Acta Math., 190 (2003), 241–306

Best uniform rational approximation of x^α on $[0,1]$

by

HERBERT R. STAHL

TFH-Berlin
Berlin, Germany

1. Introduction and main result

Many problems in approximation theory can be connected with the problem of approximating the function $|x|^\alpha$ on a set having the origin as an inner point. One of the main reasons for that is the fact that $|x|^\alpha$ can be seen as a prototype of functions that are α-Lipschitz continuous. In the present paper we are concerned with the rational approximation of the function x^α on $[0,1]$. It is not difficult to see that this approximation problem is equivalent to the approximation of $|x|^{2\alpha}$ on $[-1,1]$, and the asymptotic error estimates for both cases can easily be transferred from one to the other situation (see Theorem 2).

We start with the statement of the main result and shall then continue with a very short review of related investigations in polynomial and rational approximation. Let $\mathcal{P}_n$ denote the set of all polynomials of degree at most $n\in\mathbf{N}$ with real coefficients, $\mathcal{R}_{mn}$ the set of rational functions $\{p/q \mid p\in\mathcal{P}_m,\, q\in\mathcal{P}_n,\, q\not\equiv 0\}$, $m,n\in\mathbf{N}$, and $r^*_{mn}=r^*_{mn}(f_\alpha,[0,1];\cdot)\in\mathcal{R}_{mn}$, $m,n\in\mathbf{N}$, the *rational best approximant* to $f_\alpha:=x^\alpha$ in the uniform norm on $[0,1]$. The minimal approximation error for numerator and denominator degrees at most m and n, respectively, is denoted by

$$E_{mn}:=E_{mn}(f_\alpha,[0,1])=\|f_\alpha-r^*_{mn}\|_{[0,1]}=\inf_{r\in\mathcal{R}_{mn}}\|f_\alpha-r\|_{[0,1]} \tag{1.1}$$

with $\|\cdot\|_K$ denoting the sup-norm on $K\subseteq\mathbf{R}$. It is well known that the best approximant r^*_{mn} exists and is unique within $\mathcal{R}_{mn}$ for each $m,n\in\mathbf{N}$ (cf. [15, §§ 9.1 and 9.2], [14, Chapter 7.2] or [17, §5.1]). The central task in this paper is to prove

Research supported by the Deutsche Forschungsgemeinschaft (AZ: Sta 299/8-2).

THEOREM 1. *The limit*

$$\lim_{n\to\infty} e^{2\pi\sqrt{\alpha n}} E_{nn}(x^\alpha, [0,1]) = 4^{1+\alpha}|\sin \pi\alpha| \tag{1.2}$$

holds for each $\alpha>0$.

Since $|x|^\alpha$ is an even function, $x\in\mathbf{R}$, it is not difficult to verify that the uniquely existing best approximant $r^*_{mn}=r^*_{mn}(|x|^\alpha, [-1,1]; \cdot\,)$ is also an even function. Consequently, a substitution of x^2 by x shows that

$$E_{2m,2n}(|x|^{2\alpha}, [-1,1]) = E_{mn}(x^\alpha, [0,1]) \quad \text{for all } m,n\in\mathbf{N}, \tag{1.3}$$

and as a corollary to Theorem 1 we have

THEOREM 2. *The limit*

$$\lim_{n\to\infty} e^{\pi\sqrt{\alpha n}} E_{nn}(|x|^\alpha, [-1,1]) = 4^{1+\alpha/2}\left|\sin \tfrac{1}{2}\pi\alpha\right| \tag{1.4}$$

holds for each $\alpha>0$.

The analogue of (1.4) in polynomial approximation is connected with a conjecture by S. N. Bernstein, which, however, has been disproved in the 1980s by A. S. Varga and A. J. Carpenter with the help of high-precision numerical calculations (cf. [26], [27]). Because of its relevance for rational approximation, we will repeat some of the results that form the background of this conjecture. From Jackson's and Bernstein's theorems (cf. [15, §§ 5.5 and 5.6]) we know that the polynomial approximation error $E_{m,0}(|x|^\alpha, [-1,1])$ behaves like $\mathcal{O}(m^{-\alpha})$ as $m\to\infty$ and that the exponent $-\alpha$ in the estimate is best possible. (By $\mathcal{O}(\,\cdot\,)$ we denote Landau's big oh.) In [2] and [3] S. N. Bernstein has proved that the limit

$$\lim_{m\to\infty} m^{-\alpha} E_{m,0}(|x|^\alpha, [-1,1]) =: \beta(\alpha) \tag{1.5}$$

exists and is different from zero for each $\alpha>0$, $\alpha\notin 2\mathbf{N}$. This result is much stronger and more difficult to prove than general conclusions of Jackson's and Bernstein's theorems, where only the order of the error development is taken into consideration. In [2] the special case $\alpha=1$ of (1.5) had been studied. The existence of the constant $\beta:=\beta(1)$ has been proved there, and numerical bounds $0.278<\beta<0.286$ ([2, p. 41]) had been calculated. In this connection S. N. Bernstein raised the question, whether the value of β, which now carries the name *Bernstein constant*, could be expressed by known transcendentals. Since $1/2\sqrt{\pi}=2.82(\pm 0.0005)$ lies well inside his numerical bounds for β, he raised the question whether $\beta\overset{?}{=}1/2\sqrt{\pi}$ (cf. [2, p. 56]). This speculation is now known as Bernstein's conjecture, and it has been disproved in [26] by high-precision calculations. An answer to

Bernstein's original question about an expression of β by known transcendentals is still open. In [3] only an asymptotic formula has been proved for $\beta(\alpha)$. Numerical calculations of $\beta(\alpha)$ for a selection of values of α have been presented in [27].

There are two striking differences between polynomial and rational best approximation to the function $|x|^\alpha$ on $[-1,1]$ or $[0,1]$, on which we want to comment. Rational approximants converge much faster than the polynomial ones, which can rather impressively be seen by a comparison of the two formulae (1.4) and (1.5). It is also quite surprising that in the somewhat simpler polynomial approximation problem no explicit formula is known for the constant $\beta(\alpha)$, while in the rational case we have the comparatively simple expression on the right-hand side of formula (1.4). In the case of $\alpha=1$ we have the very simple number 8 as leading coefficient in the asymptotic error estimate, which has been proved in [19].

Bernstein's investigations [2] and [3] have been published in 1914 and 1938. The study of best rational approximation of $|x|$ was started only in 1964 by D. J. Newman's surprising (at the time) result in [16] that

$$\tfrac{1}{2}e^{-9\sqrt{n}} \leqslant E_{nn}(|x|,[-1,1]) \leqslant 3e^{-\sqrt{n}} \quad \text{for all } n=4,5,\dots . \tag{1.6}$$

The result already shows that rational approximants converge indeed much faster than the polynomial analogues.

Newman's investigation has triggered a whole series of contributions, we mention only those that contain substantial improvements of the error estimate in the uniform norm:

$$E_{nn}(x^\alpha,[0,1]) \leqslant e^{-c(\alpha)\sqrt[3]{n}}, \quad \alpha\in\mathbf{R}_+, \qquad ([6],\ 1967),$$

$$E_{nn}(x^{1/3},[0,1]) \leqslant e^{-c\sqrt{n}}, \qquad ([4],\ 1968),$$

$$E_{nn}(x^\alpha,[0,1]) \leqslant e^{-c(\alpha)\sqrt{n}}, \quad \alpha\in\mathbf{R}_+, \qquad ([8],\ 1967),$$

$$\tfrac{1}{3}e^{-\pi\sqrt{2n}} \leqslant E_{nn}(x^{1/2},[0,1]) \leqslant e^{-\pi\sqrt{n}\,(1-\mathcal{O}(n^{-1/4}))}, \qquad ([5],\ 1968),$$

$$e^{-c(\alpha)\sqrt{n}} \leqslant E_{nn}(x^\alpha,[0,1]), \quad \alpha\in\mathbf{Q}_+\backslash\mathbf{N}, \qquad ([9],\ 1972),$$

$$e^{-4\pi\sqrt{\alpha n}\,(1+\varepsilon)} \leqslant E_{nn}(x^\alpha,[0,1]) \leqslant e^{-\pi\sqrt{\alpha n}\,(1-\varepsilon)},$$
$$\alpha\in\mathbf{Q}_+\backslash\mathbf{N},\ \varepsilon>0,\ n\geqslant n_0(\alpha,\varepsilon), \qquad ([10],\ 1974),$$

$$E_{nn}(x^{1/2},[0,1]) \leqslant cne^{-\pi\sqrt{2n}}, \qquad ([30],\ 1974),$$

$$\tfrac{1}{3}e^{-\pi\sqrt{2n}} \leqslant E_{nn}(x^{1/2},[0,1]) \leqslant ce^{-\pi\sqrt{2n}}, \qquad ([31],\ 1975),$$

$$e^{-c_1(s)\sqrt{n}} \leqslant E_{nn}(\sqrt[s]{x},[0,1]) \leqslant e^{-c_2(s)\sqrt{n}}, \quad s\in\mathbf{N}, \qquad ([24],\ 1976).$$

Here $c, c(\alpha), \dots$ denote constants that are independent of n. The estimates are given only for approximation on $[0,1]$; relation (1.3) shows that these results can immediately be transferred to the problem of approximating $|x|^\alpha$ on $[-1,1]$.

The sharpest results about asymptotic error estimates for best rational approximants to $f_\alpha = x^\alpha$ in the uniform norm on $[0,1]$ have been obtained independently by T. Ganelius [7] in 1979 and by N. S. Vyacheslavov [32] in 1980. Both authors proved that for $\alpha \in \mathbf{R}_+ \setminus \mathbf{N}$ there exists a constant $c_1 = c_1(\alpha) > 0$ such that

$$\liminf_{n\to\infty} e^{2\pi\sqrt{\alpha n}} E_{nn}(x^\alpha, [0,1]) \geqslant c_1(\alpha), \tag{1.7}$$

and conversely that for each positive *rational number* $\alpha \in \mathbf{Q}_+$ there exists a constant $c_2 = c_2(\alpha)$ such that

$$\limsup_{n\to\infty} e^{2\pi\sqrt{\alpha n}} E_{nn}(x^\alpha, [0,1]) \leqslant c_2(\alpha). \tag{1.8}$$

In both investigations it could not be shown that $c_2 = c_2(\alpha)$ depends continuously on α. Thus, the estimate (1.8) remained open for $\alpha \in \mathbf{R}_+ \setminus \mathbf{Q}$. However, T. Ganelius was able to prove the somewhat weaker result

$$E_{nn}(x^\alpha, [0,1]) \leqslant c_2(\alpha) e^{-2\pi\sqrt{\alpha n} + c_3(\alpha)\sqrt[4]{n}} \quad \text{for } n \geqslant n_0(c_2(\alpha), c_3(\alpha)), \tag{1.9}$$

which holds for all $\alpha > 0$ (cf. [7]). In (1.9), $c_2(\alpha)$ and $c_3(\alpha)$ are constants depending only on α. For approximation in the L^p-norm, $1 < p < \infty$, the upper estimate (1.8) has been proved in [1] for all $\alpha > 0$; however, in the uniform norm the problem seems to have been solved only for rational α up to now.

The results (1.7)–(1.9) give the correct exponent $-2\pi\sqrt{\alpha n}$ in the error formula, but not much is said about the coefficient in front of the error formula. This problem has now been settled by Theorem 1. Like in the analogous situation in polynomial approximation, it is proved that the limit (1.2) exists and has the value given on the right-hand side of (1.2). Contrary to the estimate (1.8), the limit (1.2) holds for all $\alpha > 0$. Theorem 2 has been proved in [19] for the special case of $\alpha = 1$, which corresponds to $\alpha = \frac{1}{2}$ in Theorem 1. A simplified proof of this result has been given in [14, Chapter 8].

The investigation of strong error estimates with precise information about the leading coefficient in front of the error formula received a strong impetus from the surprising numerical results obtained by R. S. Varga, A. Ruttan and R. S. Carpenter in [26], [29] and [27]. Starting with numerical investigations of the Bernstein conjecture, R. S. Varga has developed numerical tools that are based on the Remez algorithm, Richardson extrapolation and the use of high numbers of significant digits, which allow mathematical conjectures to be checked by numerical means (for a survey of different applications, see [25]). In [28], R. S. Varga and R. S. Carpenter were the first to conjecture the concrete form of the right-hand side of (1.2). Independently, formula (1.2) was announced in [20]. The present research owes much to the impetus it received from Richard Varga's discoveries and numerical explorations.

The present paper is structured as follows: The proof of Theorem 1 will be prepared by auxiliary results in §§2–4. In §2 we prove several results about the behavior of the error function $f_\alpha - r^*_{mn}$. To a large extent these results are consequences of Chebyshev's theorem on alternation points. The results allow us to derive a rather explicit integral formula for the approximation error in §3. Besides of that in §3 results about the location of poles and zeros of the approximants $r^*_{n+1+[\alpha],n}$ are proved. In §4 we study an auxiliary function r_n. These investigations are rather technical. The proof of Theorem 1 is contained in §5. In the proof, a special logarithmic potential plays an essential role, which has already been studied in [19] and in [14, Chapter 8].

In the different sections the following mathematical tools are dominant: In §2 these are mainly results from the theory of best rational approximants, in §3 results from rational interpolation and multipoint Padé approximation, in §4 different techniques from complex analysis, and in §5 elements from potential theory.

2. Basic properties of rational best approximants

In the present section we show that the rational best approximants r^*_{mn} have maximal numerator and denominator degree. We further prove that Theorem 1 holds for all close-to-diagonal sequences if it holds for one of these sequences, and we investigate the extreme points of the error function $x^\alpha - r^*_{nm}(x)$ on $[0,1]$.

Since $r^*_{mn}(x) \equiv x^\alpha$ for $\alpha \in \mathbf{N}$ and $m > \alpha$, the limit (1.2) is trivial for $\alpha \in \mathbf{N}$, and we can assume without loss of generality that $\alpha \notin \mathbf{N}$. In the sequel we assume that $\alpha \in \mathbf{R}_+ \setminus \mathbf{N}$ is a fixed number, we set $f_\alpha := x^\alpha$.

LEMMA 1. *If the limit*

$$\lim_{n\to\infty} e^{2\pi\sqrt{\alpha n}} E_{n+k,n}(f_\alpha, [0,1]) = 4^{1+\alpha} |\sin \pi\alpha| \tag{2.1}$$

holds for one $k \in \mathbf{Z}$, then it holds for every $k \in \mathbf{Z}$.

Proof. Set $E_{mn} := E_{mn}(f_\alpha, [0,1])$. We have $E_{mn} \geqslant E_{MN}$ if $m \leqslant M$ and $n \leqslant N$. For $k_1, k_2 \in \mathbf{Z}$, $d := k_1 - k_2 > 0$, it follows that

$$\begin{aligned} e^{2\pi\sqrt{\alpha n}} E_{n+k_2,n} &\geqslant e^{2\pi\sqrt{\alpha n}} E_{n+k_1,n} \\ &\geqslant e^{2\pi\sqrt{\alpha(n+d)}} E_{(n+d)+k_2,n+d}\, e^{2\pi\sqrt{\alpha}\,(\sqrt{n}-\sqrt{n+d})}. \end{aligned} \tag{2.2}$$

Because of the estimate

$$e^{2\pi\sqrt{\alpha}\,(\sqrt{n}-\sqrt{n+d})} = e^{2\pi\sqrt{\alpha n}\,(1-\sqrt{1+d/n})} = 1 + \mathcal{O}\big(1/\sqrt{n}\big) \quad \text{as } n\to\infty, \tag{2.3}$$

it follows from the inequalities (2.2) that we have identical limits in (2.1) for $k_1>k_2$. The case $k_1<k_2$ can be treated in the same way. □

Lemma 1 shows that we can use any paradiagonal sequence $\{r^*_{n+k,n}\}_{n\in\mathbf{N}}$ in the proof of Theorem 1. It turns out that the sequence $\{r^*_{m_n n}\}_{n\in\mathbf{N}}$ with numerator degree

$$m_n := n+1+[\alpha], \quad n\in\mathbf{N}, \tag{2.4}$$

is suited best for carrying through the proof of Theorem 1. By $[\alpha]$ we denote the greatest integer not larger than α. In order to simplify notation in the sequel, the subindex m_n will be suppressed, i.e., we write r^*_n instead of $r^*_{m_n n}$.

Using estimate (2.3) and the inequalities (2.2), we see that Theorem 1 can be extended to a rather broad class of close-to-diagonal sequences. Of course, an analogous generalization of Theorem 2 holds true in the same way. We have

THEOREM 3. *For $\alpha>0$ and any sequence $\{(n_j,m_j)\in\mathbf{N}^2\,|\,j=1,2,\dots\}$ satisfying*

$$n_j+m_j\to\infty \quad \text{and} \quad |n_j-m_j| = o(\sqrt{n_j}) \quad \text{as } j\to\infty, \tag{2.5}$$

the limit

$$\lim_{j\to\infty} e^{2\pi\sqrt{\alpha n_j}} E_{m_j,n_j}(x^\alpha,[0,1]) = 4^{1+\alpha}|\sin\pi\alpha| \tag{2.6}$$

holds. By $o(\,\cdot\,)$ we denote Landau's little oh.

