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Teichmüller geodesics of infinite complexity

by

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Contents

1. Introduction

Let $\mathcal{M}_g$ denote the moduli space of Riemann surfaces of genus g, and $\mathrm{Mod}_g \cong \pi_1(\mathcal{M}_g)$ the mapping-class group of genus g. Closed geodesics for the Teichmüller metric on $\mathcal{M}_g$ correspond to pseudo-Anosov elements $\phi \in \mathrm{Mod}_g$.

Let $f: \mathbf{R} \to \mathcal{M}_g$ be a Teichmüller geodesic whose image is a closed loop representing $\phi \in \pi_1(\mathcal{M}_g)$. The complexification of this geodesic yields a holomorphic map

$$F: \mathbf{H} \to \mathcal{M}_g$$

satisfying $f(s) = F(ie^{2s})$. The map F descends to the Riemann surface

$$V_\phi = \mathbf{H}/\Gamma_\phi, \quad \Gamma_\phi = \{A \in \mathrm{Aut}(\mathbf{H}) : F(Az) = F(z)\},$$

and the induced map $V_\phi \to \mathcal{M}_g$ is generically injective.

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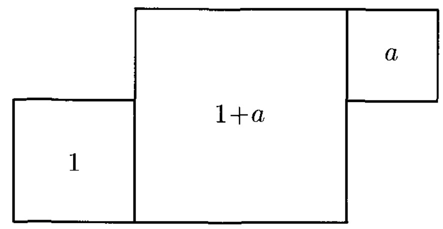

Fig. 1. The polygon $S(a)$ built from three squares.

Clearly Γ_ϕ contains the cyclic subgroup generated by the hyperbolic element A corresponding to ϕ. Accordingly, one might expect that V_ϕ is typically a cylinder, isomorphic to $\mathbf{H}/\langle A\rangle$.

Our main result shows that in fact, for genus $g=2$, the topology of V_ϕ is often much more complex.

THEOREM 1.1. *Let $\phi\in\mathrm{Mod}_2$ be a pseudo-Anosov mapping class with orientable foliations, and let*

$$V_\phi = \mathbf{H}/\Gamma_\phi \to \mathcal{M}_2$$

be the complexification of the Teichmüller geodesic representing $\phi\in\pi_1(\mathcal{M}_2)$. Then the limit set of Γ_ϕ is equal to $\partial\mathbf{H}$.

Since its limit set is $\partial\mathbf{H}$, Γ_ϕ is either a lattice or an infinitely generated group. Lattice examples are studied in [Mc]. Here we show:

THEOREM 1.2. *There exist infinitely many distinct complex geodesics $V_\phi\to\mathcal{M}_2$ with $\pi_1(V_\phi)$ infinitely generated.*

Holomorphic 1-forms. The results above admit a more precise formulation in terms of the moduli space of holomorphic 1-forms.

Let $\Omega\mathcal{M}_g\to\mathcal{M}_g$ be the bundle of pairs (X,ω), where $\omega\neq 0$ is a holomorphic 1-form on $X\in\mathcal{M}_g$. There is a natural action of $\mathrm{SL}_2(\mathbf{R})$ on $\Omega\mathcal{M}_g$ (§3) whose orbits project to complex geodesics in $\mathcal{M}_g$.

Let $\mathrm{SL}(X,\omega)\subset\mathrm{SL}_2(\mathbf{R})$ denote the stabilizer of (X,ω). In §10 we show:

THEOREM 1.3. *When X has genus two, the limit set of $\mathrm{SL}(X,\omega)$ is either the empty set, a singleton, or $\partial\mathbf{H}$.*

Now consider the polygon $S(a)\subset\mathbf{C}$ built from squares of side lengths 1, $1+a$ and a as shown in Figure 1. We also show:

THEOREM 1.4. *Let $b\in\mathbf{Q}_+$ be a positive rational such that $a=b-1+\sqrt{b^2-b+1}$ is irrational. Construct $(X,\omega)\in\Omega\mathcal{M}_2$ from $(S(a),dz)$ by identifying opposite sides. Then $\mathrm{SL}(X,\omega)$ is an infinitely generated group.*

The groups Γ_ϕ and $\mathrm{SL}(X,\omega)$ are related as follows. When the stable and unstable foliations of $\phi\in\mathrm{Mod}_g$ are orientable, they can be represented by the vertical and horizontal foliations of a holomorphic 1-form $(X,\omega)\in\Omega\mathcal{M}_g$. Then Γ_ϕ is conjugate to $\mathrm{SL}(X,\omega)$ in $\mathrm{Aut}(\mathbf{H})$, and the image of $V_\phi\subset\mathcal{M}_g$ is the same as the projection of the orbit

$$\mathrm{SL}_2(\mathbf{R})\cdot(X,\omega)\subset\Omega\mathcal{M}_g.$$

Conversely, provided it contains a hyperbolic element, $\mathrm{SL}(X,\omega)$ is conjugate to Γ_ϕ for some pseudo-Anosov element $\phi\in\mathrm{Mod}_g$.

Thus Theorems 1.1 and 1.2 follow from the two preceding results on $\mathrm{SL}(X,\omega)$.

Cusps of triangle groups. Many triangle groups occur as $\mathrm{SL}(X,\omega)$ for suitable (X,ω). By studying these groups from the perspective of complex geodesics, in the Appendix we show:

THEOREM 1.5. *Let Γ be a triangle group with signature*

$$(2,2n,\infty) \ \textit{or} \ (3,n,\infty), \quad \textit{where } n=4,5,6,$$

or

$$(2,n,\infty), \quad \textit{where } n=5.$$

Then the set of cross-ratios of the cusps of Γ coincides with $\mathbf{P}^1(K_n)-\{0,1,\infty\}$, where $K_n=\mathbf{Q}(\cos(\pi/n))$.

Note that $K_4=\mathbf{Q}(\sqrt{2})$, $K_5=\mathbf{Q}(\sqrt{5})$ and $K_6=\mathbf{Q}(\sqrt{3})$.

The cusp set was previously determined by Leutbecher for signatures $(2,n,\infty)$, $n=5,8,10,12$, and by Seibold for signatures $(3,n,\infty)$, $n=4,5,6$, using quite different methods [Le], [Se].

Dehn twists. Our results on $\mathrm{SL}(X,\omega)$ can be compared to a basic construction of pseudo-Anosov mappings from [Th, Theorem 7]. This construction starts with two systems of simple closed curves α and β filling a surface S. It yields a complex geodesic $\mathbf{H}\to\mathcal{M}_g$ stabilized by the subgroup $\langle T_\alpha,T_\beta\rangle\subset\mathrm{Mod}_g$ generated by Dehn twists on α and β.

When the complementary regions $S-(\alpha\cup\beta)$ are $4k$-gons, the complex geodesic comes from a holomorphic 1-form $(X,\omega)\in\Omega\mathcal{M}_g$, and the Dehn twists generate a subgroup of the form

$$\Gamma=\langle T_\alpha,T_\beta\rangle=\left\langle\begin{pmatrix}1&\mu\\0&1\end{pmatrix},\begin{pmatrix}1&0\\-\mu&1\end{pmatrix}\right\rangle\subset\mathrm{SL}(X,\omega)$$

for some $\mu>0$. The hyperbolic elements of Γ correspond to pseudo-Anosov elements $\phi\in\mathrm{Mod}_g$.

Typically one has $\mu>2$, in which case the limit set of Γ is a Cantor set. On the other hand, in genus two the limit set of $\mathrm{SL}(X,\omega)$ is always the full circle, by Theorem 1.1 above. So even in this explicit construction, $\mathrm{SL}(X,\omega)$ is generally much larger than its manifest subgroup $\langle T_\alpha, T_\beta\rangle$.

Galois flux. We conclude by sketching the proof of Theorem 1.3. The proof involves the dynamics of interval exchange transformations, the arithmetic of quadratic numbers, and the topology of measured foliations.

Let $K=\mathbf{Q}(\sqrt{d}\,)$ be a real quadratic field, and let $k\mapsto k'$ denote the Galois involution sending $\sqrt{d}$ to $-\sqrt{d}$.

(1) Let $f: I\to I$ be an interval exchange transformation, satisfying $f(x)=x+t_j$ on subintervals I_j forming a partition of I. Assuming that the translation lengths t_j lie in K, we define the *Galois flux* of f by

$$\mathrm{flux}(f)=\sum |I_j|\,t_j'.$$

The Galois flux measures the average growth rate of $(f^n(x))'$ for $x\in I\cap K$.

If f is uniquely ergodic, then $\mathrm{flux}(f)\neq 0$ (§2).

(2) Now consider $(X,\omega)\in\Omega\mathcal{M}_g$, and let $\mathcal{F}_\varrho$ be the measured foliation of X defined by $\varrho=\mathrm{Re}\,\omega$. Assume that ϱ has relative periods in K, meaning $\int_\gamma \varrho\in K$ for any path joining a pair of zeros of ϱ. Let ϱ' represent its Galois conjugate cohomology class. Then the flux of the first return map $f: I\to I$ for any full transversal to $\mathcal{F}_\varrho$ satisfies

$$\mathrm{flux}(f)=-\int_X \varrho\wedge\varrho'. \tag{1.1}$$

We therefore define the flux of ϱ by the expression above (§4).

(3) Let us restrict to the case where the flux of ϱ vanishes. Then $\mathcal{F}_\varrho$ cannot be uniquely ergodic.

By a result of Masur, the failure of unique ergodicity implies that the Teichmüller geodesic X_t generated by (X,ω) tends to infinity in $\mathcal{M}_g$. Therefore the length of the shortest closed geodesic on X_t is very small for $t\gg 0$. Using Diophantine properties of $\sqrt{d}$, we conclude in §6 that this short geodesic is isotopic to a loop L running along leaves of $\mathcal{F}_\varrho$.

(4) Next we specialize to the case of genus $g=2$. By analyzing the possible configurations for L, we establish the following dichotomy (§7):

Either $\mathcal{F}_\varrho$ is periodic (all its leaves are closed), or $(X,\mathcal{F}_\varrho)$ is the connected sum of a pair of tori with irrational foliations.

(5) Now impose the stronger condition that the *complex flux* vanishes, meaning that ω has relative periods in $K(i)$ and

$$\int \omega \wedge \omega' = \int \omega \wedge \bar{\omega}' = 0.$$

Then for any $s \in \mathbf{P}^1(K)$, the foliation $\mathcal{F}$ of $(X, |\omega|)$ by geodesics of slope s satisfies the dichotomy above.

Suppose that s is the slope of a geodesic joining a Weierstrass point of X to a zero of ω. Then the possibilities for the foliation are more limited:

$\mathcal{F}$ is periodic, with two cylinders whose moduli have rational ratio.

A suitable product of Dehn twists in these cylinders yields a parabolic element $A \in \mathrm{SL}(X, \omega)$ fixing $1/s$. In particular, $1/s$ belongs to the limit set. Since slopes s as above are dense in $\mathbf{P}^1(\mathbf{R})$, we conclude (§8):

The limit set of $\mathrm{SL}(X, \omega)$ *is* $\partial \mathbf{H}$.

