-2^{-k}/10)^{-1}$. It follows from Lemma 3.14 and our assumption that $Q(\Theta_R)\geqslant\frac{1}{100}$ that for any $j\in\mathbb{N}$,

$$\mathbf{P}[\mathcal{Z}_0^\sigma \mid \Phi_{r_j}, \Theta_R] \geqslant \frac{c_2}{\log R} 1_{\{Q_j\geqslant 1/100\}} 1_{\{r_j\geqslant s_0\}}$$

for some $c_2=c_2(\bar\Lambda)>0$. An appeal to (3.36) therefore implies that

$$\mathbf{P}[\mathcal{Z}_0^\sigma \mid \Phi_{r_j}, \Theta_R] \geqslant \frac{c_0 c_2}{\log R} 1_{\{Q_j\geqslant 2^{-1}/100\}} 1_{\{r_j\geqslant s_1\}}.$$

Continuing inductively, we get for every $n\in\mathbb{N}$,

$$\mathbf{P}[\mathcal{Z}_0^\sigma \mid \Phi_{r_j}, \Theta_R] \geqslant \frac{c_0^n c_2}{\log R} 1_{\{Q_j\geqslant 2^{-n}/100\}} 1_{\{r_j\geqslant s_n\}}.$$

Since

$$\sup_n s_n < 2R\biggl(1-\sum_{k=0}^\infty \frac{2^{-k}}{10}\biggr)^{-1} = \frac{5}{2}R,$$

we get for every $j\in\mathbb{N}$,

$$\mathbf{P}[\mathcal{Z}_0^\sigma \mid \Phi_{r_j}, \Theta_R] \geqslant \frac{c_0^{-\log_2 Q_j} c_2}{\log R} 1_{\{r_j \geqslant 5R/2\}} = c_2 \frac{Q_j^{-\log_2 c_0}}{\log R} 1_{\{r_j \geqslant 5R/2\}}. \tag{3.39}$$

From (3.37) we get that for every j the conditional probability that there is some $k>j$ such that $r_k>2R$, $Q_k<2Q_j$ and $Q_{k+1}>0$, given Φ_{r_j} and Θ_R, is at most $1-c_1$. Let m_n denote the number of $k\in\mathbb{N}$ such that $r_k>3R$ and $Q_k\in(2^{-n}, 2^{1-n}]$. Fix some $n\in\mathbb{N}$, and suppose that $\mathbf{P}[m_n>0|\Phi_{4R}, \Theta_R]>0$. On the event $m_n>0$, let k_n be the first k such that $Q_k\in(2^{-n}, 2^{1-n}]$. By induction and (3.37), for every $m\in\mathbb{N}$,

$$\mathbf{P}[m_n > 2m \mid m_n > 0, \Phi_{r_{k_n}}, \Theta_R] \leqslant (1-c_1)^m.$$

An appeal to the upper bound in Lemma 3.14 gives

$$\mathbf{P}[\mathcal{Z}_0^\sigma, m_n > 2m \mid m_n > 0, \Phi_{r_{k_n}}, \Theta_R] \leqslant \frac{c_3(1-c_1)^m}{\log R}$$

for some $c_3=c_3(\bar{\Lambda})$. On the other hand, (3.39) gives

$$\mathbf{P}[\mathcal{Z}_0^\sigma \mid m_n > 0, \Phi_{r_{k_n}}, \Theta_R] \geqslant \frac{c_2 c_0^n}{\log R}.$$

Comparing the last two inequalities, we get

$$\mathbf{P}[m_n > 2m \mid \mathcal{Z}_0^\sigma, \Phi_{r_0}, \Theta_R] \leqslant \frac{c_3 c_0^{-n}(1-c_1)^m}{c_2}.$$

In particular, there is an $n_0=n_0(\bar{\Lambda}, p)\in\mathbb{N}$ and a $c_4=c_4(\bar{\Lambda})$ such that

$$\mathbf{P}[\text{there exists } n \geqslant n_0 : m_n > c_4 n \mid \mathcal{Z}_0^\sigma, \Phi_{r_0}, \Theta_R] \leqslant \tfrac{1}{3}(1-p).$$

Let n_1 be the least integer larger than 3 such that $\prod_{n=n_1}^\infty (1-10\cdot 2^{1-n})^{c_4 n} > \frac{7}{8}$, and let $n_2=n_1\vee n_0$. On the event $\mathcal{Z}_0^\sigma \cap \bigcap_{n>n_2}\{m_n\leqslant c_4 n\}$ we must have some $j\in\mathbb{N}$ with $r_j>\frac{7}{2}R$ and $Q_j\geqslant 2^{-n_2}$ (because $r_{j+1}/r_j=1-10Q_j$ and $n_2\geqslant n_1$). Consequently,

$$\mathbf{P}\big[\text{there exists } j : Q_j > c_5, r_j > \tfrac{7}{2}R \,\big|\, \mathcal{Z}_0^\sigma, \Phi_{4R}, \Theta_R\big] \geqslant 1-\tfrac{1}{3}(1-p) \tag{3.40}$$

holds for some $c_5=c_5(p, \bar{\Lambda})>0$. Note that this almost achieves our goal of proving (3.35). The difference between (3.40) and (3.35) is that in the latter the radius r at which $Q(\Phi_r)$ is bounded from below is variable.

Let $\mathcal{A}$ be the event that there is a $j\in\mathbb{N}$ with $Q_j>c_5$ and $r_j>\frac{7}{2}R$, and on $\mathcal{A}$, let j_0 denote the first such j. We have from (3.39) that

$$\mathbf{P}[\mathcal{Z}_0^\sigma \mid \mathcal{A}, \Phi_{r_{j_0}}, \Theta_R] \geqslant \frac{c_2 c_5^{-\log_2 c_0}}{\log R}. \tag{3.41}$$

Fix some $s>0$ small. We now argue that

$$\mathbf{P}[\mathcal{Z}_0^\sigma \mid \mathcal{A}, \Phi_{r_{j_0}}, \Theta_R, |x^{3R}-y^{3R}|<sR] \leqslant \frac{c_7 s^{c_6}}{\log R} \tag{3.42}$$

for some positive constants c_7 and c_6 depending only on $\bar{\Lambda}$. The argument is similar to the one given in the proof of the case $k=3$ in (3.38). Let z be the midpoint of the segment $[x^{3R}, y^{3R}]$. We construct a barrier as a perturbation of the connected component of $\partial B(z, 2sR)\setminus\partial D(\mathrm{ext}_{3R}\gamma)$ that intersects $\mathfrak{B}_{3R}$. If that barrier is hit by the extension of $\mathrm{ext}_{3R}\gamma$ (which happens with probability bounded away from 1), then we condition on the extension up to that barrier, and construct another barrier at radius $4sR$, instead. We continue in this manner, constructing barriers at radii $2^n sR$ up to the least n such that $2^n s>\frac{1}{1000}$, say. Because the probability of avoiding the nth barrier given that the $(n-1)$th barrier has been reached is bounded away from 1, we find that $\mathbf{P}[\gamma\supset\beta_R|\mathcal{A}, \Phi_{r_{j_0}}, \Theta_R, |x^{3R}-y^{3R}|<sR]$ is bounded by a constant times some positive power of s. The estimate (3.42) follows by considering the behavior of S.

Suppose now that the random walk S after its first hit to $\mathfrak{B}_{r_{j_0}}$ but before its first hit to $\mathfrak{B}_{3R}$ gets within distance sR of $\mathrm{ext}_{3R}\gamma$. Then, by Lemma 2.1, conditional on S up to the first time this has happened and on $\mathcal{A}$, $\Phi_{r_{j_0}}$, Θ_R and $\mathrm{ext}_{3R}\gamma$, the conditional probability for S hitting $\mathfrak{B}_{5R/2}$ before hitting $\partial D(\mathrm{ext}_{3R}\gamma)$ is at most $c_8 s^{\zeta_1}$, for some universal constant c_8. Thus, the conditional probability for $\mathcal{Z}_0^\sigma$ is at most $c_8 s^{\zeta_1}/\log R$. Combining this with (3.42), one gets

$$\mathbf{P}[\mathcal{Z}_0^\sigma \mid \mathcal{A}, \Phi_{r_{j_0}}, \Theta_R, Q(\Phi_{3R})<s] \leqslant \frac{c_7 s^{c_6}+c_8 s^{\zeta_1}}{\log R}.$$

Comparison with (3.41) now gives

$$\begin{aligned}\mathbf{P}[Q(\Phi_{3R})<s \mid \mathcal{Z}_0^\sigma, \mathcal{A}, \Phi_{r_{j_0}}, \Theta_R] &= \frac{\mathbf{P}[Q(\Phi_{3R})<s, \mathcal{Z}_0^\sigma \mid \mathcal{A}, \Phi_{r_{j_0}}, \Theta_R]}{\mathbf{P}[\mathcal{Z}_0^\sigma \mid \mathcal{A}, \Phi_{r_{j_0}}, \Theta_R]} \\ &\leqslant \frac{\mathbf{P}[\mathcal{Z}_0^\sigma \mid \mathcal{A}, \Phi_{r_{j_0}}, \Theta_R, Q(\Phi_{3R})<s]}{\mathbf{P}[\mathcal{Z}_0^\sigma \mid \mathcal{A}, \Phi_{r_{j_0}}, \Theta_R]} \leqslant \frac{c_7 s^{c_6}+c_8 s^{\zeta_1}}{c_2 c_5^{-\log_2 c_0}}.\end{aligned}$$

Thus, we obtain, for all s sufficiently small,

$$\mathbf{P}[Q(\Phi_{3R})<s \mid \mathcal{Z}_0^\sigma, \mathcal{A}, \Phi_{4R}, \Theta_R] \leqslant \tfrac{1}{3}(1-p).$$

Taking (3.40) into account, this gives (3.35), and completes the proof of the simplified case.

The argument in the general case proceeds as follows. We define inductively two sequences r_j and $\hat{r}_j$, starting with $r_0=4R$ and $\hat{r}_0=R$. At each step j, we set

$$Q_j := Q(\Phi_{r_j})\wedge Q(\Theta_{\hat{r}_j})\wedge 1_{\{\mathbf{P}[\mathcal{Z}_0^\sigma|\Phi_{r_j},\Theta_{\hat{r}_j}]>0\}}$$

and take $r_{j+1}=(1-10Q_j)r_j$ and $\hat{r}_{j+1}=(1+10Q_j)\hat{r}_j$. The proof proceeds essentially as above. The straightforward details are left to the reader. □

COROLLARY 3.16. *There exists a constant* $c=c(\bar{\Lambda})>0$ *such that the following estimate holds. Let* $(D',\partial'_+,\partial'_-,h'_\partial,h',\gamma',v'_0)$ *satisfy the same assumptions as we have for* $(D,\partial_+,\partial_-,h_\partial,h,\gamma,v_0)$. *Let* $R>c$, *and assume that* $\mathfrak{B}_{6R}\subset D'\cap D$ *and* $v_0,v'_0\notin\mathfrak{B}_{6R}$. *Let* Θ'_R, Φ'_{4R} *and* $\mathcal{Z}_0^{\sigma}{}'$ *be the objects corresponding to* Θ_R, Φ_{4R} *and* $\mathcal{Z}_0^\sigma$ *for the system in* D' (*with the same* σ, *that is*, $\sigma'=\sigma$). *Then*

$$\mathbf{P}[\Theta_R=\vartheta\mid\mathcal{Z}_0^\sigma,\Phi_{4R}]\leqslant c\mathbf{P}[\Theta'_R=\vartheta\mid\mathcal{Z}_0^{\sigma}{}',\Phi'_{4R}] \tag{3.43}$$

holds for all ϑ *and for all* Φ_{4R} *and* Φ'_{4R} *satisfying* $\mathbf{P}[\mathcal{Z}_0^\sigma|\Phi_{4R}]>0$ *and* $\mathbf{P}[\mathcal{Z}_0^{\sigma}{}'|\Phi'_{4R}]>0$, *respectively. Consequently*, *under the same assumptions*, *there exists a coupling of the conditional laws of* Θ_R *and* Θ'_R *such that*

$$\mathbf{P}[\Theta_R=\Theta'_R\mid\mathcal{Z}_0^\sigma,\mathcal{Z}_0^{\sigma}{}',\Phi_{4R},\Phi'_{4R}]\geqslant\frac{1}{c}.$$

Proof. It is enough to prove the first claim, since the latter claim immediately follows. Let $c'>0$ be the constant denoted as c in the separation lemma (Lemma 3.15) with $p=\frac{1}{2}$. Let $\mathcal{Q}$ denote the event $Q(\Phi_{3R})\wedge Q(\Theta_{2R})\geqslant c'$. Let X be the collection of all θ such that $\Theta_{2R}=\theta$ is possible and $Q(\theta)\geqslant c'$, and let X_ϑ be the collection of all $\theta\in X$ that are compatible with $\Theta_R=\vartheta$; that is, such that $\{\Theta_{2R}=\theta$ and $\Theta_R=\vartheta\}$ is possible. In the following, $f\approx g$ will mean that f/g is contained in $[1/c,c]$ for some constant $c=c(\bar{\Lambda})>0$. By Lemma 3.15 and the choice of p,

$$\mathbf{P}[\Theta_R=\vartheta\mid\mathcal{Z}_0^\sigma,\Phi_{4R}]\approx\mathbf{P}[\Theta_R=\vartheta\mid\mathcal{Q},\mathcal{Z}_0^\sigma,\Phi_{4R}]=\sum_{\theta\in X_\vartheta}\mathbf{P}[\Theta_{2R}=\theta\mid\mathcal{Q},\mathcal{Z}_0^\sigma,\Phi_{4R}]. \tag{3.44}$$

Now, if $Q(\Phi_{3R})>c'$ and $\theta\in X$, then

$$\begin{aligned}\mathbf{P}[\Theta_{2R}=\theta\mid\mathcal{Q},\mathcal{Z}_0^\sigma,\Phi_{3R}]&=\frac{\mathbf{P}[\Theta_{2R}=\theta,\mathcal{Q},\mathcal{Z}_0^\sigma\mid\Phi_{3R}]}{\mathbf{P}[\mathcal{Z}_0^\sigma,\mathcal{Q}\mid\Phi_{3R}]}\\&=\frac{\mathbf{P}[\Theta_{2R}=\theta,\mathcal{Z}_0^\sigma\mid\Phi_{3R}]}{\mathbf{P}[\mathcal{Z}_0^\sigma,\mathcal{Q}\mid\Phi_{3R}]}\\&=\frac{\mathbf{P}[\Theta_{2R}=\theta\mid\Phi_{3R}]\mathbf{P}[\mathcal{Z}_0^\sigma\mid\Theta_{2R}=\theta,\Phi_{3R}]}{\mathbf{P}[\mathcal{Z}_0^\sigma,\mathcal{Q}\mid\Phi_{3R}]}.\end{aligned}$$

We apply Lemma 3.13 to the first factor in the numerator and Lemma 3.14 to the second factor, and get

$$\mathbf{P}[\Theta_{2R}=\theta\mid\mathcal{Q},\mathcal{Z}_0^\sigma,\Phi_{3R}]\approx\frac{p_{2R}(\theta)/\log R}{\mathbf{P}[\mathcal{Z}_0^\sigma,\mathcal{Q}\mid\Phi_{3R}]}.$$

The sum of the left-hand side over all $\theta\in X$ is 1. Consequently,

$$\mathbf{P}[\Theta_{2R}=\theta_0\mid\mathcal{Q},\mathcal{Z}_0^\sigma,\Phi_{3R}]\approx\frac{p_{2R}(\theta_0)}{\sum_{\theta\in X}p_{2R}(\theta)}.$$

By taking expectation conditioned on $\mathcal{Q}$, $\mathcal{Z}_0^\sigma$ and Φ_{4R}, it follows that the same relation holds when we replace Φ_{3R} by Φ_{4R}. We now sum over $\theta_0 \in X_\vartheta$ and invoke (3.44), to obtain

$$\mathbf{P}[\Theta_R = \vartheta \mid \mathcal{Z}_0^\sigma, \Phi_{4R}] \approx \frac{\sum_{\theta \in X_\vartheta} p_{2R}(\theta)}{\sum_{\theta \in X} p_{2R}(\theta)}.$$

This implies (3.43), and completes the proof. □

Our intermediate goal to show that the dependence between the local behavior near S_τ and the global behavior far away is now accomplished. Roughly, the next objective will be to show that it is unlikely that γ contains an arc with a very large diameter whose endpoints are both relatively close to S_τ.

For $R > r > 0$, let $\mathcal{J} = \mathcal{J}(r, R)$ denote the event that there are more than two disjoint arcs of γ connecting $\mathfrak{B}_r$ and $\partial \mathfrak{B}_R$ or that S exits $\mathfrak{B}_R$ between the time it first hits $\mathfrak{B}_r$ and τ_0. Set $\mathcal{Z}_0 := \bigcup_{\sigma=0}^5 \mathcal{Z}_0^\sigma$. Our next objective is to show that conditioned on $\mathcal{Z}_0^\sigma$ or $\mathcal{Z}_0$, $\mathcal{J}(r, R)$ is unlikely if $R \gg r > 0$. More precisely, the claim is as follows.

Lemma 3.17. *Assume* (h), (D) *and* (S). *For every* $p > 0$ *there is some* $a = a(p, \bar{\Lambda}) > 10$ *such that if* $r > 1$, $R > ar$, $v_0 \notin \mathfrak{B}_{4R} \subset D$ *and* $\mathbf{P}[\mathcal{Z}_0^\sigma \mid \Phi_R, \Theta_r] > 0$, *then*

$$\mathbf{P}[\mathcal{J}(r, R) \mid \Phi_R, \Theta_r, \mathcal{Z}_0^\sigma] < p$$

and also

$$\mathbf{P}[\mathcal{J}(r, R) \mid \Phi_R, \mathcal{Z}_0] < p.$$

One may first think that this can be proved by repeating the argument in the proof of Lemma 3.14. The difficulty in carrying out this idea is that the sets A^a and A^b described in the proof of Lemma 3.14 may extend beyond $\mathfrak{B}_{sR'}$ for large s (where R' is as in that lemma) if there are more than two disjoint arcs of $\operatorname{ext}_{R'} \gamma$ connecting $\partial \mathfrak{B}_{R' + \varepsilon' R}$ and $\partial \mathfrak{B}_{sR'}$.

Since the proof of the lemma is a bit involved and somewhat indirect, we take a few moments to give an overview of the strategy. First, it is established that under the conditioning the simple random walk S is unlikely to backtrack to $\partial \mathfrak{B}_R$ after hitting $\mathfrak{B}_r$ and before τ_0. Next, we identify a pair of arcs α_1 and α_2 that are defined from Φ_{3r} each of which has one endpoint on $\operatorname{ext}_{3r} S$ and the other on $\operatorname{ext}_{3r} \gamma$. A barriers argument is then used to show that with high conditional probability $\gamma \setminus \operatorname{ext}_{3r} \gamma$ does not hit $\alpha_1 \cup \alpha_2$. In this case, we see that $\alpha_1 \cup \alpha_2$ has an alternative definition in terms of γ and S, which is in some sense more symmetric. Next we define another pair of arcs $\tilde{\alpha}_1 \cup \tilde{\alpha}_2$, which have a similar definition as $\alpha_1 \cup \alpha_2$, except that they are defined from $\Theta_{R/3}$. Again, the same barriers argument can be used to show that with high conditional probability these arcs are not visited by $\gamma \setminus \beta_{R/3}$. In this case, these arcs have a more symmetric definition, which leads us to conclude that with high conditional probability $\alpha_1 \cup \alpha_2 = \widetilde{\alpha}_1 \cup \widetilde{\alpha}_2$. This is then used to establish that the endpoints of these arcs on γ belong to $\mathrm{ext}_{3r}\gamma$ as well as $\beta_{R/3}$. Next, we prove that these two endpoints belong to different connected components of $\gamma \setminus e_\sigma^*$. This then implies that each of the two strands of $\mathrm{ext}_{3r}\gamma$ merges with $\beta_{R/3}$, which implies that there are no more than two disjoint crossings between $\partial\mathfrak{B}_R$ and $\mathfrak{B}_{3r}$ in γ.

Proof. The second claimed inequality with p replaced by $6p$ follows from the first inequality and taking conditional expectation, since $\mathcal{Z}_0 = \bigcup_{\sigma=0}^5 \mathcal{Z}_0^\sigma$. Thus, we only need to prove the first inequality. By Lemma 3.15, there is a constant $c_0 = c_0(\bar{\Lambda}, p) > 0$ such that

$$\mathbf{P}[Q(\Phi_{3R/4}) \wedge Q(\Theta_{R/2}) \wedge Q(\Phi_{3r}) \wedge Q(\Theta_{2r}) \leqslant c_0 \mid \Phi_R, \Theta_r, \mathcal{Z}_0^\sigma] < \tfrac{1}{10} p. \tag{3.45}$$

Let r' be in the range $[\sqrt{rR}, \sqrt{rR}+1]$, chosen so that the circle $\partial B(0, r')$ does not contain any TG-vertices nor any TG*-vertices. Let S_* denote the part of the walk S from its first visit to q^{3r} until its last visit to $\hat{q}^r$ prior to τ_0. Conditional on Φ_{3r} and Θ_r, the probability that S_* exits $\mathfrak{B}_{r'/3}$ without hitting $\mathrm{ext}_{3r}\gamma$ decays to zero as $a \to \infty$ (by Lemma 2.1). On the event that this happens, let S_{**} be the initial segment of the walk S_* until it exits $\mathfrak{B}_{r'/3}$. By the proof of the upper bound in Lemma 3.14,

$$\mathbf{P}[\mathcal{Z}_0^\sigma \mid \Theta_{2r}, \Phi_{3r}, S_* \not\subset \mathfrak{B}_{r'/3}, S_{**}] \leqslant \frac{O(1)}{\log r}.$$

By the lower bound in that lemma, on the event $Q(\Phi_{3r}) \wedge Q(\Theta_{2r}) \geqslant c_0$ we have

$$O_{p,\bar{\Lambda}}(1) \mathbf{P}[\mathcal{Z}_0^\sigma \mid \Theta_{2r}, \Phi_{3r}] \geqslant \frac{1}{\log r}.$$

Consequently, a may be chosen sufficiently large so that

$$1_{\{Q(\Phi_{3r}) \wedge Q(\Theta_{2r}) \geqslant c_0\}} \mathbf{P}[S_* \not\subset \mathfrak{B}_{r'/3} \mid \Theta_{2r}, \Phi_{3r}, \mathcal{Z}_0^\sigma] < \tfrac{1}{10} p.$$

Hence, (3.45) implies that

$$\mathbf{P}[S_* \not\subset \mathfrak{B}_{r'/3} \mid \Theta_r, \Phi_R, \mathcal{Z}_0^\sigma] < \tfrac{1}{5} p. \tag{3.46}$$

Let S' be the path traced by $\mathrm{ext}_{3r} S$ from the last time in which $\mathrm{ext}_{3r} S$ was outside $\mathfrak{B}_{R/3}$ until its terminal point $q^{3r} \in \mathfrak{B}_{3r}$. Let $\widetilde{S}$ be the path traced by $\mathrm{int}_{R/3} \breve{S}$ from the last time in which $\mathrm{int}_{R/3} \breve{S}$ was inside $\mathfrak{B}_{3r}$ until its terminal point $\hat{q}^{R/3}$. Observe that S' is Φ_{3r}-measurable, $\widetilde{S}$ is $\Theta_{R/3}$-measurable, and when $S_* \subset \mathfrak{B}_{r'/3}$, we have $S' = \widetilde{S}$ as unoriented paths.

Observe that there is a connected component α of $\partial B(0,r')\setminus S'$ such that $S'\cup\alpha$ separates ∂D from $\mathfrak{B}_{3r}$. We fix such an α, and if there is more than one possible choice, we choose one in a way which depends only on S'.

On the event $Q(\Phi_{3r})>0$ each strand of $\mathrm{ext}_{3r}\gamma$ connects ∂D with $\mathfrak{B}_{3r}$, and hence $\mathrm{ext}_{3r}\gamma$ intersects α. Thus, there are precisely two connected components of $\alpha\cap D(\mathrm{ext}_{3r}\gamma)$ which have one endpoint in S' and the other in $\partial D(\mathrm{ext}_{3r}\gamma)$. Let α_1 and α_2 be these two arcs.

We now argue that

$$1_{\{Q(3r)\wedge Q(2r)\geqslant c_0\}}\mathbf{P}[\gamma\cap\alpha_1\neq\varnothing \mid \mathcal{Z}_0^\sigma,\Phi_{3r},\Theta_{2r}]<\tfrac{1}{10}p \tag{3.47}$$

if a is sufficiently large.

The basic idea of the proof of (3.47) is to construct a sequence of barriers separating $\mathfrak{B}_{3r}$ from α_1 in $D(\mathrm{ext}_{3r}\gamma)$ such that if γ hits a barrier in the sequence, the conditional probability that it will hit the next barrier is bounded away from 1.

Let D' be the connected component of $D(\mathrm{ext}_{3r}\gamma)\setminus\mathfrak{B}_{3r}$ that contains α_1. Note that D' is a simply connected domain. See Figure 3.10. Let z_{α_1} denote the endpoint of α_1 on $\partial D'$ and let ξ_1 and ξ_2 be the two connected components of $\partial D'\setminus(\{z_{\alpha_1}\}\cup\overline{\mathfrak{B}}_{3r})$. (Both have z_{α_1} as an endpoint and the other endpoint in $\partial\mathfrak{B}_{3r}$.) Note that any path in $D'\setminus\alpha_1$ connecting ξ_1 and ξ_2 separates α_1 from $\partial\mathfrak{B}_{3r}$ in $D(\mathrm{ext}_{3r}\gamma)$. For each $\varrho\in(3r+3,r'-3)$ let $A(\varrho)$ denote the connected component of $D'\setminus\overline{B(0,\varrho)}$ that contains α_1, and let $\alpha(\varrho)$ denote the connected component of $\partial A(\varrho)\cap\partial B(0,\varrho)$ that separates α_1 from $\mathfrak{B}_{3r}$ in D'. Observe that $\alpha(\varrho)$ has one endpoint on ξ_1 and the other on ξ_2. If $3r+3<\varrho<\varrho'<r'-3$, then $A(\varrho)\supset A(\varrho')$ and therefore $\alpha(\varrho')$ separates $\alpha(\varrho)$ from α_1 in $D(\mathrm{ext}_{3r}\gamma)$ and $\alpha(\varrho)$ separates $\alpha(\varrho')$ from $\mathfrak{B}_{3r}$. When $3r+3<\varrho<\varrho'<r'-3$, let $A(\varrho,\varrho')$ denote the connected component of $D'\setminus(\alpha(\varrho)\cup\alpha(\varrho'))$ whose boundary contains $\alpha(\varrho)\cup\alpha(\varrho')$.