It has already been mentioned in the introduction that the approximants r^*_n uniquely exist for all $n\in\mathbf{N}$. In the next lemma more specific properties of the approximants r^*_n will be proved.

LEMMA 2. *The approximant r^*_n has exactly the numerator degree $m_n=n+1+[\alpha]$ and the denominator degree n. The error function*

$$e_n := f_\alpha - r^*_n, \quad n\in\mathbf{N}, \tag{2.7}$$

has exactly $m_n+n+2=2n+3+[\alpha]$ extreme points η_{nj} on $[0,1]$. With an appropriate numeration we can assume that

$$0=\eta_{n,0}<\eta_{n,1}<\dots<\eta_{n,2n+2+[\alpha]}=1, \tag{2.8}$$

and we have

$$\eta^\alpha_{nj} - r^*_n(\eta_{nj}) = (-1)^{j+1+[\alpha]}\varepsilon_n \quad \text{for } j=0,\dots,2n+2+[\alpha] \tag{2.9}$$

with

$$\varepsilon_n := E_{m_n,n}(f_\alpha,[0,1]). \tag{2.10}$$

Proof. Set $r_n^* = p_n/q_n$ with p_n and q_n coprime polynomials, $m' := \deg(p_n)$ and $n' := \deg(q_n)$. The restriction of the product $q_n e_n$ to $[0,1]$ belongs to the space

$$W_n := \operatorname{span}\{1, z, ..., z^{m'}, z^\alpha, ..., z^{\alpha+n'}\}. \tag{2.11}$$

Since W_n forms a Chebyshev system on $[0,1]$ of dimension $m'+n'+2$ (see [12, Chapter 1, §3]), we conclude that $q_n e_n$ has at most $m'+n'+1$ zeros on $[0,1]$, and consequently e_n has also at most $m'+n'+1$ zeros on $[0,1]$. Therefore, the error function e_n has at most $m'+n'+2$ alternation points on $[0,1]$.

From Chebyshev's theorem about alternation points for rational best approximants (see [14, Chapter 7, Theorem 2.6] or [15, Theorem 23]) we know that there exist $m_n+n+2-d$ points satisfying the alternation condition (2.9) and d is given by

$$d = \min(m_n - m', n - n'). \tag{2.12}$$

From the earlier upper estimate it then follows that

$$m'+n'+2 \geqslant m_n+n+2-d, \tag{2.13}$$

which implies that $d \geqslant 0$, and with (2.12), it further follows that

$$d \geqslant (m_n - m') + (n - n') \geqslant 2d. \tag{2.14}$$

Hence $d=0$, $m_n = m'$, and $n = n'$.

It remains only to show that the smallest and the largest extreme points $\eta_{n,0}$ and $\eta_{n,2n+2+[\alpha]}$, respectively, are the end points of the interval $[0,1]$ and that at $z=1$ we have $e_n(1) = -\varepsilon_n$. Indeed, if one of the two points $\eta_{n,0}$ or $\eta_{n,2n+2+[\alpha]}$ were not an end point of $[0,1]$, then there would exist a constant $c \in \mathbf{R}$ such that

$$e_n - c = f_\alpha - (r_n^* + c) \tag{2.15}$$

has at least m_n+n+2 zeros in $[0,1]$. But this contradicts the fact that the restriction of $q_n(e_n - c)$ to $[0,1]$ belongs to W_n. For $z \in \mathbf{R}_+$ near infinity we have $e_n(z) < 0$. Since e_n can have no sign change on $(1, \infty)$, it follows that $e_n(1) = -\varepsilon_n$. □

As an immediate consequence of Lemma 2 we know that the error function e_n has $m_n+n+1 = 2n+2+[\alpha]$ different zeros z_{nj} in the open interval $(0,1)$; more precisely, we have

$$\eta_{n,j-1} < z_{nj} < \eta_{nj} \quad \text{for } j = 1, ..., 2n+2+[\alpha] \tag{2.16}$$

and

$$e_n(z_{nj}) = z_{nj}^\alpha - r_n^*(z_{nj}) = 0 \quad \text{for } j = 1, ..., 2n+2+[\alpha]. \tag{2.17}$$

From (2.17) we deduce that the rational best approximant $r_n^* \in \mathcal{R}_{n+1+[\alpha],n}$ interpolates f_α at the $2n+2+[\alpha]$ points z_{nj}. In the next section we shall see that this interpolation property has interesting consequences, and it allows us to prove basic properties of the rational approximant r_n^*.

3. Consequences of the interpolation property

An explicit formula for the approximation error $e_n = f_\alpha - r_n^*$ will be derived, and some information about the location of poles and zeros of the approximant r_n^* will be given. Unfortunately the location of some zeros of r_n^* remains unclear. This lack of more precise knowledge will cause a lot of additional work in §§ 4 and 5.

We denote by w_n the polynomial

$$w_n(z) := \prod_{j=1}^{2n+2+[\alpha]} (z - z_{nj}), \tag{3.1}$$

where the z_{nj} are the zeros of e_n introduced in (2.16). Since $z_{nj} \in (0,1)$, $j = 1, ..., 2n+2+[\alpha]$, we have

$$\operatorname{sign} w_n(z) = (-1)^{[\alpha]} \quad \text{for } z \in \mathbf{R}_- := \{x \in \mathbf{R} \mid x \leqslant 0\}. \tag{3.2}$$

For formula (3.3) we make the temporary assumption that $-1 < \alpha < 0$. If C is an integration path in $\mathbf{C} \setminus \mathbf{R}_-$ surrounding z, then from Cauchy's integral formula it follows that the principal branch of f_α can be represented as

$$f_\alpha(z) = \frac{1}{2\pi i} \oint_C \frac{\zeta^\alpha \, d\zeta}{\zeta - z} = \frac{\sin \pi\alpha}{\pi} \int_{-\infty}^0 \frac{|x|^\alpha \, dx}{x - z} \quad \text{for } z \in \mathbf{C} \setminus \mathbf{R}_-. \tag{3.3}$$

The second equality in (3.3) results from moving C towards $\mathbf{R}_-$. The second integral exists because of our temporary assumption that the integrand has a zero of order $1 - \alpha = 1 + |\alpha| > 1$ at infinity, and a pole of order $\alpha > -1$ at the origin.

The representation (3.3) shows that f_α is a Stieltjes function if $\alpha \in (-1, 0)$. From the standard theory of rational interpolants to Stieltjes or Markov functions we have rather detailed information about the structure of these interpolants (cf. [11] or [22, Chapter 6.1]). If $\alpha > 0$, then the last integral in (3.3) does no longer exist. But, nevertheless, we can deduce results similar to those that hold in the case of Stieltjes functions (see for more details [22, Chapters 6.1–6.3]). In the sequel we assume as before that $\alpha \in \mathbf{R}_+ \setminus \mathbf{N}$.

LEMMA 3. *Set $r_n^* = p_n / q_n$, $q_n(z) = z^n + ... \in \mathcal{P}_n$, $p_n \in \mathcal{P}_{n+1+[\alpha]}$. The denominator polynomial q_n satisfies the orthogonality relation*

$$\int_{-\infty}^0 x^j q_n(x) \frac{|x|^\alpha \, dx}{w_n(x)} = 0 \quad \textit{for } j = 0, ..., n-1. \tag{3.4}$$

The n zeros $\pi_{n,1}, ..., \pi_{nn}$ of q_n are all simple and contained in $(-\infty, 0)$. With an appropriate numeration we have

$$-\infty < \pi_{n,1} < ... < \pi_{nn} < 0. \tag{3.5}$$

For the approximation error $e_n = f_\alpha - r_n^*$ *we have the representation*

$$e_n(z) = \frac{\sin \pi\alpha}{\pi} \frac{w_n(z)}{(g_n q_n)(z)} \int_{-\infty}^0 \frac{(g_n q_n)(x)|x|^\alpha \, dx}{w_n(x)(x-z)} \quad \textit{for } z \in \mathbf{C}\setminus\mathbf{R}_-, \tag{3.6}$$

where $g_n \in \mathcal{P}_n \setminus \{0\}$ *is an arbitrary polynomial.*

Remark. Because of (3.2) the measure μ_n defined by

$$\frac{d\mu_n}{dx}(x) := \frac{\sin \pi\alpha}{\pi} \frac{|x|^\alpha}{w_n(x)} \, dx, \quad x \in \mathbf{R}_-, \tag{3.7}$$

is positive. Since w_n is of degree $2n+2+[\alpha]$, and since all zeros of w_n are contained in $(0,1)$, the mass of μ_n is finite. We have $\operatorname{supp}(\mu_n) = \mathbf{R}_-$ for all $n \in \mathbf{N}$.

Proof. The interpolation property (2.17) of r_n^* implies that the expression

$$\frac{q_n f_\alpha - p_n}{w_n}(z) \quad \text{is analytic in } \mathbf{C}\setminus\mathbf{R}_-. \tag{3.8}$$

Let C be a positively oriented, closed integration path in $\mathbf{C}\setminus\mathbf{R}_-$ surrounding all interpolation points z_{nj}, and let $g_n \in \mathcal{P}_n \setminus \{0\}$. Cauchy's integration formula yields

$$\begin{aligned} \left(g_n \frac{q_n f_\alpha - p_n}{w_n}\right)(z) &= \frac{1}{2\pi i} \oint_C g_n(\zeta) \frac{q_n(\zeta)\zeta^\alpha - p_n(\zeta)}{w_n(\zeta)} \frac{d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i} \oint_C \frac{(g_n q_n)(\zeta)\zeta^\alpha \, d\zeta}{w_n(\zeta)(\zeta - z)} - \frac{1}{2\pi i} \oint_C \frac{(g_n p_n)(\zeta) \, d\zeta}{w_n(\zeta)(\zeta - z)} \end{aligned} \tag{3.9}$$

for $z \in \operatorname{Int}(C)$. The last term on the last line of (3.9) is identically zero since the integrand is analytic outside of C and has a zero of order $\geqslant 2$ at infinity. Hence, we have

$$\left(g_n \frac{q_n f_\alpha - p_n}{w_n}\right)(z) = \frac{1}{2\pi i} \oint_C \frac{(g_n q_n)(\zeta)\zeta^\alpha \, d\zeta}{w_n(\zeta)(\zeta - z)}. \tag{3.10}$$

For any $g_n \in \mathcal{P}_n$ the integrand in (3.10) has a zero of order larger than 1 at infinity. Therefore, in (3.10) we can shrink the integration path C to $\mathbf{R}_-$, which yields

$$\left(g_n \frac{q_n f_\alpha - p_n}{w_n}\right)(z) = \frac{\sin \pi\alpha}{\pi} \int_{-\infty}^0 \frac{(g_n q_n)(x)}{w_n(x)} \frac{|x|^\alpha \, dx}{x - z} \quad \text{for } z \in \mathbf{C}\setminus\mathbf{R}_-. \tag{3.11}$$

From that formula, (3.6) follows immediately.

Taking $g_n(z) = z^{j+1}$ with $j = 0, ..., n-1$ and considering (3.11) near the origin yields

$$\frac{\sin \pi\alpha}{\pi} \int_{-\infty}^0 x^j q_n(x) \frac{|x|^\alpha \, dx}{w_n(x)} = 0 \quad \text{for } j = 0, ..., n-1, \tag{3.12}$$

which proves (3.4).

We know from (3.7) that the measure μ_n is positive. Since relation (3.12) shows that q_n is orthogonal with respect to this positive measure, it follows from the elementary theory of orthogonal polynomials that all zeros of q_n are simple and contained in the interior of $\operatorname{supp}(\mu_n)=\mathbf{R}_-$ (cf. [23, Chapter III]). This proves (3.5). □

Since we know that the best approximant r_n^* has only simple poles, we have the partial fraction representation

$$r_n^*(z) = h_n(z)+\sum_{j=1}^{n} \frac{\lambda_{nj}}{z-\pi_{nj}} \tag{3.13}$$

with h_n a polynomial of the form

$$h_n(z) = A_n z^{[\alpha]+1}+\ldots \in \mathcal{P}_{[\alpha]+1}. \tag{3.14}$$

If we multiply the error function $e_n=f_\alpha-r_n^*$ by $z-\pi_{nj}$, $j=1,\ldots,n$, and choose in formula (3.6) $g_n:=q_n/(\cdot-\pi_{nj})$, then we have

$$\begin{aligned}-\lambda_{nj} = [(z-\pi_{nj})e_n(z)]_{z=\pi_{nj}} &= \frac{\sin\pi\alpha}{\pi}\,\frac{w_n(\pi_{nj})}{q_n'(\pi_{nj})^2}\int_{-\infty}^{0}\left(\frac{q_n(x)}{x-\pi_{nj}}\right)^2\frac{|x|^\alpha\,dx}{w_n(x)}\\ &= \frac{\sin\pi\alpha}{\pi}\,w_n(\pi_{nj})\int_{-\infty}^{0} l_{nj}(x)^2\frac{|x|^\alpha}{w_n(x)}\,dx, \quad j=1,\ldots,n,\end{aligned} \tag{3.15}$$

where $l_{nj}\in\mathcal{P}_{n-1}$ is the Lagrangian basis polynomial satisfying $l_{nj}(\pi_{ni})=\delta_{ij}$ for $i,j=1,\ldots,n$. Formula (3.6) holds only for $z\notin\mathbf{R}_-$; however, the extension to $z=\pi_{nj}\in\mathbf{R}_-$ is possible from both sides of $\mathbf{R}_-$ for the specific choice of g_n. Note that the integrand in (3.6) remains bounded if z tends to π_{ni} vertically to the real line. From (3.15), the positivity of the measure μ_n in (3.7), and (3.6), it then follows that

$$(-1)^{[\alpha]+1}\lambda_{nj} > 0 \quad \text{for } j=1,\ldots,n. \tag{3.16}$$

From the error formula (3.6) we can deduce also an expression for the leading coefficient A_n in (3.14). We have

$$e_n(z) = (f_\alpha-r_n^*)(z) = z^\alpha - A_n z^{[\alpha]+1}+\mathcal{O}(z^{[\alpha]}) \quad \text{as } z\to\infty. \tag{3.17}$$

Inserting $g_n:=q_n$ into formula (3.6) and multiplying by $z^{-[\alpha]-1}$ yields

$$\begin{aligned}A_n = -[z^{-[\alpha]-1}e_n(z)]_{z=\infty} &= \frac{\sin\pi\alpha}{\pi}\lim_{z\to\infty}\frac{w_n(z)}{z^{[\alpha]+2}q_n(z)^2}\int_{-\infty}^{0}\frac{q_n(x)^2|x|^\alpha\,dx}{w_n(x)(1-x/z)}\\ &= \frac{\sin\pi\alpha}{\pi}\int_{-\infty}^{0}\frac{q_n(x)^2}{w_n(x)}|x|^\alpha\,dx,\end{aligned} \tag{3.18}$$

which implies together with (3.7) that $A_n>0$. From (3.13), (3.16) and (3.18) we deduce the next lemma.

LEMMA 4. *Let ζ_{nj}, $j=1,\ldots,n+1+[\alpha]$, be the zeros of the numerator polynomial p_n, i.e.,*

$$p_n(z) = A_n \prod_{j=1}^{n+1+[\alpha]} (z-\zeta_{nj}). \tag{3.19}$$

The zeros ζ_{nj} can be numbered in such a way that the $n+1$ first zeros of p_n lie on $\mathbf{R}_-$ and satisfy the inequalities

$$-\infty < \zeta_{n,1} < \pi_{n,1} < \zeta_{n,2} < \pi_{n,2} < \ldots < \zeta_{nn} < \pi_{nn} < \zeta_{n,n+1} < 0. \tag{3.20}$$

Remarks. (1) The inequalities in (3.20) complement those from (3.5).

(2) In the lemma nothing has been said about the location of the $[\alpha]$ zeros $\zeta_{n,n+2},\ldots,\zeta_{n,n+1+[\alpha]}$ of r_n^*, which do not appear in (3.20). It will be shown below that these zeros converge to the origin with a certain speed. It follows from the proof of Lemma 4 that for $[\alpha]$ odd there exists at least one positive zero of r_n^* on $(0,1)$.

Proof. From (3.16) we know that all coefficients λ_{nj}, $j=1,\ldots,n$, in (3.13) have the same sign for a given n. Hence, between two adjacent poles π_{nj} and $\pi_{n,j+1}$, $j=1,\ldots,n-1$, there lies at least one zero of r_n^*.

Since $A_n>0$, it follows from (3.16) and (3.14) that

$$\lambda_{n,1} r_n^*(z) > 0 \tag{3.21}$$

for $z\in\mathbf{R}_-$ near infinity, and from (3.13) and (3.21) we then deduce that r_n^* has a sign change between $-\infty$ and π_{nn}. Hence, there is at least one zero in the interval $(-\infty,\pi_{nn})$.

If we choose $g_n:=q_n$ in formula (3.6), then we deduce from the positivity of the measure μ_n defined in (3.7) that

$$e_n(z) < 0 \quad \text{for all } z\in[1,\infty). \tag{3.22}$$

From (2.8) and (2.9) we know that

$$r_n^*(0) = -e_n(0) = (-1)^{[\alpha]}\varepsilon_n. \tag{3.23}$$

This together with (3.13) and (3.16) shows that r_n^* has a sign change between $\pi_{n,1}$ and the origin. Hence, there is at least one zero in the interval $(\pi_{n,1},0)$.

