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On the density of geometrically finite Kleinian groups

by

JEFFREY F. BROCK and KENNETH W. BROMBERG

University of Chicago
Chicago, IL, U.S.A.

California Institute of Technology
Pasadena, CA, U.S.A.

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1. Introduction

In the 1970s, work of W. P. Thurston revolutionized the study of Kleinian groups and their 3-dimensional hyperbolic quotients. Nevertheless, a complete topological and geometric classification of hyperbolic 3-manifolds persists as a fundamental unsolved problem.

Even for *tame* hyperbolic 3-manifolds $N=\mathbf{H}^3/\Gamma$, where N has tractable topology (N is homeomorphic to the interior of a compact 3-manifold), the correct picture of the range of complete hyperbolic structures on N remains conjectural.

On the other hand, *geometrically finite* hyperbolic 3-manifolds are completely parameterized by an elegant deformation theory. As an approach to a general classification, Thurston proposed a program to extend this parameterization to all hyperbolic 3-manifolds with finitely generated fundamental group [T2]. A critical, and as yet unyielding obstacle is the *density conjecture*:

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CONJECTURE 1.1 (Bers–Sullivan–Thurston). *Let M be a complete hyperbolic 3-manifold with finitely generated fundamental group. Then M is a limit of geometrically finite hyperbolic 3-manifolds.*

Our main result is the following theorem.

THEOREM 1.2. *Let M be a complete hyperbolic 3-manifold with finitely generated fundamental group, incompressible ends and no cusps. Then M is an algebraic limit of geometrically finite hyperbolic 3-manifolds.*

We call M *geometrically finite* if its *convex core*, the minimal convex subset of M, has finite volume. The manifold M is the quotient of $\mathbf{H}^3$ by a *Kleinian group* Γ, a discrete, torsion-free subgroup of the orientation-preserving isometries of hyperbolic 3-space. Then $M=\mathbf{H}^3/\Gamma$ is an *algebraic limit* of $M_i=\mathbf{H}^3/\Gamma_i$ if there are isomorphisms $\varrho_i:\Gamma\to\Gamma_i$ so that after conjugating the groups Γ_i in $\mathrm{Isom}^+(\mathbf{H}^3)$ if necessary, we have $\varrho_i(\gamma)\to\gamma$ for each $\gamma\in\Gamma$. We say that M has *incompressible ends* if it is homotopy equivalent to a compact submanifold with incompressible boundary.

The *algebraic deformation space* $AH(M)$ is the collection of discrete, faithful representations $\varrho:\pi_1(M)\to\mathrm{Isom}^+(\mathbf{H}^3)$ up to conjugacy, with the topology of algebraic convergence. Marden and Sullivan proved that the interior of $AH(M)$ consists of such geometrically finite hyperbolic 3-manifolds (see [Ma] and [Su2]). Then Conjecture 1.1 predicts that the deformation space is the closure of its interior.

Theorem 1.2 generalizes the recent result of the second author [Brm1], which applies to cusp-free *singly degenerate* manifolds M with the homotopy type of a surface. In that case, the result gives a partial solution to an earlier version of Conjecture 1.1 formulated by L. Bers in [Be]. For the modern formulation, see [Su2] and [T2]. Our strategy is essentially similar here: due to work of Minsky (see [Mi4]) one need only consider the case that M has arbitrarily short geodesics; such geodesics necessarily exit an end of M. After work of Bonahon [Bon1] and Otal [Ot1], such geodesics are eventually *unknotted*: they are isotopic into a level surface in the end. This unknottedness facilitates the use of the grafting trick of [Brm1], but peculiarities of the general *doubly degenerate* case force us to develop new deformation-theoretic techniques to complete the proof.

Indeed, fundamental in the treatment of each case is the use of 3-dimensional hyperbolic cone-manifolds, namely, 3-manifolds that are hyperbolic away from a closed geodesic cone-type singularity. The theory of deformations of these manifolds that change only the cone-angle, developed by C. Hodgson and S. Kerckhoff [HK1], [HK2], [HK4], and the second author [Brm2], [Brm3], is instrumental in our study. In particular, the recent innovations of [HK2] and [HK4] have extended the theory to treat the setting of arbitrary cone-angles, whereas [HK1] treats only the case of cone-angle at most 2π

(see [HK3] for an expository account). These estimates are essential to results of [Brm1] and their generalizations here.

Though the power of cone-deformations has been amply demonstrated in the proof of the orbifold theorem and the study of hyperbolic Dehn-surgery space developed by Hodgson and Kerckhoff, we hope that the present study will suggest its wider applicability as a new tool in the study of deformation spaces of infinite-volume hyperbolic 3-manifolds.

The principal application of the cone-deformation theory here is its ability to control the geometric effect of a cone-deformation that decreases the cone-angle at the singular locus when the singular locus is a sufficiently short geodesic. Since each simple closed geodesic in a hyperbolic 3-manifold may be regarded as a "singular locus" with cone-angle 2π, we obtain control on how a geometrically finite structure with a short closed geodesic differs from the complete hyperbolic structure on the manifold with the same conformal boundary and the short geodesic removed (the resulting cusp may be viewed as a singular locus with cone-angle 0).

A central result of the paper is a *drilling theorem*, giving an example of this type of control. Here is a version applicable to complete, smooth hyperbolic structures:

THEOREM 1.3. (The drilling theorem) *Let M be a geometrically finite hyperbolic 3-manifold. For each $L>1$, there is an $l>0$ so that if c is a geodesic in M with length $l_M(c)<l$, there is an L-bi-Lipschitz diffeomorphism of pairs*

$$h\colon (M\backslash\mathbf{T}(c), \partial\mathbf{T}(c)) \longrightarrow (M_0\backslash\mathbf{P}(c), \partial\mathbf{P}(c)),$$

where $M\backslash\mathbf{T}(c)$ denotes the complement of a standard tubular neighborhood of c in M, M_0 denotes the complete hyperbolic structure on $M\backslash c$, and $\mathbf{P}(c)$ denotes a standard rank-2 cusp corresponding to c.

(See Theorem 6.2 for a more precise version.)

The drilling theorem and its algebraic antecedents in [Brm2] are reminiscent of the essential estimates needed to control the algebraic effect of other types of pinching deformations. Such estimates have been used to show (for example) the density of maximal cusps in boundaries of deformation spaces [Mc2], [CCHS]. While these estimates give algebraic control over pinching short curves on the conformal boundary, a very short geodesic in M can have large length on the conformal boundary of M. The drilling theorem, by contrast, applies to any short geodesic in M.

The drilling theorem has proven to be of general use in the study of deformation spaces of hyperbolic 3-manifolds. Indeed, Theorem 1.3 represents the main technical tool in the recent topological tameness theorems of the authors' with R. Evans and J. Souto for algebraic limits of geometrically finite manifolds, and the consequent reduction of Ahlfors' measure conjecture to Conjecture 1.1 (see [Ah] and [BBES]).

Grafting and geometric finiteness. Initially, our argument mirrors that of [Brm1], in which a singly degenerate M with arbitrarily short geodesics is first shown to be approximated by geometrically finite cone-manifolds.

The grafting construction of [Brm1] produces cone-manifolds that approximate a doubly degenerate manifold as well, but the proof that these cone-manifolds are geometrically finite is entirely different in this case. Here, we replace considerations of projective structures on surfaces with notions of convex hulls and geometric finiteness for variable (pinched) negative curvature developed by B. Bowditch and M. Anderson, after applying a theorem of Gromov and Thurston to perturb the relevant cone-metrics to smooth metrics of negative curvature.

Bounded geometry and arbitrarily short geodesics. After [Brm1], our central challenge here is to address the possibility that M is *doubly degenerate*, namely, the case for which $M \cong S \times \mathbf{R}$ and the convex core is all of M. In this case, M has two *degenerate ends*: each end has an exiting sequence of closed geodesics that are homotopic to simple curves on S. Our analysis turns on whether such geodesics can be taken to be arbitrarily short.

When each end of M has such a family of arbitrarily short geodesics, a streamlined argument exists that avoids certain technical tools developed here. We refer the reader to [BB, §3] for a discussion of the argument, which is more directly analogous to that of [Brm1]. We remark that in particular no application of Thurston's double limit theorem is required; the convergence of the relevant approximations follows directly from the cone-deformation theory.

When M is assumed to have *bounded geometry* (M has a global lower bound to its injectivity radius) and M is homotopy equivalent to a surface, Minsky's *ending lamination theorem* for bounded geometry implies Theorem 1.2 (see [Mi4, Corollary 2]). The theorem guarantees that any such M is completely determined by its *end-invariants*, asymptotic data associated to the ends of M. An application of Thurston's double limit theorem ([T1], cf. [Oh1]) and continuity of the length function for laminations (see [Bro1]) allows one to realize the end-invariants of M as those of a limit N of geometrically finite manifolds Q_n. Minsky's theorem [Mi4, Corollary 1] then implies that N is isometric to M, and thus $\{Q_n\}_{n=1}^{\infty}$ converges to M.

A persistently difficult case has been that of M with *mixed type*. In this case, one end of M has bounded geometry, the other arbitrarily short geodesics. For manifolds of mixed type, the full strength of our techniques is required to isolate the geometry of the ends from one another. Rather than breaking the argument into the above cases, however, we have presented a unified treatment that handles all cases simultaneously.

Scheme of the proof. As a guide to the reader, we briefly describe the scheme of the proof of Theorem 1.2.

I. *Reduction to surface groups.* The essential difficulties arise in the search for geometrically finite approximations to a hyperbolic 3-manifold M with the homotopy type of a surface S. Within this category, it is the *doubly degenerate* manifolds that remain after [Brm1]. Each such manifold has a positive and a negative degenerate end, given a choice of orientation.

II. *Realizing ends on a Bers boundary.* We first seek to *realize* the geometry of each end of M as that of an end of a singly degenerate limit of quasi-Fuchsian manifolds $\{Q(X,Y_n)\}_{n=1}^{\infty}$ or $\{Q(X_n,Y)\}_{n=1}^{\infty}$: given the positive end E of M, say, we seek a limit $Q=\lim_{n\to\infty} Q(X,Y_n)$ so that E admits a marking- and orientation-preserving bi-Lipschitz diffeomorphism to an end of Q. We prove that such limits can always be found (Theorem 7.2) by considering the bounded geometry case and the case when E has arbitrarily short geodesics separately.

III. *Bounded geometry.* If the end E has a lower bound to its injectivity radius, we employ techniques of Minsky to show that its *end-invariant* $\nu(E)$ has *bounded type*: any incompressible end of a hyperbolic 3-manifold with end-invariant $\nu(E)$ has a lower bound to *its* injectivity radius, whether or not the bound holds globally. After producing a limit Q on a Bers boundary with end-invariant $\nu(E)$, an application of Minsky's bounded geometry theory shows that Q realizes E in the above sense.

IV. *Arbitrarily short geodesics.* If the end E has arbitrarily short geodesics, a simultaneous grafting procedure produces a hyperbolic cone-manifold with two components in its singular locus, each with cone-angle 4π. Generalizing tameness results for variable negative curvature, we show that the simultaneous grafting is *geometrically finite*: its convex core is compact. Applying the drilling theorem (Theorem 6.2) we deform the metric back to a smooth structure rel, the conformal boundary with bounded distortion of the metric structure outside a tubular neighborhood of the singular locus. Successive simultaneous graftings give quasi-Fuchsian manifolds limiting to a manifold Q that realizes E.

V. *Asymptotic isolation.* We then prove an asymptotic isolation theorem (Theorem 8.1) which again uses the drilling theorem to show that any cusp-free doubly degenerate limit M of quasi-Fuchsian manifolds $Q(X_n,Y_n)$ has positive and negative ends E^+ and E^- so that E^+ depends only on $\{Y_n\}_{n=1}^{\infty}$ and E^- depends only on $\{X_n\}_{n=1}^{\infty}$ up to bi-Lipschitz diffeomorphism.

VI. *Conclusion.* The proof is concluded by realizing the positive end E^+ of M by the limit of $\{Q(X,Y_n)\}_{n=1}^{\infty}$, and the negative end E^- of M by the limit of $\{Q(X_n,Y)\}_{n=1}^{\infty}$, where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are determined by Theorem 7.2. Thurston's double limit theorem implies that $Q(X_n, Y_n)$ converges up to subsequence to a limit M', and thus Theorem 8.1 implies that the ends of M' admit marking-preserving bi-Lipschitz diffeomorphisms to the ends of M. By an application of Sullivan's rigidity theorem, we have $Q(X_n, Y_n) \to M$.

We conclude with two remarks.

Generalizations. The hypotheses of the theorem can be weakened with only technical changes to the argument. The clearly essential hypothesis is that M be *tame*, which is guaranteed in our setting by the assumption that M have incompressible ends (by Bonahon's theorem [Bon1]). In the setting of tame manifolds with compressible ends, the principal obstruction to carrying out our argument lies in the need for *unknotted* short geodesics, guaranteed in the incompressible setting by a result of J.P. Otal (see [Ot3] and Theorem 2.5). We expect this to be a surmountable difficulty and will take up the issue in a future paper.

The assumption that M have no parabolics is required only by our use of Minsky's ending lamination theorem for bounded geometry [Mi1], where the hyperbolic manifolds in question are assumed to have a global lower bound on their injectivity radii rather than simply a lower bound to the length of the shortest geodesic.

A reworking of Minsky's theorem to allow *peripheral* parabolics represents the only obstacle to allowing parabolics in our theorem. While such a reworking is now essentially straightforward after the techniques introduced in [Mi4], we have chosen in a similar spirit to defer these technicalities to a later paper in the interest of conveying the main ideas.

Ending laminations. We also remark that recently announced work of the first author with R. Canary and Y. Minsky [BCM] has completed Minsky's program to prove Thurston's *ending lamination conjecture* for hyperbolic 3-manifolds with incompressible ends. This result predicts (in particular) that each hyperbolic 3-manifold M equipped with a cusp-preserving homotopy equivalence from a hyperbolic surface S is determined up to isometry by its parabolic locus and its end-invariants (see [Mi5] and [BCM]).

As in the bounded geometry case, Theorem 1.2 follows from the ending lamination conjecture via an application of [T1], [Oh1] and [Bro1], so the results of [BCM] will give an alternative proof of our main theorem. We point out that the techniques employed here are independent of those of [BCM], and of a different nature. In particular, we expect the drilling theorem (Theorem 6.2) to have applications beyond the scope of this paper, and we refer the reader to [BBES] for an initial example of its application in a different context.

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2. Preliminaries

A *Kleinian group* is a discrete, torsion-free subgroup of $\mathrm{Isom}^+(\mathbf{H}^3)=\mathrm{Aut}(\widehat{\mathbf{C}})$. Each Kleinian group Γ determines a complete hyperbolic 3-manifold $M=\mathbf{H}^3/\Gamma$ as the quotient of $\mathbf{H}^3$ by Γ. The manifold M extends to its *Kleinian manifold* $N=(\mathbf{H}^3\cup\Omega)/\Gamma$ by adjoining its *conformal boundary* ∂M, namely, the quotient by Γ of the *domain of discontinuity* $\Omega\subset\widehat{\mathbf{C}}$ where Γ acts properly discontinuously. (Unless explicitly stated, all Kleinian groups will be assumed *non-elementary*.)

The *convex core* of M, which we denote by $\mathrm{core}(M)$, is the smallest convex subset of M whose inclusion is a homotopy equivalence. The complete hyperbolic 3-manifold M is *geometrically finite* if $\mathrm{core}(M)$ has a finite-volume unit neighborhood in M.

The thick-thin decomposition. The *injectivity radius* $\mathrm{inj}: M\to\mathbf{R}^+$ measures the radius of the maximal embedded metric open ball at each point of M. For $\epsilon>0$, we denote by $M^{<\epsilon}$ the *ϵ-thin part* where $\mathrm{inj}(x)<\epsilon$, and by $M^{\geqslant\epsilon}$ the *ϵ-thick part* $M\setminus M^{<\epsilon}$. By the *Margulis lemma* there is a universal constant ε so that each component T of the *thin part* $M^{<\varepsilon}$, where $\mathrm{inj}(x)<\varepsilon$, has a standard type: either T is an open solid-torus neighborhood of a short geodesic, or T is the quotient of an open horoball $B\subset\mathbf{H}^3$ by a $\mathbf{Z}$- or $\mathbf{Z}\oplus\mathbf{Z}$-parabolic group fixing B.

Curves and surfaces. Let S be a closed topological surface of genus at least 2. We denote by $\mathcal{S}$ the set of all isotopy classes of essential simple closed curves on S. The *geometric intersection number*

$$i:\mathcal{S}\times\mathcal{S}\longrightarrow\mathbf{Z}^+$$

counts the minimal number of intersections of representatives of curves in a pair of isotopy classes $(\alpha,\beta)\in\mathcal{S}\times\mathcal{S}$.

The *Teichmüller space* $\mathrm{Teich}(S)$ parameterizes marked hyperbolic structures on S: pairs (f,X) where $f:S\to X$ is a homeomorphism to a hyperbolic surface X modulo the equivalence relation that $(f,X)\sim(g,Y)$ when there is an isometry $\phi: X\to Y$ for which $\phi\circ f\simeq g$. If we allow S to have boundary, then X is required to have finite area and $f:\mathrm{int}(S)\to X$ is a homeomorphism from the interior of S to X.

We topologize Teichmüller space by the *quasi-isometric distance*

$$d_{\mathrm{qi}}((f,X),(g,Y)),$$

which is the log of the infimum over all bi-Lipschitz diffeomorphisms $\phi: X\to Y$ homotopic to $g\circ f^{-1}$ of the best bi-Lipschitz constant for ϕ (cf. [T7]). Each $\alpha\in\mathcal{S}$ has a unique geodesic representative on any surface $(f,X)\in\mathrm{Teich}(S)$ by taking the representative of the free-homotopy class of $f(\alpha)$ on X of shortest length.

To interpolate between simple closed curves in $\mathcal{S}$, Thurston introduced the *measured geodesic laminations*, $\mathcal{ML}(S)$, which may be obtained formally as the completion of the image of $\mathbf{R}^+\times\mathcal{S}$ under the map $\iota:\mathbf{R}^+\times\mathcal{S}\to\mathbf{R}^{\mathcal{S}}$ defined by $\langle\iota(t,\alpha)\rangle_\beta=ti(\alpha,\beta)$.

On a given (f,X) in Teichmüller space, a *geodesic lamination* is a closed subset of X given as a union of pairwise disjoint geodesics on S. The measured laminations $\mathcal{ML}(S)$ are then identified with *measured geodesic laminations*, pairs (λ,μ) of a geodesic lamination λ and a *transverse measure*, an association of a measure μ_α to each arc α transverse to λ so that μ_α is invariant under holonomy and finite for compact α. One obtains the *projective measured laminations* $\mathcal{PL}(S)$ as the quotient $(\mathcal{ML}(S)\setminus\{0\})/\mathbf{R}^+$. (See [T1], [FLP], [Ot2] or [Bon2] for more about geodesic and measured laminations.)

Surface groups. By $H(S)$ we denote the set of all marked hyperbolic 3-manifolds $(f:S\to M)$: i.e. complete hyperbolic 3-manifolds M equipped with homotopy equivalences $f:S\to M$, modulo the equivalence relation

$$(f:S\to M)\sim(g:S\to N)$$

if there is an isometry $\phi:M\to N$ for which $\phi\circ f\simeq g$.

Each $(f:S\to M)$ in $H(S)$ determines a representation

$$f_*=\varrho:\pi_1(S)\longrightarrow\mathrm{Isom}^+(\mathbf{H}^3),$$

well defined up to conjugacy in $\mathrm{Isom}^+(\mathbf{H}^3)=\mathrm{PSL}_2(\mathbf{C})$. We topologize $H(S)$ by the compact-open topology on the induced representations, up to conjugacy. Convergence in this sense is known as *algebraic convergence*; we equip $H(S)$ with this *algebraic topology* to obtain the space $AH(S)$, the *algebraic deformation space*.

The subset $QF(S)\subset AH(S)$ denotes the *quasi-Fuchsian locus*, namely, manifolds $(f:S\to Q)$ so that Q is bi-Lipschitz diffeomorphic to the quotient of $\mathbf{H}^3$ by a Fuchsian group. Such a quasi-Fuchsian manifold Q simultaneously uniformizes a pair $(X,Y)\in\mathrm{Teich}(S)\times\mathrm{Teich}(S)$ as its conformal boundary ∂Q, namely, the quotient of the region where the covering group $f_*(\pi_1(S))$ for Q acts properly discontinuously on $\widehat{\mathbf{C}}$. In our convention, X compactifies the negative end of $Q(X,Y)\cong S\times\mathbf{R}$ and Y compactifies the positive end ($\widehat{\mathbf{C}}$ is assumed oriented so that the resulting identification of $\widetilde{S}$ with $\widetilde{Y}\subset\widehat{\mathbf{C}}$ is orientation preserving while the identification of $\widetilde{X}$ with $\widetilde{S}$ is orientation reversing—by our convention $Q(Y,Y)$ is a Fuchsian manifold).

