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The biharmonic Neumann problem in Lipschitz domains

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1. Introduction

Let $\Omega\subset\mathbf{R}^n$, $n\geqslant 2$, be a bounded Lipschitz domain with connected boundary $\partial\Omega$. The main purpose of this article is to solve the following Neumann problem for the biharmonic equation in Lipschitz domains:

$$\Delta^2 u=0, \tag{1.1}$$

$$\begin{gathered}\nu\Delta u+(1-\nu)\frac{\partial^2 u}{\partial N^2}=f_0,\\ \frac{\partial\Delta u}{\partial N}+(1-\nu)\frac{1}{2}\frac{\partial}{\partial T_{ij}}\left(\frac{\partial^2 u}{\partial N\,\partial T_{ij}}\right)=\Lambda_0.\end{gathered}\tag{1.2}$$

Here f_0 is prescribed in an appropriate Lebesgue space $L^p(\partial\Omega)$ with respect to surface measure ds, Λ_0 is a linear functional prescribed in the dual space to the Sobolev space $W^{1,p'}(\partial\Omega)$ with respect to surface measure, and ν is a constant known as the Poisson ratio. A unique solution u (modulo linear functions) is obtained in the class of solutions with nontangential maximal function of the second-order derivatives in $L^p(\partial\Omega)$.

The letter N denotes the outer unit normal vector to the domain, and T various tangential directions to the Lipschitz boundary. The components of these vectors are not better than bounded measurable functions. If the Poisson ratio takes the value 1, the problem is not well-posed. Consequently the second- and third-order directional derivatives in (1.2) are always present. The second-order derivatives, formed with the Hessian matrix for u, do not include differentiations of the component functions. The third directional differentiation does, in the sense of distributions.

The quantities in the boundary operators depend only on the local Lipschitz geometry of the domain. Because this geometry is measured in a scale-invariant way, estimates for the Neumann problem (1.1) and (1.2) must also be scale invariant if they are to depend only on the geometry. Moreover, the boundary operators are independent of any particular choice of orientation for the rectangular coordinate system. The boundary operators are intrinsic to the geometry of the boundary.

Compact polyhedral domains of $\mathbf{R}^n$, $n\geqslant 3$, offer a similar but slightly different setting, in which solving a Neumann problem like this can be considered. The geometry there also demands scale-invariant estimates for scale-invariant equations such as (1.1). But unlike Lipschitz domains the boundaries of generic polyhedra are not locally graphs of functions, even when the boundary is (topologically) a manifold. There is no orientation of a coordinate system to exploit. That the problem (1.1) and (1.2) is formulated in a way that is independent of the local graph property of Lipschitz domains suggests that it might also be solvable in polyhedra.

The above Neumann problem has another trait, established below in Lipschitz domains, which seems to be needed in the polyhedral setting. Any exceptional subspaces of data, for which uniqueness or existence or estimates fail, not only are finite but can be described explicitly and seen to be independent of the domain. The subspace of linear solutions is a typical example. In contrast, a problem involving oblique derivatives, perhaps from a smooth vector field in order to avoid some of the nondifferentiable quantities in (1.2), is unlikely to yield such definite information.

These distinctive features are shared with the Neumann problem for Laplace's equation and have been used to study that problem in polyhedra in recent joint work with A. L. Vogel [45]. There the regions near nongraph corners and edges are decomposed dyadicly into similar Lipschitz polyhedra at all (vanishing) scales. Use is made, therefore, of the orientation invariance of the problem and the scale invariance of the Lipschitz domain estimates. The constant functions are the only exceptional space of solutions and are seen to not enter into any of the estimates. In contrast, if finite-dimensional exceptional spaces of solutions were allowed to accumulate from the local analysis of each corner and edge, one might very well conclude that the problem is Fredholm in a compact polyhedron, but such a conclusion would not seem very meaningful. After all, the Neumann problem is a variational problem. In the end only the constant solutions should be exceptional, especially in domains that would seem to present only a finite number of difficulties.

The Neumann problem here is variational. It is a straightforward generalization, to higher dimensions and nonsmooth domains, of Gustav Kirchhoff's solution to the problem of modelling small deflections of a thin elastic plate with free edges. Our most immediate source was an article by J. Giroire and J.-C. Nédélec [17], where the problem is considered in smooth planar domains and solutions studied in the class $W^{2,2}(\Omega)$. Their results, which center around obtaining integral representations with integrable kernels, are extended to polygonal domains in [28]. In this same vein the problem appears in [2, pp. 679–681]. Each of the first two articles refers to "Kirchhoff's hypothesis".

S. Agmon [1], also working in the plane with the Hilbert space approach, exhibited the inhomogeneous version of (1.1) with vanishing Neumann data and ν outside the closed interval $[-3, 1]$ as an example of a regular self-adjoint boundary value problem that fails to have its eigenvalues confined to a half-line. See §21 below. Earlier Å. Pleijel [34] had considered the eigenvalue problem with $0 \leqslant \nu < 1$, and he refers to the article of Friedrichs [15], where the boundary operators can be found on pp. 224–225. In [35, pp. 415 and 426], systems of integral equations of the second kind (singular integrals) on a smooth closed arc are proposed for the Dirichlet problem (clamped edges), the problem of supported edges, and the Neumann problem. Completely solving the systems is described as "evasive", and solutions are limited to those used in constructing Green functions after a method of H. Weyl. The classic engineering text [27, pp. 106–116 and 251–252] contains a fascinating discussion on the mathematical history of the problem solved by Kirchhoff, together with its current formulation due to Kelvin and Tait.

In this article solutions are shown to exist with derivatives up to second order that converge pointwise nontangentially a.e. (ds) and in $L^p(\partial\Omega)$. The third-order data is shown to converge in the sense of distributions (using parallel approximating boundaries) in the space $W^{-1,p}(\partial\Omega)$ dual to $W^{1,p'}(\partial\Omega)$. The analysis here is basically $p=2$, but a perturbation of all estimates to a small interval about $p=2$ is shown to depend only on the Lipschitz geometry of the domain, and solvability there also follows. The optimal range for p, which from known results will also depend on dimension, must be investigated elsewhere.

As with harmonic functions, the data for the biharmonic Neumann problem is dual to the data $(u, -\partial u/\partial N) \in W^{1,p'} \times L^{p'}$ for the corresponding Dirichlet problem. Also, as in the harmonic case, solutions to the Neumann problem coincide with those for the biharmonic *regularity* problem. This last problem is also a Dirichlet problem, but with data prescribed in a one-derivative-smoother space of functions than $W^{1,p'} \times L^{p'}$. See §17. The higher-order Dirichlet and regularity problems are well understood in Lipschitz domains [33], [8]. The analysis here shows for the first time that the variational dual to the Dirichlet problem for higher-order elliptic equations, with distributional highest-order data, can be solved in the strong sense of pointwise nontangential limits at the boundary.

This is begun with the observation that for harmonic functions in Lipschitz domains, not only are there two Dirichlet problems that can be solved in the strong pointwise sense, but there are two Neumann problems as well. The first, with data in $L^2(\partial\Omega)$, was solved by D. S. Jerison and C. E. Kenig [21]. This was extended by B. E. J. Dahlberg and C.E. Kenig to include data from the L^p-spaces, $1<p<2+\varepsilon$ [7]. The second Neumann problem takes its data in the space of bounded linear functionals $W^{-1,p}(\partial\Omega)$. Its solutions coincide with the harmonic solutions to the Dirichlet problem with data in $L^p(\partial\Omega)$, inducing an isomorphism between $W^{-1,p}(\partial\Omega)$ and $L^p(\partial\Omega)$. This is the content of Proposition 4.2, which gives a concrete and independent representation for the spaces of data used in the biharmonic Neumann problem. Its proof is an application of results on the harmonic Dirichlet, Neumann and regularity problems [6], [21], [22], [7], [41].

Next, the isomorphism makes it possible to interpret the biharmonic analogue of Jerison and Kenig's Rellich formula [21] as an energy estimate, for solutions, on the boundary in terms of the data (1.2). The simple algebraic–geometric step (6.8) that leads to the $(1-\nu)$-terms of (1.2) is the familiar decomposition of derivatives on the boundary into normal and tangential components. As shown later in the proof of Theorem 7.7, the formula (6.9) is a genuine Rellich formula, like the harmonic one, in that the regularity data and Neumann data are shown by it to be equivalent in norm. This is not the case with the earlier higher-order boundary energy estimates [43], [33] that led to solutions of the Dirichlet and regularity problems. It is notable that these earlier formulas depended on existence of conjugate solutions produced by integrations in the domain up to the boundary in a direction transverse to the boundary. This very useful idea, which originated in [39], plays no role here and is not possible in polyhedra.

The biharmonic Rellich formula is proved for $(1-n)^{-1}\leqslant\nu<1$, and the Neumann problem is solved in this same range. The customary range for plate problems seems to be $0<\nu<\frac{1}{2}$ ([20, p. 99], [14, p. 118] and [4, p. 129]). It is apparently tied to 3-dimensional considerations. In [18, p. 63] the right endpoint is said to correspond to incompressible materials, while on p. 167 the left endpoint is said to correspond to 1-dimensional motion. Both correspondences are corroborated by the nice illustration on p. 126 of [4], which shows ν measuring the relative *decrease* in diameter to relative *increase* in length when an elastic cylinder is deformed axially. If volume were to decrease when length is increased (i.e. increase if the cylinder is axially compressed), calculating $1>(1+\Delta L/L)(1+\Delta D/D)^2$ shows that infinitesimally

$$\frac{-\Delta D/D}{\Delta L/L}=\nu>\frac{1}{2}.$$

Some of the authors cited above have worked in a range up to $\nu=1$ when $n=2$. It is mathematically possible to do so, and corresponds to replacing volume with area in the above calculation. Likewise, *negative* values for the Poisson ratio are dealt with here. Physically these imply simultaneous increases in an axially deformed cylinder's diameter and length. The last chapter of [18] describes a material called *antirubber* that behaves in this way, invented by Roderic Lakes of the University of Iowa.

A priori estimates for the Neumann problem follow from the Rellich identity. Because pointwise limits for solutions to the biharmonic regularity problem are known, it would be possible by limiting arguments to identify these solutions with weak solutions to the Neumann problem obtained via the Lax–Milgram lemma in the manner of [21] and [22]. The alternative argument used here is based on the second major result of this article. The a priori estimates are applied to biharmonic layer potentials, both double and single, and invertibility on Lipschitz boundaries of layer potentials associated with higher-order equations is established for the first time. In fact, given the spaces of data and $\Omega\subset\mathbf{R}^n$, the results here seem new even for smooth domains (Theorem 11.3). Previously a systematic use of layer potentials on Lipschitz boundaries was confined to second-order equations and systems [41], [10], [13], [16], [26].

One justification for using the layer-potential approach to solve the Neumann problem is that it is theoretically possible to do so without first knowing the solution to the Dirichlet problem. By solving the integral equations for the Neumann problem one obtains a solution to the Dirichlet problem because of a duality between the boundary operators for the two problems. This is shown to work for the biharmonic layer potentials in §14, though the Dirichlet problem is already solved [9]. None of the biharmonic problems discussed here are as yet understood in polyhedra, however.

Pointwise limits of derivatives of potentials, acting on L^p-functions and linear functionals from the $W^{-1,p}$-spaces, are shown in §8 to follow from Coifman–McIntosh–Meyer [5]. The singular integrals of theirs that appear here are shown to map between the $W^{-1,p}$-spaces, in the sense of distributions. In addition, Lemma 8.1 proves the normal derivative of the classical double-layer potential to be invertible from $L^p_0(\partial\Omega)$ to $W^{-1,p}_0(\partial\Omega)$. Continuity and jump discontinuity across the boundary when the boundary operators (1.2) are applied to the biharmonic single-layer potential (9.1) are analyzed. Results precisely analogous to the classical harmonic case are obtained, resulting in a system of integral equations of the second kind, (9.6).

As in the harmonic case the method of compact operators does not apply on Lipschitz boundaries to solve (9.6). However, the equivalence in norm between Neumann and regularity data that followed from the Rellich identity allows the method of [41]. The spaces of data $\mathbf{X}^p$ analogous to the functions of mean value zero for the harmonic Neumann problem are introduced in §10, together with a reduced space for the case $\nu=(1-n)^{-1}$. The algebraic kernels of the boundary operators are precisely described and closed range established. Complete descriptions of the ranges for the various systems (interior, exterior, ν-dependence, p-dependence) for all $n\geqslant 2$ follows by continuity methods from the previously mentioned solution of the integral equations in smooth domains. For this latter result the Riesz–Schauder theory applies. However, care must be taken because not only are there identity operators and compact operators, but there are also Hilbert-transform and Riesz-transform operators involved. A perturbation argument of A. P. Calderón is used to pass from $p=2$ to $2-\varepsilon<p<2+\varepsilon$, and the complete solution of the integral equations is stated in Theorem 12.1.

As also discussed in the next section. Calderón's argument, the Rellich identity and [5] make it transparent that the estimates on solutions here depend only on p, ν and the Lipschitz geometry of the domain. The biharmonic Neumann problem is stated and solved for interior data in in this way in Theorem 13.2.

The biharmonic analogue of the classical double-layer potential is defined in the following section, and all solutions to the Dirichlet problem, $2-\varepsilon<p<2+\varepsilon$, are shown to be represented by it.

The exterior Neumann problem is resolved in Theorem 15.4, with particular attention paid to rates of decay and exceptional spaces of solutions. For example, the harmonic analogue of Theorem 15.2 would state that *in the plane a harmonic function with vanishing Neumann data and* $o(|X|)$ *at infinity must be constant.*

§§16–20 are devoted to completing the biharmonic layer-potential theory in analogy to that for harmonic potentials in all dimensions and for $2-\varepsilon<p<2+\varepsilon$. This includes analyzing both the single-layer potential and the double-layer potential as invertible operators that can be used to solve the problem of regularity for the Dirichlet problem, as well as establishing the biharmonic analogues of certain classical operator identities, e.g. (18.4). The regularity problem is described in §17, and the biharmonic single layer is shown to map onto all possible regularity data when $n\geqslant 3$ and $p=2$ in Theorem 17.5. As known from the harmonic case, potential-theoretic arguments reduce the question of range to that of uniqueness. Here is where the lack of decay of the biharmonic fundamental solution in dimensions 2, 3 and 4 becomes of concern, with Remark 16.3 showing why $n\geqslant 5$ poses no difficulties. In the low dimensions we begin by using the Kelvin transform in order to identify biharmonic analogues to the classical equilibrium distribution from potential theory. See [19], [36], [46] and Remark 4.4 below. In the harmonic case this is the density in the single layer that produces the constant solution. In the biharmonic case, linear (affine) solutions. See Definitions 16.1 and 17.1. A classical theorem of Hadamard, on the biharmonic Green function, is produced here in Lipschitz domains (Remark 16.8).

A good deal of effort is devoted to this kind of low-dimensional precision throughout the article. In polyhedral domains the harmonic Neumann problem is understood in the strong pointwise sense only in dimensions 3 and 4, dimension 3 being somewhat critical as far as estimates are concerned and dimension 4 somewhat interesting topologically [44], [45]. This is one motivating factor. Dimension 2 is examined in detail because it is in classical plate problems that biharmonic functions seem to have their one direct application to physics.

The distinction between $\nu=(n-1)^{-1}$ and $\nu>(n-1)^{-1}$ in the various theorems is slight. It is interesting, however, that below this endpoint, $p=2$ counterexamples in Lipschitz domains can be constructed. This is done in Lemmas 21.1 and 21.2. The example there also illuminates the meaning of Neumann data taken in the sense of distributions. An interesting relationship between the type of solvability here, the Lopatinskiĭ–Shapiro conditions, and classical coercivity estimates, is also indicated in that section, based on a reading of Agmon's article.

The article ends somewhat where it began, with the second Neumann problem mentioned above, solved for biharmonic functions and named *subregularity for the Neumann problem*. An example is given that uses its boundary value operators in order to formulate the Neumann problem in Lipschitz domains for a sixth-order operator.

2. Some conventions

Points of $\mathbf{R}^n$ will generally be denoted by X, Y and Z, with components $X=(X_1, ..., X_n)$. Lebesgue measure in $\mathbf{R}^n$ is written dX. Euclidean distance between sets of points will often be denoted by $\operatorname{dist}(\cdot,\cdot)$. Partial derivatives $\partial/\partial X_j$ will usually be written D_j, and the gradient operator $\nabla=(D_1, ..., D_n)$. If $\alpha=(\alpha_1, ..., \alpha_n)$ is a differentiable vector field, its divergence will be denoted $\operatorname{div}\alpha=D_1\alpha_1+...+D_n\alpha_n=D_j\alpha_j$, where repeated indices generally indicate summation notation $j=1, ..., n$. Double summation notation will also be used as in

$$\frac{\partial}{\partial T_{ij}}\left(\frac{\partial}{\partial T_{ij}}\right).$$

The Kronecker δ is denoted by δ_{ij}. The operator $\Delta=\operatorname{div}\nabla$ denotes the Laplacian and Δ^2 the bi-Laplacian. Γ^X and B^X are used for the fundamental solutions for the Laplacian and the bi-Laplacian, respectively, with pole at X (§8), and ω_n is the surface area of the unit sphere $\mathbf{S}^{n-1}\subset\mathbf{R}^n$. All second derivatives of u will be denoted by $\nabla\nabla u$, which, more precisely, will stand for the Hessian matrix of u, especially in the Hilbert–Schmidt norm $|\nabla\nabla u|^2=D_iD_ju\,D_iD_ju$. The inner product $\nabla u\cdot\nabla v=D_iu\,D_iv$ will be used, and directional derivatives $\alpha\cdot\nabla u$ or $T_{ij}\cdot\nabla u=\partial u/\partial T_{ij}$ will not necessarily mean that the vector field is of unit length. However, N always denotes an *outer unit normal* vector to a *bounded* domain. Closure of Ω is indicated by $\bar{\Omega}$, and $\bar{\Omega}^c$ denotes its open complement. Points on the boundary $\partial\Omega$ will generally be written P and Q (but sometimes X and Y). Function spaces on the boundary are with respect to surface measure ds. Generally, f, g and F denote $L^p(\partial\Omega)$-functions or $W^{1,p}(\partial\Omega)$-Sobolev functions, and Λ denotes linear functionals from the dual space $W^{-1,p'}(\partial\Omega)$ (§4). Dual exponents are defined by $p+p'=pp'$. The notation $\|\cdot\|_p$ always indicates an L^p-norm on the *boundary*. C_0^∞ denotes compactly supported infinitely differentiable functions of $\mathbf{R}^n$, while the 0-subscript in L_0^p, $W_0^{1,p}$, etc. indicates *mean value zero*. By *linear functions* is meant polynomials of degree one. *Affine* is used only when distinguishing certain equilibrium distributions.

3. Lipschitz domains

A bounded domain Ω of $\mathbf{R}^n$ is a *Lipschitz domain* if for each $Q\in\partial\Omega$ there is a rotation of the Euclidean coordinate system of $\mathbf{R}^n=\mathbf{R}^{n-1}\times\mathbf{R}$ and a neighborhood $\mathcal{N}$ of Q such that $\mathcal{N}\cap\partial\Omega$ equals the intersection of $\mathcal{N}$ with the *graph* of a real-valued Lipschitz function defined on $\mathbf{R}^{n-1}$ [6]. It follows that surface (Lebesgue) measure ds is well-defined as well as are normal vectors N a.e. (ds). See Denjoy, Rademacher and Stepanov's theorem [38, p. 250]. It is possible to quantify the *Lipschitz geometry* or *Lipschitz nature* or *character* of the domain by the (local) Lipschitz norms and the finite number of neighborhoods $\mathcal{N}$ needed to cover $\partial\Omega$ (see, for example, [33, p. 21]). In particular, *the Lipschitz character remains uniform over the domain approximation scheme below*, first shown to exist by Nečas [29]. See [41, p. 581] and [33, p. 23] for a more precise statement.

Let Ω be a bounded Lipschitz domain with normal vector field N and surface measure ds.

Definition 3.1. A *sequence of smooth* (C^∞) *approximating domains* $\Omega_j\subset\Omega$ (or $\Omega_j\supset\bar{\Omega}$) has the properties that

(i) each $\partial\Omega_j$ is homeomorphic to $\partial\Omega$, with $Q^{(j)}\in\partial\Omega_j$ mapped to $Q\in\partial\Omega$ only if $Q^{(j)}$ is contained in the nontangential approach region (see below) for Q;

(ii) Euclidean distance between points under the homeomorphism vanishes *uniformly* as $j\to\infty$;

(iii) the normal vectors $N^{(j)}$ converge, when mapped by the homeomorphisms, in every $L^p(\partial\Omega)$, $p<\infty$, and pointwise a.e. (ds) to N (and similarly for the naturally defined tangent vectors);

(iv) the Jacobian determinants under the homeomorphisms are uniformly bounded, bounded away from zero, and converge in every $L^p(\partial\Omega)$, $p<\infty$, and pointwise a.e. (ds) to 1;

(v) there are $C^\infty(\mathbf{R}^n)$-vector fields α that can be constructed to depend only on the Lipschitz geometry of Ω, and a constant $C>0$ depending only on the Lipschitz geometry, such that $\alpha\cdot N^{(j)}\geqslant C$ uniformly in j and points of $\partial\Omega_j$.

Recall that the *nontangential approach region* for each $Q\in\partial\Omega$ is

$$A(Q)=\{X\in\Omega:\operatorname{dist}(X,\partial\Omega)>\beta|X-Q|\},$$

where $0<\beta<1$ is fixed small enough depending on the Lipschitz nature of Ω. The regions are also defined for $\bar{\Omega}^c$. Given a (perhaps vector-valued) function F in Ω the *nontangential maximal function* at Q is defined by $N(F)(Q)=\sup_{X\in A(Q)}|F(X)|$, $Q\in\partial\Omega$. By *nontangential limits*, when they exist, are meant $\lim_{X\to Q;\, X\in A(Q)} F(X)$, for $Q\in\partial\Omega$.

There is an $\varepsilon>0$ depending only on the Lipschitz geometry of Ω so that existence and uniqueness for harmonic functions in the sense of nontangential limits are known, as referenced in the introduction, for the Dirichlet problem in the class $N(u)\in L^p(\partial\Omega)$, $2-\varepsilon<p\leqslant\infty$; and for the Neumann and regularity problems in the class $N(\nabla u)\in L^p(\partial\Omega)$, $1\leqslant p<2+\varepsilon$. Thus the interval $2-\varepsilon<p<2+\varepsilon$ (or $p<2+\varepsilon$ or $2-\varepsilon<p'$, etc.) will be written frequently with the understanding that the ε depends only on the Lipschitz geometry of Ω, either as shown here or as derived from these theorems on harmonic functions.

4. Representation of dual spaces for Sobolev spaces on Lipschitz boundaries

Let $\Omega\subset\mathbf{R}^n$ be a Lipschitz domain. The Sobolev spaces $W^{1,p}(\partial\Omega)$ with weak first derivatives in $L^p(\partial\Omega)$, $1\leqslant p\leqslant\infty$, can be defined in a global fashion by saying that $f\in W^{1,p}(\partial\Omega)$ if and only if there exist functions $g_{jk}\in L^p(\partial\Omega)$ so that for all $\psi\in C_0^\infty(\mathbf{R}^n)$,

$$\int_{\partial\Omega}(N_jD_k-N_kD_j)\psi f\,ds=-\int_{\partial\Omega}\psi g_{jk}\,ds,\quad 1\leqslant j<k\leqslant n,\tag{4.1}$$

and so that compatibility conditions $\left(\binom{n-1}{2}\text{ of which are independent}\right)$

$$N_l g_{jk}=N_k g_{jl}-N_j g_{kl},\quad 1\leqslant j<k<l\leqslant n,$$

are satisfied. Let Ω be a bounded domain and denote by $|\partial\Omega|$ the surface measure of its boundary. Then $W^{1,p}(\partial\Omega)$ is a Banach space with norm

$$\|f\|_{1,p}=|\partial\Omega|^{1/(1-n)}\|f\|_p+\sum_{1\leqslant j<k\leqslant n}\|g_{jk}\|_p,\tag{4.2}$$

where $\|\cdot\|_p$ will always denote the L^p-norm on the boundary $\partial\Omega$ with respect to surface measure. Because the boundary is locally a graph, it can be seen that this definition is the same as that defined by flattening the boundary to $\mathbf{R}^{n-1}$ in order to lift the Sobolev spaces defined in Euclidean space to the boundary. See [45].

