Acta Math., 192 (2004), 111–117

Solid von Neumann algebras

by

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Dedicated to Professor Masamichi Takesaki on the occasion of his 70th birthday

1. Introduction

Recall that a von Neumann algebra is said to be *diffuse* if it does not contain a minimal projection. We say that a von Neumann algebra $\mathcal{M}$ is *solid* if for any diffuse von Neumann subalgebra $\mathcal{A}$ in $\mathcal{M}$, the relative commutant $\mathcal{A}'\cap\mathcal{M}$ is injective. A solid von Neumann algebra is necessarily finite. We prove the following theorem which answers a question of Ge [G] whether the free group factors are solid.

THEOREM 1. *The group von Neumann algebra $\mathcal{L}\Gamma$ of a hyperbolic group Γ is solid.*

Recall that a factor $\mathcal{M}$ is said to be prime if $\mathcal{M}\cong\mathcal{M}_1\bar{\otimes}\mathcal{M}_2$ implies that either $\mathcal{M}_1$ or $\mathcal{M}_2$ is finite-dimensional. The existence of such factors was proved by Popa [P2] who showed that the group factors of uncountable free groups are prime. The case for countable free groups had remained open for some time, but was settled by Ge [G]. This was generalized by Ştefan [Şt] to their subfactors of finite index. Our theorem gives a further generalization. Indeed, combined with a result of Popa [P3] (Proposition 7 in this paper), we obtain the following proposition.

PROPOSITION 2. *A subfactor of a solid factor is again solid, and a solid factor is non-Γ and prime unless it is injective.*

Notably, this provides infinitely many prime II_1-factors (with the property (T)). Indeed, thanks to a theorem of Cowling and Haagerup [CH], for lattices Γ_n in $\mathrm{Sp}(1,n)$, we have $\mathcal{L}\Gamma_m\not\cong\mathcal{L}\Gamma_n$ whenever $m\neq n$. However, we are unaware of any non-injective solid factor with the Haagerup property other than the free group factor(s). This proposition also distinguishes the factor $(\mathcal{L}\mathbf{F}_\infty\bar{\otimes}L_\infty[0,1])*\mathcal{L}\mathbf{F}_\infty$ from the free group factor $\mathcal{L}\mathbf{F}_\infty$, which answers a question of Shlyakhtenko.

The author was supported by the JSPS Postdoctoral Fellowships for Research Abroad.

2. Preliminary results on reduced group C^*-algebras

For a discrete group Γ, we denote by λ (resp. ϱ) the left (resp. right) regular representation on $l_2\Gamma$, and let $C^*_\lambda\Gamma$ (resp. $C^*_\varrho\Gamma$) be the C^*-algebra generated by $\lambda(\Gamma)$ (resp. $\varrho(\Gamma)$) in $\mathbf{B}(l_2\Gamma)$ and $\mathcal{L}\Gamma=(C^*_\lambda\Gamma)''$ be its weak closure. The C^*-algebra $C^*_\lambda\Gamma$ (resp. the von Neumann algebra $\mathcal{L}\Gamma$) is called the *reduced group C^*-algebra* (resp. the *group von Neumann algebra*).

The study of C^*-norms on tensor products was initiated in the 1950s by Turumaru, but the first substantial result was obtained by Takesaki [T] who showed that the minimal tensor norm is the smallest among the possible C^*-norms on a tensor product of C^*-algebras. He also introduced the notion of nuclearity and found that the reduced group C^*-algebra $C^*_\lambda\mathbf{F}_2$ of the free group $\mathbf{F}_2$ on two generators is not nuclear. Namely, the $*$-homomorphism

$$C^*_\lambda\mathbf{F}_2\otimes C^*_\varrho\mathbf{F}_2\ni\sum_{i=1}^n a_i\otimes x_i\longmapsto\sum_{i=1}^n a_i x_i\in\mathbf{B}(l_2\mathbf{F}_2)$$

is not continuous with respect to the minimal tensor norm. Yet, Akemann and Ostrand [AO] proved the remarkable theorem that it is continuous if one composes it with the quotient map π from $\mathbf{B}(l_2\mathbf{F}_2)$ onto the Calkin algebra $\mathbf{B}(l_2\mathbf{F}_2)/\mathbf{K}(l_2\mathbf{F}_2)$. By a completely different argument, Skandalis (Théorème 4.4 in [Sk]) proved the same for all discrete subgroups in connected simple Lie groups of rank one, and Higson and Guentner (Lemma 5.2 in [HG]) for all hyperbolic groups. In summary, we have the following result.