Bers exhibited a homeomorphism

$$Q:\mathrm{Teich}(S)\times\mathrm{Teich}(S)\longrightarrow QF(S)$$

that assigns to the pair (X,Y) the quasi-Fuchsian manifold $Q(X,Y)$ simultaneously uniformizing X and Y. The manifold $Q(X,Y)$ naturally inherits a homotopy equivalence

$f: S \to Q(X,Y)$ from the marking on either of its boundary components, so the simultaneous uniformization is naturally an element of $AH(S)$.

One obtains a *Bers slice* of the quasi-Fuchsian space $QF(S)$ by fixing one factor in the product structure; we denote by

$$B_X = \{X\} \times \mathrm{Teich}(S) \subset QF(S)$$

the Bers slice of quasi-Fuchsian manifolds with X compactifying their negative ends. As one may fix the conformal boundary compactifying either the positive or negative end, we will employ the notation

$$B_X^+ = \{X\} \times \mathrm{Teich}(S) \quad \text{and} \quad B_Y^- = \mathrm{Teich}(S) \times \{Y\}$$

to distinguish the two types of slices.

If $g: M \to N$ is a bi-Lipschitz diffeomorphism between Riemannian n-manifolds, its *bi-Lipschitz constant* $L(g) \geqslant 1$ is the infimum over all L for which

$$\frac{1}{L} \leqslant \frac{|g_*(v)|}{|v|} \leqslant L$$

for all $v \in TM$.

Following McMullen (see [Mc3, §3.1]), we define the *quasi-isometric distance* on $AH(S)$ by

$$d_{\mathrm{qi}}((f_1, M_1), (f_2, M_2)) = \inf \log L(g),$$

where the infimum is taken over all orientation-preserving bi-Lipschitz diffeomorphisms $g: M_1 \to M_2$ for which $g \circ f_1$ is homotopic to f_2. If there is no such diffeomorphism in the appropriate homotopy class, then we say that (f_1, M_1) and (f_2, M_2) have infinite quasi-isometric distance. The quasi-isometric distance is lower semi-continuous on $AH(S) \times AH(S)$ ([Mc3, Proposition 3.1]).

Geometric and strong convergence. Another common and related notion of convergence of hyperbolic manifolds comes from the *Hausdorff topology*, which we now describe.

A hyperbolic 3-manifold determines a Kleinian group only up to conjugation. Equipping M with a unit orthonormal frame ω at a basepoint p (a *base-frame*) eliminates this ambiguity via the requirement that the covering projection

$$\pi: (\mathbf{H}^3, \widetilde{\omega}) \longrightarrow (\mathbf{H}^3, \widetilde{\omega})/\Gamma = (M, \omega)$$

sends the standard frame $\widetilde{\omega}$ at the origin in $\mathbf{H}^3$ to ω.

The framed hyperbolic 3-manifolds $(M_n, \omega_n) = (\mathbf{H}^3, \widetilde{\omega})/\Gamma_n$ converge *geometrically* to a *geometric limit* $(N, \omega) = (\mathbf{H}^3, \widetilde{\omega})/\Gamma_G$ if Γ_n converges to Γ_G in the geometric topology:

(1) For each $\gamma \in \Gamma_G$ there are $\gamma_n \in \Gamma_n$ with $\gamma_n \to \gamma$.

(2) If elements γ_{n_k} in a subsequence Γ_{n_k} converge to γ, then γ lies in Γ_G.

Geometric convergence has an internal formulation: (M_n,ω_n) converges to (N,ω) if for each smoothly embedded compact submanifold $K\subset N$ containing ω, there are diffeomorphisms $\phi_n: K\to(M_n,\omega_n)$ so that $\phi_n(\omega)=\omega_n$ and so that ϕ_n converges to an isometry on K in the C^∞-topology ([BP], [Mc3, Chapter 2]).

When $(f: S\to M)$ lies in $AH(S)$, a base-frame $\omega\in M$ determines a discrete, faithful representation $f_*: \pi_1(S)\to\Gamma$, where $(M,\omega)=(\mathbf{H}^3,\widetilde{\omega})/\Gamma$. Denote by $AH_\omega(S)$ the marked *framed* hyperbolic 3-manifolds $(f: S\to(M,\omega))$, i.e. framed hyperbolic 3-manifolds (M,ω) together with homotopy equivalences $f: S\to(M,\omega)$ up to isometries that preserve marking and base-frame.

The space $AH_\omega(S)$ carries the topology of convergence on generators of the induced representations f_*; the topology on $AH(S)$ is simply the quotient topology under the natural base-frame forgetting map $AH_\omega(S)\to AH(S)$. As with $AH(S)$ we will often assume an implicit marking and refer to $(M,\omega)\in AH_\omega(S)$.

Consideration of $AH_\omega(S)$ allows us to understand the relation between algebraic and geometric convergence (see [Bro3, §2]):

THEOREM 2.1. *Given a sequence $\{(f_n: S\to M_n)\}_{n=1}^\infty$ with limit $(f: S\to M)$ in $AH(S)$ there are convergent lifts $(f_n: S\to(M_n,\omega_n))$ to $AH_\omega(S)$ so that, after passing to a subsequence, (M_n,ω_n) converges geometrically to a geometric limit (N,ω) covered by M by a local isometry.*

When this local isometry is actually an isometry, we say that the convergence is *strong*.

Definition 2.2. The sequence $M_n\to M$ in $AH(S)$ converges *strongly* if there are lifts $(M_n,\omega_n)\to(M,\omega)$ to $AH_\omega(S)$ so that (M_n,ω_n) also converges geometrically to (M,ω).

Pleated surfaces. Given $M\in AH(S)$ and a simple closed curve $\alpha\in\mathcal{S}$ representing a non-parabolic conjugacy class of $\pi_1(M)$, we follow Bonahon's convention and denote by α^* the geodesic representative of α in M. To control how α^* can lie in M, Thurston introduced the notion of a *pleated surface*.

Definition 2.3. A path isometry $g: X\to N$ from a hyperbolic surface X to a hyperbolic 3-manifold N is a *pleated surface* if for each $x\in X$ there is a geodesic segment σ through x so that g maps σ isometrically to N.

Recall that the condition for g to be a *path isometry* means that g sends rectifiable arcs in X to rectifiable arcs in N of the same arc length.

When M lies in $AH(S)$, a particularly useful class of pleated surfaces arises from those that "preserve marking" in the following sense: denote by $\mathcal{PS}(M)$ the set of all pairs (g, X) where X lies in Teich(S) and $g: X \to M$ is a pleated surface with the property that $g \circ \phi \simeq f$, where ϕ is the implicit marking on X and f is the implicit marking on M.

Given a lamination $\mu \in \mathcal{ML}(S)$, we say that the pleated surface $(g, X) \in \mathcal{PS}(M)$ *realizes* μ if each geodesic leaf l in the support of μ realized as a geodesic lamination on X is mapped by g by a local isometry; alternatively, the lift $\tilde{g}: \widetilde{X} \to \mathbf{H}^3$ sends every leaf $\tilde{l}$ of the lift $\tilde{\mu}$ of μ to a complete geodesic in $\mathbf{H}^3$.

The following *bounded diameter theorem* for pleated surfaces is instrumental in Thurston's studies of geometrically tame hyperbolic 3-manifolds (see [T1, §8] or alternative versions in [Bon1] and [C1]). Since we are working in the cusp-free setting, we state the theorem in this context.

THEOREM 2.4. *Each compact subset K of the cusp-free manifold $M \in AH(S)$ has a compact enlargement K' so that if $(g, X) \in \mathcal{PS}(M)$ and $g(X) \cap K \neq \varnothing$, then $g(X)$ lies entirely in K'.*

Tame ends. Let M be a complete hyperbolic 3-manifold with finitely generated fundamental group. By a theorem of P. Scott [Sc], there is a compact submanifold $\mathcal{M} \subset M$ whose inclusion is a homotopy equivalence. By convention, given a choice of compact core $\mathcal{M}$ for M, the *ends* of M are the connected components of the complement $M \setminus \mathcal{M}$. Each end E is *cut off* by a boundary component $S \subset \partial \mathcal{M}$.

The end E is *tame* if it is homeomorphic to the product $S \times \mathbf{R}^+$, and the manifold M is *topologically tame* (or simply *tame*) if it is the interior of a compact manifold with boundary. When M is tame we can choose the compact core $\mathcal{M}$ such that each end E is tame. Manifestly, the end E depends on a choice of compact core, but as we will typically be interested in the end E only up to bi-Lipschitz diffeomorphism, we will assume such a core to be chosen in advance and address any ambiguity as the need arises.

An end E of M is *geometrically finite* if it has finite-volume intersection with the convex core of M. Otherwise it is *geometrically infinite.* By a theorem of Marden (see [Ma]), a geometrically finite end is tame. The manifold M is geometrically finite if and only if each of its ends is geometrically finite.

Tameness and Otal's theorem. One key element of our argument in §5 involves the fact that any collection of sufficiently short closed curves in $M \in AH(S)$ is *unknotted* and *unlinked*.

Given $M \in AH(S)$ the tameness theorem of Bonahon and Thurston [Bon1], [T1] guarantees the existence of a product structure $F: S \times \mathbf{R} \to M$. Otal defines a notion of 'unknottedness' with respect to this product structure as follows: a closed curve $\alpha \in M$ is *unknotted* if it is isotopic in M to a simple curve in a *level surface* $F(S \times \{t\})$. Likewise, a collection $\mathcal{C}$ of closed curves in M is *unlinked* if there is an isotopy of the collection $\mathcal{C}$

sending each member $\alpha\in\mathcal{C}$ to a distinct level surface $F(S\times\{t_\alpha\})$.

THEOREM 2.5. (Otal) *Let S be a closed surface, and let $(f: S\to M)$ lie in $AH(S)$. There is a constant $l_{\text{knot}}>0$ depending only on S so that if $\mathcal{C}$ is any collection of closed curves in M for which $l_M(\alpha^*)<l_{\text{knot}}$ for each $\alpha\in\mathcal{C}$, then the collection $\mathcal{C}$ is unlinked.*

See [Ot1] and, in particular, [Ot3, Theorem B].

Cone-manifolds. A key ingredient for our argument will be the notion of a 3-*dimensional hyperbolic cone-manifold.* Let N be a compact 3-manifold with boundary and $\mathcal{C}$ a collection of disjoint simple closed curves. A *hyperbolic cone-metric* on $(N,\mathcal{C})$ is a hyperbolic metric on the interior of $N\setminus\mathcal{C}$ whose completion is a singular metric on the interior of N. In a neighborhood of a point in $\mathcal{C}$, the metric will have the form

$$dr^2+\sinh^2 r\, d\theta^2+\cosh^2 r\, dz^2,$$

where θ is measured modulo the cone-angle α. The singular locus will be identified with the z-axis and will be totally geodesic. Note that the cone-angle will be constant along each component of the singular locus.

3. Geometric finiteness in negative curvature

In this section, defining the notion of geometric finiteness for 3-dimensional hyperbolic cone-manifolds, we will use and show its equivalence to precompactness of the set of closed geodesics in the cusp-free setting. We then go on to employ the work of Bonahon and Canary [Bon1], [C1] to show the existence of simple closed geodesics exiting any end of M that is not geometrically finite.

Geometric finiteness for cone-manifolds. When the convex core of the complete hyperbolic 3-manifold M has a finite-volume unit neighborhood, the only obstruction to the compactness of the convex core is the presence of cusps in M. In the cusped case, a slightly different definition is required. For our discussion, we consider only cusps that arise from rank-2 Abelian subgroups of the fundamental group, i.e. *rank-2 cusps.*

Definition 3.1. A 3-dimensional hyperbolic cone-manifold M is *geometrically finite without rank-1 cusps* if M has a compact core bounded by convex surfaces and tori.

In the sequel, all hyperbolic cone-manifolds we will consider will be free of rank-1 cusps. As such, we simply refer to geometrically finite manifolds without rank-1 cusps as *geometrically finite*.

Given a compact core $\mathcal{M}$ for such an M, the geometric finiteness of M is usefully rephrased as a condition on the ends of M (again, we refer to components of $M\setminus\mathcal{M}$ as the *ends* of M; they are neighborhoods of the topological ends of M).

Definition 3.2. An end E of a 3-dimensional hyperbolic cone-manifold M is *geometrically finite* if its intersection with the convex core of M has finite volume.

An end that is cut off by a torus will be a rank-2 cusp and will be entirely contained in the convex core. Since we are assuming that M does not have rank-1 cusps, each end of a geometrically finite manifold cut off by a higher genus surface will intersect the convex core in a compact set.

Then one may easily verify the following proposition.

PROPOSITION 3.3. *The 3-dimensional hyperbolic cone-manifold M is geometrically finite if and only if each end of M is geometrically finite.*

Geometrically infinite ends. An end E of M that is not geometrically finite is *geometrically infinite* or *degenerate.*

Definition 3.4. Let E be a geometrically infinite end of a 3-dimensional hyperbolic cone-manifold M, cut off by a surface S. Then E is *simply degenerate* if for any compact subset $K \subset E$ there is a simple curve α on S whose geodesic representative lies in $E \setminus K$.

In the smooth hyperbolic setting, a synonym for a simply degenerate end is a *geometrically tame* end; we use the same terminology here. The cusp-free hyperbolic cone-manifold M is *geometrically tame* if all its ends are geometrically finite or geometrically tame.

Thurston and Bonahon proved that a geometrically tame manifold M with freely indecomposable fundamental group is *topologically tame*, namely, M is homeomorphic to the interior of a compact 3-manifold. Generalizing Bonahon's work, Canary proved a general converse:

THEOREM 3.5. ([C1]) *Let M be a complete hyperbolic 3-manifold. If M is topologically tame then M is geometrically tame.*

Geometric finiteness in variable negative curvature. B. Bowditch has given a detailed analysis of how various notions of geometric finiteness for complete hyperbolic 3-manifolds and their equivalences generalize to the case of *pinched negative curvature*, namely, 3-manifolds with complete Riemannian metrics with all sectional curvatures in the interval $[-a^2, -b^2]$, where $0<b<a$.

Such a manifold is the quotient of a *pinched Hadamard manifold* X, a simply connected manifold with sectional curvatures pinched between $-a^2$ and $-b^2$, by a discrete subgroup Γ of its orientation-preserving isometries $\mathrm{Isom}^+(X)$. For our purposes, we assume that X has dimension 3. The action of Γ on X has much in common with actions of Kleinian groups on $\mathbf{H}^3$. In particular, X has a natural *ideal sphere* X_I, or *sphere at* *infinity*, which may be identified with equivalence classes of infinite geodesic rays in X, where rays are equivalent if they are asymptotic.

As in the hyperbolic setting, the action of Γ on X_I is partitioned into its *limit set* Λ, where the orbit of a (and hence any) point in X accumulates on X_I, and its *domain of discontinuity* $\Omega = X_I \setminus \Lambda$.

The *convex core* of the quotient manifold $M = X/\Gamma$ of pinched negative curvature is the quotient $\mathrm{hull}(\Lambda)/\Gamma$ of the convex hull in X of the limit set Λ by the action of Γ. Then following [Bow2] we make the following definition.

Definition 3.6. The manifold $M = X/\Gamma$ of pinched negative curvature is *geometrically finite* if the radius-1 neighborhood of the convex core has finite volume.

In the case when the manifold M has no cusps (the group Γ is free of parabolic elements; see [Bow2, §2]), this notion is equivalent to the compactness of the convex core.

Indeed, it suffices to consider the quotient of the *join* of the limit set $\mathrm{join}(\Lambda)$: the collection of all geodesics in X joining pairs of points in Λ. We apply the following theorem of Bowditch [Bow1] which follows from work of M. Anderson [An].

THEOREM 3.7. (Bowditch) *Let M be a Riemannian manifold of pinched negative curvature. Then there is a $\sigma > 0$ depending only on the pinching constants so that*

$$\mathrm{hull}(\Lambda) \subset \mathcal{N}_\sigma(\mathrm{join}(\Lambda)).$$

(Cf. [Bow2, §5.3].)

In the complete smooth hyperbolic setting, the density of the fixed points of hyperbolic isometries in $\Lambda \times \Lambda$ gives another characterization of geometric finiteness: the complete cusp-free hyperbolic 3-manifold is geometrically finite if and only if the closure of the set of closed geodesics in M is compact.

A lacuna in the various existing discussions of how features of the complete hyperbolic setting generalize to the pinched negative curvature setting is the following equivalence, which will allow us to improve the results of Canary [C1].

LEMMA 3.8. *Let M be a 3-dimensional manifold of pinched negative curvature and no cusps. Then M is geometrically finite if and only if the closure of the set of closed geodesics in M is compact.*

Proof. Let $M = X/\Gamma$, where X is a 3-dimensional pinched Hadamard manifold and Γ is a discrete subgroup of $\mathrm{Isom}^+(X)$.

Since fixed points of hyperbolic isometries are again dense in $\Lambda \times \Lambda$, it follows that lifts of closed geodesics to X are dense in $\mathrm{join}(\Lambda)$. Applying Theorem 3.7, we have

$$\mathrm{hull}(\Lambda) \subset \mathcal{N}_\sigma(\mathrm{join}(\Lambda)),$$

where σ depends only on the pinching constants for M. But if the closure of the set of closed geodesics in M is compact then the quotient $\overline{\text{join}(\Lambda)}/\Gamma$ is compact. It follows that the convex core

$$\text{core}(M) \subset \mathcal{N}_\sigma(\overline{\text{join}(\Lambda)}/\Gamma)$$

is compact. Thus M is a geometrically finite manifold of pinched negative curvature.

Conversely, since all closed geodesics in M lie in core(M), the closure of the set of closed geodesics in M is compact whenever core(M) is compact. □

COROLLARY 3.9. *Let M be a 3-dimensional hyperbolic cone-manifold with no cusps so that for every component c of the singular locus, the cone-angle at c is greater than 2π. Then M is geometrically finite if and only if the closure of the set of all closed geodesics in M is compact.*

Proof. By a standard argument (see [GT]) the assumption on the cone-angles implies that the singular hyperbolic metric on M may be perturbed to give a negatively curved metric on M that is hyperbolic away from a tubular neighborhood of the cone-locus. The result is a Riemannian manifold of pinched negative curvature $\widehat{M}$.

The smoothing $\widehat{M}$ is a new metric on M, and in this new metric each closed geodesic is a uniformly bounded distance from its geodesic representative in M. It follows that the closure of the set of closed geodesics in M is compact if and only if the closure of the set of closed geodesics in $\widehat{M}$ is compact.

If the union of all closed geodesics in $\widehat{M}$ is precompact, then $\widehat{M}$ is geometrically finite, by Lemma 3.8. It follows that the convex core for $\widehat{M}$ is compact (since $\widehat{M}$ has no cusps). For $R>0$ sufficiently large, the radius-R neighborhood of the convex core of $\widehat{M}$ gives a compact core $\mathcal{M}$ for $\widehat{M}$ bounded by convex surfaces that miss the neighborhoods where the metrics on M and $\widehat{M}$ differ. Since convexity is a local property for embedded surfaces, it follows that the surfaces $\partial\mathcal{M}$ are convex in M as well.

We conclude that if the closed geodesics are precompact in M then M has a compact core bounded by convex surfaces, so M is geometrically finite. The converse is immediate. □

We now prove the appropriate generalization of Canary's theorem in the context of hyperbolic cone-manifolds. As in the smooth case, we say that a 3-dimensional hyperbolic cone-manifold is *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold.

THEOREM 3.10. *Suppose that M is a topologically tame 3-dimensional hyperbolic cone-manifold. Assume that the cone-angle at each component of the singular locus is at* *least 2π. Then M is geometrically tame: each end E of M is either geometrically finite or simply degenerate.*

Proof. As above, we let $\widehat{M}$ be a smoothing of M to a manifold of pinched negative curvature, modifying the metric in a close neighborhood of the singular locus. Since neighborhoods of the ends are unchanged by this smoothing, it suffices to prove the theorem for $\widehat{M}$.

Let E be a geometrically infinite end of $\widehat{M}$ cut off by a surface S_0, and let K be a compact submanifold of E so that $\partial K = S_0 \sqcup S$, where S is a smooth surface in E. By a straightforward generalization of an argument in [Bon1] to the setting of pinched negative curvature, we may find a closed curve on S_0 (not necessarily simple) whose geodesic representative lies outside of K.

In [C1] a generalization of Bonahon's tameness theorem [Bon1] is applied in the context of branched covers of hyperbolic 3-manifolds. After smoothing the branching locus to obtain a manifold with pinched negative curvature that is hyperbolic outside of a compact set, Canary discusses the appropriate generalization to the main theorem of [Bon1] in this context. In [C1, §4], however, a *geometrically infinite end* E cut off by S is *defined* to be an end for which there are closed loops $\alpha_n \subset S$ whose geodesic representatives eventually lie outside of every compact subset of E. The above shows that if an end E is not geometrically finite in our sense, then it is geometrically infinite in the sense of Canary [C1, §4].

Applying the tameness theorem of [C1, Theorem 4.1], if M is a tame 3-dimensional hyperbolic cone-manifold, then all of its ends are either geometrically finite or simply degenerate. □

4. Bounded geometry

A central dichotomy in the study of ends of hyperbolic 3-manifolds lies in the distinction between hyperbolic manifolds with *bounded geometry* and those with *arbitrarily short geodesics*.

A recent theorem of Y. Minsky shows that whether a manifold $M \in AH(S)$ has bounded geometry is predicted by a comparison of its end-invariants, a collection of geodesic laminations and hyperbolic surfaces associated to the ends of M.