When Ω has connected boundary, the Sobolev spaces $W_0^{1,p}(\partial\Omega)$ of $W^{1,p}(\partial\Omega)$-functions with mean value zero will also be considered. By the Poincaré inequality the norm (4.2) on this space is equivalent to $\|f\|_{1,p}-|\partial\Omega|^{1/(1-n)}\|f\|_p$ by constants depending only on the Lipschitz geometry of Ω.

Define the vectors $T_{jk}=(0,...,0,-N_k,0,...,0,N_j,0,...,0)$, where only the jth and kth components can be nonzero, $1\leqslant j<k\leqslant n$. Put $T_{kj}=-T_{jk}$, $1\leqslant j<k\leqslant n$. The notation T_{jj} will stand for the zero vector. Denote the tangential derivatives in (4.1) by $\partial/\partial T_{jk}$.

The dual spaces $W^{-1,p'}(\partial\Omega)$ are defined to be the Banach spaces of bounded linear functionals $\Lambda: W^{1,p}(\partial\Omega)\to\mathbf{R}$ with norm $\|\Lambda\|_{-1,p'}=\sup\{\langle\Lambda,f\rangle:\|f\|_{1,p}=1\}$. Let $W_0^{-1,p'}$ likewise indicate the functionals on $W_0^{1,p}(\partial\Omega)$. When $\Lambda\in W_0^{-1,p'}$ it will always be understood that Λ maps the constant function to zero.

Definition 4.1. Let Ω be a Lipschitz domain and $\Omega_j\subset\Omega$ a sequence of smooth approximating domains (Definition 3.1). Let F be a continuous function in Ω with nontangential limits f a.e. (ds) on $\partial\Omega$. Given a harmonic function h in Ω, the normal derivative *in the sense of distributions* $\partial h/\partial N$ will be said to *act on* f *in the sense of distributions* by

$$\int_{\partial\Omega}\frac{\partial h}{\partial N}f\,ds=\lim_{j\to\infty}\int_{\partial\Omega_j}\frac{\partial h}{\partial N}F\,ds$$

if the limit exists, where N denotes the outer unit normal on the boundary over which the integral is taken, and ds denotes surface measure on the boundary over which the integral is taken.

The following proposition identifies in this way certain harmonic functions with functionals $\Lambda\in W^{-1,p'}(\partial\Omega)$. For $f\in W^{1,p}$, we will write $\langle\Lambda,f\rangle=\int_{\partial\Omega}\Lambda f\,ds$ with the understanding that in general the integral is defined only in the above sense of distributions.

When Ω is Lipschitz, the pairing of Sobolev spaces and their duals on the boundary has the following representation:

PROPOSITION 4.2. *Let $\Omega\subset\mathbf{R}^n$, $n\geqslant 2$, be a bounded Lipschitz domain with connected boundary. Let $1<p'<2+\varepsilon$ and $p+p'=pp'$, where $\varepsilon=\varepsilon(\Omega)>0$ is such that the p'-Neumann and regularity problems are uniquely solvable. Then*:

(i) *Given $\Lambda\in W_0^{-1,p}(\partial\Omega)$ there exists a unique harmonic function h in Ω with $N(h)\in L^p(\partial\Omega)$ and $h\in L_0^p(\partial\Omega)$ such that $\Lambda=\partial h/\partial N$ in the sense of distributions, and so that given any $f\in W_0^{1,p'}(\partial\Omega)$ and its Poisson extension $\mathcal{P}(f)$,*

$$\langle\Lambda,f\rangle=\int_{\partial\Omega}\Lambda f\,ds=\int_{\partial\Omega}\frac{\partial h}{\partial N}f\,ds=\int_{\partial\Omega}h\frac{\partial\mathcal{P}(f)}{\partial N}\,ds. \tag{4.3}$$

Any such h supplies a linear functional $\Lambda_h \in W_0^{-1,p}(\partial\Omega)$, and the map

$$h \longmapsto \Lambda_h : L_0^p(\partial\Omega) \longrightarrow W_0^{-1,p}(\partial\Omega) \tag{4.4}$$

is an isomorphism with bounds depending only on the Lipschitz geometry of Ω.

(ii) *For $n \geqslant 3$, given $\Lambda \in W^{-1,p}(\partial\Omega)$ there exists a unique harmonic function h in $\bar{\Omega}^c$ with $N(h) \in L^p(\partial\Omega)$ so that $\Lambda = \partial h/\partial N$ in the sense of distributions, with* (4.3) *holding for all $f \in W^{1,p'}(\partial\Omega)$, and $\mathcal{P}(f)$ uniquely determined by $\mathcal{P}(f)(X) = O(|X|^{2-n})$ at infinity.*

Any such h supplies a linear functional Λ_h, and the map

$$h \longmapsto \Lambda_h : L^p(\partial\Omega) \longrightarrow W^{-1,p}(\partial\Omega) \tag{4.5}$$

is an isomorphism with bounds depending only on the Lipschitz geometry of Ω.

(iii) *For $n=2$ everything in statement* (ii) *holds with the exception that for h with constant boundary values on $\partial\Omega$ the nontangential maximal function must be defined with truncated cones, i.e. there is a 1-dimensional subspace of harmonic functions needed for the isomorphism which behave like* $\log|X|$ *at infinity.*

Proof. For (i), referring to Definition 4.1, Green's second identity in each Ω_j, together with nontangential estimates and pointwise limits from [6], [22] and [41] for the Dirichlet and Dirichlet regularity problems justifying Lebesgue dominated convergence, shows that any h as described satisfies the third equality in (4.3) and thus supplies a bounded linear functional.

Given any Λ, the linear map $\partial\mathcal{P}(f)/\partial N \mapsto \langle \Lambda, f\rangle$, when $f \in W_0^{1,p'}(\partial\Omega)$, is a map on $L_0^{p'}(\partial\Omega)$-functions by solvability of the Dirichlet regularity problem [41], and is bounded and defined on *all* of $L_0^{p'}(\partial\Omega)$ by the solvability of the Neumann problem [21], [7]. Thus by duality of Lebesgue spaces and solvability of the Dirichlet problem [6], there is a unique h with boundary values in $L_0^p(\partial\Omega)$ that represents this map and therefore the map $f \mapsto \langle \Lambda, f\rangle$ by (4.3). The L^p-estimates for the cited harmonic boundary value problems depend only on the Lipschitz nature of Ω, as then do the bounds in the isomorphism (4.4).

For the case $n=2$ in the exterior domain, a number of facts from pp. 592–598 of [41] are useful. Let $k(g)$ denote the classical (harmonic) double-layer potential of a density g defined on the boundary. Let $s(f)$ denote the classical single layer, and let c denote a constant function. There is a unique nonnegative function f^* depending on Ω and satisfying $\|f^*\|_2 = 1$ such that $s(f^*)$ is constant in Ω and is $O(\log|X|)$ at infinity (f^* is known as the *equilibrium distribution* [46], [19]). It can happen that the constant value in Ω is zero (for *any* domain depending on how the plane is scaled). The boundary values of $s(f^*)+c$ then, for an appropriately chosen c, together with those of $s(f)$ for $f \in L_0^{p'}(\partial\Omega)$ span $W^{1,p'}(\partial\Omega)$. The single layer $s(f)$ is $O(|X|^{-1})$ at infinity. The boundary values of $k(g)$, when taken form the exterior domain, always integrate against f^* to yield zero, and $k(1)$ is identically zero in the exterior domain. For $g \in L_0^p(\partial\Omega)$, the $k(g)$ together with $s(f^*)+c$ span $L^p(\partial\Omega)$. The space $W_0^{-1,p}(\partial\Omega)$ is spanned by the $\partial k(g)/\partial N$, while

$$\frac{\partial s(f^*)}{\partial N} = f^*, \tag{4.6}$$

so that $W^{-1,p}(\partial\Omega)$ is spanned. In proving the third identity of (4.3), only the pairing of $\partial(s(f^*)+c)/\partial N$ with $s(f^*)+c$ lacks the needed decay at infinity. But no integration by parts is needed in this case. Thus (iii) follows in the same manner as did (i), and the proof of (ii) will be left to the interested reader. □

Remark 4.3. Taking h to be the constant function in (ii) and (iii) does not violate uniqueness because h will not satisfy (4.3).

Remark 4.4. In any dimension for any domain of Proposition 4.2, there is a unique function f^* with norm 1 such that the single layer of f^* is constant. This equilibrium distribution function is in (and in general no better than) $L^{p'}(\partial\Omega)$ for $p'<2+\varepsilon$ and is pointwise nonnegative. For $h \in L_0^p$ as in (i) of the proposition one can define

$$\tilde{h} = h - \frac{\int_{\partial\Omega} hf^*\,ds}{\int_{\partial\Omega} f^*\,ds}$$

and show that h and $\tilde{h}$ have equivalent norms depending only on f^* (and thus Ω). Consequently $\tilde{h} \mapsto \Lambda_{\tilde{h}} = \Lambda_h$ is an isomorphism. Biharmonic equilibrium distributions will be discussed in §16. In fact, by adapting the analysis there to the harmonic case one can see that

f^* is the Kelvin transform of the density of harmonic measure for the bounded domain obtained by reflecting $\bar{\Omega}^c$ in the unit sphere.

Remark 4.5. The statement in the proof, that $k(f)$ from the exterior integrates to zero against f^*, is true in all dimensions. By the invertibility properties of the trace of the double-layer potential (see, for example, Corollary 4.4 (iv) of [41]) this property suffices for a harmonic function h to have a layer-potential representation in the exterior domain. This leads to the following proposition:

If a harmonic function $h = O(|X|^{1-n})$ at ∞, then h is represented by a double-layer potential.

This follows because the decay and (4.6) show that h must integrate to zero against f^* by

$$\int_{\partial\Omega} h \frac{\partial s(f^*)}{\partial N}\,ds = \int_{\partial\Omega} \frac{\partial h}{\partial N} s(f^*)\,ds = 0$$

since $s(f^*)$ is constant on $\partial\Omega$ and the Neumann data for h must integrate to zero on the boundary of expanding annular domains.

Remark 4.6. Similarly,

if a harmonic function $h=O(|X|^{1-n})$ *and* $N(h)\in L^p(\partial\Omega)$, *then* $\Lambda=\partial h/\partial N\in W_0^{-1,p}(\partial\Omega)$.

Remark 4.7. The Sobolev spaces $W^{1,p'}(\partial\Omega)$ and $W_0^{1,p'}(\partial\Omega)$ for $1<p'<2+\varepsilon$ are reflexive because they can be embedded as closed subspaces of a Banach space of $\mathbf{R}^N$-valued $L^{p'}$-functions. See the functional-analytic argument, for example on p. 140 of [40]. The representation of linear functionals in Proposition 4.2 also leads to a direct argument.

Remark 4.8. For C^1-domains, $\varepsilon(\Omega)=\infty$ by [12].

It is convenient not to have the pairing (4.3) tied to the Poisson extensions of the boundary Sobolev functions.

LEMMA 4.9. *Let* $\Omega\subset\mathbf{R}^n$ *be a bounded Lipschitz domain and let* $1<p'<2+\varepsilon$. *Let* $\mathcal{N}$ *be an open neighborhood of the boundary in* $\mathbf{R}^n$. *Suppose that* f *is any* $C^1(\mathcal{N}\setminus\partial\Omega)$-*function with the properties that* f *and its gradient have nontangential limits a.e.* (ds) *on the boundary, and have nontangential maximal functions in* $L^{p'}(\partial\Omega)$. *As in Proposition 4.2 let* $\Lambda=\Lambda_h$ *be a linear functional from* $W^{-1,p}$ *and let* Ω_j *denote approximating smooth domains.*

Then

$$\langle\Lambda,f\rangle=\lim_{j\to\infty}\int_{\partial\Omega_j}\mathcal{P}(f)\frac{\partial h}{\partial N}\,ds_j=\lim_{j\to\infty}\int_{\partial\Omega_j}f\frac{\partial h}{\partial N}\,ds_j. \tag{4.7}$$

Proof. Using the interior estimate for harmonic functions that bounds $|\nabla h|$ on $\partial\Omega_j$ by $N(h)\operatorname{dist}^{-1}(\partial\Omega_j,\partial\Omega)$, the fundamental theorem of calculus that bounds $|\mathcal{P}(f)-f|$ by $N(\nabla(\mathcal{P}(f)-f))\operatorname{dist}(\partial\Omega_j,\partial\Omega)$, and Dahlberg's solution of the harmonic Dirichlet problem, we have

$$\int_{\partial\Omega_j}|\mathcal{P}(f)-f|\left|\frac{\partial h}{\partial N}\right|ds_j\leqslant C\|N(\nabla(\mathcal{P}(f)-f))\|_{p'}\|h\|_p<\infty \tag{4.8}$$

with C independent of j. Moreover, the integrand on the left is pointwise bounded by the product of the two maximal functions. It follows by dominated convergence that if ∇h is continuous at the boundary, the left-hand side vanishes in the limit. But such harmonic functions $\tilde{h}$ form a dense class in $L^p(\partial\Omega)$. Consequently,

$$\limsup_{j\to\infty}\int_{\partial\Omega_j}|\mathcal{P}(f)-f|\left|\frac{\partial h}{\partial N}\right|ds_j\leqslant\limsup_{j\to\infty}\int_{\partial\Omega_j}|\mathcal{P}(f)-f|\left|\frac{\partial h}{\partial N}-\frac{\partial\tilde{h}}{\partial N}\right|ds_j$$

for general h by the vanishing, and can then be made arbitrarily small by (4.8). □

5. Biharmonic Neumann data

Let Ω be a Lipschitz domain and let $1\leqslant p\leqslant\infty$. Let $G\in C^1(\Omega)$ with $N(G)\in L^p(\partial\Omega)$ and with nontangential limits g a.e. (ds). Let $f\in C^1(\Omega)$ with the properties that f and its gradient have nontangential limits a.e. (ds) on the boundary, and have nontangential maximal functions in $L^{p'}(\partial\Omega)$. Let Ω_i be smooth approximating domains and let T_{jk} be any of the tangent vectors defined in §4. As in Definition 4.1 we say that the *distribution* $\partial g/\partial T_{jk}$ *acts on* f *in the sense of distributions* by

$$\int_{\partial\Omega}\frac{\partial g}{\partial T_{jk}}f\,ds=\lim_{i\to\infty}\int_{\partial\Omega_i}\frac{\partial G}{\partial T_{jk}}f\,ds$$

if the limit exists. Applying the divergence theorem in each Ω_i and dominated convergence yields

$$\int_{\partial\Omega}\frac{\partial g}{\partial T_{jk}}f\,ds=-\int_{\partial\Omega}g\frac{\partial f}{\partial T_{jk}}\,ds$$

when f and G are in $C^2(\Omega)$.

It follows, given this definition, that if every $f\in W^{1,p'}(\partial\Omega)$ has an extension as described, then $\partial g/\partial T_{jk}\in W^{-1,p}(\partial\Omega)$. In agreement with Proposition 4.2 this is true for $p'<2+\varepsilon$ (i.e. $2-\varepsilon<p$) by using harmonic solutions to the regularity problem.

The *biharmonic Neumann data* for solutions to $\Delta^2u=0$ in Lipschitz domains, generalized to Euclidean spaces from the planar formulation, are then defined to be

$$M_\nu(u)=\nu\Delta u+(1-\nu)\frac{\partial^2u}{\partial N^2}\in L^p(\partial\Omega) \tag{5.1}$$

and

$$K_\nu(u)=\frac{\partial\Delta u}{\partial N}+(1-\nu)\frac{1}{2}\frac{\partial}{\partial T_{ij}}\left(\frac{\partial^2u}{\partial N\,\partial T_{ij}}\right)\in W^{-1,p}(\partial\Omega) \tag{5.2}$$

for the exponents of integrability $2-\varepsilon<p<2+\varepsilon$ when $N(\nabla\nabla u)\in L^p(\partial\Omega)$, and for the Poisson ratio $(1-n)^{-1}\leqslant\nu<1$.

Summation in (5.2) is over all $i,j=1,...,n$, and second derivatives are formed on the Hessian by, e.g., $\partial^2u/\partial N^2=N_kN_lD_kD_lu$. The factor $\frac{1}{2}$ in (5.2) is an artifact of the double summation notation. The outside tangential derivatives in $K_\nu(u)$ are taken in the sense of distributions just described, the smooth approximation scheme of Definition 3.1 allowing the second directional derivatives to be smooth inside Ω. If Ω itself is smooth, it can be given as $\{x:\Phi(x)<0\}$ with $\nabla\Phi=N$ everywhere at the boundary. The Gauss divergence theorem and change of variables then show that an integral like (summation notation)

$$\int_{\partial\Omega}\frac{\partial F}{\partial T_{ij}}\frac{\partial G}{\partial T_{ij}}\,ds$$

is invariant under rotations of the coordinate system. This and similar observations for the other types of derivatives suffice to show that *the Neumann boundary operators are formulated in a way that is intrinsic to the geometry of the boundary.*

The restrictions on p and ν are, in part, based on the existence theorems proved in this article. As shown by example in Remark 21.3, existence of solutions in Lipschitz domains with data in the above spaces must fail in general when the exponent p is larger than the upper limit shown for (5.1) and (5.2). This is in analogy to the harmonic Neumann problem and, like that problem, is not the case in smoother domains. See §11. The lower limit for p seemed natural when considering the two terms of $K_\nu(u)$ separately. But in principle it should be possible to solve the Neumann problem for

$$\frac{2(n-1)}{n+1}-\varepsilon<p<2+\varepsilon$$

given the corresponding known results on the Dirichlet and regularity problems [31] and [32], and most remarkably [37]. As proved in §21 the problem for $p=2$ must fail in general when ν is not restricted as above. It may, however, be possible to solve the Neumann problem for some p strictly below 2 depending on $-3<\nu<(1-n)^{-1}$, even though the problem for $p=2$ is not solvable for these ν.

Taken by itself the second term of $K_\nu(u)$ has norm bounded as

$$\left\|\frac{\partial}{\partial T_{ij}}\left(\frac{\partial^2 u}{\partial N\partial T_{ij}}\right)\right\|_{-1,p}\leqslant\sum_{i,j=1}^{n}\left\|\frac{\partial^2 u}{\partial N\partial T_{ij}}\right\|_p \tag{5.3}$$

when $1<p'<2+\varepsilon$. By Proposition 4.2 and Lemma 4.9, for this same range of p', if u is a solution in Ω or if $\Delta u=O(|X|^{1-n})$ as in Remark 4.6, then

$$\left|\int_{\partial\Omega}K_\nu(u)f\,ds\right|\leqslant C\|\nabla\nabla u\|_p\|\nabla_T f\|_{p'}, \tag{5.4}$$

where C depends on p, the Lipschitz nature of Ω and on ν. More generally, $\|f\|_{1,p'}$ replaces the p'-norm in (5.4) and

$$\|K_\nu(u)\|_{-1,p}\leqslant C\|\nabla\nabla u\|_p. \tag{5.5}$$

6. A biharmonic Rellich identity

Let $\Omega\subset\mathbf{R}^n$ be a bounded Lipschitz domain. Let α denote a smooth vector field of $\mathbf{R}^n$ that is transverse to $\partial\Omega$, i.e. there is a constant $C=C(\alpha,\partial\Omega)>0$ such that

$$N\cdot\alpha\geqslant C\quad\text{a.e. on }\partial\Omega. \tag{6.1}$$

Given a number $\theta\geqslant -1/n$, define the differential operators

$$L_{ij} = D_i D_j + \theta\delta_{ij}\Delta \tag{6.2}$$

for $i,j=1,\dots,n$, where δ denotes the Kronecker δ.

Let $\Delta^2 u=0$ in Ω with $N(\nabla\nabla u)\in L^2(\partial\Omega)$. Then u is a solution to the biharmonic regularity problem [43], [31], and has, together with its derivatives of first and second orders, well-defined nontangential limits a.e. on the Lipschitz boundary. Therefore with limiting arguments and summation convention, the Gauss divergence theorem in smooth approximating domains yields the first equality, while computation of derivatives yields the second in

$$\begin{aligned}
&\int_\Omega L_{ij}(\alpha\cdot\nabla u)\,L_{ij}(u)\,dX \\
&\quad= \int_{\partial\Omega} D_j(\alpha\cdot\nabla u)\left[\frac{\partial D_j u}{\partial N} + (2\theta+n\theta^2)N_j\Delta u\right] ds \\
&\qquad -(1+2\theta+n\theta^2)\int_\Omega \nabla(\alpha\cdot\nabla u)\cdot\nabla\Delta u\,dX \\
&\quad= \int_\Omega \big(L_{ij}(\alpha_k)\,D_k u\,L_{ij}(u) + 2[D_i\alpha_k D_j D_k u + \theta\delta_{ij}\nabla\alpha_k\cdot\nabla D_k u]\,L_{ij}(u)\big)\,dX \\
&\qquad + \int_\Omega \alpha\cdot\nabla(L_{ij}(u))\,L_{ij}(u)\,dX.
\end{aligned} \tag{6.3}$$

The last integral of (6.3) is equal to

$$\frac{1}{2}\int_{\partial\Omega} N\cdot\alpha\,L_{ij}(u)L_{ij}(u)\,ds - \frac{1}{2}\int_\Omega \operatorname{div}\alpha\,L_{ij}(u)L_{ij}(u)\,dX. \tag{6.4}$$

For the last integral preceding the second equality of (6.3), Lemma 4.9 implies that, in the sense of distributions,

$$\int_\Omega \nabla(\alpha\cdot\nabla u)\cdot\nabla\Delta u\,dX = \int_{\partial\Omega} \alpha\cdot\nabla u\,\frac{\partial\Delta u}{\partial N}\,ds. \tag{6.5}$$

The Poisson ratio in the range $(1-n)^{-1}\leqslant\nu<1$ is related to θ by

$$\nu = \frac{2\theta+n\theta^2}{1+2\theta+n\theta^2}, \quad \theta\geqslant -\frac{1}{n}. \tag{6.6}$$

Then (6.3), (6.4) and (6.5) yield

$$\begin{aligned}
\frac{1-\nu}{2}\int_{\partial\Omega} N\cdot\alpha\,L_{ij}(u)L_{ij}(u)\,ds = (1-\nu)\int_\Omega \big(&\tfrac{1}{2}\operatorname{div}\alpha\,L_{ij}(u)L_{ij}(u) \\
&-2[D_i\alpha_k D_j D_k u + \theta\delta_{ij}\nabla\alpha_k\cdot\nabla D_k u]\,L_{ij}(u) \\
&-L_{ij}(\alpha_k)D_k u\,L_{ij}(u)\big)\,dX - \int_{\partial\Omega}\alpha\cdot\nabla u\,\frac{\partial\Delta u}{\partial N}\,ds \\
&+\int_{\partial\Omega}\left((1-\nu)\nabla(\alpha\cdot\nabla u)\cdot\frac{\partial\nabla u}{\partial N} + \nu\nabla(\alpha\cdot\nabla u)\cdot N\Delta u\right) ds.
\end{aligned} \tag{6.7}$$

For the solid integral, $(1-\nu)\theta=n^{-1}\big(\nu-1+\sqrt{(1-\nu)(1+(n-1)\nu)}\,\big)$.