(6) Finally we turn to the proof of Theorem 1.3. Consider $(X, \omega) \in \Omega \mathcal{M}_2$ such that the limit set of $\mathrm{SL}(X, \omega)$ contains two or more points. Then there is a hyperbolic element $A \in \mathrm{SL}(X, \omega)$.

If $K = \mathbf{Q}(\mathrm{tr}(A)) \cong \mathbf{Q}$, then (X, ω) arises via a branched covering of a torus, and $\mathrm{SL}(X, \omega)$ is commensurable to $\mathrm{SL}_2(\mathbf{Z})$. In particular, its limit set is $\partial \mathbf{H}$.

Otherwise, K is a real quadratic field. Replacing (X, ω) with $B \cdot (X, \omega)$ for suitable $B \in \mathrm{SL}_2(\mathbf{R})$, we may assume that ω has relative periods in $K(i)$ and zero complex flux (§9); then the limit set is $\partial \mathbf{H}$ by the analysis above.

Notes and references. The topological theory of measured foliations, and its central role in Thurston's classification of surface diffeomorphisms, is presented in [FLP] and [Th]. Foliations which need not be transversally orientable are represented by holomorphic quadratic differentials [HM]. For more details from the complex-analytic perspective, see [Be], [St] and [Ga].

The action of $\mathrm{SL}_2(\mathbf{R})$ on $\Omega \mathcal{M}_g$ and on spaces of quadratic differentials is discussed in [Mas1], [V2] and [Ko]. The group $\mathrm{SL}(X, \omega)$ has been much studied, especially in connection with polygonal billiards; see, for example, [V3], [V4], [Wa], [KS], [Vo], [GJ], [Mc] and [MT].

The Galois flux can be considered as a projection of the Sah–Arnoux–Fathi invariant of an interval exchange transformation, defined by $\sum |I_j| \otimes t_j \in \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}$ [Ar], [V1]. The SAF-invariant is in turn related to the Kenyon–Smillie invariant of a translation surface, $J(X, \omega) \in \mathbf{R}^2 \wedge_{\mathbf{Q}} \mathbf{R}^2$, in a manner similar to (1.1) above [KS, §4]. For more on interval exchange transformations, see [Ke], [CFS, Chapter 5] and [Man, II.4].

Hubert and Schmidt have recently constructed examples of genus $g\geqslant 4$ where $\mathrm{SL}(X,\omega)$ is infinitely generated [HS]. See also [Ca], in which Boshernitzan, Calta and Eskin announce results related to Corollary 4.2 and Theorem A.1 below.

This paper is a sequel to [Mc], which provides additional background on Teichmüller geodesics. I would like to thank H. Masur, J. Smillie and the referees for many useful suggestions.

2. Galois flux and interval dynamics

Interval exchange transformations are among the simplest measure-preserving dynamical systems. When $f: I\to I$ is algebraic, the behavior of its Galois conjugates can shed light on its dynamics over $\mathbf{R}$. In this section we show:

THEOREM 2.1. *If $f: I\to I$ is a uniquely ergodic interval exchange transformation defined over a real quadratic field K, then its Galois flux is nonzero.*

Quadratic fields. Throughout this paper, $K\subset\mathbf{R}$ denotes a real quadratic field with a fixed embedding in $\mathbf{R}$. There is a unique square-free integer $d>0$ such that $K=\mathbf{Q}(\sqrt{d})$. We denote the Galois conjugate of $k=u+v\sqrt{d}$ by $k'=u-v\sqrt{d}$.

Interval exchange transformations. Let $I=[a,b)\subset\mathbf{R}$. A bijective mapping $f: I\to I$ is an *interval exchange transformation* if there is a finite partition

$$I=\bigcup I_j=\bigcup\,[a_j,b_j)$$

and a sequence $t_j\in\mathbf{R}$ such that

$$f(x)=x+t_j \quad \text{for all } x\in I_j. \tag{2.1}$$

In other words, f cuts I into subintervals and rearranges them in a new order.

Clearly f preserves Lebesgue measure μ on I. We say that f is *uniquely ergodic* if every f-invariant Borel measure on I is proportional to μ. Unique ergodicity is equivalent to the condition that every orbit of x is equidistributed, meaning that

$$\frac{1}{N}\sum_{0}^{N-1}\phi(f^n(x))\to\frac{1}{|I|}\int_I\phi(x)\,dx$$

for all bounded piecewise-continuous functions $\phi: I\to\mathbf{R}$ and all $x\in I$.

Galois flux. An interval exchange transformation f is *defined over K* if its translation lengths t_j all lie in K. In this case we define its *Galois flux* by

$$\operatorname{flux}(f)=\sum |I_j|\,t_j'.$$

Here are some basic properties of the flux.

- We have $\mathrm{flux}(f^n)=n\,\mathrm{flux}(f)$. Thus $\mathrm{flux}(f)=0$ if f is periodic.
- Let $g(x)=TfT^{-1}(x)$, where $T(x)=ax+b$, $a\in K$. Then we have

$$\mathrm{flux}(g)=N(a)\,\mathrm{flux}(f),$$

where $N(a)=aa'\in\mathbf{Q}$ is the norm of a. In particular, the flux is invariant under conjugation by translations.

- The interval exchange transformation

$$f(x)=\begin{cases} x+t & \text{for } x\in[0,1-t), \\ x+t-1 & \text{for } x\in[1-t,1) \end{cases}$$

models a rotation of the unit circle by $t\in(0,1)$. The map f is periodic when $t\in\mathbf{Q}$, and uniquely ergodic otherwise.

For $t\in K$ we have $\mathrm{flux}(f)=t'-t$. Thus the flux of a rotation vanishes if and only if f is periodic.

Proof of Theorem 2.1. Recall that the map $t\mapsto(t,t')$ sends the ring of integers $\mathcal{O}_K\subset K$ to a lattice in $\mathbf{R}^2$. Let Λ be the intersection of this lattice with $I\times\mathbf{R}$. Since $\mathcal{O}_K$ is dense in $\mathbf{R}$, $\Lambda\neq\varnothing$. Note that the number of lattice points in a rectangle satisfies

$$|\Lambda\cap I\times[-M,M]|=O(M). \tag{2.2}$$

After rescaling f, we can assume that its translation lengths satisfy $t_j\in\mathcal{O}_K$. Define $F\colon I\times\mathbf{R}\to I\times\mathbf{R}$ by

$$F(x,y)=(x+t_j,y+t_j') \quad \text{if } x\in I_j.$$

Then F preserves Λ. Moreover, the orbit $(x_n,y_n)=F^n(x_0,y_0)$ of a point in Λ satisfies $x_n=f^n(x_0)$ and

$$y_n=y_0+\sum_0^{n-1}\phi(x_j),$$

where $\phi(x)=t_j'$ for $x\in I_j$. Since f is uniquely ergodic, we have

$$\frac{y_n}{n}=\frac{y_0}{n}+\frac{1}{n}\sum_0^{n-1}\phi(x_j)\to\frac{1}{|I|}\int_I\phi(x)\,dx=\frac{1}{|I|}\sum|I_j|t_j'=\frac{\mathrm{flux}(f)}{|I|}$$

as $n\to\infty$.

Now suppose that the flux of f is zero. Then $y_n/n\to0$, so $|y_n|=o(n)$. Since the set $S_n=\{(x_0,y_0),\dots,(x_n,y_n)\}$ lies in Λ, by (2.2) above we have $|S_n|=o(n)$ as well. Therefore

$(x_i, y_i) = (x_j, y_j)$ for some $i < j$, and thus f has a periodic point, contradicting unique ergodicity. □

Growth rates. The same argument shows that the flux measures the rate of growth of Galois conjugates: for any $x \in K \cap I$, we have

$$\lim \frac{f^n(x)'}{n} = \frac{\mathrm{flux}(f)}{|I|} \tag{2.3}$$

when f is uniquely ergodic.

Periodicity. A similar theorem holds in the presence of periodic points.

THEOREM 2.2. *If f is defined over K and uniquely ergodic on its aperiodic points, then* $\mathrm{flux}(f) \neq 0$.

Proof. Let $I = P \cup A$ be the partition of I into periodic and aperiodic points. One can check that P and A are finite unions of intervals permuted by f. (Use the fact that a maximal periodic interval contains the orbit of a discontinuity of f among its endpoints.) Since $f|A$ is uniquely ergodic, the intervals $A_1, \ldots, A_n$ comprising A are permuted cyclically, and $f^n|A_i$ is uniquely ergodic for each i. Using the fact that a periodic map has zero flux, we then find:

$$\mathrm{flux}(f) = \mathrm{flux}(f^n|A_1) \neq 0$$

by the preceding theorem. □

Notes. The Galois flux can be defined for arbitrary number fields K by taking $t' \in \mathbf{R}^{r-1} \times \mathbf{C}^s$ to be the vector of values of t at the other infinite places of K. As in the quadratic case, (2.3) holds when f is uniquely ergodic.

3. Riemann surfaces and measured foliations

This section summarizes background on measured foliations from the complex perspective.

Geometry of holomorphic 1-forms. Let ω be a holomorphic 1-form on a compact Riemann surface X of genus g. We denote the space of all such 1-forms by $\Omega(X)$. Assume $\omega \neq 0$, and let $Z(\omega) \subset X$ be its zero-set. We have $|Z(\omega)| \leqslant 2g-2$.

The form ω determines a conformal metric $|\omega|$ on X, with negative curvature concentrated on $Z(\omega)$, and otherwise flat. Any two points of $(X, |\omega|)$ are joined by a unique geodesic. A geodesic is *straight* if its interior is disjoint from $Z(\omega)$. Since a straight geodesic does not change direction, its length satisfies $\int_\gamma |\omega| = |\int_\gamma \omega|$.

Saddle connections. A *saddle connection* is a straight geodesic (of positive length) that begins and ends at a zero of ω (a saddle). When X has genus $g \geqslant 2$, every essential loop on X is homotopic to a closed geodesic that is a finite union of saddle connections.

Affine structure. From ω we also obtain a branched complex affine structure on X, with local charts $\phi: U \to \mathbf{C}$ satisfying $d\phi = \omega$. These complex charts are well-defined up to translation, injective away from the zeros of ω, and of the form $\phi(z) = z^{p+1}$ near a zero of order p.

Foliations. The harmonic form $\varrho = \operatorname{Re} \omega$ determines a *measured foliation* $\mathcal{F}_\varrho$ on X.

Two points $x, y \in X$ lie on the same leaf of $\mathcal{F}_\varrho$ if and only if they are joined by a path satisfying $\varrho(\gamma'(t)) = 0$. The leaves are locally smooth 1-manifolds tangent to $\operatorname{Ker} \varrho$, coming together in groups of $2p$ at the zeros of ω of order p. The leaves are oriented by the condition $\operatorname{Im} \varrho > 0$. In a complex affine chart, $\varrho = dx$ and the leaves of $\mathcal{F}_\varrho$ are vertical lines in $\mathbf{C}$.

A *transversal* $\tau: I \to X$ for $\mathcal{F}_\varrho$ is a smooth embedding of an interval into X such that $\tau^* \varrho$ is everywhere positive on I. A *transverse measure* μ for $\mathcal{F}_\varrho$ is a measure on its space of leaves. Formally, μ is the assignment of a finite Borel measure μ_τ on I to each transversal $\tau: I \to X$, such that the measures are consistent under restriction, reparameterization and isotopy along leaves. It is conventional to require that μ_τ be nonatomic, so any individual leaf has measure zero.

