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Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation

by

Carlos E. Kenig

University of Chicago
Chicago, IL, U.S.A.

Frank Merle

Université de Cergy-Pontoise
Pontoise, France

1. Introduction

In this paper we consider the energy-critical non-linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = \pm |u|^{4/(N-2)} u, & (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ u\big|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^N), \\ \partial_t u\big|_{t=0} = u_1 \in L^2(\mathbb{R}^N). \end{cases}$$

Here the $-$ sign corresponds to the defocusing problem, while the $+$ sign corresponds to the focusing problem. The theory of the local Cauchy problem (CP) for this equation was developed in many papers, see for instance [11], [17], [25], [33], [36], [37], [39], etc. In particular, one can show that if $\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \leqslant \delta$, with δ small, there exists a unique solution with $(u, \partial_t u) \in C(\mathbb{R}; \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N))$ with the norm

$$\|u\|_{L_{xt}^{2(N+1)/(N-2)}} < \infty$$

(i.e., the solution scatters in $\dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$). See §2 of this paper for a review and an update of the results.

In the defocusing case, Struwe [42] in the radial case, when $N=3$, Grillakis [13] in the general case when $N=3$, and then Grillakis [14], Shatah–Struwe [35], [36], [37], Bahouri–Shatah [5], and Kapitanski [17], in higher dimensions, proved that this also holds for any (u_0, u_1) with $\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} < \infty$ and that (for $3 \leqslant N \leqslant 5$) for more regular (u_0, u_1) the

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solution preserves the smoothness for all time. This topic has been the subject of intense investigation. See the recent work of Tao [44] for a recent installment in it and further references.

In the focusing case, these results do not hold. In fact, the classical identity

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^N}|u(x,t)|^2\,dx=2\int_{\mathbb{R}^N}((\partial_t u)^2-|\nabla u|^2+|u(t)|^{2N/(N-2)})\,dx \tag{1.1}$$

(see the work of H. Levine [24] and also §3 and §5) was used by Levine [24] to show that if $(u_0,u_1)\in H^1\times L^2$ is such that

$$E((u_0,u_1))=\int_{\mathbb{R}^N}\left(\frac{1}{2}|\nabla u_0|^2+\frac{1}{2}|u_1|^2-\frac{(N-2)}{2N}|u_0|^{2N/(N-2)}\right)dx<0,$$

the solution must break down in finite time. Moreover,

$$W(x)=W(x,t)=\left(1+\frac{|x|^2}{N(N-2)}\right)^{-(N-2)/2}$$

is in $\dot{H}^1(\mathbb{R}^N)$ and solves the elliptic equation

$$\Delta W+|W|^{4/(N-2)}W=0,$$

so that scattering cannot always occur even for global (in time) solutions.

In this paper we initiate the detailed study of the focusing case (see also [23] for an interesting recent work in this direction). We show the following result.

THEOREM 1.1. *Let* $(u_0,u_1)\in\dot{H}^1\times L^2$, $3\leqslant N\leqslant 5$. *Assume that*

$$E((u_0,u_1))<E((W,0)).$$

Let u be the corresponding solution of the Cauchy problem, with maximal interval of existence $I=(-T_-(u_0,u_1),T_+(u_0,u_1))$ (see Definition 2.13).

(i) *If* $\int_{\mathbb{R}^N}|\nabla u_0|^2\,dx<\int_{\mathbb{R}^N}|\nabla W|^2\,dx$, *then*

$$I=(-\infty,\infty)\quad\textit{and}\quad\|u\|_{L^{2(N+1)/(N-2)}_{xt}}<\infty.$$

(ii) *If* $\int_{\mathbb{R}^N}|\nabla u_0|^2\,dx>\int_{\mathbb{R}^N}|\nabla W|^2\,dx$, *then*

$$T_+(u_0,u_1)<\infty\quad\textit{and}\quad T_-(u_0,u_1)<\infty.$$

Our proof follows the new point of view into these problems that we introduced in [19], where we obtained the corresponding result for the energy-critical non-linear Schrödinger equation for radial data. In §3 we prove some elementary variational estimates which yield the necessary coercivity for our arguments and which follows from arguments in [19]. In §4, using the work of Bahouri–Gérard [4] and the concentration compactness argument from [19], we produce a "critical element" for which scattering fails and which enjoys a compactness property because of its criticality (Propositions 4.1 and 4.2). At this point, we show a crucial orthogonality property of "critical elements" related to a second conservation law in the energy space (Propositions 4.10 and 4.11) which exploits the finite speed of propagation for the wave equation and its Lorentz invariance. This is the extra ingredient that allows us to go beyond the radial case as in [19]. In §5 and §6 we prove a rigidity theorem (Theorem 5.1), which allows us to conclude the argument. The first case of the rigidity theorem deals with infinite time of existence. This uses localized conservations laws of the type (1.1) and related ones, very much in the spirit of the corresponding localized virial identity used in [19]. The second case of the rigidity theorem deals with finite time of existence. This case is dealt with in [19] through the use of the L^2 conservation law, which is absent for the wave equation. We proceed in two stages. First we show that the solution must have self-similar behavior (Proposition 5.7). Then, in §6, following Merle–Zaag [30] and earlier work on non-linear heat equations by Giga–Kohn [10], we introduce self-similar variables and the new resulting equation, which has a monotonic energy. We then show that there exists a non-trivial asymptotic solution w^*, which solves a (degenerate) elliptic non-linear equation. Finally, using the estimates that we proved on w^* and the unique continuation principle, we show that w^* must be zero, a contradiction which gives our rigidity theorem. In §7 we prove our main theorem as a consequence of the rigidity theorem.

Finally, we would like to point out that we expect that our arguments will extend to $N \geqslant 6$, using arguments similar to those in the work of Tao–Visan [45] for the local solvability in time of the equation and the corresponding extension of the work of Bahouri–Gérard [4] (the rest of the argument is independent of the dimension).

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2. A review of linear estimates and the Cauchy problem

In this section we will review the theory of the Cauchy problem

$$\begin{cases} \partial_t^2 u - \Delta u = |u|^{4/(N-2)}u, & (x,t)\in\mathbb{R}^N\times\mathbb{R}, \\ u\big|_{t=0} = u_0 \in \dot H^1(\mathbb{R}^N), \\ \partial_t u\big|_{t=0} = u_1 \in L^2(\mathbb{R}^N), \end{cases} \tag{CP}$$

i.e. the $\dot H^1$ critical, focusing Cauchy problem for the non-linear wave equation, and some of the associated linear theory. We start out with some preliminary notation and linear estimates. Consider thus the associated linear problem

$$\begin{cases} \partial_t^2 w - \Delta w = h, & (x,t)\in\mathbb{R}^N\times\mathbb{R}, \\ w\big|_{t=0} = w_0 \in \dot H^1(\mathbb{R}^N), \\ \partial_t w\big|_{t=0} = w_1 \in L^2(\mathbb{R}^N). \end{cases} \tag{LCP}$$

The solution operator to (LCP) is given by

$$\begin{aligned} w(x,t) &= \cos(t\sqrt{-\Delta}\,)w_0 + (-\Delta)^{-1/2}\sin(t\sqrt{-\Delta}\,)w_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta}\,)}{\sqrt{-\Delta}} h(s)\,ds \\ &= S(t)((w_0,w_1)) + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta}\,)}{\sqrt{-\Delta}} h(s)\,ds. \end{aligned}$$

Lemma 2.1. (Strichartz estimates [25], [12]) *There is a constant C, independent of T, such that*

$$\begin{aligned} &\sup_{0<t<T} (\|w(t)\|_{\dot H^1} + \|\partial_t w(t)\|_{L^2}) \\ &\quad + \|w\|_{L_t^{2(N+1)/(N-1)}\dot W_x^{1/2,2(N+1)/(N-1)}} + \|\partial_t w\|_{L_t^{2(N+1)/(N-1)} W_x^{-1/2,2(N+1)/(N-1)}} \\ &\quad + \|w\|_{L_t^{2(N+1)/(N-2)} L_x^{2(N+1)/(N-2)}} + \|w\|_{L_t^{(N+2)/(N-2)} L_x^{2(N+2)/(N-2)}} \\ &\qquad \leqslant C(\|w_0\|_{\dot H^1(\mathbb{R}^N)} + \|w_1\|_{L^2(\mathbb{R}^N)} + \|h\|_{L_t^{2(N+1)/(N+3)}\dot W_x^{1/2,2(N+1)/(N+3)}}). \end{aligned}$$

Lemma 2.2. (Trace theorem) *Let w_0, w_1, h and w be as in Lemma 2.1. Then, for $|d|\leqslant\frac14$,*

$$\begin{aligned} \sup_t \left\| \nabla_x w\left(\frac{x_1-dt}{\sqrt{1-d^2}}, x', \frac{t-dx_1}{\sqrt{1-d^2}}\right)\right\|_{L^2(dx_1dx')} + \sup_t \left\| \partial_t w\left(\frac{x_1-dt}{\sqrt{1-d^2}}, x', \frac{t-dx_1}{\sqrt{1-d^2}}\right)\right\|_{L^2(dx_1dx')} \\ \leqslant C(\|w_0\|_{\dot H^1(\mathbb{R}^N)} + \|w_1\|_{L^2(\mathbb{R}^N)} + \|h\|_{L_t^1L_x^2}). \end{aligned}$$

Proof. Let $v(x,t)=U(t)f$ be given by $\hat v(\xi,t)=e^{it|\xi|}\hat f(\xi)$, with $f\in L^2$. We will show that

$$\sup_t \left\| v\left(\frac{x_1-dt}{\sqrt{1-d^2}}, x', \frac{t-dx_1}{\sqrt{1-d^2}}\right)\right\|_{L^2(dx_1dx')} \leqslant C\|f\|_{L^2},$$

which easily implies the desired estimate. But

$$v(x,t)=\int_{\mathbb{R}^N} e^{ix\cdot\xi}e^{it|\xi|}\hat{f}(\xi)\,d\xi=\int_{\mathbb{R}^N} e^{ix_1\xi_1}e^{it|\xi|}e^{ix'\cdot\xi'}\hat{f}(\xi)\,d\xi_1\,d\xi'$$
$$=\int_{\mathbb{R}^N} e^{ix_1\xi_1}e^{it\sqrt{\xi_1^2+|\xi'|^2}}e^{ix'\cdot\xi'}\hat{f}(\xi_1,\xi')\,d\xi_1\,d\xi',$$

so that

$$\begin{aligned}
v&\left(\frac{x_1-dt}{\sqrt{1-d^2}},x',\frac{t-dx_1}{\sqrt{1-d^2}}\right)\\
&=\int_{\mathbb{R}^N} e^{i(x_1-dt)\xi_1/\sqrt{1-d^2}}e^{i(t-dx_1)\sqrt{\xi_1^2+|\xi'|^2}/\sqrt{1-d^2}}e^{ix'\cdot\xi'}\hat{f}(\xi)\,d\xi_1\,d\xi'\\
&=\int_{\mathbb{R}^N} e^{ix_1(\xi_1-d|\xi|)/\sqrt{1-d^2}}e^{ix'\cdot\xi'}e^{-idt\xi_1/\sqrt{1-d^2}}e^{it|\xi|/\sqrt{1-d^2}}\hat{f}(\xi)\,d\xi_1\,d\xi'\\
&=\int_{\mathbb{R}^N} e^{ix_1(\xi_1-d|\xi|)/\sqrt{1-d^2}}e^{ix'\cdot\xi'}\hat{g}_t(\xi)\,d\xi_1\,d\xi',
\end{aligned}$$

where $\hat{g}_t(\xi)=e^{-idt\xi_1/\sqrt{1-d^2}}e^{it|\xi|/\sqrt{1-d^2}}\hat{f}(\xi)$. We now define

$$\eta_1=\frac{\xi_1-d|\xi|}{\sqrt{1-d^2}}\quad\text{and}\quad\eta'=\xi',$$

and compute

$$\left|\frac{d\eta}{d\xi}\right|=\det\begin{pmatrix}\frac{1-d\xi_1/|\xi|}{\sqrt{1-d^2}} & \frac{-d\xi_2/|\xi|}{\sqrt{1-d^2}} & \dots & \dots & \frac{-d\xi_N/|\xi|}{\sqrt{1-d^2}}\\ 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\end{pmatrix}=\left(\frac{1-d\xi_1/|\xi|}{\sqrt{1-d^2}}\right)\approx 1$$

for $|d|\leqslant\frac{1}{4}$. The result now follows from Plancherel's theorem. □

Remark 2.3. A density argument in fact shows that

$$t\longmapsto w\left(\frac{x_1-dt}{\sqrt{1-d^2}},x',\frac{t-dx_1}{\sqrt{1-d^2}}\right)\in C(\mathbb{R};\dot{H}^1(\mathbb{R}^N)),$$

and similarly for $\partial_t w$.

Remark 2.4. Let $F(u)=|u|^{4/(N-2)}u$. Then clearly, for $3\leqslant N\leqslant 6$,

$$\begin{aligned}
|F(u)|&\leqslant|u|^{(N+2)/(N-2)},\\
|(\nabla F)(u)|&\leqslant C|u|^{4/(N-2)},\\
|(\nabla F)(u)-(\nabla F)(v)|&\leqslant C|u-v|(|u|^{(6-N)/(N-2)}+|v|^{(6-N)/(N-2)}),\\
|\nabla_x(F(u(x)))-\nabla_x(F(v(x)))|&\leqslant C|u(x)|^{4/(N-2)}|\nabla u(x)-\nabla v(x)|\\
&\quad+C|\nabla v(x)|(|u|^{(6-N)/(N-2)}+|v|^{(6-N)/(N-2)})|u-v|.
\end{aligned}$$

We will also need a version of the chain rule for fractional derivatives (see [8], [21], [40] and [46]).

Lemma 2.5. *Assume that* $F(0)=F'(0)=0$ *and that for all* a *and* b,

$$|F'(a+b)| \leqslant C(|F'(a)|+|F'(b)|) \quad \text{and} \quad |F''(a+b)| \leqslant C(|F''(a)|+|F''(b)|).$$

Then, for $0<\alpha<1$,

$$\|D^\alpha F(u)\|_{L^p_x} \leqslant C\|F'(u)\|_{L^{p_1}_x}\|D^\alpha u\|_{L^{p_2}_x},$$

where $1/p=1/p_1+1/p_2$, $1<p_j<\infty$, *and*

$$\begin{aligned}\|D^\alpha(F(u)-F(v))\|_{L^p_x} &\leqslant C(\|F'(u)\|_{L^{p_1}_x}+\|F'(v)\|_{L^{p_1}_x})\|D^\alpha(u-v)\|_{L^{p_2}_x} \\ &\quad + C(\|F''(u)\|_{L^{r_1}_x}+\|F''(v)\|_{L^{r_1}_x})(\|D^\alpha u\|_{L^{r_2}_x}+\|D^\alpha v\|_{L^{r_2}_x})\|u-v\|_{L^{r_3}_x},\end{aligned}$$

where $1/p=1/r_1+1/r_2+1/r_3$, $1<r_j<\infty$, *and* $1<p<\infty$.

Remark 2.6. In our application of Lemma 2.5, we will have

$$F(u)=|u|^{4/(N-2)}u, \quad 3\leqslant N\leqslant 5,$$

and

$$\begin{aligned}F'(u) &= C_N|u|^{4/(N-2)}, \\ F''(u) &= \widetilde{C}_N \operatorname{sign}(u)|u|^{4/(N-2)-1} = \widetilde{C}_N \operatorname{sign}(u)|u|^{(6-N)/(N-2)}.\end{aligned}$$

We will choose

$$p=\frac{2(N+1)}{N+3} \quad \text{and} \quad p_2=\frac{2(N+1)}{N-1}, \quad \text{so that} \quad \frac{1}{p_1}=\frac{1}{p}-\frac{1}{p_2}=\frac{2}{N+1},$$

and

$$r_3=\frac{2(N+1)}{N-2} \quad \text{and} \quad r_2=\frac{2(N+1)}{N-1}, \quad \text{so that} \quad \frac{1}{r_1}=\frac{1}{p}-\frac{1}{r_2}-\frac{1}{r_3}=\frac{6-N}{2(N+1)}.$$

Notice that

$$p_1\frac{4}{N-2}=\frac{2(N+1)}{N-2} \quad \text{and} \quad \frac{6-N}{N-2}r_1=\frac{2(N+1)}{N-2}.$$

Let us now define the $S(I)$ and the $W(I)$ norms for an interval I by

$$\begin{aligned}\|v\|_{S(I)} &= \|v\|_{L_I^{2(N+1)/(N-2)}L_x^{2(N+1)/(N-2)}}, \\ \|v\|_{W(I)} &= \|v\|_{L_I^{2(N+1)/(N-1)}L_x^{2(N+1)/(N-1)}}.\end{aligned}$$

THEOREM 2.7. (See [33], [11], [36]) *Let* $(u_0,u_1)\in\dot H^1\times L^2$, $I\ni 0$ *be an interval and* $\|(u_0,u_1)\|_{\dot H^1\times L^2}\leqslant A$. *Then, for* $3\leqslant N\leqslant 5$, *there exists* $\delta=\delta(A)$ *such that if*

$$\|S(t)((u_0,u_1))\|_{S(I)}<\delta,$$

then there exists a unique solution u *to* (CP) *in* $\mathbb{R}^N\times I$, *with* $(u,\partial_t u)\in C(I;\dot H^1\times L^2)$, $\|D_x^{1/2}u\|_{W(I)}+\|\partial_t D_x^{-1/2}u\|_{W(I)}<\infty$ *and* $\|u\|_{S(I)}\leqslant 2\delta$. *Moreover, if*

$$(u_{0,k},u_{1,k})\to(u_0,u_1),\quad \text{as } k\to\infty,$$

in $\dot H^1\times L^2$ *(so that, for k large,* $\|S(t)((u_0,u_1))\|_{S(I)}<\delta$*), then the corresponding solutions* $(u_k,\partial_t(u_k))\to(u,\partial_t u)$, *as* $k\to\infty$, *in* $C(I;\dot H^1\times L^2)$.

Sketch of the proof. (CP) is equivalent to the integral equation

$$u(t)=S(t)((u_0,u_1))+\int_0^t\frac{\sin((t-s)\sqrt{-\Delta}\,)}{\sqrt{-\Delta}}F(u)(s)\,ds,$$

where $F(u)=|u|^{4/(N-2)}u$. We let

$$B_{a,b}=\{v \text{ on } \mathbb{R}^N\times I:\|v\|_{S(I)}\leqslant a \text{ and } \|D_x^{1/2}v\|_{W(I)}\leqslant b\}$$

and

$$\Phi_{(u_0,u_1)}(v)=S(t)((u_0,u_1))+\int_0^t\frac{\sin((t-s)\sqrt{-\Delta}\,)}{\sqrt{-\Delta}}F(v)(s)\,ds.$$

We will next choose δ, a and b so that $\Phi_{(u_0,u_1)}:B_{a,b}\to B_{a,b}$ and is a contraction there. Note that, by Lemma 2.1,

$$\|D_x^{1/2}\Phi_{(u_0,u_1)}(v)\|_{W(I)}\leqslant CA+C\|F(v)\|_{L_x^{2(N+1)/(N+3)}\dot W_x^{1/2,2(N+1)/(N+3)}}.$$

But, by Lemma 2.5, $\|D_x^{1/2}F(v)\|_{L_x^{2(N+1)/(N+3)}}$ is bounded by

$$C\|F'(v)\|_{L_x^{(N+1)/2}}\|D_x^{1/2}v\|_{L_x^{2(N+1)/(N-1)}}\leqslant C\|v\|^{4/(N-2)}_{L_x^{2(N+1)/(N-2)}}\|D_x^{1/2}v\|_{L_x^{2(N+1)/(N-1)}},$$

so that

$$\begin{aligned}\|D_x^{1/2}F(v)\|_{L_I^{2(N+1)/(N+3)}L_x^{2(N+1)/(N+3)}}&\\ &\leqslant C\|v\|^{4/(N-2)}_{L_I^{2(N+1)/(N-2)}L_x^{2(N+1)/(N-2)}}\|D_x^{1/2}v\|_{L_I^{2(N+1)/(N-1)}L_x^{2(N+1)/(N-1)}}\\ &\leqslant C\|v\|^{4/(N-2)}_{S(I)}\|D_x^{1/2}v\|_{W(I)}.\end{aligned}$$

Hence, for $v\in B_{a,b}$,

$$\|D_x^{1/2}\Phi_{(u_0,u_1)}(v)\|_{W(I)}\leqslant CA+Ca^{4/(N-2)}b.$$

Similarly, using Lemma 2.1 for the second term in $\Phi_{(u_0,u_1)}$, and the argument above, together with our assumption on (u_0,u_1) for the first term, we obtain

$$\|\Phi_{(u_0,u_1)}\|_{S(I)}\leqslant \delta+Ca^{4/(N-2)}b.$$

Next, choose $b=2AC$ and a so that $Ca^{4/(N-2)}\leqslant\frac{1}{2}$. Then,

$$\|D_x^{1/2}\Phi_{(u_0,u_1)}(v)\|_{W(I)}\leqslant b.$$

If $\delta=\frac{1}{2}a$ and $Ca^{4/(N-2)-1}b\leqslant\frac{1}{2}$ (which is possible if $N<6$) we obtain $\|\Phi_{(u_0,u_1)}(v)\|_{S(I)}\leqslant a$, so that $\Phi_{(u_0,u_1)}\colon B_{a,b}\to B_{a,b}$. Next, for the contraction, we again use Lemmas 2.1 and 2.5, to see that

$$\begin{aligned}
&\|D_x^{1/2}(\Phi_{(u_0,u_1)}(v)-\Phi_{(u_0,u_1)}(v'))\|_{W(I)}+\|\Phi_{(u_0,u_1)}(v)-\Phi_{(u_0,u_1)}(v')\|_{S(I)}\\
&\qquad\leqslant C\|D_x^{1/2}(F(v)-F(v'))\|_{L_I^{2(N+1)/(N+3)}L_x^{2(N+1)/(N+3)}}\\
&\qquad\leqslant C[(\|v\|^{4/(N-2)}_{L_I^{2(N+1)/(N-2)}L_x^{2(N+1)/(N-2)}}+\|v'\|^{4/(N-2)}_{L_I^{2(N+1)/(N-2)}L_x^{2(N+1)/(N-2)}})\\
&\qquad\qquad\quad\times\|D_x^{1/2}(v-v')\|_{L_I^{2(N+1)/(N-1)}L_x^{2(N+1)/(N-1)}}\\
&\qquad\qquad+(\|v\|^{(6-N)/(N-2)}_{L_I^{2(N+1)/(N-2)}L_x^{2(N+1)/(N-2)}}+\|v'\|^{(6-N)/(N-2)}_{L_I^{2(N+1)/(N-2)}L_x^{2(N+1)/(N-2)}})\\
&\qquad\qquad\quad\times(\|D_x^{1/2}v\|_{L_I^{2(N+1)/(N-1)}L_x^{2(N+1)/(N-1)}}+\|D_x^{1/2}v'\|_{L_I^{2(N+1)/(N-1)}L_x^{2(N+1)/(N-1)}})\\
&\qquad\qquad\quad\times\|v-v'\|_{L_I^{2(N+1)/(N-2)}L_x^{2(N+1)/(N-2)}}]\\
&\qquad\leqslant 2Ca^{4/(N-2)}\|D_x^{1/2}(v-v')\|_{W(I)}+2Ca^{(6-N)/(N-2)}2b\|v-v'\|_{S(I)},
\end{aligned}$$

and the contraction property follows for $N<6$. We then find $u\in B_{a,b}$ solving

$$\Phi_{(u_0,u_1)}(u)=u.$$

To show that $(u,\partial_t u)\in C(I;\dot H^1\times L^2)$ we use Lemma 2.1, together with the fact that $D_x^{1/2}F(u)\in L_I^{2(N+1)/(N+3)}L_x^{2(N+1)/(N+3)}$. This also shows that $\partial_t D_x^{-1/2}u\in W(I)$. The continuity statement at the end is an easy consequence of the fixed point argument, so that the proof is complete. □

Remark 2.8. $u\in L_I^{(N+2)/(N-2)}L_x^{2(N+2)/(N-2)}$, because of Lemma 2.1 and the fact that $D_x^{1/2}F(u)\in L_I^{2(N+1)/(N+3)}L_x^{2(N+1)/(N+3)}$. Note that because of this and the integral equation, the conclusion of Lemma 2.2 holds for u, provided the integrations on the left-hand side are restricted to $(x_1,x',t)\in\mathbb{R}^N\times I$ so that

$$\left(\frac{x_1-dt}{\sqrt{1-d^2}},x',\frac{t-dx_1}{\sqrt{1-d^2}}\right)\in\mathbb{R}^N\times I.$$

Remark 2.9. (Higher regularity of solutions; see for example [11]) If

$$(u_0, u_1) \in (\dot{H}^1 \cap \dot{H}^{1+\mu}, H^\mu),$$

$0 \leqslant \mu \leqslant 1$, and (u_0, u_1) satisfies the conditions in Theorem 2.7, then

$$(u, \partial_t u) \in C(I; (\dot{H}^1 \cap \dot{H}^{1+\mu}) \times H^\mu)$$

and

$$\|D_x^{1/2+\mu} u\|_{W(I)} + \|D_x^{1/2} u\|_{W(I)} + \|\partial_t D_x^{\mu-1/2} u\|_{W(I)} + \|\partial_t D_x^{-1/2} u\|_{W(I)} < \infty,$$

$\|u\|_{S(I)} \leqslant 2\delta$. (In this result we also need to use the assumption $3 \leqslant N \leqslant 5$.)

Remark 2.10. There exists $\tilde{\delta}$ such that if $\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \leqslant \tilde{\delta}$, then the conclusion of Theorem 2.7 applies to any interval I. In fact, by Lemma 2.1,

$$\|S(t)((u_0, u_1))\|_{S((-\infty,\infty))} \leqslant C\tilde{\delta},$$

and the claim follows.