Let $v=\alpha\cdot\nabla u$ in (6.7). Then the first integrand of the last integral separates into normal and tangential parts as

$$\begin{aligned} D_j v N_i D_i D_j u &= D_j v N_k N_i (N_k D_i D_j u - N_j D_i D_k u) + D_j v N_j N_i N_k D_i D_k u \\ &= N_k D_j v \frac{\partial^2 u}{\partial N \partial T_{kj}} + \frac{\partial v}{\partial N}\frac{\partial^2 u}{\partial N^2} = -N_j D_k v \frac{\partial^2 u}{\partial N \partial T_{kj}} + \frac{\partial v}{\partial N}\frac{\partial^2 u}{\partial N^2}, \end{aligned}$$

i.e.

$$\nabla v\cdot\frac{\partial \nabla u}{\partial N} = \frac{1}{2}\frac{\partial v}{\partial T_{kj}}\frac{\partial^2 u}{\partial N \partial T_{kj}} + \frac{\partial v}{\partial N}\frac{\partial^2 u}{\partial N^2}, \tag{6.8}$$

where the identities

$$-N_j D_k v \frac{\partial}{\partial T_{kj}} = N_k D_j v \frac{\partial}{\partial T_{kj}} = \frac{1}{2}\frac{\partial v}{\partial T_{kj}}\frac{\partial}{\partial T_{kj}}$$

have been used.

In the sense of distributions the Rellich formula (6.7) becomes

$$\begin{aligned} &\frac{1-\nu}{2}\int_{\partial\Omega} N\cdot\alpha\, L_{ij}(u) L_{ij}(u)\, ds \\ &\quad = \int_\Omega E_{ij}(\alpha,u,\nu) L_{ij}(u)\, dX - \int_{\partial\Omega} \alpha\cdot\nabla u\, K_\nu(u)\, ds + \int_{\partial\Omega} \frac{\partial}{\partial N}(\alpha\cdot\nabla u) M_\nu(u)\, ds, \end{aligned} \tag{6.9}$$

where the solid integral is precisely that of (6.7). The Rellich formula (6.9) holds for Ω replaced with $\bar{\Omega}^c$ when α is compactly supported.

The boundary integrals on the right-hand side of (6.9) are bounded as

$$\begin{aligned} \left|\int_{\partial\Omega} \alpha\cdot\nabla u\, K_\nu(u)\, ds\right| &\leqslant C\|\nabla u\|_{1,2}\|K_\nu(u)\|_{-1,2}, \\ \left|\int_{\partial\Omega} \frac{\partial}{\partial N}(\alpha\cdot\nabla u) M_\nu(u)\, ds\right| &\leqslant C\|\nabla u\|_{1,2}\|M_\nu(u)\|_2 \end{aligned} \tag{6.10}$$

by $K_\nu(u)\in W^{-1,2}(\partial\Omega)$ (Proposition 4.2 and (5.3)) and the Schwarz inequality.

7. A priori estimates for the Neumann problem

It will be assumed in the proofs of the following lemmas and theorems that $|\partial\Omega|$ is of unit size. First, two lemmas for the case $\nu=(1-n)^{-1}$:

LEMMA 7.1. *When $\theta=n^{-1}$ (i.e. $\nu=(1-n)^{-1}$),*

$$L_{ij}(u)L_{ij}(u)=|\nabla\nabla u|^2-\frac{1}{n}(\Delta u)^2=\sum_{\substack{i,j=1\\ i\neq j}}^{n}(D_iD_ju)^2+\frac{1}{2n}\sum_{i,j=1}^{n}(D_i^2u-D_j^2u)^2.$$

Proof. The diagonal term from the left is

$$\begin{aligned}\sum_{i=1}^{n}L_{ii}(u)^2&=-\frac{1}{n}(\Delta u)^2+\sum_{i=1}^{n}(D_i^2u)^2\\&=\sum_{i=1}^{n}[(D_i^2u)^2-(\Delta u/n)^2]\\&=\sum_{i=1}^{n}(D_i^2u+\Delta u/n)(D_i^2u-\Delta u/n)\\&=\frac{1}{n}\sum_{i,j=1}^{n}(D_i^2u+\Delta u/n)(D_i^2u-D_j^2u)\\&=\frac{1}{n}\sum_{i,j=1}^{n}(D_j^2u+\Delta u/n)(D_j^2u-D_i^2u).\end{aligned}$$

Averaging the last two expressions gives the result. □

LEMMA 7.2. *There is a constant $C=C(n)$ so that for any j and k,*

$$\left|\frac{\partial^2u}{\partial N\partial T_{jk}}\right|^2\leqslant C(|\nabla\nabla u|^2-n^{-1}(\Delta u)^2).$$

Proof. Using summation convention in i,

$$\begin{aligned}\frac{\partial^2u}{\partial N\partial T_{jk}}&=N_i(N_jD_k-N_kD_j)D_iu\\&=N_jN_k(D_k^2u-D_j^2u)+N_j\sum_{i\neq k}N_iD_iD_ku-N_k\sum_{i\neq j}N_iD_iD_ju,\end{aligned}$$

and the inequality follows from the last lemma. □

Next follows a coercive estimate on the boundary.

LEMMA 7.3. *Let $\Omega\subset\mathbf{R}^n$ be a bounded Lipschitz domain with connected boundary.*

(i) *Let $\Delta^2u=0$ in Ω with $N(\nabla\nabla u)\in L^2(\partial\Omega)$ and $\int_{\partial\Omega}\Delta u\,ds=0$. Let $(1-n)^{-1}\leqslant\nu<1$ and let L_{ij} be defined as in (6.2) and (6.6). Then there exist constants C_1 and C_2*

independent of ν, *with* C_1 *depending on the Lipschitz nature of* Ω *and the smooth vector field* α, *and* C_2 *depending on the Lipschitz nature of* Ω, *so that*

$$\int_{\partial\Omega} |\nabla\nabla u|^2\, ds \leqslant C_1 \int_{\partial\Omega} N\cdot\alpha\, L_{ij}(u)\, L_{ij}(u)\, ds + C_2 \|K_\nu(u)\|^2_{-1,2}.$$

(ii) *The same statement holds* without *the hypothesis of mean value zero when* u *is a solution in the exterior domain.*

In both (i) *and* (ii), *when* $\nu > (1-n)^{-1}$, C_2 *may be taken to be zero and the hypothesis of mean value zero in* (i) *dropped, in which case* C_1 *diverges as* $\nu \downarrow (1-n)^{-1}$.

Proof. For u in Ω and any $1 \leqslant i \leqslant n$,

$$D_i^2 u = \frac{1}{n}\Big(\Delta u + \sum_{j\neq i} (D_i^2 u - D_j^2 u)\Big).$$

By Lemma 7.1 therefore, it suffices to estimate the square norm of Δu. By Proposition 4.2, the triangle inequality and (5.3), this is bounded by the $W^{-1,2}$-norm of $K_\nu(u)$ plus the norm of the second derivatives estimated by Lemma 7.2.

Parts (ii) and (iii) of Proposition 4.2, (5.3) and the same argument, all apply to the exterior case. □

Remark 7.4. For the biharmonic function $u = |X|^2$, $M_\nu(u) = 2(\nu(n-1)+1)$ and $K_\nu(u) = 0$ on any $\partial\Omega$.

The forms $L_{ij}(u)L_{ij}(u)$ dominate $|\nabla\nabla u|^2$ pointwise as long as $\nu > (1-n)^{-1}$.

The a priori estimate that bounds second-order derivatives of solutions by their Neumann data (5.1) and (5.2) can now be shown. It will be convenient to have a notation for the norm on the space of data.

Definition 7.5. A norm on $(\Lambda, f) \in W^{-1,p}(\partial\Omega) \times L^p(\partial\Omega)$ is defined by

$$|\!|\!|\Lambda, f|\!|\!|_p = \|\Lambda\|_{-1,p} + \|f\|_p.$$

THEOREM 7.6. *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded Lipschitz domain with connected boundary, and take* $1/(1-n) \leqslant \nu < 1$.

(i) *If* $\Delta^2 u = 0$ *in* Ω *with* $N(\nabla\nabla u) \in L^2(\partial\Omega)$ *and* $\int_{\partial\Omega} \Delta u\, ds = 0$, *there exists a constant* $C = C(\partial\Omega, \nu)$ *such that*

$$\int_{\partial\Omega} |\nabla\nabla u|^2\, ds \leqslant C |\!|\!|K_\nu(u), M_\nu(u)|\!|\!|_2^2, \tag{7.1}$$

where the Neumann data on the right-hand side are defined in (5.1) *and* (5.2).

(ii) *If* $\Delta^2 u=0$ *in* $\bar{\Omega}^c$ *with* $N(\nabla\nabla u)\in L^2(\partial\Omega)$ *and decay, as* $|X|\to\infty$,

$$|\nabla u(X)| = \begin{cases} O(|X|^{2-n}) & \text{for } n>2, \\ O(|X|^{-1}) & \text{for } n=2, \end{cases} \tag{7.2}$$

then

$$\begin{aligned}\int_{\partial\Omega} |\nabla\nabla u|^2\, ds &\leqslant C \| K_\nu(u), M_\nu(u) \|_2^2 \\ +C \| K_\nu(u), M_\nu(u)\|_2^{1/2} &\left(|\partial\Omega|^{2/(1-n)} \left| \int_{\partial\Omega} u\, ds \right| + |\partial\Omega|^{1/(1-n)} \left| \int_{\partial\Omega} \nabla u\, ds \right| \right)^{3/2}.\end{aligned} \tag{7.3}$$

(iii) C *diverges as* $\nu\uparrow 1$. *By Remark* 4.4 *the mean-value hypothesis can be replaced with*

$$\int f^* \Delta u\, ds = 0.$$

Without any mean-value hypothesis, C *in* (7.1) *diverges as* $\nu\downarrow 1/(1-n)$.

Proof. When u is defined in Ω, adding a linear function to u changes neither side of (7.1). Therefore it may also be assumed that u and the gradient of u have mean value zero on the boundary, justifying any use of the Poincaré inequality there.

Using the hypothesis of mean value zero, Lemma 7.3 reduces the proof to estimating the integrals on the right-hand side of the Rellich formula (6.9). The boundary integrals of (6.9), by inequalities (6.10) together with the Poincaré inequality, are bounded above by the square root of the right-hand side of (7.1) times the square root of the left-hand side of (7.1). The Schwarz inequality applied to the solid integrals yields a bound

$$C \left(\int_\Omega |\nabla\nabla u|^2 + |\nabla u|^2\, dX \right)^{1/2} \left(\int_\Omega L_{ij}(u) L_{ij}(u)\, dX \right)^{1/2}. \tag{7.4}$$

The left integral of (7.4) can be put on the boundary by estimates for the biharmonic Dirichlet problem [9] and the Poincaré inequality applied again. The integral inside the right-hand root is equal to

$$(1-\nu)^{-1} \int_{\partial\Omega} \left(\frac{\partial u}{\partial N} M_\nu(u) - u K_\nu(u) \right) ds \tag{7.5}$$

by the same calculation used in (6.3) (Green's first identity, see (10.2) below). Again duality as in (6.10) and the Poincaré inequality yield products of the norms appearing on the left- and right-hand sides of (7.1), which suffices.

When u is defined in the exterior domain one may take the vector field α to be supported near the boundary. The solid integrals of (6.9) admit a bound like (7.4) but over a compact subregion of $\bar{\Omega}^c$ near the boundary. By the existence of pointwise limits for biharmonic solutions in the class $N(\nabla\nabla u)\in L^2(\partial\Omega)$ and the fundamental theorem of calculus, the $|\nabla u|^2$-term in the left integral corresponding to that of (7.4) can be bounded by

$$\int_{\bar{\Omega}^c} |\nabla\nabla u|^2\,dX + \int_{\partial\Omega} |\nabla u|^2\,ds. \tag{7.6}$$

The decay hypothesis (7.2) on ∇u implies by "interior" estimates ([23, p. 155]) that

$$|\nabla\nabla u(X)| \leqslant \begin{cases} C|X|^{1-n} & \text{for } n>2, \\ C|X|^{-2} & \text{for } n=2, \end{cases} \tag{7.7}$$

so that integration by parts in the solid integral of (7.6) bounds (7.6) by

$$\int_{\bar{\Omega}^c} (\Delta u)^2\,dX + \|\nabla u\|_2^2 + \|\nabla\nabla u\|_2^2 \leqslant C(\|\nabla u\|_2^2 + \|\nabla\nabla u\|_2^2).$$

The last inequality follows, for example, by the fact that the harmonic function Δu with decay from (7.7) and nontangential maximal function in L^2 admits an invertible layer-potential representation. See Remark 4.5.

The right-hand integral of (7.4) taken over the exterior domain yields (7.5) again. This is justified because the fundamental theorem of calculus, over rays from X to a compact neighborhood of $\bar{\Omega}$, and hypothesis (7.2) show that $u(X)$ is $O(\log|X|)$ at infinity, while [23] shows that three derivatives of u decays like $|X|^{-n}$. Consequently the left-hand side of (7.1) can be bounded by the fourth root of the right-hand side times the $\frac{3}{2}$-power of the square norms of u, ∇u and $\nabla\nabla u$ on the boundary. Introducing the mean values of u and its gradient, applying the Poincaré inequality and using Young's inequalities yield (7.3). □

The next theorem is the companion of the last. It was first proved in the case of bounded star-like domains in [43]. Here it will be derived from the Rellich formula (6.9).

THEOREM 7.7. *Let $\Omega\subset\mathbf{R}^n$ be a bounded Lipschitz domain with connected boundary.*

(i) *If $\Delta^2 u=0$ in Ω with $N(\nabla\nabla u)\in L^2(\partial\Omega)$, then there exists a constant C depending only on the Lipschitz nature of Ω such that*

$$\int_{\partial\Omega} |\nabla\nabla u|^2\,ds \leqslant C\int_{\partial\Omega} |\nabla_T\nabla u|^2\,ds.$$

(ii) *If $\Delta^2 u=0$ in $\bar{\Omega}^c$ with $N(\nabla\nabla u)\in L^2(\partial\Omega)$ and decay as in (7.2), then*

$$\int_{\partial\Omega} |\nabla\nabla u|^2\,ds \leqslant C\biggl(\int_{\partial\Omega} |\nabla_T\nabla u|^2\,ds + |\partial\Omega|^{2/(1-n)}\biggl|\int_{\partial\Omega}\nabla u\,ds\biggr|^2\biggr).$$

Proof. Let $\nu=\theta=0$ in (6.9). See (6.4), (6.6), (5.1) and (5.2). The integrand on the left-hand side of (6.9) can be written as

$$\frac{N\cdot\alpha}{2}\left(\frac{\partial^2 u}{\partial N^2}\right)^2+\frac{N\cdot\alpha}{2}\left[\left(|\nabla\nabla u|^2-\left|\frac{\partial\nabla u}{\partial N}\right|^2\right)+\left(\left|\frac{\partial\nabla u}{\partial N}\right|^2-\left(\frac{\partial^2 u}{\partial N^2}\right)^2\right)\right]. \tag{7.8}$$

Each of the differences of squares can be written as sums of squares of tangential derivatives. The integrand of the last integral of (6.9) can be written

$$N\cdot\alpha\left(\frac{\partial^2 u}{\partial N^2}\right)^2+\left(N_i(\alpha_j-\alpha\cdot NN_j)D_iD_ju+\frac{\partial\alpha}{\partial N}\cdot\nabla u\right)\frac{\partial^2 u}{\partial N^2}. \tag{7.9}$$

Here $\alpha_j-\alpha\cdot NN_j$ are the components of a tangent vector. Solving (6.9) for the square of the second normals,

$$\begin{aligned}\int_{\partial\Omega}&\frac{N\cdot\alpha}{2}\left(\frac{\partial^2 u}{\partial N^2}\right)^2 ds\\ &=\int_{\partial\Omega}\Bigg(\alpha\cdot\nabla u K_0(u)+\frac{N\cdot\alpha}{2}\left[\left(|\nabla\nabla u|^2-\left|\frac{\partial\nabla u}{\partial N}\right|^2\right)+\left(\left|\frac{\partial\nabla u}{\partial N}\right|^2-\left(\frac{\partial^2 u}{\partial N^2}\right)^2\right)\right]\\ &\qquad-\left(N_i(\alpha_j-\alpha\cdot NN_j)D_iD_ju+\frac{\partial\alpha}{\partial N}\cdot\nabla u\right)\frac{\partial^2 u}{\partial N^2}\Bigg)ds\\ &\qquad-\int_{\Omega}E_{ij}(\alpha,u,0)D_iD_ju\,dX,\end{aligned} \tag{7.10}$$

or, for u defined in the exterior, the solid integral is taken over a compact subset of $\bar{\Omega}$ depending on α.

The foregoing continues to hold if u is replaced by u plus any linear function, in which case the inequality to be proved in (i) is unchanged. By this device it may be assumed that the Poincaré inequality holds for u and its gradient on the boundary.

For (i), the solid integral in (7.10) (see (6.7)) is bounded by

$$\int_{\Omega}|\nabla\nabla u|^2\,dX+\int_{\partial\Omega}|\nabla u|^2\,ds.$$

This last solid integral may be replaced by (7.5) with $\nu=0$. By applying the duality (5.4) and the Schwarz and Poincaré inequalities to (7.10), (i) follows.

The decay hypothesis in (ii) shows, as in the preceding theorem, that u grows no faster than $\log|X|$ at infinity and that (5.4) holds. The decay is enough in order to treat the solid integral as in (i), but without the addition of a linear function. Consequently, duality and the Schwarz and Poincaré inequalities yield (ii). □

8. Pointwise limits of potentials at the boundary

Let $B^X = B^X(Y) = B(X-Y)$ denote the fundamental solution for Δ^2 with pole at $X \in \mathbf{R}^n$, and similarly let Γ^X denote the fundamental solution for Laplace's equation. More precisely,

$$B^X(Y) = \begin{cases} [2(n-4)(n-2)\omega_n]^{-1}|X-Y|^{4-n}, & n>4,\ n=3, \\ [-4\omega_4]^{-1}\log|X-Y|, & n=4, \\ [-8\pi]^{-1}|X-Y|^2(1-\log|X-Y|), & n=2, \end{cases} \tag{8.1}$$

where ω_n is the surface measure of the unit sphere $\mathbf{S}^{n-1}$ of $\mathbf{R}^n$. Then $\Delta B^X = \Gamma^X$. Define the potential

$$S_0\Lambda(X) = \int_{\partial\Omega} \Lambda B^X\, ds, \quad X \in \mathbf{R}^n \backslash \partial\Omega, \tag{8.2}$$

in the sense of distributions for any $\Lambda \in W^{-1,p'}(\partial\Omega)$, $1<p<2+\varepsilon$, and Ω a bounded Lipschitz domain.

Given $\Lambda \in W_0^{-1,p}(\partial\Omega)$, Proposition 4.2 associates Λ with a harmonic function defined inside Ω and with another defined in the exterior domain. The L^p-Dirichlet boundary values of the two harmonic functions differ, but are related by the classical double-layer potential according to the next lemma. This fact will be used more than once in analyzing the boundary values of biharmonic potentials like (8.2), and it clarifies Remark 4.4 above.

Denote by int and ext nontangential limits on the boundary taken from Ω and $\bar{\Omega}^c$, respectively.

LEMMA 8.1. *Let Ω and the notation $\partial h/\partial N = \Lambda_h$ be as in Proposition* 4.2. *Let* $1<p'<2+\varepsilon$. *Then the map*

$$g \longmapsto \Lambda_h : L_0^p(\partial\Omega) \longrightarrow W_0^{-1,p}(\partial\Omega)$$

is a well-defined isomorphism with bounds depending only on the Lipschitz geometry of Ω, when h is the classical double-layer potential

$$h(Y) = \int_{\partial\Omega} \frac{\partial}{\partial N}\Gamma^Y g\, ds, \quad Y \in \mathbf{R}^n \backslash \partial\Omega. \tag{8.3}$$

In particular, $\Lambda_h = \partial h^{\mathrm{int}}/\partial N$ and $\Lambda_h = \partial h^{\mathrm{ext}}/\partial N$ from Proposition 4.2 *are identical when h is as in* (8.3).

Proof. By the invertibility of layer potentials on Lipschitz boundaries [41], [7], the harmonic functions (8.3) span the codimension-1 subspace of $L^p(\partial\Omega)$-functions mentioned in Remark 4.4 (the $\tilde{h}$ there) that is isomorphic to $W_0^{-1,p}(\partial\Omega)$. Only the equality across the boundary needs to be examined.

For $Y\in\mathbf{R}^n\backslash\partial\Omega$ and each i and j, define the potentials

$$h_{ij}(Y)=\int_{\partial\Omega}N_iD_j\Gamma^Y g\,ds.$$

Summing on i, $D_jh=D_ih_{ij}$, and summing on j, $D_jh_{ij}=0$, i.e. in the sense of distributions,

$$\frac{\partial h}{\partial N}=\frac{\partial h_{ij}}{\partial T_{ji}}=-\frac{1}{2}\frac{\partial(h_{ij}-h_{ji})}{\partial T_{ij}}$$

by the interchange of indices as used in (6.8).

In (4.3) make the substitution $u(Y)=\int_{\partial\Omega}\Gamma^Y f\,ds$, $Y\in\mathbf{R}^n$, the classical single-layer potential of $f\in L^{p'}$, for $f\in W^{1,p'}$, and compute $\Lambda_h u$ using either $\partial h^{\mathrm{int}}/\partial N$ or $\partial h^{\mathrm{ext}}/\partial N$ in the sense of distributions. That is, by using either interior or exterior approximating boundaries $\partial\widetilde{\Omega}$ and transferring the tangential derivatives to u by the divergence theorem,

$$\lim_{\partial\widetilde{\Omega}}\frac{1}{2}\int_{\partial\widetilde{\Omega}}(h_{ij}-h_{ji})\frac{\partial u}{\partial T_{ij}}\,d\tilde{s}=\frac{1}{2}\int_{\partial\Omega}\text{p.v.}\int_{\partial\Omega}\frac{\partial}{\partial T_{ij}}\Gamma^Y g\,ds\;\text{p.v.}\frac{\partial}{\partial T^Y_{ij}}\int_{\partial\Omega}\Gamma^Y f\,ds\,ds(Y).$$

The limits and continuity across the boundary for these potentials with tangential derivatives on Lipschitz boundaries is known [41], and proves the lemma. □

Remark 8.2. The above proof shows that in the sense of distributions on $\partial\Omega$,

$$\frac{\partial}{\partial N^Q}\int_{\partial\Omega}\frac{\partial}{\partial N}\Gamma^Q g\,ds=-\frac{1}{2}\frac{\partial}{\partial T^Q_{ij}}\,\text{p.v.}\int_{\partial\Omega}\frac{\partial}{\partial T_{ij}}\Gamma^Q g\,ds$$

The operator is symmetric and continuous across the boundary.

Remark 8.3. The classical single-layer potential applied to the operator of Remark 8.2 becomes, on the boundary, the product of the classical double-layer potential from the exterior times the classical double-layer potential from the interior (and vice versa). By [41] and [3] this is an invertible operator from $L^p_0(\partial\Omega)$ to the $L^p(\partial\Omega)$-functions, $2-\varepsilon<p<2+\varepsilon$, that integrate to zero against the harmonic equilibrium distribution of Remark 4.4. The operator and operator inverse bounds are also shown by these references to depend only on the Lipschitz geometry of the domain. Thus the map

$$g\longmapsto\frac{1}{2}\,\text{p.v.}\int_{\partial\Omega}\frac{\partial}{\partial T^Q_{ij}}\Gamma^P(Q)\left[\text{p.v.}\int_{\partial\Omega}\frac{\partial}{\partial T_{ij}}\Gamma^Q g\,ds\right]ds(Q)$$

is an isomorphism on these codimension-1 subspaces of L^p. By [7] this is true for $2-\varepsilon<p<\infty$.