If $[\alpha]$ is odd, then it follows from $\varepsilon_n>0$ and (3.23) that $r_n^*(0)<0$ and $r_n^*(1)>0$, and therefore there exists at least one zero of r_n^* in the interval $(0,1)$. □

4. Auxiliary functions I

In the present section a function r_n will be studied which is a rational transformation of the error function e_n. This type of function played already a fundamental role in D. J. Newman's paper [16]. In the proof of Theorem 1, below, we have to use a further refined machinery, which includes a quadratic transformation that will be studied in §6. The results of the present section lay the ground work for these later investigations. Two of the four lemmas demand quite lengthy and involved proofs.

The auxiliary function r_n is defined as

$$r_n(z) := \frac{f_\alpha(z) - r_n^*(z)}{f_\alpha(z) + r_n^*(z)} = \frac{e_n(z)}{2f_\alpha(z) - e_n(z)} = \frac{1 - z^{-\alpha} r_n^*(z)}{1 + z^{-\alpha} r_n^*(z)} \tag{4.1}$$

for $z \in \mathbf{C} \setminus \mathbf{R}_-$.

In the next lemma we assemble properties of r_n which follow directly from the definition in (4.1) or from properties of the extreme points η_{nj} of the error function e_n that have been introduced and studied in Lemma 2. Note that in (2.10) we have introduced the abbreviation $\varepsilon_n := E_{n+1+[\alpha],n}(f_\alpha, [0,1])$.

LEMMA 5. *We have*

$$r_n(z) \geqslant \frac{-\varepsilon_n}{2z^\alpha + \varepsilon_n} \quad \textit{for } z \in [0,1], \tag{4.2}$$

$$r_n(z) \leqslant \frac{\varepsilon_n}{2z^\alpha - \varepsilon_n} \quad \textit{for } z \in \left[\left(\tfrac{1}{2}\varepsilon_n\right)^{1/\alpha}, 1\right]. \tag{4.3}$$

At the $2n+3+[\alpha]$ *extreme points* η_{nj} *of the error function* e_n, *the function* r_n *assumes the values*

$$r_n(\eta_{nj}) = \frac{(-1)^{j+1+[\alpha]}\varepsilon_n}{2\eta_{nj}^\alpha + (-1)^{j+[\alpha]}\varepsilon_n}, \quad j = 0, ..., 2n+2+[\alpha]. \tag{4.4}$$

At the zeros z_{nj} *of the error function* e_n, *and at the poles* π_{nj} *and the zeros* ζ_{nj} *of the approximant* r_n^*, *the function* r_n *assumes the following values*:

$$r_n(z_{nj}) = 0, \qquad j = 1, ..., 2n+2+[\alpha], \tag{4.5}$$

$$r_n(\pi_{nj}) = r_n(0) = r_n(\infty) = -1, \quad j = 1, ..., n, \tag{4.6}$$

$$r_n(\zeta_{nj}) = 1, \qquad j = 1, ..., n+1+[\alpha]. \tag{4.7}$$

The function r_n *has no other zeros in* $\mathbf{C} \setminus \mathbf{R}_-$ *than those given in* (4.5).

If K_α *denotes the disk*

$$K_\alpha := \left\{ z \in \mathbf{C} \;\middle|\; |z + i \cot \pi\alpha| < \frac{1}{|\sin \alpha\pi|} \right\} \tag{4.8}$$

(note that we have assumed $\alpha\notin\mathbf{N}$*), then we have*

$$r_n(z)\in\begin{cases}\partial K_\alpha & \textit{for } z\in\mathbf{R}_-+i0,\\ \overline{\partial K_\alpha}:=\{\bar z\mid z\in\partial K_\alpha\} & \textit{for } z\in\mathbf{R}_--i0,\end{cases}\tag{4.9}$$

where the two banks of $\mathbf{R}_-$ *are denoted by* $\mathbf{R}_-\pm i0$.

Proof. The assertions (4.2)–(4.7) follow immediately from the definition of r_n in (4.1) together with (2.8) and (2.9) in Lemma 2, and the assertions (3.17), (3.18) and (3.23). From (1.1), the error formula (3.6), and the fact that all poles of $f_\alpha-r_n^*$ are contained in $\mathbf{R}_-\cup\{\infty\}$, it follows that the function r_n has no other zeros in $\mathbf{C}\backslash\mathbf{R}_-$ than those given in (4.5).

The mapping $g:\overline{\mathbf{R}}\to\partial K_\alpha$ defined by

$$r\mapsto g(r)=\frac{1-re^{-i\pi\alpha}}{1+re^{-i\pi\alpha}}=\frac{1-r^2+2ir\sin\pi\alpha}{1+r^2+2r\cos\pi\alpha}\tag{4.10}$$

is bijective, and we have

$$g(0)=1,\quad g(1)=i\tan\left(\tfrac12\pi\alpha\right),\quad g(-1)=-i\cot\left(\tfrac12\pi\alpha\right),\quad g(\infty)=-1.\tag{4.11}$$

At the values $g(1)$ and $g(-1)$ the smallest and largest modulus on the circle ∂K_α is assumed. The assertions in (4.9) follow from a comparison of the last term in (4.1) with (4.10). Note that $r_n^*(z)\in\overline{\mathbf{R}}$ for all $z\in\mathbf{R}_-$. □

Since we know from (3.16) that all coefficients λ_{nj}, $j=1,\dots,n$, in the partial fraction representation (3.13) have identical signs, the value $r_n^*(x)$ runs through the extended real line $\overline{\mathbf{R}}$ when x is moved along the interval $(\pi_{nj},\pi_{n,j+1})$ with π_{nj} and $\pi_{n,j+1}$ two adjacent poles. From the definition of the function r_n in (4.1) and the bijectivity of the mapping (4.10) it follows that $\arg r_n(z)$ grows exactly by 2π if z is moved from π_{nj} to $\pi_{n,j+1}$ on $\mathbf{R}_-+i0$. Correspondingly, $\arg r_n(z)$ grows by 2π if z is moved in the opposite direction from $\pi_{n,j+1}$ to π_{nj} on the other bank $\mathbf{R}_--i0$ of $\mathbf{R}_-$. Because of (4.6) the same conclusions hold for the intervals $(-\infty,\pi_{nn})$ and $(\pi_{n,1},0)$, since from (3.17), (3.18) and (3.23) we know that at infinity and at the origin r_n^* is the dominant term in e_n.

The information about the poles of r_n^* established in the inequalities (3.5) of Lemma 3 together with the considerations just made show that $\arg r_n(z)$ grows by $4\pi(n+1)$ if z moves once around the boundary of the domain $\mathbf{C}\backslash\mathbf{R}_-$. This boundary consists of the two banks $\mathbf{R}_-+i0$ and $\mathbf{R}_--i0$ of $\mathbf{R}_-$. At ∞ the function r_n has the boundary value -1 for all limiting directions. From Lemma 5 we know that r_n has exactly $2n+2+[\alpha]$ simple zeros in $\mathbf{C}\backslash\mathbf{R}_-$. These are the zeros of the polynomial w_n. Since the growth of $\arg r_n(z)$ along the boundary of $\mathbf{C}\backslash\mathbf{R}_-$ is $4\pi(n+1)$, it follows from the argument principle that the function r_n has poles with a total order $[\alpha]$ in $\mathbf{C}\backslash\mathbf{R}_-$. These poles will be denoted by

$$b_{n,1}, \ldots, b_{n,[\alpha]} \in \mathbf{C}\backslash\mathbf{R}_-. \tag{4.12}$$

The precise location of the poles $b_{n,1}, \ldots, b_{n,[\alpha]}$ seems difficult to determine, but in the next lemma we shall show that they converge to the origin with a certain speed as $n \to \infty$. It turns out that the same behavior can be proved for the $[\alpha]$ zeros $\zeta_{n,n+2}, \ldots, \zeta_{n,n+1+[\alpha]}$ of the approximant r_n^* that have not been covered by assertion (3.20) in Lemma 4. As before, the approximation error $E_{n+1+[\alpha],n}(f_\alpha, [0,1])$ is denoted by ε_n.

LEMMA 6. *For each* $j=1, \ldots, [\alpha]$ *we have*

$$b_{nj} = \mathcal{O}(\varepsilon_n^{1/\alpha}) \quad \textit{as } n \to \infty, \tag{4.13}$$

$$\zeta_{n,n+1+j} = \mathcal{O}(\varepsilon_n^{1/\alpha}) \quad \textit{as } n \to \infty. \tag{4.14}$$

Proof. (i) Transformations $z \mapsto w := z/a_n$ of the independent variable z will play a fundamental role. In the first part of the proof the sequence $\{a_n\}_{n=1}^{\infty}$ will be chosen in such a way that it converges to zero slower than $\{\varepsilon_n\}_{n=1}^{\infty}$.

We start with the introduction of some technical notations. The function $B(z,x)$ is defined by

$$B(z,x) := \frac{\sqrt{z} - \sqrt{x}}{\sqrt{z} + \sqrt{\bar{x}}}, \quad z, x \in \mathbf{C}\backslash\mathbf{R}_-, \tag{4.15}$$

with $\sqrt{\cdot}$ denoting the principal branch. We have

$$\begin{aligned} |B(z,x)| &< 1 \quad \text{for } z, x \in \mathbf{C}\backslash\mathbf{R}_-, \\ |B(z,x)| &= 1 \quad \text{for } z \in \mathbf{R}_- \pm i0,\ x \in \mathbf{C}\backslash\mathbf{R}_-, \\ |B(x,x)| &= 0 \quad \text{for } x \in \mathbf{C}\backslash\mathbf{R}_-. \end{aligned} \tag{4.16}$$

If we set

$$\hat{r}_n(z) := r_n(z) Q_n(z), \quad Q_n(z) := \prod_{j=1}^{[\alpha]} B(z, b_{nj}), \tag{4.17}$$

then $\hat{r}_n$ is analytic in $\mathbf{C}\backslash\mathbf{R}_-$, and it follows from (4.9) in Lemma 5 together with (4.11) that

$$|\hat{r}_n(z)| \leqslant \max\left(\left|\tan \tfrac{1}{2}\pi\alpha\right|, \left|\cot \tfrac{1}{2}\pi\alpha\right|\right) \quad \text{for } z \in \mathbf{C}, \tag{4.18}$$

where in case of $z \in \mathbf{R}_-$ the point z can lie on each one of the two banks of $\mathbf{R}_-$.

We now assume that the sequence $\{a_n \in \mathbf{R}_+ \mid n=1, \ldots, \infty\}$ satisfies

$$\lim_{n\to\infty} a_n = 0 \tag{4.19}$$

and

$$\varepsilon_n^{1/\alpha} = o(a_n) \quad \text{as } n \to \infty \tag{4.20}$$

with $o(\cdot)$ denoting Landau's little oh. As new independent variable we define

$$w := z/a_n. \tag{4.21}$$

Functions or constants resulting from transformation (4.21) are marked by a tilde, i.e., we set

$$\tilde{r}_n(w) := r_n(a_n w), \quad \tilde{\hat{r}}_n(w) := \hat{r}_n(a_n w), \quad \tilde{b}_{nj} := b_{nj}/a_n, \quad \tilde{\eta}_{nj} := \eta_{nj}/a_n, \quad \dots . \tag{4.22}$$

For $w \in \left(\left(\frac{1}{2}\varepsilon_n\right)^{1/\alpha}/a_n, 1/a_n\right]$ we deduce from (4.3) that

$$|\tilde{r}_n(w)| \leqslant \frac{\varepsilon_n}{2(a_n w)^\alpha - \varepsilon_n} = \frac{\varepsilon_n a_n^{-\alpha}}{2w^\alpha - \varepsilon_n a_n^{-\alpha}} = \mathcal{O}(\varepsilon_n a_n^{-\alpha}) \quad \text{as } n \to \infty. \tag{4.23}$$

Because of (4.20), the estimate (4.23) implies that $\tilde{r}_n(w) = o(1)$ as $n \to \infty$ uniformly on compact subsets of $(0, \infty)$, because of (4.16) and (4.17) the function $|\tilde{r}_n|$ dominates $|\tilde{\hat{r}}_n|$ in $\mathbf{C}\setminus\mathbf{R}_-$, and because of (4.18) the function $\tilde{\hat{r}}_n$ is analytic and bounded in $\mathbf{C}\setminus\mathbf{R}_-$. It therefore follows from (4.23) that

$$\lim_{n\to\infty} \tilde{\hat{r}}_n(w) = 0 \quad \text{locally uniformly for } w \in \mathbf{C}\setminus\mathbf{R}_-. \tag{4.24}$$

From any infinite sequence $N \subseteq \mathbf{N}$ we can select an infinite subsequence, which we continue to denote by N, such that the limits

$$\tilde{b}_{nj} = b_{nj}/a_n \to \tilde{b}_j, \quad \tilde{\zeta}_{n,n+1+j} = \zeta_{n,n+1+j}/a_n \to \tilde{\zeta}_j \quad \text{as } n \to \infty,\ n \in N, \tag{4.25}$$

exist in the cordial metric for $j = 1, \dots, [\alpha]$, $\tilde{b}_j, \tilde{\zeta}_j \in \overline{\mathbf{C}}$. For the functions $\widetilde{Q}_n(w) := Q_n(a_n w)$ with Q_n defined in (4.17), we have

$$\widetilde{Q}_n(w) \to \widetilde{Q}(w) := \prod_{j=1}^{[\alpha]} B(w, \tilde{b}_j) \quad \text{as } n \to \infty,\ n \in N, \tag{4.26}$$

locally uniformly in $\mathbf{C}\setminus\mathbf{R}_-$. Note that $B(z,x)$ is invariant under scale changes, i.e., $B(z,x) = B(az, ax)$ for all $a \in \mathbf{R}_+$. From (4.17), (4.24) and (4.26), we deduce that

$$\lim_{\substack{n\to\infty \\ n\in N}} \tilde{r}_n(w) = 0 \quad \text{locally uniformly for } w \in \mathbf{C}\setminus\left(\mathbf{R}_- \cup \{\tilde{b}_j\}_{j=1}^{[\alpha]}\right). \tag{4.27}$$

From (4.27), the third term in (4.1), and the definition $\tilde{r}_n^*(w) := r_n^*(a_n w)/a_n^\alpha$, it then follows that

$$\lim_{\substack{n\to\infty \\ n\in N}} \tilde{r}_n^*(w) = w^\alpha \quad \text{locally uniformly for } w \in \mathbf{C}\setminus\mathbf{R}_-. \tag{4.28}$$

Note that $\tilde{r}_n^*$ is a rational function with all its poles in $\mathbf{R}_-$.

As an immediate consequence of (4.28) it follows that the zeros $\tilde{\zeta}_{n,n+1+j}$, $j=1, ..., [\alpha]$, of the approximant r_n^* can cluster only on $\mathbf{R}_-$. However, this is not good enough for a proof of (4.13) and (4.14), we have to prove that the $[\alpha]$ zeros $\tilde{\zeta}_{n,n+1+j}$, $j=1, ..., [\alpha]$, converge to the origin. For this we need a more detailed analysis, which will be carried out next.

(ii) Let the rational function $\tilde{r}_n^+$ be defined by the factorization

$$\tilde{r}_n^* = \tilde{r}_n^+ \tilde{p}_n^* \quad \text{with } \tilde{p}_n^*(w) := \prod_{j=1}^{[\alpha]} (w - \tilde{\zeta}_{n,n+1+j}). \tag{4.29}$$

It follows from Lemma 4 that $\tilde{r}_n^+$ is a rational function of numerator degree $n+1$ and denominator degree n having all its zeros and poles interlacing and lying on $\mathbf{R}_-$. As a consequence of the interlacing property given in (3.20), and since $A_n > 0$ in (3.19), we have

$$0 \leqslant \arg \tilde{r}_n^+(w) \leqslant \arg(w) \quad \text{for } w \in \overline{H}_+\setminus\{0\} \tag{4.30}$$

with $H_+ := \{w \mid \mathrm{Im}(w) > 0\}$. Corresponding inequalities hold for $w \in \overline{H}_-\setminus\{0\}$ with H_- denoting the lower half-plane. From (4.28), (4.29) and (4.30) it follows that necessarily we have

$$\lim_{\substack{n\to\infty \\ n\in N}} \tilde{\zeta}_{n,n+1+j} = \tilde{\zeta}_j = 0 \quad \text{for } j = 1, ..., [\alpha], \tag{4.31}$$

and

$$\lim_{\substack{n\to\infty \\ n\in N}} \arg \tilde{p}_n^*(w) = [\alpha] \arg(w) \quad \text{locally uniformly for } w \in \mathbf{C}\setminus\mathbf{R}_-. \tag{4.32}$$

Note that because of (4.31) $\arg \tilde{p}_n^*$ is well defined in $\mathbf{C}\setminus(\mathbf{R}_-\cup\{|w|\leqslant\varepsilon\})$ for any $\varepsilon > 0$ and $n \in \mathbf{N}$ sufficiently large if on (ε, ∞) we start with the principal branch of the argument function.