The foliation $\mathcal{F}_\varrho$ comes equipped with the transverse measure defined by $\mu_\tau = \tau^* \varrho$.

Unique ergodicity. If ϱ is the only transverse measure for $\mathcal{F}_\varrho$, up to scale, then $\mathcal{F}_\varrho$ is *uniquely ergodic.*

Slopes. Straight geodesics on $(X, |\omega|)$ become straight lines in $\mathbf{C}$ in any complex affine chart determined by ω. Thus $\mathcal{F}_\varrho$ can alternatively be described as the foliation of X by parallel geodesics of constant slope ∞. Similarly, $\mathcal{F}_{\operatorname{Re}(x+iy)\omega}$ gives the foliation of X by geodesics of slope x/y.

The spine. The union of all saddle connections running along leaves of $\mathcal{F}_\varrho$ is the *spine* of the foliation. The spine is a finite graph embedded in X.

Cylinders. A *cylinder* $A \subset X$ is a maximal open region swept out by circular leaves of $\mathcal{F}_\varrho$. The subsurface $(A, |\omega|)$ is isometric to a right circular cylinder of height $h(A)$ and circumference $c(A)$; its *modulus* $\operatorname{mod}(A) = h(A)/c(A)$ is a conformal invariant. Provided X is not a torus, ∂A is a union of saddle connections.

Periodicity. The foliation $\mathcal{F}_\varrho$ is *periodic* if all its leaves are compact. In this case, either X is a torus foliated by circles, or the complement of the spine of $\mathcal{F}_\varrho$ in X is a finite union of cylinders $A_1, \ldots, A_n$.

Action of $\mathrm{GL}_2^+(\mathbf{R})$. Let $\mathcal{M}_g$ denote the moduli space of compact Riemann surfaces of genus g. The space of all pairs (X,ω), consisting of a Riemann surface of genus g equipped with a nonzero holomorphic 1-form, forms a punctured vector bundle $\Omega\mathcal{M}_g\to\mathcal{M}_g$.

Let $\mathrm{GL}_2^+(\mathbf{R})$ denote the group of automorphisms of $\mathbf{R}^2$ with $\det(A)>0$. There is a natural action of $\mathrm{GL}_2^+(\mathbf{R})$ on $\Omega\mathcal{M}_g$. To define $A\cdot(X,\omega)$ for

$$A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\mathrm{GL}_2^+(\mathbf{R}),$$

consider the harmonic 1-form

$$\omega_A=(1\quad i)\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\operatorname{Re}\omega \\ \operatorname{Im}\omega\end{pmatrix} \tag{3.1}$$

on X. There is a unique complex structure with respect to which ω_A is holomorphic; its charts yield a new Riemann surface X_A, and we define $A\cdot(X,\omega)=(X_A,\omega_A)$; cf. [Mc, §3].

It is often convenient to regard ω_A as a form on X. Note that

$$\int_C \omega_A = A\left(\int_C\omega\right) \tag{3.2}$$

for any 1-cycle C on X, where A acts $\mathbf{R}$-linear on $\mathbf{C}$.

Teichmüller geodesics. Restricting to elements of the diagonal subgroup

$$\left\{G_t=\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}: t\in\mathbf{R}\right\},$$

we obtain a path $(X_t,\omega_t)=G_t\cdot(X,\omega)$ whose projection to $\mathcal{M}_g$ is the *Teichmüller geodesic* generated by (X,ω).

The geodesics on X for the metric $|\omega_t|$ are the same as those for $|\omega|$ (although their lengths may be different).

Real-affine maps and $\mathrm{SL}(X,\omega)$. The stabilizer of $(X,\omega)\in\Omega\mathcal{M}_g$ is the discrete group $\mathrm{SL}(X,\omega)\subset\mathrm{SL}_2(\mathbf{R})$.

Here is an intrinsic definition of $\mathrm{SL}(X,\omega)$. A map $\phi: X\to X$ is *real-affine* with respect to ω if, after passing to the universal cover, there is an $A\in\mathrm{GL}_2(\mathbf{R})$ and $b\in\mathbf{R}$ such that the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\tilde{\phi}"] \arrow[d, "I_\omega"'] & \widetilde{X} \arrow[d, "I_\omega"] \\ \mathbf{C} \arrow[r, "Av+b"] & \mathbf{C} \end{tikzcd}$$

commutes. Here $I_\omega(q)=\int_p^q\omega$ is obtained by integrating the lift of ω.

We denote the linear part of ϕ by $D\phi = A \in \mathrm{SL}_2(\mathbf{R})$. Then $\mathrm{SL}(X,\omega)$ is the image of the group $\mathrm{Aff}^+(X,\omega)$ of orientation-preserving real-affine automorphisms of (X,ω) under $\phi \mapsto D\phi$.

Periods. We define the (absolute) periods of a closed 1-form α on X by

$$\mathrm{Per}(\alpha) = \left\{ \int_C \alpha : C \in H_1(X;\mathbf{Z}) \right\} \subset \mathbf{C}.$$

Thus α has periods in a field $K \subset \mathbf{C}$ if and only if it represents a cohomology class in $H^1(X;K) \subset H^1(X;\mathbf{C})$.

Similarly, the *relative periods* of α are the values of $\int_C \alpha$ as C ranges over all 1-cycles with boundaries in the zero-set $Z(\alpha)$. The form α has relative periods in K if and only if it represents a cohomology class in $H^1(X, Z(\alpha); K)$.

Any nonzero form $\omega \in \Omega(X)$ has a pair of periods $p_1, p_2 \in \mathbf{C}$ that form a basis for $\mathbf{C}$ over $\mathbf{R}$; otherwise X would carry a harmonic form with trivial periods, and hence a nonconstant harmonic function.

4. Flux and foliations

In this section we apply the Galois flux to the study of measured foliations.

First return maps. Let $\mathcal{F}_\varrho$ be a measured foliation of a compact Riemann surface X, determined by $\varrho = \mathrm{Re}\,\omega$, $\omega \in \Omega(X)$.

A transversal $\tau: I \to X$ for $\mathcal{F}_\varrho$ is *normalized* if $\tau^*\varrho = dx$. Any transversal can be normalized by a change of coordinates in its domain.

A normalized transversal $\tau: I \to X$ is *full* if it crosses every leaf of X. For such a transversal the *first return map*

$$f: I \to I$$

is defined so that $\tau(f(x))$ is the first point where the positively directed leaf through $\tau(x)$ again crosses $\tau(I)$.

Since ϱ is a transverse invariant measure for $\mathcal{F}_\varrho$ and $\tau^*\varrho = dx$, the map f preserves linear measure. Moreover, f is discontinuous only when a leaf runs into a zero or an endpoint of $\tau(I)$. Thus with a suitable convention for such leaves, f is an interval exchange transformation. Compare [Man, p. 119], [St, IV.12.4].

Flux. The translation lengths of f are periods of ϱ. In fact we have

$$x - f(x) = \int_{f(x)}^{x} \tau^*\varrho = \int_\gamma \varrho, \tag{4.1}$$

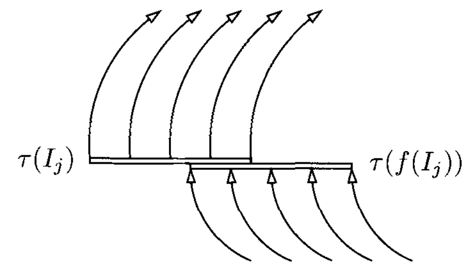

Fig. 2. Leaves of $\mathcal{F}_\varrho$ and the first return map.

where γ is a loop that runs from $\tau(x)$ to $\tau(f(x))$ along a leaf of $\mathcal{F}_\varrho$, and then returns along $\tau(I)$ to $\tau(x)$.

If the periods of ϱ lie in a real quadratic field $K\subset\mathbf{R}$—equivalently, if we have

$$[\varrho]\in H^1(X;K)\subset H^1(X;\mathbf{R})$$

on the level of cohomology—then the interval exchange $f: I\to I$ is also defined over K. In addition, $[\varrho]$ determines a unique Galois conjugate cohomology class $[\varrho']\in H^1(X;K)$, characterized by $\int_\gamma \varrho'=(\int_\gamma \varrho)'$ for all loops γ. We use ϱ' to denote any representative of the cohomology class Galois conjugate to $[\varrho]$.

The main result of this section is a formula for the flux of f.

THEOREM 4.1. *Let $\mathcal{F}_\varrho$ be a measured foliation with periods in a real quadratic field K. Then the first return map $f: I\to I$ for any full transversal $\tau: I\to X$ is defined over K, and we have*

$$\operatorname{flux}(f)=-\int_X \varrho\wedge\varrho'.$$

In particular, the flux is the same for all full transversals.

Proof. Let $I=\bigcup I_j=\bigcup [a_j,b_j)$ be a partition of I into intervals on which $f(x)=x+t_j$. Let $\gamma_j: S^1\to X$ be a smooth loop that starts at $\tau(a_j)$, runs positively along a leaf to $\tau(f(a_j))$ and then returns along $\tau(I)$ to $\tau(a_j)$. Define $C_j: I_j\times S^1\to X$ by

$$C_j(x,y)=\gamma_j(y)+x-a_j,$$

where the sum above is defined using the local complex affine structure determined by ω. The map C_j sweeps out a singular annulus on X, connecting $\tau(I_j)$ to $\tau(f(I_j))$ along the leaves of $\mathcal{F}_\varrho$ (see Figure 2). The 2-cycle $\sum C_j$ represents the fundamental class of X.

By (4.1) we have $\int_{S^1}\gamma_j^*\varrho'=-t_j'$. Since the loops $C_j(x,\cdot\,): S^1\to X$ are homologous for all $x\in I_j$, the period of ϱ' around each one is $-t_j$. Therefore we have

$$\int_X \varrho\wedge\varrho' = \sum_j \int_{I_j\times S^1} C_j^*(\varrho\wedge\varrho') = \sum_j \int_{I_j\times S^1} dx\wedge C_j^*(\varrho')$$
$$= \sum_j |I_j| \int_{S^1} \gamma_j^*(\varrho') = -\sum |I_j|\, t_j' = \mathrm{flux}(f). \qquad \square$$

Flux of a foliation. Motivated by the result above, we define the flux of a real-valued 1-form ϱ with periods in K by

$$\mathrm{flux}(\varrho) = -\int_X \varrho\wedge\varrho'.$$

COROLLARY 4.2. *If $\mathcal{F}_\varrho$ is uniquely ergodic, then* $\mathrm{flux}(\varrho)\neq 0$.

Proof. A measured foliation is uniquely ergodic if and only if its first return map is, so this follows from Theorem 2.1. $\square$

Cylinders for $\mathcal{F}_\varrho$ correspond to periodic intervals for the first return map. Thus Theorem 2.2 implies:

COROLLARY 4.3. *Let $P\subset X$ be a finite union of cylinders for $\mathcal{F}_\varrho$. If the restriction of the foliation $\mathcal{F}_\varrho$ to $X\setminus\bar{P}$ is uniquely ergodic, then* $\mathrm{flux}(\varrho)\neq 0$.