LEMMA 8.4. *In either Ω or $\bar{\Omega}^c$ the potential $S_0\Lambda$ of* (8.2) *with $\Lambda\in W^{-1,p}(\partial\Omega)$, $1<p'<2+\varepsilon$, satisfies that*

(i) $N(\nabla\nabla S_0\Lambda)\in L^p(\partial\Omega)$;

(ii) $\nabla\nabla S_0\Lambda(X)$ *has nontangential limits a.e.* (ds) *on the boundary;*

(iii) *the nontangential limits of* (ii) *taken from Ω agree a.e.* (ds) *with those taken from $\bar{\Omega}^c$.*

Further, if $\Lambda=\Lambda_h$ for a harmonic function h given by (8.3) *with $g\in L^p_0(\partial\Omega)$, then the potential and its derivatives have the representations*

(iv) $S_0\Lambda(X)=\frac{1}{2}\int_{\partial\Omega}\frac{\partial}{\partial T_{ij}}B^X\left[\text{p.v.}\int_{\partial\Omega}\frac{\partial}{\partial T_{ij}}\Gamma^Y g\,ds\right]ds(Y)$;

(v) $D_kD_lS_0\Lambda(X)=\frac{1}{2}\int_{\partial\Omega}\frac{\partial}{\partial T_{ij}}D_kD_lB^X\left[\text{p.v.}\int_{\partial\Omega}\frac{\partial}{\partial T_{ij}}\Gamma^Y g\,ds\right]ds(Y)$

for all $X\in\mathbf{R}^n\setminus\partial\Omega$ and a.e. (ds) *$X\in\partial\Omega$ when the outside integral of* (v) *is also taken in the principle value sense. In* (iv) *and* (v), *summation convention $1\leqslant i,j\leqslant n$ is implied.*

Proof. The lemma follows from the singular integral results of Coifman, McIntosh and Meyer [5] once (iv) is established. (See also [41] and [11] for continuity across the boundary.) By representing $\partial h/\partial N$ in terms of the h_{ij} as in the proof of Lemma 8.1, and applying Lemma 4.9 over approximating boundaries $\partial\tilde{\Omega}$,

$$S_0\Lambda(X)=\lim_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}}\frac{\partial h}{\partial N}B^X ds=\frac{1}{2}\int_{\partial\Omega}\text{p.v.}\,(h_{ij}-h_{ji})\frac{\partial B^X}{\partial T_{ij}}\,ds, \tag{8.4}$$

which is (iv). □

By virtue of Lemma 8.4, Proposition 4.2 and the invertibility of the classical layer potentials [41], [7], bounded operators on $\partial\Omega$ may be defined:

Definition 8.5. (i) $\mathcal{M}^0_\nu: W^{-1,p}\to L^p$, $\Lambda\mapsto M_\nu S_0\Lambda$;

(ii) $\mathcal{K}^0_\nu: W^{-1,p}\to W^{-1,p}$,

$$\Lambda\longmapsto\frac{1}{2}\left(\frac{\partial^{\text{ext}}}{\partial N}+\frac{\partial^{\text{int}}}{\partial N}\right)\Delta S_0\Lambda+\frac{1-\nu}{2}\frac{\partial}{\partial T_{ij}}\left(\frac{\partial^2 S_0\Lambda}{\partial N\partial T_{ij}}\right),$$

in the sense of distributions, with spaces defined on the boundary $\partial\Omega$ and $1<p'<2+\varepsilon$.

Recall that N always denotes the *outer* unit normal to Ω. The next lemma says that the known jump property across the boundary for the adjoint to the classical double-layer potential continues to hold when it operates on the $W^{-1,p}(\partial\Omega)$-spaces.

LEMMA 8.6. *Let*

$$h(Y)=\int_{\partial\Omega}\frac{\partial}{\partial N}\Gamma^Y g\,ds,\quad Y\in\mathbf{R}^n\backslash\partial\Omega,$$

be the double-layer potential (8.3) *for any* $g\in L^p(\partial\Omega)$, $2-\varepsilon<p<\infty$, *and let* $\Lambda=\Lambda_h$ *be as in Lemma* 8.1. *Then in the sense of distributions,*

$$\frac{\partial^{\text{ext}}}{\partial N}\Delta S_0\Lambda-\frac{\partial^{\text{int}}}{\partial N}\Delta S_0\Lambda=\Lambda. \tag{8.5}$$

Proof. Let u be the classical single-layer potential of $f\in L^{p'}$ as in the proof of Lemma 8.1. Applying the left-hand side of (8.5) to u, by the classical formula

$$\frac{\partial^{\text{ext}}u}{\partial N}-\frac{\partial^{\text{int}}u}{\partial N}=f$$

and (v) of Lemma 8.4, yields

$$\frac{1}{2}\int_{\partial\Omega}f(X)\,\text{p.v.}\int_{\partial\Omega}\frac{\partial}{\partial T_{ij}}\Gamma^X(Y)\left[\text{p.v.}\int_{\partial\Omega}\frac{\partial}{\partial T_{ij}}\Gamma^Y g\,ds\right]ds(Y)\,ds(X). \tag{8.6}$$

By the same calculations used in (8.4), the right-hand side applied to u yields

$$\frac{1}{2}\int_{\partial\Omega}\text{p.v.}\int_{\partial\Omega}\frac{\partial}{\partial T_{ij}}\Gamma^Y g\,ds\;\text{p.v.}\frac{\partial}{\partial T_{ij}}\int_{\partial\Omega}\Gamma^Y(X)f(X)\,ds(X)\,ds(Y). \tag{8.7}$$

The Fubini theorem is justified in this setting, which establishes the equality of (8.6) and (8.7). □

A computation yields

$$D_iD_jD_kB(X)=[2\omega_n]^{-1}\left(\frac{\delta_{ij}X_k+\delta_{ik}X_j+\delta_{jk}X_i}{|X|^n}-\frac{nX_iX_jX_k}{|X|^{n+2}}\right) \tag{8.8}$$

for $1\leqslant i,j,k\leqslant n$ and $n\geqslant 2$, with Kronecker δ-notation. It follows by [5] that

$$\lim_{X\to P}\int_{\partial\Omega}D_iD_jD_kB(Q-X)f(Q)\,ds(Q)=\pm\frac{1}{2}N_iN_jN_kf(P)+\text{p.v.}\int_{\partial\Omega}D_iD_jD_kB^Pf\,ds \tag{8.9}$$

for a.e. $P\in\partial\Omega$, with the plus and minus sign occurring when X approaches nontangentially from Ω and from the exterior of Ω, respectively. This may be seen by carrying out Miranda's computation for the odd homogeneous kernels (8.8) found, for example, in line 29 on p. 54 of [11] ($n(P)$ is the *inner* normal there).

Define the potential

$$S_1 f(X) = \int_{\partial\Omega} \frac{\partial}{\partial N} B^X f \, ds, \quad X \in \mathbf{R}^n \backslash \partial\Omega, \tag{8.10}$$

for any $f \in L^p(\partial\Omega)$, $1<p<\infty$, and Ω a bounded Lipschitz domain.

By (8.9), the Neumann data for (8.10) is

$$M_\nu^{\text{int}}(S_1 f)(P) = \frac{1}{2} f(P) + \nu \text{ p.v.} \int_{\partial\Omega} \frac{\partial}{\partial N} \Gamma^P f \, ds + (1-\nu) \text{ p.v.} \, N_j^P N_k^P \int_{\partial\Omega} \frac{\partial}{\partial N} D_j D_k B^P f \, ds \tag{8.11}$$

and

$$M_\nu^{\text{ext}}(S_1 f)(P) = -f(P) + M_\nu^{\text{int}}(S_1 f)(P), \quad P \in \partial\Omega, \tag{8.12}$$

from the interior and the exterior, respectively. And in the sense of distributions on $\partial\Omega$,

$$K_\nu(S_1 f)(P) = \frac{\partial}{\partial N} \int_{\partial\Omega} \frac{\partial}{\partial N} \Gamma^P f \, ds + \frac{1-\nu}{2} \frac{\partial}{\partial T_{ij}} \left(\text{p.v.} \frac{\partial^2}{\partial N \partial T_{ij}} \int_{\partial\Omega} \frac{\partial}{\partial N} B^P f \, ds \right) \tag{8.13}$$

when $2-\varepsilon<p<\infty$. Continuity across the boundary in (8.13) follows for the first term on the right by Lemma 8.1, while the principle value integral in parentheses is continuous across by (8.9).

By virtue of (8.11), (8.12) and (8.13), bounded operators on $\partial\Omega$ may be defined:

Definition 8.7. (i) $\mathcal{M}_\nu^1 \colon L^p \to L^p \colon f \mapsto M_\nu^{\text{int}}(S_1 f) - \frac{1}{2} f = M_\nu^{\text{ext}}(S_1 f) + \frac{1}{2} f$;
(ii) $\mathcal{K}_\nu^1 \colon L^p \to W^{-1,p} \colon f \mapsto K_\nu(S_1 f)$
in the sense of distributions, with spaces defined on the boundary $\partial\Omega$ and $2-\varepsilon<p<\infty$.

9. Layer-potential solutions and the integral equations of second kind

For any $\Lambda \in W^{-1,p}$ and $f \in L^p$, $2-\varepsilon<p<2+\varepsilon$, define a solution to the biharmonic equation in $\mathbf{R}^n \backslash \partial\Omega$ by

$$u(X) = S_0 \Lambda(X) - S_1 f(X), \tag{9.1}$$

with the potentials (8.2) and (8.10).

Let $\mathcal{I}$ denote the identity operator. Using the notation of Definitions 8.5 and 8.7, it follows by Lemma 8.4 (iii), (8.11) and (8.12) that

$$M_\nu^{\text{int}}(u) = \mathcal{M}_\nu^0 \Lambda - \left(\tfrac{1}{2}\mathcal{I} + \mathcal{M}_\nu^1\right) f, \tag{9.2}$$

$$M_\nu^{\text{ext}}(u) = \mathcal{M}_\nu^0 \Lambda + \left(\tfrac{1}{2}\mathcal{I} - \mathcal{M}_\nu^1\right) f. \tag{9.3}$$

By Lemma 8.6 and (8.13),

$$K_\nu^{\mathrm{int}}(u) = \left(-\tfrac{1}{2}\mathcal{I}+\mathcal{K}_\nu^0\right)\Lambda - \mathcal{K}_\nu^1 f, \tag{9.4}$$

$$K_\nu^{\mathrm{ext}}(u) = \left(\tfrac{1}{2}\mathcal{I}+\mathcal{K}_\nu^0\right)\Lambda - \mathcal{K}_\nu^1 f. \tag{9.5}$$

Thus solving the boundary value problem (1.1) and (1.2) is reduced to solving the system of integral equations of the second kind

$$T_\nu^{\mp}(\Lambda, f) = \mp\frac{1}{2}\begin{bmatrix}\Lambda\\ f\end{bmatrix} + \begin{bmatrix}\mathcal{K}_\nu^0 & -\mathcal{K}_\nu^1\\ \mathcal{M}_\nu^0 & -\mathcal{M}_\nu^1\end{bmatrix}\begin{bmatrix}\Lambda\\ f\end{bmatrix} = \begin{bmatrix}\Lambda_0\\ f_0\end{bmatrix} \tag{9.6}$$

in the space $W^{-1,p}(\partial\Omega)\times L^p(\partial\Omega)$, $2-\varepsilon<p<2+\varepsilon$. Here the minus sign corresponds to the interior problem, and the plus sign to the exterior.

Remark 9.1. When f^* is the equilibrium distribution from classical potential theory, $T_\nu^+(f^*, 0)$ does not have mean value zero (cf. the proof of Proposition 4.2 and Definition 8.5 (ii)), and therefore supplies a linear functional not in the $W_0^{-1,p}$.

Because in general the square matrix of boundary operators is noncompact for Lipschitz boundaries, the invertibility method of [41] will be used. This will be begun in the next section and completed in §12.

10. The semi-Fredholm property of the boundary operator

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$. It will be helpful to have a notation for certain closed subspaces of $W_0^{-1,p}(\partial\Omega)\times L^p(\partial\Omega)$ of codimension n and $n+1$, respectively.

Definition 10.1. In the sense of distributions, for $1<p<\infty$, let

$$\mathbf{X}^p = \mathbf{X}^p(\partial\Omega) = \left\{(\Lambda, f)\in W_0^{-1,p}(\partial\Omega)\times L^p(\partial\Omega) : \int_{\partial\Omega} \Lambda Q_j - f N_j\, ds(Q) = 0,\ j=1,\dots,n\right\}$$

and

$$\widetilde{\mathbf{X}}^p = \left\{(\Lambda, f)\in \mathbf{X}^p : \int_{\partial\Omega} \Lambda|Q|^2 - 2N\cdot Q f\, ds(Q) = 0\right\}.$$

Remark 10.2. $\mathbf{X}^p(\partial\Omega)\oplus\operatorname{span}\{(N_i, -Q_i) : i=1,\dots,n\} = W_0^{-1,p}(\partial\Omega)\times L^p(\partial\Omega)$.

Remark 10.3. If one takes $\Lambda=\partial h/\partial N$ in the condition that defines $\mathbf{X}^p$, then it follows that $\int_{\partial\Omega}(h-f)N_j\, ds=0$. Thus taking u as in (9.1) with X *exterior* to Ω implies that

$$\begin{aligned} u(X) &= \int_{\partial\Omega}\frac{\partial B^X}{\partial N}(h-f)\, ds - \int_\Omega \Gamma^X h\, dY \\ &= \int_{\partial\Omega} N_j^Q\left(D_j B^X(Q) - D_j B^X(0)\right)(h(Q)-f(Q))\, ds(Q) - \int_\Omega \Gamma^X h\, dY. \end{aligned} \tag{10.1}$$

Consequently, u is $O(|X|^{2-n})$ for $n>2$ and $O(\log|X|)$ for $n=2$ at infinity. The decay hypothesis (7.2) for $n=2$ is met.

The integration-by-parts formula (Green's first identity) for solutions u,

$$(1-\nu)\int_{\Omega} L_{ij}(v)L_{ij}(u)\,dX=\int_{\partial\Omega}\left(\frac{\partial v}{\partial N}M_{\nu}(u)-vK_{\nu}(u)\right)ds \tag{10.2}$$

for v linear when $\nu\geqslant(1-n)^{-1}$ and for $v=|X|^2$ when $\nu=(1-n)^{-1}$, yields the following lemma:

LEMMA 10.4. *The interior boundary operator T_{ν}^{-} of* (9.6) *maps $W^{-1,p}(\partial\Omega)\times L^p(\partial\Omega)$ into $\mathbf{X}^p$ when $\nu\geqslant(1-n)^{-1}$ and into $\widetilde{\mathbf{X}}^p$ when $\nu=(1-n)^{-1}$, for $1<p<\infty$.*

LEMMA 10.5. *The boundary operators of* (9.6)

(i) $T_{\nu}^{-}:\mathbf{X}^p\to\mathbf{X}^p$ *when* $\nu>(1-n)^{-1}$;

(ii) $T_{\nu}^{-}:\widetilde{\mathbf{X}}^p\to\widetilde{\mathbf{X}}^p$ *when* $\nu=(1-n)^{-1}$;

(iii) $T_{\nu}^{+}:W^{-1,p}\times L^p\to W^{-1,p}\times L^p$

are injective for $2-\varepsilon<p<2+\varepsilon$.

Proof. First consider u from (9.1) to be a solution to (1.1) and (1.2) in Ω with vanishing right-hand side. Consequently, letting $v=u$ in Green's first identity (10.2), the vanishing of the right-hand side of the identity shows that u is linear in Ω when ν is as in (i), and, by Lemma 7.1, equals $a+\mathbf{b}\cdot X+c|X|^2$ when ν is as in (ii). (The identity remains valid for $p<2$ bounded from below by a Sobolev exponent.) It follows, by continuity across the boundary, that from either domain $(u,\partial u/\partial N)$, when restricted to the boundary, is equal to $(a+\mathbf{b}\cdot Q+c|Q|^2,\mathbf{b}\cdot N+2cQ\cdot N)$, with $c=0$ in the former case.

The assumed vanishing of $T_{\nu}^{-}(\Lambda,f)$ and the jumps (9.6) yield

$$T_{\nu}^{+}(\Lambda,f)=(\Lambda,f)=(K_{\nu}^{\mathrm{ext}}(u),M_{\nu}^{\mathrm{ext}}(u)). \tag{10.3}$$

When $n\geqslant 3$, (9.1) will be $O(|X|^{3-n})$ at infinity for any $(\Lambda,f)\in W_0^{-1,p}\times L^p$ by (8.1) and Proposition 4.2, which allows extra decay to be obtained in (8.2) by writing $S_0\Lambda(X)=\int\Lambda(B^X(Q)-B^X(0))\,ds(Q)$. When $n=2$, the assumption that $(\Lambda,f)\in\mathbf{X}^p$ permits

$$u(X)=\int_{\partial\Omega}\Lambda(B^X(Q)-B^X(0)-Q_jD_jB^X(0))-f\left(\frac{\partial}{\partial N}B^X(Q)-N_j^QD_jB^X(0)\right)ds(Q), \tag{10.4}$$

which is $O(\log|X|)$ at infinity. In every dimension then, there is sufficient decay to justify Green's first identity (with $u=v$ again) in the exterior domain. By definition of the function subspaces, Definition 10.1, and the above conclusions on the boundary values of u and its derivatives, it again follows that the right-hand side of (10.2) vanishes.

For (i), u must be linear in the exterior domain, in which case its Neumann data (10.3) must vanish. Thus T_ν^- is injective. By Remark 7.4 the same is true for case (ii).

For (iii), the vanishing of $T_\nu^+(\Lambda, f)$ implies that $T_\nu^-(\Lambda, f)=(-\Lambda, -f)$, so that by Lemma 10.4 the decay required in the exterior is obtained, and similar arguments establish the result. $\square$

Remark 10.6. In the above proof it might seem possible that $a^{\text{int}}+\mathbf{b}^{\text{int}}\cdot Q+c^{\text{int}}|Q|^2=a^{\text{ext}}+\mathbf{b}^{\text{ext}}\cdot Q+c^{\text{ext}}|Q|^2$ for all $Q\in\partial\Omega$, with different interior and exterior constants, when the boundary is a sphere. But the potentials also have the normal derivative continuous across the boundary. Applying the divergence theorem to the equal normals shows that the interior and exterior constants are the same.

The proof of the next lemma is as in [41]. Here some of the details for the biharmonic case are set down.

LEMMA 10.7. *Let $\Omega\subset\mathbf{R}^n$ be a bounded Lipschitz domain with connected boundary. Then each of the boundary operators of Lemma* 10.5 *has closed range when $p=2$ and $n\geqslant 2$.*

Proof. For (i) of Lemma 10.5 it suffices to show closed range from a subspace of finite codimension. Therefore let $(\Lambda, f)\in\mathbf{X}^2$ also satisfy $\int_{\partial\Omega} ff^*\,ds=0$, where f^* is as in Remark 4.4. Let u be as in (9.1).

By (9.6),

$$|||\Lambda, f|||_2\leqslant |||T_\nu^+(\Lambda, f)|||_2+|||T_\nu^-(\Lambda, f)|||_2.$$

By (9.3) and (9.5),

$$|||T_\nu^+(\Lambda, f)|||_2=|||K_\nu^{\text{ext}}(u), M_\nu^{\text{ext}}(u)|||_2.$$

From the definitions of the Neumann data (5.1) and (5.2) along with (5.5),

$$|||K_\nu^{\text{ext}}(u), M_\nu^{\text{ext}}(u)|||_2\leqslant C\|\nabla^{\text{ext}}\nabla u\|_2.$$

By (ii) of Theorem 7.7,

$$\|\nabla^{\text{ext}}\nabla u\|_2\leqslant C\left(\|\nabla_T\nabla u\|_2+\left|\int_{\partial\Omega}\nabla u\,ds\right|\right),$$

the required decay for $n=2$ following from Remark 10.3. By (iii) of Lemma 8.4 and (8.9), $\nabla_T\nabla u$ is continuous across the boundary at a.e. (ds) point. Consequently, using the assumption on f and parts (i) and (iii) of Theorem 7.6, it has been shown that

$$|||\Lambda, f|||_2\leqslant C\left(|||T_\nu^-(\Lambda, f)|||_2+|\partial\Omega|^{1/(1-n)}\left|\int_{\partial\Omega}\nabla u\,ds\right|\right)\quad\text{for all }(\Lambda, f)\in\mathbf{X}^2,\ \int_{\partial\Omega} ff^*\,ds=0,$$

with C depending only on the Lipschitz nature of Ω.

Closed range for T_ν^- now follows by functional analytic arguments. The other cases are treated similarly. $\square$

The operators (9.6) have finite kernels and closed range. Showing that the ranges are dense in the spaces of Lemma 10.5 requires knowing this on smooth approximating domains.

11. Invertibility on smooth boundaries

When $\partial\Omega$ is smooth, the 2×2-matrix of boundary operators of (9.6) will contain compact operators as well as operators that are not compact. However, the operators $T_\nu^\mp$ can be proved to be of the form invertible + compact because $\nu<1$. This will now be done.

In trying to identify the compact operators in the smooth case, four *types* of principle-value operators are encountered. With the notation

$$\frac{\partial^3}{\partial A\,\partial B\,\partial C}U(X)=A_iB_jC_k\frac{\partial^3}{\partial X_i\,\partial X_j\,\partial X_k}U(X)$$

(summation notation) and $P,Q\in\partial\Omega$, they have kernels of the form

(i) $\dfrac{\partial}{\partial N}\Gamma^P(Q)$, the classical double-layer-potential kernel;

(ii) $\dfrac{\partial^3}{\partial N^3}B^P(Q)$;

(iii) $\dfrac{\partial^3}{\partial T_{ij}\,\partial T_{kl}\,\partial N}B^P(Q)$, where $T_{ij}=T_{kl}$ is permitted;

(iv) $\dfrac{\partial^3}{\partial N^2\partial T_{kl}}B^P(Q)$.

Here the normal and tangential vectors depend on Q, but they may be replaced with corresponding vectors that depend on P because the resulting kernels differ from the above kernels by kernels that give rise to compact operators, since the boundary is smooth.

Again, by the smoothness of the boundary, it is enough to take the boundary to be (locally) the hyperplane $X_n=0$, in order to understand which of the above kernels give rise to compact operators. By (8.8) on the hyperplane with $X\neq 0$ (i.e. $Q\neq P$) the first three are identically zero, giving rise to compact operators in the general setting. The fourth gives rise to bounded operators of Riesz transform type. Indeed by (8.8) on the hyperplane, the fourth is

$$[2\omega_n]^{-1}\frac{N_kX_l-N_lX_k}{|X|^n}=\frac{1}{2}\frac{\partial}{\partial T_{kl}}\Gamma(X).$$

Thus in general,

$$\frac{\partial^3}{\partial N^2 \partial T_{kl}} B^P(Q) = \frac{1}{2} \frac{\partial}{\partial T_{kl}} \Gamma^P(Q) + \text{compact kernels.} \tag{11.1}$$

(All of this continues to hold on C^1-boundaries. See [12], [42] or Corollary 2.2.14 of [24] for examples.)