THEOREM 3. *Let Γ be a hyperbolic group or a discrete subgroup in a connected simple Lie group of rank one. Then, the $*$-homomorphism*

$$\nu_\Gamma\colon C^*_\lambda\Gamma\otimes C^*_\varrho\Gamma\ni\sum_{i=1}^n a_i\otimes x_i\longmapsto\pi\Bigl(\sum_{i=1}^n a_i x_i\Bigr)\in\mathbf{B}(l_2\Gamma)/\mathbf{K}(l_2\Gamma)$$

is continuous with respect to the minimal tensor norm on $C^*_\lambda\Gamma\otimes C^*_\varrho\Gamma$.

The crucial ingredient in the proof was the amenability of the action of Γ on a suitable boundary which is 'small at infinity'. For the information on amenable actions, we refer the reader to the book [AR] of Anantharaman-Delaroche and Renault. Since Γ acts amenably on a compact set, $C^*_\lambda\Gamma$ is embeddable into a nuclear C^*-algebra and thus has the property (C) of Archbold and Batty (Theorem 3.6 in [AB]). Although we do not need this fact, we mention that the property (C) is equivalent to exactness by a deep theorem of Kirchberg [K]. By Effros and Haagerup's theorem (Theorem 5.1 in [EH]), the property (C) implies the local reflexivity. In summary, the following lemma is true.

LEMMA 4. *Let Γ be as above. Then, $C^*_\lambda\Gamma$ is locally reflexive, i.e., for any finite-dimensional operator system $E\subset(C^*_\lambda\Gamma)^{**}$, there is a net of unital completely positive maps $\theta_i: E\to C^*_\lambda\Gamma$ which converges to id_E in the point-weak* topology.*

3. Proof of the theorem

We recall the following principle [Ch]: If $\Psi: A\to B$ is a unital completely positive map and its restriction to a C^*-subalgebra $A_0\subset A$ is multiplicative, then we have $\Psi(axb)=\Psi(a)\Psi(x)\Psi(b)$ for any $a,b\in A_0$ and $x\in A$.

Let $\mathcal{N}\subset\mathcal{M}$ be finite von Neumann algebras with a faithful trace τ on $\mathcal{M}$. Then, there is a normal conditional expectation $\varepsilon_\mathcal{N}$ from $\mathcal{M}$ onto $\mathcal{N}$, which is defined by the relation $\tau(\varepsilon_\mathcal{N}(a)x)=\tau(ax)$ for $a\in\mathcal{M}$ and $x\in\mathcal{N}$. This implies that a von Neumann subalgebra of a finite injective von Neumann algebra is again injective. Moreover, $\varepsilon_\mathcal{N}$ is unique in the sense that any trace-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$ coincides with $\varepsilon_\mathcal{N}$. Indeed, for any $a\in\mathcal{M}$ and $x\in\mathcal{N}$, we have

$$\tau(\varepsilon'(a)x)=\tau(\varepsilon'(ax))=\tau(ax)=\tau(\varepsilon_\mathcal{N}(a)x).$$

We say that a von Neumann subalgebra $\mathcal{M}$ in $\mathbf{B}(\mathcal{H})$ satisfies the condition (AO) if there are unital ultraweakly dense C^*-subalgebras $B\subset\mathcal{M}$ and $C\subset\mathcal{M}'$ such that B is locally reflexive and the $*$-homomorphism

$$\nu: B\otimes C\ni\sum_{i=1}^n a_i\otimes x_i\longmapsto\pi\Big(\sum_{i=1}^n a_ix_i\Big)\in\mathbf{B}(\mathcal{H})/\mathbf{K}(\mathcal{H})$$

is continuous with respect to the minimal tensor norm on $B\otimes C$.

We have seen in §2 that the group von Neumann algebra $\mathcal{L}\Gamma$ satisfies the condition (AO) whenever Γ is a hyperbolic group or a discrete subgroup in a connected simple Lie group of rank one.