In this section we adapt Minsky's techniques to produce a version of these criteria which can be applied end-by-end: we show that whether or not a simply degenerate end E has bounded geometry depends only on *its* ending lamination $\nu(E)$ and not on the remaining ends.

End-invariants. When an end E of $M \in AH(S)$ is geometrically finite it admits a foliation by surfaces whose geometry is exponentially expanding, but whose conformal structures converge to that of a component, say X, of the conformal boundary of M. With the induced marking from f, X determines a point in Teichmüller space, and the asymptotic geometry of the end E is determined by this marked Riemann surface. We say that X is the *end-invariant* of the geometrically finite end E (see, e.g., [EM] and [Mi1]).

A simply degenerate end of M also has a well-defined end-invariant.

Definition 4.1. Let E be a simply degenerate end of M cut off by a surface S. Let α_n be a sequence of simple closed curves on S whose geodesic representatives α_n^* leave every compact subset of E. Then the support $|[\nu]|$ of any limit $[\nu] \in \mathcal{PL}(S)$ of α_n is the *ending lamination* of E.

By a theorem of Thurston, any two limits $[\nu]$ and $[\nu']$ in $\mathcal{PL}(S)$ satisfy

$$|\nu| = |\nu'|,$$

so $\nu(E)$ is well defined. We call the ending lamination $\nu(E)$ the *end-invariant* for the degenerate end E.

For each M in $AH(S)$ with no cusps, we will denote by ν^- and ν^+ the end-invariants of the negative and positive ends E^- and E^+ of M, respectively.

Curve complexes and projections. In [Ha], W. Harvey organized the simple closed curves on S into a complex in order to develop a better understanding of the action of the mapping class group. Recently (see [Mi4], [Mi3] and [Bro4]), his complex has become a fundamental object in the study of 3-dimensional hyperbolic manifolds.

The *complex of curves* $\mathcal{C}(S)$ is obtained by associating a vertex to each element of $\mathcal{S}$ and stipulating that $k+1$ vertices determine a k-simplex if the corresponding curves can be realized disjointly on S. Except for some sporadic low-genus cases, the same definition works for non-annular surfaces with boundary (provided that $\mathcal{S}$ is taken to represent the isotopy classes of *non-peripheral* essential simple closed curves on S), and a similar *arc-complex* can be defined for consideration of the annulus. A remarkable theorem of H. Masur and Y. Minsky establishes that the natural distance on $\mathcal{C}(S)$ obtained by making each k-simplex into a standard Euclidean simplex, turns $\mathcal{C}(S)$ into a δ-hyperbolic metric space (see [MM1] for more details).

When $Y \subset S$ is a proper essential subsurface of S, $\mathcal{C}(Y)$ is naturally a subcomplex of $\mathcal{C}(S)$. Masur and Minsky define a projection map $\pi_Y: \mathcal{C}(S) \to \mathcal{P}(\mathcal{C}(Y))$ from $\mathcal{C}(S)$ to the set of subsets of $\mathcal{C}(Y)$, by associating to each $\alpha \in \mathcal{C}(S)$ the arcs of essential intersection of α with Y, surgered along the boundary of Y to obtain simple closed curves in Y. The possible surgeries can produce curves in Y that intersect, but given any simplex $\sigma \in \mathcal{C}(S)$

the total diameter of $\pi_Y(\sigma)$ in $\mathcal{C}(Y)$ is at most 2 (see [MM2, §2, Lemma 2.3] for more details).

The *projection distance* $d_Y(\alpha,\beta)$ measures the distance from α to β relative to the subsurface Y:

$$d_Y(\alpha,\beta)=\operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(\alpha)\cup\pi_Y(\beta)).$$

Note that by the above, the projection π_Y is 2-Lipschitz, i.e. we have

$$d_Y(\alpha,\beta)\leqslant 2d_{\mathcal{C}(S)}(\alpha,\beta)$$

for any pair of vertices α and β in $\mathcal{C}(S)$.

By a result of E. Klarreich [Kl], the Gromov boundary of $\mathcal{C}(S)$ is in bijection with the possible ending laminations for a cusp-free simply degenerate end of $M\in AH(S)$. We denote this collection of geodesic laminations by $\mathcal{EL}(S)$. Given such an ending lamination ν, the projection $\pi_Y(\nu)$ can be defined just as for $\alpha\in\mathcal{C}(S)$, and $\pi_Y(\nu)$ is the limiting value of $\pi_Y(\alpha_i)$, where α_i converges to $\nu\in\partial\mathcal{C}(S)$.

If $Z\in\operatorname{Teich}(S)$ is a conformal boundary component of $M\in AH(S)$, there is a uniform upper bound to the length of the shortest geodesic on Z. Although the shortest geodesic may not be unique, the set short(Z) of shortest geodesics on Z determines a set of uniformly bounded diameter in $\mathcal{C}(S)$. Thus, given end-invariants ν^- and ν^+ for a cusp-free $M\in AH(S)$, we can compare the end-invariants in the surface Y by the quantity

$$d_Y(\nu^-,\nu^+),$$

where if $\nu^-=Z\in\operatorname{Teich}(S)$ we replace ν^- with short(Z).

Using such comparisons, the main results of [Mi3] and [Mi4] give necessary and sufficient conditions for the length of the shortest closed geodesic in M to have a lower bound $l_0>0$.

THEOREM 4.2. (Minsky) *Let $M\in AH(S)$ have no cusps and end-invariants (ν^-,ν^+). Then M has bounded geometry if and only if the supremum*

$$\sup_{Y\subset S} d_Y(\nu^-,\nu^+)$$

over all proper essential subsurfaces $Y\subset S$ is bounded above.

We deduce the following corollary.

COROLLARY 4.3. *Let the doubly degenerate manifold $M\in AH(S)$ have no cusps. If the positive end E^+ of M has bounded geometry, then any degenerate manifold Q in the Bers slice B_Y with ending lamination $\nu(E^+)$ has bounded geometry.*

Proof. Assume otherwise. Then by Minsky's theorem, there exists a family of essential subsurfaces $Y_j \subset S$ so that

$$d_{Y_j}(Y, \nu^+) \to \infty$$

as j tends to ∞. Choosing $\alpha_j \subset \partial Y_j$, we have

$$l_Q(\alpha_j) \to 0$$

by [Mi3, Theorem B]. Since the geodesic representatives α_i^* in Q exit the end of Q, any limit $[\nu]$ of α_i in $\mathcal{PL}(S)$ has intersection number zero with ν^+ by the exponential decay of the intersection number (see [T1, Chapter 9] and [Bon1, Proposition 3.4]).

We claim that the projection sequence

$$\{d_{Y_j}(\nu^-, \nu^+)\}_{j=1}^{\infty}$$

is also unbounded.

Consider the distances

$$d_{Y_j}(\nu^-, Y).$$

Then either $d_{Y_j}(\nu^-, Y)$ remains bounded, or we may pass to a subsequence so that $d_{Y_j}(\nu^-, Y) \to \infty$.

In the first case we have by the triangle inequality,

$$d_{Y_j}(Y, \nu^+) \leqslant d_{Y_j}(Y, \nu^-) + d_{Y_j}(\nu^-, \nu^+);$$

in particular, $d_{Y_j}(\nu^-, \nu^+)$ is unbounded. By the main theorem of [Mi3], it follows that the simple closed curves α_i satisfy $l_M(\alpha_i) \to 0$.

Thus, the geodesic representatives α_i^{**} of α_i in M must exit the end E^- of M, since their lengths have zero infimum. Again applying [T1, Chapter 9] and [Bon1, Proposition 3.4], we find that $[\nu]$ has intersection number zero with ν^-. It follows that $\nu^- = \nu^+$, a contradiction (the ending laminations of a cusp-free doubly degenerate manifold M in $AH(S)$ must be distinct; see [Bon1, §5] and [T1, Chapter 9]). Thus, Q has bounded geometry in this case.

If, on the other hand, $d_{Y_j}(\nu^-, Y) \to \infty$, we consider a limit Q' of quasi-Fuchsian manifolds in the Bers slice B_Y with ending lamination ν^-. Then by [Mi3] the curves α_j again have the property that $l_{Q'}(\alpha_j) \to 0$, so the geodesic representatives of α_j in Q' must exit the end of Q'. We again arrive at the contradiction $\nu^- = \nu^+$, so we may conclude again that Q has bounded geometry. $\square$

The argument motivates the following definition.

Definition 4.4. A lamination $\nu \in \mathcal{EL}(S)$ has *bounded type* if for any $\alpha \in \mathcal{C}(S)$, $\{d_{Y_j}(\alpha, \nu)\}_{j=1}^{\infty}$ is bounded over all essential subsurfaces $Y_j \subset S$.

Remark. The projection distances $d_{Y_j}(\alpha, \nu)$ are reminiscent of the continued fraction expansion of an irrational number. In the case when S is a punctured torus, this analogy is literal in the sense that simple closed curves on S are encoded by their rational slopes, and measured laminations (up to scale) are naturally the completion of the simple closed curves (see [Mi2]). In the punctured-torus setting, bounded-type laminations are encoded by *bounded-type* irrationals, namely, irrationals with uniformly bounded continued fraction expansion.

THEOREM 4.5. *Let E be a geometrically infinite tame end of a cusp-free hyperbolic 3-manifold M, and assume that there is a lower bound to the injectivity radius on E. Then there is a compact set $K \subset E$ and a manifold Q_∞ on the Bers boundary ∂B_Y so that the subset $E \setminus K$ is bi-Lipschitz diffeomorphic to the complement $E_\infty \setminus K_\infty$ of a compact subset $K_\infty \subset Q_\infty$.*

Proof. Let $\nu = \nu(E)$ be the ending lamination for E. Since the injectivity radius of E is bounded below, it follows that ν has bounded type. By an application of the continuity of the length function for laminations on $AH(S)$ [Bro2, Theorem 1.3], there exists some $Q_\infty \in \partial B_Y$ so that $\nu(Q_\infty) = \nu$.

By the previous theorem, the manifold Q_∞ has a positive lower bound to its injectivity radius since ν has bounded type. If E_∞ represents the simply degenerate end of Q_∞ for which $\nu(E_\infty) = \nu$, then E and E_∞ represent ends of two different manifolds with injectivity radius bounded below and the same ending lamination.

Applying the main theorem of [Mi1], or its generalization [Mo] if N does not have a global lower bound to its injectivity radius, the ends E and E_∞ are bi-Lipschitz diffeomorphic. □

5. Grafting in degenerate ends

In this section we describe a central construction of the paper. The *grafting* of a simply degenerate end, introduced as a technique in [Brm1], serves as the key to approximating degenerate ends of complete hyperbolic 3-manifolds by cone-manifolds.

The grafting construction. By Bonahon's theorem [Bon1], each manifold $M \in AH(S)$ is homeomorphic to $S \times \mathbf{R}$. Given a particular choice of homeomorphism $F: S \times \mathbf{R} \to M$, each simple closed curve α on S has an associated embedded *positive grafting annulus*

$$A_\alpha^+ = F(\alpha \times [0, \infty))$$

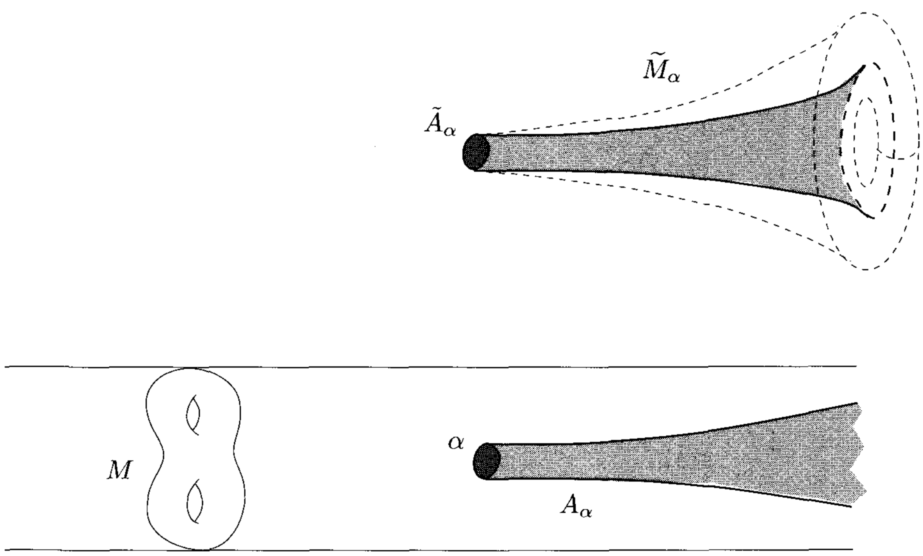

Fig. 1. Lifting the grafting annulus.

in M and a *negative grafting annulus* $A_\alpha^- = F(\alpha\times(-\infty,0])$.

Consider the solid-torus cover $\widetilde{M}_\alpha = \mathbf{H}^3/\langle\alpha^*\rangle$ obtained as the quotient of $\mathbf{H}^3$ by a representative of the conjugacy class of $F(\alpha\times\{0\})$ in $\pi_1(M)$. Let $A_\alpha = A_\alpha^+$ denote the positive grafting annulus for α. Then by the lifting theorem, A_α lifts to an annulus $\tilde{A}_\alpha$ in the cover $\widetilde{M}_\alpha$.

Let $\mathrm{Gr}^+(M,\alpha)$ denote the singular 3-manifold obtained by isometrically gluing the metric completions of

$$M\setminus A_\alpha \quad\text{and}\quad \widetilde{M}_\alpha\setminus\tilde{A}_\alpha$$

in the following way:

I. For reference, choose an orientation on the curve α. Together with the product structure F, this orientation gives a local "left" and "right" side in M to the annulus A_α corresponding to the left and right side of the curve $F(\alpha\times\{t\})$ in $F(S\times\{t\})$.

II. The metric completion of $M\setminus A_\alpha$ contains two isometric copies $\mathcal{A}_l$ and $\mathcal{A}_r$ of the annulus A_α in its metric boundary corresponding to the local left and right side of the annulus with respect to the choice of orientation of α. Likewise, the metric boundary of the metric completion of $\widetilde{M}_\alpha\setminus\tilde{A}_\alpha$ contains the two isometric copies $\tilde{\mathcal{A}}_l$ and $\tilde{\mathcal{A}}_r$ of $\tilde{A}_\alpha$ corresponding to the local left and right side of $\tilde{A}_\alpha$ in $\widetilde{M}_\alpha$.

III. The parameterization of A_α by $F|_{\alpha\times[0,\infty)}$ induces parameterizations

$$F_l\colon \alpha\times[0,\infty)\longrightarrow\mathcal{A}_l \quad\text{and}\quad F_r\colon\alpha\times[0,\infty)\longrightarrow\mathcal{A}_r$$

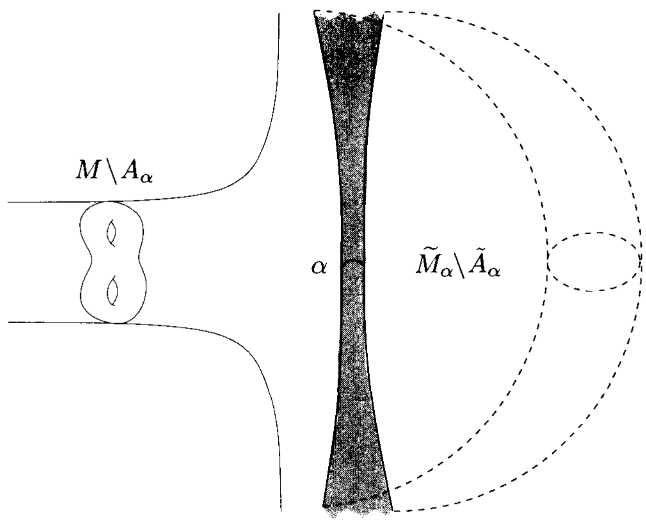

Fig. 2. The grafted end.

of the annuli $\mathcal{A}_l$ and $\mathcal{A}_r$, and

$$\widetilde{F}_l\colon \alpha\times[0,\infty)\longrightarrow\tilde{\mathcal{A}}_l \quad\text{and}\quad \widetilde{F}_r\colon \alpha\times[0,\infty)\longrightarrow\tilde{\mathcal{A}}_r$$

of the annuli $\tilde{\mathcal{A}}_l$ and $\tilde{\mathcal{A}}_r$. We obtain the grafting $\mathrm{Gr}^+(M,\alpha)$ by identifying the metric completions

$$\overline{M\setminus A_\alpha} \quad\text{and}\quad \overline{\widetilde{M}_\alpha\setminus\tilde{A}_\alpha}$$

by the mapping ϕ from the metric boundary of $M\setminus A_\alpha$ to the metric boundary of $\widetilde{M}_\alpha\setminus\tilde{A}_\alpha$ determined by setting

$$\phi(F_l(x,t))=\widetilde{F}_r(x,t) \quad\text{and}\quad \phi(F_r(x,t))=\widetilde{F}_l(x,t).$$

There is a natural projection

$$\pi\colon \mathrm{Gr}^+(M,\alpha)\longrightarrow M$$

obtained by defining π to be the identity on $M\setminus A_\alpha$ and the restriction of the natural covering map $\widetilde{M}_\alpha\to M$ on $\widetilde{M}_\alpha\setminus\tilde{A}_\alpha$, and then extending across the gluing. The projection π is a covering mapping away from α and is the two-fold branched covering map of M branched along α in a neighborhood of α.

IV. Since the gluing is isometric, the resulting grafted end has a hyperbolic metric away from a singularity along the curve α. When the curve α is a geodesic, the singularity becomes a cone-type singularity, and $\mathrm{Gr}^+(M,\alpha)$ is a hyperbolic cone-manifold homeomorphic to $S\times\mathbf{R}$ with cone-angle 4π at α.

By Otal's theorem (Theorem 2.5) any sufficiently short geodesic in $M\in AH(S)$ is unknotted, guaranteeing that the grafting construction may be applied to the geodesic itself. We now prove that grafting along a short geodesic always produces a geometrically finite end.

THEOREM 5.1. *If $(f: S\to M)\in AH(S)$ has no cusps and α is an essential simple closed curve in S for which $l_M(\alpha^*)<l_{\mathrm{knot}}$, then the positive end of the hyperbolic cone-manifold $\mathrm{Gr}^+(M,\alpha^*)$ is geometrically finite.*

Proof. Applying Otal's theorem (Theorem 2.5), α is isotopic into a level surface for any product structure on M, so we choose a homeomorphism $F: S\times\mathbf{R}\to M$ so that

(1) $F(S\times\{0\})$ is homotopic to f;

(2) $F|_{S\times\{0\}}$ *realizes* α: i.e. $F(\alpha\times\{0\})=\alpha^*$.

Let E^+ be the positive end of M. Let $M^c=\mathrm{Gr}^+(M,\alpha^*)$ and let E^c denote the positive end of M^c. Arguing by contradiction, assume that the end E^c is not geometrically finite. Then §3 guarantees that there are simple closed curves γ_k on S whose geodesic representatives in E^c eventually lie outside of every compact subset of E^c.

We choose a particular exhaustion of E^c by compact submanifolds that is adapted to an exhaustion of the original end E^+ as follows:

(1) Let K_j be an exhaustion of E^+ by the compact submanifolds

$$K_j=F(S\times[0,j]).$$

(2) Let $\widehat{K}_j$ be the lift of $K_j\backslash A$ to E^c for which the restriction of π to $\widehat{K}_j$ is an isometric embedding.

(3) Extend $\widehat{K}_j$ to a compact subset K_j^c by taking the union of $\widehat{K}_j$ with an exhaustion of $\widetilde{M}_\alpha$ by solid tori as follows: Let $A_\alpha=F(\alpha\times[0,\infty))$ again be the positive grafting annulus for α^* and let $\widetilde{F}$ be the lift of $F_{\alpha\times[0,\infty)}$ to $\widetilde{M}_\alpha$. Let V_j be an exhaustion of $\widetilde{M}_\alpha$ by closed solid tori so that V_j intersects the lift $\tilde{A}_\alpha$ of A_α to $\widetilde{M}_\alpha$ in $\widetilde{F}(\alpha\times[0,j])$. Then

$$K_j^c=\widehat{K}_j\cup V_j$$

exhausts the end E^c.

Let $\{\gamma_j\}_{j=1}^\infty\subset\{\gamma_k\}_{k=1}^\infty$ be a subsequence for which

$$\gamma_j^*\subset E^c\backslash K_j^c.$$

We claim that there is a compact subset K of M so that the projections $\pi(\gamma_j^*)$ of γ_j^* to M, all intersect K. Consider the (unique) component $\widetilde{S}_\alpha$ of the lifts of S to the solid torus $\widetilde{M}_\alpha$ for which $\pi_1(\widetilde{S}_\alpha)=\mathbf{Z}$, i.e. $\widetilde{S}_\alpha$ is the annular lift of S to $\widetilde{M}_\alpha$ that contains the curve α. The properly embedded annulus $\widetilde{S}_\alpha$ separates $\widetilde{M}_\alpha$ into two pieces, one covering the component of $M\backslash S$ containing S_0, and the other covering the noncompact portion of $E^+\backslash S$.