We now assume that (4.14) is false. Then there exists a sequence $a_n > 0$, $n \in N$, with $N \subseteq \mathbf{N}$ an infinite subsequence such that the sequence $\{a_n\}_{n\in N}$ satisfies (4.19), (4.20) and

$$a_n \leqslant \max\{|\zeta_{n,n+2}|, ..., |\zeta_{n,n+1+[\alpha]}|\} \quad \text{for all } n \in N. \tag{4.33}$$

The sequence N contains an infinite subsequence, which we continue to denote by N, such that the limits (4.25), (4.26), and consequently also the limits (4.31) and (4.32), exist.

From the inequalities (4.33) we conclude that at least one of the limits $\tilde{\zeta}_j$, $j=1,\dots,[\alpha]$, introduced in (4.31), is of modulus larger than or equal to 1. However, this contradicts the conclusions made in (4.31), and thus proves (4.14).

(iii) From definition (4.1) and from the definition of $\tilde{r}_n^*$ made just before (4.28), we immediately deduce that $\tilde{r}_n^*(\tilde{b}_{nj})=-\tilde{b}_{nj}^\alpha$ holds for each of the poles listed in (4.12), which implies that

$$\arg \tilde{r}_n^*(\tilde{b}_{nj}) = \alpha \arg \tilde{b}_{nj} \pm \pi \mod(2\pi) \quad \text{for } j=1,\dots,[\alpha]. \tag{4.34}$$

Let again $\{a_n\}_{n\in N}$ be a sequence that satisfies the assumptions (4.19) and (4.20) with $N\subseteq \mathbf{N}$ an infinite subsequence substituting $\mathbf{N}$ in (4.19) and (4.20), and assume that the limits (4.25) and (4.26) exist. From (4.29), (4.30) and (4.32) we deduce that the estimates

$$[\alpha]\arg(w) \leqslant \liminf_{\substack{n\to\infty \\ n\in N}} \arg \tilde{r}_n^*(w) \leqslant \limsup_{\substack{n\to\infty \\ n\in N}} \arg \tilde{r}_n^*(w) \leqslant (1+[\alpha])\arg(w) \tag{4.35}$$

hold for w uniformly on compact subsets of $\overline{H}_+\setminus\{0\}$. On $\overline{H}_-\setminus\{0\}$ corresponding estimates hold. The function $\arg \tilde{r}_n^*$ is well defined in $\mathbf{C}\setminus(\mathbf{R}_-\cup\{|w|\leqslant\varepsilon\})$ for any $\varepsilon>0$ and $n\in\mathbf{N}$ sufficiently large if on (ε,∞) one starts with the principal branch of the argument function. On $(-\infty,-\varepsilon)\pm i0$ the function $\arg \tilde{r}_n^*$ is defined by continuation from both sides.

From (4.34), (4.35) and the corresponding estimates in $\overline{H}_-\setminus\{0\}$, it then follows that we necessarily have

$$\lim_{\substack{n\to\infty \\ n\in N}} \tilde{b}_{nj} = \lim_{\substack{n\to\infty \\ n\in N}} b_{nj}/a_n = 0 \quad \text{for } j=1,\dots,[\alpha]. \tag{4.36}$$

Note that for finite $n\in\mathbf{N}$, it follows from Lemma 5 that $b_{nj}\notin\mathbf{R}_-$ for $j=1,\dots,[\alpha]$.

Let us now assume that (4.13) is false. Then there exists a sequence $a_n>0$, $n\in N$, with $N\subseteq\mathbf{N}$ an infinite subsequence such that the sequence $\{a_n\}_{n\in N}$ satisfies (4.19), (4.20), and we have

$$a_n \leqslant \max\{|b_{n,1}|,\dots,|b_{n,[\alpha]}|\} \quad \text{for all } n\in N. \tag{4.37}$$

As a consequence we know from (4.36) that each sequence $\{b_{nj}/a_n\}_{n\in N}$, $j=1,\dots,[\alpha]$, contains an infinite subsequence that converges to 0. However, this contradicts the inequalities (4.37), and therefore it proves (4.13). □

While the last lemma already demanded a rather involved proof, the next one will be not less complicated to prove, and in addition also its statements are rather technical and lengthy.

LEMMA 7. (i) *Any infinite sequence* $N \subseteq \mathbf{N}$ *contains an infinite subsequence, which we continue to denote by* N, *such that the limits*

$$\lim_{\substack{n\to\infty\\ n\in N}} \varepsilon_n^{-1/\alpha} z_{nj} =: \tilde{z}_j \in [0,\infty), \tag{4.38}$$

$$\lim_{\substack{n\to\infty\\ n\in N}} \varepsilon_n^{-1/\alpha} \zeta_{n,n+2-j} =: \tilde{\zeta}_j \in (-\infty,0), \tag{4.39}$$

$$\lim_{\substack{n\to\infty\\ n\in N}} \varepsilon_n^{-1/\alpha} \pi_{n,n+1-j} =: \tilde{\pi}_j \in (-\infty,0) \tag{4.40}$$

exist for $j=1,2,\ldots$, *the limits*

$$\lim_{\substack{n\to\infty\\ n\in N}} \varepsilon_n^{-1/\alpha} \zeta_{n,n+1-j} =: \tilde{a}_j \in \mathbf{C}\setminus\mathbf{R}_-, \tag{4.41}$$

$$\lim_{\substack{n\to\infty\\ n\in N}} \varepsilon_n^{-1/\alpha} b_{nj} =: \tilde{b}_j \in \mathbf{C}\setminus\mathbf{R}_- \tag{4.42}$$

exist for $j=1,\ldots[\alpha]$, *the limit*

$$\lim_{\substack{n\to\infty\\ n\in N}} r_n(\varepsilon_n^{1/\alpha} w) =: \tilde{r}(w) \tag{4.43}$$

exists in the cordial metric uniformly for w *varying on compact subsets of* $(\mathbf{C}\setminus\mathbf{R}_-)\cup(-\infty,0)\pm i0$, *and the limit*

$$\lim_{\substack{n\to\infty\\ n\in N}} \frac{1}{\varepsilon_n} r_n^*(\varepsilon_n^{1/\alpha} w) =: \tilde{r}^*(w) \tag{4.44}$$

exists in the cordial metric locally uniformly for $w\in\mathbf{C}$. *The somewhat complicated formulation* $(\mathbf{C}\setminus\mathbf{R}_-)\cup(-\infty,0)\pm i0$ *after* (4.43) *means that we consider this set as a subset of the Riemann surface associated with the multivalued function* $f_\alpha(w)=w^\alpha$.

(ii) *In* (4.38)–(4.42) *the points* z_{nj}, $j=1,\ldots,2n+2+[\alpha]$, *are the zeros of the function* r_n, *which have been investigated in Lemma* 5, *the* ζ_{nj}, $j=1,\ldots,n+1$, *and the* π_{nj}, $j=1,\ldots,n$, *are the zeros and the poles that the rational best approximant* r_n^* *has on* $\mathbf{R}_-$, *and which have been investigated in Lemmas* 3 *and* 4, *the points* $\zeta_{n,n+1+j}$, $j=1,\ldots,[\alpha]$, *are those* $[\alpha]$ *zeros of the approximant* r_n^* *from which we know that they exist in* $\mathbf{C}\setminus\mathbf{R}_-$, *and the* b_{nj}, $j=1,\ldots,[\alpha]$, *are the* $[\alpha]$ *poles of the function* r_n *in* $\mathbf{C}\setminus\mathbf{R}_-$, *which have first been mentioned in* (4.12).

(iii) *We have* $0=\tilde{z}_1=\ldots=\tilde{z}_{j_0}<\tilde{z}_{j_0+1}<\tilde{z}_{j_0+2}<\ldots$ *with an index* $j_0\in\mathbf{N}$ *that satisfies* $0\leqslant j_0\leqslant[\alpha]+1$. *Further, we have* $\ldots<\tilde{\zeta}_3<\tilde{\pi}_2<\tilde{\zeta}_2<\tilde{\pi}_1<\tilde{\zeta}_1<0$. *With respect to the* $2[\alpha]$ *limit points* $\tilde{a}_j,\tilde{b}_j$, $j=1,\ldots,[\alpha]$, *we only know that* $\tilde{a}_j,\tilde{b}_j\in\mathbf{C}\setminus(-\infty,0)$ *for* $j=1,\ldots,[\alpha]$.

(iv) *The limit function* $\tilde{r}$ *in* (4.43) *is analytic in* $\mathbf{C}\backslash(\mathbf{R}_-\cup\{\tilde{b}_1,...,\tilde{b}_{[\alpha]}\})$, *meromorphic in* $\mathbf{C}\backslash\mathbf{R}_-$, *at each* $\tilde{z}_j$, $j>j_0+1$, *it has a simple zero, it is different from zero for all* $w\in\mathbf{C}\backslash(\mathbf{R}_-\cup\{\tilde{z}_1,\tilde{z}_2,...\})$, *and it has analytic boundary values* $\tilde{r}(w)$ *for all* $w\in(\mathbf{R}_-\pm i0)$. *The boundary values* $\tilde{r}(w)$ *are contained in* ∂K_α *for* $w\in(-\infty,0)+i0$, *and contained in* $\partial\bar{K}_\alpha$ *for* $w\in(-\infty,0)-i0$.

(v) *The limit function* $\tilde{r}^*$ *in* (4.44) *is meromorphic in* $\mathbf{C}$, *it has a simple zero at each* $\tilde{\zeta}_j$, $j\in\mathbf{N}$, *a simple pole at each* $\tilde{\pi}_j$, $j\in\mathbf{N}$, *and* $[\alpha]$ *zeros at the points* $\tilde{a}_1,...,\tilde{a}_{[\alpha]}$.

(vi) *The only cluster point of the sequence* $\{\tilde{z}_j\}_{j\in\mathbf{N}}$ *is* ∞, *and the only cluster point of the two sequences* $\{\tilde{\zeta}_j\}_{j\in\mathbf{N}}$ *and* $\{\tilde{\pi}_j\}_{j\in\mathbf{N}}$ *is* $-\infty$.

(vii) *For any* $R>\infty$ *we have*

$$\limsup_{n\to\infty}\operatorname{card}\{z_{nj}\leqslant\varepsilon_n^{1/\alpha}R\mid j\in\{1,...,2n+2+[\alpha]\}\}<\infty, \tag{4.45}$$

$$\limsup_{n\to\infty}\operatorname{card}\{\eta_{nj}\leqslant\varepsilon_n^{1/\alpha}R\mid j\in\{0,...,2n+2+[\alpha]\}\}<\infty, \tag{4.46}$$

$$\limsup_{n\to\infty}\operatorname{card}\{|\zeta_{nj}|\leqslant\varepsilon_n^{1/\alpha}R\mid j\in\{1,...,n+1\}\}<\infty, \tag{4.47}$$

$$\limsup_{n\to\infty}\operatorname{card}\{|\pi_{nj}|\leqslant\varepsilon_n^{1/\alpha}R\mid j\in\{1,...,n\}\}<\infty. \tag{4.48}$$

In (4.46) *the* η_{nj}, $j=0,...,2n+2+[\alpha]$, *are the extreme points of the error function* e_n *on* $[0,1]$.

Remarks. (1) As in the proof of Lemma 6 a transformation of the form (4.21) will play a fundamental role in the proof of Lemma 7, but now it has the special form

$$w:=\varepsilon_n^{-1/\alpha}z,\quad n=1,2,..., \tag{4.49}$$

which does not satisfy condition (4.20). Transformation (4.21) is implicitly already contained in the limits (4.38) through (4.48). A comparison of the limit (4.27) with (4.43) shows that the precise form of (4.49) is crucial. If, for instance, one had used transformation (4.21) with a sequence $\{a_n\}$ satisfying (4.20) instead of transformation (4.49), then the limit function $\tilde{r}$ in (4.43) would have been identically zero, and as a consequence, most of the results of Lemma 7 could not be formulated.

(2) It follows from (4.42) that limit (4.43) holds in the ordinary metric uniformly on compact subsets of $((\mathbf{C}\backslash\mathbf{R}_-)\cup(-\infty,0)\pm i0)\backslash\{\tilde{b}_1,...,\tilde{b}_{[\alpha]}\}$.

(3) With more effort it could have been proved that the limits (4.38) through (4.44) hold for the full sequence $\mathbf{N}$ and not only for subsequences $N\subset\mathbf{N}$. However, since the results of Lemma 7 are only of technical relevance for later proofs, the necessary extra work for a proof of the stronger result has been avoided.

(4) With more effort, it could also have been proved that in the inequalities between the zeros $\tilde{z}_j$ in part (iii) of the lemma the index j_0 is equal to 0, but again such a stronger result is not needed in later proofs.

Proof. (a) We start with an investigation of the sequence of functions r_n. In the first step we deduce properties that follow rather immediately from results established in Lemmas 5 and 4.

In the same way as in the proof of Lemma 6 we denote all functions and constants that result from an application of transformation (4.49) by a tilde. Thus, we have

$$\begin{aligned} \tilde{r}_n(w) &:= r_n(\varepsilon_n^{1/\alpha} w), \\ \tilde{\eta}_{nj} &:= \eta_{nj}\, \varepsilon_n^{-1/\alpha}, \quad j=0,\dots,2n+2+[\alpha], \\ \tilde{z}_{nj} &:= z_{nj}\, \varepsilon_n^{-1/\alpha}, \quad j=1,\dots,2n+2+[\alpha], \\ \tilde{b}_{nj} &:= b_{nj}\, \varepsilon_n^{-1/\alpha}, \quad j=1,\dots,[\alpha], \\ \tilde{\zeta}_{nj} &:= \zeta_{nj}\, \varepsilon_n^{-1/\alpha}, \quad j=1,\dots,n+1+[\alpha], \\ \tilde{\pi}_{nj} &:= \pi_{nj}\, \varepsilon_n^{-1/\alpha}, \quad j=1,\dots,n. \end{aligned} \tag{4.50}$$

Under transformation (4.49), the interval $(0,1]$ in the z-variable is transformed into the interval $(0, \varepsilon_n^{-1/\alpha}]$ in the w-variable. From (4.2) and (4.3) we deduce that

$$|\tilde{r}_n(w)| \leqslant \frac{1}{2w^\alpha - 1} \quad \text{for } w \in [2^{-1/\alpha}, \varepsilon_n^{-1/\alpha}]. \tag{4.51}$$

From (4.4) and (4.50) it follows that at the transformed extreme points $\tilde{\eta}_{nj}$ we have

$$\tilde{r}_n(\tilde{\eta}_{nj}) = \frac{(-1)^{j+1+[\alpha]}}{2\tilde{\eta}_{nj}^{\alpha} + (-1)^{j+[\alpha]}}, \quad j=0,\dots,2n+2+[\alpha]. \tag{4.52}$$

As a consequence of (4.9) in Lemma 5 in combination with (4.10) and (4.11), it follows that

$$\begin{aligned} m := \min\left(\left|\tan \tfrac{1}{2}\pi\alpha\right|, \left|\cot \tfrac{1}{2}\pi\alpha\right|\right) &\leqslant |\tilde{r}_n(x \pm i0)| \\ &\leqslant \max\left(\left|\tan \tfrac{1}{2}\pi\alpha\right|, \left|\cot \tfrac{1}{2}\pi\alpha\right|\right) =: M \end{aligned} \tag{4.53}$$

for all $x \in \mathbf{R}_-$. Thus, we have a rather good knowledge of the behavior of $\tilde{r}_n$ on $\mathbf{R}_- \pm i0$ and on $[2^{-1/\alpha}, \varepsilon_n^{-1/\alpha}]$. It is immediate that $m = 1/M$. Based on (4.15), we define

$$\widetilde{Q}_n(w) := Q_n(\varepsilon_n^{1/\alpha} w) = \prod_{j=1}^{[\alpha]} B(w, \tilde{b}_{nj}). \tag{4.54}$$

From the asymptotic estimate (4.13) in Lemma 6 it follows that any infinite sequence $N \subseteq \mathbf{N}$ contains an infinite subsequence, which we continue to denote by N, such that the limits

$$\lim_{\substack{n\to\infty\\ n\in N}} \tilde{b}_{nj} = \tilde{b}_j, \quad j = 1, \ldots, [\alpha], \tag{4.55}$$

$$\lim_{\substack{n\to\infty\\ n\in N}} \widetilde{Q}_n(w) = \widetilde{Q}(w) = \prod_{j=1}^{[\alpha]} B(w, \tilde{b}_j) \tag{4.56}$$

exist and are finite. The limits (4.55) are identical with those in (4.42). The limit (4.56) holds locally uniformly in $\mathbf{C}\setminus\mathbf{R}_-$. The function $\tilde{r}_n$ is analytic in $\mathbf{C}\setminus\mathbf{R}_-$ except for the $[\alpha]$ poles at $\tilde{b}_{n,1}, \ldots, \tilde{b}_{n,[\alpha]}$. Hence, we deduce from (4.16) and (4.53) that

$$|\tilde{r}_n(w)| \leqslant \frac{M}{|\widetilde{Q}_n(w)|} \quad \text{for } w \in \mathbf{C}\setminus\mathbf{R}_-. \tag{4.57}$$

With (4.56) it follows from Montel's theorem that we can select an infinite subsequence of N, which we continue to denote by N, such that the limit

$$\lim_{\substack{n\to\infty\\ n\in N}} \tilde{r}_n(w) =: \tilde{r}(w) \tag{4.58}$$

exists locally uniformly for $w \in \mathbf{C}\setminus(\mathbf{R}_- \cup \{\tilde{b}_1, \ldots, \tilde{b}_{[\alpha]}\})$, which partially proves (4.43). The extension to a proof of uniform convergence in the cordial metric on compact subsets of $(\mathbf{C}\setminus\mathbf{R}_-) \cup (-\infty, 0) \pm i0$ will be done below at the end of step (f).