By Definition 8.7, (8.11) and the compactness of kernels (i) and (ii),

$$\mathcal{M}_\nu^1 : L^p \longrightarrow L^p \quad \text{is compact.}$$

LEMMA 11.1. *With h and g as in* (8.3) *and $\partial\Omega$ smooth, let $\Lambda = \partial h/\partial N$. Then the map $g \mapsto \Lambda \mapsto \frac{1}{8}(1+\nu)g + \mathcal{M}_\nu^0 \Lambda : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is compact.*

Proof. By (i) of Definition 8.5, the definition of Neumann data (5.1) and (v) of Lemma 8.4,

$$\begin{aligned} \mathcal{M}_\nu^0 \Lambda(P) = \text{p.v.}\, \frac{\nu}{2} \int_{\partial\Omega} \frac{\partial}{\partial T_{ij}} \Gamma^P \left[\text{p.v.} \int_{\partial\Omega} \frac{\partial}{\partial T_{ij}} \Gamma^Q g \, ds \right] ds(Q) \\ + \text{p.v.}\, \frac{1-\nu}{2} \frac{\partial^2}{\partial N^P \partial N^P} \int_{\partial\Omega} \frac{\partial}{\partial T_{ij}} B^P \left[\text{p.v.} \int_{\partial\Omega} \frac{\partial}{\partial T_{ij}} \Gamma^Q g \, ds \right] ds(Q), \end{aligned} \tag{11.2}$$

which by (11.1) differs only by compact operators from

$$\text{p.v.}\, \frac{1+\nu}{4} \int_{\partial\Omega} \frac{\partial}{\partial T_{ij}} \Gamma^P \left[\text{p.v.} \int_{\partial\Omega} \frac{\partial}{\partial T_{ij}} \Gamma^Q g \, ds \right] ds(Q).$$

By (ii) of Lemma 8.4 one may take $P \in \Omega$, apply Remark 8.2 and use Green's formula in the *exterior* domain (or by (iii) of Lemma 8.4 the roles of interior and exterior domains may be reversed) to see that this last integral is the limit, as P approaches the boundary, of

$$\frac{1+\nu}{2} \int_{\partial\Omega} \frac{\partial}{\partial N} \Gamma^P h^{\text{ext}} \, ds.$$

The trace of the integral is the trace of the classical double-layer potential from the interior acting on the trace of the classical double layer from the exterior acting on g. From the well-known jump relations for these potentials the trace of the integral is then a compact operator on g plus $-\frac{1}{4}g$. □

Because the product of a bounded linear operator with a compact operator is compact and the first map of Lemma 11.1 is bounded and invertible by Proposition 4.2 and Lemma 8.4, it follows when Λ and g are related as in the lemma that

$$\Lambda \longmapsto \tfrac{1}{8}(1+\nu)g + \mathcal{M}_\nu^0 \Lambda : W^{-1,p} \longrightarrow L^p \quad \text{is compact.}$$

The principle-value integral in formula (8.13) is also of type (iv) and may have its kernel replaced by (11.1). Since the tangential derivative in the sense of distributions is bounded from L^p to $W^{-1,p}$ and Remark 8.2 can be applied as above, a calculation and (ii) of Definition 8.7 show that

$$f \longmapsto -\frac{1+\nu}{2}\frac{\partial}{\partial N^P}\int_{\partial\Omega}\frac{\partial}{\partial N}\Gamma^P f\,ds+\mathcal{K}^1_\nu f(P)\colon L^p \longrightarrow W^{-1,p} \quad \text{is compact.}$$

LEMMA 11.2. *With h and g as in* (8.3) *and $\partial\Omega$ smooth, let $\Lambda=\partial h/\partial N$. Then the map*

$$g \longmapsto \Delta S_0\left(\frac{\partial^{\text{ext}}}{\partial N}+\frac{\partial^{\text{int}}}{\partial N}\right)\Delta S_0\Lambda\colon L^p(\partial\Omega)\longrightarrow L^p(\partial\Omega)$$

is compact, as then are

$$g \longmapsto \left(\frac{\partial^{\text{ext}}}{\partial N}+\frac{\partial^{\text{int}}}{\partial N}\right)\Delta S_0\Lambda\colon L^p(\partial\Omega)\longrightarrow W^{-1,p}(\partial\Omega),$$

$$\Lambda \longmapsto \left(\frac{\partial^{\text{ext}}}{\partial N}+\frac{\partial^{\text{int}}}{\partial N}\right)\Delta S_0\Lambda\colon W^{-1,p}(\partial\Omega)\longrightarrow W^{-1,p}(\partial\Omega).$$

Proof. Denote by

$$\tilde{\Lambda}=\left(\frac{\partial^{\text{ext}}}{\partial N}+\frac{\partial^{\text{int}}}{\partial N}\right)\Delta S_0\Lambda,$$

by Lemma 8.4 and Proposition 4.2 a linear functional on $W^{1,p'}$. Let then

$$\tilde{\Lambda}=\frac{\partial}{\partial N}\int_{\partial\Omega}\frac{\partial}{\partial N}\Gamma\tilde{g}\,ds$$

as in (8.3) for a unique L^p-function $\tilde{g}$. Take $f\in L^{p'}(\partial\Omega)$. By (v) (and (ii) and (iii)) of Lemma 8.4 and then Remark 8.2,

$$\begin{aligned}\int_{\partial\Omega} f\Delta S_0\tilde{\Lambda}\,ds(X) &= \frac{1}{2}\int_{\partial\Omega}\text{p.v.}\int_{\partial\Omega}\frac{\partial}{\partial T_{ij}}\Gamma^Y\tilde{g}\,ds\ \text{p.v.}\,\frac{\partial}{\partial T^Y_{ij}}\int_{\partial\Omega}\Gamma^Y f\,ds(X)\,ds(Y)\\ &= \int_{\partial\Omega}\tilde{\Lambda}\int_{\partial\Omega}\Gamma^Y f\,ds\,ds(Y).\end{aligned}$$

The inside integral is the classical single-layer potential. By Proposition 4.2 the sum of exterior and interior normal derivatives may be removed from $\tilde{\Lambda}$ and the single layer differentiated. By smoothness of the boundary, the well-known jump relations for derivatives of the single-layer potential and the continuity across the boundary of $\Delta S_0\Lambda$ ((iii) of Lemma 8.4), the result is a compact operator (the dual operator to the principle-value part of the trace of the classical double-layer potential; see kernel (i) listed above) acting on $f \in L^{p'}$ integrated against the L^p-function $\Delta S_0 \Lambda$. By Schauder's theorem on the dual operators of compact operators then, the first map in the statement of the lemma is identical to the map that takes g to the image of a known compact operator of L^p acting on the function $\Delta S_0 \Lambda$. The first map is therefore compact.

Proposition 4.2 and Lemma 8.6 consequently show that the second map is compact, as is then the third. □

By Definition 8.5 (ii) and Lemma 11.2, $\mathcal{K}_\nu^0$ is the sum of a compact operator plus a product of bounded operators (as used previously, $\partial/\partial T_{ij}: L^p \to W^{-1,p}$ and $\Lambda \mapsto g$: $W^{-1,p} \to L^p$) with, by Lemma 8.4 (v), a compact operator of type (iii) listed above. Consequently,

$$\mathcal{K}_\nu^0 : W^{-1,p} \longrightarrow W^{-1,p} \quad \text{is compact.}$$

Using the above four observations on compact boundary operators, the operators $T_\nu^\mp$ of Lemma 10.5 from (9.6) are seen to differ by compact operators from the operator on $W_0^{-1,p} \times L^p$ that maps

$$(\Lambda, f) \longmapsto \left(\mp\frac{1}{2}\Lambda - \frac{1+\nu}{2} \frac{\partial}{\partial N} \int \frac{\partial}{\partial N} \Gamma f \, ds, \mp\frac{1}{2} f - \frac{1+\nu}{8} g \right), \tag{11.3}$$

where

$$\Lambda = \frac{\partial}{\partial N} \int \frac{\partial}{\partial N} \Gamma g \, ds$$

as in (8.3). By the previously observed invertibility of this latter representation for Λ (Lemma 8.1), the invertibility of (11.3) is equivalent to the invertibility of

$$(g, f) \longmapsto \left(\mp\frac{1}{2} g - \frac{1+\nu}{2} f, \mp\frac{1}{2} f - \frac{1+\nu}{8} g \right),$$

which is invertible for $\nu \neq 1$ (and -3).

THEOREM 11.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth connected boundary. Then the boundary operators of Lemma* 10.5 *are invertible for* $1 < p < \infty$.

Proof. T_ν^+ equals the invertible operator (11.3) (which may be extended one more dimension by Remark 9.1) plus a compact operator defined on $W^{-1,p} \times L^p$ for every p. By the Riesz–Schauder theory and Lemma 10.5, invertibility follows for $p=2$. The dual operator $(T_\nu^+)'$ on $W^{1,p'} \times L^{p'}$ is then invertible for $p=2$. By Schauder's theorem it is of the form invertible + compact for every p. By the inclusion $L^p(\partial\Omega) \subset L^2(\partial\Omega)$ for $p>2$ and similarly for the Sobolev and dual Sobolev spaces, the operators T_ν^+ and $(T_\nu^+)'$ are injective and thus invertible for $p \geqslant 2$ and $p' \geqslant 2$, respectively. Thus T_ν^+ is invertible for all p.

The operator T_ν^- is the sum of an invertible operator plus a compact operator, each defined on $W_0^{-1,p}\times L^p$ for all p. By the Riesz–Schauder theory, the dimension of its kernel is equal to the dimension of its cokernel. By Lemma 10.5 (i) and inclusion, T_ν^- is injective on $\mathbf{X}^p$ for $p\geqslant 2$. Thus its kernel is no larger than n (Remark 10.2). By Lemma 10.4 its cokernel is not smaller than n. Thus T_ν^- is invertible on $\mathbf{X}^p$ for $p\geqslant 2$, and its kernel is in $W_0^{-1,p}\times L^p$ for all $p<\infty$. Any nonzero member ψ of this kernel is uniquely a sum of a *nonzero* element of the complementary subspace of Remark 10.2 plus another element in any $\mathbf{X}^p$, so that ψ cannot be a member of $\mathbf{X}^p$ for any $p>1$. Thus $W_0^{-1,p}\times L^p$ is also the direct sum of $\mathbf{X}^p$ and the kernel for every p. Define the compact projection $\mathcal{PR}$ to be the identity on the kernel and the zero operator on $\mathbf{X}^p$ for all p. Then $T_\nu^-+\mathcal{PR}$ is invertible on $W_0^{-1,p}\times L^p$ for $p\geqslant 2$, and now, arguing as in the first paragraph, is invertible for all p. Because T_ν^- vanishes on the kernel and maps into $\mathbf{X}^p$ while $\mathcal{PR}$ vanishes on the latter, it follows that T_ν^- is invertible on $\mathbf{X}^p$. □

12. Invertibility of biharmonic layer potentials on Lipschitz boundaries

On Lipschitz boundaries the operators of Lemma 10.5 are injective and have closed range when $p=2$ by Lemma 10.7. They have dense range in the spaces of Lemma 10.5, and so are onto those spaces, by application of Theorem 11.3. This follows because Theorem 11.3 and the standard approximation of Lipschitz domains by smooth domains make possible a method of continuity. See [41], [25], [24] and others. A perturbation from $p=2$ to an interval around 2 follows from a Hilbert space argument of A. P. Calderón [3], which uses the fact that, by interpolation, the operator norms of the Calderón–Zygmund singular integrals studied in §8, which have norms depending only on the Lipschitz geometry of Ω, vary continuously in p. An operator from (9.6) need only be adjusted so as to be an invertible composition of Calderón–Zygmund singular integrals on (or on appropriate subspaces of) $L^2\times L^2$. This can be done by composing on the left with the classical single-layer potential and the identity operator, mapping $W^{-1,p}\times L^p$ to $L^p\times L^p$, and composing on the right with the operator of Remark 8.2 and the identity, mapping $L^p\times L^p$ to $W^{-1,p}\times L^p$. The invertibility of the latter map for p follows from known classical layer-potential results and Proposition 4.2. The invertibility of the former is discussed in Remark 8.3. That the composition of these two operators with $T_\nu^\mp$ results in a composition of bounded singular integrals on L^p-spaces follows by using the lemmas and calculations of §8. The following theorem has, therefore, been proved:

THEOREM 12.1. *Let $\Omega\subset\mathbf{R}^n$, $n\geqslant 2$, be a bounded Lipschitz domain with connected boundary, and let $T_\nu^\mp(\Lambda,f)=(\Lambda_0,f_0)$ be the system of boundary integral equations of the* *second kind* (9.6). *Then the system is uniquely solvable in the spaces* $\mathbf{X}^p$ *and* $\widetilde{\mathbf{X}}^p$ *of Definition* 10.1 *and* $W^{-1,p}(\partial\Omega)\times L^p(\partial\Omega)$; *specifically, the boundary operators*

(i) $T_\nu^-:\mathbf{X}^p\to\mathbf{X}^p$ *when* $\nu>(1-n)^{-1}$;

(ii) $T_\nu^-:\widetilde{\mathbf{X}}^p\to\widetilde{\mathbf{X}}^p$ *when* $\nu=(1-n)^{-1}$;

(iii) $T_\nu^+:W^{-1,p}\times L^p\to W^{-1,p}\times L^p$

are invertible for $2-\varepsilon<p<2+\varepsilon$.

13. Solution of the interior Neumann problem

Definition 13.1. Let $(\Lambda_0,f_0)\in W^{-1,p}(\partial\Omega)\times L^p(\partial\Omega)$ and $(1-n)^{-1}\leqslant\nu<1$. Then the L^p-Neumann problem is said to be solved for data (Λ_0,f_0) by a solution u to the biharmonic equation (1.1) if

(i) $N(\nabla\nabla u)\in L^p(\partial\Omega)$;

(ii) $K_\nu(u)=\Lambda_0$ in the sense of distributions;

(iii) $M_\nu(u)=f_0$ in the sense of nontangential convergence a.e. (ds),

where the operators of (ii) and (iii) are defined in (5.2) and (5.1).

THEOREM 13.2. *Let* $\Omega\subset\mathbf{R}^n$, $n\geqslant2$, *be a bounded Lipschitz domain with connected boundary, and let* $2-\varepsilon<p<2+\varepsilon$. *Then there exist solutions defined in* Ω *to the* L^p-*Neumann problem, unique up to the addition of linear functions for any data* (Λ_0,f_0) *taken in the subspaces* $\mathbf{X}^p$ *when* $\nu>(1-n)^{-1}$, *or unique up to the addition of the quantity* $a+\mathbf{b}\cdot X+c|X|^2$ *for data taken in* $\widetilde{\mathbf{X}}^p$ *when* $\nu=(1-n)^{-1}$. *Furthermore there is a constant* C *depending on the Lipschitz nature of* Ω, *as well as* p *and* ν, *so that*

$$\|N(\nabla\nabla u)\|_p\leqslant C\|\Lambda_0,f_0\|_p$$

when u *is a solution with data* (Λ_0,f_0).

Proof. Existence is by Theorem 12.1. Uniqueness follows by Green's first identity as in the proof of Lemma 10.5.

The solution u is therefore given as in (9.1) with $(\Lambda,f)\in\mathbf{X}^p$. By Lemma 10.4 and (9.6) it follows that $T_\nu^+(\Lambda,f)\in\mathbf{X}^p$ also.

By [5], Lemma 8.4 and the isomorphism of Proposition 4.2, $\|N(\nabla\nabla u)\|_p\leqslant C\|\Lambda,f\|_p$ with C depending on the Lipschitz nature of Ω. Letting $p=2$ and following the proof of Lemma 10.7 one arrives at

$$\|N(\nabla\nabla u)\|_2\leqslant C\|\Lambda,f\|_2\leqslant C(\|\Lambda_0,f_0\|_2+\|\nabla^{\text{ext}}\nabla u\|_2).$$

Referring to the proof of part (ii) of Theorem 7.7, that the exterior Neumann data is in $\mathbf{X}^p$, now allows for the addition of any linear function to the solution. This is because by Remark 10.3, second- and third-order derivatives decay rapidly enough at infinity to be integrated in formula (10.2) against the added linear terms, and therefore the boundary integrals on expanding spheres will vanish when (10.2) is used in the exterior domain. Consequently, by the Poincaré inequality, the estimate in (ii) of Theorem 7.7 is improved to that of (i), which can be followed by (i) (and (iii)) of Theorem 7.6. Thus (given ν) C depends only on the Lipschitz geometry of Ω.

Applying Calderón's corollary [3] as in §12, the estimate

$$\|\Lambda, f\|_2 \leqslant C \|\Lambda_0, f_0\|_2 \tag{13.1}$$

can be extended to a small interval of p's depending on the Lipschitz geometry of Ω, which finishes the proof. □

Estimate (13.1) is $\|\Lambda, f\|_2 \leqslant C\|T_\nu^-(\Lambda, f)\|_2$. The proof shows that the constant depends on the constants from the Rellich identity of §6, and therefore are explicitly tied to the Lipschitz geometry of Ω. Given this starting point, Calderón's argument, as mentioned in §12, then preserves the connection to the Lipschitz geometry of the domain for the cases $2-\varepsilon<p<2+\varepsilon$. The exterior operators are treated similarly. Consequently we have the following result:

COROLLARY 13.3. *The invertible operators of Theorem* 12.1 *are isomorphisms with constants depending only on the Lipschitz geometry of* Ω.

Note. Isomorphism will always mean that a map between vector spaces is *onto* in addition to being injective.

14. Layer-potential solutions for the Dirichlet problem

Given $(F, g) \in W^{1,p'}(\partial\Omega) \times L^{p'}(\partial\Omega)$ and $(1-n)^{-1} \leqslant \nu < 1$, define the biharmonic function

$$v(X) = \int_{\partial\Omega} K_\nu(B^X) F + M_\nu(B^X) g \, ds, \quad X \in \mathbf{R}^n \backslash \partial\Omega. \tag{14.1}$$

By [5], $N(\nabla u) \in L^{p'}$, and by §8, in particular Definitions 8.5 and 8.7, (8.9), (8.13) and Remark 8.2, v has nontangential biharmonic Dirichlet boundary values a.e. (ds),

$$\left(v, -\frac{\partial v}{\partial N}\right) = \pm\frac{1}{2}(F, g) + (F, g)\begin{pmatrix} (\mathcal{K}_\nu^0)' & -(\mathcal{K}_\nu^1)' \\ (\mathcal{M}_\nu^0)' & -(\mathcal{M}_\nu^1)' \end{pmatrix} = (T_\nu^\pm)'(F, g), \tag{14.2}$$

where the plus sign corresponds to boundary values taken from the interior, while the minus sign corresponds to boundary values from the exterior, and the square matrix is the operator transpose of that in (9.6).

It was proved in [9] that the biharmonic Dirichlet problem was uniquely solvable in the class $N(\nabla u)\in L^{p'}$ in bounded Lipschitz domains of $\mathbf{R}^n$ for any data from the spaces $W^{1,p'}\times L^{p'}$ when $2-\varepsilon<p'<2+\varepsilon$. By (14.2) and (iii) of Theorem 12.1 it follows that *solutions to the biharmonic Dirichlet problem* [9] *are uniquely representable by the potentials* (14.1) *when* $2-\varepsilon<p'<2+\varepsilon$ *and* $(1-n)^{-1}\leqslant\nu<1$.

15. Solution of the exterior Neumann problem

Definition 15.1. Let $\Omega\subset\mathbf{R}^2$ be a bounded Lipschitz domain with connected boundary, and let $(1-n)^{-1}\leqslant\nu<1$. Denote by (F_ν^{ij},g_ν^{ij}) the unique solutions to the three (interior) integral equations of (14.2), $(T_\nu^+)'(F_\nu^{ij},g_\nu^{ij})=(-Q_iQ_j,N_iQ_j+N_jQ_i)$, $Q\in\partial\Omega$ and $1\leqslant i\leqslant j\leqslant 2$. Define solutions to the exterior Dirichlet problem

$$v_\nu^{ij}(X)=\int_{\partial\Omega}(K_\nu(B^X)F_\nu^{ij}+M_\nu(B^X)g_\nu^{ij})\,ds,\quad X\in\mathbf{R}^2\setminus\bar{\Omega}.$$

Define the 3-dimensional space of solutions in the exterior

$$\mathbf{Z}_\nu=\operatorname{span}\{v_\nu^{ij}(X)+X_iX_j:1\leqslant i\leqslant j\leqslant 2\}.$$

Let $N(\cdot)$ denote the nontangential maximal function with respect to uniformly truncated nontangential approach regions.

THEOREM 15.2. *Let* $\Omega\subset\mathbf{R}^2$ *be a bounded Lipschitz domain with connected boundary. Let* $\Delta^2u=0$ *in* $\mathbf{R}^2\setminus\bar{\Omega}$ *with* $N(\nabla\nabla u)\in L^2(\partial\Omega)$ *and* $u(X)=o(|X|^3)$ *at infinity. Suppose that for some* ν, $(1-n)^{-1}\leqslant\nu<1$, u *has vanishing Neumann data on* $\partial\Omega$. *Then, up to linear functions,* $u\in\mathbf{Z}_\nu$ *and is thus* $O(|X|^2)$.

Proof. Given a point X^0 not in Ω, let $A_R=A_R(X^0)=\{X:|X-X^0|<R\}\setminus\bar{\Omega}$. For any $X\in A_R$, Green's representation formula (see (10.2)) yields

$$u(X)=\int_{\partial A_R}\left(K_\nu(B^X)u-M_\nu(B^X)\frac{\partial u}{\partial N}\right)ds-\int_{\partial A_R}\left(B^XK_\nu(u)-\frac{\partial B^X}{\partial N}M_\nu(u)\right)ds.$$

Taking two derivatives, using the hypothesis on the Neumann data and evaluating at X^0,

$$\begin{aligned}D_iD_ju(X^0)&+\int_{\partial\Omega}\left(K_\nu(D_iD_jB^{X^0})u-M_\nu(D_iD_jB^{X^0})\frac{\partial u}{\partial N}\right)ds\\&=\int_{|Q-X^0|=R}\left(K_\nu(D_iD_jB^{X^0})u-M_\nu(D_iD_jB^{X^0})\frac{\partial u}{\partial N}\right)ds\\&\quad-\int_{|Q-X^0|=R}\left(D_iD_jB^{X^0}K_\nu(u)-\frac{\partial D_iD_jB^{X^0}}{\partial N}M_\nu(u)\right)ds.\end{aligned}\qquad(15.1)$$

In the last integral, $8\pi D_iD_jB^{X^0}=2N_iN_j-\delta_{ij}(1-2\log R)$. But $K_\nu(u)$ integrated against a constant on the boundary of any bounded domain is always zero (let v equal a constant in (10.2)). Therefore the $\log R$-term in the second to last integrand vanishes, and the four integrands on the circle are of the form $R^{-\alpha}o(R^\alpha)$ for $\alpha=0,1,2,3$. The representation (15.1) may be done with respect to any other fixed point X^1 taken from the exterior, and the two compared when the radii are equal. After changing variables the difference of the right-hand sides can be written over the circle $\{Q:|Q|=R\}$ with integrands as in (15.1) but with B^0 replacing B^{X^0} and $u(Q+X^0)-u(Q+X^1)$ replacing u. Consequently the integrands will be of order $R^{-\alpha}o(R^{\alpha-1})$. Letting R go to infinity, one concludes that the left-hand side of (15.1) is constant over all X^0 in the exterior domain.