LEMMA 5. *Let $B\subset\mathcal{M}$ and $C\subset\mathcal{M}'$ be unital ultraweakly dense C^*-subalgebras with B locally reflexive, and let $\mathcal{N}\subset\mathcal{M}$ be a von Neumann subalgebra with a normal conditional expectation $\varepsilon_\mathcal{N}$ onto $\mathcal{N}$. Assume that the unital completely positive map*

$$\Phi_\mathcal{N}: B\otimes C\ni\sum_{i=1}^n a_i\otimes x_i\longmapsto\sum_{i=1}^n\varepsilon_\mathcal{N}(a_i)x_i\in\mathbf{B}(\mathcal{H})$$

is continuous with respect to the minimal tensor norm on $B\otimes C$. Then $\mathcal{N}$ is injective.

Proof. Since $B\otimes_{\min}C\subset\mathbf{B}(\mathcal{H})\otimes_{\min}C$ and $\mathbf{B}(\mathcal{H})$ is injective, $\Phi_\mathcal{N}$ extends to a unital completely positive map $\Psi:\mathbf{B}(\mathcal{H})\otimes_{\min}C\to\mathbf{B}(\mathcal{H})$. Then, Ψ is automatically a C-bimodule map. Put $\psi(a)=\Psi(a\otimes1)$ for $a\in\mathbf{B}(\mathcal{H})$. Then, for every $a\in\mathbf{B}(\mathcal{H})$ and $x\in C$, we have

$$x\psi(a)=\Psi(1\otimes x)\Psi(a\otimes1)=\Psi(a\otimes x)=\Psi(a\otimes1)\Psi(1\otimes x)=\psi(a)x.$$

Hence, ψ maps $\mathbf{B}(\mathcal{H})$ into $C'=\mathcal{M}$. It follows that $\tilde{\psi}=\varepsilon_{\mathcal{N}}\psi\colon\mathbf{B}(\mathcal{H})\to\mathcal{N}$ is a unital completely positive map such that $\tilde{\psi}|_B=\varepsilon_{\mathcal{N}}|_B$. Let I be the set of all triples $(\mathcal{E},\mathcal{F},\varepsilon)$ where $\mathcal{E}\subset\mathcal{N}$ and $\mathcal{F}\subset\mathcal{N}_*$ are finite subsets and $\varepsilon>0$ is arbitrary. The set I is directed by the order relation $(\mathcal{E}_1,\mathcal{F}_1,\varepsilon_1)\leqslant(\mathcal{E}_2,\mathcal{F}_2,\varepsilon_2)$ if and only if $\mathcal{E}_1\subset\mathcal{E}_2$, $\mathcal{F}_1\subset\mathcal{F}_2$ and $\varepsilon_1\geqslant\varepsilon_2$. Let $i=(\mathcal{E},\mathcal{F},\varepsilon)\in I$ and let $E\subset\mathcal{N}$ be the finite-dimensional operator system generated by $\mathcal{E}$. We note that $E\subset\mathcal{M}=pB^{**}$ and $\varepsilon_{\mathcal{N}}^*(\mathcal{F})\subset\mathcal{M}_*=pB^*$, where $p\in B^{**}$ is the central projection supporting the identity representation of B on $\mathcal{H}$. Since B is locally reflexive (cf. Lemma 4), there is a unital completely positive map $\theta_i\colon E\to B$ such that for $a\in\mathcal{E}$ and $f\in\mathcal{F}$, we have $|\langle\theta_i(a),\varepsilon_{\mathcal{N}}^*(f)\rangle-\langle a,\varepsilon_{\mathcal{N}}^*(f)\rangle|<\varepsilon$. Take a unital completely positive extension $\bar{\theta}_i\colon\mathbf{B}(\mathcal{H})\to\mathbf{B}(\mathcal{H})$ of θ_i, and let $\sigma_i=\tilde{\psi}\bar{\theta}_i\colon\mathbf{B}(\mathcal{H})\to\mathcal{N}$. It follows that for $a\in\mathcal{E}$ and $f\in\mathcal{F}$,

$$|\langle\sigma_i(a),f\rangle-\langle a,f\rangle|=|\langle\varepsilon_{\mathcal{N}}\theta_i(a),f\rangle-\langle a,f\rangle|=|\langle\theta_i(a),\varepsilon_{\mathcal{N}}^*(f)\rangle-\langle a,\varepsilon_{\mathcal{N}}^*(f)\rangle|<\varepsilon.$$

Therefore, any cluster point, in the point-ultraweak topology, of the net $\{\sigma_i\}_{i\in I}$ is a conditional expectation from $\mathbf{B}(\mathcal{H})$ onto $\mathcal{N}$. □

We now prove Theorem 1, or more precisely the following result.