For $n\in\mathbb{N}$, let $\varrho_n:=2^n(3r+3)$, and let N be the largest n such that $8\varrho_n<r'$. Fix some $n\in\{1,...,N\}$ and some small $\delta>0$ ($\delta=\frac{1}{100}$ should do). Set $A_n^\delta:=A((1-\delta)\varrho_n,(1+\delta)\varrho_n)$. By continuity, A_n^δ contains points w such that $\mathrm{dist}(w,\xi_1)=\mathrm{dist}(w,\xi_2)$. Set

$$\eta(n):=\min\bigl\{\mathrm{dist}(w,\xi_1):w\in\overline{A_n^\delta},\mathrm{dist}(w,\xi_1)=\mathrm{dist}(w,\xi_2)\bigr\}.$$

First, assume that $\eta(n)>\delta\varrho_n$. In that case, a barrier Υ_n is defined as follows. By continuity, there is a subarc Υ' of $\alpha(\varrho_n)\subset A_n^\delta$ with endpoints z_1 and z_2 such that $\mathrm{dist}(z_j,\xi_j)=\frac{1}{10}\delta\varrho_n$, $j=1,2$, and $\mathrm{dist}(\Upsilon',\partial D')=\frac{1}{10}\delta\varrho_n$. Let z_j' be a point in ξ_j at distance $\frac{1}{10}\delta\varrho_n$ from z_j, $j=1,2$. Then we take Υ_n as the union of Υ' with the two line segments $[z_1,z_1']$ and $[z_2,z_2']$. Recall the definition of $\mathrm{dist}(\cdot,\cdot;\cdot)$ from (3.30), and note that $\mathrm{dist}(z_j',\xi_{3-j};D')>\frac{1}{10}\delta\varrho_n$, $j=1,2$, for otherwise, by continuity again, there would be a

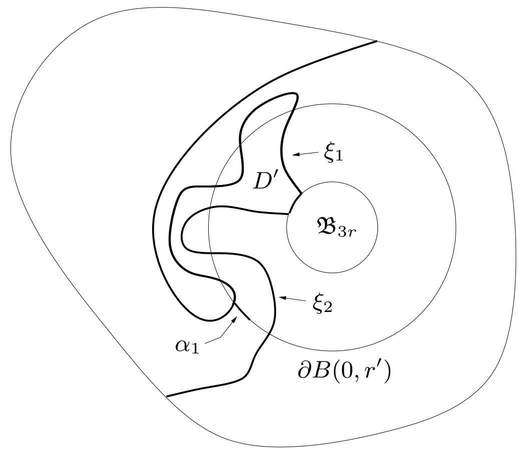

Figure 3.10. The domain D'.

point $w\in\overline{D}'$ satisfying $\operatorname{dist}(w,z_j';D')\leqslant\frac{1}{10}\delta\varrho_n$ (and therefore $w\in\overline{A_n^\delta}$) that is at equal distance from ξ_1 and from ξ_2, which would contradict our assumption $\eta(n)>\delta\varrho_n$. It easily follows that in this case Υ_n is a $\left(\frac{1}{20}\delta,2\varrho_n\right)$-barrier.

We now assume that $\eta(n)\leqslant\delta\varrho_n$. Let $w_1\in\overline{A_n^\delta}$ be a point satisfying

$$\operatorname{dist}(w_1,\xi_1)=\operatorname{dist}(w_1,\xi_2)\leqslant\delta\varrho_n.$$

For $j=1,2$, let $p_j\in\xi_j$ be a point satisfying $|w_1-p_j|=\operatorname{dist}(w_1,\partial D')$. If

$$\operatorname{dist}(p_j,\xi_{3-j};D')\geqslant\delta^2\varrho_n\quad\text{for } j=1,2,$$

then we may take as our barrier the union $[p_1,w_1]\cup[w_1,p_2]$. This will be a $\left(\frac{1}{4}\delta,2\delta\varrho_n\right)$-barrier. Otherwise, fix a point w_2 satisfying

$$\operatorname{dist}(w_2,p_j;D')\leqslant\delta^2\varrho_n\quad\text{and}\quad\operatorname{dist}(w_2,\xi_1)=\operatorname{dist}(w_2,\xi_2)\leqslant\delta^2\varrho_n$$

and consider the above construction with w_2 in place of w_1. It may happen that the construction succeeds now, and we construct a $\left(\frac{1}{4}\delta,2\delta^2\varrho_n\right)$-barrier. Otherwise, we find a point $w_3\in D'$ satisfying $\operatorname{dist}(w_3,w_1;D')\leqslant(\delta+\delta^2+\delta^3)\varrho_n$ such that

$$\operatorname{dist}(w_3,\xi_1)=\operatorname{dist}(w_3,\xi_2)\leqslant\delta^3\varrho_n.$$

We continue this procedure until some $\left(\frac{1}{4}\delta,2\delta^m\varrho_n\right)$-barrier is obtained. The procedure must terminate successfully at some finite m, for otherwise the points w_m would converge to some point in $\xi_1\cap\xi_2$ within distance $2\delta\varrho_n$ from $\overline{A_n^\delta}$, which is clearly impossible. Note that the barrier Υ_n thus constructed is contained in $A_n^{2\delta}$. Thus, when $1\leqslant n'<n\leqslant N$, $n, n'\in\mathbb{N}$, we have $\operatorname{dist}(\Upsilon_n, \Upsilon_{n'}; D')>\frac{1}{4}\varrho_n$ and Υ_n separates α_1 from $\Upsilon_{n'}$ in D'.

Suppose $n\in\{1, ..., N-1\}$. Note that (contrary to what appears in Figure 3.10, which does not show the scale of the lattice) the endpoints of Υ_n are not on γ, since $\xi_1\cup\xi_2$ are disjoint from γ, by construction. On the event $\gamma\cap\Upsilon_n\neq\varnothing$, let γ_1 and γ_2 be the two arcs of γ extending from the endpoints of ∂_+ to the first encounter with Υ_n. Now we apply Theorem 3.11 with $Y=\{\Upsilon_{n+1}\}$. Our careful construction above ensures that Υ_{n+1} is an $(\varepsilon, \operatorname{diam}\Upsilon_{n+1})$-barrier for some universal constant $\varepsilon>0$. Note that ξ_1 and ξ_2 contain vertices on which h takes the same sign. We conclude from the theorem that

$$\mathbf{P}[\gamma\cap\Upsilon_{n+1}\neq\varnothing\mid\gamma_1, \gamma_2, \Phi_{3r}, \Theta_{2r}]<1-c_1$$

for some $c_1=c_1(\bar\Lambda)>0$. The above implies that

$$\mathbf{P}[\gamma\cap\Upsilon_{n+1}\neq\varnothing\mid\gamma\cap\Upsilon_n\neq\varnothing, \Phi_{3r}, \Theta_{2r}]<1-c_1,$$

which gives

$$\mathbf{P}[\gamma\cap\alpha_1\neq\varnothing\mid\Phi_{3r}, \Theta_{2r}]\leqslant(1-c_1)^{N-1}. \tag{3.48}$$

Conditioned on γ, Φ_{3r} and Θ_{2r}, the probability of $\mathcal{Z}_0^\sigma$ is at most $O(1)/\log r$, by the proof of the upper bound in Lemma 3.14. Thus,

$$\mathbf{P}[\gamma\cap\alpha_1\neq\varnothing, \mathcal{Z}_0^\sigma\mid\Phi_{3r}, \Theta_{2r}]\leqslant\frac{O(1)(1-c_1)^{N-1}}{\log r}.$$

On the other hand, the lower bound tells us that on the event $\{Q(3r)\wedge Q(2r)\geqslant c_0\}$, we have $O_{\bar\Lambda,p}(1)\mathbf{P}[\mathcal{Z}_0^\sigma|\Phi_{3r}, \Theta_{2r}]\geqslant 1/\log r$. Thus, (3.47) follows.

Clearly, (3.47) also holds for α_2. On $\mathcal{Z}_0^\sigma$ let α_1^* and α_2^* be the two connected components of $\alpha\cap D(\gamma)$ that have one endpoint in S' and the other in $\partial D(\gamma)$. Note that when $\mathcal{Z}_0^\sigma$ holds and $\gamma\cap(\alpha_1\cup\alpha_2)=\varnothing$, we have $\alpha_1^*\cup\alpha_2^*=\alpha_1\cup\alpha_2$. Thus, (3.47) for α_1 and for α_2 together with (3.45) now gives

$$\mathbf{P}[\alpha_1^*\cup\alpha_2^*\neq\alpha_1\cup\alpha_2\mid\Phi_R, \Theta_r, \mathcal{Z}_0^\sigma]<\tfrac{3}{10}p. \tag{3.49}$$

We now follow an analog of the above argument with the roles of inside and outside switched. Observe that there is a connected component $\widetilde\alpha$ of $\partial B(0, r')\setminus\widetilde S$ such that $\widetilde S\cup\widetilde\alpha$ separates ∂D from $\mathfrak{B}_{3r}$. We fix such an $\widetilde\alpha$, and if there is more than one possible choice, we choose it in the same way in which α was chosen from S'; that is, we make sure that $\alpha=\widetilde\alpha$ if $S'=\widetilde S$ (as unoriented paths). The point is that although α is Φ_{3r}-measurable and $\widetilde\alpha$ is $\Theta_{R/3}$-measurable, we have $\alpha=\widetilde\alpha$ if $S'=\widetilde S$.

On the event $Q(\Theta_{R/3})>0$, let $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ be the two connected components of $\widetilde{\alpha}\cap D(\beta_{R/3})$ that have one endpoint on $\widetilde{S}$ and the other on $\partial D(\beta_{R/3})$. On the event $\mathcal{Z}_0^\sigma$, let $\widetilde{\alpha}_1^*$ and $\widetilde{\alpha}_2^*$ be the two connected components of $\widetilde{\alpha}\cap D(\gamma)$ that have one endpoint on $\widetilde{S}$ and the other on $\partial D(\gamma)$. Essentially the same proof which gave (3.49) now gives

$$\mathbf{P}[\widetilde{\alpha}_1^*\cup\widetilde{\alpha}_2^*\neq\widetilde{\alpha}_1\cup\widetilde{\alpha}_2 \mid \Phi_R,\Theta_r,\mathcal{Z}_0^\sigma] < \tfrac{3}{10}p. \tag{3.50}$$

But observe that when $\mathcal{Z}_0^\sigma$ and $S'=\widetilde{S}$ hold, we clearly have $\widetilde{\alpha}_1^*\cup\widetilde{\alpha}_2^*=\alpha_1^*\cup\alpha_2^*$. Also recall that $S'=\widetilde{S}$ when $S_*\subset\mathfrak{B}_{r'/3}$. Thus, from (3.46), (3.49) and (3.50), we get

$$\mathbf{P}[\widetilde{\alpha}_1\cup\widetilde{\alpha}_2\neq\alpha_1\cup\alpha_2 \text{ or } S_*\not\subset\mathfrak{B}_{r'/3} \mid \Phi_R,\Theta_r,\mathcal{Z}_0^\sigma] < \tfrac{4}{5}p.$$

Assume that $\mathcal{Z}_0^\sigma$, $\widetilde{\alpha}_1\cup\widetilde{\alpha}_2=\alpha_1^*\cup\alpha_2^*=\alpha_1\cup\alpha_2$ and $S_*\subset\mathfrak{B}_{r'/3}$ hold. It remains to show that in this case the path γ has no more than two disjoint arcs connecting $\mathfrak{B}_r$ and $\partial\mathfrak{B}_R$. Recall that $\widetilde{\alpha}_1$ has an endpoint on $\partial D(\beta_{R/3})$. This endpoint is on a TG-triangle containing a TG*-vertex $v_1\in\beta_{R/3}$. Similarly, there is a TG*-vertex $v_2\in\beta_{R/3}$ for which the TG-triangle containing it has an endpoint of $\widetilde{\alpha}_2$. From $\widetilde{\alpha}_1\cup\widetilde{\alpha}_2=\alpha_1\cup\alpha_2$, we conclude that $v_1,v_2\in\operatorname{ext}_{3r}\gamma$ as well.

Shortly, we will prove that v_1 and v_2 are in separate connected components of $\beta_{R/3}\setminus e_\sigma^*$. This implies that each connected component of $\beta_{R/3}\setminus e_\sigma^*$ intersects $\operatorname{ext}_{3r}\gamma$. Since $\beta_{R/3}\cup\operatorname{ext}_{3r}\gamma\subset\gamma$, and γ is a simple path, it easily follows that $\gamma=\beta_{R/3}\cup\operatorname{ext}_{3r}\gamma$, which implies that there are at most two disjoint crossings in γ between $\mathfrak{B}_r$ and $\partial\mathfrak{B}_R$.

It remains to prove that v_1 and v_2 are in different connected components of $\beta_{R/3}\setminus e_\sigma^*$. This will be established using planar topology arguments. Let $\widehat{\alpha}$ consist of $\widetilde{\alpha}_1\cup\widetilde{\alpha}_2$, a simple path $\widetilde{S}_0\subset\widetilde{S}$ connecting them, and short line segments (contained in the TG-triangles containing v_1 and v_2) from the endpoints of $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ on $\partial D(\beta_{R/3})$ to v_1 and v_2. Then $\widehat{\alpha}$ is a simple path and only the endpoints of $\widehat{\alpha}$ are on γ. Let $\tilde{\beta}$ be the connected component of $\beta_{R/3}\setminus\{v_1,v_2\}$ with endpoints v_1 and v_2. Then $\tilde{\beta}\cup\widehat{\alpha}$ is a simple closed path and it suffices to show that $e_\sigma^*\subset\tilde{\beta}$. If $\tilde{\beta}\cup\widehat{\alpha}$ separates 0 from ∂D, then $\tilde{\beta}$ must contain e_σ^*, because each connected component of $\gamma\setminus\{e_\sigma^*\}$ connects ∂D to TG*-vertices adjacent to 0 and is disjoint from $\widehat{\alpha}\setminus\{v_1,v_2\}$.

Suppose that $\tilde{\beta}\cup\widehat{\alpha}$ does not separate 0 from ∂D. Recall that $\widetilde{\alpha}\cup\widetilde{S}$ separates ∂D from $\mathfrak{B}_{3r}$ and therefore from 0. Since $\widetilde{S}$ itself does not separate ∂D from 0, it follows that the winding number of $\widetilde{\alpha}\cup\widetilde{S}_0$ around 0 is ±1 (depending on orientation). Since $\tilde{\beta}\cup\widehat{\alpha}$ does not separate 0 from ∂D, its winding number around 0 is zero. If we remove from the union of the two paths $\widetilde{\alpha}\cup\widetilde{S}_0$ and $\tilde{\beta}\cup\widehat{\alpha}$ all the non-trivial arcs where they agree, we get a closed curve χ, which consists of $\tilde{\beta}$, a segment of $\widetilde{\alpha}$ and the two short connecting segments near v_1 and v_2, and χ has odd winding number around 0. Consequently, it separates 0 from ∂D. But observe that $\widetilde{S}$ is disjoint from χ. Moreover, since we are assuming $S_*\subset\mathfrak{B}_{r'/3}$, it follows that $\mathrm{int}_{R/3}\breve{S}$ is also disjoint from it. But this contradicts the fact that χ separates ∂D from 0, since $\mathrm{int}_{R/3}\breve{S}$ can be extended to a path disjoint from χ and connecting 0 and ∂D. Thus, the proof is now complete. □

3.6. Coupling and limit

In this subsection, we retain our previous assumptions (h), (D) and (S) about the system D, ∂_+, ∂_-, h_∂, v_0, h, γ. Moreover, we also consider another such system D', ∂'_+, ∂'_-, h'_∂, v'_0, h', γ', which is supposed to satisfy the same assumptions. In particular, $\|h'_\partial\|_\infty\leqslant\bar{\Lambda}$. Generally, we will use $'$ to denote objects related to the system in D'. For example, $\mathcal{J}'(r,R)$ will denote the event corresponding to $\mathcal{J}(r,R)$.

Definition 3.18. Fix $R>r>0$, and suppose that $\mathfrak{B}_R\subset D\cap D'$. Consider the intersection $\mathrm{ext}_r\gamma\cap\mathfrak{B}_R$ as a collection of oriented paths, oriented so as to have vertices in $V_+(\gamma)$ on the right. We say that Φ_r and Φ'_r *match* in $\mathfrak{B}_R$ if the set of vertices in $\mathfrak{B}_R$ visited by $\mathrm{ext}_r S$ is the same as the corresponding set for S' and $\mathrm{ext}_r\gamma\cap\mathfrak{B}_R=\mathrm{ext}_r\gamma'\cap\mathfrak{B}_R$ with all the orientations agreeing or with all the orientations reversed.

We now show that if the configurations match in a big annulus, then it is likely that the interfaces agree in the inner disk; more precisely, we have the following result.

LEMMA 3.19. *For every $\delta>0$, $r\geqslant 10$ and $R^*>r+3$ there is an $R=R(\delta,r,R^*,\bar{\Lambda})>R^*$ such that the following holds. Suppose that $v'_0,v_0\notin\mathfrak{B}_R\subset D\cap D'$, $\mathbf{P}[\neg\mathcal{J}(r,R^*),\mathcal{Z}_0\,|\,\Phi_r]>0$ and $\mathbf{P}[\neg\mathcal{J}'(r,R^*),\mathcal{Z}'_0\,|\,\Phi'_r]>0$. Assume that Φ_r and Φ'_r match in $\mathfrak{B}_R$. In particular, the endpoint q^r of $\mathrm{ext}_r S$ in $\mathfrak{B}_r$ is the same as that of $\mathrm{ext}_r S'$. Let ν be the law of $\gamma_*:=\gamma\setminus\mathrm{ext}_r\gamma$ (as an unoriented path) conditioned on $\mathcal{Z}_0$, Φ_r and $\neg\mathcal{J}(r,R^*)$, and let ν' be the law of $\gamma'_*:=\gamma'\setminus\mathrm{ext}_r\gamma'$ conditioned on $\mathcal{Z}'_0$, Φ'_r and $\neg\mathcal{J}'(r,R^*)$. Then $\|\nu-\nu'\|<\delta$.*

Here, $\|\nu-\nu'\|$ denotes the total variation norm $\sum_\vartheta|\nu[\gamma_*=\vartheta]-\nu'[\gamma'_*=\vartheta]|$.

Proof. Assume that the orientation of $\mathrm{ext}_r\gamma\cap\mathfrak{B}_R$ agrees with that of $\mathrm{ext}_r\gamma'\cap\mathfrak{B}_R$. This involves no loss of generality, since we may replace ∂'_+ with ∂'_-, replace h' by $-h'$, etc.

Since we are assuming that $\mathbf{P}[\neg\mathcal{J}(r,R^*),\mathcal{Z}_0\,|\,\Phi_r]>0$, there is a path $\vartheta\subset\mathfrak{B}_{R^*}$ such that $\mathbf{P}[\gamma_*=\vartheta,\neg\mathcal{J}(r,R^*),\mathcal{Z}_0\,|\,\Phi_r]>0$. Let $\widehat{\Gamma}$ be the collection of all such ϑ, and fix some $\vartheta\in\widehat{\Gamma}$. Obviously, the length of ϑ is $O(R^*)^2$. We start extending $\mathrm{ext}_r\gamma$ starting at one of the endpoints, say x^r, and consider the conditional probability that each successive step follows ϑ, given that the previous steps follow θ and given Φ_r. Each step is decided by the sign of h on a specific vertex v. When we condition on the values of h on the neighbors of v, the conditional law of $h(v)$ is a Gaussian with some constant positive variance. It follows from (3.2) that with high probability (conditioned on the success of the previous steps) the mean of this Gaussian random variable is unlikely to be large. Thus, the probability for either sign is bounded away from zero, which means that each step is successful with probability bounded away from zero. By (3.3) it is unlikely that $h(v)$ will be very close to zero. Proposition 3.3 therefore implies that if $R>R^*$ is very large, the probability for a successful one-step extension for γ' is almost the same as for γ. Thus, we conclude that for sufficiently large $R>R^*$,

$$(1-\delta)\mathbf{P}[\gamma'_*=\vartheta \mid \Phi'_r] \leqslant \mathbf{P}[\gamma_*=\vartheta \mid \Phi_r] \leqslant (1+\delta)\mathbf{P}[\gamma'_*=\vartheta \mid \Phi'_r]$$

holds for all $\vartheta\in\widehat{\Gamma}$. It is moreover clear that

$$\mathbf{P}[\mathcal{Z}_0, \neg\mathcal{J}(r,R^*) \mid \gamma_*=\vartheta, \Phi_r] = \mathbf{P}[\mathcal{Z}'_0, \neg\mathcal{J}'(r,R^*) \mid \gamma'_*=\vartheta, \Phi'_r],$$

because under $\neg\mathcal{J}(r,R^*)$ the random walk S cannot get close to any place where $\mathrm{ext}_r\gamma$ differs from $\mathrm{ext}_r\gamma'$ between the first visit to q^r and time τ_0. Thus,

$$1-\delta \leqslant \frac{\mathbf{P}[\gamma_*=\vartheta, \mathcal{Z}_0, \neg\mathcal{J}(r,R^*) \mid \Phi_r]}{\mathbf{P}[\gamma'_*=\vartheta, \mathcal{Z}'_0, \neg\mathcal{J}'(r,R^*) \mid \Phi'_r]} \leqslant 1+\delta.$$

The lemma follows (though perhaps δ needs to be readjusted). □

The next lemma shows that given Φ_R the events $\mathcal{Z}_0^\sigma$ have comparable probabilities for different σ.

LEMMA 3.20. *As usual, assume* (h), (D) *and* (S). *There is a constant* $c=c(\bar{\Lambda})\geqslant 1$ *such that for all* R *sufficiently large and every* $\sigma,\sigma'\in\{0,1,...,5\}$ *we have*

$$\mathbf{P}[\mathcal{Z}_0^\sigma \mid \Phi_R] \leqslant c\mathbf{P}[\mathcal{Z}_0^{\sigma'} \mid \Phi_R].$$

Proof. The statement is clear when $R=100$, because in that case if it is at all possible to extend Φ_R in such a way that $\mathcal{Z}_0^\sigma$ holds, then the probability that $\mathcal{Z}_0^{\sigma'}$ holds is bounded away from zero (by a function of $\bar{\Lambda}$). (We may choose the continuations of γ and S as we please, and as long as the continuations involve a bounded number of steps, the probability for these continuations are bounded away from zero, as in the proof of Lemma 3.19.) When $R>100$, we may just condition on the corresponding extension of Φ_R up to radius 100. □

We now come to one of the main results in this section—the existence of a limiting interface.

Theorem 3.21. (Limit existence) *There is a* (*unique*) *probability measure* μ_∞ *on the space of two-sided infinite simple* TG*-*paths* $\dot\gamma$ *which is the limit of the law of* γ (*unoriented*) *conditioned on* $\mathcal{Z}_0$ *and* Φ_R, *in the following sense. Assume* (h), (D) *and* (S). *For every finite set of* TG*-*edges* E_0 *and every* $\delta>0$ *there is an* $R_0=R_0(\delta,E_0,\bar\Lambda)$ *such that if* $R>R_0$, $v_0\notin\mathfrak{B}_R\subset D$ *and* $\mathbf{P}[\mathcal{Z}_0\,|\,\Phi_R]>0$, *then*

$$|\mathbf{P}[E_0\subset\gamma\,|\,\mathcal{Z}_0,\Phi_R]-\mu_\infty[E_0\subset\dot\gamma]|<\delta.$$

Proof. Clearly, it suffices to show that, for every $r>0$, if R is sufficiently large, $v_0,v_0'\notin\mathfrak{B}_R\subset D\cap D'$, $\mathbf{P}[\mathcal{Z}_0\,|\,\Phi_R]>0$ and $\mathbf{P}[\mathcal{Z}_0'\,|\,\Phi_R']>0$, then we may couple the conditioned laws of γ given Φ_R and $\mathcal{Z}_0$, and γ' given Φ_R' and $\mathcal{Z}_0'$ such that

$$\mathbf{P}[\gamma\setminus\mathrm{ext}_r\gamma=\gamma'\setminus\mathrm{ext}_r\gamma'\,|\,\Phi_R,\mathcal{Z}_0,\Phi_R',\mathcal{Z}_0']>1-\delta. \tag{3.51}$$

(Here, the equivalence is an equivalence of unoriented paths.) Let a_n be the constant a given by Lemma 3.17 when one takes $p=\delta_n:=\frac{1}{8}2^{-n}\delta$. We define a sequence of radii $r_0,r_1,...$ inductively, as follows. Let c_1 be the constant c given by Corollary 3.16, and set $r_0:=r\vee 10\vee c_1$. Given r_n, let $\hat r_n$ be the R promised by Lemma 3.19 when we take r_n for r, δ_n for δ and a_nr_n for R^*. Finally, set $r_{n+1}=4a_n\hat r_n$. Let c_2 be the constant promised by Lemma 3.20. We assume, with no loss of generality, that $\delta<1/4c_1$. Let $N\in\mathbb{N}$ be sufficiently large so that $(1-1/72c_1c_2^2)^{N-1}<\frac{1}{2}\delta$. We will prove (3.51) on the assumption that $R>6r_N$.

The construction of the coupling is as follows. First, we choose $\Phi_{r_{N-1}}$ and $\Phi'_{r_{N-1}}$ independently according to their conditional distribution given Φ_R, $\mathcal{Z}_0$, Φ_R' and $\mathcal{Z}_0'$. We proceed by reverse induction. Suppose that $n\in[1,N-1]\cap\mathbb{N}$ and that Φ_{r_n} and Φ'_{r_n} have been determined. If Φ_{r_n} and Φ'_{r_n} match inside $\mathfrak{B}_{\hat r_n}$, then we couple γ and γ' in such a way as to maximize the probability that $\gamma\setminus\mathrm{ext}_{r_n}\gamma=\gamma'\setminus\mathrm{ext}_{r_n}\gamma'$, subject to maintaining their correct conditional distributions given the choices previously made. If they do not match, then we couple $\Phi_{r_{n-1}}$ and $\Phi'_{r_{n-1}}$ in such a way as to maximize the probability that they match in $\mathfrak{B}_{\hat r_{n-1}}$, subject to their correct conditional distributions. If Φ_{r_0} and Φ'_{r_0} have been determined, but γ and γ' have not, then we couple γ and γ' arbitrarily, subject to their correct conditional distributions.