Thus

$$u(X)+\int_{\partial\Omega}\left(K_\nu(B^X)u-M_\nu(B^X)\frac{\partial u}{\partial N}\right)ds=a+\mathbf{b}\cdot X-c_{ij}X_iX_j \tag{15.2}$$

in the exterior. The quantity $u-a-\mathbf{b}\cdot X$ may be replaced with u in (15.2) without changing the integral term because, when X is taken in the exterior, (10.2) shows that the integral term vanishes on linear functions. Consequently one may take a and $\mathbf{b}$ to be zero. Letting X approach Q in the boundary, in (15.2) and for the normal derivative of both sides in (15.2), and comparing with (14.1) and (14.2), it follows that

$$(T_\nu^+)'\left(u,-\frac{\partial u}{\partial N}\right)=c_{ij}(-Q_iQ_j,N_iQ_j+N_jQ_i).$$

Thus, up to linear functions, u is in the subspace of Definition 15.1. □

Example 15.3. For general $\Lambda\in W^{-1,p}(\partial\Omega)$ the potential $S_0\Lambda(X)$ grows in the plane like $|X|^2\log|X|$, so one is led to consider uniqueness for solutions in the class of Theorem 15.2. That the conclusion is the best one can expect can be readily seen by considering solutions to the Neumann problem outside the unit disc. For example, let $v=\log|X|$. Then

$$M_\nu(v)=(1-\nu)|Q|^{-4}(|Q|^2-2(N\cdot Q)^2),$$
$$K_\nu(v)=(1-\nu)\frac{\partial}{\partial T}(|Q|^{-4}(N\cdot Q)(T\cdot Q)),$$

so that on the unit circle $M_\nu(v)=\nu-1$ while $K_\nu(v)=0$. Combining this with Remark 7.4 it follows that outside the unit disc the solutions

$$u=(1-\nu)|X|^2+2(1+\nu)\log|X|,\quad -1\leqslant\nu<1,$$

have zero Neumann data.

Let $O(R^\alpha)$ denote asymptotic behavior as $R\to\infty$.

THEOREM 15.4. *Let $\Omega\subset\mathbf{R}^n$, $n\geqslant 2$, be a bounded Lipschitz domain with connected boundary, and let $2-\varepsilon<p<2+\varepsilon$. Then there exist solutions defined in $\mathbf{R}^n\setminus\bar{\Omega}$ to the L^p-Neumann problem for any data $(\Lambda_0, f_0)\in W^{-1,p}(\partial\Omega)\times L^p(\partial\Omega)$ with the solutions unique*

(i) *in the class $O(R^{4-n})$ when $n\geqslant 5$;*

(ii) *up to constants, in the class $O(\log R)$ when $n=4$;*

(iii) *up to linear functions, in the class $O(R)$ when $n=3$;*

(iv) *up to the functions in $\mathbf{Z}_\nu$ plus linear functions, in the class $O(R^2\log R)$ when $n=2$.*

Furthermore there is a constant C depending on the Lipschitz nature of Ω, as well as p and ν, so that

$$\|N(\nabla\nabla u)\|_p \leqslant C\|\Lambda_0, f_0\|_p$$

when u is a solution with data (Λ_0, f_0) and $n\geqslant 3$.

Proof. Existence is by Theorem 12.1, and the dependence of C on the Lipschitz geometry is as in the proof of Theorem 13.2.

Uniqueness in each case varies from the proof of Theorem 15.2 in minor ways. For example, when $n=3$, taking only the first derivatives of u in Green's representation formula suffices to obtain (15.2) with the c_{ij} equal to zero. Thus u is linear plus $O(R^{-1})$, which is enough to conclude that

$$\int_{|Q|=R}\left(K_\nu(u)u - M_\nu(u)\frac{\partial u}{\partial N}\right)ds$$

vanishes as R goes to infinity. This then allows one to conclude from Green's first identity in the exterior that u is in fact linear. $\square$

Remark 15.5. When the data is taken in the space $W_0^{-1,p}(\partial\Omega)\times L^p(\partial\Omega)$, uniqueness is obtained

(i) in the class $O(R^{3-n})$ when $n\geqslant 4$;

(ii) up to constants, in the class $O(1)$ when $n=3$;

(iii) up to linear functions, in the class $O(R\log R)$ when $n=2$.

16. The biharmonic equilibrium distribution for $n\geqslant 3$

Let $\Omega\subset\mathbf{R}^n$ be a bounded Lipschitz domain with connected boundary and suppose that Ω contains the origin. Denote by $\widetilde{\Omega}$ the bounded domain that is obtained by reflecting $\bar{\Omega}^c$ in the unit sphere under the transformation $X\mapsto X/|X|^2$.

A biharmonic Green function with nontangential L^2-estimates on all second derivatives near the boundary can be constructed for $\tilde{\Omega}$ in the standard way (see [32, pp. 388–389] and [33, pp. 23, 33 and 35]): Given $X\in\tilde{\Omega}$, let $W^X(Y)$ be the unique solution to the biharmonic Dirichlet problem with data on $\partial\tilde{\Omega}$, $W^X(Q)=B^X(Q)$ and $\partial W^X/\partial N=\partial B^X/\partial N$ a.e. $ds(Q)$. The solution of the biharmonic regularity problem for the Dirichlet problem [43] implies that $N(\nabla\nabla W^X)\in L^p(\partial\tilde{\Omega})$ for $p<2+\varepsilon$. Defining

$$G(X,Y)=B(X-Y)-W^X(Y) \tag{16.1}$$

one obtains the biharmonic Poisson integral representation for solutions in terms of Dirichlet data from $W^{1,p'}\times L^{p'}$,

$$u(X)=\int_{\partial\tilde{\Omega}}\left(uK_\nu(G^X)-\frac{\partial u}{\partial N}M_\nu(G^X)\right)ds.$$

A computation shows that $M_\nu(G^X)=\Delta G^X$ and (in the sense of distributions) $K_\nu(G^X)=\partial\Delta G^X/\partial N$.

If X is fixed in the exterior domain to $\tilde{\Omega}$, $B^X(Y)$ is a solution in $\tilde{\Omega}$. Using the Poisson representation at the origin,

$$B(X)=\int_{\partial\tilde{\Omega}}\left(B^XK_\nu(G^0)-\frac{\partial B^X}{\partial N}M_\nu(G^0)\right)ds,\quad X\in\mathbf{R}^n\setminus\bar{\tilde{\Omega}}. \tag{16.2}$$

Define $\tilde{\Lambda}=K_\nu(G^0)$ and $\tilde{f}=M_\nu(G^0)$. Then $(\tilde{\Lambda},\tilde{f})\in W^{-1,p}\times L^p$, $p<2+\varepsilon$. The biharmonic Kelvin transform takes solutions $u(X)$ to solutions $v(X)=|X|^{4-n}u(X/|X|^2)$ at reflected points. Kelvin transforming (16.2) yields

$$\begin{aligned}[2(n-4)(n-2)\omega_n]^{-1}&=\int_{\partial\Omega}B^X(Q)\tilde{\Lambda}\left(\frac{Q}{|Q|^2}\right)\frac{ds(Q)}{|Q|^{n+2}}\\&\quad+(n-4)\int_{\partial\Omega}B^X(Q)\tilde{f}\left(\frac{Q}{|Q|^2}\right)N^Q\cdot Q\frac{ds(Q)}{|Q|^{n+2}}\\&\quad+\int_{\partial\Omega}\frac{\partial B^X(Q)}{\partial N}\tilde{f}\left(\frac{Q}{|Q|^2}\right)\frac{ds(Q)}{|Q|^n}\end{aligned} \tag{16.3}$$

for $n\neq2,4$ and every $X\in\Omega$.

When $n=4$ the logarithmic fundamental solution and the fact that

$$\int_{\partial\tilde{\Omega}}\tilde{\Lambda}\,ds=\int_{\partial\tilde{\Omega}}K_\nu(G^X)\,ds=1$$

for $X\in\tilde{\Omega}$ cause (16.2) to transform to

$$\begin{aligned}&\frac{1}{4\omega_4}\int_{\partial\tilde{\Omega}}\left((\log|P|)K_\nu(G^0)(P)-\frac{\partial\log|P|}{\partial N^P}M_\nu(G^0)(P)\right)ds(P)\\&\quad=\int_{\partial\Omega}\left(B^X(Q)\tilde{\Lambda}\left(\frac{Q}{|Q|^2}\right)+\frac{\partial B^X(Q)}{\partial N}\tilde{f}\left(\frac{Q}{|Q|^2}\right)|Q|^2\right)\frac{ds(Q)}{|Q|^6}\end{aligned} \tag{16.4}$$

for all $X\in\Omega$. The constant in (16.3) is invariant under scaling. However, as with the logarithmic potential for Laplace's equation in the plane, the constant value in (16.4) is not. See Remark 16.4 below.

Definition 16.1. For $n\geqslant 3$ let $\Omega\subset\mathbf{R}^n$ be a bounded Lipschitz domain with connected boundary. Suppose that Ω contains the origin. The biharmonic equilibrium distribution for Ω is defined to be $(\Lambda^*, f^*)\in W^{-1,p}(\partial\Omega)\times L^p(\partial\Omega)$, $p<2+\varepsilon$, where

$$\Lambda^*(Q)=\left[\tilde{\Lambda}\left(\frac{Q}{|Q|^2}\right)+(n-4)\tilde{f}\left(\frac{Q}{|Q|^2}\right)N^Q\cdot Q\right]|Q|^{-n-2}$$

and

$$f^*(Q)=-\tilde{f}\left(\frac{Q}{|Q|^2}\right)|Q|^{-n}.$$

Here $\tilde{\Lambda}$ and $\tilde{f}$ are the Poisson kernels of (16.2) for the bounded domain that is the reflection of $\mathbf{R}^n\backslash\bar{\Omega}$ in the unit sphere centered at the origin.

Remark 16.2. With the bounded domains Ω and $\tilde{\Omega}$ related by reflection as above and $X=Y/|Y|^2$, $X\in\Omega$, $Q=P/|P|^2$, $Q\in\partial\Omega$, the identities below are useful when computing with the Kelvin transform. Recall that N always denotes the *outer* unit normal to a *bounded* domain.

(i) $|P-Y|=\dfrac{|Q-X|}{|Q|\,|X|}$;

(ii) $N^Q=2P\dfrac{P\cdot N^P}{|P|^2}-N^P$;

(iii) $\dfrac{\partial F(P/|P|^2)}{\partial N^P}=-|Q|^2\dfrac{\partial F(Q)}{\partial N^Q}$;

(iv) $ds(P)=\dfrac{ds(Q)}{|Q|^{2n-2}}$.

Remark 16.3. Let $W^0(Y)$, $Y\in\tilde{\Omega}$, be the solution from the construction of the Green function for $\tilde{\Omega}$ above. Put $u(X)=|X|^{4-n}W^0(X/|X|^2)$, $X=Y/|Y|^2$. Then on $\partial\Omega$, u is equal to the constant on the left-hand side of (16.3) when $n\neq 2,4$. Also,

$$D_ju(X)=(4-n)X_j|X|^{-2}u(X)+|X|^{4-n}D_iW^0(Y)(\delta_{ij}-2X_iX_j|X|^{-2})|X|^{-2},$$

so

$$\nabla u(X)=(4-n)Yu(X)+\nabla W^0(Y)|Y|^{n-2}-2Y(Y\cdot\nabla W^0(Y))|Y|^{n-4}.$$

Since $\nabla W^0(P)=\nabla B(P)=-[2(n-2)\omega_n]^{-1}P|P|^{2-n}$ for $P\in\partial\tilde{\Omega}$, it follows that ∇u vanishes on $\partial\Omega$ for $n\neq 2,4$.

When $n\geqslant 5$, u is a solution in the exterior domain to Ω that decays rapidly enough at infinity so that Green's representation formula and the vanishing of the gradient yield

$$u(X)=-\int_{\partial\Omega}K_\nu(B^X)u\,ds+\int_{\partial\Omega}\left(B^XK_\nu(u)-\frac{\partial B^X}{\partial N}M_\nu(u)\right)ds,\quad X\in\mathbf{R}^n\backslash\bar{\Omega}.$$

But as in the proof of Theorem 15.2, the first term vanishes because u is constant. The last integral and its gradient in X are continuous across the boundary. By uniqueness for the Dirichlet problem then, the integral representation is identically equal to the constant of (16.3) for $X\in\Omega$. It should follow by (16.3) that $(K_\nu(u),M_\nu(u))$ is the equilibrium distribution (Λ^*,f^*) when $n\geqslant 5$.

This is a question of uniqueness for the potential solutions $u=S_0\Lambda-S_1f$ that can be answered in the standard way. If such a u is identically zero in Ω, then $T_\nu^-(\Lambda,f)=(0,0)$. The Dirichlet data also vanishes and is continuous across the boundary. Assuming $n\geqslant 5$, Green's first identity (10.2), the vanishing Dirichlet data and the decay of u at infinity show that u vanishes in all of $\mathbf{R}^n$. Then $T_\nu^+(\Lambda,f)=(0,0)$, and by (9.6), $(\Lambda,f)=(0,0)$. This argument remains valid for an interval of $p<2$ bounded from below by a Sobolev exponent.

In particular, the equilibrium distribution is independent of the choice of origin.

Remark 16.4. Let $t\tilde{\Omega}=\{Y:t^{-1}Y\in\tilde{\Omega}\}$ and let Ω_t denote the interior of the complement of its reflection (i.e. $\Omega_t=t^{-1}\Omega$). Denote by $\tilde{\Lambda}_t$ and $\tilde{f}_t$ the Poisson kernels of (16.2) corresponding to the domains $t\tilde{\Omega}$. Then $\tilde{\Lambda}_t(P)=t^{1-n}\tilde{\Lambda}(t^{-1}P)$ and $\tilde{f}_t(P)=t^{2-n}\tilde{f}(t^{-1}P)$. Writing (16.4) for the domains Ω_t and computing the constant value on the left by a change of variables yield

$$\begin{aligned}\frac{1}{4\omega_4}\log t+\frac{1}{4\omega_4}\int_{\partial\tilde{\Omega}}\left((\log|P|)K_\nu(G^0)(P)-\frac{\partial\log|P|}{\partial N^P}M_\nu(G^0)(P)\right)ds(P)\\ =\int_{\partial\Omega_t}\left(B^X(Q)\tilde{\Lambda}_t\left(\frac{Q}{|Q|^2}\right)+\frac{\partial B^X(Q)}{\partial N}\tilde{f}_t\left(\frac{Q}{|Q|^2}\right)|Q|^2\right)\frac{ds(Q)}{|Q|^6}.\end{aligned}$$

Letting (Λ_t^*,f_t^*) denote the equilibrium distribution of Definition 16.1 for the domains Ω_t, it follows that

given a bounded Lipschitz domain $\Omega\subset\mathbf{R}^4$ with connected boundary, the potential solutions $S_0\Lambda_t^*(X)-S_1f_t^*(X)$ for each Ω_t are constant for $X\in\Omega_t$, and the constant is nonzero for every value of $t>0$ with the exception of one value.

Definition 16.5. The biharmonic single-layer potential of (Λ,f) is defined to be the solution (9.1) and is denoted by

$$S(\Lambda,f)(X)=S_0\Lambda(X)-S_1f(X).$$

THEOREM 16.6. *Let $n\geqslant 3$ and let $\Omega\subset\mathbf{R}^n$ be a bounded Lipschitz domain with connected boundary. Then the equilibrium distribution (Λ^*, f^*) of Definition 16.1 is the basis for the unique 1-dimensional subspace of $W^{-1,p}(\partial\Omega)\times L^p(\partial\Omega)$, $p<2+\varepsilon$, upon which the biharmonic single-layer potential becomes the constant solution in Ω.*

Moreover, $S(\Lambda^*, f^*)(X)$ for $X\in\Omega$ takes the constant value of (16.3) when $n\neq 4$, and the constant value of (16.4) when $n=4$. Further,

$$\int_{\partial\Omega}\Lambda^* ds\neq 0 \tag{16.5}$$

and $\|\Lambda^*, f^*\|_p\leqslant C|\partial\Omega|^{1/p-2/(n-1)}$, where C depends only on p and the Lipschitz geometry of Ω for $n\geqslant 3$.

Proof. By translation it may be assumed that Ω contains the origin. Let (Λ^*, f^*) be as in Definition 16.1.

Suppose first that there exists $(\Lambda, f)\in W_0^{-1,p}\times L^p$ so that $u=S_0\Lambda-S_1 f$ is a constant solution. The assumption on Λ provides the additional decay to conclude, as in Remark 16.3, that (Λ, f) vanishes identically. It follows that $\int_{\partial\Omega}\Lambda^* ds\neq 0$, at least in the case $n\neq 4$. When $n=4$, the vanishing of (Λ^*, f^*) implies the vanishing of $(\tilde{\Lambda}, \tilde{f})$, which contradicts (16.2). Thus the same conclusion holds for $n=4$. If now $u=S_0\Lambda-S_1 f$ is a constant solution with $\int_{\partial\Omega}\Lambda\, ds\neq 0$, it follows that (Λ, f) is a constant multiple of (Λ^*, f^*) in either case.

Now the translation invariance of both sides of (16.3) and uniqueness show that the definition of (Λ^*, f^*) is independent of the choice of origin when $n\neq 4$. The left-hand side of (16.4) is also independent of the choice of origin in Ω. This can be seen by considering a domain Ω for which the integral (see Remark 16.4) on the left of (16.4) vanishes. If (Λ^{**}, f^{**}) represents the equilibrium distribution for a translation of Ω, the translation invariance of the right-hand side of (16.4) shows that $S(\Lambda^{**}, f^{**})$ vanishes, and uniqueness then shows that (Λ^{**}, f^{**}) and (Λ^*, f^*) are scalar multiples. The formula in Remark 16.4 shows that scaling both domains by the same amount results in the same constant value $(\log t)$ for the single-layer potentials of each distribution. By translation invariance of the single-layer potentials and the 1-dimensional subspace uniqueness again, it follows that the scalar multiple was 1. Thus the equilibrium distribution and "the constant value" are well-defined.

The norm of the equilibrium distribution is bounded by the norm of the Poisson kernel $(\tilde{\Lambda}, \tilde{f})$, which depends on the Lipschitz geometry of Ω by the solvability of the Dirichlet regularity problem [43], [31], [33]. □

COROLLARY 16.7. *For a bounded Lipschitz domain $\Omega\subset\mathbf{R}^n$, $n\neq 2, 4$, with connected boundary and $X\in\Omega$, the solution W^X from the construction of the biharmonic Green function (16.1) has the property that $W^X(X)\neq 0$.*

Proof. Using Definition 16.1 and changing variables by reflection in (16.5) yields

$$\int_{\partial\Omega} \Lambda^* ds = \int_{\partial\tilde{\Omega}} \left(|P|^{4-n} K_\nu(G^0)(P) - \frac{\partial |P|^{4-n}}{\partial N^P} M_\nu(G^0)(P) \right) ds(P) \neq 0.$$

Up to a constant multiple this is $W^0(0)$ when $n \neq 2, 4$. The roles of Ω and $\tilde{\Omega}$ may be switched, and the origin replaced by X. □

Remark 16.8. In $\mathbf{R}^4$ the ball of radius $\sqrt{e}$ fails to have the property of the corollary, since for it, $W^0(X) = -(8e\omega_4)^{-1}|X|^2$. The corollary holds for $n=2$, as shown below, and as known in the classical case from a result of Hadamard. See [36, p. 140].

17. The regularity problem and the biharmonic single-layer potential

The densities (Λ, f), that are mapped by the single-layer potential $S(\Lambda, f)$ of Definition 16.5 to linear functions in a bounded Lipschitz domain Ω, can likewise be identified. Let

$$\tilde{\Lambda}_j = \frac{\partial}{\partial Z_j} K_\nu(G^Z)\Big|_{Z=0} \quad \text{and} \quad \tilde{f}_j = \frac{\partial}{\partial Z_j} M_\nu(G^Z)\Big|_{Z=0}, \quad j = 1, ..., n, \tag{17.1}$$

where the Green function is as in (16.2). Then the analogue of (16.2) is

$$-D_j B(X) = \int_{\partial\tilde{\Omega}} \left(B^X \tilde{\Lambda}_j - \frac{\partial B^X}{\partial N} \tilde{f}_j \right) ds, \quad X \in \mathbf{R}^n \setminus \bar{\tilde{\Omega}}.$$

Defining Λ_j^* and f_j^* with respect to $\tilde{\Lambda}_j$ and $\tilde{f}_j$ exactly as in Definition 16.1, the analogue of (16.3) and (16.4) is

$$[2(n-2)\omega_n]^{-1} X_j = S(\Lambda_j^*, f_j^*)(X), \quad X \in \Omega, \ j = 1, ..., n, \ n \geqslant 3. \tag{17.2}$$

As in Theorem 16.6, $\|\Lambda_j^*, f_j^*\|_p$ depends on p and the Lipschitz geometry of Ω.

Definition 17.1. The *linear equilibrium distributions* (Λ_j^*, f_j^*) are defined by (17.2).

It will be convenient to refer to all of the equilibrium distributions that have been defined as the *affine equilibrium distributions*, and to sometimes write (Λ_0^*, f_0^*) for (Λ^*, f^*).

Remark 17.2. The set of affine equilibrium distributions from Definitions 16.1 and 17.1 is a linearly independent set by (17.2) and (16.3) or (16.4), $n \geqslant 3$. Because the operator T_ν^- vanishes on this set and is injective on the subspace X^p, it follows that

$$\mathbf{X}^p(\partial\Omega) \oplus \text{span}\{(\Lambda^*, f^*), (\Lambda_i^*, f_i^*) : i = 1, ..., n\} = W^{-1,p}(\partial\Omega) \times L^p(\partial\Omega).$$

Any biharmonic function u with $N(\nabla\nabla u)\in L^p$ will take nontangential boundary values $(u,\nabla u)$ in a space $WA_2^p(\partial\Omega)$ consisting of arrays of Sobolev functions known as Whitney arrays [43], [33, p. 25]. For example, any of the solutions above to the interior or exterior Neumann problem will do this. A member of $WA_2^p(\partial\Omega)$ is written $\dot f=\{f_0,f_1,\dots,f_n\}\subset W^{1,p}(\partial\Omega)$. An array $\dot f$ is in $WA_2^p(\partial\Omega)$ if and only if for all $1\leqslant i<j\leqslant n$ the compatibility conditions

$$\frac{\partial}{\partial T_{ij}}f_0(Q)=N_i f_j(Q)-N_j f_i(Q),\quad Q\in\partial\Omega, \tag{17.3}$$

hold. The compatibility conditions show that an array $\dot f$ is the zero array if and only if f_0 and (summation notation) $N_i f_i$ both vanish. Thus WA_2^p may be identified with the subspace

$$\{(F,g):\text{there exists } \dot F\in WA_2^p \text{ such that } F_0=F \text{ and } -N_iF_i=g\} \tag{17.4}$$

of $W^{1,p}\times L^p$. Since $(u,\nabla u)$ satisfies (17.3) one can write $\dot u=(u,\nabla u)$ on $\partial\Omega$ and identify it in (17.4) with the Dirichlet data $(u,-\partial u/\partial N)$.

$WA_2^p(\partial\Omega)$ is a Banach space under the norm

$$\|\dot f\|_{2,p}=|\partial\Omega|^{2/(1-n)}\|f_0\|_p+|\partial\Omega|^{1/(1-n)}\sum_{i=1}^n\|f_i\|_p+\sum_{i=1}^n\|\nabla_T f_i\|_p.$$

The *regularity problem*, given data in WA_2^p, is to find a unique solution u in the class $N(\nabla\nabla u)\in L^p$ so that the data is assumed by $\dot u$. The problem was formulated and solved in [43] and more generally in [33]. Every solution to the regularity problem is also a solution to a Neumann problem.

Remark 17.3. The subspace (17.4) may be described independently as the set of $(F,g)\in W^{1,p}\times L^p$ such that $N_i\,\partial F/\partial T_{ij}-N_jg\in W^{1,p}$ for $j=1,\dots,n$. It can be shown that $F_0=F$ and $F_j=N_i\,\partial F/\partial T_{ij}-N_jg$ form an array $\dot F$ by using extensions of $W^{1,p}(\partial\Omega)$-functions and smooth approximating domains to justify the integrations by parts.