Complex flux. Let $K(i)\subset\mathbf{C}$ be the extension of K by $\sqrt{-1}$. Elements of $K(i)$ have the form $k=k_1+ik_2$, $k_1,k_2\in K$. Let $(k_1+ik_2)'=k_1'+ik_2'$ and $\overline{k_1+ik_2}=k_1-ik_2$. These involutions generate the Galois group $\mathbf{Z}/2\times\mathbf{Z}/2$ of $K(i)/\mathbf{Q}$.

Now suppose that $\omega\in\Omega(X)$ has periods in $K(i)$, and that its Galois conjugate $[\omega']\in H^1(X,K(i))$ satisfies

$$\int \omega\wedge\omega' = 0.$$

We then define the *complex flux* of ω by

$$\mathrm{Flux}(\omega) = -\int_X \omega\wedge\bar{\omega}'.$$

Note that $\mathrm{Flux}(k\omega)=k\bar{k}'\,\mathrm{Flux}(\omega)$ for any $k\in K(i)$.

We emphasize that, whenever we consider the complex flux, we assume that $\int\omega\wedge\omega'=0$. This condition holds, for example, if $[\omega']\in H^1(X,K(i))$ is represented by a holomorphic 1-form.

Flux at varying slopes. The flux of $\varrho=\mathrm{Re}\,\omega$ satisfies

$$\mathrm{flux}(\varrho) = -\frac{1}{4}\int_X (\omega+\bar{\omega})\wedge(\omega'+\bar{\omega}') = \frac{1}{2}\,\mathrm{Re}\,\mathrm{flux}(\omega). \tag{4.2}$$

Here we have used the condition $\int \omega\wedge\omega'=0$ to eliminate half of the cross-terms.

In particular, if the complex flux of ω vanishes, so does the real flux of ϱ. More generally we have:

THEOREM 4.4. *Let ω have periods in $K(i)$ and zero complex flux. Let $\mathcal{F}_\varrho$ be the foliation of X by geodesics of slope k, where*

$$k=k_1/k_2\in\mathbf{P}^1,\quad k_1,k_2\in K,\quad \textit{and}\quad \varrho=\operatorname{Re}(k_1+ik_2)\omega.$$

Then $\mathcal{F}_\varrho$ has periods in K, and $\operatorname{flux}(\varrho)=0$.

Periodic foliations. The next result sheds light on the meaning of the imaginary part of the complex flux. As in §3 we let $h(A)$ and $c(A)$ denote the height and circumference of a cylinder A in the $|\omega|$-metric.

THEOREM 4.5. *Consider $(X,\omega)\in\Omega\mathcal{M}_g$, $g\geqslant 2$, such that ω has periods in $K(i)$ and $\int\omega\wedge\omega'=0$. Suppose that the foliation $\mathcal{F}_\varrho$ determined by $\varrho=\operatorname{Re}\omega$ is periodic, with cylinders $(A_i)_1^n$. Then we have*

$$\sum_1^n h(A_i)\,c(A_i)'=\frac{1}{2}\operatorname{Im}\operatorname{Flux}(\omega). \tag{4.3}$$

Proof. Note that $c(A_i)=\int_L \operatorname{Im}\omega$ for any positively oriented closed leaf L contained in A_i. In particular, we have $c(A_i)\in K$ since it is a period of $\operatorname{Im}\omega$. Since A_i is swept out by parallels of L, we have $\int_{A_i}\operatorname{Re}\omega\wedge\operatorname{Im}\omega'=h(A_i)c(A_i)'$. Thus we find

$$\begin{aligned}\sum_1^n h(A_i)\,c(A_i)'&=\int_X \operatorname{Re}\omega\wedge\operatorname{Im}\omega'=\frac{1}{4i}\int_X(\omega+\bar\omega)\wedge(\omega'-\bar\omega')\\&=\frac{1}{4i}\int_X(\bar\omega\wedge\omega'-\omega\wedge\bar\omega')=\frac{1}{2}\operatorname{Im}\operatorname{Flux}(\omega),\end{aligned}$$

using the fact that $\bigcup A_i$ has full measure in X. □

Recall that the norm $N(k)=kk'$ is a rational number for any $k\in K$, and $h(A)/c(A)=\operatorname{mod}(A)$. Thus equation (4.3) yields:

COROLLARY 4.6. *Suppose that the complex flux of ω vanishes and $\mathcal{F}_{\operatorname{Re}\omega}$ is periodic. Then the moduli of its cylinders satisfy the rational linear relation*

$$\sum_1^n \operatorname{mod}(A_i)\,N(c(A_i))=0.$$

In particular, we have $n\geqslant 2$, and $\operatorname{mod}(A_1)/\operatorname{mod}(A_2)\in\mathbf{Q}$ if $n=2$.

5. Saddle connections

In this section we use the Diophantine properties of square roots to study short saddle connections along a Teichmüller geodesic.

Recall that any $(X,\omega)\in\Omega\mathcal{M}_g$ determines a path $(X_t,\omega_t)\in\Omega\mathcal{M}_g$ such that X_t moves along a Teichmüller geodesic. Let $\mathcal{F}_\varrho$ be the foliation determined by $\varrho=\operatorname{Re}\omega$. As t increases, the leaves of $\mathcal{F}_\varrho$ are contracted exponentially fast; that is, we have $\int_\gamma|\omega_t|=e^{-t}\int_\gamma|\omega|$ for any arc γ contained in a leaf.

We will show:

THEOREM 5.1. *Suppose that ϱ has relative periods in a real quadratic field K, where $\varrho=\operatorname{Re}\omega$ for $(X,\omega)\in\Omega\mathcal{M}_g$. Then there exists an $r>0$ such that for all $t\geqslant 0$ and all saddle connections γ,*

$$\int_\gamma|\omega_t|<r \quad\Rightarrow\quad \gamma \textit{ lies in a leaf of } \mathcal{F}_\varrho.$$

Proof. We begin with a definition: a *rectangle* $R\subset X$ is a region that maps homeomorphically to $[0,a]\times[0,b]$ under a suitable affine chart $\phi\colon R\to\mathbf{C}$ with $d\phi=\omega$. It is straightforward to show that X can be tiled by a finite number of rectangles. The zeros of ω are included in the rectangles' vertices. We allow a vertex of one rectangle to lie on an edge of another.

The boundaries of the rectangles, taken together, give a finite graph $G\subset X$. Each edge E of G is either horizontal or vertical; that is, writing

$$\int_E\omega=x(E)+iy(E),$$

we have $x(E)=0$ or $y(E)=0$.

Let $K=\mathbf{Q}(\sqrt{d}\,)$ where $d>0$ is a square-free integer. Since $\operatorname{Re}\omega$ has relative periods in K, we can choose the tiles so that $x(E)\in K$ for all edges E of G. In fact, after rescaling ω, we can assume that $x(E)=a(E)+b(E)\sqrt{d}$ with $a(E),b(E)\in\mathbf{Z}$.

Consider a saddle connection γ on (X,ω), with $\int_\gamma\omega=x_0+iy_0$. Letting $(x_t,y_t)=(e^tx_0,e^{-t}y_0)$, we have

$$\int_\gamma|\omega_t|=|x_t+iy_t|.$$

We may assume $x_0\neq 0$, since otherwise γ lies along a leaf of $\mathcal{F}_\varrho$.

To complete the proof, we must find a uniform lower bound for $\int_\gamma|\omega_t|$. Note that there is a lower bound $s>0$ to the length of all saddle connections on $(X,|\omega|)$. Thus we may assume $|y_0|>|x_0|$, since otherwise we have

$$\int_\gamma|\omega_t|\geqslant\int_\gamma|\omega|\geqslant s>0$$

for $t>0$.

The path γ is homotopic, *rel* its endpoints, to a chain of edges $E_1, ..., E_n$ joining a pair of vertices of G. Choose this chain so that n is minimal. Then the number of edges is controlled by the length $|x_0+iy_0|$ of γ; since $|x_0|<|y_0|$, we have $n=O(|y_0|)$. Integrating $\operatorname{Re}\omega$ along this chain, we obtain

$$x_0 = \sum_1^n x(E_i) = \sum_I a(E_i)+b(E_i)\sqrt{d} = a+b\sqrt{d}\,.$$

Since G has only a finite number of edges, $a(E_i)$ and $b(E_i)$ are $O(1)$, and thus a and b are $O(n)=O(|y_0|)$. We may assume $b\neq 0$, since otherwise we have $|x_t+iy_t|\geqslant|x_t|\geqslant|x_0|=|a|\geqslant 1$ (since $a\in\mathbf{Z}$).

Now recall that square roots are poorly approximated by rationals: there exists a $C_d>0$ such that for all $a/b\in\mathbf{Q}$ we have

$$\left|\frac{a}{b}+\sqrt{d}\right| > \frac{C_d}{b^2}$$

[HW, §11.7]. Therefore we have

$$0<C_d<\left|a+b\sqrt{d}\right||b| = O(|x_0y_0|);$$

in other words, $|x_0y_0|>r^2$ for some $r>0$ independent of γ. It follows that

$$\int_\gamma |\omega_t| = |x_t+iy_t| \geqslant \sqrt{2|x_ty_t|} = \sqrt{2|x_0y_0|} > r > 0,$$

as desired. □

6. Loops in the spine

In this section we show that arithmetic properties of ϱ imply topological properties of $\mathcal{F}_\varrho$.

Recall that the union of all the saddle connections running along leaves of $\mathcal{F}_\varrho$ is the *spine* of the foliation. We will establish:

THEOREM 6.1. *Let $\mathcal{F}_\varrho$ be a measured foliation of a compact Riemann surface X of genus $g\geqslant 2$. Suppose that ϱ has relative periods in a real quadratic field K and zero flux. Then there is a closed loop in the spine of the foliation.*

Recurrence. A Teichmüller geodesic ray $(X_t: t\geqslant 0)$ is *recurrent* if there is a compact set $B\subset\mathcal{M}_g$ such that $X_t\in B$ for arbitrarily large values of t; otherwise it is *divergent*.

Let $L(X,g)$ denote the length of the shortest closed geodesic on X in the metric g. Then a geodesic is divergent if and only if $L(X_t,g_t)\to 0$ for the hyperbolic metrics g_t on X_t [Mu].

The proof of Theorem 6.1 will use the following result from [Mas2]:

THEOREM 6.2 (Masur). *If the Teichmüller geodesic ray generated by* $(X,\omega)\in\Omega\mathcal{M}_g$ *is recurrent, then the foliation* $\mathcal{F}_{\mathrm{Re}\,\omega}$ *is uniquely ergodic.*

Proof of Theorem 6.1. Let $\varrho=\mathrm{Re}\,\omega$ and let X_t be the Teichmüller geodesic ray generated by (X,ω). Since ϱ has zero flux, $\mathcal{F}_\varrho$ is not uniquely ergodic (Corollary 4.2); hence X_t is divergent and $l_t=L(X_t,g_t)\to 0$.