The interlacing property between the transformed extreme points $\tilde{\eta}_{nj}$, $j=0, \ldots, 2n+2+[\alpha]$, and the zeros $\tilde{z}_{nj}$, $j=1, \ldots, 2n+2+[\alpha]$, of the functions $\tilde{r}_n$, which has been established in Lemma 5, will be used in the sequel at many places. It is a consequence of this property that at most $[\alpha]+1$ extreme points $\tilde{\eta}_{nj}$ and at most $[\alpha]+1$ zeros $\tilde{z}_{nj}$ can lie in the interval $[0, 1]$.

Indeed, it follows from (2.9) in Lemma 2 that if e_n has $k+1$ extreme points $\eta_{n,0}, \ldots, \eta_{n,[\alpha]}$ in the interval $[0, \varepsilon_n^{1/\alpha}]$, then the rational best approximant r_n^* has at least k zeros in this interval. From remark (2) to Lemma 4 we know that r_n^* has at most $[\alpha]$ zeros outside of $\mathbf{R}_-$. Hence, not more than $[\alpha]+1$ extreme points η_{nj} can lie in the interval $[0, \varepsilon_n^{1/\alpha}]$. With transformation (4.49) the assertion then follows for $\tilde{\eta}_{nj}$, and from the interlacing property together with $\tilde{\eta}_{n,0}=0$ the assertion follows for the zeros $\tilde{z}_{nj}$.

By choosing an infinite subsequence of N if necessary, which we continue to denote by N, we can assume that the limits (4.38), (4.39), (4.40) and (4.41) hold in the cordial metric. The limits (4.42) have already been assumed in (4.55). From Lemma 6 it follows that the limits (4.41) and (4.42) exist also in the ordinary metric if they exist in the cordial one. In case of the limits (4.38), (4.39) and (4.40) we cannot draw this conclusion at the present stage, and therefore infinity can so far not be excluded as limit point.

(b) In the next step of the proof we show that any infinite sequence $N\subseteq\mathbf{N}$ contains an infinite subsequence, which we continue to denote by N, such that the sets $\{\tilde{\eta}_{nj}\,|\,j=0,\dots,2n+2+[\alpha]\}$ have necessarily infinitely many cluster points in $[0,\infty)$ as $n\to\infty$, $n\in N$. The assertion follows from (4.52), (4.51) and an argument that is of a type used in the proof of the Phragmén–Lindelöf maximum principle. The proof will be carried out indirectly; conclusions of the results will be drawn in step (c).

Let us assume that there exist only finitely many cluster points of the sets $\{\tilde{\eta}_{nj}\,|\,j=0,\dots,2n+2+[\alpha]\}$, $n\in N$, in $(0,\infty)$. Then there exists an infinite subsequence of N, which we continue to denote by N, such that there exists $j_1\in\mathbf{N}$ and

$$\begin{aligned} &\tilde{\eta}_{nj}\to\tilde{\eta}_j\in[0,\infty) && \text{for } j=0,\dots,j_1, \quad \text{while}\\ &\tilde{\eta}_{nj}\to\infty && \text{for } j=j_1+1,\dots \end{aligned} \tag{4.59}$$

as $n\to\infty$, $n\in N$. Because of the interlacing property between the $\tilde{\eta}_{nj}$ and $\tilde{z}_{nj}$, we can further assume that there exists $j_2\in\mathbf{N}$ with $j_2=j_1$ or $j_2=j_1+1$ such that

$$\begin{aligned} &\tilde{z}_{nj}\to\tilde{z}_j\in[0,\infty) && \text{for } j=1,\dots,j_2,\\ &\tilde{z}_{nj}\to\infty && \text{for } j=j_2+1,\dots \end{aligned} \tag{4.60}$$

as $n\to\infty$, $n\in N$. With the function $B(w,x)$ introduced in (4.15) and already used in (4.54), we define

$$\tilde{g}_n(w):=\prod_{j=j_2+1}^{2n+2+[\alpha]}B(w,\tilde{z}_{nj}),\quad \tilde{h}_n(w):=\prod_{j=1}^{j_2}\frac{B(w,\tilde{z}_{nj})}{\tilde{Q}_n(w)} \tag{4.61}$$

with $\tilde{Q}_n$ defined like in (4.54). Since $\tilde{r}_n/(\tilde{g}_n\tilde{h}_n)$ is analytic and different from zero in $\mathbf{C}\backslash\mathbf{R}_-$, we deduce from (4.53) and $m=1/M$ that

$$m\leqslant\left|\frac{(\tilde{g}_n\tilde{h}_n)(w)}{\tilde{r}_n(w)}\right|\leqslant M\quad \text{for } w\in\mathbf{C}\backslash\mathbf{R}_- \tag{4.62}$$

and all $n\in N$. Note that from the definition of r_n in (4.1) together with (4.6) and (4.9) in Lemma 5 we know that there exist neighborhoods of 0 and ∞ such that r_n is continuous in the intersection of $\mathbf{C}\backslash\mathbf{R}_-$ with these neighborhoods, and it has continuous boundary values on $\mathbf{R}_-\pm i0$ for any approach from inside of the intersection of $\mathbf{C}\backslash\mathbf{R}_-$ with the neighborhoods.

It is not difficult to deduce from the definition of $B(w,x)$ in (4.15) that $1\geqslant|B(w,x)|\geqslant 1-2\sqrt{|x|/|w|}$ for $|w|\geqslant|x|$. From the estimate (4.13) in Lemma 6 (or equivalently from the convergence (4.58)) and the estimate (4.16), it follows that there exist $R>1$ and $c>0$ such that

$$\frac{1}{|\tilde{h}_n(w)|}\geqslant|\tilde{Q}_n(w)|\geqslant 1-\frac{c}{\sqrt{|w|}}\quad \text{for } |w|>R,\ w\in\mathbf{C}\backslash\mathbf{R}_-. \tag{4.63}$$

From definition (4.15) we deduce that $|B(re^{it},x)|$ is a monotonically increasing function of $|t|$ for fixed $x, r\in\mathbf{R}_+$ and $t\in(-\pi,\pi)$. It may be best for verifying this conclusion by looking at the map from $\mathbf{C}\backslash\mathbf{R}_-$ onto the half-plane $\{\mathrm{Re}(w)>0\}$. Since all zeros $\tilde{z}_{nj}$, $j=j_2+1,\dots,2n+2+[\alpha]$, of the function $\tilde{g}_n$ lie on $(0,\infty)$, the function $|\tilde{g}_n|$ is also monotonically increasing along circles, i.e., we have

$$|\tilde{g}_n(re^{it})|\leqslant|\tilde{g}_n(re^{it'})| \quad \text{if } |t|\leqslant|t'|,\ r\in\mathbf{R}_+,\ t,t'\in(-\pi,\pi). \tag{4.64}$$

We consider the function

$$H(w):=\exp\left[i\log\frac{i-\sqrt{w}}{i+\sqrt{w}}\right] \quad \text{for } w\in\mathbf{D}\backslash\mathbf{R}_-, \tag{4.65}$$

which is analytic in $\mathbf{D}\backslash\mathbf{R}_-$ with $\mathbf{D}$ denoting the unit disc $\{|w|<1\}$, has boundary values

$$|H(w)|=1 \qquad \text{for } w\in(-1,0]\pm i0, \tag{4.66}$$

$$|H(w)|=e^{-\pi/2} \quad \text{for } |w|=1,\ |\arg w|<\pi, \tag{4.67}$$

and there exists $\varepsilon>0$ such that

$$1-3\sqrt{w}\leqslant|H(w)|\leqslant 1-\sqrt{w} \quad \text{for all } w\in[0,\varepsilon]. \tag{4.68}$$

From (4.52), the estimates (4.62), (4.63), and the convergence (4.59), we conclude that

$$|\tilde{g}_n(\tilde{\eta}_{n,j_1+1})|\geqslant m\frac{|\tilde{r}_n(\tilde{\eta}_{n,j_1+1})|}{|\tilde{h}_n(\tilde{\eta}_{n,j_1+1})|}\geqslant\frac{m}{2\tilde{\eta}^{\alpha}_{n,j_1+1}-1}\left(1-\frac{c}{\sqrt{\tilde{\eta}_{n,j_1+1}}}\right)\geqslant\frac{m}{3}\tilde{\eta}^{-\alpha}_{n,j_1+1} \tag{4.69}$$

for $n\in N$ sufficiently large. We note that from (4.59) we know that $\tilde{\eta}_{n,j_1+1}\to\infty$ as $n\to\infty$, $n\in N$.

The function $\tilde{g}_n$ is analytic in the domain $\{w\in\mathbf{C}\backslash\mathbf{R}_- \mid |w|<\tilde{\eta}_{n,j_1+1}\}$, and both functions $|\tilde{g}_n|$ and $|H(\cdot/\tilde{\eta}_{n,j_1+1})|$ have boundary value 1 on the intervals $(-\tilde{\eta}_{n,j_1+1},0]\pm i0$. From (4.67), the inequalities (4.64), (4.69), and the maximum principle, we conclude that

$$|\tilde{g}_n(w)|\geqslant|H(w/\tilde{\eta}_{n,j_1+1})|^{-(2/\pi)\log((m/3)\tilde{\eta}^{-\alpha}_{n,j_1+1})} \tag{4.70}$$

for $w\in\mathbf{C}\backslash\mathbf{R}_-$ with $|w|\leqslant\tilde{\eta}_{n,j_1+1}$. Since $\tilde{\eta}_{n,j_1+1}\to\infty$ as $n\to\infty$, $n\in N$, it follows that

$$-\frac{2}{\pi}\log\left(\frac{m}{3}\tilde{\eta}^{-\alpha}_{n,j_1+1}\right)=\frac{2}{\pi}\log\frac{3}{m}+\frac{2\alpha}{\pi}\log\tilde{\eta}_{n,j_1+1}\to\infty \tag{4.71}$$

as $n\to\infty$, $n\in N$. Using the left-hand side estimate of (4.68) and the right-hand side of (4.70) yields that

$$|\tilde{g}_n(w)|\geqslant\left(1-3\frac{\sqrt{w}}{\sqrt{\tilde{\eta}_{n,j_1+1}}}\right)^{(2/\pi)\log(3/m)+(2\alpha/\pi)\log\tilde{\eta}_{n,j_1+1}} \tag{4.72}$$

for all $w\in(0,\infty)$ and $n\in N$ sufficiently large. For $w\in(0,\infty)$ fixed, we therefore have

$$\begin{aligned}\log&\left(1-\frac{\sqrt{w}}{\sqrt{\tilde{\eta}_{n,j_1+1}}}\right)^{(2/\pi)\log(3/m)+(2\alpha/\pi)\log\tilde{\eta}_{n,j_1+1}}\\ &=-\left(\frac{2}{\pi}\log\frac{3}{m}+\frac{2\alpha}{\pi}\log\tilde{\eta}_{n,j_1+1}\right)\frac{\sqrt{w}}{\sqrt{\tilde{\eta}_{n,j_1+1}}}+\mathcal{O}\left(\frac{\log\tilde{\eta}_{n,j_1+1}}{\tilde{\eta}_{n,j_1+1}}\right)\\ &=\mathcal{O}\left(\frac{\log\tilde{\eta}_{n,j_1+1}}{\sqrt{\tilde{\eta}_{n,j_1+1}}}\right)\quad\text{as } n\to\infty,\ n\in N,\end{aligned} \tag{4.73}$$

which proves

$$\lim_{\substack{n\to\infty\\ n\in N}}|\tilde{g}_n(w)|=1 \tag{4.74}$$

for all $w\in(0,\infty)$. From the definition of $B(z,x)$ in (4.15) together with (4.17), (4.61), the estimate (4.13) in Lemma 6, and the assumptions made in (4.60), we conclude that we have

$$\lim_{\substack{w\to\infty\\ w\in\mathbf{C}\setminus\mathbf{R}_-}}|\tilde{h}_n(w)|=1\quad\text{uniformly for } n\in N. \tag{4.75}$$

The limits (4.74) and (4.75) together contradicts the estimate (4.51), and thus the assertion has been proved that the sets $\{\tilde{\eta}_{nj}\mid j=0,\dots,2n+2+[\alpha]\}$ have necessarily infinitely many cluster points in $(0,\infty)$ as $n\to\infty$, $n\in N$.

(c) In the present step we shall draw some conclusions from the assertion proved in part (b). As a first consequence, we conclude that the limit function $\tilde{r}$ in (4.58) is not identically zero. Indeed, from (4.52) it follows that if we had $\tilde{r}\equiv 0$ in (4.43), then the sets $\{\tilde{\eta}_{nj}\mid j=0,\dots,2n+2+[\alpha]\}$ could have no cluster points in the interval $(0,\infty)$ as $n\to\infty$, $n\in N$, since because of (4.52) the values of $\tilde{r}$ are bounded away from zero at any point in $(0,\infty)$, at which a sequence of extreme points $\tilde{\eta}_{n,j_n}$ clusters as $n\to\infty$, $n\in N$. From part (b) we know that there exist infinitely many finite cluster points.

Further, it follows from the assertions proved in part (b), together with Hurwitz's theorem (or the argument principle) and the locally uniform convergence (4.58), that $\tilde{r}$ has infinitely many zeros in $(0,\infty)$ and no zero in $\mathbf{C}\setminus\mathbf{R}$. Indeed, between two transformed extreme points $\tilde{\eta}_{nj}$ and $\tilde{\eta}_{n,j+1}$ there always lies a zero $\tilde{z}_{nj}$ of $\tilde{r}_n$, which implies that the limit function $\tilde{r}$ has to have infinitely many zeros $\tilde{z}_j$ in $(0,\infty)$. On the other hand, all values $\tilde{r}_n(w)$, $n \in N$, are different from zero for $w \in \mathbf{C}\setminus\mathbf{R}$, which implies that $\tilde{r}$ is either identically zero or different from zero in $\mathbf{C}\setminus\mathbf{R}$. Because of (4.52), $\tilde{r}$ cannot be identically zero. It further follows that all limits (4.38) are finite, and therefore they exist also in the ordinary metric. The zeros $\tilde{z}_j$ of $\tilde{r}$ are the limit points in (4.38).

Since the function $\tilde{r}$ is analytic in $\mathbf{C}\setminus\mathbf{R}_-$ except at the $[\alpha]$ possible poles $\tilde{b}_1, ..., \tilde{b}_{[\alpha]}$, and it is not identically zero, we can conclude that the zeros $\tilde{z}_j$ of $\tilde{r}$ can have no cluster points in $(0, \infty)$. We have already earlier proved that each function $\tilde{r}_n$ can have at most $[\alpha]+1$ zeros in the interval $[0, 1]$. Hence, it follows from the locally uniform convergence (4.58) that the zeros $\tilde{z}_j$, $j=1, 2, ...$, cannot cluster at $w=0$. However, we cannot exclude that up to $[\alpha]+1$ of the first $\tilde{z}_j$ can be equal to 0. These observations prove limit (4.45) and the order relations

$$0 = \tilde{z}_1 = ... = \tilde{z}_{j_0} < \tilde{z}_{j_0+1} \leqslant \tilde{z}_{j_0+2} \leqslant ... 0 \leqslant j_0 \leqslant [\alpha]+1. \tag{4.76}$$

Analogously to the limits (4.38), we can assume that the limits

$$\lim_{\substack{n\to\infty \\ n\in N}} \varepsilon_n^{-1/\alpha} \eta_{nj} = \lim_{\substack{n\to\infty \\ n\in N}} \tilde{\eta}_{nj} =: \tilde{\eta}_j \tag{4.77}$$

exist for $j=0, 1, ...$. We have $\tilde{\eta}_j \in [0, \infty)$ for each $j \in \mathbf{N}$, and each $\tilde{\eta}_j$ satisfies relation (4.52).

With respect to the limit function $\tilde{r}$ it only remains to prove in the present step that all zeros $\tilde{z}_j$, $j > j_0$, of $\tilde{r}$ are simple, which then proves that all inequalities in (4.76) are valid in a strict sense, from which the inequalities in part (iii) of the lemma follow.

Indeed, since between two adjacent zeros $\tilde{z}_{nj}$ and $\tilde{z}_{n,j+1}$ of $\tilde{r}_n$ there lies exactly one transformed extreme point $\tilde{\eta}_{n,j+1}$, it follows from (4.52) and the locally uniform convergence in (4.58) that for $j \in \mathbf{N}$ fixed, the two sequences $\{\tilde{z}_{nj}\}_{n\in N}$ and $\{\tilde{z}_{n,j+1}\}_{n\in N}$ cannot converge to the same limit point as $n \to \infty$, $n \in N$. Therefore, all zeros $\tilde{z}_j$ of $\tilde{r}$ have to be simple. Of course, it has to be excluded that some of the $[\alpha]$ poles, which the function $\tilde{r}_n$ has at the points $\tilde{b}_{nj}$, may cancel out with zeros $\tilde{z}_{nj}$ of $\tilde{r}_n$ in the limiting case as $n \to \infty$, $n \in N$, i.e., that $\tilde{z}_j = \tilde{b}_l$ for some $j > j_0$ and $l \in \{1, ..., [\alpha]\}$. This possibility cannot be excluded by the locally uniform convergence (4.58) in $\mathbf{C}\setminus(\mathbf{R}_- \cup \{\tilde{b}_1, ..., \tilde{b}_{[\alpha]}\})$. But it will be shown at the end of step (f) that the convergence (4.58) holds locally uniformly in the cordial metric in $\mathbf{C}\setminus\mathbf{R}_-$, which implies that the poles and zeros of $\tilde{r}_n$ cannot have common limit points in $\mathbf{C}\setminus\mathbf{R}_-$.