Remark 2.11. Given $(u_0, u_1) \in \dot{H}^1 \times L^2$, there exists $I \ni 0$ such that the hypothesis of Theorem 2.7 is satisfied on I. This is clear because, by Lemma 2.1,

$$\|S(t)((u_0, u_1))\|_{S(I)} < \infty.$$

Remark 2.12. (Finite speed of propagation; see for instance [37]) Let R denote the fundamental solution of the Cauchy problem, i.e. $u=R$ solves

$$\begin{cases} (\partial_t^2 - \Delta_x) u = 0, & (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ u\big|_{t=0} = 0, \\ \partial_t u\big|_{t=0} = \delta(x), \end{cases} \tag{2.1}$$

where $\delta(x)$ is the Dirac mass at 0. Then, we can write the solution of (LCP) in the form

$$w(t) = \partial_t R(t) * w_0 + R(t) * w_1 - \int_0^t R(t-s) * h(s)\, ds,$$

where $*$ denotes convolution in the spatial variable. As is well known,

$$\operatorname{supp} R(\,\cdot\,, t) \subset \overline{B(0,t)} \quad \text{and} \quad \operatorname{supp} \partial_t R(\,\cdot\,, t) \subset \overline{B(0,t)}.$$

Thus, if

$$\operatorname{supp} u_0 \cap \overline{B(x_0, a)} = \varnothing, \ \ \operatorname{supp} u_1 \cap \overline{B(x_0, a)} = \varnothing, \ \ \operatorname{supp} h \cap \biggl(\bigcup_{0 \leqslant t \leqslant a} \overline{B(x_0, a-t)} \times \{t\} \biggr) = \varnothing,$$

then we have

$$w \equiv 0 \quad \text{on} \quad \bigcup_{0 \leqslant t \leqslant a} B(x_0, a-t) \times \{t\}.$$

These remarks have immediate consequences for the solutions of (CP) given in Theorem 2.7. In fact, suppose that (u_0,u_1) and (u_0',u_1') are data satisfying the conditions of Theorem 2.7 and such that $(u_0,u_1)=(u_0',u_1')$ in $\overline{B(x_0,a)}$. Then, the corresponding solutions u and u' agree on

$$\biggl(\bigcup_{0\leqslant t\leqslant a}\overline{B(x_0,(a-t))}\times\{t\}\biggr)\cap(\mathbb{R}^N\times I).$$

To see this, for $n\in\mathbb{N}$, define

$$u^{(n+1)}(x,t)=S(t)((u_0,u_1))+\int_0^t\frac{\sin((t-s)\sqrt{-\Delta}\,)}{\sqrt{-\Delta}}F(u^{(n)})\,ds$$

(for $n=0$, we set $u^{(0)}(x,t)=S(t)((u_0,u_1))$). We define correspondingly $(u')^{(n+1)}(x,t)$. The proof of Theorem 2.7 gives us $u=\lim_{n\to\infty}u^{(n)}$ and $u'=\lim_{n\to\infty}(u')^{(n)}$. The previous remarks allow us to show inductively that $u^{(n+1)}=(u')^{(n+1)}$ on

$$\biggl(\bigcup_{0\leqslant t\leqslant a}B(x_0,(a-t))\times\{t\}\biggr)\cap(\mathbb{R}^N\times I),$$

which establishes the claim. Typical applications of this remark are the following:

(a) If $\operatorname{supp}(u_0)\subset B(0,b)$, $\operatorname{supp}(u_1)\subset B(0,b)$ and (u_0,u_1) satisfies the hypothesis of Theorem 2.7, then

$$u(x,t)\equiv0\quad\text{on }\{(x,t):|x|>b+t,t\geqslant0\text{ and }t\in I\}.$$

(b) We can approximate solutions u in $\mathbb{R}\times I'$, $I'\Subset I$, by means of regular, compactly supported solutions, combining (a), Remark 2.9 and the last statement in Theorem 2.7.

Similar statements hold for $t<0$, for instance if $(u_0,u_1)=(u_0',u_1')$ in $\overline{B(x_0,a)}$, then u and u' agree on

$$\biggl(\bigcup_{-a\leqslant t\leqslant 0}B(x_0,(a+t))\times\{t\}\biggr)\cap(\mathbb{R}^N\times I).$$

Definition 2.13. Let $t_0\in I$. We say that u is a *solution* of (CP) in I if

$$(u,\partial_tu)\in C(I;\dot H^1\times L^2),\quad D_x^{1/2}u\in W(I),\quad u\in S(I),\quad (u,\partial_tu)\big|_{t=t_0}=(u_0,u_1)$$

and the integral equation

$$u(t)=S(t)((u_0,u_1))+\int_{t_0}^t\frac{\sin((t-s)\sqrt{-\Delta}\,)}{\sqrt{-\Delta}}F(u(s))\,ds$$

holds, with $F(u)=|u|^{4/(N-2)}u$, for $x\in\mathbb{R}^N$ and $t\in I$.

Note that if $u^{(1)}$ and $u^{(2)}$ are solutions of (CP) on I, and

$$(u^{(1)}(t_0), \partial_t u^{(1)}(t_0)) = (u^{(2)}(t_0), \partial_t u^{(2)}(t_0)),$$

then $u^{(1)} \equiv u^{(2)}$ on $\mathbb{R}^N \times I$. (See the argument in [19, Definition 2.10]). This allows one to define a maximal interval

$$I((u_0, u_1)) = (t_0 - T_-(u_0, u_1), t_0 + T_+(u_0, u_1)),$$

with $T_\pm(u_0, u_1) > 0$ where the solution is defined. If

$$T_1 > t_0 - T_-(u_0, u_1) \quad \text{and} \quad T_2 < t_0 + T_+(u_0, u_1),$$

with $t_0 \in (T_1, T_2)$, then u solves (CP) in $\mathbb{R}^N \times [T_1, T_2]$, so that

$$(u, \partial_t u) \in C([T_1, T_2]; \dot{H}^1 \times L^2)), \quad D_x^{1/2} u \in W([T_1, T_2]), \quad u \in S([T_1, T_2]),$$
$$u \in L^{(N+2)/(N-2)}([T_1, T_2]; L_x^{2(N+2)/(N-2)}) \quad \text{and} \quad \partial_t D_x^{-1/2} u \in W([T_1, T_2]).$$

Remark 2.14. If u is such that $(u, \partial_t u) \in C(I; \dot{H}^1 \times L^2)$, $\|u\|_{S(I)} \leqslant B$ and there exist u_j with $(u_j, \partial_t(u_j)) \in C(I; \dot{H}^1 \times L^2)$, $(u_j, \partial_t(u_j)) \to (u, \partial_t u)$ in $C(I; \dot{H}^1 \times L^2)$, with u_j a solution of (CP) in I together with $\|u_j\|_{S(I)} \leqslant B$, then $\|D_x^{1/2} u\|_{W(I)} < \infty$ and u is a solution of (CP) in I. This follows by showing that $\|D_x^{1/2} u_j\|_{W(I)} \leqslant B'$, where B' is independent of j. To show this, first find A so that $\sup_{t \in I} \|(u_j, \partial_t(u_j))\|_{\dot{H}^1 \times L^2} \leqslant A$, for all j. Next, partition $I = \bigcup_{k=1}^M I_k$, where I_k is such that $\|u_j\|_{S(I_k)} \leqslant \delta$, where $\delta = \delta(A)$ is to be chosen. Note that $M = M(B, \delta)$. We then use the integral equation for u_j, and the estimate

$$\|D_x^{1/2} F(u_j)\|_{L_{I_k}^{2(N+1)/(N+3)} L_x^{2(N+1)/(N+3)}} \leqslant C \delta^{4/(N-2)} \|D_x^{1/2} u_j\|_{W(I_k)}$$

(see the proof of Theorem 2.7), so that

$$\|D_x^{1/2} u_j\|_{W(I_k)} \leqslant CA + C\delta^{4/(N-2)} \|D_x^{1/2} u_j\|_{W(I_k)}.$$

Thus, for δ small, we obtain $\|D_x^{1/2} u_j\|_{W(I_k)} \leqslant 2CA$ and adding in k, we obtain the desired bound.

LEMMA 2.15. (Standard finite blow-up criterion) *If $T_+(u_0, u_1) < \infty$, then*

$$\|u\|_{S([t_0, t_0 + T_+(u_0, u_1)])} = \infty.$$

A corresponding result holds for $T_-(u_0, u_1)$.

The proof is similar to the one in [19, Lemma 2.11].

Remark 2.16. (Energy and moment identities) Let $(u_0,u_1)\in\dot H^1\times L^2$ and let $I\ni 0$ be the maximal interval of existence. Then, for $t\in I$, with

$$\frac{1}{2^*}=\frac{1}{2}-\frac{1}{N}\quad\left(2^*=\frac{2N}{N-2}\right),$$

we have

$$E((u(t),\partial_t u(t)))=\int_{\mathbb{R}^N}\left(\frac{1}{2}|\partial_t u(x,t)|^2+\frac{1}{2}|\nabla_x u(x,t)|^2-\frac{1}{2^*}|u(x,t)|^{2^*}\right)dx=E((u_0,u_1)),$$

and

$$\int_{\mathbb{R}^N}\nabla_x u(x,t)\partial_t u(x,t)\,dx=\int_{\mathbb{R}^N}\nabla u_0 u_1\,dx. \tag{2.2}$$

Proof. Let

$$e(u)(x,t)=\frac{1}{2}(\partial_t u)^2(x,t)+\frac{1}{2}|\nabla_x u(x,t)|^2-\frac{1}{2^*}|u(x,t)|^{2^*}.$$

Then, for sufficiently smooth solutions u of (CP), we have

$$\partial_t e(u)(x,t)=\sum_{j=1}^{N}\partial_{x_j}(\partial_{x_j}u(x,t)\partial_t u(x,t)), \tag{2.3}$$

as is readily seen. Now, fix any $I'\Subset I$, so that $\|u\|_{S(I')}<\infty$. By dividing $I'=\bigcup_{k=1}^{M}I_k$, with $\|u\|_{S(I_k)}\leqslant\delta(A)$, where

$$A=\sup_{t\in I'}\|(u(t),\partial_t u(t))\|_{\dot H^1\times L^2},$$

we can use Theorem 2.7 to approximate u by compactly supported solutions in $\mathbb{R}^N\times I_k$ (see Remarks 2.9 and 2.12). We then apply (2.3) and integrate by parts, and then pass to the limit, for $t\in I_k$. The proof of the second equality is similar. □

LEMMA 2.17. *Let $(u_0,u_1)\in\dot H^1\times L^2$, $\|(u_0,u_1)\|_{\dot H^1\times L^2}\leqslant A$, with maximal interval of existence $I=(-T_-(u_0,u_1),T_+(u_0,u_1))$. There exists $\varepsilon_0>0$ so that, if for some $M>0$ and $0<\varepsilon<\varepsilon_0$, we have $\int_{|x|\geqslant M}(|\nabla_x u_0|^2+|u_1|^2)\,dx\leqslant\varepsilon$, then for $t\in I_+=[0,\infty)\cap I$, we have*

$$\int_{|x|\geqslant 3M/2+t}\left(\frac{|u_0|^2}{|x|^2}+|\nabla_x u(x,t)|^2+|\partial_t u(x,t)|^2\right)dx\leqslant C\varepsilon.$$

Proof. Choose $\Psi_M\equiv 1$ for $|x|\geqslant\frac{3}{2}M$, $\Psi_M\equiv 0$ for $|x|\leqslant M$ and $|\nabla_x\Psi_M|\leqslant C/M$. Define $u_{0,M}=\Psi_M u_0$ and $u_{1,M}=\Psi_M u_1$. Because of our assumption and the Hardy inequality

$$\int_{\mathbb{R}^N}\frac{|f|^2}{|x|^2}\,dx\leqslant C\int_{\mathbb{R}^N}|\nabla f|^2\,dx,$$

we have $\|(u_{0,M}, u_{1,M})\|_{\dot{H}^1\times L^2}\leqslant C\varepsilon$. Now choose ε_0 so small that $C\varepsilon_0\leqslant\tilde{\delta}$, where $\tilde{\delta}$ is as in Remark 2.10. Then, there exists u_M solving (CP) in $I=(-\infty,\infty)$, with

$$(u_M(0), \partial_t u_M(0)) = (u_{0,M}, u_{1,M})$$

and such that

$$\sup_{t\in(-\infty,\infty)} \|(u_M(t), \partial_t u_M(t))\|_{\dot{H}^1\times L^2} \leqslant 2C\varepsilon.$$

But, by Remark 2.12, $u_M(x,t)=u(x,t)$ for $|x|\geqslant\frac{3}{2}M+t$, $t\in I_+$. The lemma follows. $\square$

Definition 2.18. Let $(v_0,v_1)\in\dot{H}^1\times L^2$ and $v(x,t)=S(t)((v_0,v_1))$, and let $\{t_n\}_{n=1}^\infty$ be a sequence, with $\lim_{n\to\infty} t_n=\bar{t}\in[-\infty,\infty]$. We say that $u(x,t)$ is a *non-linear profile* associated with $((v_0,v_1),\{t_n\}_{n=1}^\infty)$ if there exists an interval I, with $\bar{t}\in\mathring{I}$ (if $\bar{t}=\pm\infty$, then $I=[a,\infty)$ or $I=(-\infty,a]$) such that u is a solution of (CP) in I and

$$\lim_{n\to\infty} \|(u(t_n)-v(t_n), \partial_t u(t_n)-\partial_t v(t_n))\|_{\dot{H}^1\times L^2} = 0.$$

Remark 2.19. There always exists a non-linear profile associated with

$$((v_0,v_1),\{t_n\}_{n=1}^\infty).$$

The proof is similar to the one in [19, Remark 2.13], once we use the proof of Theorem 2.7 and the linear estimates

$$\sup_{t\in I} \|(w(t),\partial_t w(t))\|_{\dot{H}^1\times L^2} + \|D_x^{1/2} w\|_{W(I)} + \|w\|_{S(I)} \leqslant C\|h\|_{L_I^{2(N+1)/(N+3)}\dot{W}_x^{1/2,2(N+1)/(N+3)}},$$

where

$$w(x,t) = \int_t^\infty \frac{\sin((t-s)\sqrt{-\Delta}\,)}{\sqrt{-\Delta}} h(s)\,ds, \quad I=(a,\infty) \quad \text{and} \quad a>0,$$

which follow from [12, Proposition 3.1 (2) and (3)]. Also, as in [19, Remark 2.13], we have uniqueness of the non-linear profile and a maximal interval of existence of the non-linear profile associated with $((v_0,v_1),\{t_n\}_{n=1}^\infty)$.

THEOREM 2.20. (Long time perturbation theory; see also [18], [19] and [45]) *Let $I\subset\mathbb{R}$ be a time interval. Let $t_0\in I$, $(u_0,u_1)\in\dot{H}^1\times L^2$ and some constants $M,A,A'>0$ be given. Let $\tilde{u}$ be defined on $\mathbb{R}^N\times I$ $(3\leqslant N\leqslant 5)$ and satisfy $\sup_{t\in I}\|(\tilde{u}(t),\partial_t\tilde{u}(t))\|_{\dot{H}^1\times L^2}\leqslant A$, $\|\tilde{u}(t)\|_{S(I)}\leqslant M$ and $\|D_x^{1/2}\tilde{u}(t)\|_{W(I')}<\infty$ for each $I'\Subset I$. Assume that*

$$(\partial_t^2-\Delta_x)(\tilde{u})-F(\tilde{u})=e, \quad (x,t)\in\mathbb{R}^N\times I,$$

(*in the sense of the appropriate integral equation*) *and that*

$$\|(u_0-\tilde{u}(t_0),u_1-\partial_t\tilde{u}(t_0))\|_{\dot{H}^1\times L^2}\leqslant A',$$
$$\|D_x^{1/2}e\|_{L_I^{2(N+1)/(N+3)}L_x^{2(N+1)/(N+3)}}+\|S(t-t_0)((u_0-\tilde{u}(t_0),u_1-\partial_t\tilde{u}(t_0)))\|_{S(I)}\leqslant\varepsilon.$$

Then there exists $\varepsilon_0=\varepsilon_0(M,A,A')$ *such that, for* $0<\varepsilon<\varepsilon_0$, *there is a solution* u *of* (CP) *in* I *such that*

$$(u(t_0),\partial_t u(t_0))=(u_0,u_1),$$

with $\|u\|_{S(I)}\leqslant C(M,A,A')$ *and, for all* $t\in I$,

$$\|(u(t),\partial_t u(t))-(\tilde{u}(t),\partial_t\tilde{u}(t))\|_{\dot{H}^1\times L^2}\leqslant C(A,A',M)(A'+\varepsilon^\beta),\quad \beta>0.$$

We take this opportunity to point out that the proof of the analogous result in [19, Theorem 2.14], was incorrectly sketched in [19]. We are indebted to M. Visan and X. Zhang and to J. Holmer and S. Roudenko, for pointing this out to us. A correct proof is given in [18].

Remark 2.21. Theorem 2.20 yields the following continuity fact, which will be used later. Let $(\tilde{u}_0,\tilde{u}_1)\in\dot{H}^1\times L^2$, $\|(\tilde{u}_0,\tilde{u}_1)\|_{\dot{H}^1\times L^2}\leqslant A$, and let $\tilde{u}$ be the solution of (CP), with maximal interval of existence

$$(-T_-(\tilde{u}_0,\tilde{u}_1),T_+(\tilde{u}_0,\tilde{u}_1)).$$

Let $(u_0^{(n)},u_1^{(n)})\to(\tilde{u}_0,\tilde{u}_1)$ in $\dot{H}^1\times L^2$ and let $u^{(n)}$ be the corresponding solution of (CP), with maximal interval of existence

$$(-T_-(u_0^{(n)},u_1^{(n)}),T_+(u_0^{(n)},u_1^{(n)})).$$

Then

$$T_-(\tilde{u}_0,\tilde{u}_1)\leqslant\varliminf_{n\to\infty}T_-(u_0^{(n)},u_1^{(n)})\quad\text{and}\quad T_+(\tilde{u}_0,\tilde{u}_1)\leqslant\varliminf_{n\to\infty}T_+(u_0^{(n)},u_1^{(n)})$$

and for each $t\in(-T_-(\tilde{u}_0,\tilde{u}_1),T_+(\tilde{u}_0,\tilde{u}_1))$ we have

$$(u^{(n)}(t),\partial_t u^{(n)}(t))\to(\tilde{u}(t),\partial_t\tilde{u}(t))\quad\text{in }\dot{H}^1\times L^2.$$

Indeed, let $I\Subset(-T_-(\tilde{u}_0,\tilde{u}_1),T_+(\tilde{u}_0,\tilde{u}_1))$, so that

$$\sup_{t\in I}\|(\tilde{u}(t),\partial_t\tilde{u}(t))\|_{\dot{H}^1\times L^2}\leqslant\tilde{A}\quad\text{and}\quad\|\tilde{u}\|_{S(I)}\leqslant M<\infty.$$

We will show that, for n large, $u^{(n)}$ exists on I, that

$$\sup_{t\in I}\|(u^{(n)}(t),\partial_t u^{(n)}(t))-(\tilde{u}(t),\partial_t\tilde{u}(t))\|_{\dot{H}^1\times L^2}\leqslant C(M,\tilde{A})\|(u_0^{(n)},u_1^{(n)})-(\tilde{u}_0,\tilde{u}_1)\|_{\dot{H}^1\times L^2},$$

and, additionally, that $\|u^{(n)}\|_{S(I)}\leqslant\widetilde{M}(\tilde{A},M)$. To show this, apply Theorem 2.20, with $u=u^{(n)}$, $(u_0,u_1)=(u_0^{(n)},u_1^{(n)})$ and $e\equiv0$. If $\varepsilon_0=\varepsilon_0(M,\tilde{A},2\tilde{A})$ and n is so large that

$$\|S(t)((\tilde{u}_0-u_0^{(n)},\tilde{u}_1-u_1^{(n)}))\|_{S(I)}\leqslant\varepsilon\quad\text{and}\quad\|(\tilde{u}_0-\tilde{u}_0^{(n)},\tilde{u}_1-\tilde{u}_1^{(n)})\|_{\dot{H}^1\times L^2}\leqslant2\tilde{A},$$

then the desired conclusions follow from Theorem 2.20. Note also that if we choose $u_0^{(n)}$ and $u_1^{(n)}$ in $C_0^\infty(\mathbb{R}^N)$, the approximating solutions $u^{(n)}$ will be regular in view of Remark 2.9, and for $t\in I$ will have compact support in x, in view of Remark 2.12, and will satisfy $\|u^{(n)}\|_{S(I)}\leqslant\widetilde{M}$.

Remark 2.22. If u is a solution of (CP) in $\mathbb{R}^N\times I'$ for each $I'\Subset I$, $I=[a,\infty)$ (or $I=(-\infty,a]$), such that $\|u\|_{S(I)}<\infty$, then there exists $(u_0^+,u_1^+)\in\dot{H}^1\times L^2$ such that

$$\lim_{t\uparrow\infty}\|(u(t),\partial_t u(t))-(S(t)((u_0^+,u_1^+)),\partial_t S(t)((u_0^+,u_1^+)))\|_{\dot{H}^1\times L^2}=0.$$

See [19, Remark 2.15] and [4] for a similar proof. In our case we use the fact that

$$\|D_x^{1/2}F(u)\|_{L_I^{2(N+1)/(N+3)}L_x^{2(N+1)/(N+3)}}<\infty,$$

and the inequality used in the proof of Remark 2.19.

Remark 2.23. We recall that, since we are working in the focusing case, from the work of Levine [24], [41] we have that if $(u_0,u_1)\in H^1\times L^2$ is such that $E((u_0,u_1))<0$, then the maximal interval of existence is finite. We will return to the issue of break-down in finite time (blow-up), in the next section and at the end of the paper.

3. Variational estimates

Let

$$W(x)=W(x,t)=\left(1+\frac{|x|^2}{N(N-2)}\right)^{-(N-2)/2}$$

be a stationary solution of (CP). That is, W solves the non-linear elliptic equation

$$\Delta W+|W|^{4/(N-2)}W=0. \tag{3.1}$$

Moreover, $W\geqslant0$ and it is radially symmetric and decreasing. Note that $W\in\dot{H}^1$, but W need not belong to L^2, depending on the dimension. By invariances of equation (3.1), for

$\theta_0 \in [-\pi, \pi]$, $\lambda_0 > 0$ and $x_0 \in \mathbb{R}^N$, $W_{\theta_0, x_0, \lambda_0}(x) = e^{i\theta_0} \lambda_0^{(N-2)/2} W(\lambda_0(x - x_0))$ is still a solution of (3.1). By the work of Aubin [3] and Talenti [43], we have the following characterization of W:

$$\|u\|_{L^{2^*}} \leqslant C_N \|\nabla u\|_{L^2} \quad \text{for all } u \in \dot{H}^1; \tag{3.2}$$

moreover,

$$\begin{gathered} \text{if } \|u\|_{L^{2^*}} = C_N \|\nabla u\|_{L^2} \text{ and } u \neq 0, \text{ then} \\ \text{there exists } (\theta_0, \lambda_0, x_0) \text{ such that } u = W_{\theta_0, x_0, \lambda_0}, \end{gathered} \tag{3.3}$$

where C_N is the best constant of the Sobolev inequality (3.2) in dimension N.

Remark that

$$\int_{\mathbb{R}^N} |\nabla W|^2 \, dx = \frac{1}{C_N^N} \quad \text{and} \quad \mathcal{E}(W) = \frac{1}{N} \frac{1}{C_N^N},$$

where

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} \right) dx.$$

Indeed, the equation (3.1) gives $\int_{\mathbb{R}^N} |\nabla W|^2 \, dx = \int_{\mathbb{R}^N} |W|^{2^*} dx$. Also, (3.3) yields

$$C_N^2 \int_{\mathbb{R}^N} |\nabla W|^2 \, dx = \left(\int_{\mathbb{R}^N} |W|^{2^*} \, dx \right)^{(N-2)/N},$$

so that $C_N^2 \int_{\mathbb{R}^N} |\nabla W|^2 \, dx = \left(\int_{\mathbb{R}^N} |\nabla W|^2 \, dx \right)^{(N-2)/N}$. Hence,

$$\int_{\mathbb{R}^N} |\nabla W|^2 \, dx = \frac{1}{C_N^N} \quad \text{and} \quad \mathcal{E}(W) = \left(\frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla W|^2 \, dx = \frac{1}{N C_N^N}.$$

Lemma 3.1. *Let* $u \in \dot{H}^1(\mathbb{R}^N)$ *be such that, for* $\delta_0 > 0$,

$$\|\nabla u\|_{L^2}^2 < \|\nabla W\|_{L^2}^2 \quad \textit{and} \quad \mathcal{E}(u) \leqslant (1 - \delta_0) \mathcal{E}(W).$$

Then there exists $\bar{\delta} = \bar{\delta}(\delta_0) > 0$ *such that*

$$\|\nabla u\|_{L^2}^2 \leqslant (1 - \bar{\delta}) \|\nabla W\|_{L^2}^2 \quad \textit{and} \quad \mathcal{E}(u) \geqslant 0.$$

Proof. It is contained in [19, Lemma 3.4]. □

Corollary 3.2. *If* u *is as in Lemma 3.1, then there exists* $C_{\bar{\delta}} > 0$ *so that*

$$\int_{\mathbb{R}^N} (|\nabla u|^2 - |u|^{2^*}) \, dx \geqslant C_{\bar{\delta}} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.$$

Proof. Note that (3.2) implies that

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla u|^2-|u|^{2^*})\,dx &\geqslant \int_{\mathbb{R}^N}|\nabla u|^2\,dx - C_N^{2^*}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^{2^*/2}\\ &\geqslant \left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)\left(1-C_N^{2^*}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^{2/(N-2)}\right)\\ &\geqslant \left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)\left(1-C_N^{2^*}(1-\bar{\delta})^{1/(N-2)}\left(\int_{\mathbb{R}^N}|\nabla W|^2\,dx\right)^{2/(N-2)}\right),\end{aligned}$$

by Lemma 3.1. But

$$\left(\int_{\mathbb{R}^N}|\nabla W|^2\,dx\right)^{2/(N-2)} = \frac{1}{C_N^{2N/(N-2)}} = \frac{1}{C_N^{2^*}},$$

so that the corollary follows with $C_{\bar{\delta}}=1-(1-\bar{\delta})^{1/(N-2)}$. □

Corollary 3.3. *Let $u\in\dot{H}^1$, $\|\nabla u\|_{L^2}<\|\nabla W\|_{L^2}$. Then $\mathcal{E}(u)\geqslant 0$.*

Proof. If

$$\mathcal{E}(u)<\mathcal{E}(W)=\frac{1}{N}\frac{1}{C_N^N},$$

the claim follows from Lemma 3.1. If $\mathcal{E}(u)\geqslant\mathcal{E}(W)$, the statement is obvious. □

Remark 3.4. Let $u\in\dot{H}^1(\mathbb{R}^N)$ be such that $\mathcal{E}(u)\leqslant(1-\delta_0)\mathcal{E}(W)$. Assume that

$$\|\nabla u\|_{L^2}^2>\|\nabla W\|_{L^2}^2.$$

Then there exists $\bar{\bar{\delta}}=\bar{\bar{\delta}}(\delta_0,N)$ such that

$$\|\nabla u\|_{L^2}^2\geqslant(1+\bar{\bar{\delta}})\|\nabla W\|_{L^2}^2.$$

The proof of this is similar to the one of Lemma 3.1. See [19, Remark 3.14].