We claim that $L(X_t,|\omega_t|)$ tends to zero as well. Indeed, once l_t is small, a collar around the short geodesic provides an essential annulus $A_t\subset X_t$ with $1/\mathrm{mod}(A_t)=O(l_t)$. Now the modulus can be defined in terms of extremal length by

$$\mathrm{mod}(A_t)^{-1}=\sup_\sigma \frac{\inf_\gamma\left(\int_\gamma \sigma\right)^2}{\int_{A_t}\sigma^2},$$

where σ ranges over all conformal metrics on X_t, and γ ranges over all essential simple closed curves in A_t. Taking $\sigma=|\omega_t|$ we find

$$L(X_t,|\omega_t|)^2=O\left(\mathrm{mod}(A_t)^{-1}\int_A|\omega_t|^2\right)=O\left(l_t\int_X|\omega|^2\right),$$

and thus $L(X_t,|\omega_t|^2)\to 0$ as $t\to\infty$.

By Theorem 5.1, there is an $r>0$ such that any saddle connection of length less than r on $(X_t,|\omega_t|)$ lies in the spine. Choose t such that $L(X_t,|\omega_t|^2)<r$, and let γ be a closed geodesic with $\int_\gamma|\omega_t|<r$. Since $g\geqslant 2$ we may assume that γ is a chain of saddle connections. (Any closed geodesic disjoint from $Z(\omega)$ lies in a cylinder of parallel geodesics whose boundary is made up of saddle connections.) Each saddle connection in γ is also shorter than r, and hence γ is a closed loop in the spine of the foliation. □

Remarks. For $g=1$ the hypotheses of the theorem imply that $\mathcal{F}_\varrho$ gives a foliation of the torus by circles. Theorem 6.1 can also be deduced from properties of zero flux and results of [Bo]. Theorem 6.2 comes from [Mas2, Theorem 1.1]; note that the condition of minimality assumed there is unnecessary.

7. Genus two: foliations

In this section we specialize the analysis of foliations to the case of genus two, and establish:

THEOREM 7.1. *Let X be a surface of genus two, and let $\mathcal{F}_\varrho$ be a zero-flux measured foliation of X with a loop in its spine. Then either*

(1) *$\mathcal{F}_\varrho$ is periodic, or*

(2) *$(X,\mathcal{F}_\varrho)$ is the connected sum of a pair of tori with irrational foliations.*

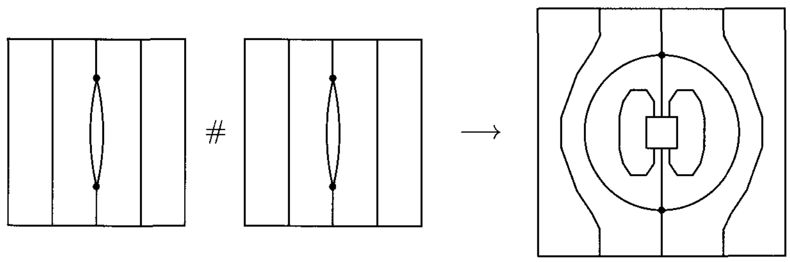

Fig. 3. Local picture of the connected sum $X_A\#Y_B$.

We will give examples of the second case in §10.

Foliations of surfaces with boundary. We begin by discussing surgery operations on measured foliations.

Let Y be a compact Riemann surface with boundary. Let $\Omega(Y)$ denote the holomorphic 1-forms on Y that reduce to real-valued forms along ∂Y. Any $\omega\in\Omega(Y)$ extends by Schwarz reflection to a holomorphic form on the double of Y across its boundary. The multiplicity of a zero of a form at $p\in\partial Y$ is defined to be the multiplicity of its extension to the double of Y.

Provided $\omega\neq 0$, $\varrho=\operatorname{Re}\omega$ determines a measured foliation $\mathcal{F}_\varrho$ of Y with each component of ∂Y contained in a leaf. The form ϱ has an even number of zeros (counted with multiplicity) on each component of ∂Y.

For example, given $(X,\alpha)\in\Omega\mathcal{M}_g$ and a compact interval A contained in the smooth part of a leaf of $\mathcal{F}_{\operatorname{Re}\alpha}$, we can slit X open along A to obtain a foliated surface with boundary X_A. The new foliation has two simple zeros along ∂X_A, one at each end of the slit.

Connected sum. Let $(X,\alpha)\in\Omega\mathcal{M}_g$ and $(Y,\beta)\in\Omega\mathcal{M}_h$ be a pair of surfaces with measured foliations $\mathcal{F}_{\operatorname{Re}\alpha}$ and $\mathcal{F}_{\operatorname{Re}\beta}$. Let $A\subset X$ and $B\subset Y$ be compact intervals along the leaves of the corresponding foliations and disjoint from their zeros. Suppose that A and B have the same length: that is, $\int_A|\alpha|=\int_B|\beta|$. Then there is a unique way to glue ∂X_A isometrically to ∂Y_B preserving the orientations of the foliations. Joining the 1-forms α and β together across the boundary, we obtain the *connected sum*

$$(Z,\gamma)=(X_A,\alpha)\#(Y_B,\beta)\in\Omega\mathcal{M}_{g+h}.$$

The foliation $\mathcal{F}_{\operatorname{Re}\gamma}$ restricts to $\mathcal{F}_{\operatorname{Re}\alpha}$ and $\mathcal{F}_{\operatorname{Re}\beta}$ on the subsurfaces $X_A, Y_B\subset Z$. See Figure 3.

Boundary contraction. Now let $\mathcal{F}_\varrho$ be a measured foliation of a surface with boundary X. Suppose that ϱ has at least one zero on each component of ∂X. Then $\mathcal{F}_\varrho$ descends

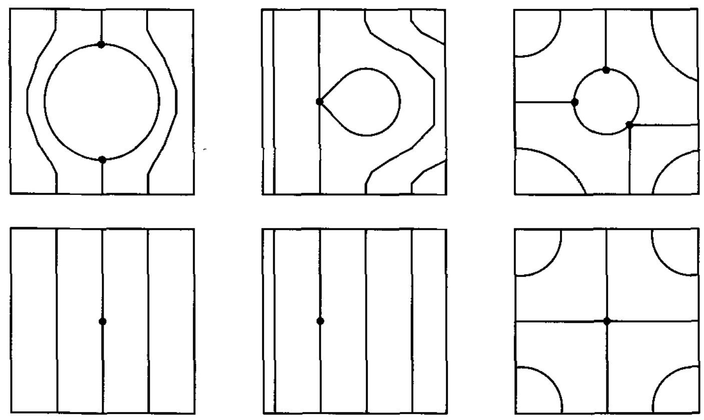

Fig. 4. Contracting the boundary.

to a measured foliation $\mathcal{G}$ of the surface $Y=X/\partial X$ obtained by collapsing each component of ∂X to a single point. A component of ∂X carrying $2p$ zeros of $\mathcal{F}_\varrho$ gives rise to a zero of order $p-1$ for $(Y, \mathcal{G})$.

This surgery is more radical than connected sum; although the quotient foliation $\mathcal{G}$ is defined by a smooth 1-form, there is no canonical complex structure on $X/\partial X$.

On the other hand, every leaf of $\mathcal{F}_\varrho$ meets the interior of X by our assumption on its zeros, and thus there is a bijection between the leaves of $\mathcal{F}_\varrho$ and $\mathcal{G}$. This implies that the dynamics is preserved by boundary contraction; for example, $\mathcal{G}$ is uniquely ergodic or periodic if and only if the same property holds for $\mathcal{F}_\varrho$.

Figure 4 shows examples of boundary contraction where the boundary contains

(i) two simple zeros;

(ii) a single double zero;

(iii) two simple zeros and one double zero.

Proof of Theorem 7.1. Since X has genus two, ϱ has two zeros $z_1, z_2 \in X$. We will assume $z_1 \neq z_2$; the case $z_1 = z_2$ is similar (but simpler). Let $\eta: X \to X$ be the hyperelliptic involution. Note that $\eta(z_1) = z_2$ and $\eta^*\varrho = -\varrho$. Thus η preserves the leaves of $\mathcal{F}_\varrho$, but reverses their orientation.

Let L be a loop in the spine of $\mathcal{F}_\varrho$. We distinguish three cases, shown in Figure 5.

(A) First suppose that L contains exactly one zero of ϱ. Then $\eta(L)$ is a loop through the other zero, disjoint from L and homologous to $-L$. Thus X splits along $L \cup \eta(L)$ into a pair of foliated surfaces with boundary, $X = X_0 \cup X_1$, of genus zero and one respectively.

The foliation of X_0 has no zeros (since its double is a torus), and therefore $\operatorname{int}(X_0)$ is a cylinder of $\mathcal{F}_\varrho$. In particular, $\mathcal{F}_\varrho | X_0$ is periodic.

There is a simple zero on each component of ∂X_1, so we obtain a foliated torus

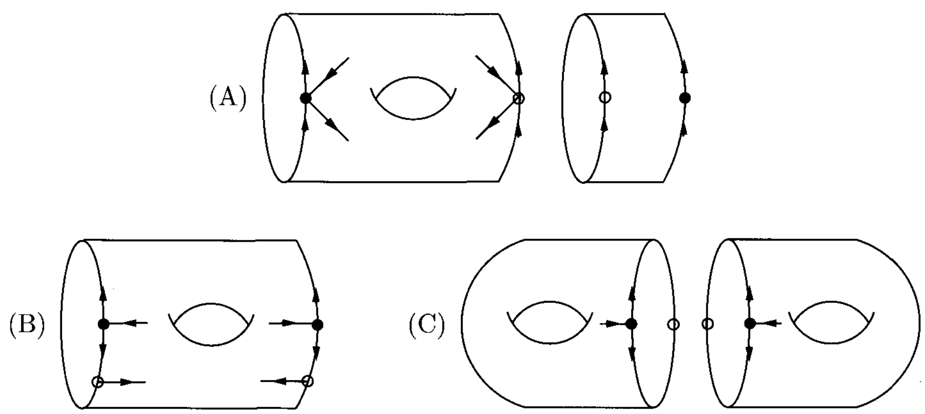

Fig. 5. Results of splitting along a loop in the spine.

$Y=X_1/\partial X_1$ upon contracting its boundary to a pair of points. Any measured foliation of a torus such as Y is either periodic or uniquely ergodic. Since the foliations of Y and X_1 are dynamically equivalent, $\mathcal{F}_\varrho|X_1$ is also periodic or uniquely ergodic.

By Corollary 4.3, a foliation with zero flux cannot be decomposed into a union of cylinders and a single uniquely ergodic component. Thus the foliation of X_1, and hence of X, is actually periodic.

(B) Now suppose that L contains both zeros of ϱ. Then $L=L_1\cup L_2$ is a pair of saddle connections. We may assume $\eta(L)=L$; indeed, if $\eta(L_1)\not\subset L$ then we can replace L with $L_1\cup\eta(L_1)$.

Suppose that L is not homologous to zero. Cutting X open along L, we obtain a surface X_1 of genus one. The map η interchanges the boundary components of X_1, so there are two zeros on each component. (The case of simple zeros is shown in Figure 5 (B).) As in case (A), upon contracting the boundary of X_1 to a pair of points we obtain a foliated torus Y. The foliation of Y is either uniquely ergodic or periodic, and the former is ruled out by the zero-flux condition. Thus the foliation of X is periodic in this case as well.