As a by-product of the interlacing property between the extreme points $\tilde{\eta}_{nj}$ and the zeros $\tilde{z}_{nj}$, we conclude that the asymptotic estimate (4.46) is a consequence of (4.45).

(d) In the next three steps we investigate the convergence behavior of the sequence of transformed rational best approximants

$$\tilde{r}_n^*(w) := \frac{1}{\varepsilon_n} r_n^*(\varepsilon_n^{1/\alpha} w), \quad n \in N, \tag{4.78}$$

and properties of its limit function $\tilde{r}^*$. In this investigation we use properties of the approximants r_n^* and its denominator polynomials $q_n \in \mathcal{P}_n$, which have been proved in Lemma 3. Further, a comparison of the convergence behavior of the sequence of approximants $\{\tilde{r}_n^*\}$ with that of the sequence $\{\tilde{r}_n\}$ will be used. This part of the proof is rather technical and lengthy.

From the boundedness (4.57), the existence of the limits (4.38) and (4.45), the properties (4.16) of the function $B(w, z)$, and the identities (4.52), we deduce that the infinite product

$$\prod_{j=1}^{\infty} B(w, \tilde{z}_j) \tag{4.79}$$

exists and is not identically zero in $\mathbf{C} \backslash \mathbf{R}_-$. Indeed, otherwise the limit function $\tilde{r}$ in (4.58) would be identically zero, but this would contradict (4.52). From (4.15) we deduce that the product (4.79) is not identically zero in $\mathbf{C} \backslash \mathbf{R}_-$ if, and only if, we have

$$\sum_{j=[\alpha]+2}^{\infty} \frac{1}{\sqrt{\tilde{z}_j}} < \infty. \tag{4.80}$$

By choosing an infinite subsequence of N, which we continue to denote by N, we can assume that the limits

$$\lim_{\substack{n \to \infty \\ n \in N}} \tilde{\pi}_{n,n+1-j} =: \tilde{\pi}_j \quad \text{and} \quad \lim_{\substack{n \to \infty \\ n \in N}} \tilde{\zeta}_{n,n+2-j} =: \tilde{\zeta}_j \tag{4.81}$$

exist in the cordial metric for $j = 1, 2, \dots$, and because of the estimates (4.14) in Lemma 6, we can further assume that the $[\alpha]$ limits

$$\lim_{\substack{n \to \infty \\ n \in N}} \tilde{\zeta}_{n,n+1+j}(w) =: \tilde{a}_j, \quad j = 1, \dots, [\alpha], \tag{4.82}$$

exist in the ordinary metric, and from (4.14) we know that $\tilde{a}_j \in \mathbf{C}$ for $j = 1, \dots, [\alpha]$. In (4.81) the possibilities $\tilde{\pi}_j = -\infty$ or $\tilde{\zeta}_j = -\infty$ cannot be excluded at the present stage. From the interlacing property (3.20) in Lemma 4 it follows that

$$-\infty \leqslant \dots \leqslant \tilde{\zeta}_3 \leqslant \tilde{\pi}_2 \leqslant \tilde{\zeta}_2 \leqslant \tilde{\pi}_1 \leqslant \tilde{\zeta}_1 \leqslant 0. \tag{4.83}$$

With (4.81) and (4.82), the limits (4.39), (4.40) and (4.41) in the lemma are proved. However, the proof of the strong inequalities between the $\tilde{\pi}_j$ and the $\tilde{\zeta}_j$, which are stated in part (iii) of the lemma, remains still open. For this purpose, and also for a complete proof of the limit (4.44), it is necessary to bring more specific properties of the approximants r_n^* into play. We start with some definitions.

Using transformation (4.49) together with definitions introduced in Lemma 3, we define

$$\begin{aligned}\tilde{e}_n(w) &:= \frac{1}{\varepsilon_n} e_n(\varepsilon_n^{1/\alpha} w) = w^\alpha - \tilde{r}_n^*(w) \\ &= \frac{\widetilde{w}_n(w)}{\tilde{q}_n(w)^2} \frac{\sin \pi\alpha}{\pi} \int_{-\infty}^0 \frac{\tilde{q}_n(x)^2}{\widetilde{w}_n(x)} \frac{|x|^\alpha \, dx}{x-w}, \quad w \in \mathbf{C}\setminus\mathbf{R}_-,\end{aligned} \tag{4.84}$$

with the polynomials $\widetilde{w}_n$ and $\tilde{q}_n$ defined by

$$\widetilde{w}_n(w) := \varepsilon_n^{-(2n+2+[\alpha])/\alpha} w_n(\varepsilon_n^{1/\alpha} w) = w^{2n+2+[\alpha]} + \dots, \tag{4.85}$$

$$\tilde{q}_n(w) := \varepsilon_n^{-n/\alpha} q_n(\varepsilon_n^{1/\alpha} w) = w^n + \dots, \tag{4.86}$$

and the polynomials w_n and q_n introduced in (3.1) and in Lemma 3, respectively. In (4.84) the last equality follows from (3.6). Further, we define

$$\tilde{I}_n(w) := \frac{\sin \pi\alpha}{\pi c_n} \int_{-\infty}^0 \frac{\tilde{q}_n(x)^2}{\widetilde{w}_n(x)} \frac{|x|^\alpha \, dx}{x-w} \tag{4.87}$$

with constants $c_n > 0$ determined by the condition

$$\tilde{I}_n(1) = -1 \quad \text{for } n \in \mathbf{N}. \tag{4.88}$$

From (3.2), (4.85) and the fact that all zeros $\tilde{z}_{nj}$ of the polynomials $\widetilde{w}_n$ are contained in $(0, \infty)$, it follows that the measures

$$d\tilde{\mu}_n(x) := \frac{\sin \pi\alpha}{\pi c_n} \frac{\tilde{q}_n(x)^2}{\widetilde{w}_n(x)} |x|^\alpha \, dx, \quad x \in \mathbf{R}_-, \; n \in N, \tag{4.89}$$

are positive and of finite mass. From standardization (4.88) and the positivity of the measures $\tilde{\mu}_n$ we deduce that for each cone $C_\varphi := \{w \in \mathbf{C} \mid \arg(w) \leqslant \varphi\}$, $\varphi < \pi$, there exists a constant c_φ such that

$$|\tilde{I}_n(w)| \leqslant c_\varphi < \infty \quad \text{for all } w \in C_\varphi \text{ and } n \in N. \tag{4.90}$$

By Montel's theorem we therefore know that there exists an infinite subsequence of N, which we continue to denote by N, such that the limit

$$\lim_{\substack{n\to\infty \\ n \in N}} \tilde{I}_n(w) =: \tilde{I}(w) \tag{4.91}$$

exists locally uniformly for $w \in \mathbf{C}\setminus\mathbf{R}_-$. From standardization (4.88) it follows that we also have $\tilde{I}(1) = -1$, and by Hurwitz's theorem we further conclude that

$$\tilde{I}(1) = -1 \quad \text{and} \quad \tilde{I}(w) \neq 0, \infty \quad \text{for all } w \in \mathbf{C}\setminus\mathbf{R}_-. \tag{4.92}$$

(e) In the present step we prove limit (4.44) in part. The complete proof will follow in step (f). We start the analysis by showing that the limit

$$\lim_{\substack{n\to\infty \\ n\in N}} c_n \frac{\widetilde{w}_n(w)}{\tilde{q}_n(w)^2} =: \tilde{g}_1(w) \tag{4.93}$$

exists locally uniformly for $w\in\mathbf{C}\setminus\{0\}$ in the cordial metric.

From (4.84), the defining properties (2.9) of the extreme points η_{nj}, and the transformations (4.50), it follows that $|\tilde{e}_n(\tilde{\eta}_{nj})|=1$ for $j=1,...,2n+2+[\alpha]$. From the fact that at most $[\alpha]+1$ of the transformed extreme points $\tilde{\eta}_{nj}$ can lie in the interval $[0,1)$, which has been proved at the end of step (a), and from the existence of the limits (4.77) together with limit (4.91) and its properties (4.92), we then deduce that

$$\lim_{\substack{n\to\infty \\ n\in N}} c_n \frac{|\widetilde{w}_n(\tilde{\eta}_{nj})|}{\tilde{q}_n(\tilde{\eta}_{nj})^2} = \frac{1}{\tilde{I}(\tilde{\eta}_j)} \quad \text{for each } j>[\alpha]+1. \tag{4.94}$$

Let $R>1$ be arbitrary, and let $j_2\in\mathbf{N}$, $j_2>[\alpha]+2$, be chosen so that $\tilde{z}_j>R$ for all $j>j_2$. We can assume that R is so large that $\tilde{z}_{[\alpha]+5}<R$. We define

$$\widetilde{w}_{n,j_2}(w) := \prod_{j=j_2}^{2n+2+[\alpha]} (w-\tilde{z}_{nj}). \tag{4.95}$$

Then for $v, v_0\in\{|w|<R\}$, v_0 fixed, the limit

$$\begin{aligned} \lim_{\substack{n\to\infty \\ n\in N}} \frac{\widetilde{w}_{n,j_2}(v)}{\widetilde{w}_{n,j_2}(v_0)} &= \lim_{\substack{n\to\infty \\ n\in N}} \prod_{j=j_2}^{2n+2+[\alpha]} \frac{v-\tilde{z}_{nj}}{v_0-\tilde{z}_{nj}} = \lim_{\substack{n\to\infty \\ n\in N}} \prod_{j=j_2}^{2n+2+[\alpha]} \left(1+\frac{v-v_0}{v_0-\tilde{z}_{nj}}\right) \\ &= \prod_{j=j_2}^{\infty} \left(1+\frac{v-v_0}{v_0-\tilde{z}_j}\right) =: \tilde{g}_{2,j_2}(v) \end{aligned} \tag{4.96}$$

exists locally uniformly for $v\in\{|w|<R\}$, and it is different from zero for $|v|<R$ if, and only if, we have

$$\sum_{j=[\alpha]+2}^{\infty} \frac{1}{\tilde{z}_j} < \infty. \tag{4.97}$$

From (4.80) we know that (4.97) holds true, and therefore the limit (4.96) holds true locally uniformly for $v\in\{|w|<R\}$, and we have

$$\tilde{g}_{2,j_2}(v) \neq 0 \quad \text{for } |v|<R. \tag{4.98}$$

Let now $j_3 \in \mathbf{N}$ be such that $0 < \tilde{\eta}_{j_3} < \tilde{\eta}_{j_3+1} < R$. From the properties (2.9) of the extreme points η_{nj} together with the transformations (4.50) and the definitions (4.84) and (4.87), we conclude that

$$\begin{aligned} -1 &= \frac{\tilde{e}_n(\tilde{\eta}_{n,j_3+1})}{\tilde{e}_n(\tilde{\eta}_{n,j_3})} = \frac{\tilde{I}_n(\tilde{\eta}_{n,j_3+1})}{\tilde{I}_n(\tilde{\eta}_{n,j_3})} \frac{\widetilde{w}_n(\tilde{\eta}_{n,j_3+1})}{\widetilde{w}_n(\tilde{\eta}_{n,j_3})} \frac{\tilde{q}_n(\tilde{\eta}_{n,j_3})^2}{\tilde{q}_n(\tilde{\eta}_{n,j_3+1})^2} \\ &= \frac{\tilde{I}_n(\tilde{\eta}_{n,j_3+1})}{\tilde{I}_n(\tilde{\eta}_{n,j_3})} \prod_{j=1}^{2n+2+[\alpha]} \left(1 + \frac{\tilde{\eta}_{n,j_3+1} - \tilde{\eta}_{n,j_3}}{\tilde{\eta}_{n,j_3} - \tilde{z}_{nj}}\right) \prod_{j=1}^{n} \left(1 - \frac{\tilde{\eta}_{n,j_3+1} - \tilde{\eta}_{n,j_3}}{\tilde{\eta}_{n,j_3+1} - \tilde{\pi}_{nj}}\right)^2. \end{aligned} \tag{4.99}$$

With the limits (4.77), (4.91), (4.38) and (4.82), it follows that also in the limiting case we have

$$-1 = \frac{\tilde{I}(\tilde{\eta}_{j_3+1})}{\tilde{I}(\tilde{\eta}_{j_3})} \prod_{j=1}^{\infty} \left(1 + \frac{\tilde{\eta}_{j_3+1} - \tilde{\eta}_{j_3}}{\tilde{\eta}_{j_3} - \tilde{z}_j}\right) \prod_{j=1}^{\infty} \left(1 - \frac{\tilde{\eta}_{j_3+1} - \tilde{\eta}_{j_3}}{\tilde{\eta}_{j_3+1} - \tilde{\pi}_j}\right)^2. \tag{4.100}$$

From (4.100) and (4.97), we conclude that besides of estimate (4.97) also the estimate $\sum_{j=1}^{\infty} |\tilde{\pi}_j - \tilde{\eta}_{j_3+1}|^{-1} < \infty$ holds true, which is equivalent to

$$\sum_{j=1}^{\infty} \frac{1}{|\tilde{\pi}_j - 1|} < \infty. \tag{4.101}$$

As one of the consequences of (4.101), we see that the sequence $\{\tilde{\pi}_j\}_{j=1}^{\infty}$ has $-\infty$ as its only cluster point, which together with (4.83) completes the proof of part (vi) in the lemma.

Let $j_4 \in \mathbf{N}$ be chosen so that $|\tilde{\pi}_j| > R$ for all $j \geqslant j_4$. We define

$$\tilde{q}_{n,j_4}(w) := \prod_{j=j_4}^{n} (w - \tilde{\pi}_{n,n+1-j}). \tag{4.102}$$

From error representation (4.84), the definitions (4.87), (4.95), (4.102), and a consideration of the transformed error function $\tilde{e}_n$ simultaneously at the two points v and $\tilde{\eta}_{j_3}$, we derive that

$$c_n \frac{\widetilde{w}_{n,j_2}(v)}{\tilde{q}_{n,j_4}(v)^2} = \prod_{j=j_2}^{2n+2+[\alpha]} \frac{v - \tilde{z}_{nj}}{\tilde{\eta}_{j_3} - \tilde{z}_{nj}} \prod_{j=j_4}^{n} \left(\frac{\tilde{\eta}_{j_3} - \tilde{\pi}_{n,n+1-j}}{v - \tilde{\pi}_{n,n+1-j}}\right)^2 \frac{\prod_{j=1}^{j_4-1} (\tilde{\eta}_{j_3} - \tilde{\pi}_{n,n+1-j})^2}{\prod_{j=1}^{j_2-1} (\tilde{\eta}_{j_3} - \tilde{z}_{nj})} \frac{\tilde{e}_n(\tilde{\eta}_{j_3})}{\tilde{I}_n(\tilde{\eta}_{j_3})}. \tag{4.103}$$

With the same arguments as applied in (4.99), (4.100) and (4.101), we deduce from (4.103) together with (4.94), (4.97) and (4.101) that the limit

$$\lim_{\substack{n\to\infty \\ n \in N}} c_n \frac{\widetilde{w}_{n,j_2}(v)}{\tilde{q}_{n,j_4}(v)^2} =: \tilde{g}_{j_2,j_4}(v) \tag{4.104}$$

exists locally uniformly for $|v|<R$, and that we have

$$\tilde{g}_{j_2,j_4}(v) \neq 0, \infty \quad \text{for all } v \in \{|w|<R\}. \tag{4.105}$$

Since the left-hand sides of (4.93) and (4.103) differ only in a finite number of factors, and since $R>1$ has been chosen arbitrarily, from the limits (4.104), (4.38), (4.39), (4.40) and (4.41), it follows that limit (4.93) exists locally uniformly in the cordial metric in $\mathbf{C}\setminus\{0\}$. Note that the poles $\tilde{\pi}_{nj}$ and zeros $\tilde{z}_{nj}$ of the functions on the left-hand side of (4.93) are lying on different sides of the origin. The limit function $\tilde{g}_1$ in (4.93) has its zeros in $\mathbf{C}\setminus\{0\}$ at the points $\tilde{z}_j$, and its poles at the points $\tilde{\pi}_j$, $j\in\mathbf{N}$.