We claim that the coupling just described achieves the bound (3.51). Let $\mathcal{M}_n$ denote the event that Φ_{r_n} and Φ'_{r_n} match inside $\mathfrak{B}_{\hat r_n}$, where $n\in\{1,...,N-1\}$, and let $\mathcal{M}$ denote the union of these events $\mathcal{M}_n$. It follows from the choice of $\hat r_n$ that if $\mathcal{M}_n$ holds and n is the largest $n'\in\{1,...,N-1\}$ with that property, then there is a coupling of the appropriately conditioned laws of γ and γ' such that

$$\begin{aligned}\mathbf{P}[\gamma\setminus\mathrm{ext}_{r_n}\gamma\neq\gamma'\setminus\mathrm{ext}_{r_n}\gamma'\,|\,\Phi_{r_n},\Phi'_{r_n},\mathcal{Z}_0,\mathcal{Z}_0']&\\ \leqslant\delta_n+\mathbf{P}[\mathcal{J}(r_n,a_nr_n)\,|\,\Phi_{r_n},\mathcal{Z}_0']+\mathbf{P}[\mathcal{J}'(r_n,a_nr_n)\,|\,\Phi'_{r_n},\mathcal{Z}_0'],&\end{aligned}$$

and hence this also holds for our coupling. By taking conditional expectation and summing over n, we get

$$\begin{aligned}\mathbf{P}[\mathcal{M}, \gamma\setminus\mathrm{ext}_r\gamma \neq \gamma'\setminus\mathrm{ext}_r\gamma' \mid \Phi_R, \mathcal{Z}_0, \Phi'_R, \mathcal{Z}'_0] \\ \leqslant \frac{1}{8}\delta + \sum_{n=1}^{N-1} (\mathbf{P}[\mathcal{J}(r_n, a_n r_n) \mid \Phi_R, \mathcal{Z}_0] + \mathbf{P}[\mathcal{J}'(r_n, a_n r_n) \mid \Phi'_R, \mathcal{Z}'_0]) \leqslant \frac{3}{8}\delta.\end{aligned}$$

(where the last inequality follows by the choice of a_n).

Now fix some $n \in \{1, ..., N-2\}$, and suppose that none of the events $\mathcal{M}_{n'}$, $n'>n$, occurs. Fix some arbitrary $\sigma \in \{0, 1, ..., 5\}$. Conditional on $\Phi_{r_{n+1}}$ and $\mathcal{Z}_0$, by the choice of c_2, there is probability at least $1/6c_2$ that $\mathcal{Z}_0^\sigma$ holds, and the same is true for the system in D'. If we additionally condition on $\mathcal{Z}_0^\sigma$ and $\mathcal{Z}_0^{\sigma\prime}$, then, by the choice of c_1, there is a coupling of the appropriate conditioned laws of $\Theta_{r_{n+1}/4}$ and $\Theta'_{r_{n+1}/4}$ such that

$$\mathbf{P}[\Theta_{r_{n+1}/4} = \Theta'_{r_{n+1}/4} \mid \Phi_{r_{n+1}}, \Phi'_{r_{n+1}}, \mathcal{Z}_0^\sigma, \mathcal{Z}_0^{\sigma\prime}] \geqslant \frac{1}{c_1}.$$

But note that if $\Theta_{r_{n+1}/4} = \Theta'_{r_{n+1}/4}$ and $\neg\big(\mathcal{J}\big(\hat{r}_n, \frac{1}{4}r_{n+1}\big) \cup \mathcal{J}'\big(\hat{r}_n, \frac{1}{4}r_{n+1}\big)\big)$ both hold (as well as $\mathcal{Z}_0^\sigma \cap \mathcal{Z}_0^{\sigma\prime}$), then $\mathcal{M}_n$ holds as well. We may then consider a coupling of the two systems which first decides the two events $\mathcal{Z}_0^\sigma$ and $\mathcal{Z}_0^{\sigma\prime}$ independently, and if both hold (which happens with probability at least $1/(6c_2)^2$), then with conditional probability at least $1/c_1$ we also have $\Theta_{r_{n+1}/4} = \Theta'_{r_{n+1}/4}$. Thus, under this coupling,

$$\begin{aligned}(6c_2)^2 \mathbf{P}[\mathcal{M}_n \mid \Phi_{r_{n+1}}, \Phi'_{r_{n+1}}, \mathcal{Z}_0, \mathcal{Z}'_0] \\ &\geqslant c_1^{-1} - \mathbf{P}\big[\mathcal{J}\big(\hat{r}_n, \tfrac{1}{4}r_{n+1}\big) \cup \mathcal{J}'\big(\hat{r}_n, \tfrac{1}{4}r_{n+1}\big) \mid \Phi_{r_{n+1}}, \Phi'_{r_{n+1}}, \mathcal{Z}_0^\sigma, \mathcal{Z}_0^{\sigma\prime}\big] \\ &\geqslant c_1^{-1} - 2\delta_n \quad \text{(by the choices of } r_{n+1} \text{ and } a_n) \\ &\geqslant \tfrac{1}{2}c_1^{-1} \quad \text{(by our assumption } 4c_1\delta < 1).\end{aligned}$$

This must hold for our coupling as well. Consequently, induction gives

$$\mathbf{P}[\neg\mathcal{M}] \leqslant \left(1 - \frac{1}{72c_2^2 c_1}\right)^{N-1},$$

which is less than $\frac{1}{2}\delta$ by the choice of N. Therefore (3.51) follows, and the proof is complete. □

3.7. Boundary values of the interface

Consider the random path $\dot{\gamma}$ whose law is the measure μ_∞ provided by Theorem 3.21. We orient $\dot{\gamma}$ so that the edges e_σ^*, $\sigma = 0, 1, ..., 5$, that are in $\dot{\gamma}$ are oriented clockwise around the hexagon $\bigcup_{\sigma=0}^{5} e_\sigma^*$ centered at 0. Let U_+ denote the set of TG-vertices adjacent to $\dot\gamma$ on its right-hand side, and let U_- denote the set of vertices adjacent to $\dot\gamma$ on its left-hand side. Using the heights interface continuity (Proposition 3.3), it is clear that given $\dot\gamma$ we may define the DGFF $\dot h$ on all of TG conditioned to be positive on U_+ and negative on U_-, as a limit of an appropriately conditioned DGFF on bounded domains. Moreover, many properties of the DGFF on bounded domains easily transfer to $\dot h$. In particular, (3.2) applies, to give $\mathbf{E}[|\dot h(0)|\,|\,\dot\gamma]=O(1)$. Set

$$\lambda := \mathbf{E}[\dot h(0)]. \tag{3.52}$$

Clearly, $0<\lambda<\infty$.

Recall that τ is the first time t such that $S_t\in\partial D(\gamma)$ and recall the notation $\operatorname{dist}(\cdot,\cdot;\cdot)$ from (3.30). In this subsection we will show that in the limit as $\operatorname{dist}(v_0,\partial D)\to\infty$ we have

$$\mathbf{E}[(\pm h(S_\tau)-\lambda)1_{\{S_\tau\in V_\pm(\gamma)\}}\,|\,\gamma]\to 0$$

in probability, under the assumption that

$$h_\partial(\partial_+\cap V_\partial)\subset[-\Lambda_0,\bar\Lambda] \quad\text{and}\quad h_\partial(\partial_-\cap V_\partial)\subset[-\bar\Lambda,\Lambda_0], \tag{∂}$$

where $\Lambda_0=\Lambda_0(\bar\Lambda)>0$ is the constant given by Lemma 3.9. The importance of the assumption (∂) is that, by Remark 3.12, it enables the application of Theorem 3.11 to barriers with endpoints on ∂D, provided that the barriers in Y_+ do not have endpoints in ∂_- and those in Y_- do not have endpoints in ∂_+. This allows us to prove the following result.

THEOREM 3.22. *Assume* (h), (D), (S) *and* (∂). *There are positive constants* $\zeta_2=\zeta_2(\bar\Lambda)$ *and* $c=c(\bar\Lambda)$ *such that the following holds true. Let* z *be any point on* ∂_+. *Then for every* $r>1$,

$$\mathbf{P}[\operatorname{dist}(z,\gamma;D)<r]\leqslant c\left(\frac{r}{\operatorname{dist}(z,\partial_-;D)}\right)^{\zeta_2}.$$

Proof. Let β_1 and β_2 be the two components of $\partial_+\setminus\{z\}$. Let $R=\operatorname{dist}(z,\partial_-;D)$. For each $\varrho\in(0,R)$, let $A(\varrho)$ denote the connected component of $B(z,\varrho)\cap D$ that has z in its boundary, and let $\alpha(\varrho)$ denote the connected component of $\partial A(\varrho)\setminus\partial D$ that separates z from ∂_- in D. Using this construction, the proof proceeds as in the proof (3.48), except that the barriers start from the outside and get closer to z, and we appeal to Remark 3.12 instead of the barriers theorem. We leave it to the reader to verify that the proof carries over with no other significant modifications. □

Our next lemma shows that it is unlikely that S_τ is adjacent to γ and is near ∂D.

LEMMA 3.23. *Assume* (h), (D), (S) *and* (∂). *For all* $p>0$ *there is some* $\delta=\delta(p,\bar{\Lambda})>0$ *such that*

$$\mathbf{P}[0<\operatorname{dist}(S_\tau,\partial D)<\delta\operatorname{dist}(v_0,\partial D)]<p.$$

Proof. Let τ_D be the first t such that $S_t\in\partial D$, and let

$$M=M(S,\gamma):=\{S_t: S_t\in\partial D(\gamma) \text{ and } 0\leqslant t<\tau_D\}.$$

We will prove the stronger statement

$$\mathbf{P}[\operatorname{dist}(M,\partial D)<\delta\operatorname{dist}(v_0,\partial D)]<p. \tag{3.53}$$

(By convention $\operatorname{dist}(\varnothing,\partial D)=\infty$.) Fix $r>0$ and set $R=\operatorname{dist}(v_0,\partial D)$. Conditioned on $\operatorname{dist}(M,\partial D)<r$, we have $\operatorname{dist}(S_{\tau_D},\gamma;D)<4r$ with probability bounded away from zero, since the random walk started at any $v\in M$ such that $\operatorname{dist}(v,\partial D)<r$ has probability bounded away from 0 to surround the closest point to v on ∂D (and therefore hit ∂D) before exiting the ball of radius $2r$ about that point. It therefore suffices to prove that

$$\mathbf{P}[\operatorname{dist}(S_{\tau_D},\gamma;D)<\delta R]<p \tag{3.54}$$

for $\delta=\delta(p,\bar{\Lambda})>0$. Let $\mathcal{A}_+=\mathcal{A}_+(\delta)$ denote the event $\operatorname{dist}(S_{\tau_D},\partial_-;D)>\delta^{1/2}R$, and similarly define $\mathcal{A}_-$ with ∂_- replaced by ∂_+. By conditioning on S_{τ_D}, Theorem 3.22 shows that

$$\mathbf{P}[\mathcal{A}_+,\operatorname{dist}(S_{\tau_D},\gamma;D)<\delta R]<\tfrac{1}{3}p$$

for an appropriate choice of δ. A symmetric argument applies on $\mathcal{A}_-$. Consequently, it is enough to prove that $\mathbf{P}[\neg(\mathcal{A}_+\cup\mathcal{A}_-)]<\frac{1}{3}p$ for an appropriate choice of δ.

Fix $r_0:=\delta^{1/2}R$, let L_{r_0} denote the set of points that lie on some path in $\overline{D}$ of diameter at most r_0 connecting ∂_+ and ∂_-, and let D_* be the connected component of v_0 in $D\setminus L_{r_0}$. (See Figure 3.11.) We now prove that

$$\partial D_*\setminus\partial D \text{ is contained in the union of two balls of radius } 2r_0. \tag{3.55}$$

Every path connecting ∂_+ and ∂_- in $\overline{D}$ must separate v_0 from x_∂ or from y_∂ in $\overline{D}$ (because, by Jordan's theorem, it separates x_∂ from y_∂ in $\overline{D}$). Let Γ_1 (respectively, Γ_2) denote the collection of paths in $\overline{D}$ of diameter at most r_0 that connect ∂_+ and ∂_- and separate x_∂ (respectively, y_∂) from v_0. Then $\partial D_*\setminus\partial D$ is contained in the union of the set of points belonging to a path in Γ_1 and the set of points belonging to a path in Γ_2. Suppose that α and α' are two paths in Γ_1, both of which intersect ∂D_*. Let β be a path connecting x_∂ with $\alpha\cup\alpha'$ in $\overline{D}$, which is disjoint from $\alpha\cup\alpha'$, except for its endpoint. If $\beta\cap\alpha\neq\varnothing$,

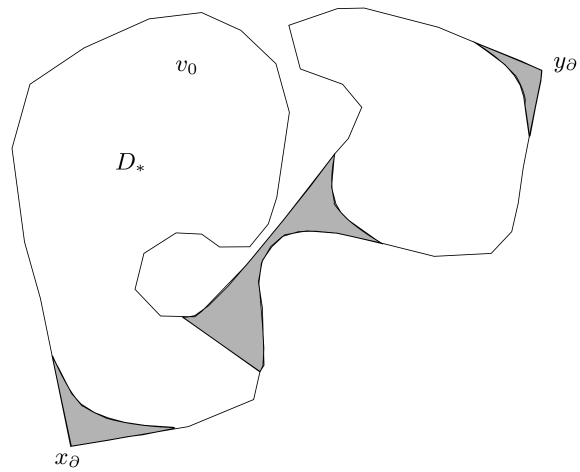

Figure 3.11. The set L_{r_0} (shaded) and D_*.

then one can connect x_∂ to v_0 in $\beta\cup\alpha\cup D_*$, and therefore $\alpha\cap\alpha'\neq\varnothing$ (since α' separates v_0 from x_∂ in $\overline{D}$). Similar reasoning applies if $\beta\cap\alpha'\neq\varnothing$. It follows that any two paths in Γ_1 that intersect ∂D_* must intersect each other, and hence the collection of all such paths is covered by the ball of radius $2r_0$ centered at any point on any such path. Since a similar argument applies to Γ_2, (3.55) follows.

By (3.55) and Lemma 2.1,

$$\mathbf{P}[\text{there exists } t\leqslant\tau_D : S_t\in L_{r_0}] < \tfrac{1}{3}p$$

for all sufficiently small $\delta>0$. Thus $\mathbf{P}[\neg(\mathcal{A}_+\cup\mathcal{A}_-)]=\mathbf{P}[S_{\tau_D}\in L_{r_0}]<\frac{1}{3}p$, and the proof is complete. □

Next, we show that S_τ is unlikely to be close to v_0 by proving the same for γ.

Lemma 3.24. *Assume* (h) *and* (D). *There are constants* $c>0$ *and* $\zeta_3>0$, *both depending only on* $\bar\Lambda$, *such that for every* $\delta>0$,

$$\mathbf{P}[\operatorname{dist}(v_0,\gamma)<\delta\operatorname{dist}(v_0,\partial D)]<c\delta^{\zeta_3}.$$

We expect that the left-hand side is bounded by $\delta^{1/2+o(1)}$, using the corresponding result [RS] for SLE(4).

Proof. Let Υ_n denote the circle of radius $2^{-n-1}\operatorname{dist}(v_0,\partial D)$ about v_0. As in the proof of (3.48), Theorem 3.11 implies that, given that γ intersects Υ_n (where $n\in\mathbb{N}$), the conditional probability that γ does not intersect Υ_{n+1} is bounded away from zero by a function of $\bar\Lambda$, provided that $2^{-n-1}\operatorname{dist}(v_0,\partial D)>10$, say. The lemma follows by induction. □

PROPOSITION 3.25. *Assume* (h), (D), (S) *and* (∂). *For every* $\varepsilon>0$ *there exists an* $R=R(\varepsilon,\bar{\Lambda})$ *such that*

$$\mathbf{E}[\mathbf{E}[(|h(S_\tau)|-\lambda)1_{\{S_\tau\notin\partial D\}}\mid\gamma]^2]<\varepsilon \tag{3.56}$$

holds, *provided that* $\mathrm{dist}(v_0,\partial D)>R$, *where* λ *is the constant given by* (3.52).

The proof is based on the simple idea that given a single instance of γ we consider two independent copies of (h,S).

Proof. Set

$$X:=\mathbf{E}[(|h(S_\tau)|-\lambda)1_{\{S_\tau\notin\partial D\}}\mid\gamma].$$

To get a handle on $\mathbf{E}[X^2]$, let (h',S') be independent of (h,S) given γ and have the same conditional law as that of (h,S) given γ. Thus, (h,S,γ) has the same law as (h',S',γ). Let $\tau':=\min\{t:S'_t\in\partial D(\gamma)\}$, $y:=\mathbf{E}[|h(S_\tau)|-\lambda|S_\tau,\gamma]$ and $y':=\mathbf{E}[|h'(S'_{\tau'})|-\lambda|S'_{\tau'},\gamma]$. Then $X^2=\mathbf{E}[yy'1_{\{S_\tau,S'_{\tau'}\notin\partial D\}}|\gamma]$ and hence

$$\mathbf{E}[X^2]=\mathbf{E}[yy'1_{\{S_\tau,S'_{\tau'}\notin\partial D\}}]. \tag{3.57}$$

Fix some $r_3\gg r_2\gg r_1\gg 0$, and assume that $\mathrm{dist}(0,\{v_0,\partial D\})>r_3$. Suppose that we condition on $\mathcal{Z}_0$; that is, on $S_\tau=0$. Then, with high conditional probability $|S'_{\tau'}|>2r_2$, and moreover $\mathrm{dist}(0,\{S'_0,S'_1,...,S'_{\tau'}\})>2r_2$. By the heights interface continuity (Proposition 3.3), given $\mathrm{ext}_{r_1}\gamma$ and $S'_{\tau'}$ and $\gamma\setminus\mathrm{ext}_{r_1}\gamma\subset\mathfrak{B}_{r_2}$, the actual choice of $\gamma\setminus\mathrm{ext}_{r_1}\gamma$ can change the value of y' by very little if $|S'_{\tau'}|>2r_2$. Thus, we conclude that $y'1_{S'_{\tau'}\notin\partial D}$ is nearly independent of y given $\mathcal{Z}_0$, $\neg\mathcal{J}(r_1,r_2)$ and $\mathrm{ext}_{r_1}\gamma$. Since y and y' are bounded, in the limit as $r_3\to\infty$, we have

$$\begin{aligned}&\mathbf{E}[yy'1_{\{S'_{\tau'}\notin\partial D\}}\mid\mathcal{Z}_0,\mathrm{ext}_{r_1}\gamma,\neg\mathcal{J}]\\&\qquad=o(1)+\mathbf{E}[y'1_{\{S'_{\tau'}\notin\partial D\}}\mid\mathcal{Z}_0,\mathrm{ext}_{r_1}\gamma,\neg\mathcal{J}]\mathbf{E}[y\mid\mathcal{Z}_0,\mathrm{ext}_{r_1}\gamma,\neg\mathcal{J}],\end{aligned} \tag{3.58}$$

where $\mathcal{J}=\mathcal{J}(r_1,r_2)$. By Lemma 3.17, $\mathbf{P}[\mathcal{J}|\mathcal{Z}_0]\to 0$ as $r_2/r_1\to\infty$. Thus

$$\mathbf{E}[y\mid\mathcal{Z}_0,\mathrm{ext}_{r_1}\gamma,\neg\mathcal{J}]-\mathbf{E}[y\mid\mathcal{Z}_0,\mathrm{ext}_{r_1}\gamma]\to 0$$

in probability as $r_2/r_1\to\infty$. A similar remark applies to the other terms in (3.58). Since y and y' are bounded, taking conditional expectation given $\mathcal{Z}_0$ in (3.58) gives

$$\mathbf{E}[yy'1_{\{S'_{\tau'}\notin\partial D\}}\mid\mathcal{Z}_0]=o(1)+\mathbf{E}[\mathbf{E}[y'1_{\{S'_{\tau'}\notin\partial D\}}\mid\mathcal{Z}_0,\mathrm{ext}_{r_1}\gamma]\mathbf{E}[y\mid\mathcal{Z}_0,\mathrm{ext}_{r_1}\gamma]\mid\mathcal{Z}_0]. \tag{3.59}$$

By the limit existence theorem (Theorem 3.21), given $\mathcal{Z}_0$ and $\mathrm{ext}_{r_1}\gamma$, near 0 the path γ_* is close in distribution to $\dot{\gamma}$ when r_1 is large. Consequently, Proposition 3.3 implies that $\mathbf{E}[|h(0)||\mathcal{Z}_0,\mathrm{ext}_{r_1}\gamma]-\lambda=o(1)$ as $r_1\to\infty$, which gives that

$$\mathbf{E}[y\mid\mathcal{Z}_0,\mathrm{ext}_{r_1}\gamma]=o(1).$$

Now (3.59) implies that

$$\mathbf{E}[yy' 1_{\{S'_{\tau'} \notin \partial D\}} \mid \mathcal{Z}_0] \to 0 \tag{3.60}$$

as $\mathrm{dist}(0, \partial D \cup \{v_0\}) \to \infty$. Lemmas 3.23 and 3.24 tell us that, for $r_0 < \infty$ fixed,

$$\mathbf{P}[S_\tau \notin \partial D, \mathrm{dist}(S_\tau, \partial D \cup \{v_0\}) < r_0] \to 0$$

as $\mathrm{dist}(v_0, \partial D) \to \infty$. Since there is nothing special about the vertex at 0, except for our assumption that $\mathrm{dist}(0, \partial D \cup \{v_0\})$ is large, we conclude from (3.60) that

$$\lim \mathbf{E}[yy' 1_{\{S_\tau, S'_{\tau'} \notin \partial D\}} \mid S_\tau] = 0$$

in probability, as $\mathrm{dist}(v_0, \partial D) \to \infty$. Now, equation (3.57) implies that $\mathbf{E}[X^2] \to 0$, since $yy' 1_{\{S_\tau, S'_{\tau'} \notin \partial D\}}$ is bounded. This gives (3.56) and completes the proof. □

Let F be the function that is equal to h_∂ on ∂D, λ on $V_+(\gamma)$, $-\lambda$ on $V_-(\gamma)$ and is discrete-harmonic on all other TG-vertices in D. Since $\mathbf{E}[h(v_0) | \gamma] = \mathbf{E}[h(S_\tau) | \gamma]$, Proposition 3.25 gives

$$\mathbf{E}[h(v_0) \mid \gamma] - F(v_0) \to 0 \tag{3.61}$$

in probability as $\mathrm{dist}(v_0, \partial D) \to \infty$.

We now need to generalize the proposition and (3.61) to apply when γ is replaced by an appropriate initial segment of γ.

Let T be some stopping time for γ started at x_∂ and let γ^T denote γ stopped at T. (Note that the relevant filtration here, the one generated by intitial segments of γ, only reveals the signs of h on vertices adjacent to these initial segments, but not the actual values of h.) Let z_T denote the vertex in $\partial D(\gamma^T)$ first visited by S, and let S^T denote the initial segment of S up to its first visit to z_T.

LEMMA 3.26. *Assume* (h), (D), (S) *and* (∂). *For every* $p_0 > 0$ *there is some* $s = s(p_0, \bar{\Lambda}) > 0$ *such that*

$$\mathbf{P}[\mathrm{dist}(z_T, \gamma \setminus \gamma^T) < s \, \mathrm{dist}(v_0, \partial D), z_T \notin \partial D] < p_0.$$

Note that we could rather easily prove the estimate with $\mathrm{dist}(z_T, \gamma \setminus \gamma^T; D(\gamma^T))$ instead of $\mathrm{dist}(z_T, \gamma \setminus \gamma^T)$, by the argument giving (3.48), but this is not sufficient for our purposes. The idea of the proof of the lemma is to first show that $S_\tau = z_T$ is usually not too unlikely given γ^T and S. Then Lemma 3.17 may be used in conjunction with the argument giving (3.48) to deduce the required result. Note that the event $S_\tau = z_T$ is the event that $\gamma \setminus \gamma^T$ is not adjacent to any vertex visited by S prior to z_T.

Proof. We first show that for every $\varepsilon>0$ there is a $p>0$ and an $R>0$, both depending only on ε and $\bar{\Lambda}$, such that

$$\mathbf{P}[\mathbf{P}[z_T = S_\tau \mid S, \gamma^T] < p] < \varepsilon \tag{3.62}$$

holds provided that $\operatorname{dist}(v_0, \partial D) > R$.

We choose $\delta=\delta(\varepsilon,\bar{\Lambda})>0$ very small. Set $r=\operatorname{dist}(v_0,\partial D)$, and assume that $r>100\delta^{-2}$. Set further $D_T:=D(\gamma^T)$. Let b_T be the point on ∂D_T near the tip of γ^T that is at equal distance from $V_+(\gamma^T)$ and $V_-(\gamma^T)$ along ∂D_T. Let ∂_+^T and ∂_-^T denote the two connected components of $\partial D_T \setminus \{y_\partial, b_T\}$ that have y_∂ and b_T as their endpoints, with ∂_+^T being the one containing vertices in ∂_+.

Let $\mathcal{A}_1$ be the event $\operatorname{dist}(z_T, \partial_+^T; D_T) \vee \operatorname{dist}(z_T, \partial_-^T; D_T) \geqslant 2\delta r$, let $\mathcal{A}_2$ be the event $\operatorname{diam}(S^T) < \delta^{-1} r$, let $\mathcal{A}_3$ be the event that the diameter of the segment of S^T after the first time at which it is distance at most $\delta^2 r$ from ∂D_T is less than $\frac{1}{2}\delta r$, and let $\mathcal{A}_4$ be the event $\operatorname{dist}(v_0, \gamma^T) \geqslant \delta^{1/2} r$.

Lemma 3.24 shows that if $\delta=\delta(\bar{\Lambda},\varepsilon)$ is sufficiently small, then $\mathbf{P}[\neg\mathcal{A}_4]<\frac{1}{4}\varepsilon$. On the other hand, Lemma 2.1 implies that, by choosing δ sufficiently small, one can ensure that $\mathbf{P}[\neg\mathcal{A}_j \mid \gamma^T] < \frac{1}{4}\varepsilon$ for $j=2,3$. We now prove the same for $j=1$. Assume that $\mathcal{A}_4$ holds. Let L be the set of points in D_T that lie on a path of diameter at most $4\delta r$ in $\overline{D}_T$ connecting ∂_+^T and ∂_-^T, and let D_* be the connected component of $D_T \setminus L$ that contains v_0 (we know that $v_0 \notin L$, since δ is small and $\mathcal{A}_4$ holds). By (3.55) applied to D_T in place of D, $\partial D_* \cap D_T$ may be covered by two balls of radius $8\delta r$. Thus, Lemma 2.1 shows that if $\delta=\delta(\varepsilon)$ is chosen sufficiently small, then $\mathbf{P}[S \text{ hits } L \text{ before } \partial D_T, \mathcal{A}_4] < \frac{1}{4}\varepsilon$. This implies that $\mathbf{P}[\neg\mathcal{A}_1, \mathcal{A}_4] < \frac{1}{4}\varepsilon$. Thus $\mathbf{P}[\neg\mathcal{A}]<\varepsilon$, where $\mathcal{A}:=\mathcal{A}_1\cap\mathcal{A}_2\cap\mathcal{A}_3\cap\mathcal{A}_4$.