Theorem 3.5. (Energy trapping) *Let u be a solution of* (CP), *with*

$$(u,\partial_t u)\big|_{t=0}=(u_0,u_1)\in\dot{H}^1\times L^2$$

and maximal interval of existence I. Assume that, for $\delta_0>0$,

$$E((u_0,u_1))\leqslant(1-\delta_0)E((W,0))\quad\textit{and}\quad\|\nabla u_0\|_{L^2}^2<\|\nabla W\|_{L^2}^2.$$

Then, there exists $\bar{\delta}=\bar{\delta}(\delta_0)$ such that, for $t\in I$, we have

$$\|\nabla_x u(t)\|_{L^2}^2\leqslant(1-\bar{\delta})\|\nabla W\|_{L^2}^2, \tag{3.4}$$

$$\int_{\mathbb{R}^N}(|\nabla_x u(t)|^2-|u(t)|^{2^*})\,dx\geqslant C_{\bar{\delta}}\int_{\mathbb{R}^N}|\nabla_x u(t)|^2\,dx, \tag{3.5}$$

$$\mathcal{E}(u(t))\geqslant 0\quad(\textit{and hence } E((u(t),\partial_t u(t)))\geqslant 0). \tag{3.6}$$

Proof. By Remark 2.16, $E((u(t),\partial_t u(t)))=E((u_0,u_1))$, $t\in I$. Also,

$$\mathcal{E}(u(t))\leqslant E((u(t),\partial_t u(t))).$$

Thus, the theorem follows from Lemma 3.1, Corollaries 3.2 and 3.3 and a continuity argument. □

COROLLARY 3.6. *Let u be as in Theorem 3.5. Then, for all $t\in I$, we have*

$$E((u(t),\partial_t u(t)))\simeq\|(u(t),\partial_t u(t))\|^2_{\dot H^1\times L^2}\simeq\|(u_0,u_1)\|^2_{\dot H^1\times L^2},$$

with comparability constants which depend only on δ_0.

Proof. For $t\in I$, we have $E((u(t),\partial_t u(t)))\leqslant\|(u(t),\partial_t u(t))\|^2_{\dot H^1\times L^2}$. Also,

$$\begin{aligned}E((u(t),\partial_t u(t)))&=\frac12\int_{\mathbb{R}^N}(\partial_t u(t))^2\,dx+\mathcal{E}(u(t))\\&=\frac12\int_{\mathbb{R}^N}(\partial_t u(t))^2\,dx+\frac12\int_{\mathbb{R}^N}(|\nabla_x u(t)|^2-|u(t)|^{2^*})\,dx\\&\quad+\left(\frac12-\frac1{2^*}\right)\int_{\mathbb{R}^N}|u(t)|^{2^*}\,dx\\&\geqslant\frac12\int_{\mathbb{R}^N}(\partial_t u(t))^2\,dx+C_{\bar\delta}\int_{\mathbb{R}^N}|\nabla_x u(t)|^2\,dx.\end{aligned}$$

Finally, $E((u(t),\partial_t u(t)))=E((u_0,u_1))\simeq\|(u_0,u_1)\|^2_{\dot H^1\times L^2}$. □

THEOREM 3.7. (Finite-time blow-up; see also Remark 2.23) *Let $(u_0,u_1)\in\dot H^1\times L^2$, $u_0\in L^2$, and let u be the solution of* (CP) *with maximal interval of existence I. Assume that $E((u_0,u_1))<E((W,0))$ and $\int_{\mathbb{R}^N}|\nabla u_0|^2\,dx>\int_{\mathbb{R}^N}|\nabla W|^2\,dx$. Then I must be a finite interval.*

Proof. Fix δ_0 positive so that $E((u_0,u_1))\leqslant(1-\delta_0)E((W,0))$. Define

$$y(t)=\int_{\mathbb{R}^N}|u(x,t)|^2\,dx.$$

We then have

$$y'(t)=2\int_{\mathbb{R}^N}u\partial_t u\,dx\quad\text{and}\quad y''(t)=2\int_{\mathbb{R}^N}((\partial_t u)^2-|\nabla_x u|^2+|u|^{2^*})\,dx.$$

(To check these identities, we proceed as in Remark 2.16, starting with data in C_0^∞ and using a limiting argument.) Let $\tilde\delta_0=\delta_0E((W,0))$, so that $E((W,0))\geqslant E((u(t),\partial_t u(t)))+\tilde\delta_0$ and hence

$$\frac1{2^*}\int_{\mathbb{R}^N}|u(t)|^{2^*}dx\geqslant\frac12\int_{\mathbb{R}^N}((\partial_t u(t))^2+|\nabla_x u(t)|^2)\,dx-E((W,0))+\tilde\delta_0,$$

so that

$$\int_{\mathbb{R}^N} |u(t)|^{2^*}\,dx \geqslant \frac{N}{N-2}\int_{\mathbb{R}^N}((\partial_t u(t))^2+|\nabla_x u(t)|^2)\,dx-2^*E((W,0))+2^*\tilde{\delta}_0.$$

But then (with $\tilde{\tilde{\delta}}_0=2\cdot 2^*\tilde{\delta}_0$), we have

$$\begin{aligned}
y''(t) &\geqslant 2\int_{\mathbb{R}^N}(\partial_t u(t))^2\,dx+\frac{2N}{N-2}\int_{\mathbb{R}^N}(\partial_t u(t))^2\,dx-2\cdot 2^*E((W,0))\\
&\qquad+\frac{2N}{N-2}\int_{\mathbb{R}^N}|\nabla_x u(t)|^2\,dx-2\int_{\mathbb{R}^N}|\nabla_x u(t)|^2\,dx+\tilde{\tilde{\delta}}_0\\
&=\frac{4(N-1)}{N-2}\int_{\mathbb{R}^N}(\partial_t u(t))^2\,dx+\frac{4}{N-2}\int_{\mathbb{R}^N}|\nabla_x u|^2\,dx-\frac{4}{N-2}\int_{\mathbb{R}^N}|\nabla W|^2\,dx+\tilde{\tilde{\delta}}_0\\
&\geqslant\frac{4(N-1)}{N-2}\int_{\mathbb{R}^N}(\partial_t u(t))^2\,dx+\tilde{\tilde{\delta}}_0
\end{aligned}$$

(by Remark 3.4 and a continuity argument). Assume now that $[0,\infty)\subset I$. Then, by our lower bound on $y''(t)$, there exists $t_0>0$ such that $y'(t_0)>0$, and hence $y'(t)>0$ for $t>t_0$. Hence, for $t>t_0$,

$$y''(t)y(t)\geqslant\frac{4(N-1)}{N-2}\biggl(\int_{\mathbb{R}^N}(\partial_t u)^2(t)\,dx\biggr)\biggl(\int_{\mathbb{R}^N}u(t)^2\,dx\biggr)\geqslant\frac{N-1}{N-2}\,y'(t)^2,$$

so that, for $t>t_0$,

$$\frac{y''(t)}{y'(t)}\geqslant\frac{N-1}{N-2}\frac{y'(t)}{y(t)}\quad\text{or}\quad(\log y'(t))'\geqslant\frac{N-1}{N-2}(\log y(t))'.$$

Hence for $t>t_0$,

$$\log y'\geqslant\frac{N-1}{N-2}\log y-C_0\quad\text{or}\quad y'(t)\geqslant\widetilde{C}_0 y^{(N-1)/(N-2)},$$

which leads to finite-time blow-up of y, because $(N-1)/(N-2)>1$. This is a contradiction which gives the result. ∎

An extension of Theorem 3.7 will be given in §7.

4. Existence and compactness of a critical element; further properties of critical elements

Let us consider the statement:

(SC) For all $(u_0,u_1)\in\dot{H}^1\times L^2$, with $\int_{\mathbb{R}^N}|\nabla u_0|^2\,dx<\int_{\mathbb{R}^N}|\nabla W|^2\,dx$ and $E((u_0,u_1))<E((W,0))$, if u is the corresponding solution of (CP) with maximal interval of existence I (see Definition 2.13) then $I=(-\infty,\infty)$ and $\|u\|_{S((-\infty,\infty))}<\infty$.

In addition, for a fixed $(u_0,u_1)\in\dot H^1\times L^2$, with $\int_{\mathbb{R}^N}|\nabla u_0|^2\,dx<\int_{\mathbb{R}^N}|\nabla W|^2\,dx$ and $E((u_0,u_1))<E((W,0))$, we say that (SC)$((u_0,u_1))$ holds if, for the corresponding solution u of (CP), with maximal interval of existence I, we have $I=(-\infty,\infty)$ and

$$\|u\|_{S((-\infty,\infty))}<\infty.$$

Note that, because of Remark 2.10, if $\|(u_0,u_1)\|_{\dot H^1\times L^2}\leqslant\tilde\delta$, then (SC)$((u_0,u_1))$ holds. Thus, in light of Corollary 3.6, there exists $\eta_0>0$ such that if (u_0,u_1) is as in (SC), and $E((u_0,u_1))\leqslant\eta_0$, then (SC)$((u_0,u_1))$ holds. Moreover, for any (u_0,u_1) as in (SC), (3.6) shows that

$$E((u_0,u_1))\geqslant 0.$$

Thus, there exists a number E_C, $\eta_0\leqslant E_C\leqslant E((W,0))$, such that, if (u_0,u_1) is as in (SC) and $E((u_0,u_1))<E_C$, then (SC)$((u_0,u_1))$ holds and E_C is optimal with this property. For the rest of this section we will assume that $E_C<E((W,0))$. Using concentration compactness ideas, following the argument in [19, §4], we prove that there exists a critical element $(u_{0,C},u_{1,C})$ at the critical level of energy E_C, so that (SC)$((u_{0,C},u_{1,C}))$ does not hold, and from the minimality, this element has a compactness property up to the symmetries of the equation (which will give rigidity in the problem). We then use the finite speed of propagation and Lorentz transformations to establish support and orthogonality properties of critical elements, which are essential to treat the non-radial case.

PROPOSITION 4.1. *There exists* $(u_{0,C},u_{1,C})\in\dot H^1\times L^2$, *with*

$$E((u_{0,C},u_{1,C}))=E_C<E((W,0))\quad\textit{and}\quad\int_{\mathbb{R}^N}|\nabla u_{0,C}|^2\,dx<\int_{\mathbb{R}^N}|\nabla W|^2\,dx$$

such that if u_C *is the solution of* (CP) *with data* $(u_{0,C},u_{1,C})$ *and with maximal interval of existence* I, $0\in\mathring{I}$, *then* $\|u_C\|_{S(I)}=\infty$.

PROPOSITION 4.2. *Assume that* u_C *is as in Proposition* 4.1 *and that* (*say*)

$$\|u_C\|_{S(I_+)}=\infty,$$

where $I_+=[0,\infty)\cap I$. *Then there exist* $x(t)\in\mathbb{R}^N$ *and* $\lambda(t)\in\mathbb{R}^+$, *for* $t\in I_+$, *such that*

$$K=\{\vec v(x,t):t\in I_+\}$$

has the property that $\overline{K}$ *is compact in* $\dot H^1\times L^2$, *where*

$$\vec v(x,t)=\left(\frac{1}{\lambda(t)^{(N-2)/2}}u_C\left(\frac{x-x(t)}{\lambda(t)},t\right),\frac{1}{\lambda(t)^{N/2}}\partial_t u_C\left(\frac{x-x(t)}{\lambda(t)},t\right)\right).$$

A similar conclusion is reached if $\|u_C\|_{S(I_-)}=\infty$, *where* $I_-=(-\infty,0)\cap I$.

The proofs of Propositions 4.1 and 4.2 are identical to the corresponding ones in [19], using Lemma 4.3 below and the results of §2, especially Theorem 2.20. We will therefore omit them.

LEMMA 4.3. (Concentration compactness) *Let* $\{(v_{0,n}, v_{1,n})\}_{n=1}^{\infty} \in \dot{H}^1 \times L^2$, *with*

$$\|(v_{0,n}, v_{1,n})\|_{\dot{H}^1 \times L^2} \leqslant A.$$

Assume that

$$\|S(t)((v_{0,n}, v_{1,n}))\|_{S((-\infty,\infty))} \geqslant \delta > 0,$$

where $\delta=\delta(A)$ *is as in Theorem 2.7. Then there exists a sequence* $\{(V_{0,j}, V_{1,j})\}_{j=1}^{\infty}$ *in* $\dot{H}^1 \times L^2$, *a subsequence of* $\{(v_{0,n}, v_{1,n})\}_{n=1}^{\infty}$ *(which we still call* $\{(v_{0,n}, v_{1,n})\}_{n=1}^{\infty}$*) and a triple* $(\lambda_{j,n}; x_{j,n}; t_{j,n}) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}$, *with*

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \to \infty$$

as $n \to \infty$, *for* $j \neq j'$ *(we say that* $(\lambda_{j,n}; x_{j,n}; t_{j,n})$ *is orthogonal if this property is satisfied) such that*

$$\|(V_{0,1}, V_{1,1})\|_{\dot{H}^1 \times L^2} > \alpha_0(A) > 0. \tag{4.1}$$

If $V_j^l(x,t) = S(t)((V_{0,j}, V_{1,j}))$, *then, given* $\varepsilon_0 > 0$, *there exists* $J = J(\varepsilon_0)$ *and*

$$\{(w_{0,n}, w_{1,n})\}_{n=1}^{\infty} \in \dot{H}^1 \times L^2,$$

such that

$$\begin{aligned} v_{0,n} &= \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{(N-2)/2}} V_j^l \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, -\frac{t_{j,n}}{\lambda_{j,n}} \right) + w_{0,n}, \\ v_{1,n} &= \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{N/2}} \partial_t V_j^l \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, -\frac{t_{j,n}}{\lambda_{j,n}} \right) + w_{1,n}, \end{aligned} \tag{4.2}$$

with $\|S(t)((w_{0,n}, w_{1,n}))\|_{S((-\infty,\infty))} \leqslant \varepsilon_0$ *for* n *large,*

$$\int_{\mathbb{R}^N} |\nabla_x v_{0,n}|^2 \, dx = \sum_{j=1}^{J} \int_{\mathbb{R}^N} |\nabla_x V_{0,j}|^2 \, dx + \int_{\mathbb{R}^N} |\nabla_x w_{0,n}|^2 \, dx + o(1), \tag{4.3}$$

$$\begin{aligned} \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla_x v_{0,n}|^2 + \frac{1}{2} |v_{1,n}|^2 \right) dx &= \sum_{j=1}^{J} \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla_x V_{0,j}|^2 + \frac{1}{2} |V_{1,j}|^2 \right) dx \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla_x w_{0,n}|^2 + \frac{1}{2} |w_{1,n}|^2 \right) dx + o(1) \end{aligned} \tag{4.4}$$

as $n \to \infty$, *and*

$$E((v_{0,n} v_{1,n})) = \sum_{j=1}^{J} E\left(\left(V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}} \right), \partial_t V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}} \right) \right) \right) + E((w_{0,n}, w_{1,n})) + o(1) \tag{4.5}$$

as $n \to \infty$.

Remark 4.4. Lemma 4.3 is due to Bahouri–Gérard [4] (see also [6], [9] and [26] for the elliptic case and [28] for the Schrödinger case). In [4] the result is proved for $N=3$, but the proof extends to all $N\geqslant 3$. Also, the norm $\|\cdot\|_{S(-\infty,\infty)}$ is replaced by $\|\cdot\|_{L_t^{(N+2)/(N-2)}L_x^{2(N+2)/(N-2)}}$ in [4], but as is mentioned in [4, p. 136], it works equally well for $\|\cdot\|_{S(-\infty,\infty)}$. See the remark on [4, p. 159] to eliminate their condition (1.6). Note that (4.3) is not explicitly stated in [4], but the proof in [4] gives it. The equality (4.3) is important for us to establish that $\int_{\mathbb{R}^N}|\nabla u_{0,C}|^2\,dx<\int_{\mathbb{R}^N}|\nabla W|^2\,dx$ in Proposition 4.1. (See also the work of Keraani [22], where the corresponding result is proved for the nonlinear Schrödinger equation and where the analogue of (4.1) is shown.) See also [19, Remark 4.8].

Corollary 4.5. *There exists a decreasing function* $g\colon(0,E_C]\to[0,\infty)$ *such that for every* (u_0,u_1) *as in* (SC), *with* $E((u_0,u_1))=E_C-\eta$, *we have*

$$\|u\|_{S((-\infty,\infty))}\leqslant g(\eta).$$

For a proof of Corollary 4.5, see [4, Corollary 2] and [22, Corollary 1.14].

We next turn our attention to further properties of critical elements.

Lemma 4.6. *Let u be a solution of* (CP), *with maximal interval of existence I. Assume that, for $t\in I^+=I\cap[0,\infty)$, there exist $x(t)\in\mathbb{R}^N$ and $\lambda(t)\in\mathbb{R}^+$ so that*

$$K=\{\vec{v}(x,t):t\in I_+\}$$

has the property that $\overline{K}$ is compact in $\dot H^1\times L^2$, where

$$\vec{v}(x,t)=\left(\frac{1}{\lambda(t)^{(N-2)/2}}u\left(\frac{x-x(t)}{\lambda(t)},t\right),\frac{1}{\lambda(t)^{N/2}}\partial_t u\left(\frac{x-x(t)}{\lambda(t)},t\right)\right).$$

Then we can choose $\tilde\lambda(t)$ and $\tilde x(t)$, continuous in I_+, so that the corresponding $\widetilde K$ has compact closure in $\dot H^1\times L^2$.

Proof. The proof given in [19, Remark 5.4] applies verbatim. □

From now on, we always use $\tilde\lambda(t)$ and $\tilde x(t)$ provided by Lemma 4.6.

Lemma 4.7. *Let u be as in Lemma 4.6 and assume that I_+ is a finite interval. After scaling, we can assume then that $I_+=[0,1)$. Then,*

$$0<\frac{C_0(K)}{1-t}\leqslant\lambda(t).$$

Proof. Consider $0<t_j\to 1$. (Because of Lemma 4.6, this suffices.) Let

$$(v_{0,j},v_{1,j})=\left(\frac{1}{\lambda(t_j)^{(N-2)/2}}u\left(\frac{x-x(t_j)}{\lambda(t_j)},t_j\right),\frac{1}{\lambda(t_j)^{N/2}}\partial_t u\left(\frac{x-x(t_j)}{\lambda(t_j)},t_j\right)\right).$$

Since $(v_{0,j}, v_{1,j})\in K$ and $\overline{K}$ is compact in $\dot{H}^1\times L^2$, there exists $C_0=C_0(K)>0$ independent of j, so that $T_+(v_{0,j}, v_{1,j})\geqslant C_0$. (Here we are using the notation in Definition 2.13.) This is an easy consequence of Theorem 2.7. Let $v_j(t)$ be the corresponding solution of (CP). Note that

$$\begin{aligned}\lambda(t_j)^{(N-2)/2}v_{0,j}(\lambda(t_j)y+x(t_j))&=u(y,t_j),\\ \lambda(t_j)^{N/2}v_{1,j}(\lambda(t_j)y+x(t_j))&=\partial_t u(y,t_j).\end{aligned}$$

Hence, by uniqueness in (CP) (see the argument in Definition 2.13), for t such that $t_j+t\leqslant T_+(u_0,u_1)=1$, we have

$$\lambda(t_j)^{(N-2)/2}v_j(\lambda(t_j)y+x(t_j),\lambda(t_j)t)=u(y,t_j+t).$$

Thus, we have $t_j+t\leqslant 1$, for all $0<\lambda(t_j)t\leqslant C_0$. But then, choose $t=C_0/\lambda(t_j)$ so that $\lambda(t_j)\geqslant C_0/(1-t_j)$, as desired. □

Lemma 4.8. *Let u be as in Lemma 4.7. Then, there exists $\bar{x}\in\mathbb{R}^N$ such that*

$$\operatorname{supp} u\subset B(\bar{x},(1-t))\quad\textit{and}\quad \operatorname{supp}\partial_t u\subset B(\bar{x},(1-t)).$$

Proof. Recall from Lemma 4.7 that $\lambda(t)\geqslant C_0(K)/(1-t)$. We claim that, for any $R_0>0$,

$$\lim_{t\uparrow 1}\int_{|x+x(t)/\lambda(t)|\geqslant R_0}(|\nabla_x u(x,t)|^2+|\partial_t u(x,t)|^2)\,dx=0.$$

Indeed, if

$$\vec{v}(x,t)=\frac{1}{\lambda(t)^{N/2}}\biggl(\nabla u\biggl(\frac{x-x(t)}{\lambda(t)},t\biggr),\partial_t u\biggl(\frac{x-x(t)}{\lambda(t)},t\biggr)\biggr),$$

then

$$\int_{|x+x(t)/\lambda(t)|\geqslant R_0}(|\nabla_x u(x,t)|^2+|\partial_t u(x,t)|^2)\,dx=\int_{|y|\geqslant\lambda(t)R_0}|\vec{v}(y,t)|^2\,dy,$$

and our claim follows from the compactness of $\overline{K}$ and the fact that $\lambda(t)\uparrow\infty$. Using this estimate, we apply Lemma 2.17 backward in time, to conclude that, for each $s\in[0,1)$ and $R_0>0$, we have

$$\lim_{t\uparrow 1}\int_{|x+x(t)/\lambda t|\geqslant 3R_0/2+(t-s)}(|\nabla_x u(x,s)|^2+|\partial_t u(x,s)|^2)\,dx=0.$$

The next step is to show that $|x(t)/\lambda(t)|\leqslant M$ for $0\leqslant t<1$. If not, we can find (in light of Lemma 4.6) $t_n\uparrow 1$ so that $|x(t_n)/\lambda(t_n)|\to\infty$. Then, for all $R>0$,

$$\{x:|x|\leqslant R\}\subset\biggl\{x:\biggl|x+\frac{x(t_n)}{\lambda(t_n)}\biggr|\geqslant\frac{3}{2}R_0+t_n\biggr\},$$

for n large enough, so that, passing to the limit in n, for $s=0$,

$$\int_{|x|\leqslant R} (|\nabla u_0|^2+|u_1|^2)\,dx=0,$$

a contradiction. Finally, pick a sequence $t_n\uparrow 1$ so that $x(t_n)/\lambda(t_n)\to -\bar{x}$. Observe that, for every $\eta_0>0$, for n large enough and for all $s\in[0,1)$,

$$\{x:|x-\bar{x}|\geqslant 1+\eta_0-s\}\subset\left\{x:\left|x+\frac{x(t_n)}{\lambda(t_n)}\right|\geqslant \frac{3}{2}R_0+(t_n-s)\right\}$$

for some $R_0(\eta_0)>0$. From this we conclude that

$$\int_{|x-\bar{x}|\geqslant 1+\eta_0-s} (|\nabla_x u(x,s)|^2+|\partial_s u(x,s)|^2)\,dx=0,$$

which gives the claim. □

Remark 4.9. After a translation, we may assume that $\bar{x}=0$. Also, since $u(\cdot\,,t)\in L^{2^*}$ for each t, the conditions $\operatorname{supp} u\subset B(0,1-t)$ and $\operatorname{supp}\nabla_x u\subset B(0,1-t)$ are equivalent.

We turn now to the next important property of u_C (at least in the non-radial situation): the second invariant of the equation for u_C is zero. We consider the cases where I_+ is a finite interval and an infinite interval.

PROPOSITION 4.10. *Assume that u_C is as in Proposition 4.2 and I_+ is a finite interval. Then*,

$$\int_{\mathbb{R}^N} \nabla u_{0,C}u_{1,C}\,dx=0.$$

Proof. By scaling, we can assume that $I_+=[0,1)$. By Lemma 4.8, we have $\operatorname{supp} u_C\subset B(0,1-t)$ and $\operatorname{supp}\partial_t u_C\subset B(0,1-t)$. Note also that for any solution u of (CP) in I, the maximal interval of existence, and for any $t\in I$, we have from (2.2) that

$$\int_{\mathbb{R}^N} \nabla_x u(t)\partial_t u(t)\,dx=\int_{\mathbb{R}^N} \nabla u_0 u_1\,dx.$$

Assume now (without loss of generality) that

$$\gamma=\int_{\mathbb{R}^N} \partial_{x_1}(u_{0,C})u_{1,C}\,dx>0.$$

We will reach a contradiction, by considering (for convenience) $u(x,t)=u_C(x,1+t)$, with $-1\leqslant t<0$. Clearly, for $-1\leqslant t<0$,

$$E((u(t),\partial_t u(t)))=E_C,\quad \int_{\mathbb{R}^N} |\nabla u(t)|^2\,dx\leqslant(1-\bar{\delta})\|\nabla W\|_{L^2}^2,\quad \gamma=\int_{\mathbb{R}^N} \partial_{x_1}u(t)\partial_t u(t)\,dx,$$

by Theorem 3.5 and our assumption above. We will consider the action of Lorentz transformations on u. (Now, $\operatorname{supp} u(\cdot,t)\subset B(0,-t)$ and $\operatorname{supp}\partial_t u(\cdot,t)\subset B(0,-t)$, for $-1\leqslant t<0$.) Thus, for $0<d<\frac{1}{4}$, consider

$$z_d(x_1,\bar{x},t)=u\left(\frac{x_1-dt}{\sqrt{1-d^2}},\bar{x},\frac{t-dx_1}{\sqrt{1-d^2}}\right), \tag{4.6}$$

where $x=(x_1,\bar{x})\in\mathbb{R}^N$, $t\in\mathbb{R}$ and $s=(t-dx_1)/\sqrt{1-d^2}$ is such that $-1\leqslant s<0$.

Note that, for this range of s and $y=(y_1,\bar{y})$ such that $(y,s)\in\operatorname{supp} u$, we have $|y|\leqslant|s|$. Thus, if $y_1=(x_1-dt)/\sqrt{1-d^2}$ and $\bar{y}=\bar{x}$, we obtain $x_1^2+|\bar{x}|^2\leqslant t^2$ in the support of z_d and $\partial_t z_d$. Fix now $-\frac{1}{2}\leqslant t<0$ and $x_1^2+|\bar{x}|^2\leqslant t^2$. Then,

$$\frac{t-dx_1}{\sqrt{1-d^2}}\geqslant\frac{(1+d)t}{\sqrt{1-d^2}}\geqslant-\frac{1}{2}\frac{1+d^2}{\sqrt{1-d^2}}\geqslant-1,$$

while

$$\frac{t-dx_1}{\sqrt{1-d^2}}\leqslant\frac{(1-d)t}{\sqrt{1-d^2}}<0.$$

Thus, for such (x,t), z_d is defined and we have $z_d(x,t)=0$, $\nabla_x z_d(x,t)=0$ and $\partial_t z_d(x,t)=0$ for $x_1^2+|\bar{x}|^2=t^2$. We extend $z_d(\cdot,t)$ to be zero for $|x|\geqslant|t|$, $-\frac{1}{2}\leqslant t<0$. An elementary calculation shows that if u is a regular solution (by a *regular solution* we will mean one as in Remark 2.9, with $\mu=1$) of

$$\partial_t^2u-\Delta u=|u|^{4/(N-2)}u \quad \text{in } \mathbb{R}^N\times[-1,0),$$

then the resulting z_d is a solution of (CP) for this equation in $-\frac{1}{2}\leqslant t<0$, $x\in\mathbb{R}^N$.