(C) Finally assume that L contains both zeros of ϱ and *is* homologous to zero. Then L consists of a pair of saddle connections with the same $|\omega|$-length, connecting z_1 to z_2 in the same direction (otherwise ω would have a nontrivial period around L). Cutting along L and regluing, we obtain a pair of foliated tori T_1 and T_2 whose connected sum is $(X,\mathcal{F}_\varrho)$.

If the foliations of T_1 and T_2 are both periodic, then $\mathcal{F}_\varrho$ is also periodic. If both foliations are uniquely ergodic, then $(X,\mathcal{F}_\varrho)$ is a connected sum of irrationally foliated tori. The mixed case (one periodic and the other uniquely ergodic) is ruled out by the zero-flux condition. □

Weierstrass points. Here is a useful way to obtain periodicity from the theorem above. Recall that the six *Weierstrass* points on a surface of genus two are the fixed points of its hyperelliptic involution.

THEOREM 7.2. *Let $(X, \mathcal{F}_\varrho)$ be a zero-flux measured foliation of a surface of genus two. Suppose that the spine of the foliation contains both a loop and a Weierstrass point. Then $\mathcal{F}_\varrho$ is periodic with at most two cylinders.*

Proof. First suppose that $\mathcal{F}_\varrho$ is periodic, with open cylinders $A_1, ..., A_n$. Each cylinder contains exactly two Weierstrass points, since it is invariant under the hyperelliptic involution. At least one of the six Weierstrass points lies in the spine (by assumption), so there are at most two cylinders.

Now suppose that $\mathcal{F}_\varrho$ is not periodic. Then, by the preceding theorem, $(X, \mathcal{F}_\varrho)$ is the connected sum of a pair of irrational foliations of tori. The spine of such a foliation is a circle on which the hyperelliptic involution acts by 180° rotation. Thus there are no Weierstrass points in the spine, contrary to assumption. □

8. Genus two: parabolics

In this section we study the foliations of varying slope attached to a fixed holomorphic 1-form. Throughout we fix the data of

- a real quadratic field $K \subset \mathbf{R}$,
- a compact Riemann surface X of genus two, and
- a holomorphic 1-form $\omega \neq 0$ in $\Omega(X)$,

such that

- ω has relative periods in $K(i)$ and zero complex flux.

We will show that the limit set of $\mathrm{SL}(X, \omega)$ is the full circle at infinity.

Dehn twists. Here is a general method for constructing elements of $\mathrm{SL}(X, \omega)$.

LEMMA 8.1. *Let $\mathcal{F}_\varrho$, $\varrho = \operatorname{Re} \omega$, be a periodic measured foliation of X with cylinders $A_1, ..., A_n$. Assume that the moduli $m_i = \operatorname{mod}(A_i)$ have rational ratios, and let $t = \operatorname{lcm}(m_1^{-1}, ..., m_n^{-1})$. Then there is a parabolic element*

$$D\phi = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in \mathrm{SL}(X, \omega),$$

represented by an affine automorphism $\phi: X \to X$ preserving the leaves of $\mathcal{F}_\varrho$.

Proof. Let t be the least common multiple of $\operatorname{mod}(A_i)^{-1}$, and let $n_i = t / \operatorname{mod}(A_i)$. Let $\phi | A_i$ be the affine automorphism of A_i that shears along the leaves of $\mathcal{F}_\varrho$ and effects an n_ith power of a Dehn twist. Then

$$D\phi|A_i = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad \text{and} \quad \phi|\partial A_i = \mathrm{id},$$

so these maps fit together to give the desired automorphism of X (cf. [V3, 2.4], [Mc, Lemma 9.7]). □

Action of $\mathrm{SL}(X,\omega)$. As usual

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$$

acts on $\mathbf{R}^2$ and on $\mathbf{H}\cup\mathbf{P}^1(\mathbf{R})$ by $(x,y)\mapsto(ax+by,cx+dy)$ and by $z\mapsto(az+b)/(cz+d)$. Let

$$A = D\phi \in \mathrm{SL}(X,\omega) \subset \mathrm{SL}_2(\mathbf{R})$$

be the linear part of a real-affine automorphism $\phi: X\to X$. Suppose that ϕ preserves the foliation $\mathcal{F}_\varrho$ with leaves of slope c/b defined by $\varrho=\mathrm{Re}\, a\omega$, $a=b+ic$. Then (c,b) is an eigenvector for A, and thus A fixes

$$\frac{c}{b} = \text{the slope of } a = \frac{1}{\text{the slope of } \mathcal{F}_\varrho}$$

in $\mathbf{P}^1(\mathbf{R})$.

Consolidating the preceding results, we can now establish:

THEOREM 8.2. *Let $\mathcal{F}_\varrho$ be the foliation of X determined by $\varrho=\mathrm{Re}(k\omega)\neq 0$ with $k\in K(i)$. Then either $\mathcal{F}_\varrho$ is periodic, or $(X,\mathcal{F}_\varrho)$ is the connected sum of a pair of irrationally foliated tori.*

Proof. By our assumptions on ω, the form ϱ has relative periods in K and vanishing flux (Theorem 4.4). By Theorem 6.1, $\mathcal{F}_\varrho$ has a closed loop in its spine, and hence it is periodic or a connected sum by Theorem 7.1. □

THEOREM 8.3. *Suppose that $\mathcal{F}_\varrho$ as above contains a leaf connecting a Weierstrass point of X to a zero of ϱ. Then*

(1) *$\mathcal{F}_\varrho$ is periodic with two cylinders, satisfying* $\mathrm{mod}(A_1)/\mathrm{mod}(A_2)\in\mathbf{Q}$, *and*

(2) $\mathrm{SL}(X,\omega)$ *contains a parabolic element A fixing $1/s\in\mathbf{P}^1(\mathbf{R})$, where s is the slope of $\mathcal{F}_\varrho$.*

Proof. By assumption there is a straight geodesic segment S running along a leaf of $\mathcal{F}_\varrho$ from a Weierstrass point p to a zero z of ϱ. Then either S or $S\cup\eta(S)$ is a saddle connection for (X, ω) (where η is the hyperelliptic involution). In either case, the spine of $\mathcal{F}_\varrho$ contains both a loop (by the argument above) and a Weierstrass point.

By Theorem 7.2, $\mathcal{F}_\varrho$ is periodic with at most two cylinders. The vanishing of the complex flux implies the rational relation $\sum \operatorname{mod}(A_i) N(h_i) = 0$; thus $\mathcal{F}_\varrho$ has exactly two cylinders, with $\operatorname{mod}(A_1)/\operatorname{mod}(A_2) \in \mathbf{Q}$ (Corollary 4.6).

By Lemma 8.1, a suitable product of Dehn twists in each cylinder gives rise to an affine automorphism with parabolic derivative $A = D\phi \in \mathrm{SL}(X, \omega)$. Since ϕ preserves the leaves of $\mathcal{F}_\varrho$, A fixes $1/s$. $\square$

THEOREM 8.4. *Let (X, ω) be a holomorphic 1-form in genus two, with periods in $K(i)$ and zero complex flux. Then the limit set of the Fuchsian group $\mathrm{SL}(X, \omega)$ is the full circle at infinity.*

Proof. Let p be a Weierstrass point of X, and let S be an oriented straight geodesic on $(X, |\omega|)$ joining p to $Z(\omega)$. Since ω has relative periods in $K(i)$ and $S \cup \eta(S)$ joins a pair of zeros, we have

$$\int_S \omega = \frac{1}{2} \int_{S - \eta(S)} \omega = k_1 + i k_2 \in K(i).$$

Thus S has slope $s = k_2/k_1$ and lies along a leaf of $\mathcal{F}_\varrho$, $\varrho = \operatorname{Re}((k_2 + i k_1)\omega)$. By the preceding theorem, $1/s$ is the fixed point of a parabolic element $A \in \mathrm{SL}(X, \omega)$, and hence belongs to the limit set.

Since the slopes of straight geodesics from p to $Z(\omega)$ fill out a dense subset of $\mathbf{P}^1(\mathbf{R})$, the limit set of $\mathrm{SL}(X, \omega)$ is the full circle. $\square$

9. Trace fields

In this section we establish a criterion for the vanishing of the complex flux, valid for a Riemann surface of any genus.

THEOREM 9.1. *Let $\mathrm{SL}(X, \omega)$ be an infinite group with real quadratic trace field K. Then after replacing (X, ω) by $B \cdot (X, \omega)$ for suitable $B \in \mathrm{GL}_2^+(\mathbf{R})$, the form ω has relative periods in $K(i)$ and zero complex flux.*

This theorem permits the following more invariant formulation of the results of §8.

THEOREM 9.2. *Let X have genus two and suppose that $\mathrm{SL}(X, \omega)$ is an infinite group with real quadratic trace field. Then:*

(1) *The limit set of $\mathrm{SL}(X, \omega)$ is the full circle at infinity.*

(2) *Let $\mathcal{F}$ be a foliation of $(X, |\omega|)$ by geodesics with the same slope as a period of ω. Then either $\mathcal{F}$ is periodic, or $(X, \mathcal{F})$ is the connected sum of a pair of irrationally foliated tori.*

(3) *If s is the slope of a saddle connection passing through a Weierstrass point, then*

(a) *the foliation by geodesics of slope s is periodic with two cylinders, satisfying* $\operatorname{mod}(A_1)/\operatorname{mod}(A_2)\in\mathbf{Q}$, *and*

(b) $\mathrm{SL}(X,\omega)$ *contains a parabolic element fixing* $1/s$.

Similarly the results of §4 and §6 yield:

THEOREM 9.3. *Let X have genus $g\geqslant 2$, and suppose that $\mathrm{SL}(X,\omega)$ is an infinite group with real quadratic trace field. Let $\mathcal{F}$ be a foliation of X by geodesics with the same slope as a period of ω. Then $\mathcal{F}$ fails to be uniquely ergodic, and there is a loop in the spine of $\mathcal{F}$.*

Trace fields. The *trace field* of $\Gamma\subset\mathrm{SL}_2(\mathbf{R})$ is given by $K=\mathbf{Q}(\operatorname{tr}\Gamma)\subset\mathbf{R}$. The degree of $K/\mathbf{Q}$ is bounded by the genus g of X [Mc, 5.1].

We begin by studying the impact of hyperbolic elements $A\in\mathrm{SL}(X,\omega)$ on the periods and flux of ω. Every such element has the form $A=D\phi$, where ϕ is a pseudo-Anosov real-affine automorphism of (X,ω).

THEOREM 9.4. *If $\mathrm{SL}(X,\omega)$ contains a hyperbolic element, then the relative and absolute periods of ω span the same vector space over $\mathbf{Q}$.*

Proof. Let γ be an oriented path joining a pair of zeros of ω, and let $p=\int_\gamma\omega$ be the corresponding relative period. We will show that p is in the span of the absolute periods, $V=\operatorname{Per}(\omega)\otimes\mathbf{Q}\subset\mathbf{C}$.