THEOREM 6. *A finite von Neumann algebra $\mathcal{M}$ satisfying the condition* (AO) *is solid.*

Proof. Let $\mathcal{A}$ be a diffuse von Neumann subalgebra in $\mathcal{M}$. Passing to a subalgebra if necessary, we may assume that $\mathcal{A}$ is abelian and prove the injectivity of $\mathcal{N}=\mathcal{A}'\cap\mathcal{M}$. It suffices to show that $\Phi_{\mathcal{N}}$ in Lemma 5 is continuous on $B\otimes_{\min}C$. Since $\mathcal{A}$ is diffuse, it is generated by a unitary $u\in\mathcal{A}$ such that $\lim_{k\to\infty}u^k=0$ ultraweakly. Indeed, every diffuse abelian von Neumann algebra with separable predual is $*$-isomorphic to $L^\infty[0,1]$, and the unitary element $u(t)=e^{2\pi it}\in L^\infty[0,1]$ has the required property (where i here, but not elsewhere, is the imaginary unit). Let $\Psi_n(a)=n^{-1}\sum_{k=1}^n u^k a u^{-k}$ for $a\in\mathbf{B}(\mathcal{H})$, and let $\Psi\colon\mathbf{B}(\mathcal{H})\to\mathbf{B}(\mathcal{H})$ be its cluster point in the point-ultraweak topology. It is not hard to see that $\Psi|_{\mathcal{M}}$ is a trace-preserving conditional expectation onto $\mathcal{N}$ and hence $\Psi|_{\mathcal{M}}=\varepsilon_{\mathcal{N}}$. It follows that for any $\sum_{i=1}^n a_i\otimes x_i\in\mathcal{M}\otimes\mathcal{M}'$, we have

$$\Psi\Big(\sum_{i=1}^n a_ix_i\Big)=\sum_{i=1}^n\varepsilon_{\mathcal{N}}(a_i)x_i=\Phi_{\mathcal{N}}\Big(\sum_{i=1}^n a_i\otimes x_i\Big).$$

Since $\lim_{k\to\infty}u^k=0$ ultraweakly, we have $\mathbf{K}(\mathcal{H})\subset\ker\Psi$. This implies that $\Psi=\widetilde{\Psi}\pi$ for some unital completely positive map $\widetilde{\Psi}\colon\mathbf{B}(\mathcal{H})/\mathbf{K}(\mathcal{H})\to\mathbf{B}(\mathcal{H})$. Since ν in the condition (AO) is continuous on $B\otimes_{\min}C$, so is $\Phi_{\mathcal{N}}=\widetilde{\Psi}\nu$. □

The following proposition was communicated to us by Popa [P3]. The author is grateful to him for allowing us to present it here.

PROPOSITION 7. *Assume that the type* II_1 *factor* $\mathcal{M}$ *(with separable predual) contains a non-injective von Neumann subalgebra* $\mathcal{N}_0$ *such that* $\mathcal{N}_0'\cap\mathcal{M}^\omega$ *is a diffuse von Neumann algebra, where* $\mathcal{M}^\omega$ *is an ultrapower algebra of* $\mathcal{M}$. *Then there exists a non-injective von Neumann subalgebra* $\mathcal{N}_1\subset\mathcal{M}$ *such that* $\mathcal{N}_1'\cap\mathcal{M}$ *is diffuse.*

Proof. Replacing it with a subalgebra if necessary, we may assume that the non-injective von Neumann subalgebra $\mathcal{N}_0$ is generated by a finite set $\{x_1, x_2, ..., x_m\}$. By Connes' characterizations of injectivity (Theorem 5.1 in [Co]), it follows that there exists $\varepsilon>0$ such that if $\{x_1', ..., x_m'\}\subset\mathcal{M}$ are so that $\|x_i'-x_i\|_2\leqslant\varepsilon$ then $\{x_i'\}_i$ generates a non-injective von Neumann subalgebra in $\mathcal{M}$. Indeed, if there existed injective von Neumann algebras $\mathcal{B}_k\subset\mathcal{M}$ such that $\lim_k\|x_i-\varepsilon_{\mathcal{B}_k}(x_i)\|_2=0$ for all i, then for any $\sum_{j=1}^n a_j\otimes y_j\in\mathcal{N}_0\otimes\mathcal{M}'$, we would have