We will now show that the z_d we defined in (4.6) is a solution of (CP) in $\mathbb{R}^N\times\left[-\frac{1}{2},0\right)$. To this end, fix $\varepsilon_0>0$ and consider $-\frac{1}{2}\leqslant t\leqslant-\varepsilon_0$, $x\in\mathbb{R}^N$. Note that in this range we have, on $\operatorname{supp} z_d$, that $-1\leqslant s\leqslant-3\varepsilon_0/\sqrt{15}$. Note also that since the $S([-1,-3\varepsilon_0/\sqrt{15}\,])$ norm of u is finite, and $u\in L^{(N+2)/(N-2)}_{[-1,-3\varepsilon_0/\sqrt{15}\,]}L_x^{2(N+2)/(N-2)}$ (see Definition 2.13), in light of Remark 2.8, we have $(z_d,\partial_t(z_d))\in C\left(\left[-\frac{1}{2},-\varepsilon_0\right];\dot{H}^1\times L^2\right)$. Also, if we let

$$J=\left|\det\frac{\partial(y,s)}{\partial(x,t)}\right|,$$

then $J\equiv1$ and hence, if $D_{\varepsilon_0}=\mathbb{R}^N\times\left[-\frac{1}{2},-\varepsilon_0\right]$ and $\widetilde{D}_{\varepsilon_0}=\Phi(D_{\varepsilon_0})$, where $\Phi(x,t)=(y,s)$, then

$$\begin{aligned}\int_{D_{\varepsilon_0}}|z_d(x,t)|^{2(N+1)/(N-1)}\,dx\,dt&=\int_{\widetilde{D}_{\varepsilon_0}}|u(y,s)|^{2(N+1)/(N-1)}\,dy\,ds\\&\leqslant\int_{-1\leqslant s\leqslant-3\varepsilon_0/\sqrt{15}}|u(y,s)|^{2(N+1)/(N-1)}\,dy\,ds\leqslant C_{\varepsilon_0}.\end{aligned}$$

Moreover, pick $u_{0,j}\in C_0^\infty(B(0,3\varepsilon_0/\sqrt{15}\,))$ and $u_{1,j}\in C_0^\infty(B(0,3\varepsilon_0/\sqrt{15}\,))$ with

$$(u_{0,j},u_{1,j})\to\left(u\left(-\frac{3}{\sqrt{15}}\varepsilon_0\right),\partial_s u\left(-\frac{3}{\sqrt{15}}\varepsilon_0\right)\right)\quad\text{in } \dot H^1\times L^2.$$

Let u_j be the solution of (CP), defined for $s<-3\varepsilon_0/\sqrt{15}$. Note that, because of Remark 2.21, we know that, for j large, u_j is a solution of (CP) for $-1\leqslant s<-3\varepsilon_0/\sqrt{15}$,

$$\|(u_j,\partial_s u_j)\|_{C((-1,-3\varepsilon_0/\sqrt{15}\,),\dot H^1\times L^2)}\leqslant C$$

and

$$\|u_j\|_{S((-1,-3\varepsilon_0/\sqrt{15}\,))}+\|u_j\|_{L^{(N+2)/(N-2)}_{[-1,-3\varepsilon_0/\sqrt{15}]}L_x^{2(N+1)/(N-2)}}\leqslant \widetilde C_{\varepsilon_0}.$$

Also, by Remark 2.9, u_j is regular for $s\in[-1,-3\varepsilon_0/\sqrt{15}\,]$ and, by Remark 2.12, we have $\operatorname{supp} u_j(\,\cdot\,,s)\subset B(0,|s|)$ for $-1\leqslant s\leqslant -3\varepsilon_0/\sqrt{15}$. If we now consider $z_{j,d}$ given by (4.6) with u replaced by u_j, the $z_{j,d}$ are solutions of (CP) in $-\frac12\leqslant t\leqslant -\varepsilon_0$. Moreover, from the proof of Remark 2.21 and the proof that $(z_d,\partial_t(z_d))\in C\big(\big[-\frac12,-\varepsilon_0\big];\dot H^1\times L^2\big)$, we can conclude that $(z_{j,d},\partial_t(z_{j,d}))\to(z_d,\partial_t z_d)$ in $C\big(\big[-\frac12,-\varepsilon_0\big];\dot H^1\times L^2\big)$ and similarly that $\|z_{j,d}\|_{S([-1/2,-\varepsilon_0])}\leqslant C_{\varepsilon_0}$. Now, from Remark 2.14, it follows that z_d is a solution of (CP) for $t\in\big[-\frac12,-\varepsilon_0\big]$. Since $\varepsilon_0>0$ is arbitrary, we conclude that $T_+\big(z_d\big(-\frac12\big),\partial_t z_d\big(-\frac12\big)\big)\geqslant 0$. But, since $\operatorname{supp} z_d,\partial_t z_d\subset\{x:|x|\leqslant|t|\}$ for any $t\in\big[-\frac12,0\big)$, either $T_+\big(z_d\big(-\frac12\big),\partial_t z_d\big(-\frac12\big)\big)=0$, or $z_d\equiv 0$. Because $u\not\equiv 0$, it is easy to see that $z_d\not\equiv 0$.

We have, by Remark 2.16, that

$$\int_{-1/2}^{-1/4} E((z_d(t),\partial_t z_d(t)))\,dt=\frac14 E\left(\left(z_d\left(-\frac12\right),\partial_t z_d\left(-\frac12\right)\right)\right). \tag{4.7}$$

We are now going to calculate the derivative in d of the left-hand side. Note that

$$\begin{aligned}\partial_{x_1} z_d&=-\frac{d}{\sqrt{1-d^2}}\,\partial_s u+\frac{1}{\sqrt{1-d^2}}\,\partial_{y_1}u,\\ \partial_{\bar x} z_d&=\partial_{\bar y}u,\\ \partial_t z_d&=\frac{1}{\sqrt{1-d^2}}\,\partial_s u-\frac{d}{\sqrt{1-d^2}}\,\partial_{y_1}u.\end{aligned}$$

Thus, the left-hand side of (4.7) equals I_1+I_2, where

$$\begin{aligned}I_1&=\int_{-1/2}^{-1/4}\int_{\mathbb R^N}\left(\frac12\left(\frac{1+d^2}{1-d^2}((\partial_s u)^2+(\partial_{y_1}u)^2)+|\nabla_{\bar y}u|^2\right)-\frac{1}{2^*}|u|^{2^*}\right)dx_1\,d\bar x\,dt,\\ I_2&=-\frac{2d}{1-d^2}\int_{-1/2}^{-1/4}\int_{\mathbb R^N}\partial_{y_1}u\,\partial_s u\,dx_1\,d\bar x\,dt.\end{aligned}$$

Also notice that, for regular f, of compact support,

$$
\begin{aligned}
&\frac{\partial}{\partial d}\int f\left(\frac{x_1-dt}{\sqrt{1-d^2}},\bar{x},\frac{t-dx_1}{\sqrt{1-d^2}}\right)dx_1\,d\bar{x}\\
&=\frac{1}{1-d^2}\int\frac{-t}{(1-d^2)^{1/2}}\left(\frac{\partial f}{\partial y_1}\left(\frac{x_1-dt}{\sqrt{1-d^2}},\bar{x},\frac{t-dx_1}{\sqrt{1-d^2}}\right)-d\frac{\partial f}{\partial s}\left(\frac{x_1-dt}{\sqrt{1-d^2}},\bar{x},\frac{t-dx_1}{\sqrt{1-d^2}}\right)\right)dx_1\,d\bar{x}\\
&\quad+\frac{1}{1-d^2}\int\frac{-x_1}{(1-d^2)^{1/2}}\left(\frac{\partial f}{\partial s}\left(\frac{x_1-dt}{\sqrt{1-d^2}},\bar{x},\frac{t-dx_1}{\sqrt{1-d^2}}\right)-d\frac{\partial f}{\partial y_1}\left(\frac{x_1-dt}{\sqrt{1-d^2}},\bar{x},\frac{t-dx_1}{\sqrt{1-d^2}}\right)\right)dx_1\,d\bar{x}\\
&=\frac{1}{1-d^2}\int -t\frac{\partial f}{\partial x_1}\,dx_1\,d\bar{x}-\frac{1}{1-d^2}\int x_1\frac{\partial f}{\partial t}\,dx_1\,d\bar{x}\\
&=-\frac{1}{1-d^2}\frac{\partial}{\partial t}\int x_1 f\,dx_1\,d\bar{x},
\end{aligned}
$$

where the integrations are over $\mathbb{R}^N$. Hence,

$$
\begin{aligned}
\frac{\partial}{\partial d}I_1(d)=&\int_{-1/2}^{-1/4}\int_{\mathbb{R}^N}\frac{1}{2}\frac{4d}{(1-d^2)^2}((\partial_s u)^2+(\partial_{y_1}u)^2)\,dx_1\,d\bar{x}\,dt\\
&-\frac{1}{1-d^2}\int_{\mathbb{R}^N}\frac{x_1}{2}\left(\frac{1+d^2}{1-d^2}((\partial_s u)^2+(\partial_{y_1}u)^2)+|\nabla_{\bar{y}}u|^2-\frac{1}{2^*}|u|^{2^*}\right)dx_1\,d\bar{x}\bigg|_{t=-1/2}^{-1/4}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial d}I_2(d)=&-\frac{2}{(1-d^2)^2}\int_{-1/2}^{-1/4}\int_{\mathbb{R}^N}\partial_{y_1}u\partial_s u\,dx_1\,d\bar{x}\,dt\\
&+\frac{2d}{1-d^2}\int_{\mathbb{R}^N}-\frac{x_1}{1-d^2}\partial_{y_1}u\partial_s u\,dx_1\,d\bar{x}\bigg|_{t=-1/2}^{-1/4}.
\end{aligned}
$$

(This computation can be justified for z_d by approximating z_d, using the fundamental theorem of calculus in d, since all the terms make sense for z_d.)

But then

$$
\begin{aligned}
\frac{\partial}{\partial d}(I_1(d)+I_2(d))\bigg|_{d=0}&=-\int_{\mathbb{R}^N}x_1 e(u)\,dx_1\,d\bar{x}\bigg|_{t=-1/2}^{t=-1/4}-2\int_{-1/2}^{-1/4}\int_{\mathbb{R}^N}\partial_{y_1}u\partial_s u\,dx_1\,d\bar{x}\,dt\\
&=\tfrac{1}{4}\gamma-\tfrac{1}{2}\gamma=-\tfrac{1}{4}\gamma
\end{aligned}
$$

in light of (2.2) and (2.3).

Also, by (4.7), $I_1(0)+I_2(0)=\frac{1}{4}E_C$, so that, for d small, we have

$$
E\big(\big(z_d\big(-\tfrac{1}{2}\big),\partial_t z_d\big(-\tfrac{1}{2}\big)\big)\big)<E_C,
$$

since $\gamma>0$.

Finally, since $E_C<E((W,0))$ and $\|\nabla u(-1)\|_{L^2}^2<\|\nabla W\|_{L^2}^2$, because of Theorem 3.5, we have that, for $-1\leqslant s\leqslant 0$, $\|\nabla_x u(\cdot\,,s)\|_{L^2}^2\leqslant(1-\bar\delta)\|\nabla W\|_{L^2}^2$, $\bar\delta>0$. We now consider

$$\int_{-1/2}^{-1/4}\int_{\mathbb{R}^N}|\nabla_x z_d(x,t)|^2\,dx\,dt.$$

A change of variables, together with the calculation of $\partial_{x_1}z_d$ and $\partial_{\bar x}z_d$, show that

$$\lim_{d\downarrow 0}\int_{-1/2}^{-1/4}\int_{\mathbb{R}^N}|\nabla_x z_d(x,t)|^2\,dx\,dt=\int_{-1/2}^{-1/4}\int_{\mathbb{R}^N}|\nabla_y u(y,s)|^2\,dy\,ds\leqslant\frac{1}{4}(1-\bar\delta)\|\nabla W\|_{L^2}^2.$$

But then, for d small,

$$\int_{-1/2}^{-1/4}\int_{\mathbb{R}^N}|\nabla_x z_d u(x,t)|^2\,dx\,dt\leqslant\frac{1}{4}\left(1-\frac{\bar\delta}{2}\right)\|\nabla W\|_{L^2}^2.$$

Thus, there exists $t_0=t_0(d)\in\left(-\frac{1}{2},-\frac{1}{4}\right)$ such that, for d small,

$$\int_{\mathbb{R}^N}|\nabla_x z_d(x,t_0)|^2\,dx<\|\nabla W\|_{L^2}^2\quad\text{and}\quad E\left(\left(z_d\left(-\tfrac{1}{2}\right),\partial_t z_d\left(-\tfrac{1}{2}\right)\right)\right)<E((W,0)).$$

By Theorem 3.5, we have, for all d small, $\left\|\nabla_x z_d\left(x,-\frac{1}{2}\right)\right\|_{L^2}^2<\|\nabla W\|_{L^2}^2$. Since the interval of existence of z_d is finite, this contradicts the definition of E_C taking $d>0$ small, and thus $\gamma=0$. □

PROPOSITION 4.11. *Let u_C be as in Proposition 4.2 and $I_+=[0,\infty)$. Assume in addition that for $t>0$, $\lambda(t)>A_0>0$. Then,*

$$\int_{\mathbb{R}^N}\nabla u_{0,C}\,u_{1,C}\,dx=0.$$

Proof. Because of Proposition 4.10, we can assume that $T_-(u_0,u_1)=\infty$. To abbreviate the notation, let us write $u(x,t)=u_C(x,t)$. Again, without loss of generality, if the conclusion does not hold, we can assume that $\gamma=\int_{\mathbb{R}^N}\partial_{y_1}u_0u_1\,dy>0$ and hence, by (2.2), for all $s\in\mathbb{R}$ we have

$$\int_{\mathbb{R}^N}\partial_{y_1}u(s)\partial_s u(s)\,dy=\gamma>0.$$

We will see that this assumption leads to a contradiction. We first start out by showing that, given $\varepsilon>0$, there exists $R_0(\varepsilon)>0$ such that, for all $s\geqslant 0$, we have

$$\int_{|y+y(s)/\lambda(s)|\geqslant R_0(\varepsilon)}\left(|\partial_s u|^2+|\nabla_y u|^2+\frac{|u|^2}{|y|^2}+|u|^{2^*}\right)dy\leqslant\varepsilon. \tag{4.8}$$

In fact, by compactness of $\overline{K}$, given $\varepsilon>0$, there exists $\widetilde{R}_0=\widetilde{R}_0(\varepsilon)>0$ such that, for all $s\in[0,\infty)$,

$$\int_{|y+y(s)/\lambda(s)|\geqslant \widetilde{R}_0/\lambda(s)} \left(|\partial_s u|^2+|\nabla_y u|^2+\frac{|u|^2}{|y|^2}+|u|^{2^*} \right) dy \leqslant \varepsilon.$$

Since $\lambda(s)\geqslant A_0$, $R_0(\varepsilon)=\widetilde{R}_0(\varepsilon)/A_0$ does the job.

Next, we show that, as a consequence of (4.8), we have good bounds for $|y(s)/\lambda(s)|$:

$$\text{for } M>0 \text{ we have } \left|\frac{y(s)}{\lambda(s)}\right| \leqslant s+M \text{ for all } s\in[0,\infty). \tag{4.9}$$

To verify (4.9), recall that, since $E((u_0,u_1))=E_C>0$, (u_0,u_1) is not identically 0, thus we have, because of Corollary 3.6,

$$\inf_{s\geqslant 0} \int_{\mathbb{R}^N} (|\nabla_y u(y,s)|^2+|\partial_s u(y,s)|^2)\, dy \geqslant C\|(u_0,u_1)\|^2_{\dot{H}^1\times L^2}=B_0>0.$$

Then, use (4.8) to choose $M_0>0$ so that

$$\int_{|y+y(s)/\lambda(s)|\geqslant M_0} (|\nabla u|^2+|\partial_s u|^2)\, dy \leqslant \frac{B_0}{2}, \quad s\in[0,\infty),$$

to conclude that

$$\int_{|y+y(s)/\lambda(s)|\leqslant M_0} (|\nabla u|^2+|\partial_s u|^2)\, dy \geqslant \frac{B_0}{2}, \quad s\in[0,\infty).$$

Now recall, from Lemma 2.17, that there exists $\varepsilon_0>0$ such that, if for some $M_1>0$ we have

$$\int_{|y|>M_1} \left(|\nabla_y u_0|^2+|u_1|^2+\frac{|u_0|^2}{|y|^2} \right) dy \leqslant \varepsilon, \tag{4.10}$$

then

$$\int_{|y|\geqslant 3M_1/2+s} (|\nabla_y u(y,s)|^2+|\partial_s u(y,s)|^2)\, dy \leqslant C\varepsilon,$$

whenever $0<\varepsilon<\varepsilon_0$ and $s\geqslant 0$. Since we can assume, without loss of generality, that $y(0)=0$ and $\lambda(0)=1$, in light of (4.8) we can always achieve (4.10). We will show that we can choose ε so small that $|y(s)/\lambda(s)|\leqslant s+3\max\{M_0,M_1\}$. Suppose, on the contrary, that $|y(s)/\lambda(s)|\geqslant s+3\max\{M_0,M_1\}$. If $|y+y(s)/\lambda(s)|\leqslant M_0$, then

$$|y|\geqslant s+3\max\{M_0,M_1\}-M_0\geqslant s+2\max\{M_0,M_1\}\geqslant s+2M_1.$$

But then,

$$\frac{B_0}{2}\leqslant \int_{|y+y(t)/\lambda(s)|\leqslant M_0} (|\nabla_y u|^2+|\partial_s u|^2)\, dy \leqslant \int_{|y|\geqslant s+2M_1} (|\nabla_y u|^2+|\partial_s u|^2)\, dy \leqslant C\varepsilon,$$

by (4.10). If $C\varepsilon<\frac{1}{2}B_0$, we reach a contradiction, which establishes (4.9). Because of the lack of compact support, the argument in the proof of Proposition 4.10 does not apply verbatim. The idea in what follows is to use a rescaling, to concentrate the solution near the origin.

Having (4.8) and (4.9) at our disposal, we now define, for $R>0$ and $d>0$,

$$z_{d,R}(x_1,\bar{x},t)=u_R\left(\frac{x_1-dt}{\sqrt{1-d^2}},\bar{x},\frac{t-dx_1}{\sqrt{1-d^2}}\right), \tag{4.11}$$

where

$$u_R(y_1,\bar{y},s)=R^{(N-2)/2}u(Ry_1,R\bar{y},Rs).$$

Note that u_R is a solution of (CP) in $\mathbb{R}^N\times\mathbb{R}$, that $E((u_R(0),\partial_s u_R(0)))=E_C$ and that there exists $\bar{\delta}>0$ such that

$$\int_{\mathbb{R}^N}|\nabla_y u_R(y,s)|^2\,dy\leqslant(1-\bar{\delta})\int_{\mathbb{R}^N}|\nabla W|^2\,dy.$$

We also have $\sup_{s\in\mathbb{R}}\|(u_R,\partial_s u_R)\|_{\dot{H}^1\times L^2}\leqslant A$ and $\|u_R\|_{S((0,\infty))}=\infty$. Moreover, we will use the fact that, when (x,t) are in a compact set, the identity

$$\partial_t e(z_{d,R})(x,t)=\sum_{j=1}^N\partial_{x_j}(\partial_{x_j}z_{d,R}\,\partial_t z_{d,R})$$

holds, which can be shown by approximating u_R by compactly supported regular solutions and making the observation that the corresponding $z_{d,R}$ are then solutions of (CP) on finite-time intervals.

We now prove the following fact:

$$\begin{gathered}\text{There exists } d_0>0 \text{ such that, for } 0<d<d_0\\ \int_1^2\int_{3\leqslant|x|\leqslant 8}(|\nabla_x z_{d,R}|^2+|\partial_t z_{d,R}|^2+|z_{d,R}|^{2^*})\,dx\,dt\leqslant\eta_1(R,d),\\ \text{where } \eta_1(R,d)\xrightarrow{R\to\infty}0 \text{ uniformly in } d<d_0.\end{gathered} \tag{4.12}$$

To establish (4.12), we use the change of variables $\Phi(x,t)=(y,s)$, where

$$y_1=\frac{x_1-dt}{\sqrt{1-d^2}},\quad \bar{y}=\bar{x}\quad\text{and}\quad s=\frac{t-dx_1}{\sqrt{1-d^2}}.$$

Then, for d small, we have, after changing variables, that the left-hand side of (4.12) is bounded by

$$\int_{1-1/8}^{2+1/8}\int_{3-1/8\leqslant|y|\leqslant 8+1/8}(|\nabla_y u_R|^2+|\partial_s u_R|^2+|u_R|^{2^*})\,dy\,ds,$$

which, after rescaling, becomes

$$\frac{1}{R}\int_{(1-1/8)R}^{(2+1/8)R}\int_{(3-1/8)R\leqslant|y|\leqslant(8+1/8)R}(|\nabla_y u|^2+|\partial_s u|^2+|u|^{2^*})\,dy\,ds.$$

But, by (4.9), we have $|y(s)/\lambda(s)|\leqslant\big(2+\frac{1}{8}\big)R+M$ for $0\leqslant s\leqslant\big(2+\frac{1}{8}\big)R$, so that, for R large, $\big\{y:|y|\geqslant\big(3-\frac{1}{8}\big)R\big\}\subset\big\{y:|y+y(s)/\lambda(s)|\geqslant\frac{1}{2}R\big\}$, and our claim then follows from (4.8).

We now pick $\theta_1=\theta_1(\alpha)\in C_0^\infty(\{\alpha:|\alpha|<5\})$, such that $\theta_1\equiv1$ on $|\alpha|<4$ and $0\leqslant\theta_1\leqslant1$, and define $\theta(x)=\theta_1(x_1)\theta_1(|\bar{x}|)$. Note that $\theta(x)\equiv1$ on $|x|\leqslant4$ and $\operatorname{supp}\theta\subset\{x:|x|\leqslant\sqrt{50}\,\}$. Our next task is to study

$$J(d)=\int_1^2\int_{\mathbb{R}^N}\theta^2 e(z_{d,R})(x_1,\bar{x},t)\,dx_1\,d\bar{x}\,dt.$$

Using the calculations in the proof of Proposition 4.10, we see that

$$\begin{aligned}
\frac{\partial}{\partial d}J(d)&=\int_1^2\int_{\mathbb{R}^N}\theta^2\frac{\partial}{\partial d}e(z_{d,R})\,dx_1\,d\bar{x}\,dt\\
&=\int_1^2\int_{\mathbb{R}^N}\theta^2\frac{1}{2}\frac{4d}{(1-d^2)^2}((\partial_s u_R)^2+(\partial_{y_1}u_R)^2)\,dx_1\,d\bar{x}\,dt\\
&\quad-\frac{2}{(1-d^2)^2}\int_1^2\int_{\mathbb{R}^N}\theta^2\partial_{y_1}u_R\cdot\partial_s u_R\,dx_1\,d\bar{x}\,dt\\
&\quad-\frac{2}{1-d^2}\int_1^2\int_{\mathbb{R}^N}t\theta_1^2(|\bar{x}|)\frac{\partial\theta_1}{\partial x_1}(x_1)\theta_1(x_1)\\
&\qquad\times\bigg(\frac{1-d^2}{1+d^2}\frac{1}{2}((\partial_{y_1}u_R)^2+(\partial_s u_R)^2)+\frac{1}{2}|\nabla_{\bar{y}}u_R|^2-\frac{1}{2^*}|u_R|^{2^*}\bigg)\,dx_1\,d\bar{x}\,dt\\
&\quad+\frac{2d}{1-d^2}\frac{2}{1-d^2}\int_1^2 t\theta_1^2(|\bar{x}|)\frac{\partial\theta_1}{\partial x_1}(x_1)\theta_1(x_1)\partial_{y_1}u_R\cdot\partial_s u_R\,dx_1\,d\bar{x}\,dt\\
&\quad-\frac{1}{1-d^2}\int_1^2\int_{\mathbb{R}^N}x_1\theta_1^2(x_1)\theta_1^2(|\bar{x}|)\,\partial_t e(z_{d,R})\,dx_1\,d\bar{x}\,dt\\
&=A+B+C+D+E.
\end{aligned}$$

Note that $A=d\eta_2(d,R)$, where $|\eta_2(d,R)|\leqslant C$ uniformly in d and R. Because of (4.12), we have $C=\eta_3(d,R)$, where $\eta_3(d,R)\xrightarrow{R\to\infty}0$, uniformly in $d<d_0$, and $D=d\eta_4(d,R)$, where $\eta_4(d,R)\xrightarrow{R\to\infty}0$, uniformly in $d<d_0$. In light of the calculation preceding (4.12), we have

$$E=\frac{1}{1-d^2}\int_1^2\int_{\mathbb{R}^N}\theta^2(x)\partial_{x_1}z_{d,R}\cdot\partial_t z_{d,R}\,dx_1\,d\bar{x}\,dt+\eta_5(d,R),$$

where $\eta_5(d,R)\xrightarrow{R\to\infty}0$, uniformly in $d<d_0$ (we have integrated by parts and used (4.12)).

We will now calculate B. For this, we change variables, $\Phi(x,t)=(y,s)$ as before. Then, $\theta^2(x)=\theta_1^2(x_1)\theta_1^2(|\bar{x}|)$, $x_1=(y_1+ds)/\sqrt{1-s^2}$ and

$$\theta^2(\Phi^{-1}(y,s))=\theta_1^2\biggl(\frac{y_1+ds}{\sqrt{1-d^2}}\biggr)\theta_1^2(|\bar{y}|).$$

Note that

$$\theta_1^2\biggl(\frac{y_1+ds}{\sqrt{1-d^2}}\biggr)=\theta_1^2\biggl(y_1\sqrt{1-d^2}+\frac{d}{\sqrt{1-d^2}}(s+dy_1)\biggr).$$

Thus, since in our domain of integration we have $\sqrt{1-d^2}\leqslant dy_1+s\leqslant 2\sqrt{1-d^2}$, for $0\leqslant d\leqslant d_0$, d_0 small, we have

$$\theta_1^2\biggl(y_1\sqrt{1-d^2}+\frac{d}{\sqrt{1-d^2}}(s+dy_1)\biggr)-\theta_1^2\bigl(y_1\sqrt{1-d^2}\,\bigr)=O(d)(\theta_1^2)'\bigl(y_1\sqrt{1-d^2}+\eta O(d)\bigr),$$

where $|\eta|\leqslant 1$.

Note that $\operatorname{supp}(\theta_1^2)'(\alpha)\subset\{\alpha:4\leqslant|\alpha|\leqslant 5\}$, so that, for d_0 small, this can only be non-zero for $3+\frac{1}{4}\leqslant|y_1|\leqslant 5+\frac{1}{4}$. Using a similar argument for $\theta_1^2\bigl(y_1\sqrt{1-d^2}\,\bigr)-\theta_1^2(y_1)$, and the argument used in the proof of (4.12), we see that B equals

$$-\frac{2}{(1-d^2)^2}\iint_{\sqrt{1-d^2}\leqslant dy_1+s\leqslant 2\sqrt{1-d^2}}\theta^2(y)\partial_{y_1}u_R\partial_s u_R\,dy\,ds+d\eta_6(d,R),$$

where $|\eta_6(d,R)|\xrightarrow{R\to\infty}0$, uniformly for $d<d_0$.