We now complete the proof of inequality (3.62) by showing that the event

$$\mathbf{P}[S_\tau = z_T \mid S, \gamma^T] < p$$

is contained in $\neg\mathcal{A}$ if $p=p(\bar{\Lambda},\delta)>0$ is chosen sufficiently small. The latter is equivalent to showing that the random variable $\mathbf{P}[S_\tau=z_T \mid S,\gamma^T]$ is bounded away from zero on $\mathcal{A}$ by a function of δ and $\bar{\Lambda}$.

Suppose that $\mathcal{A}$ holds, and that $z_T \in \partial_+^T$. Let ξ_1 and ξ_2 be the two connected components of $\partial_+^T \setminus \{z_T\}$. The construction of Υ_n in the proof (3.48) shows that there is a path Υ_0 connecting ξ_1 and ξ_2 in $B(z_T, \delta r) \setminus B\big(z_T, \frac{1}{2}\delta r\big)$ that separates z_T from ∂_-^T in D_T such that Υ_0 is an $(s, \operatorname{diam}\Upsilon_0)$-barrier for (D, γ_T) for every $s\in(0,\delta']$, where $\delta'\in(0,1)$ is a universal constant. (The assumption that $\mathcal{A}_1$ holds is used here.) If S^T never visits a vertex adjacent to Υ_0, then Υ_0 separates S^T from ∂_-^T. In this situation, if $\gamma\setminus\gamma^T$ does not hit Υ_0, then S^T does not visit a vertex adjacent to it, and therefore $z_T=S_\tau$. Thus, Theorem 3.11 (or Remark 3.12) applies to give the needed lower bound on $\mathbf{P}[S_\tau=z_T \mid S,\gamma^T]$ when S^T does not visit a vertex adjacent to Υ_0.

Suppose now that S^T does visit vertices adjacent to Υ_0. We can then construct a path Υ whose image is Υ_0 as well as all the boundaries of hexagons visited by S^T that are not separated from ∂_-^T by Υ_0. Since we are assuming that $\mathcal{A}_2$ holds, $\operatorname{diam}\Upsilon\leqslant 2\delta^{-1}r$. As $\mathcal{A}_3$ holds, $\operatorname{dist}(\Upsilon\setminus\Upsilon_0,\partial D_T)>\frac{1}{2}\delta^2 r$ and therefore also $\operatorname{diam}\Upsilon_0>\frac{1}{2}\delta^2 r$. Consequently, Υ is a $\left(\frac{1}{2}\delta'\delta^3, 2\delta^{-1}r\right)$-barrier. Now Theorem 3.11 and Remark 3.12 may be used again to give a similar lower bound on $\mathbf{P}[S_\tau=z_T\,|\,S,\gamma^T]$. As a similar argument applies when $S_\tau\in\partial_-^T$, the proof of (3.62) is now complete.

We now choose $\varepsilon=\frac{1}{9}p_0$ and take a $p>0$ and $R>0$ depending only on ε and $\bar\Lambda$ and satisfying (3.62). Let a be such that the estimate given in Lemma 3.17 holds with the p there replaced by $\frac{1}{9}p_0 p$. Let $\delta=\delta(p_0,\bar\Lambda)>0$ be sufficiently small so that $\delta^{-1}>R\vee a$. We assume that $r>10\delta^{-5}$. For $z\in D$, let $\mathcal{J}_z$ denote the event that there are more than two disjoint arcs in γ joining the two circles $\partial B(z,\delta^4 r)$ and $\partial B(z,\delta^5 r)$, and let $\mathcal{J}_z^T$ denote the event that there are more than two such arcs in γ^T. By the choice of a and δ, the probability that $\operatorname{dist}(S_\tau,\partial D\cup\{v_0\})>4\delta^4 r$ and $\mathcal{J}_{S_\tau}$ holds is at most $\frac{1}{9}p_0 p$. Consequently, the same bound applies for the probability that $\operatorname{dist}(z_T,\partial D\cup\{v_0\})>4\delta^4 r$, $z_T=S_\tau$ and $\mathcal{J}_{z_T}^T$ holds. Thus, as the events $\operatorname{dist}(z_T,\partial D\cup\{v_0\})>4\delta^4 r$ and $\mathcal{J}_{z_T}^T$ are (γ^T,S)-measurable,

$$\begin{aligned}\tfrac{1}{9}p_0 p &\geqslant \mathbf{P}[z_T=S_\tau,\operatorname{dist}(z_T,\partial D\cup\{v_0\})>4\delta^4 r,\mathcal{J}_{z_T}^T]\\ &=\mathbf{E}[\mathbf{P}[z_T=S_\tau,\operatorname{dist}(z_T,\partial D\cup\{v_0\})>4\delta^4 r,\mathcal{J}_{z_T}^T\mid\gamma^T,S]]\\ &=\mathbf{E}[\mathbf{P}[z_T=S_\tau\mid\gamma^T,S]1_{\{\operatorname{dist}(z_T,\partial D\cup\{v_0\})>4\delta^4 r\}}1_{\mathcal{J}_{z_T}^T}].\end{aligned}$$

By (3.62) and our choice of ε, we therefore have

$$\mathbf{P}[\operatorname{dist}(z_T,\partial D\cup\{v_0\})>4\delta^4 r,\mathcal{J}_{z_T}^T]\leqslant\tfrac{2}{9}p_0. \tag{3.63}$$

Let $\mathcal{H}$ denote the event $(\gamma\setminus\gamma^T)\cap\partial B(z_T,\delta^5 r)\neq\varnothing$. Now condition on γ^T and S such that $\operatorname{dist}(z_T,\partial D)>\delta r$ and $\neg\mathcal{J}_{z_T}^T$ holds. Suppose also that $\operatorname{dist}(z_T,b_T)>\delta^3 r$. Then there are precisely two connected components of $B(z_T,\delta^4 r)\cap D_T$ that intersect $\partial B(z_T,\delta^5 r)$. Let $w_1,w_2\in\partial B(z_T,\delta^5 r)\cap D_T$ be points in each of these two connected components. By constructing barriers as in the proof of (3.48), it is easy to see that if $\delta=\delta(p_0,\bar\Lambda)$ is sufficiently small, then

$$\mathbf{P}[\operatorname{dist}(w_j,\gamma\setminus\gamma^T;D(\gamma^T))\leqslant\delta^4 r\mid\gamma^T,S]\leqslant\tfrac{1}{9}p_0$$

for $j=1,2$. Now note that if $\operatorname{dist}(w_j,\gamma\setminus\gamma^T;D(\gamma^T))>\delta^4 r$ for $j=1,2$, then $\neg\mathcal{H}$ holds. Consequently,

$$\mathbf{P}[\mathcal{H},\neg\mathcal{J}_{z_T}^T,\operatorname{dist}(z_T,b_T)>\delta^3 r,\operatorname{dist}(z_T,\partial D)>\delta r]\leqslant\tfrac{2}{9}p_0.$$

We combine this with (3.63), and get

$$\mathbf{P}[\mathcal{H}, \operatorname{dist}(v_0, z_T) > 4\delta^4 r, \operatorname{dist}(z_T, b_T) > \delta^3 r, \operatorname{dist}(z_T, \partial D) > \delta r] \leqslant \tfrac{4}{9} p_0. \tag{3.64}$$

Since $r = \operatorname{dist}(v_0, \partial D)$, provided that we take $\delta = \delta(p_0, \bar{\Lambda}) > 0$ sufficiently small, Lemma 3.24 gives $\mathbf{P}[\operatorname{dist}(v_0, z_T) \leqslant \delta r] < \frac{1}{9} p_0$, Lemma 2.1 gives $\mathbf{P}[\operatorname{dist}(v_0, z_T) > \delta r, \operatorname{dist}(z_T, b_T) \leqslant \delta^2 r] < \frac{1}{9} p_0$ and (3.53) gives $\mathbf{P}[\operatorname{dist}(z_T, \partial D) \leqslant \delta r, z_T \notin \partial D] < \frac{1}{9} p_0$. These last three estimates may be combined with (3.64), to yield $\mathbf{P}[\mathcal{H}, z_T \notin \partial D] \leqslant \frac{7}{9} p_0$, which completes the proof. □

We now prove the analog of (3.61) with γ^T replacing γ. Let F_T denote the function that is $+\lambda$ on $V_+(\gamma^T)$, $-\lambda$ on $V_-(\gamma^T)$, equal to h_∂ on TG-vertices in ∂D, and is discrete-harmonic at all other vertices in $\bar{D}$.

Proposition 3.27. *Assume* (h), (D), (S) *and* (∂). *Then*

$$\mathbf{E}[h(v_0) \mid \gamma^T] - F_T(v_0) \to 0$$

in probability as $\operatorname{dist}(v_0, \partial D) \to \infty$ *while* $\bar{\Lambda}$ *is held fixed.*

Proof. Fix $\varepsilon > 0$ and set $r := \operatorname{dist}(v_0, \partial D)$. We have, by the heights interface continuity (Proposition 3.3),

$$\big|\mathbf{E}[h(z_T) \mid \gamma, z_T] - \mathbf{E}[h(z_T) \mid \gamma^T, z_T]\big| < \varepsilon$$

if $\operatorname{dist}(z_T, \gamma \setminus \gamma_T) > R_0$, where $R_0 = R_0(\varepsilon, \bar{\Lambda})$. Therefore, Lemma 3.26 with $p_0 = \varepsilon / O_{\bar{\Lambda}}(1) > 0$ gives

$$\mathbf{E}\big[\big|\mathbf{E}[h(z_T) \mid \gamma, z_T] - \mathbf{E}[h(z_T) \mid \gamma^T, z_T]\big|\big] < 2\varepsilon \tag{3.65}$$

when $r > s^{-1} R_0$, and s is as given by the lemma.

(Note that $\mathbf{E}[h(z_T) | \gamma, z_T] = \mathbf{E}[h(z_T) | \gamma^T, z_T]$ when $z_T \in \partial D$.)

In the following, we will use a parameter $\delta > 0$. The notation $o(1)$ will be shorthand for any quantity g satisfying $\lim_{\delta \to 0} \lim_{r \to \infty} |g| = 0$ while $\bar{\Lambda}$ is fixed. Let Z be a maximal set of TG-vertices in $D \cap B(v_0, 2\delta^{-1} r)$ such that the distance between any two such vertices is at least $\delta^2 r$ and the distance between any such vertex to ∂D is at least $\delta^3 r$. Then $|Z| = O(\delta^{-6})$. By (3.61) (with each $u \in Z$ in place of v_0), we therefore have

$$\mathbf{P}[\text{there exists } u \in Z : \mathbf{E}[h(u) - F(u) \mid \gamma]^2 > \varepsilon] = o(1). \tag{3.66}$$

Let t_0 be the first $t \in \mathbb{N}$ such that $\operatorname{dist}(S_t, \partial D(\gamma^T)) \leqslant \delta r$. By Lemma 2.1,

$$\mathbf{P}[\operatorname{dist}(z_T, S_{t_0}; D) \geqslant \delta^{1/2} r] = o(1). \tag{3.67}$$

By (3.54),

$$\mathbf{P}[\operatorname{dist}(z_T, \gamma; D) < \delta^{1/3} r, z_T \in \partial D] = o(1),$$

while Lemma 3.26 gives

$$\mathbf{P}[\operatorname{dist}(z_T, \gamma\setminus\gamma_T) < \delta^{1/3} r, z_T \notin \partial D] = o(1).$$

Thus,

$$\mathbf{P}[\operatorname{dist}(z_T, \gamma\setminus\gamma_T; D) < \delta^{1/3} r] = o(1).$$

This and (3.67) imply that

$$\mathbf{P}\big[\operatorname{dist}(S_{t_0}, \gamma\setminus\gamma^T; D) < \tfrac{1}{2}\delta^{1/3} r\big] = o(1). \tag{3.68}$$

Since $\operatorname{dist}(S_{t_0}, \partial D(\gamma^T)) < \delta r$, this and Lemma 2.1 imply that with probability $1-o(1)$ the L^1 norm of the difference between the discrete-harmonic measure from S_{t_0} on $\partial D(\gamma^T)$ and the discrete-harmonic measure from S_{t_0} on $\partial D(\gamma)$ is $o(1)$. Because $\mathbf{E}[h(S_{t_0}) | \gamma, S_{t_0}]$ is the average of $\mathbf{E}[h(z)|\gamma]$, where z is selected according to the discrete-harmonic measure on $\partial D(\gamma)$ from S_{t_0} and similarly for γ^T, we conclude from the above and (3.65) that

$$\mathbf{E}[h(S_{t_0}) \mid \gamma^T, S_{t_0}] - \mathbf{E}[h(S_{t_0}) \mid \gamma, S_{t_0}] = o(1)$$

in probability. Now, Lemma 2.1 implies that $\mathbf{P}[\operatorname{dist}(S_{t_0}, v_0) > \delta^{-1} r] = o(1)$. On the event $\operatorname{dist}(S_{t_0}, v_0) \leqslant \delta^{-1} r$, fix some $z_0 \in Z$ within distance $\delta^2 r$ from S_{t_0} (if there is more than one such z_0, let z_0 be chosen uniformly at random among these given (h, γ, S)). By the discrete Harnack principle (Lemma 2.2) and (3.68), we have

$$\mathbf{E}[h(S_{t_0}) \mid \gamma, S_{t_0}] - \mathbf{E}[h(z_0) \mid \gamma, z_0] = o(1)$$

in probability, which in conjunction with (3.66) yields $\mathbf{E}[h(S_{t_0}) | \gamma, S_{t_0}] - F(z_0) = o(1)$. The discrete Harnack principle now implies that $F(z_0) - F(S_{t_0}) = o(1)$ in probability, and (3.68) gives $F(S_{t_0}) - F_T(S_{t_0}) = o(1)$ in probability. Consequently,

$$\mathbf{E}[h(S_{t_0}) \mid \gamma^T, S_{t_0}] - F_T(S_{t_0}) = o(1)$$

in probability. Since $\mathbf{E}[h(\cdot) | \gamma^T]$ and F_T are discrete-harmonic in $D(\gamma^T)$, the proposition follows. □

We can now prove the height gap theorem.

Theorem 3.28. *Assume* (h), (D), (S) *and* (∂). *As above, let T denote a stopping time for γ, let $\gamma^T := \gamma[0, T]$, let D_T be the complement in D of the closed triangles meeting $\gamma[0, T)$. Let h_T denote the restriction of h to $V \cap \partial D_T$ and let v_0 be some vertex in D. Then*

$$\mathbf{E}[h(v_0) \mid \gamma^T, h_T] - F_T(v_0) \to 0$$

in probability as $\operatorname{dist}(v_0, \partial D) \to \infty$ *while $\bar\Lambda$ is held fixed, where F_T is as in Proposition* 3.27.

Proof. Set

$$X := \mathbf{E}[h(v_0)\,|\,\gamma^T, h_T] - \mathbf{E}[h(v_0)\,|\,\gamma^T].$$

By Proposition 3.27, it suffices to show that $X \to 0$ in probability as $\operatorname{dist}(v_0, \partial D) \to \infty$. For $v \in V \cap \partial D_T$, let a_v denote the conditional probability that a simple random walk started at v_0 first hits ∂D_T at v, given γ^T. Then

$$X = \sum_{v \in V \cap \partial D_T} a_v (h(v) - \mathbf{E}[h(v)\,|\,\gamma^T]).$$

Consequently,

$$X^2 = \sum_{v,u} a_v a_u (h(v) - \mathbf{E}[h(v)\,|\,\gamma^T])(h(u) - \mathbf{E}[h(u)\,|\,\gamma^T]).$$

Now Corollary 3.5 implies that it suffices to show that for every $r > 0$,

$$\sum_{v,u} a_v a_u 1_{\{|v-u|<r\}} 1_{\{v,u \notin V_\partial\}} \to 0$$

in probability. This follows by Lemmas 2.1 and 3.24. □

4. Recognizing the driving term

In this section we use a technique introduced in [LSW4] and used again in [SS] in order to show that the driving term for the Loewner evolution given by the DGFF interface with boundary values $-a$ and b converges to the driving term of $\mathrm{SLE}(4; a/\lambda - 1, b/\lambda - 1)$ if $a, b \in [-\Lambda_0, \bar{\Lambda}]$. The reader unfamiliar with this method is advised to first learn the technique from [SS, §4] or [LSW4, §3.3]. The account in [SS] is closer to the present setup and somewhat simpler, but some parts of the argument there are referred back to [LSW4].

The present argument is more involved than those of the above mentioned papers, because we prove convergence to an instance of $\mathrm{SLE}(\varkappa; \varrho_1, \varrho_2)$ rather than just plain SLE(4). The main added difficulty comes from the fact that the drift term in $\mathrm{SLE}(\varkappa; \varrho_1, \varrho_2)$ becomes unbounded as W_t comes close to the force points. These difficulties disappear if $a = b = \lambda$, in which case the convergence is to ordinary SLE and the argument giving the convergence of the driving term to scaled Brownian motion is easily established with minor adaptations of the established method. We therefore forego dwelling on this simpler case, and move on to the more general setting, assuming that the reader is already familiar with the fundamentals of the method.

4.1. About the definition of SLE($\varkappa; \varrho_1, \varrho_2$)

Throughout this section, given a Loewner evolution defined by a continuous W_t, we let x_t and y_t be defined as in §1.4 by

$$x_t := \sup\{g_t(x) : x < 0 \text{ and } x \notin K_t\} \quad \text{and} \quad y_t := \inf\{g_t(x) : x > 0 \text{ and } x \notin K_t\},$$

and we make use of the definition of SLE($\varkappa; \varrho_1, \varrho_2$) by means of the SDE (1.5). As we mentioned in §1.4, some subtlety is involved in extending the definition of SLE($\varkappa; \varrho_1, \varrho_2$) beyond times when W_t hits the force points, and in starting the process from the natural initial values $x_0 = W_0 = y_0 = 0$. This is closely related to the issues involved in defining the Bessel process, which we presently recall.

The Bessel process Z_t of dimension $\delta > 0$ and initial value $x \neq 0$ satisfies the SDE

$$dZ_t = \frac{\delta - 1}{2Z_t}\, dt + dB_t, \quad Z_0 = x, \tag{4.1}$$

which we also write in integral form as

$$Z_t = x + \int_0^t \frac{\delta - 1}{2Z_s}\, ds + B_t - B_0, \tag{4.2}$$

up until the first time t for which $Z_t = 0$. When defining Z_t for all times, this SDE is awkward to work with directly since the drift blows up whenever Z_t gets close to zero (and some of the standard existence and uniqueness theorems for SDE solutions, as given, e.g., in [RY], do not apply in this situation). However, for every $\delta > 0$, the square of the Bessel process Z_t^2 turns out to satisfy an SDE whose drift remains bounded and for which existence and uniqueness of solutions follow easily from standard theorems. For this reason, many authors construct the Bessel process by first defining the square of the Bessel process via an SDE that it satisfies and then taking its square root [RY]. (Recall also that when $\delta \leqslant 1$ the Bessel process itself does not satisfy (4.2) at all without a principal value correction. Even when $1 < \delta < 2$, which, as we will see below, is the case that corresponds to SLE($\varkappa; \varrho$) that hit the boundary and can be continued after hitting the boundary, the solution to (4.2) is not unique unless we restrict attention to non-negative solutions.)

The formal definition for SLE($\varkappa; \varrho$) with one force point (i.e., $\varrho_1 = \varrho$ and $\varrho_2 = 0$) was given in [LSW3]. It was observed there that in this case, (1.5) implies that the process $W_t - x_t$ satisfies the same SDE as the Bessel process of dimension $\delta = 1 + 2(\varrho + 2)/\varkappa$ up until the first time t for which $W_t = x_t$. Thus, to define SLE($\varkappa; \varrho$), the paper [LSW3] starts with a constant multiple of a Bessel process Z_t of the appropriate dimension and defines the evolution of the force point x_t by $x_t = x_0 + \int_0^t (2/Z_s)\, ds$ and the driving term by $W_t = x_t + Z_t$.

Defining $\mathrm{SLE}(\varkappa;\varrho_1,\varrho_2)$ is a slightly more delicate matter since neither W_t-x_t nor y_t-W_t is exactly a Bessel process (although each one is quite close to a Bessel process when the other force point is relatively far away). Although this is not a very difficult issue, it seems that there does not yet exist, in the literature, an adequate definition of $\mathrm{SLE}(\varkappa;\varrho_1,\varrho_2)$ that is valid beyond the time that the driving term hits a force point. Since we prove the convergence to $\mathrm{SLE}(\varkappa;\varrho_1,\varrho_2)$, we have to define it.

The approach we adopt is basically similar to the way in which the Bessel process (and hence $\mathrm{SLE}(\varkappa;\varrho)$) is usually defined: we pass to a coordinate system in which the corresponding SDE becomes tractable. We will describe the coordinate change we use in §4.2. Within this new coordinate system, we then prove the convergence of the Loewner driving parameters of our discrete processes to those of the corresponding $\mathrm{SLE}(\varkappa;\varrho_1,\varrho_2)$ in §4.3 and §4.4. Then §4.5 describes the reverse coordinate transformation and use it to give a formal definition of $\mathrm{SLE}(\varkappa;\varrho_1,\varrho_2)$, Definition 4.14.

We remark that there are many equivalent ways to define $\mathrm{SLE}(\varkappa;\varrho_1,\varrho_2)$ (for example, one can probably show directly that (1.5) has a unique strong solution for which $x_t\leqslant W_t\leqslant y_t$ for all t), but ours seems most efficient given that the coordinate change also simplifies the proofs in §4.3 and §4.4.

4.2. A coordinate change

In this subsection, we recall a different coordinate system for Loewner evolutions, which is virtually identical to the setup used in [LSW2, §3]. Suppose that $\gamma:[0,\infty)\to\overline{\mathbb{H}}$ is a continuous simple path that starts at $\gamma(0)=0$, does not hit $\mathbb{R}\setminus\{0\}$, satisfies $\lim_{t\to\infty}|\gamma(t)|=\infty$ and is parameterized by half-plane capacity from ∞. Let $g_t:\mathbb{H}\setminus\gamma[0,t]\to\mathbb{H}$ be the conformal map satisfying the hydrodynamic normalization at ∞, let $W_t=g_t(\gamma(t))$ be the corresponding Loewner driving term. Loewner's theorem says that g_t satisfies Loewner's chordal equation (1.3). Now we introduce a 1-parameter family of maps $G_t^*:\mathbb{H}\setminus\gamma[0,t]\to\mathbb{H}$ satisfying the normalization for $t>0$,

$$G_t^*(\infty)=\infty,\quad G_t^*((0,\infty))=(1,\infty)\quad\text{and}\quad G_t^*((-\infty,0))=(-\infty,-1).$$

That is,

$$G_t^*(z)=\frac{2g_t(z)-x_t-y_t}{y_t-x_t},\tag{4.3}$$

where x_t and y_t (as defined earlier) are the two images under g_t of 0 and $x_t<y_t$. Set

$$W_t^*:=G_t^*(\gamma(t))=\frac{2W_t-x_t-y_t}{y_t-x_t}.$$

By differentiating (4.3) and using (1.3) and (1.4) it is immediate to verify that G_t^* satisfies

$$\frac{dG_t^*(z)}{dt} = \frac{8}{(y_t-x_t)^2}\,\frac{1-G_t^*(z)^2}{(G_t^*(z)-W_t^*)(1-(W_t^*)^2)}.$$

We now define a new time parameter

$$s(t) = \log(y_t-x_t) = \log 2 - \log(G_t^*)'(\infty).$$

It is easy to verify that s is continuous and monotone increasing and $s((0,\infty))=(-\infty,\infty)$. Set $G_s=G_t^*$ and $\widetilde{W}_s=W_t^*$ when $s=s(t)$. Differentiation gives

$$ds = \frac{8(y_t-x_t)^{-2}\,dt}{1-(W_t^*)^2}. \tag{4.4}$$

Consequently, this change of time variable allows us to write the ODE satisfied by G as

$$\frac{dG_s(z)}{ds} = \frac{1-G_s(z)^2}{G_s(z)-\widetilde{W}_s}, \tag{4.5}$$

where all the terms come from the new coordinate system. Later, in §4.5, we explain how to go back to the standard chordal coordinate system.

4.3. The Loewner evolution of the DGFF interface

In addition to our previous assumptions (h) and (D) about the domain D and the boundary conditions, we now add the assumption that

(ab) there are constants a and b such that $h_\partial=b$ on ∂_+, $h_\partial=-a$ on ∂_- and

$$\min\{a,b\} > -\Lambda_0,$$

where $\Lambda_0>0$ is given by Lemma 3.9 with $\bar{\Lambda}:=\max\{|a|,|b|\}$.
In this case, clearly (∂) holds. In the following, a and b will be considered as constants, and the dependence of various constants on a and b will sometimes be suppressed (for example, when using the $O(\cdot)$ notation).

Let $\phi: D\to\mathbb{H}$ be a conformal map that corresponds ∂_+ with the positive real ray. Let γ be the zero-height interface of h joining the endpoints of ∂_+, and let γ^ϕ denote the image of γ under ϕ. Now, γ^ϕ satisfies the assumptions in the previous subsection. Consequently, we may parameterize γ^ϕ according to the time parameter $s=s(t)$ and consider the conformal maps $G_s: \mathbb{H}\setminus\gamma^\phi(-\infty,s]\to\mathbb{H}$ as defined in §4.2. As above, we set

$\widetilde{W}_s = G_s(\gamma^\phi(s))$ and have the differential equation (4.5). Our goal now is to determine the limit of the law of $\widetilde{W}$ as $\mathrm{rad}_{\phi^{-1}(i)}(D) \to \infty$. Set for $x \in [-1,1]$,

$$q_1(x) := 2(1-x^2) \quad \text{and} \quad q_2(x) := -\frac{\lambda+a}{2\lambda}(x-1) - \frac{\lambda+b}{2\lambda}(x+1). \tag{4.6}$$

We extend the definitions of q_1 and q_2 to all of $\mathbb{R}$ by taking each q_j to be constant in each of the two intervals $(-\infty,-1]$ and $[1,\infty)$. Consider the SDE

$$dY_s = q_2(Y_s)\,ds + q_1(Y_s)^{1/2}\,dB_s, \tag{4.7}$$

where B is a standard 1-dimensional Brownian motion. A weak solution is known to exist (see [KS, §5.4.D]). We also recall that the weak solution is strong and pathwise unique (see [RY, §IX, Theorems 1.7 and 3.5]).