That the "angle", of the images under the single-layer potential of the subspace $\mathbf{X}^p$ and its complementary subspace of equilibrium distributions (see Remark 17.2), will be controlled by the Lipschitz geometry of the domain is the purpose of the following lemma:

LEMMA 17.4. *Let T_0 and T_1 be bounded linear operators on a Banach space $L^p=X\oplus X^c$, and suppose that there is a constant C so that*

$$\begin{aligned} f^*\in X^c &\implies \|f^*\|\leqslant C\|T_0f^*\|,\\ f^*\in X^c &\implies T_1f^*=0,\\ f\in X &\implies \|f\|\leqslant C\|T_1f\|. \end{aligned}$$

Then there is a constant C' depending only on C and $\|T_0\|$ so that

$$\|f^*+f\| \leqslant C'(\|T_0(f^*+f)\|+\|T_1(f^*+f)\|).$$

Proof. Let $A>1$ and suppose that $\|f^*\|\leqslant A\|f\|$. Then

$$\|f^*+f\| \leqslant (1+A)\|f\| \leqslant (1+A)C\|T_1 f\| = (1+A)C\|T_1(f^*+f)\|.$$

Otherwise $\|f^*\|>A\|f\|$, which implies that $(A-1)\|f\|\leqslant\|f^*+f\|$ and

$$\begin{aligned}\|f^*+f\| &\leqslant (1+A^{-1})\|f^*\| \leqslant (1+A^{-1})C\|T_0 f^*\| \\ &\leqslant (1+A^{-1})C\|T_0(f^*+f)\| + \frac{1+A^{-1}}{A-1}C\|T_0\|\,\|f^*+f\|.\end{aligned}$$

Now A is chosen to hide the last term. □

THEOREM 17.5. *For $n\geqslant 3$ let $\Omega\subset\mathbf{R}^n$ be a bounded Lipschitz domain with connected boundary, and let S denote the biharmonic single-layer potential of Definition* 16.5. *If $n\neq 4$, the map*

$$\begin{aligned}(S,\nabla S)\colon W^{-1,2}(\partial\Omega)\times L^2(\partial\Omega) &\longrightarrow WA_2^2(\partial\Omega),\\ (\Lambda,f) &\longmapsto \dot{S}(\Lambda,f)\end{aligned}$$

is an isomorphism with constants depending only on the Lipschitz geometry of Ω.

When $n=4$ the constants also depend on a scaling factor, with the isomorphism failing for the exceptional domains of Remark 16.4. *The isomorphism holds, with constants depending only on the Lipschitz geometry of Ω, when S maps from $W_0^{-1,2}\times L^2$ to the codimension-*1 *subspace of WA_2^2-functions $\dot{F}$ satisfying $\int_{\partial\Omega}(\Lambda^* F_0 - f^* N_j F_j)\,ds=0$, where (Λ^*, f^*) is the equilibrium distribution for Ω.*

Proof. As in the proof of Lemma 10.7, a suitable bound

$$|\!|\!|\Lambda,f|\!|\!|_2 \leqslant C(\|\nabla^{\mathrm{ext}}\nabla S(\Lambda,f)\|_2+\|\nabla^{\mathrm{int}}\nabla S(\Lambda,f)\|_2)$$

follows from (9.6). If $(\Lambda,f)\in\mathbf{X}^2$, then, as in the proof of Theorem 13.2, (ii) of Theorem 7.7 improves so that $|\!|\!|\Lambda,f|\!|\!|_2\leqslant C\|\nabla_T\nabla S(\Lambda,f)\|_2$. When $n\neq 4$ any affine equilibrium distribution satisfies, by its construction, $|\!|\!|\Lambda^*,f^*|\!|\!|_2\leqslant C\|\dot{S}(\Lambda^*,f^*)\|_{2,2}$. By Lemma 17.4 (and [5]) the bounds for the isomorphism are proved.

That the single-layer map is injective follows from Theorem 16.6 and its proof.

Given data $\dot{F}$, the unique solution u to the biharmonic regularity problem with data $\dot{F}$ is supplied by Theorem 2 of [33]. By Lemma 10.4 and Theorem 12.1 there is a

$(\Lambda, f)\in \mathbf{X}^2$ so that $T_\nu^-(\Lambda, f)=(K_\nu^{\text{int}}(u), M_\nu^{\text{int}}(u))$, and consequently $S(\Lambda, f)$ and u differ by a linear function. The affine equilibrium distributions constructed above show that the map is onto.

Modifications establish the case $n=4$. □

Remark 17.6. By construction, the operator $(S, -\partial S/\partial N)$ maps the affine equilibrium distributions onto $\operatorname{span}\{(1,0),(Q_i,-N_i): i=1,\dots,n\}$, the vectors which integrate to zero against $\mathbf{X}^p$ (Definition 10.1). Thus if $(\Lambda, f)\in\mathbf{X}^p$ and if (Λ_i^*, f_i^*) is any equilibrium distribution, an application of the Fubini theorem (in the sense of distributions) shows that

$$\int_{\partial\Omega}\left(S(\Lambda,f)\Lambda^*-\frac{\partial}{\partial N}S(\Lambda,f)f^*\right)ds=\int_{\partial\Omega}\left(S(\Lambda^*,f^*)\Lambda-\frac{\partial}{\partial N}S(\Lambda^*,f^*)f\right)ds=0.$$

Defining

$$\mathbf{Y}^p(\partial\Omega)=\left\{\dot F\in WA_2^p:\int_{\partial\Omega}(F_0\Lambda_j^*-N_iF_if_j^*)\,ds=0,\ j=0,\dots,n\right\}$$

and identifying it with its subspace of (17.4), Theorem 17.5 and Remark 17.2 show that $(S,-\partial S/\partial N)\colon W^{-1,2}\times L^2\to WA_2^2$ splits into isomorphisms mapping $\mathbf{X}^2$ *onto* $\mathbf{Y}^2$ and mapping $\operatorname{span}\{(\Lambda^*,f^*),(\Lambda_i^*,f_i^*): i=1,\dots,n\}$ *onto* $\operatorname{span}\{(1,0),(Q_i,-N_i): i=1,\dots,n\}$.

18. Regularity and the biharmonic double-layer potential

Definition 18.1. Given ν and $(F,g)\in W^{1,p}\times L^p$, the *biharmonic double-layer potential* is defined by (14.1) and is written

$$\mathsf{D}_\nu(F,g)(X)=\int_{\partial\Omega}(K_\nu(B^X)F+M_\nu(B^X)g)\,ds.$$

By Green's representation formula any solution u to the interior Neumann problem can be written in terms of its Dirichlet and Neumann data

$$u(X)=\mathsf{D}_\nu\left(u,-\frac{\partial u}{\partial N}\right)(X)-S(K_\nu(u),M_\nu(u))(X),\quad X\in\Omega. \tag{18.1}$$

In addition to the Dirichlet data of the double-layer potential being bounded on $W^{1,p}\times L^p$, the trace of the double layer also maps WA_2^p to WA_2^p, $1<p<\infty$, by the lemmas of §8. Or solvability of the regularity problem [43] and (18.1) conveniently give the result for $2-\varepsilon<p<2+\varepsilon$.

Any solution to the exterior Neumann problem can be written

$$u(X)+r(X)=-\mathsf{D}_\nu\left(u,-\frac{\partial u}{\partial N}\right)(X)+S(K_\nu(u),M_\nu(u))(X),\quad X\in\mathbf{R}^n\setminus\bar\Omega, \tag{18.2}$$

where $r(X)$ is a constant, linear function etc., according to the cases (ii)–(iv) of Theorem 15.4. This is seen by applying the arguments of the proof of Theorem 15.2. If, however, $u=S(\Lambda,f)$ then $r(X)$ vanishes. For then by (9.6), $(K_\nu^{\text{ext}}(u)-\Lambda, M_\nu^{\text{ext}}(u)-f)=(K_\nu^{\text{int}}(u), M_\nu^{\text{int}}(u))$. So, combining the single-layer terms from the left and right of (18.2), the identity

$$\mathsf{D}_\nu\left(S(\Lambda,f), -\frac{\partial S(\Lambda,f)}{\partial N}\right)(X)=S(K_\nu^{\text{int}}(S(\Lambda,f)), M_\nu^{\text{int}}(S(\Lambda,f)))(X), \tag{18.3}$$

for X in the exterior, is just Green's first identity in Ω. Computing the Dirichlet data from the exterior yields, by (14.2) and (9.6), the operator identity

$$(T_\nu^-)'\left(S, -\frac{\partial}{\partial N}S\right)=\left(ST_\nu^-, -\frac{\partial}{\partial N}ST_\nu^-\right) \tag{18.4}$$

acting on (Λ, f). (This is in exact analogy to the classical harmonic case. See, for example, the top of [41, p. 590].) The right-hand side is, by Theorem 12.1 (and Corollary 13.3), an isomorphism of $\mathbf{X}^p$ followed by $(S, -\partial S/\partial N)$, which is, by Theorem 17.5, an isomorphism from $\mathbf{X}^2$ onto the subspace $\mathbf{Y}^2$ of Remark 17.6. Consequently the Lipschitz dependence for the bounds of these isomorphisms, when $\nu>(1-n)^{-1}$, shows that

$$(T_\nu^-)'\colon \mathbf{Y}^2 \longrightarrow \mathbf{Y}^2$$

is an isomorphism with constants depending only on the Lipschitz geometry of Ω.

The same analysis starting with (18.1) yields

$$(T_\nu^+)'\left(S, -\frac{\partial}{\partial N}S\right)=\left(ST_\nu^+, -\frac{\partial}{\partial N}ST_\nu^+\right),$$

and

$$(T_\nu^+)'\colon WA_2^2 \longrightarrow WA_2^2$$

is an isomorphism with constants depending only on the Lipschitz geometry of Ω.

There is no exceptional case when $n=4$ because $(T_\nu^+)'$ maps the linear functions onto the linear functions (Green's identity), and is also an isomorphism of $\mathbf{Y}^2$ with the norm $\|\nabla_T\nabla \mathsf{D}_\nu(F_0, -N_iF_i)\|_2$ equivalent to $\|\dot{F}\|_{2,2}$ there (by the operator identity). Since $\nabla_T\nabla$ vanishes on the linear functions, Lemma 17.4 applies as in the proof of Theorem 17.5.

19. Potentials for the regularity problem $2-\varepsilon<p<2+\varepsilon$

Let u and v be biharmonic functions in Ω. Fixing any derivative D_l and using the identity $(\partial/\partial N)D_l=(\partial/\partial T_{il})D_i+N_l\Delta$ one obtains the integration-by-parts formula

$$\int_{\partial\Omega}\left(\frac{\partial}{\partial N}D_l\Delta v\, u-\frac{\partial}{\partial N}D_l\nabla v\cdot\nabla u\right)ds=\int_{\partial\Omega}\left(D_iD_jv\frac{\partial}{\partial T_{il}}D_ju-\frac{\partial}{\partial N}\Delta v\, D_lu\right)ds.$$

Introducing another derivative further yields

$$\int_{\partial\Omega}\left(\frac{\partial}{\partial N}D_kD_l\Delta v\, u-\frac{\partial}{\partial N}D_kD_l\nabla v\cdot\nabla u\right)ds \\ =\int_{\partial\Omega}\left(D_iD_jD_kv\,\frac{\partial}{\partial T_{il}}D_ju+D_i\Delta v\,\frac{\partial}{\partial T_{ik}}D_lu\right)ds. \tag{19.1}$$

Let $\nu=0$. With X in the exterior in (18.3) and taking two derivatives of the left-hand side in X, formula (19.1) shows that

$$\frac{\partial^2}{\partial X_k\partial X_l}\mathsf{D}_0\left(S,-\frac{\partial S}{\partial N}\right)(X)=\int_{\partial\Omega}\left(D_iD_jD_kB^X\frac{\partial}{\partial T_{il}}D_jS+D_i\Gamma^X\frac{\partial}{\partial T_{ik}}D_lS\right)ds. \tag{19.2}$$

Letting $v=\mathsf{D}_0(S,-\partial S/\partial N)$ and $S=S(\Lambda,f)$, (5.5) and (19.2) show that

$$|||K_0^{\text{ext}}(v),M_0^{\text{ext}}(v)|||_p\leqslant C\|\nabla\nabla v\|_p\leqslant C\|\nabla_T\nabla S\|_p \tag{19.3}$$

for $2-\varepsilon<p<\infty$. On the other hand, the right-hand side of (18.3) shows that $T_0^+T_0^-(\Lambda,f)=(K_0^{\text{ext}}(v),M_0^{\text{ext}}(v))$, i.e. by Theorem 12.1 and (19.3),

$$|||(\Lambda,f)|||_p\leqslant C|||T_0^+T_0^-(\Lambda,f)|||_p\leqslant C\|\nabla_T\nabla S(\Lambda,f)\|_p \tag{19.4}$$

for all $(\Lambda,f)\in\mathbf{X}^p$, $2-\varepsilon<p<2+\varepsilon$, with the constants depending only on p and the Lipschitz geometry of Ω.

THEOREM 19.1. *When $n\geqslant3$ the statements of Theorem 17.5 extend to $2-\varepsilon<p<2+\varepsilon$, i.e.*

$$(S,\nabla S): W^{-1,p}(\partial\Omega)\times L^p(\partial\Omega)\longrightarrow WA_2^p(\partial\Omega)$$

is an isomorphism with bounds depending only on p and the Lipschitz geometry of Ω.

Proof. Estimate (19.4), the corresponding p-bounds for the affine equilibrium distributions (as in the proof of Theorem 17.5) and Lemma 17.4 yield

$$|||(\Lambda,f)|||_p\leqslant C\|\dot{S}(\Lambda,f)\|_{2,p}$$

for all (Λ,f). The map is therefore injective. This estimate, the surjectivity for $p=2$ and a limiting argument show that it is onto. □

Remark 19.2. A duality argument as in [41] is also possible. This would use solvability of the biharmonic Dirichlet problem and the area integral estimates of [30] or [8]. As a consequence the single layer can be shown to be invertible for $1<p<2+\varepsilon$ when $n=3$ because of [31], and for $2(n-1)/(n+1)-\varepsilon<p<2+\varepsilon$ when $n\geqslant4$ by the recent result of Z. Shen [37].

The identity (18.3) and its interior analogue show the following result:

COROLLARY 19.3. *For $2-\varepsilon<p<2+\varepsilon$ and $n\geqslant 3$,*

$$(T_\nu^+)'\colon WA_2^p \longrightarrow WA_2^p$$

and, when $\nu>(1-n)^{-1}$,

$$(T_\nu^-)'\colon \mathbf{Y}^p \longrightarrow \mathbf{Y}^p$$

are isomorphisms with bounds depending only on p and the Lipschitz geometry of Ω.

20. Equilibrium and regularity in the plane

An analysis of the single- and double-layer potentials, their relation to equilibrium distributions, and their application to the regularity problem, can also be done in the plane. The logarithmic fundamental solution and the low dimension bring out more of the technical problems already seen in dimensions 3 and 4. A few comments will be made.

Defining $(\Lambda_0^*, f_0^*)=(\Lambda^*, f^*)$, when $n=2$ as in Definition 16.1 with respect to $(\tilde\Lambda_0, \tilde f_0)=(\tilde\Lambda, \tilde f)$ in (16.2), does not produce the equilibrium distribution. It turns out that when defined in this way, $(\Lambda_0^*, f_0^*)\in\mathbf{X}^p$, so that if $S(\Lambda_0^*, f_0^*)$ were any linear function in Ω, then the argument of Lemma 10.5 would imply that (Λ_0^*, f_0^*) was identically zero. Instead consider all the densities (Λ_i^*, f_i^*), $i=0,1,2$, (Definition 16.1) together with a fourth $(\Lambda_\Delta^*, f_\Delta^*)$ that is defined by the formulas of Definition 16.1 with respect to Poisson kernels in the same way as (17.1), but with the Laplacian in Z replacing the first-order derivatives. In fact, let $\alpha=\alpha(D_Z)$ denote a differential operator in Z. Then the densities and the derivatives of Poisson kernels can be indexed by their associated α. Calculating with the Kelvin transform leads to

$$\begin{aligned}\int_{\partial\Omega}\Lambda_\alpha^*\,ds &= \int_{\partial\Omega}\left(\tilde\Lambda_\alpha\left(\frac{Q}{|Q|^2}\right)-2N^Q\cdot Q\tilde f_\alpha\left(\frac{Q}{|Q|^2}\right)\right)\frac{ds(Q)}{|Q|^4}\\ &= \int_{\partial\tilde\Omega}(|P|^2\tilde\Lambda_\alpha(P)-2N^P\cdot P\tilde f_\alpha(P))\,ds(P) = \alpha(D_Z)|Z|^2|_{Z=0}\end{aligned}$$

and similarly

$$\int_{\partial\Omega}(Q_j\Lambda_\alpha^*-N_j^Q f_\alpha^*)\,ds(Q)=\int_{\partial\tilde\Omega}(P_j\tilde\Lambda_\alpha(P)-N_j^P\tilde f_\alpha(P))\,ds(P)=\alpha(D_Z)Z_j|_{Z=0}.$$

One concludes that

$$\Lambda_i^*\in W_0^{-1,p}(\partial\Omega),\quad i=0,1,2,$$

$$\int_{\partial\Omega}\Lambda_\Delta^*\,ds=4$$

and

$$
\begin{gathered}
(\Lambda_0^*, f_0^*) \in \mathbf{X}^p, \\
\int_{\partial\Omega} (Q_j \Lambda_i^* - N_j^Q f_i^*)\, ds(Q) = \delta_{ij}, \quad 1 \leqslant i, j \leqslant 2, \\
\int_{\partial\Omega} (Q_j \Lambda_\Delta^* - N_j^Q f_\Delta^*)\, ds(Q) = 0.
\end{gathered}
$$

In particular, the other three densities are not in $\mathbf{X}^p$. The single-layer potentials are similarly calculated as

$$
\begin{aligned}
&\int_{\partial\Omega} \left(B^X \Lambda_\alpha^* - \frac{\partial}{\partial N} B^X f_\alpha^* \right) ds \\
&\qquad = -\frac{1}{8\pi} \int_{\partial\tilde{\Omega}} [(|P|^2|X|^2 - 2P\cdot X + 1) \log |P|]\, \tilde{\Lambda}_\alpha \, ds(P) \\
&\qquad\quad + \frac{1}{8\pi} \int_{\partial\tilde{\Omega}} \frac{\partial}{\partial N^P} [(|P|^2|X|^2 - 2P\cdot X + 1) \log |P|]\, \tilde{f}_\alpha \, ds(P) \\
&\qquad\quad - \frac{1}{8\pi} \alpha(D_Z) \big[(|Z|^2|X|^2 - 2Z\cdot X + 1)(1 - \log \big| Z|X| - X|X|^{-1} \big|) \big] \big|_{Z=0}.
\end{aligned}
$$

As argued above, if the quadratic term on the right for $S(\Lambda_0^*, f_0^*)$ were to vanish, then so would (Λ_0^*, f_0^*), which is not possible by Definition 16.1. Since

$$
\int_{\partial\tilde{\Omega}} (|P|^2 \tilde{\Lambda}_\alpha(P) - 2N^P \cdot P \tilde{f}_\alpha(P))\, ds(P) = 0,
$$

one consequence of the nonvanishing of the quadratic term is that $W^0(0) \neq 0$ for $n=2$, extending Corollary 16.7.

The other use for the nonvanishing of the quadratic term is that (Λ_0^*, f_0^*) can be used in linear combination with the other densities to remove any of the other quadratic terms, resulting in a space of three affine equilibrium distributions complementing the space $\mathbf{X}^p$ in $W^{-1,p} \times L^p$ for (most) planar domains. As in Remark 16.4, exceptional domains occur by scaling because of the logarithmic fundamental solution. With the conventions there and letting m denote the order of the differential operator α, the various Poisson kernels scale as

$$
\tilde{\Lambda}_{\alpha t}(P) = t^{-1-m} \tilde{\Lambda}_\alpha(t^{-1}P) \quad \text{and} \quad \tilde{f}_{\alpha t}(P) = t^{-m} \tilde{f}_\alpha(t^{-1}P).
$$

The corresponding single-layer potentials for the domains Ω_t with $X\in\Omega_t$ satisfy

$$\begin{aligned}t^m\int_{\partial\Omega_t}&\left(B^X\Lambda^*_{\alpha t}-\frac{\partial}{\partial N}B^Xf^*_{\alpha t}\right)ds\\&=-\frac{1}{8\pi}\int_{\partial\tilde\Omega}[(t^2|P|^2|X|^2-2tP\cdot X+1)(\log|P|+\log t)]\,\tilde\Lambda_\alpha\,ds(P)\\&\quad+\frac{1}{8\pi}\int_{\partial\tilde\Omega}\frac{\partial}{\partial N^P}[(t^2|P|^2|X|^2-2tP\cdot X+1)(\log|P|+\log t)]\,\tilde f_\alpha\,ds(P)\\&\quad-\frac{t^m}{8\pi}\alpha(D_Z)\big[(|Z|^2|X|^2-2Z\cdot X+1)\big(1-\log\big|Z|X|-X|X|^{-1}\big|\big)\big]\big|_{Z=0}.\end{aligned}$$

Because the functions $|X|^2$, X_1, X_2 and 1 are biharmonic, the integrals without the $\log|P|$-term (with the $\log t$-term) are explicitly computable. Taking α equal to $D^0=I$, $\frac{1}{2}D_1$, $\frac{1}{2}D_2$ and $\frac{1}{4}\Delta$, and replacing tX with X, the result on the right is four linear combinations of the functions $|X|^2$, X_1, X_2 and 1. Because $(\Lambda^*_{0t},f^*_{0t})\in\mathbf{X}^p$ while the others are not and $S(\Lambda^*_{0t},f^*_{0t})$ does not vanish, the coefficient matrix is singular precisely when these linear combinations fail to span the linear (affine) functions that complement $\mathbf{X}^p$. This in turn corresponds to $\log t$ being an eigenvalue of the matrix for $t=1$. There can be at most four scalings in which the single layer fails to be injective. However, an investigation of the disc example only turns up one.

With this said, versions of Theorems 17.5 and 19.1 hold for $n=2$. Bounds depending only on the Lipschitz geometry prevail there when S is restricted to $\mathbf{X}^p$, and also in general for the extension of Corollary 19.3 to $n=2$.

21. On the range for the Poisson ratio ν

When $\nu=1$ the boundary operators for the biharmonic Neumann problem are related by the Dirichlet-to-Neumann map for harmonic functions. They cannot map onto any subspace of $W^{-1,p}\times L^p$ that has finite codimension. They fail to satisfy the classical Lopatinskiĭ–Shapiro conditions for a regular boundary value problem. This condition is also not met for $\nu=-3$. A consequence was the noninvertibility of (11.3) for this value of ν. Another consequence comes by considering solutions $u=yh+H$ in the upper half-plane, where h and H are harmonic functions with sufficient decay. Then at $y=0$, $M_{-3}(u)=2h_y+4H_{yy}$ and $K_{-3}(u)=-2h_{yy}-4H_{yyy}$, the Dirichlet-to-Neumann map again.

S. Agmon [1] showed, in the plane, that the quadratic forms associated with the bi-Laplacian and the boundary operators (5.1) and (5.2), when $\nu=1$ and $\nu=-3$, cannot be coercive over $C^2(\bar\Omega)$, i.e. the inequality

$$\int_\Omega((1-\nu)|\nabla\nabla u|^2+\nu(\Delta u)^2)\,dX\geqslant c\int_\Omega|\nabla\nabla u|^2\,dX-c_0\int_\Omega u^2\,dX \tag{21.1}$$

does not hold over all $u\in C^2(\bar{\Omega})$ for any positive constants c and c_0 when ν takes these two values.