Consider now the integral

$$\begin{aligned}\iint_{\sqrt{1-d^2}\leqslant s\leqslant 2\sqrt{1-d^2}}\theta^2(y)\partial y_1 u_R\partial_s u_R\,dy\,ds&=\iint_{\sqrt{1-d^2}\leqslant s\leqslant 2\sqrt{1-d^2}}\partial y_1 u_R\partial_s u_R\,dy\,ds\\&\quad+\iint_{\sqrt{1-d^2}\leqslant s\leqslant 2\sqrt{1-d^2}}(\theta^2(y)-1)\partial y_1 u_R\partial_s u_R\,dy\,ds.\end{aligned}$$

The first term equals $\gamma\sqrt{1-d^2}$, because of (2.2) and scaling, while, in light of the support property of $\theta^2(y)-1$ and the proof of (4.12), the second term equals $\eta_7(d,R)$, with $|\eta_7(d,R)|\xrightarrow{R\to\infty}0$, uniformly for $d<d_0$.

Finally,

$$\begin{aligned}
&\iint_{\sqrt{1-d^2}\leqslant dy_1+s\leqslant 2\sqrt{1-d^2}} \theta^2(y)\partial_{y_1}u_R\partial_s u_R\,dy\,ds \\
&\qquad\qquad -\iint_{\sqrt{1-d^2}\leqslant s\leqslant 2\sqrt{1-d^2}} \theta^2(y)\partial_{y_1}u_R\partial_s u_R\,dy\,ds \\
&\qquad = \int_{y_1>0}\int_{2\sqrt{1-d^2}-dy_1}^{2\sqrt{1-d^2}} \theta^2(y)\partial_{y_1}u_R\partial_s u_R\,ds\,dy \\
&\qquad\quad +\int_{y_1<0}\int_{2\sqrt{1-d^2}}^{2\sqrt{1-d^2}-dy_1} \theta^2(y)\partial_{y_1}u_R\partial_s u_R\,ds\,dy \\
&\qquad\quad +\int_{y_1>0}\int_{\sqrt{1-d^2}-dy_1}^{\sqrt{1-d^2}} \theta^2(y)\partial_{y_1}u_R\partial_s u_R\,ds\,dy \\
&\qquad\quad +\int_{y_1<0}\int_{\sqrt{1-d^2}}^{\sqrt{1-d^2}-dy_1} \theta^2(y)\partial_{y_1}u_R\partial_s u_R\,ds\,dy \\
&\qquad = \tilde{A}+\widetilde{B}+\widetilde{C}+\widetilde{D}.
\end{aligned}$$

We will estimate $\tilde{A}$, the others being similar. In our region of integration, we have $|y_1|\leqslant 5$. We make, in the s integral, the change of variable $h=\big(2\sqrt{1-d^2}-s\big)/d$. We then have, in our region of integration, $0\leqslant h\leqslant y_1$. Thus,

$$\begin{aligned}
|\tilde{A}| &\leqslant 2d\int_{y_1>0}\int_0^{y_1}\theta^2(y)\big|\partial_{y_1}u_R\big(y,2\sqrt{1-d^2}-dh\big)\big|\,\big|\partial_s u_R\big(y,2\sqrt{1-d^2}-dh\big)\big|\,dh\,dy \\
&\leqslant 2d\int_0^5\int_{\mathbb{R}^N}\theta^2(y)\big|\partial_{y_1}u_R\big(y,2\sqrt{1-d^2}-dh\big)\big|\,\big|\partial_s u_R\big(y,2\sqrt{1-d^2}-dh\big)\big|\,dy\,dh \\
&\leqslant Cd.
\end{aligned}$$

We thus have

$$B=-\frac{2}{(1-d^2)^2}\big(\gamma\sqrt{1-d^2}+\eta_7(d,R)+d\eta_8(d,R)\big),$$

where $|\eta_8(d,R)|\leqslant C$, uniformly in d and R.

Finally, using the formulas after (4.7) and the same argument, together with the previous estimate for E, we obtain

$$E=\frac{1}{1-d^2}(\gamma+\eta_7(d,R)+d\eta_8(d,R)).$$

Next, we recall that for fixed R, $u_R\in L_I^{(N+2)/(N-2)}L_x^{2(N+2)/(N-2)}$, for any compact time interval. From this and Lemma 2.2 we see that $\theta(x)z_{d,R}(x,t)$ is in $C([1,2];\dot{H}^1\times L^2)$. Fix now $t_0\in[1,2]$ and recall, from the beginning of the proof, that

$$\partial_t e(z_{d,R})(x,t)=\sum_{j=1}^N\partial_{x_j}(\partial_{x_j}z_{d,R}\partial_t z_{d,R}).$$

Hence,

$$
\begin{aligned}
\int_{\mathbb{R}^N} &\theta^2(x)e(z_{d,R})(x,t_0)\,dx \\
&= \int_1^2\int_{\mathbb{R}^N}\theta^2(x)e(z_{d,R})(x,t)\,dx\,dt \\
&\quad +\int_1^2\int_{\mathbb{R}^N}\theta^2(x)\int_t^{t_0}\sum_{j=1}^N \partial_{x_j}(\partial_{x_j}(z_{d,R})\partial_t z_{d,R})\,d\alpha\,dx\,dt \\
&= \int_1^2\int_{\mathbb{R}^N}\theta^2(x)e(z_{d,R})(x,t)\,dx\,dt \\
&\quad -\sum_{j=1}^N\int_1^2\int_{\mathbb{R}^N}\int_t^{t_0}\partial_{x_j}(\theta^2(x))\partial_{x_j}(z_{d,R})(x,\alpha)\partial_t z_{d,R}(x,\alpha)\,d\alpha\,dx\,dt.
\end{aligned}
$$

Because of (4.12), the second term equals $\eta_9(R,d,t_0)$, with $\eta_9(R,d,t_0)\xrightarrow{R\to\infty}0$, uniformly in $t_0\in[1,2]$ and $0\leqslant d\leqslant d_0$. Thus, if

$$
E(t_0,d,R)=\int_{\mathbb{R}^N}\theta^2(x)e(z_{d,R})(x,t_0)\,dx,
$$

we have (using our previous estimates)

$$
E(t_0,d,R)=J(0)-\gamma d+d^2\eta(d,R)+\mu(d,R,t_0), \tag{4.13}
$$

where $|\eta(d,R)|\leqslant C$, uniformly in R large and $0<d<d_0$, and $|\mu(d,R,t_0)|\xrightarrow{R\to\infty}0$, uniformly in $0<d<d_0$ and $1<t_0<2$. Also, using (4.12) once more,

$$
J(0)=E_C+\tilde\eta(R),
$$

where $|\tilde\eta(R)|\xrightarrow{R\to\infty}0$.

We now need to consider

$$
\begin{aligned}
\int_1^2\int_{\mathbb{R}^N}\theta^2(x)|\nabla_x z_{d,R}(x,t)|^2\,dx\,dt = \int_1^2\int_{\mathbb{R}^N}\theta^2(x)\biggl(&\frac{1}{1-d^2}|\partial_{y_1}u_R|^2+|\nabla_{\bar y}u_R|^2 \\
&-\frac{2d}{1-d^2}\partial_{y_1}u_R.\partial_s u_R+\frac{d^2}{1-d^2}|\partial_s u_R|^2\biggr)\,dx\,dt.
\end{aligned}
$$

The arguments used before to calculate B easily yield that the right-hand side equals

$$
\int_{\mathbb{R}^N}\int_{\sqrt{1-d^2}}^{2\sqrt{1-d^2}}\theta^2(y)|\nabla_y u_R|^2\,ds\,dy+O(d),
$$

where $O(d)$ is uniform in R, i.e.,

$$\int_1^2 \int_{\mathbb{R}^N} \theta^2(x)|\nabla_x z_{d,R}(x,t)|^2\,dx\,dt = \int_{\sqrt{1-d^2}}^{2\sqrt{1-d^2}} \int_{\mathbb{R}^N} \theta^2(y)|\nabla_y u_R|^2\,dy\,ds + O(d). \tag{4.14}$$

Define now $h_{d,R}(x,t)=\theta(x)z_{d,R}(x,t)$. Then,

$$|\nabla_x h_{d,R}(x,t)|^2 = \theta^2|\nabla_x z_{d,R}|^2 + |\nabla\theta|^2|z_{d,R}|^2 + 2\theta\nabla\theta\cdot\nabla z_{d,R} z_{d,R},$$

and note that the last two terms are supported in $3\leqslant|x|\leqslant 8$. Also,

$$|h_{d,R}|^{2^*} = \theta^2(x)|z_{d,R}|^{2^*} + (|\theta|^{2^*} - |\theta|^2)|z_{d,R}|^{2^*},$$

and the last term is supported in $3\leqslant|x|\leqslant 8$.

We are now able to conclude the proof. Choose d_0 such that for $0<d<d_0$, uniformly in R, we have

$$\int_1^2 \int_{\mathbb{R}^N} \theta^2|\nabla_x z_{d,R}|^2\,dx\,dt \leqslant \left(1-\frac{\bar{\delta}}{2}\right)\int_{\mathbb{R}^N} |\nabla W|^2\,dx,$$

which we can do because of (4.14). Let $1+\bar{\bar{\delta}} = \left(1-\frac{1}{4}\bar{\delta}\right)/\left(1-\frac{1}{2}\bar{\delta}\right)$. Let $S_1=S_1(d,R)$ be the set of all $t\in[1,2]$ such that

$$\int_{\mathbb{R}^N} \theta^2(x)|\nabla_x z_{d,R}|^2(x,t)\,dx \leqslant (1+\bar{\bar{\delta}})\left(1-\frac{\bar{\delta}}{2}\right)\int_{\mathbb{R}^N} |\nabla W|^2\,dx = \left(1-\frac{\bar{\delta}}{4}\right)\int_{\mathbb{R}^N} |\nabla W|^2\,dx.$$

Then $|S_1|\geqslant\bar{\bar{\delta}}/(1+\bar{\bar{\delta}})$, for all $0<d\leqslant d_0$ and $R>0$. Next, choose d_1 small and $R>R_0(d_1)$ such that, for all $t_0\in[1,2]$, $E(t_0,d,R)\leqslant E_C-\frac{1}{2}\gamma d_1$. In addition, we can choose $d_1\leqslant d_0$. This is possible in view of (4.13). Now, for $\varepsilon>0$ to be chosen, find $R_1(\varepsilon)$ so large that for $R\geqslant R_1(\varepsilon)$ we have $\eta_1(R,d_1)\leqslant\varepsilon$, where η_1 is as in (4.12).

Consider next the set $S_2=S_2(R,d_1,\varepsilon,M)$ of all $t\in[1,2]$ such that

$$\int_{3\leqslant|x|\leqslant 8} (|\nabla_x z_{d,R}|^2 + |\partial_t z_{d,R}|^2 + |z_{d,R}|^{2^*})\,dx \leqslant M\varepsilon.$$

Because of (4.12), $|S_2|\geqslant 1-1/M$, and if we choose $M=M_{\bar{\delta}}$ so large that $(1-1/M_{\bar{\delta}})+\bar{\bar{\delta}}/(1+\bar{\bar{\delta}})>1$, we can find $t_0=t_0(R,\varepsilon)\in S_1\cap S_2$. We then have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla_x h_{d,R}(t_0)|^2\,dx &\leqslant \int_{\mathbb{R}^N} \theta^2|\nabla z_{d,R}(t_0)|^2\,dx + CM\varepsilon \\ &\leqslant \left(1-\frac{\bar{\delta}}{4}\right)\int_{\mathbb{R}^N} |\nabla W|^2\,dx + CM\varepsilon \leqslant \left(1-\frac{\bar{\delta}}{8}\right)\int_{\mathbb{R}^N} |\nabla W|^2\,dx, \end{aligned} \tag{4.15}$$

if we choose $CM\varepsilon\leqslant\frac{1}{8}\bar{\delta}\int_{\mathbb{R}^N}|\nabla W|^2\,dx$ and $R\geqslant R_1(\varepsilon)$. Also,

$$\int_{\mathbb{R}^N} e(h_{d,R})(t_0)\,dx \leqslant \int_{\mathbb{R}^N} \theta^2 e(z_{d,R})(t_0)\,dx + C\varepsilon M \leqslant E_C - \frac{\gamma d_1}{2} + C\varepsilon M, \tag{4.16}$$

for $R\geqslant R_0(d_1)$ and $R\geqslant R_1(\varepsilon)$. If we now choose $C\varepsilon M\leqslant\frac{1}{4}\gamma d_1$, we have

$$\int_{\mathbb{R}^N} e(h_{d,R})(t_0)\,dx \leqslant E_C-\frac{\gamma d_1}{4}, \tag{4.17}$$

for all $R>\max\{R_0(d_1),R_1(\varepsilon)\}$ and $\varepsilon=\varepsilon(\gamma,d_1,\bar{\delta})>0$. Let us now consider $w_R(x,t)$ to be the solution of (CP) with data $(h_{d_1,R}(t_0),\partial_t h_{d_1,R}(t_0))$ at $t=t_0$. In light of the definition of E_C, $w_R(x,t)$ exists for all time and satisfies, in view of Corollary 4.5,

$$\iint_{\mathbb{R}^N} |w_R(x,t)|^{2(N+1)/(N-2)}\,dx\,dt\leqslant C_{d_1,\gamma}, \tag{4.18}$$

uniformly for all $R>\max\{R_0(d_1),R_1(\varepsilon)\}$.

Next, observe that, by finite speed of propagation (Remark 2.12), $w_R(x,t)=z_{d,R}(x,t)$ on $\bigcup_{-2\leqslant t\leqslant 1}B(0,2+t)\times\{t\}$. To justify the application of Remark 2.12, we approximate (u_0,u_1) and hence $(u_{0,R},u_{1,R})$ by $(u_{0,R}^{(j)},u_{1,R}^{(j)})$ which are in $C_0^\infty\times C_0^\infty$. The resulting $u_R^{(j)}$ exists on any finite-time interval, for j large by Remark 2.21, and the corresponding $z_{d,R}^{(j)}$ are now solutions of (CP) on each finite-time interval. We then have, for j large, $w_R^{(j)}=z_{d,R}^{(j)}$ on the required set, and a passage to the limit (since x and t are in fixed bounded sets, we can apply Lemma 2.2), gives the required identity. But then,

$$\iint_{\bigcup_{-2\leqslant t\leqslant 1}B(0,2+t)\times\{t\}} |z_{d_1,R}|^{2(N+1)/(N-2)}\,dx\,dt\leqslant C_{d_1,\gamma}.$$

We now use our change of variables $(y,s)=\Phi(x,t)$, and observe that (for d_1 small enough)

$$\Phi\biggl(\bigcup_{-2\leqslant t\leqslant 1}B(0,2+t)\times\{t\}\biggr)\supset\bigl\{(y,s):0\leqslant s\leqslant\tfrac{1}{4}\text{ and }|y|\leqslant\tfrac{1}{4}\bigr\}.$$

But then, we obtain

$$\int_0^{1/4}\int_{|y|\leqslant 1/4}|u_R|^{2(N+1)/(N-2)}\,dy\,ds\leqslant C_{d_1,\gamma}$$

for all $R\geqslant\max\{R_0(d_1),R_1(\varepsilon)\}$. If we now rescale the above interval, we find that for all $R\geqslant\max\{R_0(d_1),R_1(\varepsilon)\}$,

$$\int_0^{R/4}\int_{|y|\leqslant R/4}|u|^{2(N+1)/(N-2)}\,dy\,ds\leqslant C_{d_1,\gamma}.$$

But, since we have $\int_{s\geqslant 0}\int_{\mathbb{R}^N}|u|^{2(N+1)/(N-2)}\,dy\,ds=\infty$, we reach a contradiction, which establishes the proposition. $\square$

5. Rigidity theorem. Part 1: Infinite-time interval and self-similarity for finite-time intervals

In this and the following section we will prove the following result.

THEOREM 5.1. *Let* $(u_0,u_1)\in\dot H^1\times L^2$ *be such that*

$$E((u_0,u_1))<E((W,0)),\quad \int_{\mathbb{R}^N}|\nabla u_0|^2\,dx<\int_{\mathbb{R}^N}|\nabla W|^2\,dx\quad\textit{and}\quad \int_{\mathbb{R}^N}\nabla u_0u_1\,dx=0.$$

Let u *be the solution of* (CP) *with* $(u(0),\partial_tu(0))=(u_0,u_1)$, *with maximal interval of existence* $(-T_-(u_0,u_1),T_+(u_0,u_1))$. *Assume that there exist* $\lambda(t)>0$ *and* $x(t)\in\mathbb{R}^N$, *for* $t\in[0,T_+(u_0,u_1))$, *with the property that if* K *is the set*

$$\left\{\vec v(x,t)=\left(\frac{1}{\lambda(t)^{(N-2)/2}}u\left(\frac{x-x(t)}{\lambda(t)},t\right),\frac{1}{\lambda(t)^{N/2}}\partial_tu\left(\frac{x-x(t)}{\lambda(t)},t\right)\right):t\in[0,T_+(u_0,u_1))\right\},$$

then $\overline{K}$ *is compact in* $\dot H^1\times L^2$.

Then, $T_+(u_0,u_1)<\infty$ *is impossible.*

Moreover, if $T_+(u_0,u_1)=\infty$ *and we assume that* $\lambda(t)\geqslant A_0>0$, *for* $t\in[0,\infty)$, *we must have* $u\equiv0$.

Remark 5.2. This theorem shows the rigidity of (CP) for optimal small data (consider the solution $u(x,t)=W(x)$ of (CP)). The momentum condition is the ingredient which allows us to treat the non-radial situation and is always true for a radial solution. Lemma 4.6 implies that we can choose $x(t)$ and $\lambda(t)$ continuous in $[0,T_+(u_0,u_1))$. Its proof also shows that we can preserve the property $\lambda(t)\geqslant A_0>0$.

We next turn to the proof of Theorem 5.1 in the case when

$$T_+(u_0,u_1)=\infty,\quad \lambda(t)\geqslant A_0.$$

Assume that $(u_0,u_1)\not\equiv(0,0)$. Because of Corollary 3.6, we have $E((u_0,u_1))=E>0$ and $\sup_{t>0}\|(\nabla u,\partial_tu)\|_{L^2}\leqslant CE$ as well as, from Theorem 3.5,

$$\int_{\mathbb{R}^N}(|\nabla_xu(t)|^2-|u(t)|^{2^*})\,dx\geqslant C_{\bar\delta}\int_{\mathbb{R}^N}|\nabla_xu(t)|^2\,dx \tag{5.1}$$

and

$$\alpha\int_{\mathbb{R}^N}(\partial_tu)^2\,dx+(1-\alpha)\int_{\mathbb{R}^N}(|\nabla_xu(t)|^2-|u(t)|^{2^*})\,dx\geqslant C_\alpha E \tag{5.2}$$

for $0<\alpha<1$.

We will also be applying (4.8), which gives the following:

$$\text{Given } \varepsilon>0, \text{ there exists } R_0(\varepsilon)>0 \text{ such that, for all } t\geqslant 0,$$
$$\int_{|x+x(t)/\lambda(t)|\geqslant R_0(\varepsilon)}\left(|\partial_t u|^2+|\nabla_x u|^2+\frac{|u|^2}{|x|^2}+|u|^{2^*}\right)dx\leqslant \varepsilon E. \tag{5.3}$$

(Here we use the assumptions $\lambda(t)\geqslant A_0>0$ and $E>0$.)

We will next summarize some algebraic properties that will be needed in the sequel. Let us fix $\phi\in C_0^\infty(\mathbb{R}^N)$, $\phi\equiv 1$ for $|x|\leqslant 1$, $\phi\equiv 0$ for $|x|\geqslant 2$, and also define, for $R>0$,

$$\phi_R(x)=\phi\left(\frac{x}{R}\right) \quad \text{and} \quad \psi_R(x)=x\phi\left(\frac{x}{R}\right).$$

We will set

$$r(R)=\int_{|x|\geqslant R}\left(\frac{|u|^2}{|x|^2}+|u|^{2^*}+|\nabla u|^2+|\partial_t u|^2\right)dx.$$

Lemma 5.3. *The following identities hold for all* $t\geqslant 0$:

(i) $\displaystyle \partial_t\int_{\mathbb{R}^N}\left(\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla_x u|^2-\frac{1}{2^*}|u|^{2^*}\right)dx=0;$

(ii) $\displaystyle \partial_t\int_{\mathbb{R}^N}\nabla u\partial_t u\,dx=0;$

(iii) $\displaystyle \partial_t\int_{\mathbb{R}^N}\psi_R(x)\cdot\nabla u\,\partial_t u\,dx=-\frac{N}{2}\int_{\mathbb{R}^N}(\partial_t u)^2\,dx+\frac{N-2}{2}\int_{\mathbb{R}^N}(|\nabla_x u|^2-|u|^{2^*})\,dx +O(r(R));$

(iv) $\displaystyle \partial_t\int_{\mathbb{R}^N}\phi_R u\partial_t u\,dx=\int_{\mathbb{R}^N}(\partial_t u)^2\,dx-\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\int_{\mathbb{R}^N}|u|^{2^*}\,dx+O(r(R));$

(v) $\displaystyle \partial_t\int_{\mathbb{R}^N}\psi_R\left(\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla_x u|^2-\frac{1}{2^*}|u|^{2^*}\right)dx=-\int_{\mathbb{R}^N}\nabla u\partial_t u\,dx+O(r(R)).$

Note that (i) is Remark 2.16, (ii) is (2.2), (v) follows from (2.3), (iv) follows from the arguments in the proof of Theorem 3.7 and (iii) follows by an integration by parts (and a limiting argument).

We now will prove the lemmas crucial for our purpose. Recall that we may assume $x(0)=0$.

Lemma 5.4. *There exist* $\varepsilon_1>0$ *and* $C>0$ *such that, if* $\varepsilon\in(0,\varepsilon_1)$, *there exists* $R_0(\varepsilon)$ *so that if* $R>2R_0(\varepsilon)$, *then there exists* $t_0=t_0(R,\varepsilon)$, $0\leqslant t_0\leqslant CR$, *with the property that for all* $0<t<t_0$ *we have* $|x(t)/\lambda(t)|<R-R_0(\varepsilon)$ *and* $|x(t_0)/\lambda(t_0)|=R-R_0(\varepsilon)$.

Proof. Since $x(0)=0$, $\lambda(t)\geqslant A_0>0$, if the conclusion failed we would have, for all $0<t<CR$ (where C is large) $|x(t)/\lambda(t)|<R-R_0(\varepsilon)$. Let

$$z_R(t)=\int_{\mathbb{R}^N}\psi_R(x)\cdot\nabla_x u\partial_t u\,dx+\left(\frac{N}{2}-\alpha\right)\int_{\mathbb{R}^N}\phi_R u\partial_t u\,dx, \quad 0<\alpha<1.$$

Then, by Lemma 5.3 and (5.2), we have

$$\begin{aligned} z_R'(t) = &-\frac{N}{2}\int_{\mathbb{R}^N}(\partial_t u)^2\,dx+\frac{N-2}{2}\int_{\mathbb{R}^N}(|\nabla u|^2-|u|^{2^*})\,dx+O(r(R)) \\ &+\left(\frac{N}{2}-\alpha\right)\left(\int_{\mathbb{R}^N}(\partial_t u)^2\,dx-\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\int_{\mathbb{R}^N}|u|^{2^*}dx\right)+O(r(R)) \\ = &-\alpha\int_{\mathbb{R}^N}(\partial_t u)^2\,dx-(1-\alpha)\int_{\mathbb{R}^N}(|\nabla u|^2\,dx-|u|^{2^*})\,dx+O(r(R)) \\ \leqslant &-C_\alpha E+O(r(R)). \end{aligned}$$

But, for $|x|\geqslant R$, we have $|x+x(t)/\lambda(t)|\geqslant R_0(\varepsilon)$, by our assumption, so that, by (5.3), $|r(R)|\leqslant\widetilde{C}\varepsilon E$. Now, choose ε so small that $z_R'(t)\leqslant-\frac{1}{2}C_\alpha E$. Note that $|z_R(t)|\leqslant\widetilde{C}_1 RE$, so that, integrating in t between 0 and CR, we get

$$CR\frac{C_\alpha}{2}E\leqslant 2\widetilde{C}_1 RE.$$

This is a contradiction for C large. □

Note that, in the radial case, we have $x(t)=0$ (see [19]) and a contradiction follows from Lemma 5.4. This proof is the counterpart of the local virial identity proof used in [19] for the non-linear Schrödinger equation.

Lemma 5.5. *There exist $\varepsilon_2>0$, $R_1(\varepsilon)>0$ and $C_0>0$ such that if $R>R_1(\varepsilon)$ and $t_0=t_0(R,\varepsilon)$ is as in Lemma 5.4, then for $0<\varepsilon<\varepsilon_2$,*

$$t_0(R,\varepsilon)\geqslant\frac{C_0 R}{\varepsilon}.$$

Proof. Let for $t\in[0,t_0]$,

$$y_R(t)=\int_{\mathbb{R}^N}\psi_R(x)e(u)(x,t)\,dx.$$

Since $\int_{\mathbb{R}^N}\nabla u_0 u_1\,dx=0$, by Lemma 5.3, (ii) and (v), we have $|y_R'(t)|=O(r(R))$. Since

$$\left|x+\frac{x(t)}{\lambda(t)}\right|\geqslant R-(R-R_0(\varepsilon))=R_0(\varepsilon),$$

for $0<t<t_0$ and $|x|\geqslant R$, we have, integrating in t,

$$|y_R(t_0)-y_R(0)|\leqslant\widetilde{C}\varepsilon E t_0.$$

On the one hand, by (5.3), we have

$$|y_R(0)|\leqslant\widetilde{C}R_0(\varepsilon)E+O(Rr(R_0(\varepsilon)))\leqslant\widetilde{C}E(R_0(\varepsilon)+\varepsilon R).$$

On the other hand,

$$|y_R(t_0)| \geqslant \left| \int_{|x+x(t_0)/\lambda(t_0)|\leqslant R_0(\varepsilon)} \psi_R e(u)(t_0)\,dx \right| - \left| \int_{|x+x(t_0)/\lambda(t_0)|\geqslant R_0(\varepsilon)} \psi_R e(u)(t_0)\,dx \right|.$$

In the first integral, $|x|\leqslant|x+x(t_0)/\lambda(t_0)|+|x(t_0)/\lambda(t_0)|\leqslant R$, so that $\psi_R(x)=x$. Note also that the second integral is bounded by $MR\varepsilon E$. Hence,

$$|y_R(t_0)| \geqslant \left| \int_{|x+x(t_0)/\lambda(t_0)|\leqslant R_0(\varepsilon)} x e(u)(t_0)\,dx \right| - MR\varepsilon E.$$

But $\int_{|x+x(t_0)/\lambda(t_0)|\leqslant R_0(\varepsilon)} x e(u)(t_0)\,dx$ equals

$$-\frac{x(t_0)}{\lambda(t_0)} \int_{|x+x(t_0)/\lambda(t_0)|\leqslant R_0(\varepsilon)} e(u)(t_0)\,dx + \int_{|x+x(t_0)/\lambda(t_0)|\leqslant R_0(\varepsilon)} \left(x + \frac{x(t_0)}{\lambda(t_0)} \right) e(u)(t_0)\,dx,$$

that is,

$$\begin{aligned} -\frac{x(t_0)}{\lambda(t_0)} \int e(u)(t_0)\,dx + \frac{x(t_0)}{\lambda(t_0)} \int_{|x+x(t_0)/\lambda(t_0)|\geqslant R_0(\varepsilon)} e(u)(t_0)\,dx& \\ + \int_{|x+x(t_0)/\lambda(t_0)|\leqslant R_0(\varepsilon)} \left(x + \frac{x(t_0)}{\lambda(t_0)} \right) e(u)(t_0)\,dx&. \end{aligned}$$

The first term is, in absolute value, $(R-R_0(\varepsilon))E$, while the last two are bounded in absolute value by $\widetilde{C}(R-R_0(\varepsilon))\varepsilon E+\widetilde{C}R_0(\varepsilon)E$. We then find

$$|y_R(t_0)| \geqslant (R-R_0(\varepsilon))E(1-\widetilde{C}\varepsilon) - MR\varepsilon E - \widetilde{C}R_0(\varepsilon)E.$$

The quantity on the right exceeds $\frac{1}{4}RE$, if for $0<\varepsilon<\varepsilon_2$ we have $(1-\widetilde{C}\varepsilon-M\varepsilon)\geqslant\frac{1}{2}$ and for $R>R_1(\varepsilon)$ we have $\frac{1}{4}R\geqslant(1+\widetilde{C})R_0(\varepsilon)$.