The following conclusions, which will be used in the next step, follow rather immediately from limit (4.93). Because of (4.91), (4.92) and (4.93), the limit

$$\lim_{\substack{n\to\infty\\ n\in N}} \tilde{e}_n(w) =: \tilde{e}(w) \tag{4.106}$$

exists locally uniformly for $w\in\mathbf{C}\setminus\mathbf{R}_-$, and we have

$$\tilde{e}(w) \neq 0 \quad \text{for all } w\in\mathbf{C}\setminus(\mathbf{R}_-\cup\{\tilde{z}_1, \tilde{z}_2, \ldots\}).$$

From (4.93) we deduce that the limit

$$\lim_{\substack{n\to\infty\\ n\in N}} \frac{\sin\pi\alpha}{\pi c_n} \frac{\tilde{q}_n(v)^2}{\widetilde{w}_n(v)} |v|^\alpha = \tilde{g}_2(v) := \frac{\sin\pi\alpha}{\pi} \frac{|v|^\alpha}{\tilde{g}_1(v)} \tag{4.107}$$

holds locally uniformly for $v\in(-\infty,0)$, and because of (3.2), we have

$$\tilde{g}_2(v) > 0 \quad \text{for all } v\in(-\infty,0)\setminus\{\tilde{\pi}_1, \tilde{\pi}_2, \ldots\}. \tag{4.108}$$

Since we know from (4.105) that the limit function $\tilde{g}_{j_2,j_4}$ in (4.104) is different from zero, it follows that for every $\varepsilon>0$ there exists a constant $c_\varepsilon<\infty$ such that

$$\frac{\tilde{q}_n(v)^2|v|^\alpha}{c_n|\widetilde{w}_n(v)|} \leqslant c_\varepsilon |v|^{\alpha-[\alpha]-1}|v-\tilde{\pi}_{nn}|^2 \quad \text{for } v\in(-\varepsilon,0) \tag{4.109}$$

and all $n\in N$. Indeed, limit (4.104) together with the fact that at most $[\alpha]+1$ zeros $\tilde{z}_{nj}$ of the polynomials $\widetilde{w}_n$ can lie in the interval $(0,1]$ implies inequality (4.109).

From (4.101) and the interlacing property (4.83), we conclude that we have

$$\sum_{j=1}^\infty \frac{1}{|\tilde{\zeta}_j-1|} < \infty. \tag{4.110}$$

From (4.106) we derive that the limit

$$\lim_{\substack{n\to\infty\\ n\in N}} \tilde{r}_n^*(w) = w^\alpha - \tilde{e}(w) \tag{4.111}$$

holds true locally uniformly for $w\in\mathbf{C}\setminus\mathbf{R}_-$. With the help of estimate (4.110) and limit (4.111) we can deduce a weak and preliminary version of limit (4.44).

Indeed, let analogously to $\tilde{q}_{n,j_4}$ in (4.102) the monic polynomial $\tilde{p}_{n,j_4}$ be defined as

$$\tilde{p}_{n,j_4}(w) := \prod_{j=j_4+1}^{n+1} (w - \tilde{\zeta}_{n,n+2-j}) \tag{4.112}$$

with j_4 chosen as in (4.102). Let $w_0\in(0,\infty)$ be such that $w_0^\alpha - \tilde{e}(w_0)\neq 0$, and let v be arbitrarily chosen from $\{|w|<R\}$. By considering simultaneously the two points v and w_0, we can show as in (4.103) and (4.104) that because of (4.101), (4.110) and (4.111), the limit

$$\lim_{\substack{n\to\infty\\ n\in N}} \frac{\tilde{p}_{n,j_4}(v)}{\tilde{q}_{n,j_4}(v)} =: \tilde{g}_{3,j_4}(v) \tag{4.113}$$

exists locally uniformly for $|v|<R$, and we have

$$\tilde{g}_{3,j_4}(v) \neq 0, \infty \quad \text{for all } v\in\{|w|<R\}. \tag{4.114}$$

Since $\tilde{r}_n^*$ differs from $\tilde{p}_{n,j_4}/\tilde{q}_{n,j_4}$ only in a finite number of linear factors, it follows from (4.113), the limits (4.81) and the arbitrary choice of $R>1$ that the limit

$$\lim_{\substack{n\to\infty\\ n\in N}} \tilde{r}_n^*(w) =: \tilde{r}^*(w) \tag{4.115}$$

exists locally uniformly for $w\in\mathbf{C}\setminus\{\tilde{\pi}_1, \tilde{\pi}_2, \ldots\}$.

By this last conclusion we have proved limit (4.44) partially. A complete proof has to establish locally uniform convergence in the cordial metric throughout $\mathbf{C}$. For this aim it is necessary to show that all limit points $\tilde{\pi}_1, \tilde{\pi}_2, \ldots$, and $\tilde{\zeta}_1, \tilde{\zeta}_2, \ldots$, are pairwise different, which is equivalent to the assertion that in (4.83) strong inequalities hold true.

(f) In order to prove strong inequalities in (4.83), we use properties of the approximants r_n^* and its denominator polynomials q_n which have been established in Lemma 3, and also some properties which have been stated immediately after the proof of Lemma 3. All these properties are consequences of the fact that the approximants r_n^* have been identified as rational interpolants of the function f_α.

From (3.4) in Lemma 3 we know that

$$\int_{-\infty}^{0} \frac{\tilde{q}_n(x) g_n(x)}{\widetilde{w}_n(x)} |x|^\alpha \, dx = 0 \tag{4.116}$$

for all polynomials $g_n \in \mathcal{P}_{n-1}$.

Let us assume first that

$$\tilde{\pi}_j = \tilde{\pi}_{j+1} \quad \text{for some } j \in \mathbf{N} \text{ with } \tilde{\pi}_j < 0. \tag{4.117}$$

From (4.107) and the limits (4.81), it then follows that

$$\lim_{\substack{n\to\infty\\ n\in N}} \frac{\sin \pi\alpha}{\pi c_n} \frac{\tilde{q}_n(v)^2 |v|^\alpha}{\tilde{w}_n(v)(v-\tilde{\pi}_{n,n+1-j})(v-\tilde{\pi}_{n,n-j})} = \frac{\tilde{g}_2(v)}{(v-\tilde{\pi}_j)^2} \tag{4.118}$$

locally uniformly for $v \in (-\infty, 0)$. From (4.81) and (4.107), we know that $\tilde{g}_2$ has a zero of order at least 4 at $\tilde{\pi}_j$. Therefore, we have

$$\lim_{\substack{n\to\infty\\ n\in N}} \max_{\tilde{\pi}_{n,n-j} \leqslant v \leqslant \tilde{\pi}_{n,n+1-j}} \frac{\tilde{q}_n(v)^2 |v|^\alpha}{c_n |\widetilde{w}_n(v)| (v-\tilde{\pi}_{n,n+1-j})(v-\tilde{\pi}_{n,n-j})} = 0. \tag{4.119}$$

From (4.116) and the fact that $\tilde{q}_n/(\cdot - \tilde{\pi}_{n,n+1-j})(\cdot - \tilde{\pi}_{n,n-j})$ is a polynomial of degree $n-2$, we conclude that

$$\lim_{\substack{n\to\infty\\ n\in N}} \frac{\sin \pi\alpha}{\pi c_n} \int_{-\infty}^0 \frac{\tilde{q}_n(x)^2 |x|^\alpha \, dx}{\tilde{w}_n(x)(x-\tilde{\pi}_{n,n+1-j})(x-\tilde{\pi}_{n,n-j})} = 0. \tag{4.120}$$

On the other hand, from Fatou's lemma, (4.118), (4.119) and (4.108), we deduce that

$$\liminf_{\substack{n\to\infty\\ n\in N}} \frac{\sin \pi\alpha}{\pi c_n} \int_{-\infty}^0 \frac{\tilde{q}_n(x)^2 |x|^\alpha \, dx}{\widetilde{w}_n(x)(x-\tilde{\pi}_{n,n+1-j})(x-\tilde{\pi}_{n,n-j})} \geqslant \int_{-\infty}^0 \frac{\tilde{g}_2(x)\, dx}{(x-\tilde{\pi}_j)^2} > 0. \tag{4.121}$$

The contradiction between (4.120) and (4.121) shows that assumption (4.117) is wrong, and we have proved that $\tilde{\pi}_{j+1} < \tilde{\pi}_j$ for all $j \in \mathbf{N}$ with $\tilde{\pi}_j < 0$.

Let us now assume that

$$\tilde{\pi}_1 = 0. \tag{4.122}$$

Because of (4.109) and the limits (4.81), we then conclude analogously to (4.119) that

$$\lim_{\substack{n\to\infty\\ n\in N}} \max_{\tilde{\pi}_{nn} \leqslant v \leqslant 0} \frac{\tilde{q}_n(v)^2 |v|^\alpha}{c_n |\widetilde{w}_n(v)(v-\tilde{\pi}_{nn})|} = 0. \tag{4.123}$$

Since $\tilde{q}_n/(\cdot - \tilde{\pi}_{nn})$ is a polynomial of degree $n-1$, we can derive a contradiction to (4.123) in the same way as done in (4.120) and (4.121), which shows that assumption (4.122) is false, and it is proved that $\tilde{\pi}_1 < 0$. Together with the earlier conclusion, we thus have shown that

$$\ldots < \tilde{\pi}_j < \ldots < \tilde{\pi}_2 < \tilde{\pi}_1 < 0. \tag{4.124}$$

In order to prove that the limit points $\tilde{\zeta}_1, \tilde{\zeta}_2, \ldots$ and $\tilde{a}_1, \ldots, \tilde{a}_{[\alpha]}$ in (4.81) and (4.82) are different from the limit points $\tilde{\pi}_1, \tilde{\pi}_2, \ldots$, we consider the asymptotic behavior of the residua $\tilde{\lambda}_{nj}$ of the transformed approximants $\tilde{r}_n^*$ at its poles $\tilde{\pi}_{nj}$, $j=1, \ldots, n$. From (3.13) we deduce the representation

$$\tilde{r}_n^*(w) = \frac{1}{\varepsilon_n} r_n^*(\varepsilon_n^{1/\alpha} w) = \tilde{h}_n(w) + \sum_{j=1}^{n} \frac{\tilde{\lambda}_{nj}}{w - \tilde{\pi}_{nj}}, \tag{4.125}$$

and from (3.15) we further derive that

$$\tilde{\lambda}_{nj} = \varepsilon_n^{-1-1/\alpha} \lambda_{nj} = \frac{-\sin \pi\alpha}{\pi} \frac{\widetilde{w}_n(\tilde{\pi}_{nj})}{\tilde{q}_n'(\tilde{\pi}_{nj})^2} \int_{-\infty}^{0} \left(\frac{\tilde{q}_n(v)}{v - \tilde{\pi}_{nj}} \right)^2 \frac{|v|^\alpha \, dv}{\widetilde{w}_n(v)} \tag{4.126}$$

for $j=1, \ldots, n$. The asymptotic behavior of the residua $\tilde{\lambda}_{n,n+1-j}$ will be studied for $n \to \infty$, $n \in N$, and $j \in \mathbf{N}$ fixed. We use tools that have already been applied in (4.118), (4.119) and (4.121), but now we use Lebesgue's theorem on dominated convergence instead of Fatou's lemma.

From (4.81) we know that $\lim_{n\to\infty, n\in N} \tilde{\pi}_{n,n+1-j} = \tilde{\pi}_j$, and from (4.107) and (4.124), it follows that the limit

$$\lim_{\substack{n\to\infty \\ n\in N}} \frac{\sin \pi\alpha}{\pi c_n} \left(\frac{\tilde{q}_n(v)}{v - \tilde{\pi}_{n,n+1-j}} \right)^2 \frac{|v|^\alpha}{\widetilde{w}_n(v)} = \frac{\tilde{g}_2(v)}{(v - \tilde{\pi}_j)^2} \tag{4.127}$$

holds locally uniformly for $v \in (-\infty, 0)$. By Lebesgue's theorem on dominated convergence, it follows from (4.127) that

$$\lim_{\substack{n\to\infty \\ n\in N}} \frac{\sin \pi\alpha}{\pi c_n} \int_{-\infty}^{0} \left(\frac{\tilde{q}_n(x)}{x - \tilde{\pi}_{n,n+1-j}} \right)^2 \frac{|x|^\alpha \, dx}{\widetilde{w}_n(x)} = \int_{-\infty}^{0} \frac{\tilde{g}_2(x) \, dx}{(x - \tilde{\pi}_j)^2} < \infty. \tag{4.128}$$

Indeed, near the origin an integrable upper bound for the integrand in (4.128) is provided by (4.109). On the lower end of $\mathbf{R}_-$, we have the estimate

$$\left(\frac{\tilde{q}_n(x)}{x - \tilde{\pi}_{n,n+1-j}} \right)^2 \frac{|x|^\alpha}{|\widetilde{w}_n(x)|} \leqslant \frac{\tilde{q}_n(x)^2}{|x-1|} \frac{|x|^\alpha}{|\widetilde{w}_n(x)|} \quad \text{for } x < \tilde{\pi}_j - 1, \tag{4.129}$$

which shows that the integrand in (4.128) is dominated by that in (4.87). We note that the integrands in (4.87) and (4.128) are both non-negative, and integral (4.87) is standardized by (4.88).

From the limits (4.93), (4.104), (4.107), together with the properties (4.105) and (4.124), it follows that the limit function $\tilde{g}_2$ in (4.107) and (4.127) has a zero of order exactly 2 at the point $\tilde{\pi}_j$. Therefore we have

$$\left. \frac{\tilde{g}_2(v)}{(v - \tilde{\pi}_j)^2} \right|_{v = \tilde{\pi}_j} = \frac{1}{2} \tilde{g}_2''(\tilde{\pi}_j) \neq 0. \tag{4.130}$$

From (4.130) and (4.127), we then deduce that

$$\lim_{\substack{n\to\infty\\ n\in N}} c_n \frac{\widetilde{w}_n(\tilde{\pi}_{n,n+1-j})}{\tilde{q}_n'(\tilde{\pi}_{n,n+1-j})^2} = 2\frac{\sin\pi\alpha}{\pi}\frac{|\tilde{\pi}_j|^\alpha}{\tilde{g}_2''(\tilde{\pi}_j)}, \tag{4.131}$$

and with (4.126), (4.128) and (4.130), it further follows that the limit

$$\lim_{\substack{n\to\infty\\ n\in N}} \tilde{\lambda}_{n,n+1-j} = -2\frac{\sin\pi\alpha}{\pi}\frac{|\tilde{\pi}_j|^\alpha}{\tilde{g}_2''(\tilde{\pi}_j)}\int_{-\infty}^0 \frac{\tilde{g}_2(x)\,dx}{(x-\tilde{\pi}_j)^2} \neq 0,\infty \tag{4.132}$$

exists and is different from 0 and ∞.

From (4.132), (4.125), the limits (4.81), (4.82), and the strong inequalities in (4.124), we deduce that the two sets $\{\tilde{a}_1, ..., \tilde{a}_{[\alpha]}, \tilde{\zeta}_1, \tilde{\zeta}_2, ...\}$ and $\{\tilde{\pi}_1, \tilde{\pi}_2, ...\}$ are disjoint, since otherwise some of the residua $\tilde{\lambda}_{n,n+1-j}$ had to converge to zero as $n\to\infty$. Hence, we have proved

$$... < \tilde{\zeta}_{j+1} < \tilde{\pi}_j < \tilde{\zeta}_j < ... < \tilde{\pi}_1 < \tilde{\zeta}_1 < 0. \tag{4.133}$$

The last inequality in (4.133) follows from (2.8) and (2.9) in Lemma 2.

Since the two sets $\{\tilde{a}_1, ..., \tilde{a}_{[\alpha]}, \tilde{\zeta}_1, \tilde{\zeta}_2, ...\}$ and $\{\tilde{\pi}_1, \tilde{\pi}_2, ...\}$ are disjoint, it follows that limit (4.115) holds not only in $\mathbf{C}\setminus\{\tilde{\pi}_1, \tilde{\pi}_2, ...\}$; instead it holds in the cordial metric locally uniformly throughout $\mathbf{C}$, which proves limit (4.44).

From the extended validity of limit (4.115), we can then derive an extension of limit (4.58). Actually, we shall prove slightly more than stated in the lemma.

Let $\mathcal{R}$ denote the Riemann surface over $\mathbf{C}\setminus\{0\}$ which is defined by analytic continuation of the function $f_\alpha(w)=w^\alpha$. Like the function f_α, so also the function $\tilde{r}_n$ can be lifted to $\mathcal{R}$. We shall use the same notation for functions defined on $\mathbf{C}$ or on $\mathcal{R}$. From (4.1) and transformation (4.49) we know that

$$\tilde{r}_n(w) = \frac{w^\alpha - \tilde{r}_n^*(w)}{w^\alpha + \tilde{r}_n^*(w)} \quad \text{and} \quad \tilde{r}(w) = \frac{w^\alpha - \tilde{r}^*(w)}{w^\alpha + \tilde{r}^*(w)}. \tag{4.134}$$

Since the functions $\tilde{r}_n$ and $\tilde{r}$ are Möbius transforms of $\tilde{r}_n^*$ and $\tilde{r}^*$, respectively, the convergence in the cordial metric, which has been proved for limit (4.115), implies that the limit

$$\lim_{\substack{n\to\infty\\ n\in N}} \tilde{r}_n(w) = \tilde{r}(w) \tag{4.135}$$

holds true also locally uniformly for $w\in\mathcal{R}$ in the cordial metric. Since the set $(\mathbf{C}\setminus\mathbf{R}_-)\cup(-\infty,0)\pm i0$ can be embedded into $\mathcal{R}$, limit (4.43) follows from (4.135).