THEOREM 4.1. *There is a time-stationary solution* $Y: (-\infty,\infty) \to [-1,1]$ *of* (4.7). *Moreover, for every finite* $S>1$ *and* $\varepsilon>0$, *there is an* $R_0 = R_0(S,\varepsilon)$ *such that if* $R := \mathrm{rad}_{\phi^{-1}(i)}(D) > R_0$ *and the assumptions* (h), (D) *and* (ab) *hold, then there is a coupling of* Y_s *with* h *such that*

$$\mathbf{P}[\sup\{|Y_s - \widetilde{W}_s| : s \in [-S,S]\} > \varepsilon] < \varepsilon.$$

The following proposition is key in the proof of the theorem. In essence, it states that $\widetilde{W}_s$ satisfies a discrete version of (4.7). Let $\mathcal{F}_s$ be the σ-field generated by $(\widetilde{W}_r : r \leqslant s)$. (Note that, although the filtration defining $\widetilde{W}_s$ is discrete, there is no problem in considering $\mathcal{F}_s$ for arbitrary s, though the behavior of $\widetilde{W}_r$ for r in some neighborhood of s might be determined by $\mathcal{F}_s$.)

PROPOSITION 4.2. *Assume* (h), (D) *and* (ab). *Fix some* $S>1$ *large and some* $\delta,\eta>0$ *small. There is a constant* $C>0$, *depending only on* a, b *and* S, *and there is a function* $R_0 = R_0(S,\delta,\eta)$, *depending only on* a, b, S, δ *and* η, *such that the following holds. If* $R := \mathrm{rad}_{\phi^{-1}(i)}(D) > R_0$ *and* s_0 *and* s_1 *are two stopping times for* $\widetilde{W}_s$ *such that almost surely* $-S \leqslant s_0 \leqslant s_1 \leqslant S$, $\Delta s := s_1 - s_0 \leqslant \delta^2$ *and* $\sup_{s\in[s_0,s_1]} |\widetilde{W}_s - \widetilde{W}_{s_0}| \leqslant \delta$, *then the following two estimates hold with probability at least* $1-\eta$:

$$\bigl|\mathbf{E}[\Delta\widetilde{W} - q_2(\widetilde{W}_{s_0})\Delta s \mid \mathcal{F}_{s_0}]\bigr| \leqslant C\delta^3, \tag{4.8}$$

$$\bigl|\mathbf{E}[(\Delta\widetilde{W})^2 - q_1(\widetilde{W}_{s_0})\Delta s \mid \mathcal{F}_{s_0}]\bigr| \leqslant C\delta^3, \tag{4.9}$$

where $\Delta\widetilde{W} := \widetilde{W}_{s_1} - \widetilde{W}_{s_0}$.

To prepare for the proof of the proposition, we need the following easy lemma. The first two statements in this lemma should be rather obvious to anyone with a solid background on conformal mappings.

LEMMA 4.3. *Set $c_1=c_1(S)=100e^S$. There are finite constants $c_2=c_2(S)>0$ and $R_0=R_0(S)>0$, depending only on S, such that if $R:=\operatorname{rad}_{\phi^{-1}(i)}(D)>R_0$ and if $z\in\mathbb{H}$ satisfies $5c_1\geqslant \operatorname{Im} z\geqslant c_1\geqslant|\operatorname{Re} z|$, then the following holds true*:

(1) $\operatorname{rad}_{\phi^{-1}(z)}(D)>c_2R$;

(2) *there is a* TG*-vertex* $v\in D$ *satisfying* $|\phi(v)-z|<\frac{1}{100}|z|$;

(3) $\operatorname{Im} G_s(z)\geqslant\frac{1}{2}e^{-s}c_1$ *for* $s\in[-S,S]$ (*and in particular*, $G_s(z)$ *is well defined in that range*);

(4) $|G_s(z)-2e^{-s}z|\leqslant 2$ *for* $s\in[-S,S]$.

Proof. Consider the conformal map $\psi(z)=(z-i)/(z+i)$ from $\mathbb{H}$ onto the unit disk $\mathbb{U}$ taking i to 0 and set $f=(\psi\circ\phi)^{-1}$. The Schwarz lemma applied to the map $z\mapsto f^{-1}(f(0)+Rz)$ restricted to $\mathbb{U}$ gives $1/|f'(0)|=|(f^{-1})'(f(0))|\leqslant 1/R$. Thus $|f'(0)|\geqslant R$. For a fixed c_1 the set of possible z is a compact subset of $\mathbb{H}$, and its image under ψ is a compact subset of $\mathbb{U}$. Consequently, the Koebe distortion theorem (see, e.g., [P, Theorem 1.3]) implies that $|f'(\psi(z))|\geqslant c_2'R$ for some c_2' depending only on c_1. Now the Koebe $\frac{1}{4}$-theorem (see, e.g., [P, Corollary 1.4]) gives that $\operatorname{rad}_{f(\psi(z))}(D)>c_2R$ for some c_2 depending on c_1. This takes care of statement (1).

Let B be the open disk of radius $\frac{1}{200}|z|$ about z. Clearly, $B\subset\mathbb{H}$. We conclude from $|f'(\psi(z))|\geqslant c_2'R$ that $|(\phi^{-1})'(z)|>c_2''R$ for some c_2'' depending only on c_1. Thus, the Koebe $\frac{1}{4}$-theorem implies that

$$\operatorname*{rad}_{\phi^{-1}(z)}(\phi^{-1}(B))\geqslant\tfrac{1}{4}c_2''\operatorname{rad}(B)R.$$

Consequently, statement (2) holds once $R_0>4/c_2''\operatorname{rad}(B)$. This takes care of (2), because $1/\operatorname{rad}(B)$ is bounded by a function of c_1.

It is easy to check that (3) follows from (4). It remains to prove the latter. Let x_t and y_t be as in §4.2. Note that $x_t<W_t<y_t$ for all $t>0$ and $\lim_{t\searrow 0}x_t=W_0=0=\lim_{t\searrow 0}y_t$. Therefore, (1.4) implies that $x_t<0<y_t$ for all $t>0$. By (1.4),

$$\partial_t(y_t-x_t)=\frac{2(y_t-x_t)}{(y_t-W_t)(W_t-x_t)}\geqslant\frac{8}{y_t-x_t}.$$

Therefore, $\partial_t((y_t-x_t)^2)\geqslant 16$, which gives

$$e^{2s(t)}\geqslant 16t. \tag{4.10}$$

Observe, by (1.3), that

$$\partial_t((\operatorname{Im} g_t(z))^2)\geqslant -4.$$

Thus, $(\operatorname{Im} g_t(z))^2\geqslant(\operatorname{Im} z)^2-4t\geqslant c_1^2-4t$. Another appeal to (1.3) now gives

$$|g_t(z)-z|\leqslant\frac{2t}{\sqrt{c_1^2-4t}}.$$

By (4.3) and the definition of $s(t)$, the above gives

$$|G_s(z)-2e^{-s}z| \leqslant \left|\frac{x_t+y_t}{y_t-x_t}\right| + \frac{2te^{-s}}{\sqrt{c_1^2-4t}}.$$

Now, the first summand on the right-hand side is at most 1, because $y_t>0>x_t$. The second summand is also at most 1 in the range $s\in[-S,S]$, by (4.10) and the choice of c_1. This completes the proof of the lemma. □

Proof of Proposition 4.2. With the notation of Lemma 4.3, let $z_j:=2c_1 i+\frac{1}{2}c_1 j$ for $j=0,1$, and let v_j be a TG-vertex satisfying condition (2) of the lemma with z_j in place of z. Then $z_j':=\phi(v_j)$ satisfies in turn the assumptions required for z in the lemma. For $k=0,1$, note that there is a stopping time T_k for γ such that $\phi\circ\gamma(T_k)=\gamma^\phi(s_k)$. Fix $j\in\{0,1\}$ and set

$$X_k = X_k(j) := \mathbf{E}[h(v_j)\mid\mathcal{F}_{s_k}], \quad k=0,1.$$

Clearly,

$$\mathbf{E}[X_1\mid\mathcal{F}_{s_0}] = X_0. \tag{4.11}$$

Recall the definition of the function F_T from Proposition 3.27. Let $\mathcal{A}_k$ be the event $|X_k-F_{T_k}(v_j)|\geqslant\delta^5$. By that proposition and the fact that z_j' satisfies condition (1) of Lemma 4.3, if R is chosen sufficiently large then $\mathbf{P}[\mathcal{A}_k]<\frac{1}{4}\eta\delta^5$. Since $F_{T_k}(v_j)=O_{a,b}(1)$, and likewise $X_k=O_{a,b}(1)$ by (3.2), we get

$$\left|\mathbf{E}[X_1-F_{T_1}(v_j)\mid\mathcal{F}_{s_0}]\right| \leqslant \delta^5+O_{a,b}(1)\mathbf{P}[\mathcal{A}_1\mid\mathcal{F}_{s_0}].$$

Let $\tilde{\mathcal{A}}$ be the event that $\mathbf{P}[\mathcal{A}_1|\mathcal{F}_{s_0}]>\delta^5$. Then $\mathbf{P}[\tilde{\mathcal{A}}]<\frac{1}{4}\eta$ (since we are assuming $\mathbf{P}[\mathcal{A}_1]<\frac{1}{4}\eta\delta^5$) and we have

$$\left|\mathbf{E}[X_1-F_{T_1}(v_j)\mid\mathcal{F}_{s_0}]\right| \leqslant O_{a,b}(\delta^5) \quad \text{on } \neg\tilde{\mathcal{A}}.$$

Thus we have, from (4.11),

$$\mathbf{E}[F_{T_1}(v_j)\mid\mathcal{F}_{s_0}]-F_{T_0}(v_j) = O_{a,b}(\delta^5) \quad \text{on } \neg(\tilde{\mathcal{A}}\cup\mathcal{A}_0). \tag{4.12}$$

Let H_k be the bounded function that is harmonic (not discrete-harmonic) in $D\setminus\gamma[0,T_k]$, has boundary values b on ∂_+, $-a$ on ∂_-, $+\lambda$ on the right-hand side of $\gamma[0,T_k]$, and $-\lambda$ on the left-hand side of $\gamma[0,T_k]$. We claim that the difference $H_k(v_j)-F_{T_k}(v_j)$ is small if $\operatorname{dist}(v_j,\partial D\cup\gamma[0,T_k])$ is large. Indeed, this easily follows by coupling the simple random walk on TG to stay with high probability relatively close to a Brownian motion and using (3.54) and Lemmas 3.23 and 2.1 to show that with high probability the boundary value sampled by the hitting point of the Brownian motion is the same as that sampled by the hitting vertex of the simple random walk. Now, Lemma 3.24 guarantees that if R is sufficiently large, then with high probability $\operatorname{dist}(v_j, \partial D\cup\gamma[0,T_k])$ is large as well. Consequently, if R is chosen sufficiently large, we have $\mathbf{P}[|H_k(v_j)-F_{T_k}(v_j)|\geqslant\delta^5]<\frac{1}{4}\eta\delta^5$. Let $\mathcal{B}_k$ be the event $|H_k(v_j)-F_{T_k}(v_j)|\geqslant\delta^5$, let $\widetilde{\mathcal{B}}$ be the event $\mathbf{P}[\mathcal{B}_1\,|\,\mathcal{F}_{s_0}]\geqslant\delta^5$ and let $\bar{\mathcal{A}}:=\mathcal{A}_0\cup\tilde{\mathcal{A}}\cup\mathcal{B}_0\cup\widetilde{\mathcal{B}}$. Note that $\mathbf{P}[\bar{\mathcal{A}}]<\eta$. The above proof of (4.12) from (4.11) now gives

$$\mathbf{E}[H_1(v_j)\,|\,\mathcal{F}_{s_0}]-H_0(v_j)=O_{a,b}(\delta^5)\quad\text{on } \neg\bar{\mathcal{A}}. \tag{4.13}$$

Now, the point is that $H_k(v_j)$ can easily be expressed analytically in terms of $Z_k=Z_{k,j}:=G_{s_k}(z'_j)=G_{s_k}(\phi(v_j))$ and $\widetilde{W}_{s_k}$. Indeed, conformal invariance implies that the harmonic measure of ∂_+ in $D\setminus\gamma[0,T_k]$ from v_j is the same as the harmonic measure of $[1,\infty)$ from Z_k, which is $1-\arg(Z_k-1)/\pi$, because $G_s\circ\phi$ corresponds ∂_+ with $[1,\infty)$. Likewise, the harmonic measure of the right-hand side of $\gamma[0,T_k]$ is

$$\frac{\arg(Z_k-1)-\arg(Z_k-\widetilde{W}_{s_k})}{\pi}.$$

Similar expressions hold for the harmonic measure of the left-hand side of $\gamma[0,T_k]$ and of ∂_-. These give

$$\pi(H_k(v_j)-b)=(\lambda-b)\arg(Z_k-1)-2\lambda\arg(Z_k-\widetilde{W}_{s_k})+(\lambda-a)\arg(Z_k+1). \tag{4.14}$$

Recall that $Z_k=G_{s_k}(z'_j)$. By (4.5), we have in the interval $s\in[s_0,s_1]$,

$$G_s(z'_j)=Z_0+\int_{s_0}^{s}\frac{1-G_r(z'_j)^2}{G_r(z'_j)-\widetilde{W}_r}\,dr. \tag{4.15}$$

Note that conditions (3) and (4) of Lemma 4.3 imply that the integrand is $O_S(1)$. Therefore $G_s(z'_j)-Z_0=O_S(\Delta s)=O_S(\delta^2)$ for $s\in[s_0,s_1]$. Moreover, we have $|\widetilde{W}_s-\widetilde{W}_{s_0}|\leqslant\delta$ in that range. Thus, it follows from (4.15) and condition (3) of the lemma that

$$Z_1-Z_0=\Delta s\left(\frac{1-Z_0^2}{Z_0-\widetilde{W}_{s_0}}+O_S(\delta)\right)=\Delta s\frac{1-Z_0^2}{Z_0-\widetilde{W}_{s_0}}+O_S(\delta^3). \tag{4.16}$$

We will now write an expression for $H_1(v_j)-H_0(v_j)$ and then use (4.13) to complete the proof. Let us first look at the term $\arg(Z_k-\widetilde{W}_{s_k})$ on the right-hand side of (4.14), and see how it changes from $k=0$ to $k=1$. For this purpose, we expand $\log(Z-\widetilde{W})$ in Taylor series up to first order in $Z-Z_0$ and up to second order in $\widetilde{W}-\widetilde{W}_{s_0}$ (since $Z_1-Z_0=O_S(\delta^2)$ while $\widetilde{W}_{s_1}-\widetilde{W}_{s_0}=O(\delta)$), as follows:

$$\begin{aligned}\arg(Z_1-\widetilde{W}_{s_1})-\arg(Z_0-\widetilde{W}_{s_0})&=\operatorname{Im}\bigl(\log(Z_1-\widetilde{W}_{s_1})-\log(Z_0-\widetilde{W}_{s_0})\bigr)\\&=\operatorname{Im}\left(\frac{Z_1-Z_0}{Z_0-\widetilde{W}_{s_0}}-\frac{\widetilde{W}_{s_1}-\widetilde{W}_{s_0}}{Z_0-\widetilde{W}_{s_0}}-\frac{(\widetilde{W}_{s_1}-\widetilde{W}_{s_0})^2}{2(Z_0-\widetilde{W}_{s_0})^2}\right)+O_S(\delta^3).\end{aligned}$$

Similar (but simpler) expansions apply to the other arguments in (4.14). We use these expansions as well as (4.14) and (4.16), to write

$$\begin{aligned}\pi(H_1(v_j)-H_0(v_j)) = (\lambda-b)\operatorname{Im}\frac{\Delta s(1-Z_0^2)}{(Z_0-\widetilde{W}_{s_0})(Z_0-1)} \\ -2\lambda\operatorname{Im}\frac{\Delta s(1-Z_0^2)-\Delta\widetilde{W}(Z_0-\widetilde{W}_{s_0})-\frac{1}{2}(\Delta\widetilde{W})^2}{(Z_0-\widetilde{W}_{s_0})^2} \\ +(\lambda-a)\operatorname{Im}\frac{\Delta s(1-Z_0^2)}{(Z_0-\widetilde{W}_{s_0})(Z_0+1)}+O_S(\delta^3).\end{aligned}$$

With the abbreviations $\tilde{x}:=\operatorname{Re}(Z_0-\widetilde{W}_{s_0})$ and $y:=\operatorname{Im}Z_0$, the above simplifies to

$$\begin{aligned}&\frac{\pi}{\lambda}(H_1(v_j)-H_0(v_j)) \\ &\quad=\frac{2y}{\tilde{x}^2+y^2}\left(\frac{\tilde{x}}{\tilde{x}^2+y^2}(\Delta s q_1(\widetilde{W}_{s_0})-(\Delta\widetilde{W})^2)+\Delta s q_2(\widetilde{W}_{s_0})-\Delta\widetilde{W}\right)+O_S(\delta^3).\end{aligned}$$

We know from (4.13) that on $\neg\bar{\mathcal{A}}$ the conditioned expectation given $\mathcal{F}_{s_0}$ of the left-hand side is $O_{a,b}(\delta^5)$. Since $(\tilde{x}^2+y^2)/y=O_S(1)$ (by statement (4) of Lemma 4.3), we have on $\neg\bar{\mathcal{A}}$,

$$\mathbf{E}\left[\frac{\tilde{x}}{\tilde{x}^2+y^2}\big(\Delta s q_1(\widetilde{W}_{s_0})-(\Delta\widetilde{W})^2\big)+\Delta s q_2(\widetilde{W}_{s_0})-\Delta\widetilde{W}\;\middle|\;\mathcal{F}_{s_0}\right]=O_{a,b,S}(\delta^3). \tag{4.17}$$

Now, this is valid for z'_j, with $j=0,1$. The choice of j only affects the left-hand side in the term $\tilde{x}/(\tilde{x}^2+y^2)$. By the choice of the points z'_j and by statement (4) of Lemma 4.3, the factor $\tilde{x}/(\tilde{x}^2+y^2)=\operatorname{Re}((Z_0-\widetilde{W}_{s_0})^{-1})$ differs between the two z'_j by an amount that is bounded away from zero by a constant depending on S. Subtracting the above relation (4.17) for z'_0 from that of z'_1, we therefore get (4.9) on $\neg\bar{\mathcal{A}}$. When this is used in conjunction with (4.17) again, one obtains (4.8) on $\neg\bar{\mathcal{A}}$. This concludes the proof of the proposition. □

4.4. Approximate diffusions

In this subsection we embark on the general study of random processes satisfying the conclusions of Proposition 4.2 and show that the proposition essentially characterizes the macroscopic behavior of the process. As one of the referees of this paper pointed out, one can try to do this more "traditionally" by proving tightness of the driving term and characterizing the subsequential fine mesh limit using the appropriate martingale problem. However, our approach is somewhat different (though not necessarily better).

Motivated by the proposition, we say that a continuous random $W:[0,S]\to[-1,1]$ is a (C,δ,η)*-approximate* (q_1,q_2)*-diffusion* if it satisfies the conclusion of the proposition; namely, for every pair of stopping times s_0 and s_1 such that almost surely $0\leqslant s_0\leqslant s_1\leqslant S$, $\Delta s:=s_1-s_0\leqslant\delta^2$ and $\sup_{s\in[s_0,s_1]}|W_s-W_{s_0}|\leqslant\delta$ we have with probability at least $1-\eta$ that (4.8) and (4.9) hold with W in place of $\widetilde{W}$.

LEMMA 4.4. *Suppose that* $a,b\geqslant-\lambda$ *and that* Y_s *satisfies* (4.7), *where* q_1 *and* q_2 *are given by* (4.6). *Suppose that* $Y_0\in[-1,1]$ *almost surely. Then there is a* $C>0$ *such that* Y_s *is a* $(C,\delta,0)$*-approximate* (q_1,q_2)*-diffusion for every* $\delta\in(0,1)$ *and in every time interval* $[0,S]$.

Proof. First, note that $q_1(x)=0$ and $xq_2(x)\leqslant0$ for $|x|\geqslant1$. This clearly implies that $\{Y_s:s\geqslant0\}\subset[-1,1]$ almost surely. Now fix some $\delta>0$ and two stopping times $s_0\leqslant s_1$ satisfying the assumptions in the definition of approximate diffusions. Let $\mathcal{F}_{s_0}$ denote the σ-field generated by $(Y_s:s\leqslant s_0)$. Then

$$Y_s-Y_{s_0}=\int_{s_0}^{s}q_2(Y_r)\,dr+\int_{s_0}^{s}q_1(Y_r)^{1/2}\,dB_r.$$

The second summand is a martingale, and therefore

$$\mathbf{E}[Y_{s_1}-Y_{s_0}\,|\,\mathcal{F}_{s_0}]=\mathbf{E}\left[\int_{s_0}^{s_1}q_2(Y_s)\,ds\,\middle|\,\mathcal{F}_{s_0}\right].$$

Since $|Y_s-Y_{s_0}|\leqslant\delta$ for $s\in[s_0,s_1]$ and q_2 is a Lipschitz function, we conclude that

$$\mathbf{E}[Y_{s_1}-Y_{s_0}\,|\,\mathcal{F}_{s_0}]=\mathbf{E}[\Delta s q_2(Y_{s_0})\,|\,\mathcal{F}_{s_0}]+O(\delta)\mathbf{E}[\Delta s\,|\,\mathcal{F}_{s_0}].$$

Thus, Y_s satisfies (4.8).

We now use Itô's formula to calculate $(Y_{s_1}-Y_{s_0})^2$:

$$\begin{aligned}(Y_{s_1}-Y_{s_0})^2&=\int_{s_0}^{s_1}2(Y_s-Y_{s_0})\,dY_s+\langle Y\rangle_{s_1}-\langle Y\rangle_{s_0}\\&=\int_{s_0}^{s_1}2(Y_s-Y_{s_0})q_2(Y_s)\,ds+\int_{s_0}^{s_1}2(Y_s-Y_{s_0})q_1(Y_s)^{1/2}\,dB_s+\int_{s_0}^{s_1}q_1(Y_s)\,ds.\end{aligned}$$

The left summand is $O(\delta^3)$ and the middle summand is a martingale and therefore its expectation given $\mathcal{F}_{s_0}$ is zero. Thus

$$\mathbf{E}[(Y_{s_1}-Y_{s_0})^2\,|\,\mathcal{F}_{s_0}]=\mathbf{E}\left[\int_{s_0}^{s_1}q_1(Y_s)\,ds\right]+O(\delta^3)=\mathbf{E}[\Delta s q_1(Y_{s_0})]+O(\delta^3),$$

because q_1 is Lipschitz. This shows that Y_s satisfies (4.9), and completes the proof. $\square$

PROPOSITION 4.5. *Fix $S>2$. Let $q_1, q_2\colon[-1,1]\to\mathbb{R}$ be defined as in* (4.6), *where we assume that $a, b>-\lambda$. Suppose that $W^1\colon[0,S]\to[-1,1]$ is a (C,δ,η)-approximate (q_1,q_2)-diffusion and that $W^2\colon[0,S]\to[-1,1]$ is a solution of* (4.7) *with the same q_1 and q_2 and $W^1(0)=W^2(0)$ almost surely. Also assume that $\eta<\delta^5/S^2$. Then there is a coupling of W^1 and W^2 such that $\sup_{s\in[0,S-1]}|W_s^1-W_s^2|\to 0$ in probability as $\delta\to 0$, while C is fixed. Namely, for every $\varepsilon>0$ there is a $\delta_0>0$, depending only on a, b, S, C and ε such that $\sup_{s\in[0,S-1]}|W_s^1-W_s^2|\leqslant\varepsilon$ with probability at least $1-\varepsilon$ if $\delta<\delta_0$.*

A useful tool in the proof of the proposition is the following lemma.

LEMMA 4.6. *Let $W\colon[0,S]\to[-1,1]$ be a (C,δ,η)-approximate (q_1,q_2)-diffusion, and let τ_0 and τ_1 be two stopping times for W satisfying $0\leqslant\tau_0\leqslant\tau_1\leqslant S$. Assume that $C\delta<\frac{1}{2}$. Let $f\colon[-1,1]\to\mathbb{R}$ be a function whose second derivative is Lipschitz with Lipschitz constant* 1 *and which satisfies $\|f'\|_\infty, \|f''\|_\infty\leqslant 1$. Set*

$$Lf(x):=\tfrac{1}{2}q_1(x)f''(x)+q_2(x)f'(x).$$

Then there is a stopping time τ_1' satisfying $\tau_0\leqslant\tau_1'\leqslant\tau_1$ almost surely and $\mathbf{P}[\tau_1'\neq\tau_1]\leqslant\eta$ such that

$$\mathbf{E}\left[f(W_{\tau_1'})-f(W_{\tau_0})-\int_{\tau_0}^{\tau_1'}Lf(W_s)\,ds\,\middle|\,\mathcal{F}_{\tau_0}\right]=O(C+1)\delta\mathbf{E}[\delta^2+\tau_1'-\tau_0\mid\mathcal{F}_{\tau_0}].$$

Moreover, in the above the function f may be random, provided that it is $\mathcal{F}_{\tau_0}$-measurable.