Thus,

$$\tfrac{1}{4}RE - \widetilde{C}E(R_0(\varepsilon)+\varepsilon R) \leqslant \widetilde{C}\varepsilon E t_0,$$

which yields the result for $0<\varepsilon<\varepsilon_2'$ and $R>R_1'(\varepsilon)$. □

Proof of Theorem 5.1, *in the case when* $T_+(u_0,u_1)=\infty$. By Lemma 5.4, we have $t_0(R,\varepsilon)\leqslant CR$ for $0<\varepsilon<\varepsilon_1$ and $R>2R_0(\varepsilon)$, while, by Lemma 5.5, for $0<\varepsilon<\varepsilon_2$, $R>R_1(\varepsilon)$ and $t_0(R,\varepsilon)\geqslant C_0R/\varepsilon$. Hence, for $R>\max\{2R_0(\varepsilon),R_1(\varepsilon)\}$, with $\varepsilon<\min\{\varepsilon_1,\varepsilon_2\}$, we have $C_0R/\varepsilon\leqslant CR$, which is a contradiction for ε small. □

We now turn to the start of the analysis of the case $T_+(u_0,u_1)<\infty$. By scaling, we can assume, without loss of generality, that

$$T_+(u_0,u_1)=1.$$

Recall, from Lemma 4.7, that

$$\lambda(t) \geqslant \frac{C_0(K)}{1-t} \tag{5.4}$$

and, from Lemma 4.8, that (after translation in x)

$$\operatorname{supp} u(\,\cdot\,,t) \subset B(0,1-t) \quad \text{and} \quad \operatorname{supp} \partial_t u(\,\cdot\,,t) \subset B(0,1-t). \tag{5.5}$$

LEMMA 5.6. *Let u be as above. Then, there is $C_1(K)>0$ such that*

$$\frac{C_1(K)}{1-t} \geqslant \lambda(t).$$

Proof. Assume this is not true. In light of Lemma 4.6, there exist $t_n \uparrow 1$, such that $\lambda(t_n)(1-t_n)\uparrow\infty$. Consider now

$$z(t) = \int_{\mathbb{R}^N} x\nabla u \partial_t u\, dx + \left(\frac{N}{2}-\alpha\right)\int_{\mathbb{R}^N} u\partial_t u\, dx, \quad 0<\alpha<1,$$

which is defined for $0\leqslant t<1$ (recall (5.5)). In view of Lemma 5.3, (iii) and (iv), we have

$$z'(t) = -\alpha\int_{\mathbb{R}^N} (\partial_t u)^2\, dx - (1-\alpha)\int_{\mathbb{R}^N} (|\nabla_x u|^2 - |u|^{2^*})\, dx.$$

Because of Corollary 3.6 ($u\not\equiv 0$, since $T_+(u_0,u_1)=1$), we have $E((u_0,u_1))=E>0$,

$$\sup_{0<t<1} \|(\nabla u, \partial_t u)\|_{L^2} \leqslant CE$$

and

$$\alpha\int_{\mathbb{R}^N} (\partial_t u)^2\, dx + (1-\alpha)\int_{\mathbb{R}^N} (|\nabla_x u|^2 - |u|^{2^*})\, dx \geqslant C_\alpha E.$$

Then, we have

$$z'(t) \leqslant -C_\alpha E, \quad 0<t<1.$$

Moreover, condition (5.5) and Hardy's inequality give that $z(t)\xrightarrow{t\to 1}0$. Also, the assumption $\int_{\mathbb{R}^N} \nabla u_0 u_1\, dx=0$ and Lemma 5.3 (ii) give that $\int_{\mathbb{R}^N} \nabla u \partial_t u\, dx=0$, $0\leqslant t<1$.

Note that, integrating in t, $z(t)\geqslant C_\alpha E(1-t)$. We have

$$\frac{z(t_n)}{1-t_n} = \frac{1}{1-t_n}\int_{\mathbb{R}^N}\left(x+\frac{x(t_n)}{\lambda(t_n)}\right)\nabla u\, \partial_t u\, dx + \left(\frac{N}{2}-\alpha\right)\frac{1}{1-t_n}\int_{\mathbb{R}^N} u\, \partial_t u\, dx \geqslant C_\alpha E.$$

We will show that

$$\frac{z(t_n)}{1-t_n} \to 0, \tag{5.6}$$

yielding a contradiction. In fact, given $\varepsilon>0$,

$$\frac{1}{1-t_n}\int_{|x+x(t_n)/\lambda(t_n)|\leqslant\varepsilon(1-t_n)}\left|x+\frac{x(t_n)}{\lambda(t_n)}\right||\nabla u(t_n)|\,|\partial_t u(t_n)|\,dx\leqslant C\varepsilon E.$$

Next, note that

$$\left|\frac{x(t_n)}{\lambda(t_n)}\right|\leqslant 2(1-t_n). \tag{5.7}$$

In fact, if (5.7) is not true, then $B(-x(t_n)/\lambda(t_n),1-t_n)\cap B(0,1-t_n)=\varnothing$, so that

$$\int_{B(-x(t_n)/\lambda(t_n),1-t_n)}|\nabla u(x,t_n)|^2\,dx=0,$$

while

$$\begin{aligned}\int_{|x+x(t_n)/\lambda(t_n)|\geqslant 1-t_n}|\nabla u(x,t_n)|^2\,dx&=\int_{|\lambda(t_n)x+x(t_n)|\geqslant\lambda(t_n)(1-t_n)}|\nabla u(x,t_n)|^2\,dx\\&=\frac{1}{\lambda(t_n)^N}\int_{|y|\geqslant\lambda(t_n)(1-t_n)}\left|\nabla u\left(\frac{y-x(t_n)}{\lambda(t_n)},t_n\right)\right|^2 dx\to 0\end{aligned}$$

as $n\to\infty$, by compactness of $\overline{K}$, since $\lambda(t_n)(1-t_n)\to\infty$. But then,

$$E((u(x,t_n),\partial_t u(x,t_n)))\to 0$$

(arguing for $\partial_t u$ in a similar way) which is a contradiction to $E>0$, and thus establishing (5.7). But then,

$$\begin{aligned}&\frac{1}{1-t_n}\int_{|x+x(t_n)/\lambda(t_n)|\geqslant\varepsilon(1-t_n)}\left|x+\frac{x(t_n)}{\lambda(t_n)}\right||\nabla u(x,t_n)|\,|\partial_t u(x,t_n)|\,dx\\&\quad\leqslant 3\int_{|x+x(t_n)/\lambda(t_n)|\geqslant\varepsilon(1-t_n)}|\nabla u(x,t_n)|\,|\partial_t u(x,t_n)|\,dx\\&\quad\leqslant\frac{3}{\lambda(t_n)^N}\int_{|y|\geqslant\varepsilon(1-t_n)\lambda(t_n)}\left|\nabla u\left(\frac{y-x(t_n)}{\lambda(t_n)},t_n\right)\right|\left|\partial_t u\left(\frac{y-x(t_n)}{\lambda(t_n)},t_n\right)\right|dy\to 0,\end{aligned}$$

as $n\to\infty$, by compactness of $\overline{K}$, and the assumption that $\lambda(t_n)(1-t_n)\uparrow\infty$. This shows (5.6) for the first term in $z(t_n)/(1-t_n)$. The second one gives the same result, using the same argument, the fact that

$$\frac{1}{1-t_n}\int_{\mathbb{R}^N}|u(t_n)|\,|\partial_t u(t_n)|\,dx\leqslant\frac{1}{1-t_n}\int_{\mathbb{R}^N}\left|x+\frac{x(t_n)}{\lambda(t_n)}\right|\frac{|u(x,t_n)|}{|x+x(t_n)/\lambda(t_n)|}|\partial_t u(x,t_n)|\,dx,$$

and Hardy's inequality. $\square$

PROPOSITION 5.7. *Let* (u_0,u_1) *be as in Theorem* 5.1, *with* $T_+(u_0,u_1)=1$. *Then*

$$\operatorname{supp}\nabla u, \partial_t u \subset B(0,1-t)$$

and

$$\vec{K}=\{(1-t)^{N/2}(\nabla u((1-t)x,t),\partial_t u((1-t)x,t)):0<t<1\}$$

has compact closure in $L^2(\mathbb{R}^N)^N\times L^2(\mathbb{R}^N)$.

Proof. We first claim that

$$(1-t)^{N/2}(\nabla u((1-t)(x-x(t)),t),\partial_t u((1-t)(x-x(t)),t))$$

has compact closure in $L^2(\mathbb{R}^N)^N\times L^2(\mathbb{R}^N)$. This is because $C_0(K)\leqslant(1-t)\lambda(t)\leqslant C_1(K)$, and if $\overline{K}$ is compact, then

$$K_1=\{\lambda^{N/2}\vec{v}(\lambda x):\vec{v}\in\vec{K} \text{ and } c_0\leqslant\lambda\leqslant c_1\}$$

also has the property that $\overline{K}_1$ is compact. Next, let

$$\tilde{v}(x,t)=(1-t)^{N/2}(\nabla u((1-t)x,t),\partial_t u((1-t)x,t)),$$

so that $\tilde{v}(x,t)=\vec{v}(x+x(t),t)$, where

$$\vec{v}(x,t)=(1-t)^{N/2}(\nabla u((1-t)(x-x(t)),t),\partial_t u((1-t)(x-x(t)),t)).$$

Note that, by (5.5), $\operatorname{supp}\vec{v}(\,\cdot\,,t)\subset\{x:|x-x(t)|\leqslant 1\}$. The fact that $E>0$, the compactness of $\vec{v}(x,t)$ and preservation of energy now imply that $|x(t)|\leqslant C$. But if

$$K_2=\{\vec{v}(x+x_0):\vec{v}\in K_1 \text{ and } |x_0|\leqslant C\},$$

then $\overline{K}_2$ is also compact and hence the proposition follows. □

6. Rigidity theorem. Part 2: Self-similar variables and conclusion of the proof of the rigidity theorem

In this section our point of departure is Proposition 5.7.

For this case, in [19], we proved an extra decay estimate which allowed us to use the L^2 invariance and get a contradiction.

Following Merle and Zaag ([30], see also [1]) we will introduce self-similar variables to show that a solution as in Proposition 5.7 cannot exist. Merle and Zaag considered the case of power non-linearities $|u|^{p-1}u$ which have $p\leqslant 1+4/(N-1)$, while here we consider the energy-critical case $p=1+4/(N-2)$. Nevertheless, many of the calculations in [30] also apply to our case, and one can use an extra Lyapunov function. We remark that a similar structure exists in the case of non-linear heat equations, as has been used by Giga and Kohn [10] and others ([29]).

Again here, we obtain some extra decay estimates which allow us to reduce to an elliptic problem with no solution.

We now set,

$$y=\frac{x}{1-t} \quad \text{and} \quad s=-\log(1-t), \quad 0\leqslant t<1,$$

and define

$$w(y,s,0)=(1-t)^{(N-2)/2}u(x,t)=e^{-s(N-2)/2}u(e^{-s}y,1-e^{-s}). \tag{6.1}$$

Note that $w(y,s,0)$ is defined for $0\leqslant s<\infty$, and that $\operatorname{supp} w(\,\cdot\,,s,0)\subset\{y:|y|\leqslant 1\}$. We also consider, for $\delta>0$ small,

$$y=\frac{x}{1+\delta-t}, \quad s=-\log(1+\delta-t),$$

and

$$w(y,s,\delta)=(1+\delta-t)^{(N-2)/2}u(x,t)=e^{-s(N-2)/2}u(e^{-s}y,1+\delta-e^{-s}). \tag{6.2}$$

Note that $w(y,s,\delta)$ is defined for $0\leqslant s<\log 1/\delta$, and that

$$\operatorname{supp} w(\,\cdot\,,\delta)\subset\left\{y:|y|\leqslant\frac{e^{-s}-\delta}{e^{-s}}=\frac{1-t}{1+\delta-t}\leqslant 1-\delta\right\}.$$

The w solve, in their domain of definition, the equation (see [30])

$$\partial_s^2 w=\frac{1}{\varrho}\operatorname{div}(\varrho\nabla w-\varrho(y\cdot\nabla w)y)-\frac{N(N-2)}{4}w+|w|^{4/(N-2)}w-2y\nabla\partial_s w-(N-1)\partial_s w, \tag{6.3}$$

where $\varrho=(1-|y|^2)^{-1/2}$.

LEMMA 6.1. *For $\delta>0$ fixed and for $s\in[0,\log 1/\delta)$, the following hold*:

(i)

$$\operatorname{supp} w(\,\cdot\,,s,\delta)\subset\{y:|y|\leqslant(e^{-s}-\delta)/e^{-s}\leqslant 1-\delta\},$$
$$\operatorname{supp} \partial_s w(\,\cdot\,,s,\delta)\subset\{y:|y|\leqslant(e^{-s}-\delta)/e^{-s}\leqslant 1-\delta\};$$

(ii) $w(\,\cdot\,,s,\delta)\in H_0^1(B_1)$ *and*

$$\int_{\mathbb{R}^N}|w|^{2^*}\,dy\leqslant C, \quad \int_{\mathbb{R}^N}|\nabla_y w|^2\,dy<\int_{\mathbb{R}^N}|\nabla W|^2\,dy,$$
$$\int_{\mathbb{R}^N}\left(|w|^2+\frac{|w|^2}{(1-|y|^2)^2}\right)dy\leqslant C \quad \text{and} \quad \int_{\mathbb{R}^N}|\partial_s w|^2\,dy\leqslant C;$$

(iii)
$$\int_{\mathbb{R}^N} (|\nabla w|^2+|\partial_s w|^2+|w|^2+|w|^{2^*})\log\frac{1}{1-|y|^2}\,dy \leqslant C\log\frac{1}{\delta};$$

(iv)
$$\int_{\mathbb{R}^N} (|\nabla w|^2+|\partial_s w|^2+|w|^2+|w|^{2^*})(1-|y|^2)^{-1/2}\,dy \leqslant \frac{C}{\delta^{1/2}}.$$

Proof. The first part of (i) was pointed out after (6.2). For the second part, we have, using the notation in (6.2),

$$\begin{aligned}\partial_s w(y,s,\delta) = -\tfrac{1}{2}(N-2)e^{-s(N-2)/2}&u(e^{-s}y, 1+\delta-e^{-s})\\ &+e^{-s}e^{-s(N-2)/2}\partial_t u(e^{-s}y,1+\delta-e^{-s})\\ &-e^{-s}e^{-s(N-2)/2}y\cdot\nabla u(e^{-s}y,1+\delta-e^{-s}),\end{aligned} \tag{6.4}$$

and (i) follows from (5.5).

Part (ii) follows from the support property of w, which gives $w(\cdot\,,s,\delta)\in H_0^{1,2}(B_1)$, a change of variables in y and (3.4), Sobolev embedding and Corollary 3.6, the Hardy inequality ([7], for example) and (6.4).

For (iii) and (iv), note that on $\operatorname{supp} w$ and $\operatorname{supp}\partial_s w$, we have

$$1-|y|^2 \geqslant 1-(1-\delta e^s)^2 = 2\delta e^s-\delta^2 e^{2s} \geqslant \delta,$$

for δ small, $0\leqslant s<\log 1/\delta$. □

For $w(y,s,\delta)$, $\delta>0$, as above, we now define (see [30])

$$\widetilde{E}(w(s)) = \int_{B_1}\left(\frac{(\partial_s w)^2+|\nabla w|^2-(y\cdot\nabla w)^2}{2}+\frac{N(N-2)}{8}w^2-\frac{(N-2)}{2N}|w|^{2^*}\right)\frac{dy}{(1-|y|^2)^{1/2}}. \tag{6.5}$$

PROPOSITION 6.2. *Let* $w=w(y,s,\delta)$, $\delta>0$, *be as above. Then, for*

$$0<s_1<s_2<\log\frac{1}{\delta},$$

the following identities hold:

(i)
$$\widetilde{E}(w(s_2))-\widetilde{E}(w(s_1)) = \int_{s_1}^{s_2}\int_{B_1}\frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}}\,dy\,ds;$$

(ii)
$$\begin{aligned}\frac{1}{2}\int_{B_1}&\left(\partial_s w w-\frac{1+N}{2}w^2\right)\frac{dy}{(1-|y|^2)^{1/2}}\bigg|_{s_1}^{s_2}\\ &= -\int_{s_1}^{s_2}\widetilde{E}(w(s))\,ds+\frac{1}{N}\int_{s_1}^{s_2}\int_{B_1}\frac{|w|^{2^*}}{(1-|y|^2)^{1/2}}\,dy\,ds\\ &\quad+\int_{s_1}^{s_2}\int_{B_1}\left((\partial_s w)^2+\partial_s w y\cdot\nabla w+\frac{\partial_s w w|y|^2}{1-|y|^2}\right)\frac{dy\,ds}{(1-|y|^2)^{1/2}};\end{aligned}$$

(iii) $\lim_{s\to\log(1/\delta)}\widetilde{E}(w(s))\leqslant E=E((u_0,u_1))$.

Proof. For (i), see the proof of Lemma 2.1 in [30]. For (ii), see the proof of (11) in [30]. We turn to the proof of (iii). We analyze term by term, using the notation in (6.2):

$$\begin{aligned}\int_{B_1}\frac{w^2}{(1-|y|^2)^{1/2}}\,dy &= \int_{|y|<(1-t)/(1+\delta-t)}(1+\delta-t)^{N-2}|u((1+\delta-t)y,t)|^2\frac{dy}{(1-|y|^2)^{1/2}}\\ &\leqslant C\int_{|x|<1-t}(1+\delta-t)^{-2}|u(x,t)|^2\frac{dx}{\delta^{1/2}}\\ &\leqslant \frac{C}{\delta^{1/2}(1+\delta-t)^2}\biggl(\int_{|x|<(1-t)}|u(x,t)|^{2^*}\,dx\biggr)^{2/2^*}(1-t)^{2/N}\xrightarrow{t\to1}0\end{aligned}$$

and

$$\begin{aligned}\int_{B_1}\frac{|w|^{2^*}}{(1-|y|^2)^{1/2}}\,dy &= \int_{|y|<(1-t)/(1+\delta-t)}(1+\delta-t)^{N}|u((1+\delta-t)y,t)|^{2^*}\frac{dy}{(1-|y|^2)^{1/2}}\\ &= \int_{|x|<(1-t)}|u(x,t)|^{2^*}\frac{dx}{(1-|y|^2)^{1/2}}.\end{aligned}$$

Recall that $|y|^2=|x|^2/(1+\delta-t)^2$, and assume that $1-\varepsilon\delta\leqslant t\leqslant 1$. Then, we have

$$\frac{1}{\varepsilon+1}\leqslant(1-|y|^2)^{1/2}\leqslant 1,$$

since $|x|\leqslant 1-t$. Thus,

$$\int_{B_1}\frac{|w|^{2^*}}{(1-|y|^2)^{1/2}}\,dy\geqslant\int_{|x|<1-t}|u(x,t)|^{2^*}\,dx,$$

and a similar computation gives that

$$\int_{B_1}\frac{|\nabla w|^2}{(1-|y|^2)^{1/2}}\,dy\leqslant\frac{1}{(1+\varepsilon)^{1/2}}\int_{|x|\leqslant1-t}|\nabla u|^2\,dx.$$

Also,

$$\begin{aligned}\int_{B_1}(y\cdot\nabla w)^2\frac{dy}{(1-|y|^2)^{1/2}} &= \int_{|x|\leqslant(1-t)}\frac{|x\cdot\nabla_x u(x,t)|^2}{(1+\delta-t)^2}\frac{dx}{(1-|y|^2)^{1/2}}\\ &\leqslant\frac{1}{1+\varepsilon}\int_{|x|\leqslant(1-t)}|\nabla_x u(x,t)|^2\,dx\,\frac{|1-t|^2}{(1+\delta-t)^2}\xrightarrow{t\to1}0.\end{aligned}$$

With these computations and (6.4), we see that

$$\lim_{t\to1}\frac12\int|\partial_s w|^2\frac{dy}{(1-|y|^2)^{1/2}}=\frac12\int|\partial_t u|^2\,dx,$$

which combined with the previous calculations yields (iii). □

COROLLARY 6.3. *For* $s\in[0,\log(1/\delta))$, *we have*

$$-\frac{C}{\delta^{1/2}}\leqslant\widetilde{E}(w(s))\leqslant E.$$

Proof. The first statement follows from Proposition 6.2, (i) and (iii), while the second one follows from Lemma 6.1 (iv) and (6.5). □

Using space-time estimates, we now obtain our first improvement of the space decay of w.

LEMMA 6.4. *For* $\delta>0$, *we have*

$$\int_0^1\int_{\mathbb{R}^N}\frac{|\partial_s w|^2}{1-|y|^2}\,dy\,ds\leqslant C\log\frac{1}{\delta}.$$

Proof. We start out with the readily verified identity

$$\begin{aligned}\frac{d}{ds}\int_{\mathbb{R}^N}&\left(\frac{1}{2}(\partial_s w)^2+\frac{1}{2}(|\nabla w|^2-(y\cdot\nabla w)^2)+\frac{(N-2)N}{8}w^2-\frac{N-2}{2N}|w|^{2^*}\right)(-\log(1-|y|^2))\,dy\\&+\int_{\mathbb{R}^N}(\log(1-|y|^2)+2)y\cdot\nabla w\,\partial_s w\,dy-\int_{\mathbb{R}^N}\log(1-|y|^2)(\partial_s w)^2\,dy-2\int_{\mathbb{R}^N}(\partial_s w)^2\,dy\\=-2&\int_{\mathbb{R}^N}\frac{(\partial_s w)^2}{1-|y|^2}\,dy.\end{aligned}$$

We now integrate between 0 and 1, and change signs. In the estimate of the left-hand side, we can drop the term $\int_{\mathbb{R}^N}\log(1-|y|^2)(\partial_s w)^2\,dy$, since it is negative. The d/ds term, and the $\int_0^1\int_{\mathbb{R}^N}(\partial_s w)^2\,dy\,ds$ term are controlled by Lemma 6.1 (using that $-\log(1-|y|^2)\leqslant C\log(1/\delta)$). It remains to bound

$$\begin{aligned}&\left|\int_0^1\int_{\mathbb{R}^N}(\log(1-|y|^2)+2)y\cdot\nabla w\,\partial_s w\,dy\,ds\right|\\&\qquad\leqslant\left(\int_0^1\int_{\mathbb{R}^N}\frac{|\partial_s w|^2}{1-|y|^2}\,dy\,ds\right)^{1/2}\left(\int_0^1\int_{\mathbb{R}^N}(1-|y|^2)|\log(1-|y|^2)+2|^2\,|\nabla w|^2\,dy\,ds\right)^{1/2}.\end{aligned}$$

The second factor is bounded because of Lemma 6.1 (ii). The proof is concluded by using the inequality $ab\leqslant\varepsilon a^2+b^2/\varepsilon$. □

LEMMA 6.5. *For* $\delta>0$, *we have*

(i)

$$\int_0^1\int_{B_1}\frac{|w|^{2^*}}{(1-|y|^2)^{1/2}}\,dy\,ds\leqslant C\left(\log\frac{1}{\delta}\right)^{1/2},$$

(ii) $\widetilde{E}(w(1))\geqslant -C|\log(1/\delta)|^{1/2}$.

Proof. We will use Proposition 6.2 (ii) to handle (i). We have

$$\frac{1}{N}\int_0^1\int_{B_1}\frac{|w|^{2^*}}{(1-|y|^2)^{1/2}}\,dy\,ds=\frac{1}{2}\int_{B_1}\biggl(\partial_s ww-\frac{1+N}{2}w^2\biggr)\frac{dy}{(1-|y|^2)^{1/2}}\biggr|_0^1 +\int_0^1\widetilde{E}(w(s))\,ds -\int_0^1\int_{B_1}\biggl((\partial_s w)^2+\partial_s wy\cdot\nabla w+\partial_s w\frac{w|y|^2}{1-|y|^2}\biggr)\frac{dy\,ds}{(1-|y|^2)^{1/2}}.$$

By Proposition 6.2, (i) and (iii), the second term on the right-hand side is bounded by E. The first term on the right-hand side is bounded using Lemma 6.1 (ii) and Cauchy–Schwarz' inequality. For the third term, because of the sign, we only need to consider the last two summands, which are bounded in absolute value by

$$\begin{aligned}\biggl|\int_0^1\int_{B_1}&\frac{|\partial_s w|}{(1-|y|^2)^{1/2}}\biggl(\frac{|w|}{1-|y|^2}+|\nabla w|\biggr)\,dy\,ds\biggr|\\ &\leqslant 2\biggl(\int_0^1\int_{B_1}\frac{|\partial_s w|^2}{1-|y|^2}\,dy\,ds\biggr)^{1/2}\biggl(\int_0^1\int_{B_1}\frac{w^2}{(1-|y|^2)^2}+|\nabla w|^2\,dy\,ds\biggr)^{1/2}\\ &\leqslant C\biggl(\log\frac{1}{\delta}\biggr)^{1/2},\end{aligned}$$

because of Lemma 6.1 (ii) and Lemma 6.4. This establishes (i).

To prove (ii), we first consider $\int_0^1\widetilde{E}(w(s))\,ds$, which is bounded from below by $-C(\log(1/\delta))^{1/2}$, by (i). The monotonicity of $\widetilde{E}$ (Proposition 6.2 (i)) concludes the proof of (ii). □

We now obtain our second improvement of decay on w.