Since $\alpha\subset K_j^c$ for all sufficiently large j, we may throw away a finite number of γ_j to guarantee that $\alpha\neq\gamma_j$ for all j. Thus, the geodesic γ_j^* intersects $\widetilde{S}_\alpha$ if and only if we have

$$i(\gamma_j,\alpha)\neq 0.$$

The geodesic γ_j^* projects isometrically by π to the geodesic representative of γ_j in M; for convenience, we denote the latter by $\pi(\gamma_j^*)$. Let X_j be a pleated surface realizing γ_j in M with the property that if $i(\gamma_j,\alpha)=0$ then X_j realizes α as well.

If γ_j^* intersects $\widetilde{S}_\alpha$ in M^c, then the geodesic $\pi(\gamma_j^*)$ intersects S in M, so the pleated surface X_j intersects S. If, on the other hand, γ_j^* does not intersect $\widetilde{S}_\alpha$, then X_j realizes α, so X_j also intersects S. By Theorem 2.4, there is a compact subset $K\subset M$ so that we have

$$X_j\subset K$$

for all j. In particular, it follows that we have

$$\pi(\gamma_j^*)\subset K$$

for all j.

Note, however, that there is a $j'>j$ so that γ_j^* intersects $\widehat{K}_{j'}\backslash\widehat{K}_j$, since otherwise γ_j^* would lie entirely in $\widetilde{M}_\alpha\backslash\tilde{A}_\alpha$, which would imply that γ_j is isotopic to α. Choosing j sufficiently large to guarantee that

$$K\subset K_j,$$

we then obtain a contradiction, since

$$\pi(\widehat{K}_{j'}\backslash\widehat{K}_j)\cap K=\varnothing.$$

We conclude that the end E^c is geometrically finite. $\square$

We introduce one further piece of notation for later use. If α and β represent simple closed curves whose geodesic representatives lie in E^- and E^+, respectively, we can perform grafting of E^- along the negative grafting annulus for α in M, and grafting of E^+ along the positive grafting annulus for β in M simultaneously. We denote by $\mathrm{Gr}^{\pm}(M,\alpha,\beta)$ this *simultaneous grafting along α and β.*

6. Geometric inflexibility of cone-deformations

In this section we establish the estimates on cone-deformations necessary to obtain geometric control away from the singular locus. The main result of this section, Theorem 6.2, harnesses the deformation theory of cone-manifolds originally developed in [HK1] and [Brm3], and improved upon in [HK2], [HK4] and [Brm2] to obtain the necessary control.

Theorem 6.2 is a geometric formulation of similar analytic results in [Brm2] that control the change of projective structure associated to an end of a cone-manifold M under a deformation that changes the cone-angle. Here, we have replaced control over the change in projective structure, which suffices for applications in [Brm1], with bi-Lipschitz control over the metric on M itself. We refer the reader to [HK3] for an expository account of the necessary recent developments in the cone-deformation theory.

Let E be an end of a geometrically finite cone-manifold that is cut off by a surface S with genus $\geqslant 2$. Then the hyperbolic structure on $E=S\times\mathbf{R}^+$ naturally extends to a conformal structure on $S\times\{\infty\}$. The hyperbolic structure on E is locally modelled on $\mathbf{H}^3$, while the conformal structure is modelled on $\widehat{\mathbf{C}}$.

More concretely, $\mathbf{H}^3$ is compactified by $\widehat{\mathbf{C}}$, and $\mathrm{PSL}_2(\mathbf{C})$ acts continuously on the compactification as hyperbolic isometries of $\mathbf{H}^3$ and as projective transformations of $\widehat{\mathbf{C}}$. Then $S\times(0,\infty]$ has an atlas of charts to $\mathbf{H}^3\cup\widehat{\mathbf{C}}$ whose transition maps are restrictions of elements of $\mathrm{PSL}_2(\mathbf{C})$. These charts will map points in $S\times\mathbf{R}^+$ to $\mathbf{H}^3$ and points in $S\times\{\infty\}$ to $\widehat{\mathbf{C}}$. Restricted to $S\times\mathbf{R}^+$ the charts will form an atlas for the end E, and on $S\times\{\infty\}$ the charts will define a conformal structure on S. We refer to S with this conformal structure as the *conformal boundary* of E.

The following theorem appears in [Brm2].

THEOREM 6.1. *Given $\alpha>0$ there exists $l>0$ such that the following holds: Let M_α be a geometrically finite hyperbolic cone-manifold with no rank-1 cusps, singular locus $\mathcal{C}$ and cone-angle α at each component $c\subset\mathcal{C}$. If the tube radius R about each component c in $\mathcal{C}$ is at least* $\operatorname{arsinh}\sqrt{2}$ *and the total length $l_{M_\alpha}(\mathcal{C})$ of $\mathcal{C}$ in M_α satisfies $l_{M_\alpha}(\mathcal{C})<l$, then there is a 1-parameter family M_t of cone-manifolds with fixed conformal boundary and cone-angle $t\in[0,\alpha]$ at each $c\subset\mathcal{C}$.*

The main result of this section allows us to control the geometric effect of the 1-parameter cone-deformation when the singular locus is sufficiently short.

THEOREM 6.2. (The drilling theorem) *Given $\alpha>0$ and $L>1$, there exists $l>0$ so that the following holds: If M_α is a hyperbolic cone-manifold satisfying the hypotheses of the previous theorem, and M_t the corresponding 1-parameter family of cone-manifolds with $t\in[0,\alpha]$, then if $l_{M_\alpha}(\mathcal{C})<l$, there is for each t a standard neighborhood $\mathbf{T}_t(\mathcal{C})$ of the*

singular locus $\mathcal{C}$ and an L-bi-Lipschitz diffeomorphism of pairs

$$h_t: (M_\alpha \backslash \mathbf{T}_\alpha(\mathcal{C}), \partial \mathbf{T}_\alpha(\mathcal{C})) \longrightarrow (M_t \backslash \mathbf{T}_t(\mathcal{C}), \partial \mathbf{T}_t(\mathcal{C}))$$

so that h_t extends to a homeomorphism $\bar{h}_t: M_\alpha \to M_t$ for each $t \in (0, \alpha]$.

As we will see, the standard neighborhood $\mathbf{T}_t(\mathcal{C})$ will be a component $\mathbf{T}_t^\epsilon(\mathcal{C})$ of the *Margulis ϵ-thin part* of M_t containing $\mathcal{C}$. In fact, for each $\epsilon>0$ less than the appropriate *Margulis constant* (which will depend in general on the tube radius and cone-angle of each component of the singular locus) there are l and h_t satisfying the theorem for $\mathbf{T}_t(\mathcal{C}) = \mathbf{T}_t^\epsilon(\mathcal{C})$.

Background. Before proving Theorem 6.2 we review the necessary background. Let N be a 3-manifold with boundary (we allow N to be non-compact). Let g be a hyperbolic metric on the interior of N that extends to a conformal structure on each component of ∂N; here, the metric g need not be complete, but the conformal structures compactify the ends where the metric is complete.

Let $\widetilde{N}$ denote the universal cover of N and let $\pi: \widetilde{N} \to N$ denote the covering projection. Then g lifts to a metric $\tilde{g}$ on the universal cover $\operatorname{int}(\widetilde{N})$ of $\operatorname{int}(N)$, and the conformal structures on ∂N lift to conformal structures on $\partial \widetilde{N}$. There is a map

$$\mathsf{Dev}: \widetilde{N} \longrightarrow \mathbf{H}^3 \cup \widehat{\mathbf{C}}$$

that is a local isometry on $\operatorname{int}(\widetilde{N})$ and conformal on $\partial \widetilde{N}$.

Furthermore, there is a representation

$$\varrho: \pi_1(N) \longrightarrow \mathrm{PSL}_2(\mathbf{C})$$

with the property that

$$\mathsf{Dev}(\gamma(p)) = \varrho(\gamma)\, \mathsf{Dev}(p) \tag{6.1}$$

for each $p \in \widetilde{N}$ and each deck transformation $\gamma \in \pi_1(N)$.

The map Dev is called the *developing map* for the metric g and is determined up to postcomposition with elements of $\mathrm{PSL}_2(\mathbf{C})$ acting on $\mathbf{H}^3 \cup \widehat{\mathbf{C}}$. Changing the developing map by postcomposition changes the corresponding representation by conjugation in $\mathrm{PSL}_2(\mathbf{C})$.

A smooth family g_t of such metrics on N determines a smooth family of developing maps

$$\mathsf{Dev}_t: \widetilde{N} \longrightarrow \mathbf{H}^3 \cup \widehat{\mathbf{C}}.$$

The developing maps Dev_t determine a time-dependent vector field v_t on $\widetilde{N}$, where $v_t(p)$ is the pullback by Dev_t of the tangent vector to the path $\mathsf{Dev}_t(p)$ at time t; i.e.

$$(\mathsf{Dev}_t)_*(v_t(p)) = \frac{d\mathsf{Dev}_t}{dt}(p).$$

We call v_t the *derivative* of the family of developing maps Dev_t.

A *Killing field* on a Riemannian manifold N is a vector field whose local flow is an isometry for the Riemannian metric on N. By differentiating (6.1) we have that for each t and for any $\gamma\in\pi_1(N)$, the vector field

$$v_t - \gamma^* v_t$$

is a Killing field on $\mathrm{int}(\widetilde{N})$ for the Riemannian metric g_t. A vector field with this automorphic property is called *automorphic for the metric* g_t, or g_t*-automorphic.* Note that a g_t-automorphic vector field v_t need not arise as the derivative of a family of developing maps.

Let $\hat{g}_t$ be a family of metrics on N for which there are diffeomorphisms $f_t\colon N\to N$ isotopic to the identity satisfying $(f_t)^* g_t = \hat{g}_t$, i.e.

$$f_t\colon (N, \hat{g}_t) \longrightarrow (N, g_t)$$

is an isometry. Then there will be a corresponding family of developing maps $\widehat{\mathsf{Dev}}_t$ with derivative $\hat{v}_t$, a time-dependent vector field on $\widetilde{N}$.

The lifts $\tilde{f}_t\colon \widetilde{N}\to\widetilde{N}$ of f_t to the universal cover allow us to compare v_t and $\hat{v}_t$: since the maps f_t are isometries from $(N,\hat{g}_t)$ to (N, g_t) one sees that the difference

$$(\tilde{f}_t)_*\hat{v}_t - v_t$$

restricts to the sum of a $\pi_1(N)$*-equivariant* vector field and a Killing field on $\mathrm{int}(\widetilde{N})$. In fact, $\widehat{\mathsf{Dev}}_t$ can be chosen so that $(\tilde{f}_t)_*\hat{v}_t - v_t$ is a $\pi_1(N)$-equivariant vector field: one simply needs to alter $\widehat{\mathsf{Dev}}_t$ by postcomposition with a family of elements in $\mathrm{PSL}_2(\mathbf{C})$.

The derivative of the family of developing maps Dev_t is a g_t-automorphic vector field; we now seek to integrate a g_t-automorphic vector field v_t to obtain a family of developing maps Dev_t, reversing the above process.

THEOREM 6.3. *Let g_t be a smoothly varying family of metrics,* Dev_t *the corresponding family of developing maps and v_t the derivative of* Dev_t*. Let w_t be a smoothly varying uniquely integrable family of g_t-automorphic vector fields on $\widetilde{N}$, tangent to the boundary $\partial\widetilde{N}$, such that $w_t - v_t$ is equivariant. For any subset $U\subset N$ contained in a compact subset of N, there exists a family of metrics $\hat{g}_t$ on N, developing maps* $\widehat{\mathsf{Dev}}_t$

for $\hat{g}_t$, $\hat{g}_t$-automorphic vector fields $\hat{v}_t$ on $\widetilde{N}$, and diffeomorphisms $f_t: N \to N$ isotopic to the identity so that

(1) $(f_t)^* g_t = \hat{g}_t$;

(2) *$\hat{v}_t$ is the derivative of the developing maps $\widehat{\mathsf{Dev}}_t$;*

(3) $(f_t)_* \hat{v}_t = w_t$ *on* $\pi^{-1}(U)$.

Proof. We first prove that each $p \in N$ has a neighborhood U such that the theorem holds on U for t near 0.

Let $\tilde{p} \in \pi^{-1}(p)$ and choose nested neighborhoods $V \subset V' \subset V''$ of $\tilde{p}$ such that both Dev_0 and π restricted to V'' are embeddings. Then there exists an $\epsilon' > 0$ such that for $|t| < \epsilon'$ the image $\mathsf{Dev}_t(V'')$ contains $\mathsf{Dev}_0(V')$. We can then choose an ϵ'' with $0 < \epsilon'' \leqslant \epsilon'$ such that for $|t| < \epsilon''$ there exists a flow

$$\phi_t: \mathsf{Dev}_0(V) \longrightarrow \mathbf{H}^3 \cup \widehat{\mathbf{C}}$$

so that ϕ_t is the flow of the time-dependent vector fields $(\mathsf{Dev}_t)_* w_t$.

Given $q \in V$ define

$$\widehat{\mathsf{Dev}}_t(q) = \phi_t \circ \mathsf{Dev}_0(q).$$

Then we claim that $\widehat{\mathsf{Dev}}_t$ can be extended to a developing map on all of $\widetilde{N}$ for each t.

To see this, we first note that for $|t| < \epsilon''$, we have

$$\widehat{\mathsf{Dev}}_t(V) \subset \mathsf{Dev}_t(V''),$$

so we may define an embedding $f_t: V \to \widetilde{N}$ by

$$h_t = \mathsf{Dev}_t^{-1} \circ \widehat{\mathsf{Dev}}_t.$$

Since V'' is disjoint from its translates, there exists a family of embeddings $f_t: \pi(V) \to N$ so that

$$\tilde{f}_t|_V = h_t$$

on V. We extend f_t to a smooth family of diffeomorphisms of all of N so that f_0 is the identity. Then by setting $U = \pi(V)$ and

$$\widehat{\mathsf{Dev}}_t = \mathsf{Dev}_t \circ \tilde{f}_t$$

we obtain the desired extension.

Now let $U \subset N$ be any subset of N contained in a compact subset K of N. We again establish the theorem for U and for t near 0. By the above, and the compactness of K, there exist a finite collection of open sets $U_i \subset N$ and an $\epsilon > 0$ so that $\bigcup_i U_i$ covers K, and the theorem holds on each U_i for $|t|<\epsilon$. Let Dev^i be the resulting developing maps for each U_i.

By the uniqueness of flows, we may define $\widehat{\mathsf{Dev}}_t$ by

$$\widehat{\mathsf{Dev}}_t|_{\pi^{-1}(U_i)} = \mathsf{Dev}^i_t|_{\pi^{-1}(U_i)},$$

and we have the theorem for U and all $|t|<\epsilon$.

It follows that we may now define $\widehat{\mathsf{Dev}}_t$ on some open interval (a,b). Applying the theorem at $t=b$ we have corresponding triples $(g'_t, \mathsf{Dev}'_t, v'_t)$ satisfying the conclusions of the theorem on U for $t\in(b-\epsilon', b+\epsilon')$ for some $\epsilon'>0$. Let $f'_t\colon (N, g'_t)\to(N, g_t)$ be the corresponding diffeomorphisms.

The developing maps Dev'_t satisfy

$$\mathsf{Dev}'_t = \mathsf{Dev}_t \circ \tilde{f}'_t.$$

Thus we have $\mathsf{Dev}_t = \mathsf{Dev}'_t \circ (\tilde{f}'_t)^{-1}$, and therefore setting

$$\widehat{\mathsf{Dev}}_t = \mathsf{Dev}'_t \circ (\tilde{f}'_t)^{-1} \circ \tilde{f}_t,$$

Dev_t extends $\widehat{\mathsf{Dev}}_t$ over a neighborhood of $t=b$. Arguing similarly for $t=a$, the set of t-values on which $\widehat{\mathsf{Dev}}_t$ may be defined is open, closed and non-empty, and therefore $\widehat{\mathsf{Dev}}_t$ can be defined for all t. The proof is complete. □

Let g_t be a smooth family of Riemannian metrics on N. We define vector-valued 1-forms η_t by the formula

$$\frac{dg_t(x,y)}{dt} = 2g_t(x, \eta_t(y)).$$

The symmetry of g_t implies that η_t is self-adjoint. We define a pointwise norm of η_t by choosing an orthonormal frame field $\{e_1, e_2, e_3\}$ for the g_t-metric and setting

$$\|\eta_t\|^2 = \sum_{i,j=1}^{3} g_t(\eta_t(e_i), \eta_t(e_j)).$$

Note that

$$g_t(x, \eta_t(x)) \leqslant \|\eta_t\| \, g_t(x,x).$$

Given two metrics g and $\hat{g}$ we define the bi-Lipschitz constant at each point $p\in N$ by

$$\operatorname{bilip}_p(g,\hat{g}) = \sup\left\{ K \geqslant 1 \;\middle|\; \frac{1}{K} \leqslant \sqrt{\frac{\hat{g}(x,x)}{g(x,x)}} \leqslant K \text{ for all } x\in T_pN,\ x\neq 0 \right\}.$$

A bound on $\|\eta_t\|$ for all $t\in[0,a]$ gives a bound on $\operatorname{bilip}_p(g_0,g_a)$. In particular,

$$\left|\frac{dg_t(x,x)}{dt}\right|\leqslant 2\|\eta_t\|\,g_t(x,x),$$

and integrating we have

$$g_a(x,x)\leqslant e^{2Ka}g_0(x,x)$$

if $\|\eta_t\|\leqslant K$ for all $t\in[0,a]$. This implies that

$$\operatorname{bilip}_p(g_0,g_a)\leqslant e^{aK}.$$

The families of metrics we will examine will always be the pullback of some fixed metric g by the flow ϕ_t of a time-dependent vector field v_t. In this case we can relate η_t to the covariant derivative of v_t. More precisely, if $g_t=\phi_t^*g$ then

$$\eta_t=\operatorname{sym}\nabla^t v_t$$

where ∇^t is the Riemannian connection for the g_t-metric and $\operatorname{sym}\nabla^t$ is the symmetric part of the covariant derivative. This follows from the fact that

$$\frac{dg_t(x,y)}{dt}=\mathcal{L}_{v_t}g_t(x,y)=g(\nabla_x^t v_t,y)+g(x,\nabla_y^t v_t)=2g(x,\operatorname{sym}\nabla_y^t v_t).$$

Our vector fields will also be divergence free and harmonic. For our purposes v is harmonic if it is divergence free and $\operatorname{curl}\operatorname{curl}v=-v$. Note that our curl is half the usual curl and is chosen to agree with the definition given in [HK1]. We also refer there for motivation for this definition of harmonic. Note that $\operatorname{curl}v$ will also be a divergence-free, harmonic vector field.

We use ∇^t to define an operator D_t on the space of vector-valued k-forms by the formula

$$D_t=\sum_{i=1}^{3}\omega^i\wedge\nabla_{e_i}^t,$$

where the e_i are an orthonormal frame field with coframe ω^i. The formal adjoint of D_t is then

$$D_t^*=\sum_{i=1}^{3}i(e_i)\nabla_{e_i},$$

where $i(e_i)$ is contraction.

Let $w_t=\operatorname{curl}v_t$. In §2 of [HK1] it is shown that

$$\operatorname{sym}\nabla^t w_t=*D_t\eta_t=\beta_t$$

and $D_t^*\eta_t=0$. Bounds on the norms of η_t and β_t will allow us to control the geodesic curvature of a smooth curve in N.

PROPOSITION 6.4. *Let $\gamma(s)$ be a smooth curve in N and let $C(t)$ be the geodesic curvature of γ at $\gamma(0)=p$ in the g_t-metric. For each $\epsilon>0$ there exists a $K>0$ depending only on ϵ, a and $C(0)$ such that $|C(a)-C(0)|\leqslant\epsilon$ if $\|\eta_t(p)\|\leqslant K$, $\|\beta_t(p)\|\leqslant K$ and $D_t^*\eta_t=0$ for all $t\in[0,a]$.*

Proof. We assume that $\gamma(s)$ is a unit-speed parameterization in the g_0-metric. In the g_t-metric we reparameterize such that $\gamma_t(s)=\gamma(h_t(s))$ is a unit-speed parameterization. Let

$$V(t)=\nabla^t_{\gamma_t'}\gamma_t'(0).$$

Then

$$C(t)^2=g_t(V(t),V(t)).$$

Differentiating we have

$$C(t)C'(t)=g_t(V(t),V'(t))+g_t(V(t),\eta_t(V(t))). \tag{6.2}$$

The result will follow if we can bound $C'(t)$; we accomplish this via a calculation in local coordinates. We choose our coordinates so as to bound the derivative at $t=0$.

To write the various tensors in local coordinates (x_1,x_2,x_3) we let $e_i=\partial/\partial x_i$, define functions g_{ij} by $g_t(e_i,e_j)=g_{ij}(t)$, and let the g^{ij} be chosen such that $(g_{ij})(g^{ij})=\mathrm{id}$. We similarly define η_i^j by the formula $\eta_t(e_i)=\sum_{j=1}^3\eta_i^j(t)e_j$. The β_i^j are defined in the same way. Recall that the Christoffel symbols Γ_{ij}^k satisfy the formula

$$\Gamma_{ij}^k=\frac{1}{2}\sum_{m=1}^{3}\left(\frac{\partial g_{im}}{\partial x_j}+\frac{\partial g_{jm}}{\partial x_i}-\frac{\partial g_{ij}}{\partial x_m}\right)g^{mk}.$$

We can choose our local coordinates such that at a point p in N we have $g_{ij}(0)=\delta_i^j$ and $\Gamma_{ij}^k(0)=0$. Note that

$$\frac{1}{2}\frac{dg_{ij}}{dt}=\sum_{k=1}^{3}\eta_i^k g_{kj},$$

so with this choice of coordinates we have

$$\eta_i^j(0)=\frac{1}{2}\frac{dg_{ij}}{dt}(0)$$

and

$$\frac{d\Gamma_{ij}^k}{dt}(0)=\frac{\partial\eta_i^k}{\partial x_j}(0)+\frac{\partial\eta_k^j}{\partial x_i}(0)-\frac{\partial\eta_i^j}{\partial x_k}(0) \tag{6.3}$$

at p.