Proof. We inductively define the stopping times s_j as follows. Set $s_0:=\tau_0$, and $s_{j+1}:=\min\{s\geqslant s_j: s=s_j+\delta^2$ or $|W_s-W_{s_j}|=\delta$ or $s_j=\tau_1\}$. If there is a $j\in\mathbb{N}$ such that W does not satisfy (4.8) or (4.9) for the stopping times (s_j, s_{j+1}) in place of (s_0, s_1), then let n be the minimal such j. (Note that the event that W does not satisfy (4.8) or (4.9) for (s_j, s_{j+1}) is $\mathcal{F}_{s_j}$-measurable.) Otherwise, let n be the minimal j such that $s_j=\tau_1$. Note that (s_n, s_{n+1}) is a pair of stopping times and they do not satisfy both (4.8) and (4.9) unless $s_n=\tau_1$. Consequently,

$$\mathbf{P}[s_n\neq\tau_1]\leqslant\eta. \tag{4.18}$$

Since $|W_{s_{j+1}}-W_{s_j}|\leqslant\delta$ using a Taylor series for f around W_{s_j} we have

$$f(W_{s_{j+1}})-f(W_{s_j})=f'(W_{s_j})\Delta_jW+\tfrac{1}{2}f''(W_{s_j})(\Delta_jW)^2+O(\delta^3),$$

where $\Delta_jW:=W_{s_{j+1}}-W_{s_j}$. We may use (4.8) and (4.9) to estimate the conditioned expectation of Δ_jW and $(\Delta_jW)^2$ given $\mathcal{F}_{s_j}$ and get for $j<n$

$$\mathbf{E}[f(W_{s_{j+1}})-f(W_{s_j})\mid\mathcal{F}_{s_j}]=Lf(W_{s_j})\mathbf{E}[s_{j+1}-s_j\mid\mathcal{F}_{s_j}]+O(1+C)\delta^3.$$

By our assumptions about f, this may also be written as

$$\mathbf{E}\left[f(W_{s_{j+1}})-f(W_{s_j})-\int_{s_j}^{s_{j+1}} Lf(W_s)\,ds\,\middle|\,\mathcal{F}_{s_j}\right]=O(1+C)\delta^3.$$

We sum this over j from 0 to $n-1$, then take expectations conditioned on F_{τ_0}, to obtain

$$\mathbf{E}\left[f(W_{s_n})-f(W_{\tau_0})-\int_{\tau_0}^{s_n} Lf(W_s)\,ds\,\middle|\,\mathcal{F}_{\tau_0}\right]=O(1+C)\delta^3\mathbf{E}[n\,|\,\mathcal{F}_{\tau_0}]. \tag{4.19}$$

Now fix some $j\in\mathbb{N}$. On the event $j+1<n$, we have $(\Delta_j W)^2=\delta^2$ or $s_{j+1}-s_j=\delta^2$. Therefore,

$$\mathbf{E}[((\Delta_j W)^2+s_{j+1}-s_j)1_{\{j<n\}}\,|\,\mathcal{F}_{s_j}]\geqslant\delta^2\mathbf{P}[j+1<n\,|\,\mathcal{F}_{s_j}].$$

By (4.9), this gives

$$\mathbf{E}[(1+q_1(W_{s_j}))(s_{j+1}-s_j)1_{\{j<n\}}\,|\,\mathcal{F}_{s_j}]\geqslant\delta^2\mathbf{P}[j+1<n\,|\,\mathcal{F}_{s_j}]-C\delta^3 1_{\{j<n\}}.$$

We take expectation conditioned on $\mathcal{F}_{\tau_0}$ and use the fact that q_1 is bounded, to obtain

$$O(1)\mathbf{E}[(s_{j+1}-s_j)1_{\{j<n\}}\,|\,\mathcal{F}_{\tau_0}]\geqslant\delta^2\mathbf{P}[j+1<n\,|\,\mathcal{F}_{\tau_0}]-C\delta^3\mathbf{P}[j<n\,|\,\mathcal{F}_{\tau_0}].$$

We sum this over all $j\in\mathbb{N}$, to get

$$O(1)\mathbf{E}[s_n-\tau_0\,|\,\mathcal{F}_{\tau_0}]\geqslant\delta^2(1-C\delta)\mathbf{E}[n-1\,|\,\mathcal{F}_{\tau_0}]-C\delta^3.$$

By our assumption that $C\delta<\frac{1}{2}$, this implies that

$$O(1)\mathbf{E}[\delta^2+s_n-\tau_0\,|\,\mathcal{F}_{\tau_0}]\geqslant\delta^2\mathbf{E}[n\,|\,\mathcal{F}_{\tau_0}].$$

When combined with (4.19), this gives

$$\mathbf{E}\left[f(W_{s_n})-f(W_{\tau_0})-\int_{\tau_0}^{s_n} Lf(W_s)\,ds\,\middle|\,\mathcal{F}_{\tau_0}\right]=O(1+C)\delta\mathbf{E}[\delta^2+s_n-\tau_0\,|\,\mathcal{F}_{\tau_0}].$$

By (4.18), this completes the proof with $\tau_1'=s_n$. □

The next lemma bounds the expected time that W_s spends close to ±1.

LEMMA 4.7. *Let W be a (C,δ,η)-approximate (q_1,q_2)-diffusion $W:[0,S]\to[-1,1]$, where q_1 and q_2 are given by (4.6), $C\delta<\frac{1}{2}$ and $b>-\lambda$. Suppose that $S\geqslant1$. Given any $\varepsilon>0$ there is some $x_0<1$, $\delta'>0$ and $\eta'>0$ all depending only on ε, a, b and S such that if $\delta<\delta'$ and $\eta<\eta'$, then*

$$\mathbf{E}\left[\int_0^S 1_{\{W_s>x_0\}}\,ds\right]<\varepsilon.$$

A similar statement holds for the set of times such that W_s is near -1, provided that $a>-\lambda$.

Proof. Set $\mu(A):=\mathbf{E}\big[\int_0^S 1_{\{W_s\in A\}}\,ds\big]$. Note that $q_2(1)<0$. Fix some $y_0\in[0,1)$ such that $q_2(x)\leqslant\frac{1}{2}q_2(1)$ throughout $[y_0,1]$ and set $y_n:=1-(1-y_0)2^{-n}$. Let $f(x)$ be the twice continuously differentiable function that is zero on $[-1,y_0]$ and satisfies

$$f''(x)=\begin{cases}\min\{x-y_0,y_1-x\}, & \text{on } [y_0,y_1],\\ 0, & \text{on } [y_1,1].\end{cases}$$

We apply Lemma 4.6 to f with $\tau_0=0$ and $\tau_1=S$. Clearly $Lf(x)=0$ in $[-1,y_0]$. On the interval $[y_0,y_1]$, we have $f''(x)\leqslant\frac{1}{2}(y_1-y_0)$, $f'(x)\geqslant 0$ and $q_2(x)<0$. Consequently,

$$Lf(x)\leqslant\tfrac{1}{4}q_1(x)(y_1-y_0)\leqslant(1-x)(y_1-y_0)\leqslant(1-y_0)^2 \quad \text{on } [y_0,y_1].$$

On the interval $[y_1,1]$, we have $f''(x)=0$, $f'(x)=\frac{1}{4}(y_1-y_0)^2$ and $q_2(x)\leqslant\frac{1}{2}q_2(1)<0$. Consequently, $Lf(x)\leqslant -c(1-y_0)^2$, where $c>0$ depends only on $q_2(1)$. Also note that

$$|f(W_S)-f(W_0)|\leqslant\sup_x f(x)-\inf_x f(x)=f(1)-0\leqslant(1-y_0)^3.$$

Therefore, Lemma 4.6 gives

$$(1-y_0)^2\mu([y_0,y_1))-c(1-y_0)^2\mu([y_1,1])\geqslant-(1-y_0)^3+O(C+1)\delta S+O(S)\eta.$$

We assume that δ' and η' are sufficiently small so that the right-hand side is larger than $-2(1-y_0)^3$. Then we get

$$c\mu([y_1,1])\leqslant 2(1-y_0)+\mu([y_0,y_1)).$$

This implies that

$$\mu([y_1,1])\leqslant 2(1-y_0)+\frac{\mu([y_0,1])}{1+c}.$$

A similar inequality applies to y_n and y_{n+1}. Induction therefore gives

$$\mu[y_n,1]\leqslant\frac{\mu([y_0,1])}{(1+c)^n}+\sum_{j=0}^{n-1}2(1-y_j)<\frac{\mu([y_0,1])}{(1+c)^n}+4(1-y_0),$$

provided that δ' and η' are smaller than some functions of n, S, C and y_0. Consequently, we first choose y_0 such that in addition to the requirements stated in the beginning of the proof, $4(1-y_0)\leqslant\frac{1}{2}\varepsilon$. Then we take n sufficiently large so that $(1-c)^{-n}S<\frac{1}{2}\varepsilon$. Then δ' and η' are determined. This proves the first claim. The second one follows by symmetry. □

The next lemma estimates the conditional expectation and conditional second moment of the time it takes W_s to move a distance of δ beyond its location at a stopping time.

Lemma 4.8. *Let W be a (C,δ,η)-approximate (q_1,q_2)-diffusion $W:[0,S]\to[-1,1]$, where q_1 and q_2 are given by* (4.6) *and $S>1$. Let $x_0\in(0,1)$. There is a function $\delta_0>0$, depending only on x_0, C, a and b such that the following holds if $\delta<\delta_0\wedge\frac{1}{2}$ and $\eta<\delta^5/S^2$. Let τ_0 be a stopping time for W and let $\tau_1:=\inf\{s\geqslant\tau_0:s=S$ or $|W_s-W_{\tau_0}|=\delta\}$. Let $\mathcal{A}$ denote the event $\{\tau_0<S-\frac{1}{2}\}\cap\{|W_{\tau_0}|<x_0\}$. Then*

$$\mathbf{E}\left[\left|\mathbf{E}[\tau_1-\tau_0\,|\,\mathcal{F}_{\tau_0}]-\frac{\delta^2}{q_1(W_{\tau_0})}\right|1_{\mathcal{A}}\right]=O_{x_0}(C+1)\delta^3. \tag{4.20}$$

Moreover,

$$\mathbf{E}[\mathbf{E}[(\tau_1-\tau_0)^2\,|\,\mathcal{F}_{\tau_0}]1_{\mathcal{A}}]=O_{x_0}(\delta^4). \tag{4.21}$$

Proof. Let $f(x)=\frac{1}{4}(x-W_{\tau_0})^2$, and let L be as in Lemma 4.6. Then

$$Lf(x)=\tfrac{1}{4}q_1(W_{\tau_0})+O(\delta)$$

for $x\in[W_{\tau_0}-\delta,W_{\tau_0}+\delta]$. Therefore, Lemma 4.6 gives

$$\mathbf{E}\big[f(W_{\tau_1'})-\tfrac{1}{4}q_1(W_{\tau_0})(\tau_1'-\tau_0)\,\big|\,\mathcal{F}_{\tau_0}\big]=O(C+1)\delta\mathbf{E}[\delta^2+\tau_1'-\tau_0\,|\,\mathcal{F}_{\tau_0}].$$

That is,

$$\mathbf{E}[f(W_{\tau_1'})\,|\,\mathcal{F}_{\tau_0}]-\tfrac{1}{4}(q_1(W_{\tau_0})+O(C+1)\delta)\mathbf{E}[\tau_1'-\tau_0\,|\,\mathcal{F}_{\tau_0}]=O(C+1)\delta^3. \tag{4.22}$$

By choosing δ_0 sufficiently small, we make sure that $O(C+1)\delta<\frac{1}{2}q_1(W_{\tau_0})$ on $\mathcal{A}$. Since $|f(W_{\tau_1'})|\leqslant\frac{1}{4}\delta^2$, the above gives

$$\mathbf{P}[\tau_1'=S\,|\,\mathcal{F}_{\tau_0}]1_{\mathcal{A}}=\frac{O(\delta^2)}{q_1(W_{\tau_0})}.$$

Recall that $f(W_{\tau_1'})=\frac{1}{4}\delta^2$ unless $\tau_1'=S$ or $\tau_1'<\tau_1$. Therefore, on $\mathcal{A}$,

$$\mathbf{E}[f(W_{\tau_1'})\,|\,\mathcal{F}_{\tau_0}]=\tfrac{1}{4}\delta^2+O_{x_0}(\delta^4)+O(\delta^2)\mathbf{P}[\tau_1'<\tau_1\,|\,\mathcal{F}_{\tau_0}].$$

We plug this and

$$\mathbf{E}[\tau_1'-\tau_1\,|\,\mathcal{F}_{\tau_0}]=O(S)\mathbf{P}[\tau_1'\neq\tau_1\,|\,\mathcal{F}_{\tau_0}]$$

into (4.22), simplify, and get

$$\delta^2-(q_1(W_{\tau_0})+O(C+1)\delta)\mathbf{E}[\tau_1-\tau_0\,|\,\mathcal{F}_{\tau_0}]=O_{x_0}(C+1)\delta^3+O(S)\mathbf{P}[\tau_1'<\tau_1\,|\,\mathcal{F}_{\tau_0}] \tag{4.23}$$

on $\mathcal{A}$. Now (4.20) follows by dividing (4.23) by $q_1(W_{\tau_0})+O(C+1)\delta$, taking expectation and recalling that $\mathbf{P}[\tau_1'\neq\tau_1]\leqslant\eta$.

Now define $t_n := \tau_1 \wedge (\tau_0 + 16n\delta^2/q_1(W_{\tau_0}))$. If δ_0 is sufficiently small, then $Lf(x) > \frac{1}{2}Lf(W_{\tau_0}) = \frac{1}{8}q_1(W_{\tau_0})$ throughout $[W_{\tau_0} - \delta, W_{\tau_0} + \delta]$ on the event $\mathcal{A}$. Thus, we get by applying Lemma 4.6 to the stopping times t_n and t_{n+1},

$$\tfrac{1}{8}(q_1(W_{\tau_0}) - O(C+1)\delta)\mathbf{E}[t'_{n+1} - t_n \mid \mathcal{F}_{t_n}] \leqslant \tfrac{1}{4}\delta^2 + O(C+1)\delta^3,$$

where t'_{n+1} is the stopping time provided by the lemma. Again, on $\mathcal{A}$ we may assume that $O(C+1)\delta < \frac{1}{2}q_1(W_{\tau_0}) \wedge \frac{1}{4}$. Thus,

$$\tfrac{1}{16}q_1(W_{\tau_0})\mathbf{E}[t'_{n+1} - t_n \mid \mathcal{F}_{t_n}] \leqslant \tfrac{1}{2}\delta^2,$$

which implies that

$$\mathbf{P}\left[t'_{n+1} = t_n + \frac{16n\delta^2}{q_1(W_{\tau_0})} \,\middle|\, \mathcal{F}_{t_n}\right] \leqslant \frac{1}{2}.$$

But if $t'_{n+1} \neq t_n + 16n\delta^2/q_1(W_{\tau_0})$, then $t'_{n+1} = \tau_1$ or $t'_{n+1} \neq t_{n+1}$. If $\mathcal{B}_n$ denotes the event that $t'_j = t_j$ for all $j = 1, ..., n$, then induction gives

$$\mathbf{P}[t_n \neq \tau_1, \mathcal{B}_n \mid \mathcal{F}_{\tau_0}]1_{\mathcal{A}} \leqslant 2^{-n}.$$

Lemma 4.6 gives $\mathbf{P}[t'_{n+1} \neq t_{n+1}] \leqslant \eta$ and therefore $\mathbf{P}[\neg\mathcal{B}_n] \leqslant n\eta$. (In fact, it is not hard to get the better estimate $\mathbf{P}[\neg\mathcal{B}_n] \leqslant \eta$.) Consequently, for $n \in \mathbb{N}$,

$$\mathbf{P}\left[\tau_1 - \tau_0 > \frac{16n\delta^2}{q_1(W_{\tau_0})} \,\middle|\, \mathcal{F}_{\tau_0}\right]1_{\mathcal{A}} \leqslant 2^{-n} + \mathbf{P}[\neg\mathcal{B}_n \mid \mathcal{F}_{\tau_0}].$$

The above applies with $\tilde{n} := n \wedge \lceil -\log_2 \eta \rceil$ in place of n, and hence

$$\mathbf{P}\left[\tau_1 - \tau_0 > \frac{16n\delta^2}{q_1(W_{\tau_0})} \,\middle|\, \mathcal{F}_{\tau_0}\right]1_{\mathcal{A}} \leqslant \mathbf{P}\left[\tau_1 - \tau_0 > \frac{16\tilde{n}\delta^2}{q_1(W_{\tau_0})} \,\middle|\, \mathcal{F}_{\tau_0}\right]1_{\mathcal{A}} \leqslant 2^{-\tilde{n}} + \mathbf{P}[\neg\mathcal{B}_{\tilde{n}} \mid \mathcal{F}_{\tau_0}].$$

We multiply both sides by $2(n+1)(16\delta^2/q_1(W_{\tau_0}))^2$, and sum over n from $n=0$ to the least m such that $16m\delta^2/q_1(W_{\tau_0}) \geqslant S$. The result on the left-hand side bounds

$$\mathbf{E}[(\tau_1 - \tau_0)^2 \mid \mathcal{F}_{\tau_0}]1_{\mathcal{A}}.$$

Consequently, the required bound (4.21) follows by taking expectations and using our assumed upper bound for η. □

The following lemma shows that when we discretize the approximate diffusion the resulting random walk has transition probabilities that can be well estimated from q_1 and q_2 away from the boundary.

Lemma 4.9. *Fix some* $x_0\in(0,1)$. *Let* $W:[0,S]\to[-1,1]$ *be a* (C,δ,η)*-approximate* (q_1,q_2)*-diffusion, where* q_1 *and* q_2 *are given by* (4.6), $S\geqslant 1$ *and* $\eta\leqslant\delta^5/S$. *Set*

$$Z:=\{k\delta:k\in\mathbb{Z} \text{ and } |k\delta|<x_0\},$$

$s_0:=\inf\{s\geqslant 0:W_s\in Z \text{ or } s=S\}$ *and inductively*

$$s_{n+1}:=\inf\{s\geqslant s_n:W_{s_n}\neq W_s\in Z \text{ or } s=S\}.$$

Also set $X_n:=W_{s_n}$ *and* $Z^0:=Z\setminus\{\min Z,\max Z\}$. *Let*

$$p_n^\pm:=\mathbf{P}[X_{n+1}=X_n\pm\delta\mid\mathcal{F}_{s_n}]$$

and

$$r_n^\pm:=\frac{1}{2}\pm\delta\frac{q_2(X_n)}{2q_1(X_n)}.$$

There is a $\delta_0>0$, *depending only on* C, x_0, a *and* b, *such that if* $\delta<\delta_0$, *then for all* $n\in\mathbb{N}$,

$$\mathbf{E}[|p_n^\pm-r_n^\pm|1_{\{s_n<S-1/2\}}1_{\{X_n\in Z^0\}}]\leqslant O_{x_0}(C+1)\delta^2.$$

Proof. We now use a different test function:

$$f(x):=\alpha(x-X_n)^2+\beta(x-X_n),$$

where $\alpha:=-q_2(X_n)\beta/q_1(X_n)$ and $\beta:=|q_1(X_n)/6q_2(X_n)|\wedge\frac{1}{3}$ with $\beta=\frac{1}{2}$ if $q_2(X_n)=0$. The choice of α and β above is tailored to give $Lf(X_n)=0$ and $|4\alpha|+|\beta|\leqslant 1$. The latter implies that $\|f''\|_\infty,\|f'\|_\infty\leqslant 1$ when f is restricted to the interval $[-1,1]$. We now apply Lemma 4.6 again with this f and stopping times s_n and s_{n+1}. Note that $Lf=O(\delta)$ in the interval $[X_n-\delta,X_n+\delta]$. Hence $Lf(W_s)=O(\delta)$ for $s\in[s_n,s_{n+1}]$. Lemma 4.6 gives

$$\mathbf{E}[f(W_{s'_{n+1}})-f(W_{s_n})\mid\mathcal{F}_{s_n}]=O(C+1)\delta\mathbf{E}[\delta^2+s'_{n+1}-s_n\mid\mathcal{F}_{s_n}],$$

on the event $X_n\in Z^0$, where s'_{n+1} is the stopping time produced by the lemma. The above may be written

$$\begin{aligned}(\beta\delta+\alpha\delta^2)p_n^++(-\beta\delta+\alpha\delta^2)p_n^-=-&\mathbf{E}[1_{\{W_{s'_{n+1}}\notin Z\}}f(W_{s'_{n+1}})\mid\mathcal{F}_{s_n}]\\ &+O(C+1)\delta\mathbf{E}[\delta^2+s'_{n+1}-s_n\mid\mathcal{F}_{s_n}].\end{aligned}$$

Set $p_n^0:=\mathbf{P}[W_{s'_{n+1}}\notin\{X_n-\delta,X_n+\delta\}|\mathcal{F}_{s_n}]$. Then $1\geqslant p_n^++p_n^-\geqslant 1-p_n^0$. Hence, the above gives

$$(2p_n^+-1)\beta+\alpha\delta=O(p_n^0)+O(C+1)\delta^2+O(C+1)\mathbf{E}[s_{n+1}-s_n\mid\mathcal{F}_{s_n}],$$

which, by the definitions of α and r_n^+ may be rewritten

$$2\beta(p_n^+ - r_n^+) = O(p_n^0) + O(C+1)\delta^2 + O(C+1)\mathbf{E}[s_{n+1} - s_n \mid \mathcal{F}_{s_n}]. \tag{4.24}$$

By (4.20) and our assumption $\eta \leqslant \delta^5/S$, we have

$$\mathbf{E}[(s_{n+1} - s_n)1_{X_n \in Z^0} 1_{s_n < S-1/2}] = O_{x_0}(\delta^2), \tag{4.25}$$

provided that δ_0 is sufficiently small. Since $W_{s'_{n+1}} \notin Z$ only when $s'_{n+1} \neq s_{n+1}$ or $s'_{n+1} = s_{n+1} = S$, on the event $\{X_n \in Z^0\} \cap \{s_n < S - \frac{1}{2}\}$,

$$\begin{aligned} \mathbf{E}[p_n^0 \mid \mathcal{F}_{s_n}] &\leqslant \mathbf{P}[s'_{n+1} \neq s_{n+1} \mid \mathcal{F}_{s_n}] + \mathbf{P}\left[s'_{n+1} \geqslant s_n + \tfrac{1}{2} \;\middle|\; \mathcal{F}_{s_n}\right] \\ &\leqslant \mathbf{P}[s'_{n+1} \neq s_{n+1} \mid \mathcal{F}_{s_n}] + 2\mathbf{E}[s_{n+1} - s_n \mid \mathcal{F}_{s_n}]. \end{aligned} \tag{4.26}$$

Note that $\beta^{-1} = O_{x_0}(1)$ and $\mathbf{P}[s'_{n+1} \neq s_{n+1}] \leqslant \eta$. Hence, we now obtain the result for p_n^+ by taking expectation on the event $\{X_n \in Z^0\} \cap \{s_n < S - \frac{1}{2}\}$ in (4.24) and using (4.25) and (4.26). A symmetric argument applies to p_n^- and r_n^-, and the proof is complete. □

Next, we show that the time parameterization of W can be well approximated by a function of the discretized walk trajectory.

LEMMA 4.10. *Assume the setting and notation of Lemma* 4.9 *in addition to* $\delta < \delta_0$. *For* $n \in \mathbb{N}$ *let* t_n *denote the time spent up to time* s_n *in segments* $[s_j, s_{j+1}]$ *such that* $X_j = W_{s_j} \in Z^0$; *that is,*

$$t_n := \sum_{j=0}^{n-1} 1_{\{X_j \in Z^0\}}(s_{j+1} - s_j).$$

Also let

$$\sigma_n := \sum_{j=0}^{n-1} 1_{\{X_j \in Z^0\}} \frac{\delta^2}{q_1(X_j)}.$$

Let $N_0 := \min\{n \in \mathbb{N} : s_n \geqslant S - \frac{1}{2}\}$. *Then for all* $n \in \mathbb{N}$,

$$\mathbf{E}[\max\{|\sigma_j - t_j| : j = 1, ..., n \wedge N_0\}] \leqslant O_{x_0}(C+1)(\delta^2 n^{1/2} + \delta^3 n). \tag{4.27}$$

Proof. Let $v_j := (s_{j+1} - s_j)1_{\{X_j \in Z^0\}} 1_{\{j < N_0\}}$, $u_j := \mathbf{E}[v_j \mid \mathcal{F}_{s_j}]$ and

$$w_j := \frac{\delta^2}{q_1(X_j)} 1_{\{X_j \in Z^0\}} 1_{\{j < N_0\}}.$$

Now, $M_n := \sum_{j=0}^{n-1}(v_j - u_j)$ is clearly a martingale. Consequently, Doob's maximal inequality for L^2 martingales [RY, II.1.6] gives

$$\mathbf{E}[\max\{|M_j| : j = 1, ..., n\}]^2 \leqslant O(1)\mathbf{E}[M_n^2] = O(1) \sum_{j=0}^{n-1} \mathbf{E}[(v_j - u_j)^2].$$

Since $u_j = \mathbf{E}[v_j | \mathcal{F}_{s_j}]$, we have $\mathbf{E}[(v_j - u_j)^2] \leqslant \mathbf{E}[v_j^2]$. By Lemma 4.8, $\mathbf{E}[v_j^2]$ is bounded by the right-hand side of (4.21). Now, the right-hand side of (4.20) bounds $\mathbf{E}[|u_j - w_j|]$. The result follows by our assumption $\eta \leqslant \delta^5/S^2$, since for every $m \leqslant n \wedge N_0$,

$$|\sigma_m - t_m| = \left| \sum_{j=0}^{m-1} (v_j - w_j) \right| \leqslant |M_m| + \sum_{j=0}^{m-1} |u_j - w_j| \leqslant \max\{|M_j| : j = 1, ..., n\} + \sum_{j=0}^{n-1} |u_j - w_j|. \qquad \square$$

Proof of Proposition 4.5. By Lemma 4.4, W^2 is a $(C', \delta, 0)$-approximate (q_1, q_2)-diffusion for some fixed constant $C' > 0$ and every $\delta > 0$. We may assume, with no loss of generality, that $C \geqslant C'$. Let $\varepsilon > 0$. Let $x_0' \in (1-\varepsilon, 1)$ satisfy Lemma 4.7 with this given ε, and assume that δ is sufficiently small so that that lemma is valid. Take $x_0 = \frac{1}{2}(1 + x_0')$. Let Z and Z^0 be as in Lemma 4.9, let s_j^k be the corresponding stopping times introduced there for W^k and let $p_{k,j}^{\pm}$ denote the random transition probabilities for W^k defined there. Also abbreviate $X_j^k := W_{s_j}^k$. Let $\mathcal{F}_s^k$ denote the filtration of W^k, $k = 1, 2$. Let $Y_j^k = 1$ if $X_{j+1}^k - X_j^k = \delta$, $Y_j^k = -1$ if $X_{j+1}^k - X_j^k = -\delta$ and $Y_j^k = 0$ if $|X_{j+1}^k - X_j^k| \neq \delta$. Then $\mathbf{P}[Y_j^k = \pm 1 | \mathcal{F}_{s_j^k}^k] = p_{k,j}^{\pm}$ and $\mathbf{P}[Y_j^k = 0 | \mathcal{F}_{s_j^k}^k] = 1 - p_{k,j}^+ - p_{k,j}^-$ if $X_j^k \in Z^0$.

For the coupling of W^1 and W^2 we use an independent identically distributed sequence U_j of uniform random variables in $[0,1]$. The coupling proceeds as follows. Up to their corresponding stopping times s_0^k, $k = 1, 2$, let them run independently. Inductively, we suppose that the coupling has been constructed up to their corresponding stopping times s_j^k, $k = 1, 2$. For each $k = 1, 2$, we take

$$Y_j^k = \begin{cases} 1, & \text{if } U_j \leqslant p_{k,j}^+, \\ -1, & \text{if } U_j \geqslant 1 - p_{k,j}^-, \\ 0, & \text{if } U_j \in (p_{k,j}^+, 1 - p_{k,j}^-). \end{cases}$$

(In other words, we try to match up $X_{j+1}^1 - X_j^1$ with $X_{j+1}^2 - X_j^2$ as much as possible.) These choices respect the correct conditional distributions for these variables. Now we sample the restriction of W^1 to $[s_j^1, s_{j+1}^1]$ and the restriction of W^2 to $[s_j^2, s_{j+1}^2]$ independently of their corresponding conditional distribution given $(\mathcal{F}_{s_j}^1, Y_j^1)$ and $(\mathcal{F}_{s_j}^2, Y_j^2)$, respectively. This completes the description of the coupling.