$$\begin{aligned}\Big\|\sum_{j=1}^n a_j y_j\Big\|_{\mathbf{B}(\mathcal{H})} &\leqslant \liminf_{k\to\infty}\Big\|\sum_{j=1}^n \varepsilon_{\mathcal{B}_k}(a_j)y_j\Big\|_{\mathbf{B}(\mathcal{H})}\\ &= \liminf_{k\to\infty}\Big\|\sum_{j=1}^n \varepsilon_{\mathcal{B}_k}(a_j)\otimes y_j\Big\|_{\mathcal{B}_k\otimes_{\min}\mathcal{M}'} \leqslant \Big\|\sum_{j=1}^n a_j\otimes y_j\Big\|_{\mathcal{N}_0\otimes_{\min}\mathcal{M}'},\end{aligned}$$

which would imply $C^*(\mathcal{N}_0,\mathcal{M}')\cong\mathcal{N}_0\otimes_{\min}\mathcal{M}'$ and the injectivity of $\mathcal{N}_0$ (cf. the proof of Lemma 5).

Since $\mathcal{N}_0'\cap\mathcal{M}^\omega$ is diffuse, it follows by induction that there exists a sequence of mutually commuting, τ-independent two-dimensional abelian $*$-subalgebras $\mathcal{A}_n\subset\mathcal{M}$, with minimal projections of trace $\frac{1}{2}$, such that

$$\|\varepsilon_{\mathcal{A}_{n+1}'\cap\mathcal{M}}(x_i)-x_i\|_2<\frac{\varepsilon}{2^{n+1}}\quad\text{for all } i.$$

But if we let $\mathcal{B}_n=\mathcal{A}_1\vee\mathcal{A}_2\vee...\vee\mathcal{A}_n$ then we also have

$$\|\varepsilon_{\mathcal{B}_n'\cap\mathcal{M}}(x_i)-\varepsilon_{\mathcal{B}_n'\cap\mathcal{M}}(\varepsilon_{\mathcal{A}_{n+1}'\cap\mathcal{M}}(x_i))\|_2<\frac{\varepsilon}{2^{n+1}}\quad\text{for all } i.$$

Since $\varepsilon_{\mathcal{B}_n'\cap\mathcal{M}}\circ\varepsilon_{\mathcal{A}_{n+1}'\cap\mathcal{M}}=\varepsilon_{\mathcal{B}_{n+1}'\cap\mathcal{M}}$, if we let $\mathcal{A}=\bigvee_n^\infty\mathcal{B}_n$ and take into account that $\varepsilon_{\mathcal{A}'\cap\mathcal{M}}=\lim_{n\to\infty}\varepsilon_{\mathcal{B}_n'\cap\mathcal{M}}$ (see e.g. [P1]), then by triangle inequalities we get

$$\|x_i-\varepsilon_{\mathcal{A}'\cap\mathcal{M}}(x_i)\|_2\leqslant\varepsilon\quad\text{for all } i.$$

Thus, if we take $\mathcal{N}_1$ to be the von Neumann algebra generated by

$$x_i'=\varepsilon_{\mathcal{A}'\cap\mathcal{M}}(x_i),\quad 1\leqslant i\leqslant m,$$

then $\mathcal{N}_1$ satisfies the required conditions. □

4. Remark

We note the possibility that, for a hyperbolic group Γ, the $*$-homomorphism

$$\mathcal{L}\Gamma\otimes C^*_\varrho\Gamma\ni\sum_{i=1}^n a_i\otimes x_i\longmapsto\pi\left(\sum_{i=1}^n a_i x_i\right)\in\mathbf{B}(l_2\Gamma)/\mathbf{K}(l_2\Gamma)$$

may be continuous with respect to the minimal tensor norm. If this is the case, then it would follow that a von Neumann subalgebra $\mathcal{N}\subset\mathcal{L}\Gamma$ is injective if and only if

$$C^*(\mathcal{N},C^*_\varrho\Gamma)\cap\mathbf{K}(l_2\Gamma)=\{0\},$$

which would reprove our results (modulo Theorem 2.1 in [Co]).

Acknowledgment. The author would like to thank Professor Sorin Popa for showing him Proposition 7, and Professor Nigel Higson for providing him with the references [HG] and [Sk]. This research was carried out while the author was visiting the University of California at Los Angeles under the support of the Japanese Society for the Promotion of Science Postdoctoral Fellowships for Research Abroad. He gratefully acknowledges the kind hospitality of the UCLA.

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Received February 19, 2003