We need to write the β_i^j as derivatives of the η_i^j. To do this we note that by definition

$$\eta_t = \sum_{i,j=1}^{3} \eta_i^j(t) e_j \otimes \omega^i.$$

Then direct calculation gives

$$D_0\eta_0 = \sum_{i,j,k=1}^{3} \frac{\partial \eta_i^j}{\partial x_k}(0) e_j \otimes \omega^k \wedge \omega^i$$

at p. This implies that

$$\beta_i^j(0) = \frac{\partial \eta_{i+2}^j}{\partial x_{i+1}}(0) - \frac{\partial \eta_{i+1}^j}{\partial x_{i+2}}(0) \tag{6.4}$$

at p, where on the right-hand side of this formula the indices are measured mod 3. From this we see that a bound on $\|\beta\|$ gives a bound on the difference

$$\frac{\partial \eta_i^j}{\partial x_k}(0) - \frac{\partial \eta_k^j}{\partial x_i}(0).$$

Another direct calculation in local coordinates gives

$$D_0^*\eta_0 = \sum_{i,j=1}^{3} \frac{\partial \eta_i^j}{\partial x_i}(0) e_j. \tag{6.5}$$

Therefore, if $D_0^*\eta_0=0$ we have $(\partial \eta_i^j/\partial x_i)(0)=0$.

Combining (6.3), (6.4) and (6.5) we have

$$\frac{d\Gamma_{11}^1}{dt}(0) = 0, \quad \frac{d\Gamma_{11}^2}{dt}(0) = \beta_3^1(0), \quad \frac{d\Gamma_{11}^3}{dt}(0) = -\beta_2^1(0). \tag{6.6}$$

We can now return to our smooth curve $\gamma(s)=(x_1(s), x_2(s), x_3(s))$. Assume that $\gamma(0)=p$ and $\gamma'(0)=e_1$. Define $h_1(t)=(dh_t/ds)(0)$ and $h_2(t)=(d^2h_t/ds^2)(0)$. By our choice of γ we have

$$V(t) = h_1(t)^2 \sum_{i=1}^{3} \left(\frac{d^2 x_i}{ds^2}(0) + \Gamma_{11}^i(t) \right) e_i + h_2(t) e_1.$$

To bound $C'(0)$ we need to bound $g_0(V(0), V'(0))$ and $g_0(V(0), \eta_0(V(0)))$. For the second term we have

$$|g_0(V(0), \eta_0(V(0)))| \leqslant C^2(0)\|\eta_0\|.$$

To bound the first term we note that $V(0)$ is perpendicular to $\gamma'(0)$. Therefore, if we let $V_\perp(t)$ be the sum of the e_2- and e_3-terms of $V(t)$, we have $g_0(V(0), V'(0))= g_0(V(0), V_\perp'(0))$. Differentiating we have

$$V_\perp'(0) = \sum_{i=2,3} \left(2h_1'(0) \frac{d^2 x_i}{ds^2}(0) + \frac{d\Gamma_{11}^i}{dt}(0) \right) e_i.$$

We need to bound $h_1'(0)$. This is accomplished by differentiating the formula

$$g_t(\gamma_t'(s), \gamma_t'(s)) = 1.$$

Note that when $s=0$ the formula becomes

$$h_1(t)^2 g_{11}(t) = 1,$$

and differentiating with respect to t yields $h_1'(0) = -\eta_1^1(0)$. To bound the derivative of the Christoffel symbols we use (6.6). Therefore

$$|V_\perp'(0)|_0 \leqslant 2C(0)\|\eta_0\| + \|\beta_0\|,$$

which in turn implies that

$$|g_0(V(0), V'(0))| \leqslant C(0)(2C(0)\|\eta_0\| + \|\beta_0\|)$$

and

$$|C'(0)| \leqslant 3C(0)\|\eta_0\| + \|\beta_0\|.$$

Since we could choose coordinates to calculate the derivative for any t we have

$$|C'(t)| \leqslant 3C(t)\|\eta_t\| + \|\beta_t\|. \tag{6.7}$$

Therefore if we choose K small enough, and if $\|\eta_t(p)\|$ and $\|\beta_t(p)\|$ are less than K, then integrating (6.7) implies that $|C(a) - C(0)| \leqslant \epsilon$. □

We remark that a similar statement holds for subsurfaces of N. In particular, if γ lies on a subsurface S then the metrics g_t induce a metric on S. If we use this induced metric on S to measure the geodesic curvature of γ then the conclusion of Proposition 6.4 still holds.

For a vector field v the symmetric traceless part of ∇v is the *strain* of v and measures the conformal distortion of the metric pulled back by the flow of v. If v is divergence free then ∇v will be traceless, so $\eta = \operatorname{sym} \nabla v$ is a strain field. If v is harmonic we say that η is also harmonic. To prove Theorem 6.2 we need the following mean-value inequality for harmonic strain fields (the result is due to Hodgson and Kerckhoff, see [Brm2, Theorem 9.9] for an exposition).

THEOREM 6.5. *Let η be a harmonic strain field on a ball B_R of radius R centered at p. Then we have*

$$\|\eta(p)\| \leqslant \frac{3\sqrt{2\operatorname{vol}(B_R)}}{4\pi f(R)} \sqrt{\int_{B_R} \|\eta\|^2 \, dV},$$

where

$$f(R)=\cosh(R)\sin(\sqrt{2}R)-\sqrt{2}\sinh(R)\cos(\sqrt{2}R)$$

and $R<\pi/\sqrt{2}$.

We now return to the concrete situation of interest: We assume given triples $(g_t, \mathsf{Dev}_t, v_t)$, a family of cone-metrics, developing maps and g_t-automorphic vector fields v_t so that v_t is the derivative of Dev_t, where $t\in[0,\alpha]$ denotes the cone-angle at the singular locus of g_t. To apply Theorem 6.5, we will invoke the following Hodge theorem of Hodgson and Kerckhoff [HK1] and its generalization [Brm3] to the geometrically finite setting.

THEOREM 6.6. (The Hodge theorem) *Given the triple* $(g_t, \mathsf{Dev}_t, v_t)$ *there exists a smooth, time-dependent, divergence-free, harmonic,* g_t*-automorphic vector field* w_t *so that for each* $t\in[0,\alpha]$ *we have*

(1) w_t *is tangent to* $\partial\widetilde{N}$;

(2) *the restriction of* w_t *to* $\partial\widetilde{N}$ *is conformal;*

(3) w_t-v_t *is an equivariant vector field.*

Combined with Theorem 6.3 the Hodge theorem has the following corollary.

COROLLARY 6.7. *Let* M_t *be the* 1*-parameter family of cone-metrics given by Theorem* 6.1. *There exists a* 1*-parameter family of cone-metrics* g_t *on* N *such that* $M_t=(N,g_t)$ *and* η_t *is a harmonic strain field outside a small tubular neighborhood of the singular locus and the rank-*2 *cusps.*

Below, we will estimate the L^2-norm of $\eta_t=\operatorname{sym}\nabla w_t$ outside of a tubular neighborhood of a short component of the singular locus. This, together with Theorem 6.5, will give us the necessary control metrics g_t outside of the thin part. Before obtaining this control, we must normalize the picture in a neighborhood of the singular locus.

In general, the Margulis lemma does not apply to cone-manifolds. If, however, there is a uniform lower bound R to the tube radius of each component of the singular locus and an upper bound α on all cone-angles, a thick–thin decomposition exists exactly analogous to that of the smooth hyperbolic setting (see [HK2] and [Brm2]). In particular, there exists $\epsilon_{R,\alpha}$ such that the $\epsilon_{R,\alpha}$-thin part $M^{\leqslant\epsilon_{R,\alpha}}$ of a hyperbolic cone-manifold M consists of tubes about short geodesics (including the singular locus) and cusps.

In our situation, we have assumed that the singular locus of M_α has tube radius at least $\operatorname{arsinh}\sqrt{2}$. It is shown in [HK2] that this tube radius will not decrease as the cone-angle decreases. Therefore we fix

$$\varepsilon=\epsilon_{\operatorname{arsinh}\sqrt{2},\alpha}.$$

Given a non-parabolic homotopy class $[\gamma]$ of a closed curve γ in M_α, it will be convenient to consider the family of embedded tubes with core the geodesic representative of γ as the cone-angle varies. For this, we use the following notation: If γ is a homotopically non-trivial closed curve in N with $l_{M_t}(\gamma) \leqslant \epsilon \leqslant \varepsilon$, we denote by $\mathbf{T}_t^\epsilon(\gamma)$ the component of $M_t^{<\epsilon}$ that contains the geodesic representative of γ in the g_t-metric. We will often need to refer to the union of the tubes about the singular locus $\mathcal{C}$. For this reason we set

$$\mathbf{T}_t^\epsilon(\mathcal{C}) = \bigcup_{c\in\mathcal{C}} \mathbf{T}_t^\epsilon(c).$$

Occasionally we will make statements about a generic Margulis tube without reference to a particular tube in the cone-manifolds M_t. We simply refer to such a generic ϵ-Margulis tube as $\mathbf{T}^\epsilon$.

THEOREM 6.8. *Given $\epsilon>0$, there are $l>0$ and $K>0$ such that if $l_{M_t}(\mathcal{C}) \leqslant l$ then we have the L^2-bound*

$$\int_{M_t \setminus \mathbf{T}_t^\epsilon(\mathcal{C})} \|\eta_t\|^2 + \|{*D_t\eta_t}\|^2 \leqslant K^2 l_{M_t}(\mathcal{C})^2.$$

To apply Theorems 6.5 and 6.8 to bound the pointwise norms $\|\eta_t(p)\|$ and $\|{*D_t\eta_t(p)}\|$ we need to control the injectivity radius of p and the distance from p to $\mathbf{T}_t^\epsilon(\mathcal{C})$. To bound these two quantities we use the following estimates of R. Brooks and J. Matelski on the geometry of equidistant tori about a short geodesic (see [BM]). Their original result only applies to tubes about non-singular geodesics. The extension to tubes about components of the singular locus is straightforward.

THEOREM 6.9. (Brooks–Matelski) *Given $\epsilon \in [0, \varepsilon]$, there are two continuous positive functions d_ϵ^u and d_ϵ^l on $[0, \varepsilon]$ with $d_\epsilon^u(\delta) \to 0$, as $\delta \to \epsilon$, and $d_\epsilon^l(\delta) \to \infty$, as $\delta \to 0$, so that given $\delta \in [0, \varepsilon]$ the distance between the boundaries of $\mathbf{T}^\epsilon$ and $\mathbf{T}^\delta$ satisfies*

$$d_\epsilon^l(\delta) \leqslant d(\partial\mathbf{T}^\epsilon, \partial\mathbf{T}^\delta) \leqslant d_\epsilon^u(\delta).$$

Given Riemannian manifolds (M, g) and (N, g'), a diffeomorphism

$$h: (M, g) \longrightarrow (N, g')$$

is *L-bi-Lipschitz* if we have the bound

$$\sup_{p\in M} \operatorname{bilip}_p(h^*g', g) \leqslant L.$$

It is worth noting that we will often be interested in the case when $M = N$ and h is the identity.

Our L^2-bound on η_t, together with the above mean-value inequality for harmonic strain fields (Theorem 6.5) and Proposition 6.4, readily gives the following corollary.

COROLLARY 6.10. *For any $\epsilon>0$, $\delta>0$, $C>0$ and $L>1$, there exists $l>0$ so that if $l_{M_\alpha}(\mathcal{C})<l$ then the following holds: Let W be a subset of N, $\gamma(s)$ a smooth curve in W and $C(t)$ the geodesic curvature of γ in the g_t-metric at $\gamma(0)$. If*

$$W\subset M_t^{\geqslant\epsilon}$$

for all $t\in[t_0,\alpha]$ and $C(0)\leqslant C$ then the identity map

$$\mathrm{id}\colon (W,g_\alpha)\longrightarrow (W,g_{t_0})$$

is L-bi-Lipschitz and $|C(0)-C(A)|\leqslant\delta$.

To apply the corollary we need to show that the thick part of M_α does not become too thin in M_t, while the thin part does not become too thick.

THEOREM 6.11. *Given an $\epsilon_1>0$ there exist an $\epsilon_0>0$ and $l>0$ so that if the length $l_{M_\alpha}(\mathcal{C})<l$ then we have*

$$M_\alpha^{\geqslant\epsilon_1}\subset \operatorname{int} M_t^{\geqslant\epsilon_0} \tag{6.8}$$

and

$$M_t^{\geqslant\epsilon_1}\subset \operatorname{int} M_\alpha^{\geqslant\epsilon_0} \tag{6.9}$$

for all $t\in[0,\alpha]$.

Proof. By Theorem 6.9 we can choose $\epsilon_0>0$ so that

$$d(\partial M_t^{\leqslant\epsilon_0},\partial M_t^{\leqslant\epsilon_1/2})\geqslant 3\epsilon_1.$$

The set A of t such that (6.8) holds is open in $[0,\alpha]$. To prove (6.8) we will show that if $l_{M_\alpha}(\mathcal{C})$ is sufficiently short then A is closed. Let a be a point in the closure of A. By continuity we have

$$M_\alpha^{\geqslant\epsilon_1}\subseteq M_a^{\geqslant\epsilon_0}.$$

Either a is in A and we have proven (6.8), or $M_\alpha^{\geqslant\epsilon_1}\cap M_a^{\leqslant\epsilon_0}$ is non-empty.

We work by contradiction and assume that $q\in M_\alpha^{\geqslant\epsilon_1}\cap M_a^{\leqslant\epsilon_0}$. Let B be a ball of radius ϵ_1 in the g_α-metric with q in ∂B and center p. We also assume that B is contained in $M_\alpha^{\geqslant\epsilon_1}$. By Corollary 6.10 there exists an l such that if $l_{M_\alpha}(\mathcal{C})\leqslant l$ then the inclusion map

$$\iota\colon (M_\alpha^{\geqslant\epsilon_1},g_\alpha)\longrightarrow (M_a^{\geqslant\epsilon_0},g_a)$$

is 2-bi-Lipschitz. This implies that $p\in M_a^{\geqslant\epsilon_2/2}$ while $d(p,q)$ is less than $2\epsilon_1$ in the g_a-metric. By our choice of ϵ_0, however, we have $d(p,\partial M_a^{\leqslant\epsilon_0})\geqslant 3\epsilon_1$, which contradicts our assumption that q lies in $M_a^{\leqslant\epsilon_0}$, proving (6.8).

The inclusion (6.9) is proved similarly. $\square$

Before we continue we need to fix some constants. First choose $\epsilon_2 < \varepsilon$ such that Theorem 6.9 implies that $d(\partial M_t^{\varepsilon}, \partial M_t^{\epsilon_2}) > 1$. Next choose $\epsilon_1 < \epsilon_2$ such that $d(\partial M_t^{\epsilon_2}, \partial M_t^{\epsilon_1}) > 2\epsilon_2$. Finally choose $\epsilon_0 < \epsilon_1$ and l_0 to satisfy the conditions of Theorem 6.11. This implies that if $l_{M_\alpha}(\mathcal{C}) \leqslant l_0$ then the inclusion map

$$\iota: (M_\alpha^{\geqslant \epsilon_1}, g_\alpha) \longrightarrow (N, g_t)$$

is an L-bi-Lipschitz diffeomorphism to its image, where L only depends on $l_{M_\alpha}(\mathcal{C})$ and $L \to 1$ as $l_{M_\alpha}(\mathcal{C}) \to 0$. The remainder of this section will be spent on extending this map to all of $M_\alpha \backslash \mathbf{T}_\alpha^{\epsilon_2}(\mathcal{C})$ in a uniformly bi-Lipschitz way.

Theorem 6.2 will easily follow from the next result.

THEOREM 6.12. *Let $V \subset N$ be either*

(1) *the ε-Margulis tube $\mathbf{T}_\alpha^{\varepsilon}(\gamma)$ about a geodesic γ with $l_{M_\alpha}(\gamma) < \epsilon_1$, or*

(2) *a rank-2 cusp component $\mathbf{P}_\alpha^{\varepsilon}$ of $M_\alpha^{\leqslant \varepsilon}$.*

For each $L > 1$ there exists an $l > 0$ such that if $l_{M_\alpha}(\mathcal{C}) \leqslant l$ then for all $t \leqslant \alpha$ there exists an L-bi-Lipschitz embedding

$$\phi_t: (V, g_\alpha) \longrightarrow (N, g_t)$$

such that ϕ_t is the identity on a neighborhood of ∂V.

We will prove the theorem via a sequence of lemmas. For simplicity, these lemmas will treat the case of the Margulis tube; the rank-2 cusp case admits a far simpler direct proof and is also a limiting case of these arguments.

We first fix more notation: focusing our attention on a single short geodesic γ, let $W = \mathbf{T}_\alpha^{\epsilon_2}(\gamma)$ and $T = \partial W$.

LEMMA 6.13. *For each $d > 0$ there exists an $l > 0$ such that if $l_{M_\alpha}(\mathcal{C}) \leqslant l$ then T is contained in the d-neighborhood $\mathcal{N}_d(\partial \mathbf{T}_t^{\epsilon_2}(\gamma))$ of $\partial \mathbf{T}_t^{\epsilon_2}(\gamma)$ and W contains $\mathbf{T}_t^{\epsilon_1}(\gamma)$ for all $t \leqslant \alpha$.*

Proof. The tubes $\mathbf{T}_t^{\epsilon_1}(\gamma)$ will vary continuously in N as t varies. Since $W \supset \mathbf{T}_\alpha^{\epsilon_1}(\gamma)$, if T is in $M_t^{> \epsilon_1}$ for all t then $W \supset \mathbf{T}_t^{\epsilon_1}(\gamma)$ for all t. For d sufficiently small, $\mathcal{N}_d(\partial \mathbf{T}_t^{\epsilon_2}(\gamma))$ will be contained in $M_t^{> \epsilon_1}$. In particular, the first conclusion implies the second.

By Theorem 6.9 there exists an $L > 0$ such that if the injectivity radius in the g_t-metric of all points in T lies in the interval $[\epsilon_2/L, L\epsilon_2]$ then T is contained in the d-neighborhood of $\partial \mathbf{T}_t^{\epsilon_2}(\gamma)$.

Let B be a ball of radius ϵ_2 in the g_α-metric centered at a point $p \in T$. Since B is contained in $M_\alpha^{\geqslant \epsilon_1}$ we have $B \subset M_t^{\geqslant \epsilon_0}$ for all t by Theorem 6.11. By Corollary 6.10

we can choose an $l>0$ such that if $l_{M_\alpha}\leqslant l$ then the identity map restricted to B is L-bi-Lipschitz from the g_α-metric to the g_t-metric. In particular, in the g_t-metric there is a ball of radius ϵ_2/L centered at p and contained in B; i.e. p has injectivity radius greater than ϵ_2/L for all t. On the other hand, if p has injectivity radius greater than $L\epsilon_2$ in the g_t-metric then there exists a ball B' of radius greater than $L\epsilon_2$ in the g_t-metric centered at p. Reversing the process above, this implies that p has injectivity radius greater than ϵ_2 in the g_α-metric. This contradiction implies that the injectivity radius at p is bounded above and below by $L\epsilon_2$ and ϵ_2/L, respectively, for all t. $\square$

LEMMA 6.14. *There exists an $l>0$ such that if $l_{M_\alpha}(\mathcal{C})\leqslant l$ then T is convex in the g_t-metric for all $t\leqslant\alpha$.*

Proof. To show that T is convex we need to show that every smooth curve σ on T has non-zero geodesic curvature in the g_t-metric at every point on σ. Since the tube $\mathbf{T}^{\epsilon_2}_\alpha(\gamma)$ has radius uniformly bounded below by some $R>0$, every smooth curve on T has geodesic curvature greater than $\tanh R$ in the g_α-metric. The result then follows from Corollary 6.10. $\square$

We denote by

$$\pi_t: T \longrightarrow \mathbf{T}^{\epsilon_2}_t(\gamma)$$

the radial projection mapping. More explicitly, each point $p\in T$ lies on a geodesic ray which is perpendicular to the core of $\mathbf{T}^{\epsilon}_t(\gamma)$ in the g_t-metric. This ray intersects $\partial\mathbf{T}^{\epsilon_2}_t(\gamma)$ in a unique point p', and we set $\pi_t(p)=p'$.