Let $N := \min\{n : s_n^1 \vee s_n^2 \geqslant S - \frac{1}{2}\}$. Let $\mathcal{A}_j$ be the event $\{X_j^1, X_j^2 \in Z^0\}$ and set

$$Q_j := 1_{\mathcal{A}_j} |Y_j^1 - Y_j^2| \quad \text{and} \quad \widehat{Q}_n := \sum_{j=0}^{n} Q_j 1_{\{j<N\}}.$$

Note that on $\neg\mathcal{A}_j$ we have $|X_{j+1}^1 - X_{j+1}^2| \leqslant |X_j^1 - X_j^2|$ unless $s_{j+1}^1 \vee s_{j+1}^2 = S$. Moreover, $|X_0^1 - X_0^2| \leqslant \delta$ and when $Y_j^k = 0$ we have $s_{j+1}^k = S$. Consequently,

$$|X_n^1 - X_n^2| 1_{\{n<N\}} \leqslant \delta + \delta \widehat{Q}_{n-1}. \tag{4.28}$$

We now proceed to estimate $\theta_j := \mathbf{E}[Q_j 1_{\{j<N\}}]$. Clearly,

$$\mathbf{P}[Q_j \neq 0 \mid \mathcal{F}^1_{s^1_j} \vee \mathcal{F}^2_{s^2_j}] \leqslant (|p^+_{1,j} - p^+_{2,j}| + |p^-_{1,j} - p^-_{2,j}|) 1_{\mathcal{A}_j}.$$

Let $r^{\pm}_{k,j}$ be the $r^{\pm}_j$ in Lemma 4.9 corresponding to the process W^k. Then

$$|p^{\pm}_{1,j} - p^{\pm}_{2,j}| \leqslant |p^{\pm}_{1,j} - r^{\pm}_{1,j}| + |r^{\pm}_{1,j} - r^{\pm}_{2,j}| + |r^{\pm}_{2,j} - p^{\pm}_{2,j}|.$$

By that lemma,

$$\mathbf{E}[|p^{\pm}_{k,j} - r^{\pm}_{k,j}| 1_{\mathcal{A}_j} 1_{\{j<N\}}] \leqslant O_{x_0}(C+1)\delta^2.$$

Using the expression given for $r^{\pm}_{k,j}$, we deduce that

$$|r^{\pm}_{1,j} - r^{\pm}_{2,j}| \leqslant O_{x_0}(\delta) |X^1_j - X^2_j|.$$

Thus, we get

$$\theta_n = \mathbf{E}[Q_n 1_{\{n<N\}}] \leqslant O_{x_0}(C+1)\delta^2 + O_{x_0}(\delta) \mathbf{E}[|X^1_n - X^2_n| 1_{\{n<N\}}].$$

In conjunction with (4.28), this gives

$$\theta_n \leqslant O_{x_0}(C+1)\delta^2 \left(1 + \sum_{j=0}^{n-1} \theta_j\right).$$

Induction therefore implies

$$\theta_n \leqslant O_{x_0}(C+1)\delta^2 (1 + O_{x_0}(C+1)\delta^2)^n.$$

Taking note of (4.28), we infer that

$$\mathbf{E}[\max_{j<n\wedge N} |X^1_j - X^2_j|] \leqslant \delta + O_{x_0}(C+1)\delta^3 n (1 + O_{x_0}(C+1)\delta^2)^n. \tag{4.29}$$

Now let $m_0 := \lceil 4S\delta^{-2} \max\{q_1(x) : |x| \leqslant x_0\} \rceil$. Observe that at least one of every two consecutive $j \in \mathbb{N}$ satisfies $X^k_j \in Z^0$ or $s^k_j = S$. Consequently, $m_0 < N$ implies that $\sigma^k_{m_0} \geqslant 2S$ for $k = 1, 2$, where σ^k_j denotes the σ_j from Lemma 4.10 corresponding to W^k. Note that in that lemma $t_j \leqslant s_j \leqslant S$. Therefore, taking $n = m_0$ in (4.27) implies that

$$\mathbf{P}[N > m_0] \leqslant O_{x_0}(C+1) S^{-1} (\delta^2 m_0^{1/2} + \delta^3 m_0) = O_{x_0}(C+1)\delta.$$

Set $X^* := \max_{j<N} |X^1_j - X^2_j|$. The above and (4.29) with $n = m_0$ imply that

$$\mathbf{P}[X^* > \delta^{1/2} \text{ or } N > m_0] \leqslant O_{x_0,C,S}(\delta^{1/2}). \tag{4.30}$$

Now for each $s \leqslant S$ let $J(s) := \min\{j \in \mathbb{N} : s_j^1 \geqslant s\}$. Then

$$\sup_{s\in[0,S-1]} |W_s^1 - W_s^2| \leqslant \sup_{s\in[0,S-1]} |W_s^1 - X_{J(s)}^1| + \sup_{s\in[0,S-1]} |X_{J(s)}^1 - X_{J(s)}^2| + \sup_{s\in[0,S-1]} |W_s^2 - X_{J(s)}^2|. \tag{4.31}$$

First, it is clear that

$$\sup_{s\in[0,S-1]} |W_s^1 - X_{J(s)}^1| \leqslant \varepsilon + \delta.$$

(The left-hand side is usually at most δ but can be as large as $1 - x_0 + \delta$ if, for example, $X_{J(s)-1} = \max Z = x_0$.) We leave aside, for now, the estimation of the second summand in (4.31) and consider the last. Set

$$t^* := \sup_{s\in[0,S-1]} |s - s_{J(s)}^2|.$$

Since $X_{J(s)}^2 = W_{s_{J(s)}^2}^2$, we have

$$\sup_{s\in[0,S-1]} |W_s^2 - X_{J(s)}^2| \leqslant \sup_{s\in[0,S-1]} \sup_{t\in[0,t^*]} |W_s^2 - W_{s+t}^2|. \tag{4.32}$$

Observe from (4.27) that for $k = 1, 2$,

$$\max_{j\leqslant N\wedge m_0} \{|\sigma_j^k - t_j^k|\} \to 0$$

in probability as $\delta \to 0$, where t_j^k is the t_j of Lemma 4.10 corresponding to W^k. Now the choice of x_0 (via Lemma 4.7) implies that

$$\mathbf{P}[\max_{j\in\mathbb{N}} |s_0^k + t_j^k - s_j^k| > \sqrt{\varepsilon}\,] < \sqrt{\varepsilon}.$$

Lemmas 4.7 and 4.8 imply that $\mathbf{E}[s_0^k] \leqslant \varepsilon \vee O_{x_0,C}(\delta^2)$. Consequently, we have

$$\max_{j\leqslant N\wedge m_0} \{|\sigma_j^k - s_j^k|\} \to 0 \tag{4.33}$$

in probability as $\varepsilon, \delta \to 0$. Now,

$$\begin{aligned} |\sigma_n^1 - \sigma_n^2| &\leqslant \sum_{j=0}^{n-1} \left| 1_{\{X_j^1 \in Z^0\}} \frac{\delta^2}{q_1(X_j^1)} - 1_{\{X_j^2 \in Z^0\}} \frac{\delta^2}{q_1(X_j^2)} \right| \\ &\leqslant \sum_{j=0}^{n-1} \left| \frac{\delta^2}{q_1(X_j^1)} - \frac{\delta^2}{q_1(X_j^2)} \right| + \sum_{k=1}^{2} \sum_{j=0}^{n-1} 1_{\{X_j^k \notin Z^0\}} \frac{\delta^2}{q_1(X_j^k)}. \end{aligned}$$

The right-hand side is monotone non-decreasing in n. When $n \leqslant N \wedge m_0$ the first sum is at most $O_{x_0}(\delta^2) m_0 X^*$. It is easy to see that for $n = N \wedge m_0$ the iterated sum on the right tends to 0 in probability: this follows from the proof of (4.33), because if we replace x_0 by $x_0 + \delta$, the terms appearing in this iterated sum are included in σ_n. We now get, from (4.30),

$$\max_{j \leqslant N} |\sigma_j^1 - \sigma_j^2| \to 0 \quad \text{in probability as } \varepsilon, \delta \to 0.$$

By (4.33), this also gives

$$\max_{j \leqslant N} |s_j^1 - s_j^2| \to 0 \quad \text{in probability as } \varepsilon, \delta \to 0.$$

In particular, (4.30) and (4.33) imply $\sup_{j \leqslant N-1}(s_{j+1}^k - s_j^k) \to 0$ in probability, because $\sigma_{j+1}^k - \sigma_j^k \leqslant O_{x_0}(\delta^2)$. One consequence is that $\mathbf{P}[J(S-1) \geqslant N] \to 0$. Furthermore, we now have

$$t^* \leqslant \sup_{s \in [0, S-1]} |s - s_{J(s)}^1| + \sup_{s \in [0, S-1]} |s_{J(s)}^2 - s_{J(s)}^1| \to 0$$

in probability. Now (4.32) implies that $\sup_{s \in [0, S-1]} |W_s^2 - X_{J(s)}^2| \to 0$ in probability, because the right-hand side in (4.32) is smaller than $(t^*)^{1/3}$ with probability going to 1 as $t^* \to 0$, since W^2 is a solution of (4.7). This takes care of the last summand on the right-hand side of (4.31).

The middle summand on the right-hand side of (4.31) also tends to 0 in probability because, as we have seen, $\mathbf{P}[J(S-1) < N] \to 1$ and $X^* \to 0$ in probability. This completes the proof. □

Proof of Theorem 4.1. Let $x_1, x_2 \in [-1, 1]$ be two arbitrary points, and let

$$Y_s^1, Y_s^2 \colon [0, \infty) \to [-1, 1]$$

be two independent solutions of (4.7) (with respect to two independent Brownian motions) which start at x_1 and x_2, respectively. We claim that $s_= := \min\{s : Y_s^1 = Y_s^2\} < \infty$ almost surely. The argument is quite standard. Suppose without loss of generality that $x_2 > x_1$. By Lemma 4.7, it is unlikely that Y_s^2 stays very close to 1 for a long time and unlikely that Y_s^1 stays very close to -1 for a long time. It is therefore easy to conclude from Lemmas 4.8 and 4.9 that there are constants $s_0, c_0 > 0$ (which do not depend on x_1 or x_2) such that $\mathbf{P}[Y_{s_0}^2 < 0] > c_0$ and $\mathbf{P}[Y_{s_0}^1 > 0] > c_0$. This implies that $\mathbf{P}[s_= < s_0] > c_0^2$. By strong uniqueness of solutions of (4.7), it follows that the solutions are Markov and have stationary transition probabilities. Consequently, we get by induction $\mathbf{P}[s_= > n s_0] < (1 - c_0^2)^n$ for all $n \in \mathbb{N}$, which proves that $s_= < \infty$ almost surely.

We now argue that $\mathbf{P}[s_= < \infty] = 1$ and the uniqueness in law of solutions of (4.7) implies that for every Borel subset $A \subset [-1,1]$ the limit

$$\mu(A) := \lim_{r\to\infty} \mathbf{P}[Y_r^1 \in A]$$

exists. We may couple a solution started at some time $s_0 < r$ such that $r - s_0$ is a large constant to be independent of Y_s^1 until the first time in $[s_0, \infty)$ in which they meet and to agree with Y_s^1 afterwards. Because these solutions are likely to meet prior to time r, it follows that $\mathbf{P}[Y_r^1 \in A]$ is close to the probability that the solution started at time s_0 is in A at time r, proving the existence of μ. (In our setting, μ may be explicitly described. Its density with respect to the Lebesgue measure is proportional to $(1+x)^{(b-\lambda)/2\lambda}(1-x)^{(a-\lambda)/2\lambda}$.) Since solutions of (4.7) are Markov, a solution

$$\widetilde{Y} : [0,\infty) \to [-1,1]$$

of (4.7) such that the distribution of $\widetilde{Y}_0$ is given by μ is time-stationary. To get a time-stationary solution $Y : (-\infty,\infty) \to [-1,1]$, we may take the weak limit of time-translations of $\widetilde{Y}$.

Now let S' be much larger than S. By Proposition 4.2 with S' instead of S and Proposition 4.5 translated to start at time $-S'$ and an appropriate choice of the S appearing there, we may couple $\widetilde{W}_s$ so that with probability close to 1 it stays close to a solution W_s^2 of (4.7) starting at $\widetilde{W}_{-S'}$ throughout $[-S', S']$. We may at the same time couple W_s^2 so that with high probability it agrees with Y_s inside the interval $[-S, S]$. Then with high probability $\widetilde{W}_s$ stays close to Y_s in $[-S, S]$, which concludes the proof of the theorem. □

Remark 4.11. At this point, it may be worthwhile to point out which properties of the functions q_1 and q_2 played a part in the proof. The only properties that are essential for the above proof are that q_1 and q_2 are both Lipschitz continuous in $[-1,1]$, that $q_1 > 0$ in $(-1,1)$ and $q_1 = 0$ on $\{-1,1\}$, and that $q_2(1) < 0 < q_2(-1)$.

4.5. Back to chordal

In §4.2, we described the transition from the chordal Loewner system to the setup with the points ± 1 fixed. We now describe the reverse transformation. We start with some continuous $Y : (-\infty,\infty) \to [-1,1]$. Set

$$w_s := \frac{e^s}{2} Y_s + \frac{1}{2}\int_{-\infty}^{s} e^u Y_u \, du. \tag{4.34}$$

Also define

$$t^*(s) := \frac{1}{8}\int_{-\infty}^{s} e^{2u}(1-Y_u^2)\,du, \quad s^*(t) := \sup\{s\in(-\infty,\infty) : t^*(s)\leqslant t\}.$$

Now set $\widehat{Y}_t := w_{s^*(t)}$ for $t>0$, $\widehat{Y}_0 := 0$, and observe that $\widehat{Y}$ is continuous provided that there is no non-trivial time interval in which $Y\in\{\pm 1\}$.

LEMMA 4.12. *If $Y_s = \widetilde{W}_s$ is defined from W_t as in §4.2, then $\widehat{Y}_t = W_t$.*

Proof. By the definition of $s(t)$ in §4.2, we have $e^s = y_t - x_t$. Consequently, (4.4) implies that $\partial_t(t^*(s(t)))=1$. Since $s(0)=-\infty$ and $t^*(-\infty)=0$, it follows that $t^*(s(t))=t$ for all $t\geqslant 0$ and $s^*(t^*(s))=s$ for all $s\in(-\infty,\infty)$. Next, (1.4) and the definition of W^* give

$$\partial_t(y_t+x_t) = \frac{8W_t^*}{(y_t-x_t)(1-(W_t^*)^2)}.$$

Since $y_0=0=x_0$ and $\widetilde{W}_{s(t)} = W_t^*$, this implies that

$$y_t+x_t = \int_0^t \frac{8\widetilde{W}_{s(r)}}{(y_r-x_r)(1-\widetilde{W}_{s(r)}^2)}\,dr = \int_{-\infty}^{s(t)} e^u \widetilde{W}_u\,du,$$

where the second equality follows by a change of variable. Now $\widehat{Y}_t = w_{s(t)} = W_t$ follows from the definition of W_t^*. The proof of the lemma is therefore complete. □

We now discuss the behavior of solutions of (4.7) in the chordal coordinate system, but generalize to the case $\varkappa\neq 4$.

LEMMA 4.13. *Let $\tilde{a}, \tilde{b}>0$, let $Y : (-\infty,\infty)\to[-1,1]$ be a solution of*

$$dY_s = (-\tilde{a}(Y_s-1)-\tilde{b}(Y_s+1))\,ds + \sqrt{\tfrac{1}{2}\varkappa(1-Y_s^2)}\,dB_s,$$

and let $\widehat{Y}_t$ be the corresponding process, as described following (4.34)*. Then on any time interval which avoids $\{t : Y_{s^*(t)}\notin\{\pm 1\}\}$, the process $\widehat{Y}_t$ satisfies the SDE of the driving term for* SLE$(\varkappa; 2(\tilde{a}-1), 2(\tilde{b}-1))$:

$$d\widehat{Y}_t = \frac{2(\tilde{a}-1)}{\widehat{Y}_t - x_t} + \frac{2(\tilde{b}-1)}{\widehat{Y}_t - y_t} + \sqrt{\varkappa}\,d\widehat{B}_t$$

for some Brownian motion $\widehat{B}_t$, where x_t and y_t satisfy (1.4) *with $\widehat{Y}_t$ in place of W_t.*

Proof. Let $\beta_s = \int_{-\infty}^s e^u Y_u \, du$, $y_s^* := \frac{1}{2}(\beta_s + e^s)$ and $x_s^* := \frac{1}{2}(\beta_s - e^s)$. Itô's formula and the definition of t^* give

$$\begin{aligned} dw_s &= e^s Y_s \, ds + \tfrac{1}{2} e^s \sqrt{\tfrac{1}{2}\varkappa(1-Y_s^2)} \, dB_s + \tfrac{1}{2} e^s (-\tilde{a}(Y_s-1) - \tilde{b}(Y_s+1)) \, ds \\ &= \sqrt{\varkappa (t^*)'(s)} \, dB_s + 4e^{-s} \left(\frac{\tilde{a}-1}{Y_s+1} + \frac{\tilde{b}-1}{Y_s-1} \right) (t^*)'(s) \, ds \\ &= \sqrt{\varkappa (t^*)'(s)} \, dB_s + 2 \left(\frac{\tilde{a}-1}{w_s - x_s^*} + \frac{\tilde{b}-1}{w_s - y_s^*} \right) (t^*)'(s) \, ds. \end{aligned}$$

Now set

$$\widehat{B}_t = \int_{-\infty}^{s^*(t)} \sqrt{(t^*)'(u)} \, dB_u.$$

Then $\widehat{B}_t$ is clearly a continuous martingale. Since also $\langle \widehat{B} \rangle_t = \int_{-\infty}^{s^*(t)} (t^*)'(u) \, du = t$, we find that $\widehat{B}$ is a Brownian motion with respect to t. The above formula for dw gives

$$d\widehat{Y}_t = \sqrt{\varkappa} \, d\widehat{B}_t + 2 \left(\frac{\tilde{a}-1}{\widehat{Y}_t - x_t} + \frac{\tilde{b}-1}{\widehat{Y}_t - y_t} \right) dt,$$

where $x_t := x^*_{s^*(t)}$ and $y_t := y^*_{s^*(t)}$. Now

$$\partial_t x_t = \frac{\partial_s x_s^*}{(t^*)'(s)} = \frac{2}{x_t - \widehat{Y}_t},$$

and similarly for y_t. This concludes the proof. □

As mentioned at the beginning of this section (§4), existence and uniqueness of solutions to the usual SDE defining SLE($\varkappa; \varrho_1, \varrho_2$) have not been proved beyond times when the driving term W_t meets the force points. We now offer the following.

Definition 4.14. If $\tilde{a}, \tilde{b} > 0$, then the Loewner equation driven by the process $\widehat{Y}$ of Lemma 4.13 is called SLE($\varkappa; 2(\tilde{a}-1), 2(\tilde{b}-1)$) (starting from $(0, 0_-, 0_+)$).

4.6. Loewner driving term convergence

In this section, we complete the proof of Theorem 1.3. The theorem will follow quite easily from Theorem 4.1.

Proof of Theorem 1.3. Fix $T, \varepsilon, \varepsilon_0, \varepsilon' > 0$. Let Y_s and $\widetilde{W}_s$ be coupled as in Theorem 4.1, but with ε' in place of ε. Let $\widehat{Y}_t$, $t^*(s)$ and $s^*(t)$ be defined from Y_s as in the beginning of §4.5. Since the interior of the set of times for which $Y_s \in \{-1, 1\}$ is empty, it follows that there is some positive $t_0>0$ such that with probability at least $1-\varepsilon_0$ we have $t^*(1)-t^*(0)>t_0$. Because Y_s is stationary, it follows that

$$\mathbf{P}[t^*(s+1)-t^*(s)>e^{2s}t_0]\geqslant 1-\varepsilon_0.$$

In particular, there is some $S_0>0$ such that $\mathbf{P}[t^*(S_0)>T+1]\geqslant 1-\varepsilon_0$. Fix S_0 satisfying this and additionally $e^{-S_0}<\frac{1}{2}\varepsilon_0$.

Let

$$\tilde{t}(s):=\frac{1}{8}\int_{-\infty}^{s}e^{2u}(1-\widetilde{W}_u^2)\,du,$$

which is the equivalent of $t^*(s)$ with $\widetilde{W}$ replacing Y. It is clear that if $\varepsilon'=\varepsilon'(S_0,\varepsilon_0)$ is sufficiently small and

$$\sup\bigl\{|\widetilde{W}_s-Y_s|:s\in[-S_0,S_0]\bigr\}<\varepsilon', \tag{4.35}$$

then for every $s\in[-\infty,S_0]$ the right-hand side in (4.34) differs from the corresponding quantity where $\widetilde{W}$ replaces Y by at most ε_0. Lemma 4.12 then gives

$$\sup\{|\widehat{W}_{\tilde{t}(s)}-W_{t^*(s)}|:s\leqslant[0,S_0]\}<\varepsilon_0,$$

where W_t is the chordal driving term for SLE$(4;a/\lambda-1,b/\lambda-1)$ and $\widehat{W}$ is the chordal driving term for $\phi_D\circ\gamma$. (Here, we also use the fact that $s^*(t^*(s))=s$.) If we assume (4.35) with ε' sufficiently small, we also get $\sup\{|t^*(s)-\tilde{t}(s)|:s\leqslant S_0\}<\varepsilon_0$. Let T_0 be the obvious upper bound for $\tilde{t}$ and t^* in $(-\infty,S_0]$; that is, $T_0:=\frac{1}{16}e^{2S_0}$. Also set

$$M(\varepsilon_0):=\sup\{|W_{t_0}-W_{t_1}|:0\leqslant t_0\leqslant t_1\leqslant t_0+\varepsilon_0\leqslant T_0+\varepsilon_0\}.$$

Since W_t is almost surely continuous, $M(\varepsilon_0)\to 0$ in probability as $\varepsilon_0\to 0$. Now the triangle inequality $|\widehat{W}_{\tilde{t}(s)}-W_{\tilde{t}(s)}|\leqslant|\widehat{W}_{\tilde{t}(s)}-W_{t^*(s)}|+|W_{t^*(s)}-W_{\tilde{t}(s)}|$ shows that when (4.35) holds we have $\sup\{|\widehat{W}_{\tilde{t}(s)}-W_{\tilde{t}(s)}|:s\leqslant S_0\}<\varepsilon_0+M(\varepsilon_0)$. Hence,

$$\sup\{|\widehat{W}_t-W_t|:t\in[0,\tilde{t}(S_0)]\}<\varepsilon_0+M(\varepsilon_0).$$

But we have seen that $t^*(S_0)$ is very likely to be larger than $T+1$ and that

$$|t^*(S_0)-\tilde{t}(S_0)|<\varepsilon_0$$

when (4.35) holds. By Theorem 4.1, when r_D is large (4.35) holds with high probability. This concludes the proof. □

4.7. Carathéodory convergence

For $K\subset\overline{\mathbb{H}}$ let $N_\varepsilon(K)$ denote the ε-neighborhood of K in $\overline{\mathbb{H}}$, and let $\overline{\mathbb{H}}_\varepsilon(K)$ denote the unbounded connected component of $\overline{\mathbb{H}}\setminus N_\varepsilon(K)$. Set

$$d_{\mathrm{CKC}}(K,K') := \inf\{\varepsilon>0 : K\cap\overline{\mathbb{H}}_\varepsilon(K') = \varnothing = K'\cap\overline{\mathbb{H}}_\varepsilon(K)\}.$$

It is easy to see that d_{CKC} is a metric on the collection of compact connected $K\subset\overline{\mathbb{H}}$ such that $\mathbb{H}\setminus K$ is connected. This metric is related to the Carathéodory kernel convergence topology, which is of central importance in the theory of conformal mappings.

Let K_t and K'_t denote the evolving hulls corresponding to two Loewner evolutions generated by continuous driving terms W_t and W'_t, respectively (as defined in §1.4). Such evolving hulls are also sometimes called *Loewner chains.* We set, for $T\geqslant 0$,

$$d^T_{\mathrm{CKC}}(K,K') := \sup_{t\in[0,T]} d_{\mathrm{CKC}}(K_t,K'_t).$$

The following is a simple lemma relating uniform convergence of driving terms to d_{CKC}-convergence of the corresponding Loewner chains.

LEMMA 4.15. *The Loewner transform $W\mapsto K$ is a continuous map from the space of continuous paths W with the topology of uniform convergence to the space of Loewner chains with d^T_{CKC}-convergence.*

In other words, for every $\varepsilon>0$, every $T>0$ and every $W:[0,T]\to\mathbb{R}$ continuous, there is some $\delta=\delta(\varepsilon,T,W)>0$ such that if $\widetilde{W}:[0,T]\to\mathbb{R}$ is continuous and satisfies $\sup_{t\in[0,T]}|W_t-\widetilde{W}_t|<\delta$, then the corresponding Loewner chains satisfy $d^T_{\mathrm{CKC}}(K,\widetilde{K})<\varepsilon$.

This lemma is similar in spirit to [L4, Proposition 4.47]. As is well known, $\widetilde{K}_T\to K_T$ in the Hausdorff metric does not follow from $\widetilde{W}\to W$ uniformly in $[0,T]$.

Proof. Fix $T>0$. Suppose that $W^n\to W$ uniformly in $[0,T]$. Let K^n denote the Loewner chain corresponding to W^n and let $g^{(n)}_t:\mathbb{H}\setminus K^n_t\to\mathbb{H}$ denote the corresponding Loewner evolution. Fix some $t_0\in[0,T]$. Since $\operatorname{diam} K^n_{t_0}$ is clearly bounded by a function of T and $\|W^n\|_\infty$ [L4, Lemma 4.13], the closure of $\{K^n_{t_0}: n\in\mathbb{N}_+\}$ is compact with respect to the Hausdorff metric on non-empty compact subsets of $\overline{\mathbb{H}}$. Consider some integer sequence $n_j\to\infty$ for which the Hausdorff limit $K':=\lim_{j\to\infty} K^{n_j}_{t_0}$ exists. If $z\in\overline{\mathbb{H}}\setminus K_{t_0}$, then there is a neighborhood U of z in $\overline{\mathbb{H}}$ such that $U\cap K^n_{t_0}=\varnothing$ for all sufficiently large n, by the continuity of solutions of ODE's in the vector field specifying the ODE. It follows that $z\notin K'$, and hence $K'\subset K_{t_0}$.