LEMMA 6.6. *For $\delta>0$, we have*

$$\int_1^{(\log(1/\delta))^{3/4}}\int_{B_1}\frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}}\,dy\,ds\leqslant C\biggl(\log\frac{1}{\delta}\biggr)^{1/2}.$$

Proof. Because of Proposition 6.2 (i), we have

$$\int_1^{(\log(1/\delta))^{3/4}}\int_{B_1}\frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}}\,dy\,ds=\widetilde{E}\biggl(w\biggl(\biggl(\log\frac{1}{\delta}\biggr)^{3/4}\biggr)\biggr)-\widetilde{E}(w(1))\leqslant E+C\biggl(\log\frac{1}{\delta}\biggr)^{1/2},$$

where we have used Corollary 6.3 and Lemma 6.5 (ii). □

COROLLARY 6.7. *For each $\delta>0$, there exists $\bar{s}_\delta\in(1,(\log(1/\delta))^{3/4})$ such that*

$$\int_{\bar{s}_\delta}^{\bar{s}_\delta+(\log(1/\delta))^{1/8}}\int_{B_1}\frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}}\,dy\,ds\leqslant\frac{2C}{(\log(1/\delta))^{1/8}}.$$

Proof. Split the interval

$$\left(1, \left(\log \frac{1}{\delta}\right)^{3/4}\right)$$

into disjoint intervals of length $(\log(1/\delta))^{1/8}$. The number of such intervals is of the order of $(\log(1/\delta))^{5/8}$. For at least one of such intervals, $(\bar{s}_\delta, \bar{s}_\delta+(\log(1/\delta))^{1/8})$, with $\bar{s}_\delta\in(1,(\log(1/\delta))^{3/4})$, we must have

$$\int_{\bar{s}_\delta}^{\bar{s}_\delta+(\log(1/\delta))^{1/8}}\int_{B_1}\frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}}\,dy\,ds \leqslant \frac{2C(\log(1/\delta))^{1/2}}{(\log(1/\delta))^{5/8}} = \frac{2C}{(\log(1/\delta))^{1/8}},$$

where C is the constant in Lemma 6.6, which proves the corollary. □

Remark 6.8. Let $\bar{s}_\delta=-\log(1+\delta-\bar{t}_\delta)$. Note that

$$\left|\frac{1-\bar{t}_\delta}{1+\delta-\bar{t}_\delta}-1\right| = \frac{\delta}{1+\delta-\bar{t}_\delta} = \frac{\delta}{e^{-\bar{s}_\delta}} \leqslant \delta^{1/4} \xrightarrow{\delta\to 0} 0.$$

Let us now reduce the time evolution problem to a stationary problem in the w variable (i.e. self-similar solutions). Pick $\delta_j\downarrow 0$, so that

$$((1-\bar{t}_{\delta_j})^{N/2}\nabla u((1-\bar{t}_{\delta_j})y,\bar{t}_{\delta_j}),(1-\bar{t}_{\delta_j})^{N/2}\partial_t u((1-\bar{t}_{\delta_j})y,\bar{t}_{\delta_j}))\to(\nabla u_0^*,u_1^*)$$

in L^2. This is possible by Proposition 5.7. Note that, because of Remark 6.8 and the compact closure of $\overrightarrow{K}$ in Proposition 5.7, we also have that

$$\begin{aligned}((1+\delta_j-\bar{t}_{\delta_j})^{N/2}\nabla u((1+\delta_j-\bar{t}_{\delta_j})y,\bar{t}_{\delta_j}),(1+\delta_j-\bar{t}_{\delta_j})^{N/2}\partial_t u((1+\delta_j-\bar{t}_{\delta_j})y,\bar{t}_{\delta_j}))\\ \to(\nabla u_0^*,u_1^*)\end{aligned}$$

in L^2. Let now u_j^* and u^* be solutions of (CP) with data

$$((1+\delta_j-\bar{t}_{\delta_j})^{(N-2)/2}u((1+\delta_j-\bar{t}_{\delta_j})y,\bar{t}_{\delta_j}),(1+\delta_j-\bar{t}_{\delta_j})^{N/2}\partial_t u((1+\delta_j-\bar{t}_{\delta_j})y,\bar{t}_{\delta_j}))$$

and (u_0^*,u_1^*), respectively, in a time interval $[0,T^*]$, independent of j, which we take to have $T^*<1$. By uniqueness in the (CP), we have

$$u_j^*(y,\tau)=(1+\delta_j-\bar{t}_{\delta_j})^{(N-2)/2}u((1+\delta_j-\bar{t}_{\delta_j})y,\bar{t}_{\delta_j}+(1+\delta_j-\bar{t}_{\delta_j})\tau). \tag{6.6}$$

Note that $\operatorname{supp} u_j^*(\cdot,\tau)\subset\{y:|(1+\delta_j-\bar{t}_{\delta_j})y|\leqslant 1-\bar{t}_{\delta_j}-(1+\delta_j-\bar{t}_{\delta_j})\tau\}$ and hence

$$|y|\leqslant\frac{1-\bar{t}_{\delta_j}}{1+\delta_j-\bar{t}_{\delta_j}}-\tau<1-\tau$$

on the support of $u_j^*(\cdot,\tau)$. Similarly,

$$\operatorname{supp}\partial_\tau u_j^*(\cdot,\tau)\subset\left\{y:|y|\leqslant\frac{1-\bar t_{\delta_j}}{1+\delta_j-\bar t_{\delta_j}}-\tau<1-\tau\right\}.$$

Let us compare the solutions in the self-similar variables. Recall from (6.2) that if $s=-\log(1+\delta_j-t)$, then

$$w(y,s,\delta_j)=(1+\delta_j-t)^{(N-2)/2}u((1+\delta_j-t)y,t).$$

Define now τ by $t=\bar t_{\delta_j}+(1+\delta_j-\bar t_{\delta_j})\tau$, so that $1+\delta_j-t=(1+\delta_j-\bar t_{\delta_j})(1-\tau)$. Define also $s=-\log((1+\delta_j-\bar t_{\delta_j})(1-\tau))$. We then have

$$w(y,s,\delta_j)=((1+\delta_j-\bar t_{\delta_j})(1-\tau))^{(N-2)/2}u((1+\delta_j-\bar t_{\delta_j})(1-\tau)y,\bar t_{\delta_j}+(1+\delta_j-\bar t_{\delta_j})\tau).\quad(6.7)$$

If we now set

$$s'=-\log(1-\tau),\quad y'=\frac{y}{1-\tau}\quad\text{and}\quad w_j^*(y',s')=(1-\tau)^{(N-2)/2}u_j^*(y,\tau),$$

then w_j^* is a solution of (6.3), for $0<\tau<T^*$. But, because of (6.6) and (6.7),

$$\begin{aligned}w_j^*(y',s')&=(1-\tau)^{(N-2)/2}(1+\delta_j-\bar t_{\delta_j})^{(N-2)/2}u((1+\delta_j-\bar t_{\delta_j})y,\bar t_{\delta_j}+(1+\delta_j-\bar t_{\delta_j})\tau)\\&=w(y',s,\delta_j),\end{aligned}$$

where

$$s=-\log(1+\delta_j-t)=-\log((1+\delta_j-\bar t_{\delta_j})(1-\tau))=-\log(1+\delta_j-\bar t_{\delta_j})-\log(1-\tau)=\bar s_{\delta_j}+s',$$

i.e.,

$$w_j^*(y',s')=w(y',\bar s_{\delta_j}+s',\delta_j).\quad(6.8)$$

Consider also

$$w^*(y',s')=(1-\tau)^{(N-2)/2}u^*(y,\tau).$$

We clearly have $\operatorname{supp}u^*(\cdot,\tau)\subset\{y:|y|\leqslant(1-\tau)\}$ and w^* solves (6.3) for $0<\tau<T^*$. Also, recall that $(u_j^*(\cdot,\tau),\partial_\tau u_j^*(\cdot,\tau))\to(u^*(\cdot,\tau),\partial_\tau u^*(\cdot,\tau))$ in $\dot H^1\times L^2$, uniformly for $\tau\in[0,T^*]$, by continuity in (CP). But then if $0\leqslant\tau\leqslant\frac12 T^*=\widetilde T$ and $0\leqslant s'\leqslant-\log(1-\widetilde T)$, we have that

$$(w_j^*(\cdot,s'),\partial_{s'}w_j^*(\cdot,s'))\xrightarrow{j\to\infty}(w^*(\cdot,s'),\partial_{s'}w^*(\cdot,s'))$$

in $\dot H_0^1\times L^2$, uniformly for $0\leqslant s'\leqslant-\log(1-\widetilde T)$. But, by (6.8), we have

$$(w(y',\bar s_{\delta_j}+s',\delta_j),\partial_{s'}w(y',\bar s_{\delta_j}+s',\delta_j))\xrightarrow{j\to\infty}(w^*(\cdot,s'),\partial_{s'}w^*(\cdot,s')),\quad(6.9)$$

in $\dot H_0^1\times L^2$, uniformly in $0\leqslant s'\leqslant-\log(1-\widetilde T)$, and w^* is a solution of (6.3) and

$$\operatorname{supp}(w^*(\cdot,s'),\partial_{s'}w^*(\cdot,s'))\subset\{y:|y|\leqslant1\}.$$

LEMMA 6.9. *Let w^* be as above. Then,*

$$w^*(y',s')=w^*(y') \quad \textit{and} \quad w^*\not\equiv 0.$$

Proof. Let $S=-\log(1-\widetilde{T})$ and choose j large. Then

$$\int_0^S\int_{B_1}\frac{(\partial_{s'}w^*(y',s'))^2}{(1-|y'|^2)^{3/2}}\,dy'\,ds' \leqslant \varliminf_{j\to\infty}\int_0^S\int_{B_1}\frac{(\partial_{s'}w(y',\bar{s}_{\delta_j}+s',\delta_j))^2}{(1-|y'|^2)^{3/2}}\,dy'\,ds'$$

by (6.9). The right-hand side is bounded by

$$\varliminf_{j\to\infty}\int_{\bar{s}_{\delta_j}}^{S+\bar{s}_{\delta_j}}\int_{B_1}\frac{(\partial_{s'}w(y',s',\delta_j))^2}{(1-|y'|^2)^{3/2}}\,dy'\,ds' \leqslant 2C\lim_{j\to\infty}\frac{1}{(\log(1/\delta_j))^{1/8}}=0,$$

by Corollary 6.7. This shows that $w^*(y',s')=w^*(y')$.

To show that $w^*\not\equiv 0$, assume, on the contrary, that $w^*\equiv 0$. Then, by (6.8) and (6.9), we would have $\nabla_{y'}w(y',\bar{s}_{\delta_j},\delta_j)\to 0$ in $L^2(\mathbb{R}^N)$, so that

$$(1+\delta_j-\bar{t}_{\delta_j})^{N/2}\nabla_y u((1+\delta_j-\bar{t}_{\delta_j})y,\bar{t}_{\delta_j})\to 0$$

in $L^2(\mathbb{R}^N)$. Because of Corollary 3.6, we have, for $0<t<1$,

$$\int_{B_1}(|\nabla u(x,t)|^2+|\partial_t u(x,t)|^2)\,dx \geqslant CE>0.$$

But,

$$\int_{B_1}|\nabla u(x,\bar{t}_{\delta_j})|^2\,dx=\int_{\mathbb{R}^N}|(1+\delta_j-\bar{t}_{\delta_j})^{N/2}\nabla_y u((1+\delta_j-\bar{t}_{\delta_j})y,\bar{t}_{\delta_j})|^2\,dy\to 0,$$

so, for j large, we obtain

$$\int_{B_1}|\partial_t u(x,\bar{t}_{\delta_j})|^2\,dx \geqslant \frac{CE}{2}. \tag{6.10}$$

But, by (6.9) and the fact that $\partial_{s'}w^*(\,\cdot\,,s')=0$, we see that $\partial_s w(y',\bar{s}_{\delta_j},\delta_j)\to 0$ in $L^2(\mathbb{R}^N)$. We now use formula (6.4), which gives

$$\begin{aligned}\partial_s w(y',\bar{s}_{\delta_j},\delta_j)=&-\tfrac{1}{2}(N-2)(1+\delta_j-\bar{t}_{\delta_j})^{(N-2)/2}u((1+\delta_j-\bar{t}_{\delta_j})y',\bar{t}_{\delta_j})\\&+(1+\delta_j-\bar{t}_{\delta_j})^{N/2}\partial_t u((1+\delta_j-\bar{t}_{\delta_j})y',\bar{t}_{\delta_j})\\&-(1+\delta_j-\bar{t}_{\delta_j})^{N/2}y'\nabla u((1+\delta_j-\bar{t}_{\delta_j})y',\bar{t}_{\delta_j}).\end{aligned}$$

From our assumption, we see that, since $|y'|\leqslant 1$, the L^2 norm of the last term goes to 0. The same can be said for the L^2 norm of the first term, by Sobolev embedding. But this contradicts (6.10), so that $w^*\not\equiv 0$. $\square$

Proposition 6.10. *Let w^* be as above. Then*, $w^* \in H_0^1(B_1)$,

$$\int_{B_1} \frac{|w^*(y)|^2}{(1-|y|^2)^2}\,dy < \infty$$

and w^* solves the (*degenerate*) *elliptic equation*

$$\frac{\operatorname{div}(\varrho\nabla w^* - \varrho(y\cdot\nabla w^*)y)}{\varrho} - \frac{N(N-2)}{4}w^* + |w^*|^{4/(N-2)}w^* = 0, \tag{6.11}$$

where $\varrho(y)=(1-|y|^2)^{-1/2}$. *Moreover*, $w^*\not\equiv 0$ *and*

$$\int_{\mathbb{R}^N} \frac{|w^*(y)|^{2^*}}{(1-|y|^2)^{1/2}}\,dy + \int_{\mathbb{R}^N} \frac{|\nabla w^*(y)|^2-(y\cdot\nabla w^*(y))^2}{(1-|y|^2)^{1/2}}\,dy < \infty. \tag{6.12}$$

Remark 6.11. We will see that (6.12) are the critical estimates which allow us to conclude the proof.

Proof. It only remains to prove (6.12). Because of (6.9) and Lemma 6.9, to bound the first term in (6.12) it suffices to show that

$$\int_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S}\int_{B_1} \frac{|w(y',s',\delta_j)|^{2^*}}{(1-|y'|^2)^{1/2}}\,dy'\,ds' \leqslant C,$$

where C is independent of j. In order to show this, we use Proposition 6.2 (ii), so that

$$\begin{aligned}
&\frac{1}{N}\int_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S}\int_{B_1} \frac{|w(y',s',\delta_j)|^{2^*}}{(1-|y'|^2)^{1/2}}\,dy'\,ds' \\
&\quad= \int_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S} \widetilde{E}(w(s'))\,ds' + \frac{1}{2}\int_{B_1}\biggl(\partial_s w w - \frac{1+N}{2}w^2\biggr)\frac{dy'}{(1-|y'|^2)^{1/2}}\biggr|_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S} \\
&\qquad - \int_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S}\int_{B_1}\biggl((\partial_s w)^2 + \partial_s w y'\cdot\nabla w + \frac{w\partial_s w|y'|^2}{1-|y'|^2}\biggr)\frac{dy'\,ds'}{(1-|y'|^2)^{1/2}}.
\end{aligned}$$

The first term of the right-hand side is bounded by Corollary 6.3, the second one by Lemma 6.1 (ii). To bound the last one we only need to estimate the last two summands. We bound the last summand, using Cauchy–Schwarz' inequality, by

$$\biggl(\int_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S}\int_{B_1}\frac{w^2}{(1-|y'|^2)^2}\,dy'\,ds'\biggr)^{1/2}\biggl(\int_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S}\int_{B_1}\frac{|\partial_s w|^2}{1-|y'|^2}\,dy'\,ds'\biggr)^{1/2} \leqslant C\biggl(\log\frac{1}{\delta_j}\biggr)^{-1/16},$$

by Lemma 6.1 (ii) and Corollary 6.7.

The second-last summand, by Cauchy–Schwarz' inequality, is bounded by

$$\biggl(\int_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S}\int_{B_1}|\nabla w|^2\,dy'\,ds'\biggr)^{1/2}\biggl(\int_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S}\int_{B_1}\frac{|\partial_s w|^2}{1-|y'|^2}\,dy'\,ds'\biggr)^{1/2},$$

which can be estimated similarly. This proves the first estimate. To prove the gradient estimate, use Corollary 6.3 and the previous proof to conclude that

$$\int_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S}\int_{B_1}(|\nabla w(y',s',\delta_j)|^2-(y'\cdot\nabla w(y',s',\delta_j))^2)\frac{dy'\,ds'}{(1-|y'|^2)^{1/2}}\leqslant C.$$

Using (6.9) and Lemma 6.9, this leads to

$$\int_{B_1}(|\nabla w^*|^2-(y'\cdot\nabla w^*)^2)\frac{dy'}{(1-|y'|^2)^{1/2}}\leqslant C,$$

which concludes the proof. □

The contradiction which finishes the proof of Theorem 5.1 is then provided by the following elliptic result.

PROPOSITION 6.12. *Let* $w\in H_0^1(B_1)$ *be such that*

(i)

$$\int_{\mathbb{R}^N}\frac{|w(y)|^2}{(1-|y|^2)^2}\,dy<\infty$$

(*a consequence of* $w\in H_0^1(B_1)$);

(ii)

$$\int_{\mathbb{R}^N}\frac{|w(y)|^{2^*}}{(1-|y|^2)^{1/2}}\,dy+\int_{\mathbb{R}^N}\frac{|\nabla w(y)|^2-(y\cdot\nabla w(y))^2}{(1-|y|^2)^{1/2}}\,dy<\infty;$$

(iii) *w satisfies the* (*degenerate*) *elliptic equation* (6.11).

Then, $w\equiv 0$.

Proof. We write again equation (6.11), with $\varrho=(1-|y|^2)^{-1/2}$:

$$\frac{\operatorname{div}(\varrho\nabla w-\varrho(y\cdot\nabla w)y)}{\varrho}-\frac{N(N-2)}{4}w+|w|^{4/(N-2)}w=0. \tag{6.13}$$

Consider first the linear part

$$Lw=\frac{\operatorname{div}(\varrho\nabla w-\varrho(y\cdot\nabla w)y)}{\varrho}=\frac{\operatorname{div}(\varrho(I-y\otimes y)\nabla w)}{\varrho}.$$

For $|y|<1-\delta$, $\delta>0$, L is a second-order elliptic operator with smooth coefficients. Thus, the well-known argument of Trudinger [47] shows that $w\in L^\infty(B_{1-\delta})$ and therefore $w\in C^2(B_{1-\delta})$, where $B_{1-\delta}=\{y:|y|<1-\delta\}$, for each $\delta>0$. From this and the classical unique continuation theorem of Aronszajn, Krzywicki and Szarski (see [2] and [15, §17.2]) we see that if $w\equiv 0$ on $1-\delta<|y|<1$, then we have $w\equiv 0$.

In order to establish this for $1-\delta<|y|<1$, it is convenient to write our equation in polar coordinates (r,θ), $0<r<\infty$, $\theta\in S^{N-1}$. In these coordinates, (6.13) becomes (with $y=r\theta$)

$$(1-r^2)^{1/2}\frac{\partial}{\partial r}(1-r^2)^{1/2}\frac{\partial w}{\partial r}+\frac{1}{r^2}\Delta_\theta w+\frac{(N-1)}{r}(1-r^2)\frac{\partial w}{\partial r}=\frac{N(N-2)}{4}w-|w|^{4/(N-2)}w, \tag{6.14}$$

where Δ_θ denotes the spherical Laplacian on S^{N-1}.

For $1-\delta<r<1$, we perform the change of variables $v(s,\theta)=w(r(s),\theta)$, with

$$r(s)=1-\frac{(1-s)^2}{4}.$$

For suitable $\tilde{\delta}$, we have $1-\tilde{\delta}\leqslant s\leqslant 1$, when $1-\delta\leqslant r\leqslant 1$. Also,

$$r'(s)=\frac{1-s}{2}\quad\text{and}\quad\frac{r'(s)}{(1-r(s))^{1/2}}=1.$$

Since

$$(1+r(s))^{1/2}\frac{\partial}{\partial s}v(s,\theta)=(1-r^2(s))^{1/2}\frac{\partial w}{\partial r}(r(s),\theta),$$

v satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial s}(1+r(s))^{1/2}\frac{\partial v}{\partial s}+\frac{1}{(1+r(s))^{1/2}}\frac{1}{r(s)^2}\Delta_\theta v+\frac{N-1}{r(s)}(1-r(s)^2)^{1/2}\frac{\partial v}{\partial s}&\\ =\frac{N(N-2)}{4(1+r(s))^{1/2}}v-\frac{|v|^{4/(N-2)}v}{(1+r(s))^{1/2}}&.\end{aligned}\tag{6.15}$$

The advantage of (6.15) is that it is elliptic, not degenerate elliptic, near $s=1$ (or $r=1$). Moreover, since $1+r(s)$ is bounded above and below away from 0 and smooth, the coefficients in (6.15) are smooth. We now turn to some estimates for v, for $1-\tilde{\delta}\leqslant s\leqslant 1$ and $\theta\in S^{N-1}$.

We first claim that

$$\int_{1-\tilde{\delta}}^1\int_{S^{N-1}}|v(s,\theta)|^{2^*}\,d\theta\,ds<\infty. \tag{6.16}$$

In fact, the integral in (6.16) equals

$$\int_{1-\delta}^1\int_{S^{N-1}}|w(r,\theta)|^{2^*}\frac{d\theta\,dr}{(1-r)^{1/2}},$$

which is finite by virtue of (ii).

Next, we notice that, for $1-\delta\leqslant|y|\leqslant 1$,

$$|\nabla_\theta w(y)|\simeq\left|\nabla w-\left(\frac{y}{|y|}\cdot\nabla w\right)\frac{y}{|y|}\right|$$

and

$$|\nabla w|^2-(y\cdot\nabla w)^2=\left(\frac{1}{|y|^2}-1\right)(y\cdot\nabla w)^2+\left|\nabla w-\frac{y}{|y|}.\nabla w\frac{y}{|y|}\right|^2$$
$$=(1-|y|^2)\left(\frac{y}{|y|}\cdot\nabla w\right)^2+\left|\nabla w-\frac{y}{|y|}\cdot\nabla w\frac{y}{|y|}\right|^2.$$

Thus, since $w\in H_0^1(B_1)$, (ii) holds, we see that

$$\int_{1-\delta}^1\int_{S^{N-1}}|\nabla_\theta w(r,\theta)|^2\frac{d\theta\,dr}{(1-r)^{1/2}}<\infty,$$

and hence

$$\int_{1-\tilde\delta}^1\int_{S^{N-1}}|\nabla_\theta v(s,\theta)|^2\,d\theta\,ds<\infty. \tag{6.17}$$

Next, we show that

$$\int_{1-\tilde\delta}^1\int_{S^{N-1}}\left|\frac{\partial v}{\partial s}(s,\theta)\right|^2\frac{d\theta\,ds}{1-s}<\infty. \tag{6.18}$$

This estimate, combined with $v\in H_0^{1,2}(B_1)$, is the one that forces v to vanish, since it means that the Cauchy data for the solution v of (6.15) vanishes. This is a consequence of the fact that $w\in H_0^1(B_1)$ and the degeneracy of (6.13). On the other hand, (6.16) and (6.17) show that we are dealing with a "standard solution" to (6.15). To obtain (6.18), change variables. The integral equals

$$\begin{aligned}\int_{1-\tilde\delta}^1\int_{S^{N-1}}\left|\frac{\partial w}{\partial r}(r(s),\theta)\right|^2\frac{|r'(s)|^2}{1-s}\,ds\,d\theta&=\int_{1-\tilde\delta}^1\int_{S^{N-1}}\left|\frac{\partial w}{\partial r}(r(s),\theta)\right|^2\frac{|r'(s)|}{2}\,d\theta\,ds\\&=\int_{1-\delta}^1\int_{S^{N-1}}\left|\frac{\partial w}{\partial r}(r,\theta)\right|^2d\theta\,\frac{dr}{2}\\&\leqslant C\int_{1-\delta}^1\int_{S^{N-1}}\left|\frac{\partial w}{\partial r}(r,\theta)\right|^2d\theta\,dr.\end{aligned}$$

Finally, a similar argument, using (i), shows that

$$\int_{1-\tilde\delta}^1\int_{S^{N-1}}\frac{|v(s,\theta)|^2}{(1-s)^3}\,d\theta\,ds<\infty. \tag{6.19}$$

Once we have the estimates (6.16)–(6.19), we define

$$\tilde v(s,\theta)=\begin{cases}v(s,\theta)&\text{for } 1-\tilde\delta<s<1,\\0&\text{for } 1<s<2.\end{cases} \tag{6.20}$$

Since $v(s,\theta)\in H_0^1(ds\,d\theta)$, for $1-\tilde{\delta}<s<1$, in light of (6.17), (6.18) and (6.19), $\tilde{v}\in H^1(ds\,d\theta)$, $1-\tilde{\delta}<s<2$, $\theta\in S^{N-1}$. We claim that $\tilde{v}$ solves (6.15) for $1-\tilde{\delta}<s<2$: To show this, let $\eta(s,\theta)$ be a test function. Let $\mu_\varepsilon(s)$ be a smooth approximation of the characteristic function of $s<1$. We have to show that

$$\iint_{S^{N-1}}(1+r(s))^{1/2}\frac{\partial\tilde{v}}{\partial s}\frac{\partial\eta}{\partial s}\,d\theta\,ds=\lim_{\varepsilon\downarrow 0}\iint_{S^{N-1}}(1+r(s))^{1/2}\frac{\partial v}{\partial s}\frac{\partial}{\partial s}(\eta\mu_\varepsilon)\,d\theta\,ds.$$

But, this reduces to showing that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\left|\iint_{S^{N-1}}\eta(1+r(s))^{1/2}\frac{\partial v}{\partial s}\frac{\partial}{\partial s}\mu_\varepsilon\,d\theta\,ds\right| &\leqslant \frac{C}{\varepsilon}\int_{1-2\varepsilon}^{1-\varepsilon}\int|\eta|(1+r(s))^{1/2}\left|\frac{\partial v}{\partial s}\right|d\theta\,ds\\ &\leqslant C\int_{1-2\varepsilon}^{1-\varepsilon}\int\left|\frac{\partial v}{\partial s}\right|\frac{d\theta\,ds}{1-s}\xrightarrow{\varepsilon\to 0}0,\end{aligned}$$

because of (6.18). We can now apply Trudinger's argument in the critical case [47] to $\tilde{v}$, to show that $\tilde{v}\in C^2(\{s:1-\tilde{\delta}<s<2\}\times S^{N-1})$. Once we have this, $\tilde{v}\equiv 0$ on $\{s:1-\tilde{\delta}<s<2\}$, because of the fact that $\tilde{v}\equiv 0$ for $1<s<2$ and the unique continuation theorem of [1]. (See also [15, §17.2].) From this, we conclude that $w\equiv 0$, as desired. □

Remark 6.13. One can skip the use of Trudinger's argument in [47] and use directly the more delicate unique continuation theorem of [16], or rather its variable coefficient version, due to C. Sogge [38] and T. Wolff [48].