LEMMA 6.15. *For each $L>1$ and $\delta>0$ there exists an $l>0$ such that if $l_{M_\alpha}(\mathcal{C})\leqslant l$ then*

(1) *the radial projection π_t is an L-bi-Lipschitz diffeomorphism;*

(2) *if σ is a geodesic in the Euclidean metric on T induced by the g_α-metric then $\pi_t(\sigma)$ has curvature bounded by δ in the Euclidean metric on $\partial\mathbf{T}^{\epsilon_2}_t(\gamma)$ induced by the g_t-metric.*

Proof. Given $p\in T$, let P be the hyperbolic plane in the g_t-metric tangent to T at p, and let r be the radial geodesic through p. By Lemma 6.13 a bound $l_{M_\alpha}(\mathcal{C})$ gives a bound on $d(p,\pi_t(p))$, and as $l_{M_\alpha}(\mathcal{C})\to 0$ we have $d(p,\pi_t(p))\to 0$. We show that a bound on $l_{M_\alpha}(\mathcal{C})$ gives a bound on the angle $\angle(\mathsf{r},P)$ between r and P, and as $l_{M_\alpha}(\mathcal{C})\to 0$ we have $\left|\angle(\mathsf{r},P)-\frac{1}{2}\pi\right|\to 0$.

To control $\angle(\mathsf{r},P)$ we make the following observation. Let

$$W'=\mathbf{T}^{\epsilon_2}_t(\gamma)\setminus\mathcal{N}_d(\partial\mathbf{T}^{\epsilon_2}_t(\gamma)).$$

By Lemma 6.13 we have $W \supset W'$, and since T is convex P and W' are disjoint. On the other hand, $p \in P$ is within $2d$ of W', and the tube W' has definite radius. Elementary hyperbolic geometry then gives the desired bound.

Next we remark that if $\angle(\mathsf{r}, P) \neq 0$ then π_t is a diffeomorphism at p. If π_t is a local diffeomorphism at each point in T, it is a covering map. Since T and $\partial \mathbf{T}_t^{\epsilon_2}(\gamma)$ are homotopic in the complement of the core geodesic, π_t must be a global diffeomorphism. To finish the proof of (1) we note that bounds on $d(p, \pi_t(p))$ and $\left|\angle(\mathsf{r}, P) - \frac{1}{2}\pi\right|$ along with a lower bound on the tube radius of W' bound the bi-Lipschitz constant of π_t at p.

Next we control the curvature of the curve $\bar{\sigma} = \pi_t \circ \sigma$. We give the tube $\mathbf{T}_t^{\varepsilon}(\gamma)$ cylindrical coordinates (r, θ, z) so that $g_t|_{\mathbf{T}_t^{\varepsilon}(\gamma)}$ is given by the Riemannian metric

$$dr^2 + \sinh^2 r \, d\theta^2 + \cosh^2 r \, dz^2,$$

where r measures the hyperbolic distance from the core geodesic of $\mathbf{T}_t^{\varepsilon}(\gamma)$. We then let $\sigma(s) = (r(s), \theta(s), z(s))$ be a unit-speed parameterization of σ.

We begin the proof of (2) with some preliminary remarks.

The bound on $\left|\angle(\mathsf{r}, P) - \frac{1}{2}\pi\right|$ described above gives a bound on $r'(s)$. We also note that by the remark after Proposition 6.4, σ will be almost geodesic on (T, g_t). In particular, $\nabla^t_{\sigma'} \sigma'$ will be nearly orthogonal to T and hence nearly radial. That is, we can bound $\angle(\nabla^t_{\sigma'} \sigma'(s), \partial/\partial r)$. Putting this all together, for any $\epsilon > 0$ if $l_{M_\alpha}(\mathcal{C})$ is sufficiently small, then $|r'(s)| \leqslant \epsilon$ and

$$\angle\left(\nabla^t_{\sigma'} \sigma'(s), \frac{\partial}{\partial r}\right) \leqslant \epsilon.$$

If $\mathbf{T}_\alpha^{\epsilon_2}(\gamma)$ has radius R_α then the curvature of σ in the g_α-metric will be less than $\coth R_\alpha$. By Corollary 6.10, for any $C > 0$, if $l_{M_\alpha}(\mathcal{C})$ is sufficiently small then the curvature of σ in the g_t-metric will be less than $\coth R_\alpha + C$. The curvature is the length of $\nabla^t_{\sigma'} \sigma'$ in the g_t-metric, so we have

$$|\nabla^t_{\sigma'} \sigma'|_t \leqslant \coth R_\alpha + C.$$

Combining this with our bound $\angle(\nabla^t_{\sigma'} \sigma', \partial/\partial R) \leqslant \epsilon$ we have

$$\left| g_t\left(\nabla^t_{\sigma'} \sigma', \frac{1}{\sinh r} \frac{\partial}{\partial \theta}\right)\right| \leqslant \epsilon(\coth R_\alpha + C).$$

Direction calculation gives

$$\begin{aligned} \nabla^t_{\sigma'} \sigma'(s) = {} & (r''(s) - \sinh r(s) \cosh r(s)(\theta'(s)^2 + z'(s)^2)) \frac{\partial}{\partial r} \\ & + (\theta''(s) + 2 \coth r(s) r'(s) \theta'(s)) \frac{\partial}{\partial \theta} \\ & + (z''(s) + 2 \tanh r(s) r'(s) z'(s)) \frac{\partial}{\partial z}, \end{aligned}$$

so

$$\left|g_t\left(\nabla^t_{\sigma'}\sigma', \frac{1}{\sinh r}\frac{\partial}{\partial\theta}\right)\right| = |\sinh r\theta'' + 2\cosh r r'\theta'| \leqslant \epsilon(\coth R_\alpha + C).$$

Since σ has unit speed we have $|\cosh r\theta'| \leqslant |\coth r|$. Combining this with the fact that $|r'| < \epsilon$ we have

$$|\sinh r\theta''| \leqslant \epsilon(2\coth r + \coth R_\alpha + C).$$

By a similar method we also see that

$$|\cosh r z''| \leqslant \epsilon(2\tanh r + \coth R_\alpha + C).$$

We can now bound the curvature of $\bar\sigma$ on $(\partial\mathbf{T}^{\epsilon_2}_t(\gamma), g_t)$. We first note that if $\overline{\nabla}^t$ is the Riemannian connection for the Euclidean metric on $\partial\mathbf{T}^{\epsilon_2}_t(\gamma)$ then

$$\overline{\nabla}^t_{\bar\sigma'}\bar\sigma'(s) = \theta''(s)\frac{\partial}{\partial\theta} + z''(s)\frac{\partial}{\partial z}.$$

Since $\bar\sigma$ does not necessarily have unit speed, the length of $\overline{\nabla}^t_{\bar\sigma'}\bar\sigma'(s)$ does not necessarily give the geodesic curvature of $\bar\sigma$. If, however, $\bar\sigma(h(s))$ is a unit-speed reparameterization of $\bar\sigma$ then $h'(s)$ is close to 1 and $h''(s)$ is small.

Thus, the geodesic curvature of $\bar\sigma$ on $(\partial\mathbf{T}^{\epsilon_2}_t(\gamma), g_t)$ is approximately the norm of $\overline{\nabla}^t_{\bar\sigma'}\bar\sigma'(s)$, for which we have the bound

$$|\overline{\nabla}^t_{\bar\sigma'}\bar\sigma'(s)|^2 = \sinh^2 r(\theta'')^2 + \cosh^2(z'')^2 \leqslant 2\epsilon^2(2\coth r + \coth R_\alpha + C)^2.$$

The right-hand side tends to zero as $l_{M_\alpha}(\mathcal{C}) \to 0$, completing the proof. $\square$

We now return to the situation at hand, and recall that the inclusion

$$\iota: (M^{\geqslant\epsilon_1}_\alpha, g_\alpha) \longrightarrow (N, g_t)$$

is an L-bi-Lipschitz diffeomorphism to its image. Given the short geodesic γ with length

$$l_{M_\alpha}(\gamma) \leqslant \epsilon_1,$$

we now show that the inclusion map can be modified to a map ϕ_t on the collar $\mathbf{T}^\varepsilon_\alpha(\gamma) \setminus \mathbf{T}^{\epsilon_2}_\alpha(\gamma)$ that is bi-Lipschitz and sends $\partial\mathbf{T}^{\epsilon_2}_\alpha(\gamma)$ to $\partial\mathbf{T}^{\epsilon_2}_t(\gamma)$.

LEMMA 6.16. *For each $L>1$ there exists an $l>0$ such that if $l_{M_\alpha}(\mathcal{C}) \leqslant l$ then there exists an embedding*

$$\phi_t: (\mathbf{T}^\varepsilon_\alpha(\gamma) \setminus \mathbf{T}^{\epsilon_2}_\alpha(\gamma), g_\alpha) \longrightarrow (N, g_t)$$

such that

(1) *ϕ_t is the identity in a neighborhood of $\partial\mathbf{T}^\varepsilon_\alpha(\gamma)$;*

(2) *ϕ_t is L-bi-Lipschitz from the g_α-metric to the g_t-metric;*

(3) *$\phi_t = \pi_t$ on T;*

(4) *$d\phi_t$ sends each unit normal vector to $\partial\mathbf{T}^{\epsilon_2}_\alpha(\gamma)$ to a unit normal vector to $\partial\mathbf{T}^{\epsilon_2}_t(\gamma)$.*

Proof. Using a smooth bump function, it is straightforward to extend the projection π_t to a map

$$\phi_t(p,r) = (p, s(r)),$$

where if (p,r) lies in T then $(p,s(r)) = \pi_t(p)$, and so that ϕ_t is the identity on $\partial\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma)$. Since by Lemma 6.13, for any $d>0$ we may choose $l>0$ so that the Margulis tube $\mathbf{T}^{\epsilon_2}_t(\gamma)$ has boundary $\partial\mathbf{T}^{\epsilon_2}_t(\gamma)$ lying within distance d of $T = \partial\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma)$ in N, $s(r)$ may be chosen so that ϕ_t is L-bi-Lipschitz.

Since radial geodesics in $\mathbf{T}^{\epsilon}_{\alpha}(\gamma)$ make small angle with $\partial\mathbf{T}^{\epsilon_2}_t(\gamma)$ (as in the proof of Lemma 6.15), a further small modification in a neighborhood of T ensures that $d\phi_t$ sends unit normal vectors to $\partial\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma)$ to unit normal vectors to $\partial\mathbf{T}^{\epsilon_2}_t(\gamma)$. $\square$

The shape of the Margulis tube $\mathbf{T}^{\epsilon}(\gamma)$ about a short geodesic γ in a hyperbolic 3-manifold varies continuously with the complex length of the core curve γ in the smooth bi-Lipschitz topology (cf. [Mi2, Lemma 6.2]). By [Brm2, Theorem 4.3], when the singular locus is sufficiently short one can control the derivative of the complex length of a bounded-length closed geodesic in M_t (for the analogous statement for the Teichmüller parameter for a rank-2 cusp, see [Brm2, Theorem 7.3]). Together, these estimates yield the following lemma:

LEMMA 6.17. *For each $L>0$ there exists an $l>0$ such that if $l_{M_\alpha}(\mathcal{C}) \leqslant l$ then there exists an L-bi-Lipschitz diffeomorphism*

$$\psi_t \colon \mathbf{T}^{\epsilon_2}_{\alpha}(\gamma) \longrightarrow \mathbf{T}^{\epsilon_2}_t(\gamma)$$

so that ψ_t restricts to $\partial\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma)$ by an affine map with respect to the Euclidean structures on $\partial\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma)$ and $\partial\mathbf{T}^{\epsilon_2}_t(\gamma)$, and $d\psi_t$ sends unit normal vectors to $\partial\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma)$ to unit normal vectors to $\partial\mathbf{T}^{\epsilon_2}_t(\gamma)$.

Applying Lemma 6.16, the final step in our argument will be the following lemma.

LEMMA 6.18. *Assume that $\mathbf{T}^{\epsilon_2}$ is an ϵ_2-Margulis tube whose core geodesic has length less than ϵ_0. Let*

$$\phi \colon \partial\mathbf{T}^{\epsilon_2} \longrightarrow \partial\mathbf{T}^{\epsilon_2}$$

be a diffeomorphism isotopic to the identity. Then for each $K>1$ there are $L>1$ and $\delta>0$ so that if ϕ is L-bi-Lipschitz and sends geodesics in the intrinsic metric on $\partial\mathbf{T}^{\epsilon_2}$ to arcs of geodesic curvature at most δ in the intrinsic metric on $\partial\mathbf{T}^{\epsilon_2}$, then ϕ extends to a K-bi-Lipschitz diffeomorphism

$$\Phi \colon \mathbf{T}^{\epsilon_2} \longrightarrow \mathbf{T}^{\epsilon_2}.$$

Proof. If $\mathbf{T}^{\epsilon_2}$ were instead a rank-2 cusp component $\mathbf{P}^{\epsilon_2}$ of $M^{\leqslant\epsilon_2}$, it would admit a natural parameterization $T\times\mathbf{R}^+$ where $(x,d)\in T\times\mathbf{R}^+$ represents the point at depth d along the inward-pointing normal to T at $x\in T$. Since the radial projection $\pi_d\colon T\to T_d$ from T to the torus T_d at depth d given by

$$\pi_d(x,0)=(x,d)$$

is conformal, the radial extension

$$\Phi(x,d)=(\phi(x),d)$$

is readily seen to be bi-Lipschitz, with bi-Lipschitz constant L.

For the Margulis tube $\mathbf{T}^{\epsilon_2}$ the situation is very similar away from a neighborhood of the core geodesic. Indeed, after removing the core geodesic, the tube has a natural product structure $T\times\mathbf{R}^+$, where $(x,r)\in T\times\mathbf{R}^+$ is now a point at radius r from γ. If $\mathbf{T}^{\epsilon_2}$ has radius R, the radial projection

$$\pi_r(x,R)=(x,r),$$

for $r\leqslant R$, is uniformly *quasi-conformal* for $r\geqslant 1$, and we will see that the radial extension

$$\Phi(x,r)=(\phi(x),r)$$

is uniformly bi-Lipschitz for $r\geqslant 1$ as a result. For $r\in(0,1)$, however, the radial extension Φ over $\mathbf{T}^{\epsilon_2}$ will not in general be bi-Lipschitz, nor will it in general extend to the core geodesic.

To correct these problems, we will construct an isotopy ϕ_r of ϕ to the identity and define our extension Φ by

$$\Phi(x,r)=(\phi_r(x),r). \tag{6.10}$$

The isotopy ϕ_r will be the flow of a time-dependent family of vector fields v_r on T. If T_r denotes the equidistant torus at radius r from γ, the extension Φ will be bi-Lipschitz if each ϕ_r is uniformly bi-Lipschitz on T_r and the vector field v_r has uniformly bounded size on T_r.

Using the intrinsic Euclidean metric on T_r, we identify all tangent spaces $T_x(T_r)$, $x\in T_r$, with $\mathbf{R}^2$ via parallel translation. In particular, for each x we view $d\phi_x$ as a linear map of $\mathbf{R}^2$ to itself. We choose an orthonormal framing $\{e_1,e_2\}$ such that e_1 and e_2 are tangent to the directions of principal curvature of T_r. Then $|d\phi-\mathrm{id}|_r$ is the maximum of the absolute value of the entries of the matrix $d\phi-\mathrm{id}$ written in terms of this basis.

Let $\mathbf{T}^{\epsilon_2}$ have radius R so that $T_R=\partial\mathbf{T}^{\epsilon_2}$. We first claim that given any $\delta'>0$ there are $\delta>0$ and $L>1$ such that $|d\phi-\mathrm{id}|_R\leqslant\delta'$. This is equivalent to showing that ϕ has small bi-Lipschitz constant and small "twisting". That is, we need to show that the angle between any v and $d\phi(v)$ is small for any tangent vector v. We have assumed that ϕ is L-bi-Lipschitz, so we only need to bound the twisting.

Let α be the shortest geodesic on T_R through x. Then the length of α is bounded by $C\epsilon_2$, where C is a universal constant, so the length of $\phi(\alpha)$ is bounded by $LC\epsilon_2$. By our assumption the geodesic curvature of $\phi(\alpha)$ is bounded by δ. Since $\phi(\alpha)$ is homotopic to α, there is some point y on α such that if v is tangent to α at y then the angle between v and $d\phi(v)$ is 0. The bounds on the curvature and the length of $\phi(\alpha)$ imply that the tangent to $\phi(\alpha)$ is nearly parallel to α everywhere. Therefore, for any vector v tangent to α, v and $d\phi(v)$ make a small angle. Since L is close to 1, ϕ is nearly conformal, so for any tangent vector v the angle between v and $d\phi(v)$ is small, and thus ϕ has small twisting.

We now consider a linear homotopy of ϕ to the identity constructed as follows. After normalizing by an isometry of $\mathbf{T}^{\epsilon_2}$, we may assume that ϕ fixes a point $p\in T_R$. We then identify the universal cover $\widetilde{T}_R$ with $\mathbf{R}^2$ so that the intrinsic metric on T_R lifts to the Euclidean metric on $\mathbf{R}^2$. We let $\tilde{\phi}$ be a lift of ϕ that fixes a point and hence a lattice. Let $\tilde{\phi}_t$ be the homotopy of $\tilde{\phi}$ to the identity given by

$$\tilde{\phi}_t(\vec{x})=(1-t)\tilde{\phi}(\vec{x})+t\vec{x}$$

for each $\vec{x}$ in $\mathbf{R}^2$. This homotopy is equivariant by the action of the covering translation group for T_R and therefore descends to a homotopy ϕ_t of T_R. Direct computation shows that if $|d\phi-\mathrm{id}|_R\leqslant\delta'$ then $|d\phi_t-\mathrm{id}|_R\leqslant(1-t)\delta'$. In particular, for δ' sufficiently small, ϕ_t is a local, and hence global, diffeomorphism for all t. In the T_r-metric another computation shows that

$$|d\phi-\mathrm{id}|_r\leqslant\frac{\tanh R}{\tanh r}\delta'.$$

Therefore for any $\delta''\geqslant 0$ we can choose L and δ such that $|d\phi_t-\mathrm{id}|\leqslant\delta''$ in the T_r-metric for all $r\geqslant 1$. A bound on $|d\phi_t-\mathrm{id}|_r$ determines a bound on the bi-Lipschitz constant. In particular, for any $L'>1$ we can choose L and δ so that ϕ_t is L'-bi-Lipschitz in the T_r-metric.

Let v_t be the time-dependent family of vector fields whose flow is ϕ_t. The norm of v_t in the T_r-metric is bounded by the supremum of the distance between x and $\tilde{\phi}(x)$ in the T_r-metric. If D_r is the diameter of T_r, this distance is bounded by $D_r\epsilon'$, where ϵ' tends to zero as the bi-Lipschitz constant L' tends to 1. But D_r is universally bounded for $1\leqslant r\leqslant 2$, so for these values of r the norm of v_t is bounded on T_r.

To finish the proof, we must reparameterize our isotopy to obtain the desired smoothness for the extension Φ. Let s be a smooth function on $[0,R]$ with

$$s(r)=\begin{cases}1, & r\leqslant 1,\\ \text{monotonically decreasing}, & 1\leqslant r\leqslant 2,\\ 0, & 2\leqslant r\leqslant R.\end{cases}$$

Note that the derivative of s is bounded (and independent of R). We now abuse notation and redefine the isotopy ϕ_r as the projection to T of the isotopy of $\tilde{\phi}$ to the identity given by the formula

$$\tilde{\phi}_r(\vec{x})=(1-s(r))\tilde{\phi}(\vec{x})+s(r)\vec{x}$$

for $r\in[0,R]$. With this notation, ϕ_r is the flow of the time-dependent vector field

$$w_r=s'(r)v_{s(r)},$$

which is again bounded in the T_r-metric for all r, and zero for $r\leqslant 1$ and $r\geqslant 2$. It follows that the extension Φ of ϕ given by (6.10) is K-bi-Lipschitz, where K depends only on L and δ. The proof is complete. $\square$

We now prove Theorem 6.12 by combining Lemmas 6.16, 6.17 and 6.18.

Proof of Theorem 6.12. Let $L>1$ and $\delta>0$ be given. Then by Lemma 6.16 there is an $l>0$ so that if $l_{M_\alpha}(\mathcal{C})\leqslant l$ then for each closed geodesic γ with $l_{M_\alpha}(\gamma)<\epsilon_1$ we have the corresponding mapping

$$\phi_t\colon(\mathbf{T}^{\varepsilon}_{\alpha}(\gamma)\backslash\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma),g_\alpha)\longrightarrow(N,g_t).$$

Then ϕ_t restricts to an L-bi-Lipschitz diffeomorphism

$$\phi_t|_{\partial\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma)}\colon\partial\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma)\longrightarrow\partial\mathbf{T}^{\epsilon_2}_{t}(\gamma)$$

so that geodesics on $\partial\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma)$ map to arcs of geodesic curvature bounded by δ on $\partial\mathbf{T}^{\epsilon_2}_{t}(\gamma)$.

Assume that l also satisfies the hypotheses of Lemma 6.17, and let

$$\psi_t\colon\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma)\longrightarrow\mathbf{T}^{\epsilon_2}_{t}(\gamma)$$

be the L-bi-Lipschitz diffeomorphism guaranteed by Lemma 6.17.