Let $A=D\phi$ be a hyperbolic element of $\mathrm{SL}(X,\omega)$. Replacing ϕ by a ϕ^n for suitable $n>0$, we can assume that ϕ fixes the zeros of ω. In particular, ϕ fixes the endpoints of γ, so $\gamma-\phi(\gamma)$ is a 1-cycle on X. By equation (3.2) we have

$$q=\int_{\gamma-\phi(\gamma)}\omega=(I-A)(p)\in V.$$

Since A is hyperbolic, $(I-A)$ is invertible on $\mathbf{C}$. But $A(V)=V$, and therefore $p=(I-A)^{-1}(q)$ also lies in V. □

THEOREM 9.5. *If $\mathrm{SL}(X,\omega)$ contains a hyperbolic element A, then*

$$V=\operatorname{Per}(\omega)\otimes\mathbf{Q}\subset\mathbf{C}$$

is a 2-dimensional vector space over $L=\mathbf{Q}(\operatorname{tr}A)\subset\mathbf{R}$.

Proof. Let $A=D\phi$ be a hyperbolic element in $\mathrm{SL}(X,\omega)$, let $t=\operatorname{tr}(A)$, let $T=\phi^*$ acting on $H^1(X,\mathbf{R})$, and let

$$S=\operatorname{Ker}(tI-T-T^{-1}).$$

It is known that the eigenvalues $\lambda^{\pm 1}$ of A are also simple eigenvalues of T [Mc, Theorem 5.3]; since $t=\lambda+\lambda^{-1}$, we have $\dim(S)=2$. Moreover, S is spanned by $[\operatorname{Re}\omega]$ and $[\operatorname{Im}\omega]$, since

$$(\phi^*+(\phi^*)^{-1})(\omega)=\operatorname{tr}(A)\omega=t\omega.$$

The rational periods V can be identified with the image of $H_1(X,\mathbf{Q})$ in S^*.

Let

$$S(L)=S\cap H^1(X,L).$$

Since rational cycles define elements of $S(L)^*=\operatorname{Hom}(S(L),L)$, we can alternatively describe V as the image of the natural map

$$\pi: H_1(X,\mathbf{Q})\to S(L)^*.$$

Letting $T_*=\phi_*$ acting on $H_1(X,\mathbf{Q})$, we have

$$\pi((T_*+T_*^{-1})^n x)=t^n\pi(x).$$

Thus V is already a vector space over $L=\mathbf{Q}(t)$, so it coincides with the image of the natural extension of π to $H_1(X,L)$. But this extension is the composition of the isomorphism $H_1(X,L)\cong H^1(X,L)^*$ and the surjection $H^1(X,L)^*\to S(L)^*$, so we have $V\cong S(L)^*\cong L^2$. □

COROLLARY 9.6. *The traces of any two hyperbolic elements in* $\operatorname{SL}(X,\omega)$ *generate the same field over* $\mathbf{Q}$.

THEOREM 9.7. *If* $\operatorname{SL}(X,\omega)$ *contains a hyperbolic element such that* $K=\mathbf{Q}(\operatorname{tr}(A))$ *is real quadratic, and* ω *has periods in* $K(i)$, *then the complex flux of* ω *is zero.*

Proof. Let $A=D\phi$, let $t=\operatorname{tr}(A)$ and let $T=\phi^*$ acting on $H^1(X,\mathbf{R})$ as before. Then we have a pair of 2-dimensional eigenspaces

$$S\oplus S'\subset H^1(X,\mathbf{R})$$

on which $U=T+T^{-1}$ acts with eigenvalues $t,t'\in K$ respectively. Since U is self-adjoint, S and S' are orthogonal under the cup product.

The eigenspace S is spanned by $\operatorname{Re}\omega$, $\operatorname{Im}\omega$. These forms actually lie in $H^1(X,K)\cap S$, since ω has periods in $K(i)$. The Galois conjugate of any form $\varrho\in H^1(X,K)\cap S$ satisfies $U\varrho'=t'\varrho'$, and hence belongs to S'. In particular, $(\operatorname{Re}\omega)'$ and $(\operatorname{Im}\omega)'$ are orthogonal to $\operatorname{Re}\omega$ and $\operatorname{Im}\omega$. This shows

$$\int\omega\wedge\omega'=\int\omega\wedge\bar{\omega}'=0,$$

and thus ω has zero complex flux. □

Branched covers of tori. We say that $(X,\omega)\in\Omega\mathcal{M}_g$ *arises via a torus* if there is a Riemann surface $E=\mathbf{C}/\Lambda$ of genus one, and a holomorphic map $f:X\to E$, such that f is branched over torsion points on E and $\omega=f^*(dz)$. In this case, $\mathrm{SL}(X,\omega)$ is commensurable to $\mathrm{SL}_2(\mathbf{Z})$ (up to conjugation), and its trace field is $\mathbf{Q}$ (cf. [GJ]). Here is a converse:

THEOREM 9.8. *If* $\mathrm{SL}(X,\omega)$ *contains a hyperbolic element with* $\mathrm{tr}(A)\in\mathbf{Q}$, *then* (X,ω) *arises via a torus.*

Proof. Since $L=\mathbf{Q}(\mathrm{tr}(A))=\mathbf{Q}$, we have $\mathrm{Per}(\omega)\otimes\mathbf{Q}\cong\mathbf{Q}^2$ by Theorem 9.5 above. Thus $\mathrm{Per}(\omega)\cong\mathbf{Z}^2$ is a lattice in $\mathbf{C}$, and $E=\mathbf{C}/\mathrm{Per}(\omega)$ is a complex torus.

We may assume that X has genus $g\geqslant 2$, since the case $g=1$ is immediate. Let p be a zero of ω, and define $f:X\to E$ by $f(q)=\int_p^q\omega$. By construction, $f^*(dz)=\omega$. By Theorem 9.4, the relative periods of ω lie in $\mathrm{Per}(\omega)\otimes\mathbf{Q}$, so f is branched over torsion points on E. Therefore (X,ω) arises via a torus. $\square$

Proof of Theorem 9.1. Assume that $\mathrm{SL}(X,\omega)$ is infinite with real quadratic trace field K. Since $K\neq\mathbf{Q}$, there is a hyperbolic element $A\in\mathrm{SL}(X,\omega)$. If $\mathrm{tr}(A)\in\mathbf{Q}$ then (X,ω) arises via a torus and the trace field of $\mathrm{SL}(X,\omega)$ is $\mathbf{Q}$, contrary to assumption. Thus $K=\mathbf{Q}(\mathrm{tr}(A))$.

As remarked in §3, ω has a pair of periods p_1,p_2 that are linearly independent over $\mathbf{R}$. By Theorem 9.5, the rational periods span a 2-dimensional vector space over K, and thus

$$V=\mathrm{Per}(\omega)\otimes\mathbf{Q}=Kp_1\oplus Kp_2\subset\mathbf{C}.$$

By Theorem 9.4, ω has relative periods in V.

Now let $B\in\mathrm{GL}_2^+(\mathbf{R})$ be a real-linear map sending $\{p_1,p_2\}$ to $\{1,i\}$, and let

$$(X_B,\omega_B)=B\cdot(X,\omega).$$

Then by equation (3.2), the relative periods of ω_B lie in $B(V)=K(i)$. The group $\mathrm{SL}(X_B,\omega_B)$ is simply a conjugate of $\mathrm{SL}(X,\omega)$, so it still contains a hyperbolic element whose trace generates K. Thus ω_B has vanishing complex flux by Theorem 9.7. $\square$

Genus two. Using Theorem 8.4, similar arguments establish:

THEOREM 9.9. *For any* $(X,\omega)\in\Omega\mathcal{M}_2$ *and real quadratic field* K, *the following conditions are equivalent*:

(1) *The trace field of* $\mathrm{SL}(X,\omega)$ *is* K.

(2) *The Jacobian of* X *admits real multiplication by* K *with* ω *as an eigenform, and the zeros* (z_1,z_2) *of* ω *satisfy* $\int_{z_1}^{z_2}\omega\in\mathrm{Per}(\omega)\otimes\mathbf{Q}$.

(3) *After replacing* (X,ω) *by* $B\cdot(X,\omega)$ *for suitable* $B\in\mathrm{GL}_2^+(\mathbf{R})$, ω *has relative periods in* $K(i)$ *and zero complex flux.*

COROLLARY 9.10. *Suppose that* $(X,\omega)\in\Omega\mathcal{M}_2$ *has periods in* $K(i)$, *zero complex flux, and that* ω *has a double zero. Then* $\mathrm{SL}(X,\omega)$ *is a lattice.*

Proof. By the preceding theorem, $\mathrm{Jac}(X)$ admits real multiplication with ω as an eigenform. Since ω also has a double zero, $\mathrm{SL}(X,\omega)$ is a lattice by Theorems 1.3 and 7.2 of [Mc]. □

Notes. See [Mc] for more on Jacobians with real multiplication, and [GJ], [EO] for the case where (X,ω) arises via a torus. Theorems 9.4 and 9.5 above also appear in [KS, Appendix].

10. Limit sets and infinitely generated groups

We can now establish our main result on the possible limit sets for $\mathrm{SL}(X,\omega)$ in genus two, and give examples where this group is infinitely generated.

THEOREM 10.1. *For any* $(X,\omega)\in\Omega\mathcal{M}_2$, *the limit set of* $\mathrm{SL}(X,\omega)$ *is either the empty set, a singleton, or the full circle at infinity.*

Proof. Suppose that the limit set contains two or more points. Then $\mathrm{SL}(X,\omega)$ is an infinite group, containing a hyperbolic element A. If the trace of A is rational, then (X,ω) arises from a torus (Theorem 9.8), so $\mathrm{SL}(X,\omega)$ is commensurable to a conjugate of $\mathrm{SL}_2(\mathbf{Z})$ and its limit set is the full circle.

On the other hand, if $\mathrm{tr}(A)$ is irrational, then the trace field of $\mathrm{SL}_2(X,\omega)$ is a real quadratic field K, and the limit set of $\mathrm{SL}(X,\omega)$ is the full circle by Theorem 9.2. □

Lattices. A discrete group $\Gamma\subset\mathrm{SL}_2(\mathbf{R})$ is a *lattice* if $\mathrm{SL}_2(\mathbf{R})/\Gamma$ has finite volume. It is known that Γ is lattice if and only if Γ is finitely generated and its limit set is the full circle. By [V3, 2.11] we have:

THEOREM 10.2 (Veech). *If* $\mathrm{SL}(X,\omega)$ *is a lattice, then for any* $a\in\mathbf{C}^*$ *the foliation* $\mathcal{F}_{\mathrm{Re}\,a\omega}$ *is either periodic or uniquely ergodic.*

If X has genus two and the limit set of $\mathrm{SL}(X,\omega)$ is the full circle, then $\mathrm{SL}(X,\omega)$ is finitely generated (and hence a lattice), provided

- its trace field is $\mathbf{Q}$ (Theorem 9.8), or
- ω has exactly one zero [Mc, Corollary 1.4].