Remark 6.14. For this part of the argument, no size or energy conditions are needed. In addition, in the radial case, Lemma 6.1 and 1-dimensional Sobolev inequalities give that $\widetilde{E}(w(0))$ is bounded in absolute value, which allows us to reduce directly to the elliptic problem.

The results in this section yield the contradiction which completes the proof of Theorem 5.1.

7. Main theorem

In this section we establish our main result (see [32] and [34] for the subcritical case, where energy controls yield the result).

THEOREM 7.1. *Let* $(u_0,u_1)\in\dot{H}^1\times L^2$, $3\leqslant N\leqslant 5$. *Assume that* $E((u_0,u_1))<E((W,0))$. *Let u be the corresponding solution of the Cauchy problem, with maximal interval of existence* $I=(-T_-(u_0,u_1),T_+(u_0,u_1))$. (*See Definition* 2.13.)

(i) *If* $\int_{\mathbb{R}^N}|\nabla u_0|^2\,dx<\int_{\mathbb{R}^N}|\nabla W|^2\,dx$, *then* $I=(-\infty,\infty)$ *and* $\|u\|_{L^{2(N+1)/(N-2)}_{xt}}<\infty$.

(ii) *If* $\int_{\mathbb{R}^N}|\nabla u_0|^2\,dx>\int_{\mathbb{R}^N}|\nabla W|^2\,dx$, *then* $T_+(u_0,u_1)<\infty$ *and* $T_-(u_0,u_1)<\infty$.

Remark 7.2. The equality $\int_{\mathbb{R}^N}|\nabla u_0|^2\,dx=\int_{\mathbb{R}^N}|\nabla W|^2\,dx$ is incompatible with the energy condition from (3.2). (Indeed, in this case $E((u_0,u_1))\geqslant E((W,0))$.)

Proof. To establish (i), we argue by contradiction. If (i) is not true, then E_C, defined in §4, must satisfy $\eta_0\leqslant E_C<E((W,0))$. Let u_C be as in Proposition 4.2 and assume that I_+ is finite. Then, by Proposition 4.10, $\int_{\mathbb{R}^N}\nabla u_{0,C}u_{1,C}\,dx=0$. But then we reach a contradiction from Theorem 5.1. If I_+ is infinite and $\lambda(t)\geqslant A_0>0$, then Proposition 4.11 shows that $\int_{\mathbb{R}^N}\nabla u_{0,C}u_{1,C}\,dx=0$, and Theorem 5.1 gives $u_C\equiv 0$, a contradiction because $E((u_C,\partial_t u_C))=E_C\geqslant\eta_0$.

To conclude the proof, we need to reduce to the case $\lambda(t)>A_0>0$, for $t\geqslant 0$, using the argument in the proof of [19, Theorem 5.1] (see also [27] for a similar proof). Recall that $E((u_C,\partial_t u_C))=E_C\geqslant\eta_0>0$. Because of Lemma 4.6, we may assume that there exist $t_n\uparrow\infty$ so that $\lambda(t_n)\to 0$. After possibly redefining $\{t_n\}_{n=1}^{\infty}$, we may assume that

$$\lambda(t_n)\leqslant\inf_{t\in[0,t_n]}\lambda(t).$$

From Proposition 4.2, we get

$$(w_{0,n}(x),w_{1,n}(x))=\left(\frac{1}{\lambda(t_n)^{(N-2)/2}}u_C\left(\frac{x-x(t_n)}{\lambda(t_n)},t_n\right),\frac{1}{\lambda(t_n)^{N/2}}\partial_t u_C\left(\frac{x-x(t_n)}{\lambda(t_n)},t_n\right)\right)\to(w_0,w_1)$$

in $\dot H^1\times L^2$. Note that $E((w_0,w_1))=E_C$. Moreover, $\int_{\mathbb{R}^N}|\nabla w_0|^2\,dx<\int_{\mathbb{R}^N}|\nabla W|^2\,dx$, by the corresponding properties of u_C and Theorem 3.5. Let $w_0(x,\tau)$, $\tau\in(-T_-(w_0,w_1),0]$, be the corresponding solution of (CP). If $T_-(w_0,w_1)<\infty$, then Propositions 4.2 and 4.10 yield $\int_{\mathbb{R}^N}\nabla w_0w_1\,dx=0$, and Theorem 5.1 and Proposition 4.2 give a contradiction. Hence $T_-(w_0,w_1)=\infty$. Let $w_n(x,\tau)$ be the solution of (CP), with data $(w_{0,n}(x),w_{1,n}(x))$, $\tau\in(-T_-(w_{0,n},w_{1,n}),0]$. Because of Remark 2.21, $\varliminf_{n\to\infty}T_-(w_{0,n},w_{1,n})=\infty$, and for any $\tau\in(-\infty,0]$ we have

$$(w_n(x,\tau),\partial_\tau w_n(x,\tau))\to(w_0(x,\tau),\partial_\tau w_0(x,\tau))$$

in $\dot H^1\times L^2$. Note that, by uniqueness in (CP), for $0\leqslant t_n+\tau/\lambda(t_n)$,

$$w_n(x,\tau)=\frac{1}{\lambda(t_n)^{(N-2)/2}}u_C\left(\frac{x-x(t_n)}{\lambda(t_n)},t_n+\frac{\tau}{\lambda(t_n)}\right). \tag{7.1}$$

Let $\tau_n=-\lambda(t_n)t_n$, and note that

$$\varliminf_{n\to\infty}(-\tau_n)=\varliminf_{n\to\infty}t_n\lambda(t_n)\geqslant T_-(w_0,w_1)=\infty,$$

so that for all $\tau\in(-\infty,0]$, for n large, we have

$$0\leqslant t_n+\frac{\tau}{\lambda(t_n)}\leqslant t_n.$$

In fact, if $-\tau_n\to-\tau_0<\infty$, then

$$\begin{aligned} w_n(x,\tau_n)&=\frac{1}{\lambda(t_n)^{(N-2)/2}}u_C\left(\frac{x-x(t_n)}{\lambda(t_n)},0\right),\\ \partial_\tau w_n(x,\tau_n)&=\frac{1}{\lambda(t_n)^{N/2}}\partial_t u_C\left(\frac{x-x(t_n)}{\lambda(t_n)},0\right),\end{aligned}$$

would converge to $(w_0(x,\tau_0),\partial_\tau w_0(x,\tau_0))$ in $\dot H^1\times L^2$, with $\lambda(t_n)\to0$, which is a contradiction from the fact that $(u_{0,C},u_{1,C})\not\equiv(0,0)$ and $(w_0,w_1)\not\equiv(0,0)$.

Next, note that we must have $\|w_0\|_{S((-\infty,0))}=\infty$. Otherwise, by Theorem 2.20, for n large, $T_-(w_{0,n},w_{1,n})=\infty$ and $\|w_n\|_{S((-\infty,0))}\leqslant M$, uniformly in n, which, in view of (7.1), contradicts $\|u_C\|_{S((0,\infty))}=\infty$.

Fix now $\tau\in(-\infty,0]$. For n sufficiently large, $t_n+\tau/\lambda(t_n)\geqslant0$ and $\lambda(t_n+\tau/\lambda(t_n))$ is defined. Let

$$\begin{aligned}&\left(\frac{1}{\lambda(t_n+\tau/\lambda(t_n))^{(N-2)/2}}u_C\left(\frac{x-x(t_n+\tau/\lambda(t_n))}{\lambda(t_n+\tau/\lambda(t_n))},t_n+\frac{\tau}{\lambda(t_n)}\right),\right.\\ &\qquad\left.\frac{1}{\lambda(t_n+\tau/\lambda(t_n))^{N/2}}\partial_t u_C\left(\frac{x-x(t_n+\tau/\lambda(t_n))}{\lambda(t_n+\tau/\lambda(t_n))},t_n+\frac{\tau}{\lambda(t_n)}\right)\right)\\ &=\left(\frac{1}{\tilde\lambda_n(\tau)^{(N-2)/2}}w_n\left(\frac{x-\tilde x_n(\tau)}{\tilde\lambda_n(\tau)},\tau\right),\frac{1}{\tilde\lambda_n(\tau)^{N/2}}\partial_\tau w_n\left(\frac{x-\tilde x_n(\tau)}{\tilde\lambda_n(\tau)},\tau\right)\right)\in K,\end{aligned}$$

with

$$\tilde\lambda_n(\tau)=\frac{\lambda(t_n+\tau/\lambda(t_n))}{\lambda(t_n)}\geqslant1\quad\text{and}\quad\tilde x_n(\tau)=x\left(t_n+\frac{\tau}{\lambda(t_n)}\right)-x(t_n)\cdot\tilde\lambda_n(\tau).\tag{7.2}$$

Now, since

$$\frac{1}{\lambda_n^{N/2}}\vec v\left(\frac{x-x_n}{\lambda_n}\right)\xrightarrow{n\to\infty}\vec{\tilde v}$$

in L^2, with either $\lambda_n\to0$, $\lambda_n\to\infty$, or $|x_n|\to\infty$ implies that $\vec{\tilde v}\equiv0$, we see that (since $E_C>0$) we may assume, after passing to a subsequence, that $\tilde\lambda_n(\tau)\to\tilde\lambda(\tau)$, $1\leqslant\tilde\lambda(\tau)<\infty$, and $\tilde x_n(\tau)\to\tilde x(\tau)\in\mathbb{R}^N$. But then

$$\left(\frac{1}{\tilde\lambda_n(\tau)^{(N-2)/2}}w_0\left(\frac{x-\tilde x_n(\tau)}{\tilde\lambda_n(\tau)},\tau\right),\frac{1}{\tilde\lambda_n(\tau)^{N/2}}\partial_\tau w_0\left(\frac{x-\tilde x_n(\tau)}{\tilde\lambda_n(\tau)},\tau\right)\right)\in\overline{K}.$$

But then, by Proposition 4.11 and Theorem 5.1, we have $(w_0,w_1)=(0,0)$, contradicting $E_C=E((w_0,w_1))$. This proves (i).

For (ii), note that if $u_0 \in L^2$, this is the result in Theorem 3.7. The proof of the general case is a modification of that of Theorem 3.7. Similar arguments in the context of radial solutions of the non-linear Schrödinger equation have been used before, see for instance the work of Ogawa and Tsutsumi [31]. Let $A = \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} > 0$. Recall that (from Lemma 2.17 and its proof) there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, there exists $M_0 = M_0(\varepsilon)$ so that

$$\int_{|x| \geqslant M_0 + t} \left(|\nabla_x u(x,t)|^2 + |\partial_t u(x,t)|^2 + |u(x,t)|^{2^*} + \frac{|u(x,t)|^2}{|x|^2} \right) dx \leqslant \varepsilon,$$

for $t \in [0, T_+(u_0, u_1))$. Assume that $T_+(u_0, u_1) = \infty$, to reach a contradiction.

Let $f(\tau)$ be a solution of the differential inequality (with $f \geqslant 0$)

$$f'(\tau) \geqslant B f(\tau)^{(N-1)(N-2)}, \quad f(0) = 1. \tag{7.3}$$

Then, the time of blow-up for f is τ_*, with $\tau_* \leqslant K_N B^{-1}$.

Consider now, for R large, $\phi \in C_0^\infty(B_2)$, with $\phi \equiv 1$ on $|x| < 1$ and $0 \leqslant \phi \leqslant 1$,

$$y_R(t) = \int_{\mathbb{R}^N} u^2(x,t) \phi\left(\frac{x}{R}\right) dx.$$

Then,

$$y_R'(t) = 2 \int_{\mathbb{R}^N} u \partial_t u \phi\left(\frac{x}{R}\right) dx,$$

and, using the notation in Lemma 5.3, we have that

$$y_R''(t) = 2 \int_{\mathbb{R}^N} ((\partial_t u)^2 - |\nabla_x u|^2 + |u|^{2^*})\, dx + O(r(R)).$$

Arguing as in the proof of Theorem 3.7, we find that

$$y_R''(t) \geqslant 2\left(1 + \frac{N}{N-2}\right) \int_{\mathbb{R}^N} (\partial_t u)^2 \phi\left(\frac{x}{R}\right) dx + \tilde{\tilde{\delta}}_0 + O(r(R)).$$

Choose now ε_1 small and $M_0 = M_0(\varepsilon_1)$ as above, so that, for $R > 2M_0$, $O(r(R)) \leqslant \varepsilon_1$, $\varepsilon_1 \leqslant \frac{1}{2}\tilde{\tilde{\delta}}_0$. We then have, for $0 < t < \frac{1}{2}R$,

$$y_R''(t) \geqslant \tfrac{1}{2}\tilde{\tilde{\delta}}_0 \quad \text{and} \quad y_R''(t) \geqslant 2\left(1 + \frac{N}{N-2}\right) \int_{\mathbb{R}^N} (\partial_t u)^2 \phi\left(\frac{x}{R}\right) dx. \tag{7.4}$$

Note also that

$$y_R(0) \leqslant C M_0^2 A^2 + \varepsilon_1 R^2 \quad \text{and} \quad |y_R'(0)| \leqslant C M_0 A^2 + \varepsilon_1 R. \tag{7.5}$$

Let

$$T=\frac{4CM_0A^2+2\varepsilon_1R+2R\sqrt{\varepsilon_1}}{\tilde{\tilde{\delta}}_0}.$$

Then, if $T<\frac{1}{2}R$,

$$y_R'(T)\geqslant\tfrac{1}{2}T\tilde{\tilde{\delta}}_0+y_R'(0)\geqslant 2CM_0A^2+\varepsilon_1R+R\sqrt{\varepsilon_1}-CM_0A^2-\varepsilon_1R=CM_0A^2+R\sqrt{\varepsilon_1}.$$

Thus, there exists $0<t_1<T$ such that $y_R'(t_1)=CM_0A^2+R\sqrt{\varepsilon_1}$ and, for $0<t<t_1$, we have $y_R'(t)<CM_0A^2+R\sqrt{\varepsilon_1}$. Note that, in light of (7.6), $y_R'(t)>y_R'(t_1)>0$, $t>t_1$ ($t<\frac{1}{2}R$), and also

$$y_R(t_1)\leqslant y_R(0)+\int_0^{t_1}y_R'\,dt\leqslant y_R(0)+t_1(CM_0A^2+R\sqrt{\varepsilon_1}\,)=y_R(0)+t_1y_R'(t_1).$$

We next estimate T. We first choose ε_1 so small that

$$\frac{2\varepsilon_1}{\tilde{\tilde{\delta}}_0}+\frac{2\sqrt{\varepsilon_1}}{\tilde{\tilde{\delta}}_0}\leqslant\frac{1}{32K_N},$$

where K_N is the constant defined at the beginning of the proof, and R is so large that

$$\frac{4CM_0A^2}{\tilde{\tilde{\delta}}_0}\leqslant\frac{R}{16K_N}.$$

We then have $T\leqslant R/8K_N$. We can also ensure that $T\leqslant\frac{1}{8}R$. Thus,

$$y_R(t_1)\leqslant CM_0^2A^2+\varepsilon_1R^2+\frac{R}{8K_N}y_R'(t_1).$$

If we now use the argument in the proof of Theorem 3.7, for the function

$$\tilde{y}_R(\tau)=y_R(t_1+\tau),\quad 0\leqslant\tau\leqslant\tfrac{1}{4}R,$$

in light of (7.6), we see that, for $0<\tau<\frac{1}{4}R$, we have that

$$\log(\tilde{y}_R'(\tau))'\geqslant\frac{N-1}{N-2}\log(\tilde{y}_R(\tau))',$$

so that, by integration,

$$\frac{\tilde{y}_R'(\tau)}{\tilde{y}_R'(0)}\geqslant\left(\frac{\tilde{y}_R(\tau)}{\tilde{y}_R(0)}\right)^{(N-1)/(N-2)}\qquad\text{for } 0\leqslant\tau\leqslant\tfrac{1}{4}R.$$

Thus, if $f(\tau)=\tilde{y}_R(\tau)/\tilde{y}_R(0)$ and $B=\tilde{y}_R'(0)/\tilde{y}_R(0)=y_R'(t_1)/y_R(t_1)$, we have that f is a solution of (7.3) for $0\leqslant\tau\leqslant\frac{1}{4}R$. Hence, we must have

$$\frac{R}{4}\leqslant\frac{y_R(t_1)}{y_R'(t_1)}K_N\leqslant\frac{K_N(CM_0^2A^2+\varepsilon_1R^2)}{y_R'(t_1)}+\frac{R}{8},$$

or

$$\frac{1}{8}\leqslant\frac{K_N(CM_0^2A^2+\varepsilon_1R^2)}{CM_0A^2R+R^2\sqrt{\varepsilon_1}}=\frac{K_N(CM_0^2A^2/R^2+\varepsilon_1)}{CM_0A^2/R+\sqrt{\varepsilon_1}}\leqslant\frac{K_NM_0}{R}+K_N\sqrt{\varepsilon_1}.$$

By taking $K_N\sqrt{\varepsilon_1}<\frac{1}{32}$, and $K_NM_0/R<\frac{1}{32}$, we reach a contradiction, which gives the proof of (ii). $\square$

To conclude, let us give some corollaries of our main results similarly to the nonlinear Schrödinger case (for full proofs see [20], [21] and the arguments below).

Corollary 7.3. *Let* $(u_0,u_1)\in \dot H^1\times L^2$, $3\leqslant N\leqslant 5$. *Assume that*

$$E((u_0,u_1))<E((W,0)) \quad \text{and} \quad \int_{\mathbb{R}^N}|\nabla u_0|^2\,dx<\int_{\mathbb{R}^N}|\nabla W|^2\,dx.$$

Then, the solution u of the Cauchy problem (CP) *with data* (u_0,u_1) *at* $t=0$ *has time interval of existence* $I=(-\infty,\infty)$, *and there exists* $(u_{0,\pm},u_{1,\pm})$ *in* $\dot H^1\times L^2$ *such that if we denote by* $v_\pm(t)$ *the solutions of* (LCP) *corresponding to these initial data, we have*

$$\lim_{t\to\pm\infty}\|(u(t),\partial_t u(t))-(v_\pm(t),\partial_t v_\pm(t))\|_{\dot H^1\times L^2}=0.$$

Moreover, if we define δ_0 *so that* $E((u_0,u_1))\leqslant(1-\delta_0)E((W,0))$, *there exists a function* $M(\delta_0)$ *so that* $\|u\|_{S((-\infty,\infty))}\leqslant M(\delta_0)$.

Let us give now a different version of the main result.

Corollary 7.4. *Let* $(u_0,u_1)\in\dot H^1\times L^2$ *and assume that*

$$\int_{\mathbb{R}_N}(|\nabla u(t)|^2+|\partial_t u(t)|^2)\,dx\leqslant\int_{\mathbb{R}^N}|\nabla W|^2\,dx-\delta_0$$

for all $t\in(-T_-(u_0,u_1),T_+(u_0,u_1))$, *for* $\delta_0>0$. *Then, the solution u of* (CP) *with data* u_0 *at* $t=0$ *has time interval of existence* $I=(-\infty,\infty)$, $\|u\|_{S((-\infty,\infty))}<\infty$.

Corollary 7.5. *Let* $3\leqslant N\leqslant 5$, $(u_0,u_1)\in\dot H^1\times L^2$ *(no size restrictions) be such that* $T_+(u_0,u_1)<\infty$ *and, for all* $t\in[0,T_+(u_0,u_1))$, $\int_{\mathbb{R}^N}(|\nabla u(t)|^2+|\partial_t u(t)|^2)\,dx\leqslant C_0$. *Then, we have for* $x_1(t)$ *and* $x_2(t)$, *and for all* $R>0$,

$$\begin{aligned}\varliminf_{t\to T_+(u_0,u_1)}\int_{|x-x_1(t)|\leqslant R}(|\nabla u(t)|^2+|\partial_t u(t)|^2)\,dx&\geqslant\frac{2}{N}\int_{\mathbb{R}^N}|\nabla W|^2\,dx\\ \varlimsup_{t\to T_+(u_0,u_1)}\int_{|x-x_2(t)|\leqslant R}(|\nabla u(t)|^2+|\partial_t u(t)|^2)\,dx&\geqslant\int_{\mathbb{R}^N}|\nabla W|^2\,dx.\end{aligned}$$

Proof. Let t_n be an arbitrary sequence such that $t_n\to T_+=T_+(u_0,u_1)$. Since

$$\int_{\mathbb{R}^N}(|\nabla u(t_n)|^2+|\partial_t u(t_n)|^2)\,dx\leqslant C_0$$

and $\|S(t)((u(t_n),\partial_t u(t_n)))\|_{S([t_n,T_+])}>\delta_0$ (where $\delta_0=\delta(C_0)$ is defined in Theorem 2.7), Lemma 4.3 gives, up to a subsequence, a decomposition of $(u(t_n),\partial_t u(t_n))$ such that (4.2)–(4.5) hold with

$$\|S(t)((w_{0,n},w_{1,n}))\|_{S((-\infty,\infty))}\leqslant\tfrac14\delta(2C_0).$$

Applying Theorem 2.20 and a pigeonhole argument (as in [20]), after taking a further subsequence in n and possibly reordering in j, we see that we can find J_0 such that for $j=1, .., J_0$ we have

$$\left\| \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right)\right\|_{S([0,T_n^+])} \to \infty,$$

and for $j=J_0+1, .., J$ we have, for some $\widetilde{C}_0$,

$$\left\| \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right)\right\|_{S([0,T_n^+])} \leqslant \widetilde{C}_0,$$

where U_j is the non-linear profile associated with

$$\left(\left(V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right), \partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right), \left\{-\frac{t_{j,n}}{\lambda_{j,n}}\right\}\right),$$

and if $T_{+,j}=T_+(U_j, \partial_t U_j)$ (possibly $T_{+,j}=\infty$), we have

$$T_n^+ = \min_{1\leqslant j\leqslant J_0} \{T_+-t_n, T_{+,j}\lambda_{j,n}+t_{j,n}\}$$

and for a sequence $\tau_k'\to T_{+,1}$ as $k\to\infty$, for all $j=1, .., J_0$ and for all n we have

$$\|U_1\|_{S([-t_{1,n}/\lambda_{1,n},(t_{1,n}+\tau_k'\lambda_{1,n}-t_{1,n})/\lambda_{1,n}])} \geqslant \|U_j\|_{S([-t_{j,n}/\lambda_{j,n},(t_{1,n}+\tau_k'\lambda_{1,n}-t_{j,n})/\lambda_{j,n}])}$$

(note that, applying Theorem 2.20, up to a subsequence, one can choose U_1 with such a property).

By scaling, we have $-t_{1,n}/\lambda_{1,n}\leqslant C$ and $\tau_k'\lambda_{1,n}+t_{1,n}\geqslant -C$. From Theorem 2.20, one can see that

$$\begin{gathered}
\varlimsup_{n\to\infty} (t_n+\tau_k'\lambda_{1,n}+t_{1,n}) < T_+ \quad \text{for all } k, \\
\lim_{n\to\infty} (t_n+\tau_k'\lambda_{1,n}+t_{1,n}) \to T_+ \quad \text{as } k\to\infty, \\
\frac{1}{C} \leqslant \frac{T_+-t_n}{\lambda_{1,n}} \quad \text{and} \quad -\frac{t_{1,n}}{\lambda_{1,n}}+\frac{T_+-t_n}{\lambda_{1,n}} \geqslant -C.
\end{gathered} \tag{7.6}$$

Using Corollary 7.4 for U_1, there is a sequence $\tau_k\to T_{+,1}$ such that

$$\varliminf_{k\to\infty} \int_{\mathbb{R}^N} (|\nabla U_1(\tau_k)|^2+|\partial_t U_1(\tau_k)|^2)\, dx \geqslant \int_{\mathbb{R}^N} |\nabla W|^2\, dx.$$

Using (7.6), (4.2), the fact that for all k there is k' such that $\tau_k\leqslant\tau_{k'}'$ and orthogonality arguments (as in [20], for example), we obtain, for all $R>0$,

$$\varliminf_{k\to\infty} \lim_{n\to\infty} \int_{|x-x_{1,n}|\leqslant R\lambda_{1,n}\log(1/\lambda_{1,n})} (|\nabla u(t_{k,n},x)|^2+|\partial_t u(t_{k,n},x)|^2)\, dx \geqslant \int_{\mathbb{R}^N} |\nabla W|^2\, dx,$$

where $t_{k,n}=t_n+t_{1,n}+\tau_k\lambda_{1,n}$, which gives the limsup result.

For the liminf result, note that

$$E((U_1,U_{1t}))\geqslant E((W,0))=\frac{1}{N}\int_{\mathbb{R}^N}|\nabla W|^2\,dx,$$

because of Theorem 1.1, which gives

$$\int_{\mathbb{R}^N}(|\nabla U_1(t)|^2+|\partial_t U_1(t)|^2)\,dx\geqslant\int_{\mathbb{R}^N}\frac{2}{N}|\nabla W|^2\,dx,\quad t\in(-T_-(U_1,U_{1t}),T_+(U_1,U_{1t})).$$

(Note that if $E((U_1,U_{1t}))<E((W,0))$ we obtain, from Theorem 1.1, the stronger inequality $\int_{\mathbb{R}^N}(|\nabla U_1(t)|^2+|\partial_t U_1(t)|^2)\,dx\geqslant\int_{\mathbb{R}^N}|\nabla W|^2\,dx$.) The finite speed of propagation (Lemma 2.17) now gives, uniformly in n, that

$$\varliminf_{R\to\infty}\int_{|x|\leqslant|t_{1,n}|/\lambda_{1,n}+R}\left(\left|\nabla U_1\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right)\right|^2+\left|\partial_t U_1\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right)\right|^2\right)dx\geqslant\frac{2}{N}\int_{\mathbb{R}^N}|\nabla W|^2\,dx.$$

From (4.2)–(4.4) and using the fact that for all $R_0,R>0$,

$$\frac{R_0}{\lambda_{1,n}}\geqslant\frac{|t_{1,n}|}{\lambda_{1,n}}+R,$$

for n large, we obtain that, for all $R_0>0$,

$$\varliminf_{n\to\infty}\int_{|x-x_{1,n}|\leqslant R_0}(|\nabla u(t_n)|^2+|\partial_t u(t_n)|^2)\,dx\geqslant\frac{2}{N}\int_{\mathbb{R}^N}|\nabla W|^2\,dx,$$

which gives the liminf result. □

Remark 7.6. The proof of the limsup result applies verbatim to the non-linear Schrödinger equation, thus completing the sketch of the proof of the limsup part of the result in [19, Corollary 5.18]. For the proof of the liminf result, we relied here on the finite speed of propagation. The corresponding result for the non-linear Schrödinger equation, claimed also in [19, Corollary 5.18] is not fully proved there and its validity remains an open question. We are grateful to R. Killip, M. Visan and X. Zhang for pointing this out.

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Carlos E. Kenig
Department of Mathematics
University of Chicago
Chicago, IL 60637
U.S.A.
cek@math.uchicago.edu

Frank Merle
Département de Mathématiques
Université de Cergy-Pontoise
FR-95302 Cergy-Pontoise
France
merle@math.u-cergy.fr

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