Since

$$\psi_t\colon\partial\mathbf{T}^{\epsilon_2}_{\alpha}(\gamma)\longrightarrow\partial\mathbf{T}^{\epsilon_2}_{t}(\gamma)$$

is affine, the composition

$$\phi_t\circ\psi_t^{-1}|_{\partial\mathbf{T}^{\epsilon_2}_{t}(\gamma)}\colon\partial\mathbf{T}^{\epsilon_2}_{t}(\gamma)\longrightarrow\partial\mathbf{T}^{\epsilon_2}_{t}(\gamma)$$

sends geodesics on $\partial\mathbf{T}_t^{\epsilon_2}(\gamma)$ to arcs of curvature bounded by δ, and is L^2-bi-Lipschitz.

Since $L>1$ and δ were arbitrary, given any $K>1$ we may choose L and δ sufficiently small so that Lemma 6.18 provides a K-bi-Lipschitz extension

$$\Phi_t\colon \mathbf{T}_t^{\epsilon_2}(\gamma) \longrightarrow \mathbf{T}_t^{\epsilon_2}(\gamma)$$

of $\phi_t \circ \psi_t^{-1}|_{\partial\mathbf{T}_t^{\epsilon_2}(\gamma)}$ over $\mathbf{T}_t^{\epsilon_2}(\gamma)$.

Since we have

$$\phi_t|_{\partial\mathbf{T}_\alpha^{\epsilon_2}(\gamma)} = \Phi_t^{-1} \circ \psi_t|_{\partial\mathbf{T}_\alpha^{\epsilon_2}(\gamma)},$$

the composition

$$\Phi_t^{-1} \circ \psi_t\colon \mathbf{T}_\alpha^{\epsilon_2}(\gamma) \longrightarrow \mathbf{T}_t^{\epsilon_2}(\gamma)$$

gives a KL-bi-Lipschitz extension of ϕ_t over $\mathbf{T}_\alpha^{\epsilon_2}(\gamma)$.

As remarked, we may apply exactly analogous versions of Lemmas 6.16, 6.17 and 6.18 for the rank-2 cusp case to complete the proof. □

Proof of Theorem 6.2. Given any $L>1$, we may choose $l>0$ satisfying the hypotheses of Theorem 6.12 so that the inclusion

$$\iota\colon (M_\alpha^{\geqslant \epsilon_1}, g_\alpha) \longrightarrow (N, g_t)$$

is an L-bi-Lipschitz diffeomorphism to its image from the g_α-metric to the g_t-metric. After decreasing l if necessary, Theorem 6.2 follows by applying Theorem 6.12 to extend ι to an L-bi-Lipschitz diffeomorphism f_t by extending over each component of $M_\alpha^{\leqslant\varepsilon}$ other than $\mathbf{T}^\varepsilon(\mathcal{C})$.

Finally, for each component $c\subset\mathcal{C}$ we may apply the argument of Lemma 6.16 to each component of $\mathbf{T}^\varepsilon(\mathcal{C})$ to modify the restriction

$$f_t|_{M_\alpha \setminus \mathbf{T}_\alpha^{\epsilon_2}(\mathcal{C})}$$

on $\mathbf{T}_\alpha^\varepsilon(c)\setminus\mathbf{T}_\alpha^{\epsilon_2}(c)$ so that $f_t(\partial\mathbf{T}_\alpha^{\epsilon_2}(c))=\partial\mathbf{T}_t^{\epsilon_2}(c)$. The resulting L-bi-Lipschitz diffeomorphism

$$h_t\colon M_\alpha\setminus\mathbf{T}_\alpha^{\epsilon_2}(\mathcal{C}) \longrightarrow M_t\setminus\mathbf{T}_t^{\epsilon_2}(\mathcal{C})$$

proves Theorem 6.2. □

We remark that the choice of ϵ_2 was arbitrary, and there exists $l>0$ satisfying the theorem for any choice of ϵ_2.

Constants. For future reference, choosing some $L>1$ and letting $l>0$ denote the corresponding constant so that the conclusions of Theorem 6.2 apply, we take as a threshold constant

$$l_0 = \min\{l_{\text{knot}}, l\}.$$

Then for any $M\in AH(S)$ and any geodesic $\gamma\subset M$ with $l_M(\gamma)<l_0$, we may graft M along γ, and we may decrease the cone-angle along γ with L-bi-Lipschitz distortion of the metric outside of a standard tube about γ.

7. Realizing ends on a Bers boundary

In this section we define a notion of *realizability* for simply degenerate ends of hyperbolic 3-manifolds on Bers boundaries for Teichmüller space:

Definition 7.1. Let E be a simply degenerate end of a hyperbolic 3-manifold M. If E admits a marking- and orientation-preserving bi-Lipschitz diffeomorphism to an end E' of a manifold Q lying on the boundary of a Bers slice, we say that E is *realized on a Bers boundary by* Q.

We prove a general realizability result for ends of manifolds $M\in AH(S)$.

THEOREM 7.2. (Ends are realizable.) *Let $M\in AH(S)$ have no cusps. Then each degenerate end E of M is realized on a Bers boundary by a manifold Q.*

Proof. As usual, the proof breaks into cases.

Case I: *the end E has arbitrarily short geodesics.* Let $\{\gamma_n\}_{n=0}^{\infty}$ be a collection of arbitrarily short geodesics in M. Pass to a subsequence so that each γ_n has length less than l_0, and so that for each $n\geqslant 1$ the geodesic γ_n is isotopic out the end E in the complement of γ_0.

Consider the simultaneous graftings

$$M_n^c = \mathrm{Gr}^{\pm}(\gamma_0, \gamma_n, M).$$

The manifolds M_n^c are 3-dimensional hyperbolic cone-manifolds with cone-angles 4π at each component γ_0 and γ_n of the singular locus.

Since we have

$$l_M(\gamma_n) < l_0$$

for each n, we may apply Theorem 6.2 to decrease the cone-angles at γ_0 and γ_n to 2π. The result is a smooth, geometrically finite hyperbolic 3-manifold homotopy equivalent to S, namely, a quasi-Fuchsian manifold Q. We let X_n and Y_n in Teich(S) be the surfaces simultaneously uniformized by Q, so that $Q=Q(X_n,Y_n)$.

We note that the conformal boundary component that arises from negative grafting along γ_0 does not change with n; there is a single $X\in\mathrm{Teich}(S)$ so that $X_n=X$. It follows that sending the cone-angles of M_n^c to 2π gives the sequence $\{Q(X,Y_n)\}_{n=1}^{\infty}$ lying in the Bers slice B_X.

Passing to a subsequence and extracting a limit $Q\in\partial B_X$, we claim that the manifold Q realizes the end E on the Bers boundary ∂B_X.

To see this, let U_n be the union of the Margulis tubes

$$U_n=\mathbf{T}^{\varepsilon}(\gamma_0)\sqcup\mathbf{T}^{\varepsilon}(\gamma_n)$$

in M. Let

$$F: S\times\mathbf{R}\longrightarrow M$$

be a product structure for M as in §5 so that γ_n is a simple curve on $F(S\times\{n\})$. We consider an exhaustion of the end E by compact submanifolds $K_n=F(S\times[t_0,t_n])$, where $[t_0,t_n]\subset[0,\infty)$ is an interval chosen so that

$$K_n\cap U_n=\varnothing.$$

Letting $F(\gamma_0\times(-\infty,0])$ be the negative grafting annulus for γ_0 and $F(\gamma_n\times[n,+\infty))$ be the positive grafting annulus for γ_n, K_n admits a marking-preserving isometric embedding ι_n to the subset $\iota_n(K_n)=K_n'\subset M_n^c$.

Let $U_n'\subset Q(X,Y_n)$ denote the union

$$U_n'=\mathbf{T}^{\varepsilon}(\gamma_0)\sqcup\mathbf{T}^{\varepsilon}(\gamma_n),$$

and let

$$h_n:(M_n^c\setminus U_n,\partial U_n)\longrightarrow(Q(X,Y_n)\setminus U_n',\partial U_n')$$

be the uniformly bi-Lipschitz diffeomorphisms furnished by Theorem 6.2. For each integer j, there is an n_j so that the mappings $\varphi_n=h_n\circ\iota_j$ are uniformly bi-Lipschitz embeddings of K_j into $Q(X,Y_n)$ for all $n>n_j$.

By Ascoli's theorem, the embeddings φ_n converge to a uniformly bi-Lipschitz embedding on K_j into the limit Q of $Q(X,Y_n)$ after passing to a subsequence. Diagonalizing, we have a uniformly bi-Lipschitz embedding of E into Q, so E is realized by Q on the Bers boundary ∂B_X.

Case II: *the end E has bounded geometry.* By our application of Minsky's bounded geometry results (Theorem 4.5), if $Q\in\partial B_X$ has end-invariant $\nu(E_Q)=\nu(E)$, then Q realizes the end E on the Bers boundary ∂B_X. Letting μ be any measure on $\nu(Q)$ and choosing weighted simple closed curves $t_n\gamma_n$ so that $t_n\gamma_n\to\mu$, we let $Y_n\in\mathrm{Teich}(S)$ be any surface for which $l_{Y_n}(\gamma_n)<1/n$. Then applying [Bro1, Theorem 2], any accumulation point Q of $Q(X,Y_n)$ has the property that $\nu(E_Q)=\nu(E)$. This completes the proof. $\square$

Applying results of [Bro1] and [Bro2], we deduce a corollary that will be essential to ensure convergence of our candidate geometrically finite approximations for M.

COROLLARY 7.3. *Let $Q=\lim_{n\to\infty} Q(X,Y_n)$ be a realization of the positive degenerate end E of $M\in AH(S)$. Then for any convergent subsequence $Y_n\to[\mu]$ in Thurston's compactification $\mathcal{PL}(S)$, we have $\nu(E)=|\mu|$.*

Proof. The corollary is a direct consequence of Theorem 6.1 of [Bro2]. □

8. Asymptotic isolation of ends

Theorem 7.2 guarantees that each degenerate end of a manifold $M\in AH(S)$ can be realized on a Bers boundary. In the doubly degenerate case, realizations of E^- and E^+ by

$$Q^- = \lim_{n\to\infty} Q(X_n, Y) \quad \text{and} \quad Q^+ = \lim_{n\to\infty} Q(X, Y_n)$$

suggest the candidate approximations $Q(X_n, Y_n)$ for the original manifold M.

The construction raises the natural question: to what extent do the ends of a limit N of quasi-Fuchsian manifolds $Q(X_n, Y_n)$ depend on the *pair* of surfaces (X_n, Y_n)? In this section we isolate the effect of the negative surfaces X_n on the positive end of N, and likewise for the negative end.

THEOREM 8.1. (Asymptotic isolation of ends.) *Let $Q(X_n,Y_n)\in AH(S)$ be a sequence of quasi-Fuchsian manifolds converging algebraically to the cusp-free limit manifold N. Then, up to marking- and orientation-preserving bi-Lipschitz diffeomorphism, the positive end of N depends only on the sequence $\{Y_n\}_{n=1}^\infty$, and the negative end of N depends only on the sequence $\{X_n\}_{n=1}^\infty$.*

Proof. Consider such a sequence $Q(X_n, Y_n)$ converging to N. By the assumption that N is cusp free, the convergence of $Q(X_n, Y_n)$ to N is strong (see [T1] or [AC2, Corollary G]).

We show that the positive end E^+ depends only on the sequence $\{Y_n\}_{n=1}^\infty$ up to bi-Lipschitz diffeomorphism preserving orientation and marking; the same argument applies to the negative end E^-. In other words, given another sequence $\{X_n'\}_{n=1}^\infty$ in Teich(S) for which the quasi-Fuchsian manifolds $Q(X_n', Y_n)$ converge to N', the positive end $(E^+)'$ of N' admits a marking- and orientation-preserving bi-Lipschitz diffeomorphism with E^+.

If the end E^+ is geometrically finite, it is well known that its associated conformal boundary component Y determines E^+ up to bi-Lipschitz diffeomorphism (see [EM] and [Mi1]). Thus we may assume that E^+ is degenerate.

If all closed geodesics in N have length at least $\frac{1}{2}l_0$ then N has a global lower bound to its injectivity radius since N has no cusps. It follows from Theorem 4.5 that there is a hyperbolic manifold $Q\in\partial B_X$ in the boundary of the Bers slice B_X so that the positive end

E_Q^+ of Q is degenerate, and there is an orientation- and marking-preserving bi-Lipschitz diffeomorphism

$$h: E^+ \longrightarrow E_Q^+.$$

By Minsky's bounded geometry theorem (Theorem 4.2), the bi-Lipschitz diffeomorphism type of the end E_Q^+ depends only on $\nu(E_Q^+)$. Since E^+ and E_Q^+ are bi-Lipschitz diffeomorphic, we have $\nu(E^+)=\nu(E_Q^+)$. If $[\mu]$ is any limit of $\{Y_n\}_{n=1}^\infty$ in $\mathcal{PL}(S)$, it follows from [Bro2, Theorem 6.1] that either $\nu(E^+)=|\mu|$ or $\nu(E^-)=|\mu|$. The fact that $\nu(E^+)=|\mu|$ is due to Thurston (see [T1, §9.2]), but the argument makes delicate use of interpolations of pleated surfaces through $Q(X_n, Y_n)$. We present an alternative argument in which a geodesic in the complex of curves plays a similar role to that of the carefully chosen path in $\mathcal{ML}(S)$ from Thurston's original argument.

By strong convergence, there are compact cores $\mathcal{M}_n \subset Q(X_n, Y_n)$ converging geometrically to a compact core $\mathcal{M} \subset N$ so that each compact core separates the ambient manifold into positive and negative ends. It follows that if $\nu(E^-)=|\mu|$ then there are simple closed curves γ_n with $[\gamma_n]\to[\mu]$ for which the geodesic representative γ_n^* of γ_n in $Q(X_n, Y_n)$ always lies in the negative end of $Q(X_n, Y_n)$ for n sufficiently large.

Let $\beta_n \in \mathcal{ML}(S)$ denote the bending lamination for the convex-core boundary component of $Q(X_n, Y_n)$ that faces Y_n (see [T1] and [EM]). By a theorem of M. Bridgeman [Bri, Proposition 2], there is a $K>0$ so that the lengths $l_{Y_n}(\beta_n)$ are uniformly bounded, and since $Y_n \to [\mu]$ in $\mathcal{PL}(S)$ there are $\mu_n \in \mathcal{ML}(S)$ with $[\mu_n]\to[\mu]$ so that $l_{Y_n}(\mu_n)$ are also uniformly bounded (see [T4, Theorem 2.2] and [T6, §9]). It follows that the projective classes $[\beta_n]$ converge up to subsequence to a limit $[\mu']$ with $i(\mu', \mu)=0$, so we have $|\mu|=|\mu'|$.

Given $\delta>0$, we may construct a nearly straight train track τ_n carrying β_n (see [T1, §8] or [Bro1, Lemma 5.2]) so that leaves of β_n lie within δ of τ_n, as do the geodesic representatives of sufficiently close approximations to β_n by simple closed curves. Diagonalizing, there are simple closed curves η_n with $[\eta_n]\to[\mu']$ so that the geodesic representatives η_n^* of η_n in $Q(X_n, Y_n)$ lie in the positive ends of $Q(X_n, Y_n)$ for all n sufficiently large.

Since $|\mu|=|\mu'|$ in $\mathcal{EL}(S)$, we have that the curves γ_n and η_n converge to the same point in the Gromov boundary $\partial\mathcal{C}(S)=\mathcal{EL}(S)$ (see [Kl, Theorem 1.4]). Joining γ_n to η_n by a geodesic g_n in $\mathcal{C}(S)$, it follows from the δ-hyperbolicity of $\mathcal{C}(S)$ and the definition of its Gromov boundary that the entire geodesic g_n converges to $|\mu|$ in $\partial\mathcal{C}(S)$. (The Gromov inner product $\langle\alpha_n|\zeta_n\rangle$ of any pair of points α_n and ζ_n along the geodesic g_n is equal to that of its endpoints $\langle\gamma_n|\eta_n\rangle$, which tends to infinity with n since γ_n and η_n both converge to $|\mu|$. It follows that the geodesics g_n *converge at infinity* to the limit $|\mu|$ of their endpoints; see for example [BH, §III.H, Definition 3.12].)

Thus, for any sequence α_n of curves so that α_n corresponds to a vertex of g_n, we have $[\alpha_n]\to[\mu]$. Since successive pairs of curves along g_n have zero intersection, we may realize each such pair by a pleated surface, one of which, say Z_n, must intersect $\mathcal{M}_n$. The sequence of pleated surfaces Z_n in $Q(X_n, Y_n)$ has a limit Z_∞ in N (after passing to a subsequence, see [T3, Proposition 5.9] and [CEG]) realizing μ, a contradiction.

It follows that $\nu(E^+)=|\mu|$, and we conclude that the bi-Lipschitz diffeomorphism type of the end E^+ depends only on the sequence $\{Y_n\}_{n=1}^\infty$ in the case that all closed geodesics in N have length at least $\frac{1}{2}l_0$.

If N has a closed geodesic γ^* for which $l_N(\gamma^*)<\frac{1}{2}l_0$, then there is an I so that for all $n>I$, we have

$$l_{Q(X_n,Y_n)}(\gamma) < l_0.$$

Otal's theorem (Theorem 2.5) implies that γ^* is the geodesic representative of a simple closed curve γ on S, and that γ^* is unknotted in each $Q(X_n, Y_n)$. It follows that the complement $Q(X_n, Y_n)\setminus\mathbf{T}^\epsilon(\gamma)$ of the Margulis tube for γ in $Q(X_n, Y_n)$ has the homeomorphism type

$$Q(X_n, Y_n)\setminus\mathbf{T}^\varepsilon(\gamma)\cong S\times\mathbf{R}\setminus\mathcal{N}(\gamma\times\{0\}),$$

where $\mathcal{N}(\gamma\times\{0\})$ denotes an embedded solid-torus neighborhood of $\gamma\times\{0\}$ in $Q(X_n, Y_n)$.

We assume for simplicity that the curve γ is non-separating on S; the separating case presents no new subtleties. For reference let

$$T = S\setminus\mathrm{collar}(\gamma)$$

be the essential subsurface in the complement of a standard open annular collar of γ.

Applying Theorem 6.2, we may send the cone-angle at γ to zero keeping the conformal boundary fixed, to obtain a manifold $M_\gamma(X_n, Y_n)$ with a rank-2 cusp $\mathbf{P}(\gamma)$ at γ and a uniformly bi-Lipschitz diffeomorphism of pairs

$$h_n\colon (Q(X_n, Y_n)\setminus\mathbf{T}^\varepsilon(\gamma), \partial\mathbf{T}^\varepsilon(\gamma)) \longrightarrow (M_\gamma(X_n, Y_n)\setminus\mathbf{P}(\gamma), \partial\mathbf{P}(\gamma))$$

so that h is marking-preserving on the ends of $Q(X_n, Y_n)$, in a sense to be made precise presently.

For simplicity of notation, let $Q_n=Q(X_n, Y_n)$ and $M_n=M_\gamma(X_n, Y_n)$. We pause to elaborate briefly on the structure of M_n. Since the geodesic γ^* is unknotted, we may choose a product structure $F_n\colon S\times\mathbf{R}\to Q_n$ so that $\gamma^*=F_n(\gamma\times\{0\})$ and so that there is a standard tubular neighborhood

$$V_\gamma = \mathrm{collar}(\gamma)\times\left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

of $\gamma\times\{0\}$ in $S\times\mathbf{R}$ with $F_n(V_\gamma)=\mathbf{T}^\varepsilon(\gamma)$.

We take $\mathcal{M}_n = F(S\times[-1,1])$ as a compact core for Q_n, and we denote by $E_n^+ = F(S\times(1,\infty))$ and $E_n^- = F(S\times(-\infty,-1))$ the positive and negative ends of Q_n with respect to $\mathcal{M}_n$. The diffeomorphism h_n sends $\mathcal{M}_n\setminus\mathbf{T}^\varepsilon(\gamma)$ to a relative compact core for M_n, and sends E_n^+ and E_n^- to ends of M_n. We will refer to $h_n(E_n^+)$ and $h_n(E_n^-)$ as the *positive* and *negative* ends of M_n. The positive and negative ends of M_n are implicitly marked by the compositions

$$h_n \circ F(S\times\{1\}) \quad \text{and} \quad h_n \circ F(S\times\{-1\}).$$

It follows from the fact that the cone-deformation preserves the conformal boundary that in this marking the positive ends of Q_n and M_n are compactified by the same surface $Y_n \in \mathrm{Teich}(S)$, and the negative ends of Q_n and M_n are compactified by the same surface $X_n \in \mathrm{Teich}(S)$.

Let $\omega_n \in Q_n\setminus\mathbf{T}^\varepsilon(\gamma)$ be a choice of base-frame for Q_n so that the lifts (Q_n,ω_n) to $AH_\omega(S)$ converge strongly to (N,ω). Let $\omega_n' = h_n(\omega_n)$ be base-frames for M_n. By Gromov's compactness theorem, we may pass to a subsequence and extract a geometric limit $(h_\infty,(Q_\infty,\omega_\infty),(M_\infty,\omega_\infty'))$ of the triples $(h_n,(Q_n,\omega_n),(M_n,\omega_n'))$, where h_∞ is a uniformly bi-Lipschitz diffeomorphism between smooth submanifolds of Q_∞ and M_∞ obtained as a limit of the uniformly bi-Lipschitz diffeomorphisms h_n.