Denote the form on the left of (21.1) (the form of (10.2), (6.2) and (6.6)) by Q_ν, for all real ν. By pointwise comparison of the integrands with those for -3 and 1, Agmon also showed that Q_ν, for $\nu>1$ and for $\nu<-3$, fails to satisfy the coercive estimate. In the same spirit the (pointwise) inequalities $-Q_\nu\leqslant\frac{1}{3}\nu Q_{-3}$ when $\nu\geqslant 1$ and $-Q_\nu\leqslant-\nu Q_1$ when $\nu\leqslant-3$ show that $-Q_\nu$ is not coercive outside the open interval $(-3,1)$.

It is a boundary version of the coercive estimate, viz. Lemma 7.3, that permits us to deduce a priori estimates for the inhomogeneous Neumann problem from the Rellich identity (6.9). The pointwise nonnegativity of the forms Q_ν from the interval $[(1-n)^{-1},1)$ is used in that lemma. The lack of strict positivity in the left endpoint case leads to the same results as the others from this interval, except on a 1-dimensional subspace of data. Thus it differs from the right endpoint case.

In the plane this leaves the interval $(-3,-1)$ to be discussed. Like the other cases outside the interval $[(1-n)^{-1},1]$ pointwise nonnegativity fails. However, Agmon showed that the classical coercive estimate (21.1) *does* hold for this interval. It will now be shown that our results must nevertheless fail for $-3<\nu<-1$ in Lipschitz domains.

In the complex plane with $0<\theta<\pi$, let $\Omega(\theta)=\{z:|\arg z|<\theta \text{ and } 0<|z|<1\}$ and let $\Delta(\theta)=\{z:|\arg z|=\theta\}\cap\partial\Omega(\theta)$. Given real numbers β, γ and q define the (complex) biharmonic functions

$$\Phi_q(z)=\beta z^q+\gamma\bar{z}z^{q-1},\quad |\arg z|<\pi.$$

LEMMA 21.1. *Fix $\nu\in(-3,-1)$. Then*

(i) *there exists $\theta_0\in\left(\frac{1}{2}\pi,\pi\right)$ and a nontrivial pair of coefficients (β,γ) such that*

$$u(z)=\operatorname{Im}\Phi_{3/2}(z)$$

satisfies

$$M_\nu(u)=0=K_\nu(u)\quad \text{on } \Delta(\theta_0) \text{ pointwise;} \tag{21.2}$$

(ii) *in the sense of distributions, $K_\nu(u)$ equals the linear functional*

$$\Lambda_0\in W^{-1,p}(\partial\Omega(\theta_0)),\quad p<2,$$

which acts on any $F\in W^{1,p'}(\partial\Omega(\theta_0))$ by

$$\begin{aligned}\langle\Lambda_0,F\rangle=\frac{1-\nu}{2}&\left(3\beta\cos\frac{3}{2}\theta_0-\gamma\cos\frac{1}{2}\theta_0\right)(F(e^{-i\theta_0})-F(e^{i\theta_0}))\\&+\int_{-\theta_0}^{\theta_0}\left(-\frac{9(1-\nu)}{8}\beta\sin\frac{3}{2}\theta+\left(1+\frac{1-\nu}{8}\right)\gamma\sin\frac{1}{2}\theta\right)F(e^{i\theta})\,d\theta;\end{aligned}\tag{21.3}$$

moreover, $(K_\nu(u),M_\nu(u))\in\mathbf{X}^p(\partial\Omega(\theta_0))$, $p<2$;

(iii) *there exists* $\varepsilon>0$, *a strictly decreasing function* $q=q(\theta)$ *and nontrivial pairs* $(\beta,\gamma)=(\beta(\theta),\gamma(\theta))$ *such that*

$$u_q(z)=\operatorname{Im}\Phi_q(z)$$

satisfies (21.2) *for each* $\theta\in(\theta_0-\varepsilon,\theta_0+\varepsilon)$.

Proof. Take the unit tangent vector T to be a rotation by $\frac{1}{2}\pi$ of the outer unit normal vector N, and let $z=re^{i\theta}$. Then, for example, on the half of $\Delta(\theta)$ with *negative* argument one has

$$\frac{\partial}{\partial N}=-\frac{1}{r}\frac{\partial}{\partial\theta},\quad \frac{\partial}{\partial T}=\frac{\partial}{\partial r},$$

$$\frac{\partial^2}{\partial T\partial N}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right),\quad \frac{\partial^2}{\partial T^2}=\frac{\partial^2}{\partial r^2},\quad \frac{\partial^2}{\partial N^2}=\frac{1}{r}\frac{\partial}{\partial r}+\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)^2,$$

and finally

$$M_\nu(\Phi_q)=r^{q-2}(1-q)[(1-\nu)\beta qe^{iq\theta}+\gamma((1-\nu)q-4)e^{i(q-2)\theta}]$$

and

$$K_\nu(\Phi_q)=-ir^{q-3}(1-q)(2-q)[(1-\nu)\beta qe^{iq\theta}+\gamma((1-\nu)(q-2)+4)e^{i(q-2)\theta}].$$

Setting the imaginary parts equal to zero yields

$$0=\beta(1-\nu)q\sin q\theta+\gamma((1-\nu)q-4)\sin(q-2)\theta$$

and

$$0=\beta(1-\nu)q\cos q\theta+\gamma((1-\nu)(q-2)+4)\cos(q-2)\theta.$$

The same equations are obtained for the other half of $\Delta(\theta)$. Viewing these as two linear equations in β and γ, nontrivial solutions are obtained when the determinant $(1-\nu)qD(q,\theta)=0$, where

$$D(q,\theta)=(3+\nu)\sin(2q-2)\theta+(1-\nu)(q-1)\sin 2\theta. \tag{21.4}$$

$D\left(\frac{3}{2},\theta\right)$ vanishes when

$$\cos\theta=\frac{3+\nu}{\nu-1}. \tag{21.5}$$

This proves (i).

Let $\theta_0\in\left(\frac{1}{2}\pi,\pi\right)$ be the solution to (21.5). On $r=1$ and $-\theta_0<\theta<\theta_0$, the pointwise value for $K_\nu(u)$ when $q=\frac{3}{2}$ is the parenthetic integrand of (21.3). In addition,

$$\frac{\partial^2}{\partial T\partial N}u=\frac{1}{4}\left(3\beta\cos\frac{3}{2}\theta-\gamma\cos\frac{1}{2}\theta\right)$$

there. However,

$$\frac{\partial^2}{\partial T\partial N}=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\quad\text{on } r=1,$$

so that there is a sign change at both corners $\theta=\pm\theta_0$ for the part of the Neumann data involving these derivatives. The third derivative must, therefore, be understood in the sense of distributions. ($\partial\Delta u/\partial N$ is pointwise bounded, however.) Take $F\in W^{1,p'}(\partial\Omega(\theta_0))$ supported outside a ball containing the origin. Let $0\leqslant\Xi_\varepsilon\leqslant 1$ be a smooth function supported in ε-balls about the two corners, equal to 1 at the corners, and satisfying $\varepsilon\|\nabla\Xi_\varepsilon\|_\infty\leqslant C$ independent of $\varepsilon>0$. Then the pointwise behaviors and one application of the divergence theorem yield

$$\begin{aligned}\lim_{j\to\infty}\int_{\partial\Omega_j}\mathcal{P}(F)K_\nu(u)\,ds_j=-(1-\nu)\int_{\partial\Omega(\theta_0)}\frac{\partial}{\partial T}(\Xi_\varepsilon F)\frac{\partial^2}{\partial N\partial T}u\,ds \qquad (21.6)\\ +\int_{\partial\Omega(\theta_0)}\Xi_\varepsilon F\frac{\partial}{\partial N}\Delta u\,ds+\int_{\partial\Omega(\theta_0)\setminus\Delta(\theta_0)}(1-\Xi_\varepsilon)FK_\nu(u)\,ds.\end{aligned}$$

Integration by parts over each piece of the boundary in the first integral, the absolute continuity of F, and letting ε vanish, yields the first term of (21.3). The second integral vanishes in ε and the third yields the rest of (21.3). For $F\in W^{1,p'}(\partial\Omega(\theta_0))$ supported in $\Delta(\theta_0)$, let the Ξ_ε be as above for ε-balls about the origin. Denote by F_ε the average $(2\varepsilon)^{-1}\int_{|Q|<\varepsilon}F(Q)\,ds(Q)$ for each ε-ball. By $K_\nu(u)\in W_0^{-1,p}(\partial\Omega_j)$ and the pointwise vanishing on $\Delta(\theta_0)$,

$$\lim_{j\to\infty}\left|\int_{\partial\Omega_j}\mathcal{P}(F)K_\nu(u)\,ds_j\right|=\lim_{j\to\infty}\left|\int_{\partial\Omega_j}\Xi_\varepsilon\mathcal{P}(F-F_\varepsilon)K_\nu(u)\,ds_j\right|\leqslant\|\Xi_\varepsilon(F-F_\varepsilon)\|_{1,p'},$$

where the inequality depends on $N(\nabla\nabla u)\in L^p$, which holds for $p<2$ only. By the Poincaré inequality this is bounded by

$$C\biggl(\int_{|Q|<\varepsilon}\left|\frac{\partial}{\partial T}F\right|^{p'}ds\biggr)^{1/p'}$$

with C independent of ε. This establishes (21.3). A computation yields

$$M_\nu(u)=\frac{3(1-\nu)}{4}\beta\sin\frac{3}{2}\theta+\frac{3+5\nu}{4}\gamma\sin\frac{1}{2}\theta\quad\text{on } r=1.$$

This, together with (21.3) and choosing $\beta=\frac{1}{3}(1+2\cos\theta_0)\sin\frac{1}{2}\theta_0$ and $\gamma=\sin\frac{3}{2}\theta_0$ as a solution to the two linear equations in β and γ, can be used to show that $(K_\nu(u),M_\nu(u))$ meets the algebraic condition to be in $\mathbf{X}^p$ (Definition 10.1) for $p<2$.

To prove (iii), implicit differentiation of $D(q,\theta)=0$ at $\left(\frac{3}{2},\theta_0\right)$ and substituting $\cos\theta_0$ for $(3+\nu)(\nu-1)$ show that

$$q'(\theta_0)=\frac{-\frac{1}{2}\tan^2\theta_0}{\theta_0-\tan\theta_0}.$$

The implicit function theorem yields (iii). □

By (21.3) the linear functional Λ_0 from (ii) of the lemma is in $W^{-1,p}(\partial\Omega(\theta_0))$ for all $p\leqslant\infty$. Thus given a $\nu\in(-3,-1)$ the solution $u_{3/2}$ from (i) of the lemma *formally* has Neumann data in $\mathbf{X}^p(\partial\Omega(\theta_0))$ for any p, but because $N(\nabla\nabla u_{3/2})\in L^p(\partial\Omega(\theta_0))$ for $p<2$ only, $u_{3/2}$ cannot be a biharmonic solution of Definition 13.1 for this data when $p\geqslant 2$. If it were true that the Neumann problem of Definition 13.1 were solvable for any data in $\mathbf{X}^2$ when $\nu\in(-3,-1)$, then the solution u that takes the same data as $u_{3/2}$ would satisfy $N(\nabla\nabla u)\in L^p$ for $p\leqslant 2$. Thus uniqueness would fail for $p<2$. But under the assumption of $\mathbf{X}^2$-solvability uniqueness for $p<2$ can be proved, thus contradicting $\mathbf{X}^2$-solvability:

LEMMA 21.2. *Let $\Omega\subset\mathbf{R}^n$ be a bounded Lipschitz domain, and choose ν to be any real number.* Assume that the biharmonic Neumann problem of Definition 13.1 for ν can be solved for all data in $\mathbf{X}^2(\partial\Omega)$. *Let $2(n-1)/(n+1)\leqslant p$ and let u be a solution in Ω with vanishing ν-Neumann data.*

Then $N(\nabla\nabla u)\in L^p$ implies that u is a linear function.

Proof. The vanishing of the data and Green's representation formula still give

$$u(X)=\int_{\partial\Omega}\left(K_\nu(B^X)u-M_\nu(B^X)\frac{\partial u}{\partial N}\right)ds,\quad X\in\Omega. \tag{21.7}$$

By translation it may be assumed that $\int_{\partial\Omega}Q_j\,ds=0$ for all j. Fix X. Because the double-layer potential is the identity on linear functions, the Neumann values of the fundamental solution satisfy

$$(K_\nu(B^X)(Q)-|\partial\Omega|^{-1}-|\Omega|^{-1}N^Q\cdot X,\,M_\nu(B^X)(Q))\in\mathbf{X}^2(\partial\Omega).$$

Let v be a solution in Ω for this data. Then the right-hand side of (21.7) can be rewritten in terms of v, u and the newly introduced linear terms. Green's second identity for two solutions then shows that

$$u(X)=\frac{1}{|\partial\Omega|}\int_{\partial\Omega}u\,ds+\frac{1}{|\Omega|}\int_\Omega\nabla u(Y)\,dY\cdot X.$$

The two assumptions on the nontangential maximal functions justify the use of the divergence theorem by way of domain approximations, existence of pointwise limits, Lebesgue dominated convergence and the Sobolev embedding theorem when $p<2$. □

Given any $\nu\in(-3,-1)$ a bounded Lipschitz domain can be constructed with a countable number of corners of angle $2\theta_0$ in such a way that a singular solution from Lemma 21.1 exists for each such corner and in the domain. Any finite linear combination of such solutions provides data for which there can be no solution by the same argument as above. The $\mathbf{X}^2$-Neumann problem for ν cannot in general be Fredholm, therefore. Part (iii) of Lemma 21.1 shows that the same can be said for the $\mathbf{X}^p$-problem for p near 2.

However, letting $\theta=\pi$ in (21.4) and setting $D=0$ yields $q=\frac{3}{2}$ as a solution again, indicating that the decreasing function in (ii) of Lemma 21.1 obtains a minimum value for $\theta>\theta_0$ below which there are no singular solutions. It is an open problem whether or not the $\mathbf{X}^p$-problem can be solved for an interval of p's strictly below 2 when the problem for $p=2$ is not solvable.

Remark 21.3. The Φ_q also give counterexamples to the solvability for $p>2$ of the Neumann problem when $n=2$ and $-1\leqslant\nu<1$. We have

$$D\left(\tfrac{3}{2},\pi\right)=0 \quad \text{and} \quad \frac{\partial}{\partial q}D\left(\tfrac{3}{2},\pi\right)=-2\pi(3+\nu).$$

Thus $q(\theta)$ can be implicitly defined with $q(\pi)=\frac{3}{2}$, and then $q'(\pi)=-(1+\nu)/(3+\nu)\pi<0$ when $-1<\nu$. When $\nu=-1$, $q'(\pi)=q''(\pi)=0$ and $q'''(\pi)=-3/2\pi$, which implies $q'(\theta)<0$ for $\theta<\pi$ in a neighborhood of π. Thus in both cases there are singular solutions u_q, for each $q>\frac{3}{2}$ in a neighborhood of $\frac{3}{2}$, defined in corresponding domains $\Omega(\theta)$, $\theta<\pi$. This means that given any $p_0>2$ close enough to 2 there is a Lipschitz domain with singular solution u such that $N(\nabla\nabla u)\in L^p$ for all $p<p_0$, but $N(\nabla\nabla u)\notin L^{p_0}$. Now the argument is the same as that given between Lemmas 21.1 and 21.2 with p_0 playing the role of 2, except that $\mathbf{X}^2$-uniqueness obviates the need for another Lemma 21.2.

Remark 21.4. The use of reentrant corners to produce singular solutions is not necessary. By considering the real part of the Φ_q, examples in convex domains are produced. The plus sign between the two terms of (21.4) changes to a minus, so that (21.5) changes sign.

Remark 21.5. Singular solutions for $p=2$ when $\nu\leqslant-3$ can also be derived from the Φ_q. But none for $\nu\geqslant1$, it seems.

22. Subregularity for the Neumann problem

Let $2-\varepsilon<p<2+\varepsilon$ and denote the dual space to the Whitney array space $WA_2^{p'}(\partial\Omega)$ by $WA_{-2}^{p}(\partial\Omega)$. Let v be a solution to the biharmonic Dirichlet problem with data

$(v, -\partial v/\partial N) \in W^{1,p} \times L^p$. Then formally the Neumann data for v becomes a linear functional $\mathbf{L}$ acting on arrays $\dot{f} \in WA_2^{p'}(\partial\Omega)$ by

$$\langle \mathbf{L}, \dot{f} \rangle = \int_{\partial\Omega} (K_\nu(v) f_0 - M_\nu(v) N_j f_j)\, ds. \tag{22.1}$$

Denote by $\mathcal{P}_2(\dot{f})$ the solution to the biharmonic regularity problem with data $\dot{f}$. Interpreted in the sense of distributions, $\mathbf{L}$ becomes a bounded linear functional by

$$\langle \mathbf{L}, \dot{f} \rangle = \int_{\partial\Omega} \left(v K_\nu(\mathcal{P}_2(\dot{f})) - \frac{\partial v}{\partial N} M_\nu(\mathcal{P}_2(\dot{f})) \right) ds$$

and

$$\langle \mathbf{L}, \dot{f} \rangle \leqslant \|\nabla v\|_p \|\!| K_\nu(\mathcal{P}_2(\dot{f})), M_\nu(\mathcal{P}_2(\dot{f})) \|\!|_{p'} \leqslant C \|\nabla v\|_p \|\nabla\nabla \mathcal{P}_2(\dot{f})\|_{p'} \leqslant C \|\nabla v\|_p \|\dot{f}\|_{2,p'},$$

where the last inequality is by solvability of the biharmonic regularity problem. Thus $\mathbf{L}$ satisfies $\|\mathbf{L}\| \leqslant C \|\nabla v\|_p$.

Given any $\mathbf{L} \in WA_{-2}^p(\partial\Omega)$, finding a biharmonic v in the class $N(\nabla v) \in L^p$ such that (22.1) holds for all $\dot{f} \in WA_2^{p'}(\partial\Omega)$, and obtaining the opposite inequality, can be thought of as solving the (biharmonic) problem of *subregularity for the Neumann problem.*

Hence, suppose that $\mathbf{L} \in WA_{-2}^p(\partial\Omega)$ is given. If $\dot{f} \in WA_2^{p'}(\partial\Omega)$ then $\mathcal{P}_2(\dot{f})$ has Neumann data in $W^{-1,p'} \times L^{p'}$ by solvability of the regularity problem. Conversely, given data $(\Lambda, g) \in W^{-1,p'} \times L^{p'}$ there is a unique $\nabla\nabla\mathcal{P}_2(\dot{f})$, by solvability of the Neumann problem, so that $\mathcal{P}_2(\dot{f})$ has Neumann data (Λ, g) and

$$\|\nabla\nabla\mathcal{P}_2(\dot{f})\|_{p'} \leqslant C \|\!|\Lambda, g\|\!|_{p'}. \tag{22.2}$$

Consequently, the map

$$\begin{aligned} V : W^{-1,p'} \times L^{p'} &\longrightarrow \mathbf{R}, \\ (\Lambda, g) &\longmapsto \langle \mathbf{L}, \dot{f} \rangle \end{aligned}$$

is a bounded linear functional by $\langle \mathbf{L}, \dot{f} \rangle \leqslant \|\mathbf{L}\| \, \|\dot{f}\|_{2,p'} \leqslant C \|\mathbf{L}\| \, \|\!|\Lambda, g\|\!|_{p'}$. Since $W^{1,p} \times L^p$ is reflexive (Remark 4.7), V is represented by a unique solution v to the biharmonic Dirichlet problem so that in the sense of distributions

$$\langle V, (\Lambda, g) \rangle = \int_{\partial\Omega} \left(v\Lambda - \frac{\partial v}{\partial N} g \right) ds = \langle \mathbf{L}, \dot{f} \rangle.$$

The last equality then holds for all $\dot{f}$ by the solvability of the regularity problem. It is a consequence of the Hahn–Banach theorem ([47, p. 108]) and (22.2) that

$$\|\nabla v\| \leqslant \sup\{\langle \mathbf{L}, \dot{f} \rangle : \|\!|\Lambda, g\|\!|_{p'} = 1\} \leqslant C \|\mathbf{L}\| \, \|\dot{f}\|_{2,p'} \leqslant C \|\mathbf{L}\|.$$

Thus the solution of the fourth boundary value problem, subregularity for the Neumann problem, follows from the solvability of the other three and from functional-analytic arguments.

This subregularity result is the analogue of Proposition 4.2, and so should bring us to the Neumann problem for sixth-order elliptic operators. As a concluding example consider the equation $\Delta^3 u=0$ and the Dirichlet form $Q(w,u)=\int_{\partial\Omega} D_iD_jD_kw\,D_iD_jD_ku\,dX$ that would correspond to $\nu=0$. Dirichlet data for solutions is prescribed in the space $WA_2^{p'}\times L^{p'}$ ([33]). Suppose that $N(\nabla\nabla w)\in L^{p'}$ and the ordered pair $(\dot{w},\partial^2 w/\partial N^2)\in WA_2^{p'}\times L^{p'}$, where $\dot{w}=\langle w,\nabla w\rangle$. Suppose also that $N(\nabla\nabla\nabla u)\in L^p$, so that Δu is a solution to the biharmonic Dirichlet problem in the class $N(\nabla\Delta u)\in L^p$. The divergence theorem in smooth approximating domains using Green's identity (10.2) and a couple of applications of (6.8) yields

$$\begin{aligned} Q(w,u)=\int_{\partial\Omega}\left(K_0(\Delta u)w-M_0(\Delta u)\frac{\partial w}{\partial N}\right)ds-\frac{1}{2}\int_{\partial\Omega}D_kw\frac{\partial}{\partial T_{ij}}\left(\frac{\partial^2}{\partial N\partial T_{ij}}D_ku\right)ds\\ -\frac{1}{2}\int_{\partial\Omega}D_kw\frac{\partial}{\partial T_{ij}}\left(N_k\frac{\partial^3}{\partial N^2\partial T_{ij}}u\right)ds+\int_{\partial\Omega}\frac{\partial^2 w}{\partial N^2}\frac{\partial^3 u}{\partial N^3}\,ds. \end{aligned}$$

The first integral is the pairing (22.1). The second and third are pairings of tangential $W^{-1,p}$-distributions with components of $\dot{w}$.

As described in §17 the Dirichlet data is equivalently described as

$$\left(w,-\frac{\partial w}{\partial N},\frac{\partial^2 w}{\partial N^2}\right)$$

on the boundary of a Lipschitz domain. Further decomposing the gradient of w into normal and tangential components yields a triple of Neumann data for the triharmonic equation,

$$K_0(\Delta u)+\frac{1}{2}\frac{\partial}{\partial T_{lk}}\left(N_l\frac{\partial}{\partial T_{ij}}\left(\frac{\partial^2}{\partial N\partial T_{ij}}D_ku\right)\right)+\frac{1}{2}\frac{\partial}{\partial T_{lk}}\left(N_l\frac{\partial}{\partial T_{ij}}\left(N_k\frac{\partial^3}{\partial N^2\partial T_{ij}}u\right)\right),\tag{TH}$$

$$M_0(\Delta u)+\frac{1}{2}N_k\frac{\partial}{\partial T_{ij}}\left(\frac{\partial^2}{\partial N\partial T_{ij}}D_ku\right)+\frac{1}{2}N_k\frac{\partial}{\partial T_{ij}}\left(N_k\frac{\partial^3}{\partial N^2\partial T_{ij}}u\right),\tag{EE}$$

$$\frac{\partial^3 u}{\partial N^3}\tag{ND}$$

in $W^{-2,p}\times L^p$.

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