Infinitely generated groups. We now show that when X has genus two and ω has a pair of distinct zeros, it may happen that $\mathrm{SL}(X,\omega)$ is an *infinitely generated group.*

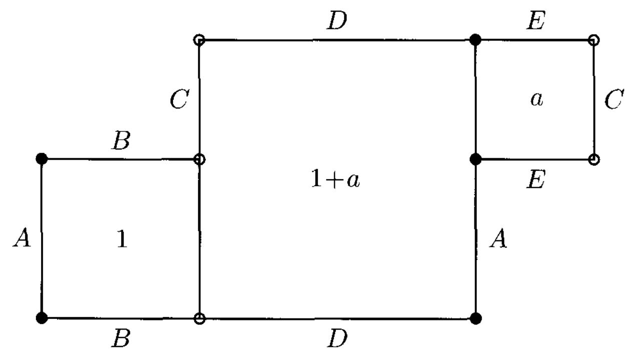

Fig. 6. The polygon $S(a)$ and its gluing pattern.

That is, $\mathrm{SL}(X,\omega)$ may fail to be a lattice, even though its limit set is the full circle at infinity. The construction we give admits many variations.

Consider the S-shaped polygon $S(a)\subset\mathbf{C}$ shown in Figure 6. This polygon can be cut into three squares of side lengths 1, $1+a$ and a.

Let X be the Riemann surface obtained by gluing together opposite sides of $S(a)$ in the pattern indicated by capital letters in Figure 6. The form $dz|S(a)$ descends to a form $\omega\in\Omega(X)$. The ten 'vertices' of $S(a)$, shown as round dots in Figure 6, descend in two groups of five to yield the two distinct zeros of ω on X.

THEOREM 10.3. *Let $b\in\mathbf{Q}_+$ be a positive rational such that*

$$a=b-1+\sqrt{b^2-b+1}$$

is irrational. Construct $(X,\omega)\in\Omega\mathcal{M}_2$ from $(S(a),dz)$ by identifying opposite sides. Then $\mathrm{SL}(X,\omega)$ is an infinitely generated group whose limit set is the full circle.

Proof. The foliation of $S(a)$ by vertical lines covers the foliation $\mathcal{F}_{\mathrm{Re}\,\omega}$ of X. This foliation is periodic, with three cylinders of modulus one coming from the three squares forming $S(a)$. Thus we have

$$P_1=\begin{pmatrix}1&0\\1&1\end{pmatrix}\in\mathrm{SL}(X,\omega)$$

by Lemma 8.1.

Similarly, $S(a)$ splits into a pair of horizontal rectangles with dimensions $1\times(2+a)$ and $a\times(1+2a)$. Thus the horizontal foliation $\mathcal{F}_{\mathrm{Im}\,\omega}$ of X is periodic with two cylinders, satisfying $\mathrm{mod}(A_1)=(2+a)^{-1}$ and $\mathrm{mod}(A_2)=a(1+2a)^{-1}$. The value of a is chosen so that

$$\frac{\mathrm{mod}(A_2)}{\mathrm{mod}(A_1)}=\frac{a(2+a)}{1+2a}=b$$

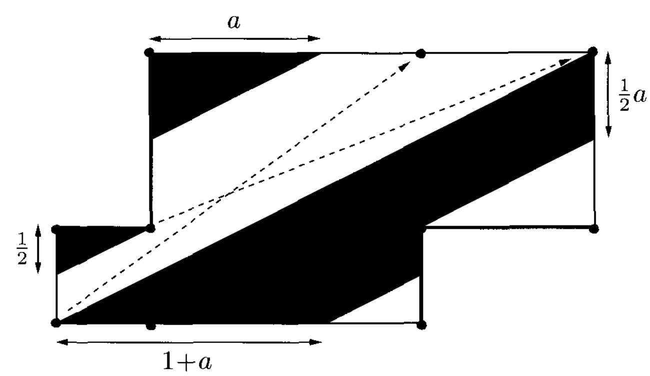

Fig. 7. The foliation with slope $\frac{1}{2}$ gives the connected sum of a pair of tori with irrational foliations.

is rational. By Lemma 8.1 again, we have

$$P_2 = \begin{pmatrix} 1 & n(2+a) \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(X, \omega)$$

for some integer $n>0$.

The product $P_1 P_2$ is hyperbolic, so the limit set of $\mathrm{SL}(X,\omega)$ is the full circle by Theorem 10.1.

To see that $\mathrm{SL}(X,\omega)$ is infinitely generated, consider the foliation of $S(a)$ by lines of slope $\frac{1}{2}$. This foliation covers $\mathcal{F}_\varrho$, $\varrho=\mathrm{Re}((1+2i)\omega)$.

As shown in Figure 7, $\mathcal{F}_\varrho$ has a pair of saddle connections that cut X into a pair of tori, shaded white and black in the figure. The dotted lines in the figure give cycles on the white torus T, with periods $p_1=(1+2a, a)$ and $p_2=(2+a, 1+a)$ in $\mathbf{R}^2=\mathbf{C}$. We have $\mathrm{Per}(\omega|T)\otimes\mathbf{Q}=\mathbf{Q}p_1\oplus\mathbf{Q}p_2$.

By a direct computation, we find $y/x=1/a$ for the solution to the equation $xp_1+yp_2=(2,1)$. Since a is irrational, this shows that $\omega|T$ has no period of slope $\frac{1}{2}$. Thus $\mathcal{F}_\varrho$ restricts to an irrational foliation of the subtorus T. Since the leaves in T are neither closed nor dense in X, $\mathcal{F}_\varrho$ is neither periodic nor uniquely ergodic.

By virtue of the Veech dichotomy (Theorem 10.2), $\mathrm{SL}(X,\omega)$ cannot be a lattice. But its limit set is the full circle, so $\mathrm{SL}(X,\omega)$ is infinitely generated. □

Dynamics at slopes $\frac{1}{2}$, $\frac{1}{3}$ *and* 1. The foliations of (X,ω) as above exhibit a variety of behaviors as their slopes vary.

- For slope $\frac{1}{2}$, the preceding proof shows that $(X,\mathcal{F})$ is the connected sum of a pair of tori with irrational foliations.
- For slope $\frac{1}{3}$, one can check that $\mathcal{F}$ is periodic, with three cylinders whose moduli have irrational ratios. This provides another proof that $\mathrm{SL}(X,\omega)$ is not a lattice.

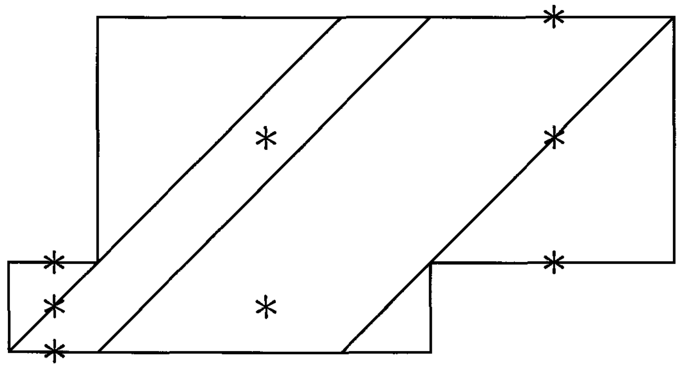

Fig. 8. The spine of the foliation of slope 1 contains a pair of Weierstrass points.

- Finally for slope 1 there is a saddle connection through a Weierstrass point, and thus $\mathcal{F}$ is a periodic foliation with two cylinders of rational ratio. (In fact, $\operatorname{mod}(A_1) = 2\operatorname{mod}(A_2)$.) The spine of this foliation, lifted to $S(a)$, is shown in Figure 8. The lifts of the six Weierstrass points on X are marked by $*$'s.

The regular 10*-gon.* In conclusion we remark that examples where $\mathrm{SL}(X,\omega)$ is infinitely generated appear to be ubiquitous in genus two.

In fact, suppose that ω has two distinct zeros and that the trace field of $\mathrm{SL}(X,\omega)$ is irrational. Then either $\mathrm{SL}(X,\omega)$ is a lattice, or it is infinitely generated. But at present, the only known cases where a lattice arises are those associated to the regular 10-gon as in [V3].

A. Appendix: Cusps of triangle groups

In this section we apply the dynamics of measured foliations to determine the cusps of certain triangle groups. The two discussions are connected by the following result.

THEOREM A.1. *If $\mathrm{SL}(X,\omega)$ is a lattice with real quadratic trace field K, then the set of cross-ratios of its cusps coincides with $\mathbf{P}^1(K)-\{0,1,\infty\}$.*

Proof. By Theorem 9.1 we may assume that ω has relative periods in $K(i)$ and zero complex flux. Then the slopes of periods of ω coincide with $\mathbf{P}^1(K)$. To complete the proof, we will show that the cusps of $\mathrm{SL}(X,\omega)$ also coincide with $\mathbf{P}^1(K)$.

Let s be a cusp. Then $(X,|\omega|)$ has a closed geodesic of slope $1/s$ [V3, 2.4], and thus $1/s$ is the slope of a period of ω. Therefore the cusps are contained in $\mathbf{P}^1(K)$.

Conversely, let $s=k_1/k_2\in\mathbf{P}^1(K)$. Then the form $\varrho=\operatorname{Re}(k_1+ik_2)\omega$ has periods in K and zero flux. By Corollary 4.2, the foliation $\mathcal{F}_\varrho$ of X by geodesics of slope $1/s$ is not uniquely ergodic. By the Veech dichotomy, $\mathcal{F}_\varrho$ is actually periodic, and by [V3, 2.11], s is a cusp. □

Triangle groups. Let $T\subset\mathbf{H}$ be a triangle with internal angles $(\pi/a,\pi/b,\pi/c)$. Let $\Gamma\subset\mathrm{SL}_2(\mathbf{R})$ be the orientation-preserving subgroup of the group generated by reflections in the sides of T. Then Γ is a *triangle group* with *signature* (a,b,c).

We allow T to have one or more vertices at infinity, in which case the corresponding entry in the signature is ∞. The cusps of Γ coincide with the orbits of the ideal vertices of T.

It is known that many triangle groups are commensurable to groups of the form $\mathrm{SL}(X,\omega)$. Using this fact we will show:

THEOREM A.2. *Let $\Gamma\subset\mathrm{SL}_2(\mathbf{R})$ be a triangle group with signature*

$$(2,2n,\infty) \text{ or } (3,n,\infty), \quad \text{where } n=4,5,6,$$

or

$$(2,n,\infty), \quad \text{where } n=5.$$

Then the set of cross-ratios of cusps of Γ coincides with $\mathbf{P}^1(K_n)-\{0,1,\infty\}$, where K_n is the field $\mathbf{Q}(\cos(\pi/n))$.

Proof. By [V3], the group $\mathrm{SL}(X,\omega)$ associated to billiards in a regular $2n$-gon is a triangle group of signature (n,∞,∞). These groups have trace field $K_n=\mathbf{Q}(\cos(\pi/n))$, which is real quadratic for $n=4,5,6$. Since the (n,∞,∞) triangle group has index two in the $(2,2n,\infty)$ triangle group, we conclude by Theorem A.1 that the cusps of the latter are $\mathbf{P}^1(K_n)$.

Similarly, by considering billiards in certain triangles, [Wa] shows that the $(3,n,\infty)$ triangle groups with $n\geqslant 4$ arise in the form $\mathrm{SL}(X,\omega)$. For $n=4,5,6$ these groups also have trace field K_n, and the same argument applies.

Finally [V3] shows that the $(2,5,\infty)$ triangle group, with trace field K_5, arises from billiards in a regular pentagon. □

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