Since the convergence of $Q_n \to N$ is strong, we have $Q_\infty = N$, and h_∞ has domain $N\setminus\mathbf{T}^\varepsilon(\gamma)$. By the above discussion, the mappings h_n induce the markings on the positive ends of M_n, so h_∞ gives a marking- and orientation-preserving uniformly bi-Lipschitz diffeomorphism of the positive end E^+ of N with the positive end E_M^+ of M_∞.

The covering Q_γ of M_∞ corresponding to $\pi_1(E_M^+)$ lies in $AH(S)$ and has the following properties:

(1) Q_γ has positive end isometric to E_M^+;

(2) Q_γ has cuspidal thin part $P_\gamma \subset Q_\gamma$, a rank-1 cusp whose inclusion on π_1 is conjugate to $\langle\gamma\rangle$;

(3) the pared submanifold $Q_\gamma\setminus P_\gamma$ has a negative end

$$E_\gamma^- \cong T\times(-\infty,0].$$

We claim that the end E_γ^- of the pared submanifold $Q_\gamma\setminus P_\gamma$ is geometrically finite.

Since it suffices to show that the cover $\widetilde{Q}_\gamma$ of Q_γ corresponding to $\pi_1(E_\gamma^-)\cong\pi_1(T)$ is quasi-Fuchsian, assume otherwise. Then there is some degenerate end E of the *pared submanifold* of $\widetilde{Q}_\gamma$, namely, the complement $\widetilde{Q}_\gamma\setminus\widetilde{P}_\gamma$ of the cuspidal thin part of $\widetilde{Q}_\gamma$. Since the manifold Q_γ is not a surface bundle over the circle, the covering theorem (see [T1] and [C2]) implies that the end E has a neighborhood U that covers a degenerate end of M_∞ finite-to-one. But the manifold M_∞ has exactly two degenerate ends, each homeomorphic to a product of the larger surface S with a half-line.

It follows that the negative end of Q_γ is geometrically finite. Let $Z\in\mathrm{Teich}(T)$ denote the associated conformal boundary component. Choose a surface $X\in\mathrm{Teich}(S)$ so that $l_X(\gamma)<\frac{1}{4}l_0$. By a theorem of Bers (see [Be, Theorem 3] or [Mc1, Proposition 6.4]), then, the manifolds $Q(X,Y_n)$ have the property that

$$l_{Q(X,Y_n)}(\gamma)<\tfrac{1}{2}l_0$$

for all n.

Performing the same process as above with $X_n=X$, we arrive at a manifold Q'_γ whose positive end $\breve{E}^+$ is bi-Lipschitz diffeomorphic to the positive end of the limit Q_∞ of $Q(X,Y_n)$ after passing to a subsequence. The negative end of Q'_γ has conformal boundary surface $Z'\in\mathrm{Teich}(T)$.

The manifold Q_γ is the cover associated to the positive end of

$$\lim_{n\to\infty} M_\gamma(X_n,Y_n)=M_\infty,$$

and the manifold Q'_γ is the cover associated to the positive end of the limit

$$\lim_{n\to\infty} M_\gamma(X,Y_n).$$

But letting $Q_\gamma(n)$ be the cover of $M_\gamma(X_n,Y_n)$ associated to Y_n and letting $Q_\gamma(n)'$ be the cover of $M_\gamma(X,Y_n)$ associated to Y_n, we have

$$\partial Q_\gamma(n)=Z_n\sqcup Y_n \quad\text{and}\quad \partial Q_\gamma(n)'=Z'_n\sqcup Y_n,$$

where Z_n converges to Z in $\mathrm{Teich}(T)$ and Z'_n converges to Z' in $\mathrm{Teich}(T)$. Letting $K_n=d_T(Z_n,Z'_n)$ be the Teichmüller distance from Z_n to Z'_n, the manifolds $Q_\gamma(n)$ and $Q_\gamma(n)'$ have quasi-isometric distance

$$d_{\mathrm{qi}}(Q_\gamma(n),Q_\gamma(n)')<K'_n,$$

where K'_n depends only on K_n (see e.g. [Mc3, Theorem 2.5]). By lower semi-continuity of the quasi-isometric distance on $AH(S)$ [Mc3, Proposition 3.1] we have a marking-preserving bi-Lipschitz diffeomorphism

$$\Phi\colon Q_\gamma\longrightarrow Q'_\gamma.$$

Restricting the diffeomorphism Φ to the positive ends of Q_γ and Q'_γ, it follows that the positive end of N is bi-Lipschitz diffeomorphic to the positive end of

$$Q_\infty=\lim_{n\to\infty} Q(X,Y_n).$$

Since for any surface $X'\in\mathrm{Teich}(S)$, the manifolds in the sequences $Q(X,Y_n)$ and $Q(X',Y_n)$ are uniformly bi-Lipschitz diffeomorphic, the bi-Lipschitz diffeomorphism type of E^+ does not depend on X (by another application of [Mc3, Proposition 3.1]), so the theorem follows. □

9. Proof of the main theorem

In this section, we assemble our results to give the proof of our main approximation theorem (Theorem 1.2). The proof naturally breaks into cases based on the homotopy type of the manifold we wish to approximate. We first treat the following case.

THEOREM 9.1. *Let $M \in AH(S)$ be cusp free. Then M is an algebraic limit of quasi-Fuchsian manifolds.*

Proof. By Theorem 7.2 (or the main theorem of [Brm1]), it suffices to consider the case when M is doubly degenerate. Let $F: S \times \mathbf{R} \to M$ be a smooth product structure on M, and let $\mathcal{M} = F(S \times [-1,1])$ denote a compact core for M. Let

$$E^- = F(S \times (-\infty, -1)) \quad \text{and} \quad E^+ = F(S \times (1, \infty))$$

denote the positive and negative ends of M.

By Theorem 7.2, the ends E^+ and E^- are realizable by manifolds

$$Q^+ = \lim_{n\to\infty} Q(X, Y_n) \quad \text{and} \quad Q^- = \lim_{n\to\infty} Q(X_n, Y)$$

on the Bers boundaries ∂B_X^+ and ∂B_Y^-.

Moreover, if ν^+ and ν^- are the end-invariants for M, Corollary 7.3 guarantees that after passing to a subsequence we may assume that we have the convergence

$$X_n \to [\mu^-] \quad \text{and} \quad Y_n \to [\mu^+]$$

in Thurston's compactification $\mathcal{PL}(S)$, and that the support of μ^- and μ^+ is given by

$$|\mu^-| = \nu^- \quad \text{and} \quad |\mu^+| = \nu^+.$$

Since ν^- and ν^+ are the end-invariants of the doubly degenerate manifold M, it follows that μ^- and μ^+ *bind the surface*: for any simple closed curve α, we have

$$i(\alpha, \mu^-) + i(\alpha, \mu^+) > 0.$$

Thus, by Thurston's *double limit theorem* ([T4, Theorem 4.1], see also [Ot2]), the sequence

$$\{Q_n\}_{n=1}^{\infty} = \{Q(X_n, Y_n)\}_{n=1}^{\infty}$$

converges after passing to a subsequence.

By an application of the continuity of the length function [Bro1, Theorem 2] we may pass to a subsequence so that the sequence $Q(X_n, Y_n)$ converges to a limit $M' \in AH(S)$

for which ν^+ and ν^- are the end-invariants of M'. It follows that the limit M' is doubly degenerate.

Applying Theorem 8.1, the ends of M' are geometrically isolated: the positive end of M' is bi-Lipschitz diffeomorphic to the positive end of Q^+, and the negative end of M' is bi-Lipschitz diffeomorphic to the negative end of Q^-. Since Q^+ and Q^- realize the ends of M, it follows that there is a smooth product structure

$$F': S\times\mathbf{R} \longrightarrow M'$$

so that

(1) F' decomposes M' into a compact core $\mathcal{M}'=F(S\times[-1,1])$ and ends $(E^-)'=F(S\times(-\infty,-1))$ and $(E^+)'=F(S\times(1,\infty))$;

(2) there are marking-preserving bi-Lipschitz diffeomorphisms

$$h^+: E^+ \longrightarrow (E^+)' \quad \text{and} \quad h^-: E^- \longrightarrow (E^-)'$$

between the positive and negative ends of M and M' so that

$$h^+(F(x,t)) = F'(x,t) \quad \text{and} \quad h^-(F(x,t)) = F'(x,t).$$

By compactness of $\mathcal{M}$, the extension

$$h: M \longrightarrow M'$$

of h^+ and h^- across $\mathcal{M}$ defined by setting $h(F(x,t))=F'(x,t)$ is a single marking-preserving bi-Lipschitz diffeomorphism from M to M'. Applying Sullivan's rigidity theorem [Su1], it follows that h is homotopic to an isometry, and we may conclude that $Q(X_n, Y_n)$ converges to M. □

To complete the proof of Theorem 1.2, we now treat the case when M is a general infinite-volume complete hyperbolic 3-manifold with incompressible ends and no cusps. The theorem in this case essentially follows directly from the case when M has the homotopy type of a surface; there are two issues to which we alert the reader:

(1) In the surface case, we applied Thurston's double limit theorem to show that the approximations converge up to subsequence. Here, an analogous compactness result is necessary ([Oh2, Theorem 2.4], cf. [T5]).

(2) Examples constructed by Anderson and Canary (see [AC1]) illustrate that homeomorphism type need not persist under algebraic limits of hyperbolic 3-manifolds. Since we control the ends of M_n, and thence the peripheral structure of $\pi_1(M_n)$, we may prevent such a topological cataclysm by an application of Waldhausen's theorem [W], [He, Theorem 13.7].

THEOREM 9.2. *Let M be an infinite-volume complete hyperbolic 3-manifold with incompressible ends and no cusps. Then M is an algebraic limit of geometrically finite manifolds.*

Proof. By Theorem 9.1, we may assume that M is not homotopy equivalent to a surface. By Bonahon's theorem, there is a compact 3-manifold N so that

$$M \cong \mathrm{int}(N).$$

Thus, M lies in $AH_0(N)$, the subset of $AH(N)$ consisting of marked hyperbolic 3-manifolds $(f\colon N \to M)$ for which $f|_{\mathrm{int}(N)}$ is homotopic to a homeomorphism.

As in the outline, we approximate the tame manifold M end-by-end, and combine the approximations into one sequence of geometrically finite hyperbolic manifolds M_n homeomorphic to M that converge to M.

Let $\mathcal{M}$ be a compact core for M and let E be a degenerate end of $M \setminus \mathcal{M}$. Let Q be the cover of M corresponding to $\pi_1(E)$. Then Q lies in $AH(S)$, where $S = \overline{E} \cap \mathcal{M}$. Assume that M is oriented so that E lifts to the positive end $\widetilde{E}$ of Q in the cover.

By the covering theorem (see [T1, Chapter 9] or [C2, Main Theorem]), we claim that the manifold Q is a singly degenerate manifold with no cusps and degenerate end $\widetilde{E}$. To see this, note first that M has no cusps, so neither does Q. The only alternative is then that Q is doubly degenerate, which implies that the covering $Q \to M$ is finite-to-one, since M is not a surface bundle over the circle. But if Q covers M finite-to-one, the manifold M is itself homotopy equivalent to a surface finitely covered by S.

The geometrically finite locus $GF_0(N) = AH_0(N)$ consists of geometrically finite hyperbolic 3-manifolds M homeomorphic to $\mathrm{int}(N)$. Realizing the degenerate ends of M on the appropriate Bers boundaries, we will obtain surfaces that determine candidate approximations for M in the interior of $AH_0(N)$. The interior of $AH_0(N)$ is typically denoted by $MP_0(N)$, the *minimally parabolic* structures on $\mathrm{int}(N)$, namely, geometrically finite structures on $\mathrm{int}(N)$ with only finite-volume cusps. Since N is assumed to have no torus boundary components, $MP_0(N)$ is simply the cusp-free hyperbolic structures on $\mathrm{int}(N)$ in the case at hand.

Choosing a reference compact core $\mathcal{M}$ for M, let $\mathcal{E}_1, \ldots, \mathcal{E}_p$ denote the geometrically finite ends of M, and let $E_1, \ldots, E_q$ denote its simply degenerate ends. Let $S_j \subset \partial N$ denote the boundary component of N lying in the closure of the end $\mathcal{E}_j$, $j = 1, \ldots, p$, and let $T_k \subset \partial N$ denote the boundary component of N lying in the closure of the end E_k, where $k = 1, \ldots, q$. Then the space $MP_0(N)$ admits the parameterization

$$MP_0(N) = \mathrm{Teich}(\partial N) = \prod_{j=1}^{p} \mathrm{Teich}(S_j) \times \prod_{k=1}^{q} \mathrm{Teich}(T_k)$$

(see [Kr, Theorem 14], [T4, Theorem 1.3] or [Mc1]).

Let $\{Y_k(n)\}_{n=1}^{\infty}\subset\mathrm{Teich}(T_k)$ denote the sequences of surfaces obtained from Theorem 7.2 so that the limit

$$Q_k=\lim_{n\to\infty}Q(Y_k(0),Y_k(n))$$

realizes the end E_k, $k=1,...,p$. Let

$$h_k:\widehat{E}_k\longrightarrow E_k$$

be the marking-preserving bi-Lipschitz diffeomorphism from the positive end of Q_k to E_k coming from Theorem 7.2. Let X_j be the conformal boundary component compactifying $\mathcal{E}_j$ for $j=1,...,p$, and let

$$g_j:\widehat{\mathcal{E}}_j\longrightarrow\mathcal{E}_j$$

be the marking-preserving bi-Lipschitz diffeomorphism from the end of the Fuchsian manifold $Q(X_j,X_j)$.

In the above parameterization, we let M_n be determined by

$$M_n=(X_1,...,X_p,Y_1(n),...,Y_p(n))\in\mathrm{Teich}(\partial N).$$

Let $f_n:N\to M_n$ be the implicit homotopy equivalences marking the manifolds M_n. We claim that the sequence M_n converges up to subsequence in $AH(N)$.

Corollary 7.3 guarantees that $Y_k(n)$ may be chosen to converge to a limit $[\mu_k]\in\mathcal{PL}(S)$ with support $|\mu_k|=\nu_k$, where $\nu_k=\nu(E_k)$. Thus, there are simple closed curves γ_n on T_k and positive real weights t_n so that $t_n\gamma_n$ converges in $\mathcal{ML}(T_k)$ to μ_k, and the lengths $l_{Y_k(n)}(t_n\gamma_n)$ remain bounded. Applying a theorem of K. Ohshika generalizing Thurston's compactness theorem (see [Oh2, Theorem 2.4] and [T5]) we conclude that the sequence M_n converges after passing to a subsequence. We pass to a subsequence so that M_n converges algebraically to M_∞, and geometrically to a manifold M_G covered by M_∞ by a local isometry $\pi:M_\infty\to M_G$. Let $f_\infty:N\to M_\infty$ denote the marking on M_∞.

We will exhibit a compact core $\mathcal{M}_\infty$ for M_∞ so that any homotopy equivalence $f'_\infty:N\to\mathcal{M}_\infty$ homotopic to f_∞ is homotopic to a homeomorphism, and the ends $M_\infty\setminus\mathcal{M}_\infty$ are exactly the ends $\widehat{E}_k$ and $\widehat{\mathcal{E}}_j$, where $k=1,...,p$ and $j=1,...,p$.

First consider the covers $\mathcal{Q}_j(n)$ of M_n corresponding to $\pi_1(X_j)$. The manifolds $\mathcal{Q}_j(n)$ range in the Bers slice B_{X_j}, and the surface X_j persists as a conformal boundary component of the algebraic and geometric limits $\mathcal{Q}_j(\infty)$ and $\mathcal{Q}_j^G$ of $\{\mathcal{Q}_j(n)\}_{n=1}^{\infty}$ (after potentially passing to a further subsequence; see e.g. [KT, Proposition 2.3]).

The end $\mathcal{E}_j(n)$ of $\mathcal{Q}_j(n)$ cut off by the boundary of a smooth neighborhood of the convex-core boundary component facing X_j embeds in the covering projection to M_n.

The ends $\mathcal{E}_j(n)$ converge geometrically to the geometrically finite end $\mathcal{E}_j(\infty)$ of the geometric limit Q_j^G, and this end is compactified by X_j. It follows that there are marking-preserving smooth embeddings φ_j^n of $\mathcal{E}_j(\infty)$ into M_n that converge C^∞ to an isometric embedding $\varphi_j: \mathcal{E}_j(\infty) \to M_G$.

The isometric embedding φ_j is marking-preserving in that

$$(\varphi_j)_*|_{\pi_1(S_j)} = \pi_* \circ (f_\infty)_* \circ (\iota_j)_*,$$

where $\iota_j: S_j \to N$ is the inclusion map. Thus, the end $\mathcal{E}_j(\infty)$ lifts to an end $\bar{\mathcal{E}}_j$ of the algebraic limit M_∞.

Consider, on the other hand, covers $Q_k(n)$ of M_n corresponding to $Y_k(n)$. The manifolds $Q_k(n) \subset AH(S)$ converge algebraically to $Q_k(\infty)$, and there is a measured lamination μ_k so that $l_{Q_k(n)}(\mu_k) \to 0$. The measured lamination μ_k "fills" the surface T_k: for any essential simple closed curve γ on T_k, we have $i(\gamma, \mu_k) \neq 0$. Applying [Bro1, Theorem 2], the support $|\mu_k| = \nu_k$ is an ending lamination for $Q_k(\infty)$, which implies that $Q_k(\infty)$ has a degenerate end $E_k(\infty)$ with ending lamination ν_k. By an argument using the covering theorem and [JM, Lemma 3.6], the end $E_k(\infty)$ embeds in the geometric limit Q_k^G (see [AC2, Proposition 5.2]). Just as above, then, we have a limiting isometric embedding $\phi_k: E_k(\infty) \to M_\infty$ which is marking-preserving in the sense that

$$(\phi_k)_*|_{\pi_1(T_k)} = \pi_* \circ (f_\infty)_* \circ (i_k)_*,$$

where $i_k: T_k \to N$ is the inclusion map. Let $\overline{E}_k$ denote this end of M_∞.

By an application of Waldhausen's theorem (see [W] and [He, Theorem 13.7]), the homotopy equivalence $f_\infty: N \to M_\infty$ is homotopic to a homeomorphism to a compact core $\mathcal{M}_\infty$ for M_∞ that cuts off the geometrically finite ends $\bar{\mathcal{E}}_j$ and the simply degenerate ends $\overline{E}_k$.

We claim that the ends $\bar{\mathcal{E}}_j$ and $\overline{E}_k$ have no cusps. Since $\bar{\mathcal{E}}_j$ is a geometrically finite end compactified by the closed surface X_j, there is no isotopy class in $\bar{\mathcal{E}}_j$ with arbitrarily short length. Likewise, $\overline{E}_k$ is a degenerate end with ending lamination $\nu_k = |\mu_k|$ that fills the surface T_k. By results of Thurston and Bonahon (see [T1, Chapter 9] and [Bon1, Proposition 3.4]), if $\gamma \subset T_k$ is a simple closed curve represented by a cusp in $\overline{E}_k$ (γ has representatives with arbitrarily short representatives in the end $\overline{E}_k$) then we have

$$i(\gamma, \mu_k) = 0,$$

contradicting the fact that ν_k fills the surface.

It follows that M_∞ is cusp free, since any element $g \in \pi_1(N)$ for which $f_\infty(g)$ is parabolic must have arbitrarily short representatives in its free-homotopy class exiting some end $\bar{\mathcal{E}}_j$ or $\overline{E}_k$ of M_∞. By a theorem of Anderson and Canary [AC2, Corollary G] we may conclude that M_n converges strongly to M_∞.

It follows that each sequence of covers $\mathcal{Q}_j(n)$ or $Q_k(n)$ converges strongly to a limit on a Bers boundary; otherwise the limit would be doubly degenerate, which is ruled out once again by the covering theorem.

It follows that the geometric limit of the quasi-Fuchsian manifolds $\mathcal{Q}_j(n)$ or $Q_k(n)$ is uniformly bi-Lipschitz diffeomorphic to the quasi-Fuchsian manifolds (either $Q(X_j, X_j)$ or $Q(Y_k(0), Y_k(n)))$ whose limit realizes the corresponding ends $\mathcal{E}_j$ or E_k. Thus each end of M_∞ admits a marking-preserving bi-Lipschitz diffeomorphism to the end of the corresponding realization in a Bers boundary.

As these ends admit marking-preserving bi-Lipschitz diffeomorphisms to the ends of M, we may extend the corresponding bi-Lipschitz diffeomorphisms across the compact cores to obtain a single marking-preserving bi-Lipschitz diffeomorphism

$$\Psi: M_\infty \longrightarrow M.$$

Since the corresponding conformal structures compactifying the geometrically finite ends of M_∞ and M are the same, we may apply Sullivan's theorem [Su1] to conclude that Ψ is homotopic to an isometry, and we have

$$M = M_\infty = \lim_{n\to\infty} M_n.$$

As the manifolds M_n are geometrically finite, the proof is complete. □

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JEFFREY F. BROCK
Department of Mathematics
University of Chicago
5734 S. University Avenue
Chicago, IL 60637
U.S.A.

KENNETH W. BROMBERG
Department of Mathematics
California Institute of Technology, 253-37
Pasadena, CA 91125
U.S.A.

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