With the intention of reaching a contradiction, suppose that there is some point $z\in\partial K_{t_0}\setminus(K'\cup\mathbb{R})$. Let z' be a point in $\mathbb{H}\setminus K_{t_0}$ satisfying $|z-z'|<\frac{1}{3}\operatorname{dist}(z',K'\cup\mathbb{R})$. The above argument shows that $\lim_{j\to\infty} g_{t_0}^{(n_j)}(z')=g_{t_0}(z')\in\mathbb{H}$. On the other hand, for all sufficiently large j we have $\operatorname{dist}(z', K_{t_0}^{n_j})>2|z-z'|$. Now the Koebe distortion theorem (e.g., [P, Corollary 1.4]) applied to the restriction of $g_{t_0}^{(n_j)}$ to the disk of radius $2|z-z'|$ about z' (once with z' and again with z) shows that $\operatorname{Im} g_{t_0}^{(n_j)}(z)\geqslant\frac{3}{16}\operatorname{Im} g_{t_0}^{(n_j)}(z')$. However, since $z\in K_{t_0}$, for every $\varepsilon>0$ there is some first $t_1\in[0,t_0)$ such that $\operatorname{Im} g_{t_1}(z)\leqslant\varepsilon$. The convergence argument above shows that for all arbitrarily large n, $\operatorname{Im} g_{t_1}^{(n)}(z)\leqslant 2\varepsilon$. Since $\operatorname{Im} g_t^{(n)}(z)$ decreases monotonically in t, it follows that $\operatorname{Im} g_{t_0}^{(n)}(z)\leqslant 2\varepsilon$ for all sufficiently large n. This contradicts our previous conclusion

$$\operatorname{Im} g_{t_0}^{(n_j)}(z) \geqslant \tfrac{3}{16}\operatorname{Im} g_{t_0}^{(n_j)}(z') \to \tfrac{3}{16}\operatorname{Im} g_{t_0}(z') > 0,$$

and proves that $K'\supset\partial K_{t_0}\setminus\mathbb{R}$.

Let $\widetilde{K}$ be the union of K' and the bounded connected components of $\overline{\mathbb{H}}\setminus K'$. The above implies that $\widetilde{K}\supset K_{t_0}\setminus\mathbb{R}$. Now note that $K_{t_0}\setminus\mathbb{R}$ is dense in K_{t_0}. (This follows from the easy direction (2) $\Rightarrow$ (1) in [LSW1, Theorem 2.6] and from the fact that $(\mathbb{H}\cap K_t)\setminus K_{t'}$ is non-empty when $t>t'$.) Consequently, $\widetilde{K}=K_{t_0}$, which implies that $d_{\mathrm{CKC}}(K_t^n, K_t)\to 0$ for every fixed $t\in[0,T]$.

Note that the above proof also gives $\lim_{s\to t} d_{\mathrm{CKC}}(K_s,K_t)=0$ for $t\in[0,T]$ and s tending to t in $[0,T]$. Thus, K_t is continuous in t with respect to d_{CKC}. Since $d_{\mathrm{CKC}}(L,\tilde{L})\leqslant d_{\mathrm{CKC}}(L',\tilde{L})\vee d_{\mathrm{CKC}}(L'',\tilde{L})$ when $L'\subset L\subset L''$, the d_{CKC}^T convergence easily follows from the pointwise convergence, from continuity of K_t and from monotonicity of K_t^n in t. □

4.8. Improving the convergence topology

In this subsection we complete the proof of Theorem 1.2. There are examples showing that the convergence of the Loewner driving term does not imply the uniform convergence of the paths parameterized by capacity. (See [LSW4, §3.4].) Therefore, we will need to apply other considerations. Before embarking on the proof, we note that when $a,b\geqslant\lambda$ the trace of $\mathrm{SLE}(4;a/\lambda-1,b/\lambda-1)$ is a simple path that does not hit $\mathbb{R}$, except at its starting point. Indeed, note first that the force points are moving monotonically away from one another. By comparison with a Bessel process, for example, it is easy to see that the trace does not hit the real line at any time $t>0$. It also does not hit itself, since $t\mapsto g_s(\gamma(t+s))$ has law that is mutually absolutely continuous with the path of SLE(4).

LEMMA 4.16. *Let $T>0$ and let $W_n\colon[0,T]\to\mathbb{R}$ be a sequence of continuous functions converging uniformly to a function $W\colon[0,T]\to\mathbb{R}$. Suppose that each W_n is the driving term of a Loewner evolution of a path $\gamma_n\colon[0,T]\to\overline{\mathbb{H}}$, and W is the driving term of a Loewner evolution of a simple path $\gamma\colon[0,T]\to\overline{\mathbb{H}}$ satisfying $\gamma(0,T]\cap\mathbb{R}=\varnothing$. Then $\lim_{n\to\infty}\sup_{t\in[0,T]} d_{\mathrm{H}}(\gamma_n[0,t],\gamma[0,t])=0$, where d_{H} denotes the the Hausdorff metric.*

Proof. First note that $\operatorname{diam}\gamma_n[0,T]$ is bounded, because $\|W_n\|_\infty$ is bounded. Fix some $t\in[0,T]$, and let Γ denote a subsequential Hausdorff limit of $\gamma_n[0,t]$. It suffices to prove that $\Gamma=\gamma[0,t]$. By Lemma 4.15, we know that for every $\varepsilon>0$ we have $\Gamma\cap\overline{\mathbb{H}}_\varepsilon(\gamma[0,t])=\varnothing$. Since γ is a simple path satisfying $\gamma(0,T]\cap\mathbb{R}=\varnothing$, it follows that $\bigcup_{\varepsilon>0}\overline{\mathbb{H}}_\varepsilon(\gamma[0,t])=\mathbb{H}\setminus\gamma[0,t]$, which implies that $\Gamma\subset\mathbb{R}\cup\gamma[0,t]$. Fix some $z_1\in\mathbb{R}\setminus\gamma[0,t]=\mathbb{R}\setminus\{\gamma(0)\}$. By the continuity in W and z of the solutions of Loewner's equation (1.3), it follows that there is a neighborhood V of z_1 such that $V\cap\gamma_n[0,T]=\varnothing$ for all sufficiently large n. This implies that $z_1\notin\Gamma$, and hence $\Gamma\subset\gamma[0,t]$.

Now let $t'\in[0,t]$. By Lemma 4.15 again, for every $\varepsilon>0$ and every n sufficiently large, $\gamma(t')\notin\overline{\mathbb{H}}_\varepsilon(\gamma_n[0,t])$, which means that every path connecting $\gamma(t')$ to ∞ in $\overline{\mathbb{H}}$ must come within distance ε from $\gamma_n[0,t]$. Thus, every such path must intersect Γ. Since $\Gamma\subset\gamma[0,t]$ is closed, this implies that $\gamma(t')\in\Gamma$. Therefore, $\Gamma=\gamma[0,t]$; that is,

$$\lim_{n\to\infty} d_{\mathrm{H}}(\gamma_n[0,t],\gamma[0,t])=0.$$

Since $\gamma_n[0,t]$ and $\gamma[0,t]$ are monotone increasing in t and $\gamma[0,t]$ is continuous in t with respect to d_{H}, it easily follows that $\lim_{n\to\infty}\sup_{t\in[0,T]} d_{\mathrm{H}}(\gamma_n[0,t],\gamma[0,t])=0$. □

Here is an outline of the main ideas going into the proof of Theorem 1.2. Let γ^ϕ be the path $\phi\circ\gamma$ parameterized by half-plane capacity. The main step in the proof is to show that if we fix $T>0$, we have $\sup_{t\in[0,T]}|\gamma^\phi(t)-\gamma_{\mathrm{SLE}}(t)|\to0$ in probability. By Theorem 1.3 and Lemma 4.16, we get $\sup_{t\in[0,T]} d_{\mathrm{H}}(\gamma^\phi[0,t],\gamma_{\mathrm{SLE}}[0,t])\to0$ in probability (since γ_{SLE} is a simple path). We only need to rule out the possibility that γ^ϕ has significant (and fast) backtracking along γ_{SLE}. This is ruled out by invoking Lemma 3.17 and observing that the ϕ-image of the place where simple random walk (starting from a vertex near $\phi^{-1}(i)$) hits $\partial D(\gamma)$ can be close to any fixed segment of $\gamma_{\mathrm{SLE}}(0,T]$.

Proof of Theorem 1.2. Let γ^ϕ be the path $\phi\circ\gamma$ parameterized by half-plane capacity. Let $\delta,T>0$ and $r_D=\operatorname{rad}_{\phi^{-1}(i)}(D)$. Let W denote the Loewner driving term of γ_{SLE}. Since γ_{SLE} is almost surely a simple path, Lemma 4.16 implies that for every $\varepsilon>0$ there is some $\varepsilon'=\varepsilon'(\varepsilon,\gamma_{\mathrm{SLE}})>0$ such that if $\widetilde{W}$ is the driving term of a continuous path $\widetilde{\gamma}$ and $\sup_{t\in[0,T]}|\widetilde{W}_t-W_t|<\varepsilon'$, then $\sup_{t\in[0,T]} d_{\mathrm{H}}(\gamma_{\mathrm{SLE}}[0,t],\widetilde{\gamma}[0,t])<\varepsilon$. Moreover, it is not hard to see that ε' can be chosen as a measurable function of W. Hence, Theorem 1.3 implies that for every $\varepsilon_0>0$, if r_D is larger than some function of ε_0, δ, T, a and b, then there is a coupling of h and $\mathrm{SLE}(4;a/\lambda-1,b/\lambda-1)$ such that

$$\varrho:=\sup_{t\in[0,T]} d_{\mathrm{H}}(\gamma^\phi[0,t],\gamma_{\mathrm{SLE}}[0,t])<\varepsilon_0$$

with probability at least $1-\delta$. Without yet specifying ε_0, we assume that indeed $\gamma^\phi[0,T]$ and γ_{SLE} are so coupled. Let $\mathcal{A}_0$ be the event that $\varrho<\varepsilon_0$. Then $\mathbf{P}[\mathcal{A}_0]\geqslant1-\delta$.

Let $0<t_0<t_1<t_2<t_3\leqslant T$. We will show that under this coupling, if r_D is large, then with high probability $\gamma^\phi(t)$ is close to $\gamma_{\mathrm{SLE}}[t_0,T]$ for every $t\in[t_3,T]$. This will then imply that $\sup_{t\in[0,T]}|\gamma_{\mathrm{SLE}}(t)-\gamma^\phi(t)|$ is small.

Since γ_{SLE} is almost surely a simple path disjoint from $\{i\}$, there is a constant $s_0=s_0(t_0,t_1,t_2,t_3,T,\delta)>0$ such that $\mathbf{P}[\mathcal{A}_1]\geqslant 1-\delta$, where $\mathcal{A}_1$ is the event that

(1) the harmonic measure of $\gamma_{\mathrm{SLE}}[t_1,t_2]$ from i with respect to $\mathbb{H}\setminus\gamma_{\mathrm{SLE}}[0,\infty)$ is at least s_0,

(2) $\operatorname{dist}(\mathbb{R},\gamma_{\mathrm{SLE}}[t_0,T])>s_0$,

(3) $\operatorname{dist}(\gamma_{\mathrm{SLE}}[0,t_j],\gamma_{\mathrm{SLE}}[t_{j+1},T])>s_0$ for $j=0,1,2$,

(4) $\operatorname{diam}\gamma_{\mathrm{SLE}}[0,T]<1/s_0$, and

(5) $i\in\overline{\mathbb{H}}_{s_0}(\gamma_{\mathrm{SLE}}[0,T])$.

(It is tedious, but straightforward, to check that $\mathcal{A}_1$ is measurable.)

Consider a simple random walk S independent of h starting at a TG-vertex closest to $\phi^{-1}(i)$. Let τ_T be the first time t when $S(t)\in\partial D(\phi^{-1}\circ\gamma^\phi[0,T])$, and, as in §3.7, let $z_T:=S(\tau_T)$. We claim that for every $\varepsilon_1>0$, if r_D is sufficiently large and ε_0 is sufficiently small, then

$$\mathbf{P}[\operatorname{dist}(\phi(z_T),\gamma_{\mathrm{SLE}}[t_1,t_2])<\varepsilon_1 \mid \gamma_{\mathrm{SLE}},\gamma^\phi[0,T]]>\tfrac12 s_0 \quad \text{on } \mathcal{A}_0\cap\mathcal{A}_1. \tag{4.36}$$

To prove (4.36), first observe that conditional on γ_{SLE} such that $\mathcal{A}_1$ holds, a 2-dimensional Brownian motion $\widehat{S}$ started at i has probability at least s_0 to first hit $\gamma_{\mathrm{SLE}}\cup\mathbb{R}$ in $\gamma_{\mathrm{SLE}}[t_1,t_2]$. Moreover, if this happens, the Brownian motion is likely to stay within a compact subset $L\subset\mathbb{H}$ before hitting $\gamma_{\mathrm{SLE}}[t_1,t_2]$ and not to come arbitrarily close to γ_{SLE} far from its hitting point. On compact subsets of $\mathbb{H}$, the map $r_D^{-1}\phi^{-1}$ distorts distances by a bounded factor, by the Koebe distortion theorem [P, Theorem 1.3 and Corollary 1.4]. Since ϕ^{-1} takes a Brownian motion to a monotonically time-changed Brownian motion, by taking r_D large we may couple $r_D^{-1}S$ and $r_D^{-1}\phi^{-1}\circ\widehat{S}$ to stay arbitrarily close (until $\phi^{-1}\circ\widehat{S}$ hits ∂D) with high probability, up to a time change. Assuming that ϱ is arbitrarily small and taking (5) into account, we find that on $\mathcal{A}_1$ and given $(\gamma_{\mathrm{SLE}},\gamma^\phi[0,T])$, with conditional probability at least $\frac56 s_0$ the random walk gets to a vertex v where $\operatorname{dist}(\phi(v),\gamma_{\mathrm{SLE}}[t_1,t_2])$ is arbitrarily small before time τ_T. Now, on the event $\mathcal{A}_0$, $\gamma^\phi[0,T]$ has to be close by, and so we find from Lemma 2.1 that on $\mathcal{A}_0\cap\mathcal{A}_1$ and given $(\gamma_{\mathrm{SLE}},\gamma^\phi[0,T])$, with conditional probability at least $\frac23 s_0$ we have $\operatorname{dist}(\phi(z_T),\gamma_{\mathrm{SLE}}[t_1,t_2])$ arbitrarily small. This proves (4.36).

We take $\gamma^\phi[T,\infty)$ independent of γ_{SLE} given $\gamma^\phi[0,T]$ in the coupling of γ_{SLE} with γ. As in §3.5, let τ be the hitting time of S on $\partial D(\gamma)$. In (3.62) we choose $\varepsilon=\delta$, and get a corresponding $p>0$. Let $\mathcal{A}_2$ denote the event

$$\mathbf{P}[z_T=S_\tau \mid S,\gamma^\phi[0,T]]\geqslant p.$$

Then (3.62) reads $\mathbf{P}[\mathcal{A}_2]\geqslant 1-\delta$. In conjunction with (4.36) and $\mathbf{P}[\mathcal{A}_j]\geqslant 1-\delta$, $j=0,1$, this implies that

$$\mathbf{P}\big[\mathbf{P}[\operatorname{dist}(\phi(S_\tau),\gamma_{\mathrm{SLE}}[t_1,t_2])<\varepsilon_1 \mid \gamma_{\mathrm{SLE}},\gamma^\phi[0,T]]>\tfrac{1}{2}ps_0\big]\geqslant 1-3\delta. \tag{4.37}$$

Conditional on S_τ, on the event $\mathcal{A}_1\cap\{\operatorname{dist}(\phi(S_\tau),\gamma_{\mathrm{SLE}}[t_1,t_2])<\varepsilon_1\}$ we invoke Lemma 3.17 with S_τ translated to 0 where the p in that lemma is chosen as $\frac{1}{2}ps_0\delta$ and the R is taken to be s_2r_D, where $s_2=s_2(s_0)>0$ is a small constant depending only on s_0. Note that the assumption necessary for the lemma that $\operatorname{dist}(S_\tau,\{v_0\}\cup\partial D)>4R$ holds by (2), (4) and (5) in the definition of $\mathcal{A}_1$ and the fact that the distance distortion of $r_D^{-1}\phi^{-1}$ is bounded on compact subsets of $\mathbb{H}$. Let $\tilde a$ denote the a provided by the lemma, which is a function of s_0, a, b and δ. Set $r=R/(\tilde a+1)$. Then the lemma together with (4.37) imply that with probability $1-O(\delta)$ there is within distance ε_1 from $\gamma_{\mathrm{SLE}}[t_1,t_2]$ the ϕ-image of a vertex $v\in\partial D(\gamma)$ such that $\phi^{-1}\circ\gamma^\phi[0,T]$ has precisely two disjoint crossings of the annulus $\{z:r\leqslant|z-v|\leqslant R\}$. If this happens, let z_1 be a point in $\gamma_{\mathrm{SLE}}[t_1,t_2]$ closest to such a $\phi(v)$.

Let r' denote the lower bound we get on $\{|\phi(v)-\phi(z)|:|v-z|\geqslant r\}$ which follows from the bounded distortion of $r_D\phi$. We may also assume that $|\phi(v)-\phi(z)|\leqslant\frac{1}{8}s_0$ when $|v-z|\leqslant R$. Now take $\varepsilon_1=\frac{1}{4}r'$ and let $s_3=s_3(\delta,r')\in\big(0,\frac{1}{8}r'\big)$ be so small that with probability at least $1-\delta$ for every ball of radius $\frac{1}{4}r'$ centered at a point $z_0\in\gamma_{\mathrm{SLE}}[t_0,T]$ the distance outside of the ball $B\big(z_0,\frac{1}{4}r'\big)$ between the two connected components of $\mathbb{R}\cup(\gamma_{\mathrm{SLE}}[0,T]\setminus\{z_0\})$ is at least s_3. Note that when this is the case, every path connecting these two components outside of $B\big(z_0,\frac{1}{4}r'\big)$ has to intersect $\overline{\mathbb{H}}_{s_3/3}(\gamma_{\mathrm{SLE}}[0,T])$. Consequently, if additionally $\varrho<\frac{1}{3}s_3$, then $\gamma^\phi[0,T]$ cannot contain an arc whose endpoints are within distance $\frac{1}{4}s_3$ of these two components, unless the arc visits the ball $B\big(z_0,\frac{1}{4}r'\big)$. If ϱ is sufficiently small, then there is some $t_3'\leqslant t_3$ such that $|\gamma_{\mathrm{SLE}}(t_3)-\gamma^\phi(t_3')|<\frac{1}{4}s_3$. Now choose $z_0:=z_1$, for the previous paragraph. The path $\gamma^\phi[0,t_3']$ must pass through the ball $B\big(z_0,\frac{1}{4}r'\big)$, and therefore $\phi^{-1}\circ\gamma^\phi[0,t_3']$ contains two disjoint crossings of the annulus $\{z:r\leqslant|z-v|\leqslant R\}$. If we assume that $\phi^{-1}\circ\gamma^\phi[0,T]$ has no more than two disjoint crossings of this annulus (which happens with probability at least $1-O(\delta)$), it follows that for $t\in[t_3,T]$ the point $\gamma^\phi(t)$ is closer to $\gamma_{\mathrm{SLE}}[t_0,T]$ than to $\gamma_{\mathrm{SLE}}[0,t_0]\cup\mathbb{R}$. Since in the above δ, t_0 and t_3 are arbitrary subject to the constraint $0<t_0<t_3$ and $\delta>0$, the claimed uniform convergence in $[0,T]$ follows.

To prove convergence in law with respect to the uniform d_* metric, it suffices to show that for every radius $r_1>0$ there is some $r_2>r_1$ such that γ^ϕ is unlikely to return to $B(0,r_1)$ after its first exit from $B(0,r_2)$. For this proof, we will use the conformal invariance of extremal length (see [A]).

Fix some $r_1>0$. The extremal length of the collection of arcs in the half-annulus $A:=\{z\in\overline{\mathbb{H}}:r_1<|z|<r_2\}$, which connect $\mathbb{R}_+$ with $\mathbb{R}_-$, tends to zero as $r_2\to\infty$. Let $A':=$

$\phi^{-1}(A)$, $\partial_j := \phi^{-1}(\{z\in\overline{\mathbb{H}} : |z| = r_j\})$, $j=1,2$, $L := \mathrm{dist}(\partial_1, \partial_2; A')$ and $L' := \mathrm{dist}(\partial_+, \partial_-; A')$. By conformal invariance of extremal length, it follows that $L'/L \to 0$ as $r_2 \to \infty$, uniformly in D. (Otherwise, the metric which is equal to the Euclidean metric in the ball of radius $3L$ centered on a point in an arc of length at most $2L$ from ∂_1 to ∂_2 in A', and is zero outside this ball, contradicts the extremal length going to zero.)

Let $\beta\subset A'$ be an arc of diameter at most $2L'$ connecting ∂_+ and ∂_-. Let

$$L_1 := \mathrm{dist}(\beta, \partial_1; A') \quad \text{and} \quad L_2 := \mathrm{dist}(\beta, \partial_2; A').$$

Since $L'/L\to 0$ as $r_2\to\infty$, we have $L_1 \geqslant \frac{1}{3}L$ or $L_2 \geqslant \frac{1}{3}L$ if r_2 is sufficiently large. Suppose first that $L_2 \geqslant \frac{1}{3}L$. Let s_0 denote the first time such that $|\phi\circ\gamma(s_0)| = r_2$. Then there are two connected components β_1 and β_2 of $\beta\setminus\gamma[0, s_0]$ such that $\gamma[0, s_0]\cup\beta_1\cup\beta_2$ separates $\phi^{-1}(B(0, r_1))$ from y_∂ in D. Now the proof of Theorem 3.22 shows that

$$\mathbf{P}[\gamma\cap\beta_1 \neq \varnothing \mid \gamma[0, s_0]] < \delta$$

if L_2/L' is sufficiently large. (Hence when $L_2 \geqslant \frac{1}{3}L$ and r_2 is sufficiently large.) A similar estimate holds with β_2. Thus, with probability at most $O(\delta)$, γ^ϕ contains two disjoint crossings of the annulus $r_1 \leqslant |z| \leqslant r_2$. On the other hand, if $L_2 < \frac{1}{3}L$ and $L_1 \geqslant \frac{1}{3}L$, then we may apply the same argument to the reversal of γ (or else slightly modify the way the analog of Theorem 3.22 is proved) to reach the same conclusion. This completes the proof. □

5. Other lattices

In this section we describe the modifications necessary to adapt the proofs of Theorems 1.3 and 1.2 to the more general framework of Theorem 1.4.

Before we go into the actual proof, a few words need to be said about the properties of the weighted random walk on $\mathfrak{G}$ and its convergence to Brownian motion. Fix some vertex v_0 in $\mathfrak{G}$, and let V_0 denote its orbit under the group generated by the two translations T_1 and T_2 preserving $\mathfrak{G}$. If the walk starts at v_0, then a new Markov chain is obtained by looking at the sequence of vertices in V_0 that the walk visits. A simple path reversal argument shows that for this new Markov chain the transition probability from v to u is the same as the transition probability from u to v, for every pair of vertices $v, u\in V_0$. Also observe that the $\mathbb{R}^2$-length of a single step has an exponential tail. This is enough to show that the Markov chain on V_0, rescaled appropriately in time and space, converges to a linear image of Brownian motion (and it is not hard to verify that the linear transformation is non-singular). Moreover, the few properties of the simple random walk on TG that we have used in the course of the paper are easily verified for this Markov chain on V_0 and easily translated to the weighted walk on $\mathfrak{G}$.

Proof of Theorem 1.4. Very few changes are needed to adapt the proof. Let $\mathfrak{G}$ denote the original lattice consisting of only edges of positive weight, and let $\overline{\mathfrak{G}}$ denote the triangulation of $\mathfrak{G}$, as described in §1.5. Let $\overline{\mathfrak{G}}^*$ denote the planar dual of $\overline{\mathfrak{G}}$.

The statement and proof of Lemma 3.1 requires some changes, because in the more general setup it is not true that every vertex adjacent to an interface on the right has a $\mathfrak{G}$-neighbor on the left of the interface (and similarly in the other direction). Thus, in the revised version of the lemma, the assumption that each vertex in V_+ neighbors with a vertex in $V_-\cup V_\partial$ and every vertex in V_- neighbors with a vertex in $V_+\cup V_\partial$ needs to be replaced by the assumption that for some constant m, depending on the lattice, for every vertex $v\in V_+$ the $\mathfrak{G}$-graph-distance from v to $V_-\cup V_\partial$ is at most m, and symmetrically for vertices in V_-. This change requires a few extra lines in the proof of (3.1). Let M_j be the maximum of $\mathbf{E}[e^{h(v)}\,|\,\mathcal{K}]$ for vertices in V_+ at $\mathfrak{G}$-distance at most j from $V_-\cup V_\partial$. Every vertex at $\mathfrak{G}$-distance $j>0$ from $V_-\cup V_\partial$ has a $\mathfrak{G}$-neighbor at $\mathfrak{G}$-distance $j-1$ from $V_-\cup V_\partial$. Therefore, the proof of (3.8) now gives

$$M_j \leqslant O(1) M_m^c M_{j-1}^{1-c} + O(1),$$

where $c<1$ is some constant depending only on the lattice and its edge weights. We can certainly drop the trailing additive $O(1)$. Induction on j now gives

$$M_j \leqslant O(1)^{q_j} M_m^{c\,q_j} M_0^{(1-c)^j},$$

where $q_j=1+(1-c)+...+(1-c)^{j-1}=(1-(1-c)^j)/c$. When $j=m$, this reads

$$M_m^{(1-c)^m} \leqslant O(1)^{q_m} M_0^{(1-c)^m},$$

Clearly, $M_0\leqslant e^{\bar\Lambda}$, and the bound $M_m=O_{m,\bar\Lambda}(1)$ follows. A corresponding bound clearly also holds for $\mathbf{E}[e^{-h(v)}\,|\,\mathcal{K}]$ when $v\in V_-$. The remainder of the proof of the analog of Lemma 3.1 proceeds without difficulty.

The proof of Lemma 3.2 needs to be similarly adapted, but essentially the same argument works.

The next point which requires adaptation is the definition of $\mathcal{Z}_0^\sigma$ in §3.5. Let Σ denote the set of pairs (v,e^*), where v is a vertex in $\mathfrak{G}$ and e^* is an edge in $\overline{\mathfrak{G}}^*$ that is dual to one of the edges incident with v in $\overline{\mathfrak{G}}$. If $\sigma=(v,e^*)\in\Sigma$, let $\mathcal{Z}^\sigma$ denote the event that the first vertex adjacent to γ that S hits is v and moreover $e^*\in\gamma$. Let Σ' be a collection of elements of Σ, one from each orbit under the group generated by the translations T_1 and T_2 preserving $\mathfrak{G}$. Let $\mathcal{Z}_0:=\bigcup_{\sigma\in\Sigma'}\mathcal{Z}^\sigma$. The proof then proceeds essentially unchanged, with $\mathcal{Z}^\sigma$ in place of $\mathcal{Z}_0^\sigma$ and with the modified definition for $\mathcal{Z}_0$. □

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