We have $\tilde{r}^*(w)\in\overline{\mathbf{R}}$ for all $w\in\mathbf{R}$. From (4.134) and the validity of limit (4.44) in the cordial metric in $\mathbf{C}$, it then is immediate that

$$\tilde{r}_n(w) \in \begin{cases} \partial K_\alpha & \text{for } w\in(-\infty,0)+i0, \\ \overline{\partial K_\alpha} & \text{for } w\in(-\infty,0)-i0, \end{cases} \tag{4.136}$$

with K_α the disc defined in (4.8).

With the completion of the proof of limit (4.43) the proof of the lemma is completed. □

The last lemma in the present section contains information about the behavior of the function r_n in the domain $\mathbf{C}\setminus\mathbf{R}_-$ away from the origin.

LEMMA 8. *There exists a constant $R>0$ such that for $n\in\mathbf{N}$ sufficiently large we have*

$$|r_n(\varrho e^{it})| \leqslant |r_n(\varrho e^{it'})| \quad \text{for } t,t'\in[-\pi,\pi],\ |t|\leqslant|t'|,\ \varrho\geqslant R\varepsilon_n^{1/\alpha}, \tag{4.137}$$

and

$$r_n(\varrho e^{it}) \in \begin{cases} \mathrm{Cl}(K_\alpha) & \text{for } 0\leqslant t\leqslant\pi,\ \varrho\geqslant R\varepsilon_n^{1/\alpha}, \\ \mathrm{Cl}(\bar{K}_\alpha) & \text{for } -\pi\leqslant t\leqslant 0,\ \varrho\geqslant R\varepsilon_n^{1/\alpha}, \end{cases} \tag{4.138}$$

with $\mathrm{Cl}(\cdot)$ *denoting the closure, and the disc K_α has been defined in* (4.8) *of Lemma* 5.

For the error function $e_n=f_\alpha-r_n^*$, we have the monotonicity

$$0 > x^{-\alpha}e_n(x) > (x')^{-\alpha}e_n(x') \quad \text{for } 1<x<x'. \tag{4.139}$$

Remark. The existence of the poles $b_{n,1},\ldots,b_{n,[\alpha]}$ in case of $\alpha>1$ shows that at least estimate (4.138) cannot hold for all $z\in\mathbf{C}\setminus\mathbf{R}_-$ if $\alpha>1$.

Proof. (i) We start with a proof of (4.137). As in the proof of Lemma 7, we use transformation (4.49), and based on this transformation, the notations introduced in (4.50). Thus, for instance, $\tilde{r}_n$ is defined by $\tilde{r}_n(w):=r_n(\varepsilon_n^{1/\alpha}w)$. An important piece of the proof of relation (4.137) is the verification of the inequality

$$\frac{\partial}{\partial\varrho}\arg\tilde{r}_n(\varrho e^{it}) \leqslant 0 \quad \text{for all } \varrho\geqslant R,\ n\geqslant n_0,\ t\in[0,\pi]. \tag{4.140}$$

Note that contrary to the function $\arg\tilde{r}_n$, its derivative in (4.140) is single-valued in any domain in which $\tilde{r}_n$ is analytic and different from zero.

The function $\tilde{r}_n$ has poles at $\tilde{b}_{nj}$, $j=1,\ldots,[\alpha]$, and zeros at $\tilde{z}_{nj}$, $j=1,\ldots,2n+2+[\alpha]$. We define

$$\widetilde{Q}_n(w) := \prod_{j=1}^{[\alpha]} \frac{w-\tilde{b}_{nj}}{w} = \prod_{j=1}^{[\alpha]}\left(1-\frac{\tilde{b}_{nj}}{w}\right), \tag{4.141}$$

$$\widetilde{G}_n(w) := \prod_{j=1}^{2n+2+[\alpha]} (w-\tilde{z}_{nj}) \tag{4.142}$$

and

$$\hat{r}_n(w) := \tilde{r}_n(w)\frac{\widetilde{Q}_n(w)}{\widetilde{G}_n(w)}. \tag{4.143}$$

The rational function $\widetilde{Q}_n$ in (4.141) has similarities with the function (4.54), but both functions are different. For the next steps of the analysis it is important that the quotient $\widetilde{Q}_n/\widetilde{G}_n$ has no sign change on $\mathbf{R}_-$. Indeed, since $\tilde{b}_{nj}\notin\mathbf{R}_-$, $j=1,...,[\alpha]$, and since all $\tilde{z}_{nj}\in(0,\infty)$, it follows that

$$w^{[\alpha]}\frac{\widetilde{Q}_n(w)}{\widetilde{G}_n(w)} > 0 \quad \text{for } w\in\mathbf{R}_-. \tag{4.144}$$

The function $\hat{r}_n$ is analytic and different from zero in $\mathbf{C}\backslash\mathbf{R}_-$. From Lemma 5 we know that it has analytic continuations to $(-\infty,0)+i0$ and $(-\infty,0)-i0$.

From (4.144) we know that $\arg\hat{r}_n(w)=\arg\tilde{r}_n(w)+(-1)^{[\alpha]}\pi$ for $w\in\mathbf{R}_-+i0$, and from (4.9) in Lemma 5 and the discussion after Lemma 5, we further know that $\arg\hat{r}_n(w)$ is monotonically increasing for $w\in(-\infty,0)+i0$ and monotonically decreasing for $w\in(-\infty,0)-i0$. From (4.1), (4.141) and (4.144), we conclude that

$$w^{[\alpha]}\hat{r}_n(w)|_{w=0} < 0. \tag{4.145}$$

From (4.1) it further follows that

$$\arg((-1)^{[\alpha]+1}\hat{r}_n(w+i0)) = -\arg((-1)^{[\alpha]+1}\hat{r}_n(w-i0)) \tag{4.146}$$

for $w\in(-\infty,0)$. At the origin $w=0$, the function $\arg((-1)^{[\alpha]+1}\hat{r}_n(w))$ has a jump about $\pi[\alpha]$ if this function is considered with an argument running along the two banks $\mathbf{R}_-+i0$ and $\mathbf{R}_--i0$ of $\mathbf{R}_-$.

The two functions $\arg\hat{r}_n$ and $\arg((-1)^{[\alpha]+1}\hat{r}_n)$ are harmonic in $\mathbf{C}\backslash\mathbf{R}_-$ and have harmonic extensions to $(-\infty,0)\pm i0$. Also the expression

$$\varrho\frac{\partial}{\partial\varrho}\arg(\hat{r}_n(\varrho e^{it})) = \varrho\frac{\partial}{\partial\varrho}\arg((-1)^{[\alpha]+1}\hat{r}_n(\varrho e^{it})), \quad w=\varrho e^{it},$$

is harmonic in $\mathbf{C}\backslash\mathbf{R}_-$, which can easily be seen by mapping $\mathbf{C}\backslash\mathbf{R}_-$ conformally onto the strip $\{v\,|\,|\mathrm{Im}(v)|<\pi\}$. From the monotonicity of $\arg(\hat{r}_n)$ on $\mathbf{R}_-+i0$ and $\mathbf{R}_--i0$, it follows that $\varrho(\partial/\partial\varrho)\arg(\hat{r}_n(\varrho e^{i\pi}))<0$ and $\varrho(\partial/\partial\varrho)\arg(\hat{r}_n(\varrho e^{-i\pi}))>0$ for $\varrho\in(\infty,0)$. From this observation together with the symmetry property (4.146) and the harmonicity of $\varrho(\partial/\partial\varrho)\arg(-\hat{r}_n(\varrho e^{it}))$ in $\mathbf{C}\backslash\mathbf{R}_-$, we conclude that

$$\begin{aligned}
&\frac{\partial}{\partial\varrho}\arg(\hat{r}_n(\varrho e^{it})) < 0 \quad \text{for } t\in(0,\pi],\ \varrho>0,\\
&\frac{\partial}{\partial\varrho}\arg(\hat{r}_n(\varrho e^{it})) = 0 \quad \text{for } t=0,\ \varrho>0,\\
&\frac{\partial}{\partial\varrho}\arg(\hat{r}_n(\varrho e^{it})) > 0 \quad \text{for } t\in[-\pi,0),\ \varrho>0.
\end{aligned} \tag{4.147}$$

It follows from definition (4.143) that we have to study also the behavior of the arguments of the functions $\widetilde{Q}_n$ and $\widetilde{G}_n$ if we want to verify (4.140). For this purpose we use the identity

$$\frac{\partial}{\partial \varrho}\arg(\varrho e^{it}-b)=\frac{-b\sin t}{|\varrho e^{it}-b|^2}, \tag{4.148}$$

which holds for $b\in(0,\infty)$, $\varrho\in(0,\infty)\setminus\{b\}$, $t\in[-\pi,\pi]$, and the inequality

$$\left|\frac{\partial}{\partial \varrho}\arg\left(\left(1-\frac{b}{\varrho}e^{-it}\right)\left(1-\frac{\bar{b}}{\varrho}e^{-it}\right)\right)\right|\leqslant\frac{2|b|(\varrho+|b|)^2|\sin t|}{|\varrho e^{it}-b|^2|\varrho e^{it}-\bar{b}|^2}, \tag{4.149}$$

which holds for $b\in\mathbf{C}\setminus\mathbf{R}$, $\varrho>0$ and $t\in(0,\pi]$. Both relations will be verified only after (4.155), below.

With the help of (4.148) and (4.149) we show that there exist $R>0$ and $n_0\in\mathbf{N}$ such that

$$\left|\frac{\partial}{\partial \varrho}\arg\widetilde{Q}_n(\varrho e^{it})\right|\leqslant-\frac{\partial}{\partial \varrho}\arg\widetilde{G}_n(\varrho e^{it})\quad\text{for } \varrho\geqslant R,\ n\geqslant n_0,\ t\in(0,\pi]. \tag{4.150}$$

Indeed, it follows from the definition of the transformed poles $\tilde{b}_{nj}$, $j=1,...,[\alpha]$, in (4.12) that we either have $\tilde{b}_{nj}\in(0,\infty)$ or the $\tilde{b}_{nj}$ appear in conjugated pairs $\{\tilde{b}_{nj},\bar{\tilde{b}}_{nj}\}$. From (4.13) in Lemma 6, we know that there exists $R>0$ such that $|\tilde{b}_{nj}|<R$ for all $j=1,...,[\alpha]$ and $n\in\mathbf{N}$. Using estimate (4.149) for the conjugated pairs $\{\tilde{b}_{nj},\bar{\tilde{b}}_{nj}\}$, and identity (4.148) for the poles $\tilde{b}_{nj}\in(0,\infty)$, we deduce from (4.141), (4.148) and (4.149) that there exist $R>0$, $n_0\in\mathbf{N}$ and a constant $c<\infty$ such that

$$\left|\frac{\partial}{\partial \varrho}\arg\widetilde{Q}_n(\varrho e^{it})\right|\leqslant\frac{c}{\varrho^2}|\sin t|\quad\text{for } \varrho\geqslant R,\ n\geqslant n_0,\ t\in[-\pi,\pi]. \tag{4.151}$$

On the other hand, from (4.142) and (4.148) we deduce that

$$\frac{\partial}{\partial \varrho}\arg\widetilde{G}_n(\varrho e^{it})=-\sin t\sum_{j=1}^{2n+2+[\alpha]}\frac{\tilde{z}_{nj}}{|\varrho e^{it}-\tilde{z}_{nj}|^2}\leqslant-\sin t\sum_{j=1}^{2n+2+[\alpha]}\frac{\tilde{z}_{nj}}{|\varrho+\tilde{z}_{nj}|^2} \tag{4.152}$$

for all $\varrho\geqslant R$ and $t\in(0,\pi]$. With the limits (4.38) in Lemma 7, we conclude that

$$\varrho^2\sum_{j=1}^{2n+2+[\alpha]}\frac{\tilde{z}_{nj}}{|\varrho+\tilde{z}_{nj}|^2}=\sum_{j=1}^{2n+2+[\alpha]}\frac{\tilde{z}_{nj}}{|1+\tilde{z}_{nj}/\varrho|^2}\geqslant\frac{1}{4}\sum_{\tilde{z}_{nj}\leqslant\varrho}\tilde{z}_{nj}\to\frac{1}{4}\sum_{\tilde{z}_j\leqslant\varrho}\tilde{z}_j\quad\text{as } n\to\infty. \tag{4.153}$$

Since from the inequalities in part (iii) of Lemma 7 together with limit (4.38) in Lemma 7, we know that there exist infinitely many points $\tilde{z}_j<\infty$, $j\in\mathbf{N}$, we further conclude that

$$\frac{1}{4}\sum_{\tilde{z}_j\leqslant\varrho}\tilde{z}_j\to\infty\quad\text{as } n\to\infty. \tag{4.154}$$

From the relations (4.151) through (4.154), it then follows that (4.150) has to hold true. From (4.150) together with (4.143) and (4.147), we then deduce inequality (4.140).

By using the Cauchy–Riemann differential equations in polar coordinates, it follows that (4.140) implies

$$\frac{\partial}{\partial t}\log|\tilde{r}_n(\varrho e^{it})|\geqslant 0 \quad \text{for } \varrho\geqslant R,\ n\geqslant n_0,\ t\in(0,\pi], \tag{4.155}$$

with $R>0$, and $n_0\in\mathbf{N}$, chosen as in (4.140). This inequality proves (4.137) for $t,t'\in[0,\pi]$. Since the function r_n is of real type, the same inequality follows for $t,t'\in[-\pi,0]$.

For a completion of the proof of (4.137) it is still necessary to verify the two relations (4.148) and (4.149). Identity (4.148) follows rather immediately from considering the derivative of $\log(\varrho e^{it}-b)$ and a subsequent taking of the imaginary part. Let now $b=|b|e^{i\beta}\in\mathbf{C}\setminus\mathbf{R}$. Proceeding as in the verification of (4.148) we arrive at

$$\begin{aligned}\operatorname{Im}\frac{\partial}{\partial\varrho}\arg\left(\left(1-\frac{b}{\varrho}e^{-it}\right)\left(1-\frac{\bar{b}}{\varrho}e^{-it}\right)\right)&=\frac{-|b|\sin(t-\beta)}{|\varrho-|b|e^{-i(t-\beta)}|^2}+\frac{-|b|\sin(t+\beta)}{|\varrho-|b|e^{-i(t+\beta)}|^2}\\&=\frac{-2|b|(\varrho+|b|)^2\cos\beta\sin t+2\varrho|b|^2\sin(2t)}{|\varrho-|b|e^{-i(t-\beta)}|^2\,|\varrho-|b|e^{-i(t+\beta)}|^2}.\end{aligned} \tag{4.156}$$

From (4.156) the estimate (4.149) follows rather directly by trigonometric inequalities.

(ii) We now come to the proof of the relations (4.138). Let $R_1>0$ be so large that (4.137) holds true for all $n\geqslant n_0\in\mathbf{N}$. Then for the $[\alpha]$ poles of $\tilde{r}_n$ in $\mathbf{C}\setminus\mathbf{R}_-$, we have $|\tilde{b}_{nj}|<R_1$, $j=1,\dots,[\alpha]$, and the function $\tilde{r}_n$ is analytic in the domain $D_{R_1}:=\mathbf{C}\setminus(\mathbf{R}_-\cup\{|w|\leqslant R_1\})$.

Knowing that the limits (4.39) and (4.40) exist and that the limit (4.43) exists locally uniformly on $\mathbf{R}_-+i0$, we conclude from the discussion of the behavior of the function $\arg\tilde{r}_n$ on $\mathbf{R}_-+i0$ after the proof of Lemma 5 that we can choose $R>R_1$ such that $\arg\tilde{r}_n$ grows by more than 2π on the interval $[-R,-R_1]+i0$ for each $n\geqslant n_0$.

Let K_α be the disc defined by (4.8) in Lemma 5, and let $v_\alpha\in\partial K_\alpha$ be the point which lies closest to the origin. This point is unique if $\alpha+0.5\notin\mathbf{N}$, and it is not difficult to verify that

$$\arg(v_\alpha)=\tfrac{1}{2}\pi(-1)^{[2\alpha]}\quad\operatorname{mod}(2\pi). \tag{4.157}$$

In case of $\alpha+0.5\in\mathbf{N}$, the point v_α is no longer unique, since in this case $K_\alpha=\{|w|<1\}$; however, we can assume that (4.157) holds true.

For each $n\geqslant n_0$, we can choose $R_{(n)}$ such that $R_1\leqslant R_{(n)}\leqslant R$ and $\arg\tilde{r}_n(R_{(n)}e^{i\pi})=\frac{1}{2}\pi(-1)^{[2\alpha]}\operatorname{mod}(2\pi)$. Since $\tilde{r}_n(R_{(n)}e^{it})$ approaches ∂K_α as $t\to\pi-0$, it follows from the monotonicity (4.137) that at a point that lies nearest to the origin, we have

$$\tilde{r}_n(R_{(n)}e^{it})\in\operatorname{Cl}(K_\alpha)\quad\text{for all } t\in(0,\pi]. \tag{4.158}$$

From (4.9) in Lemma 5, we know that

$$\tilde{r}_n(\varrho e^{i(\pi-0)}) \in \partial K_\alpha \quad \text{for } R_1 \leqslant \varrho \leqslant \infty. \tag{4.159}$$

From (4.2) and (4.3) in Lemma 5, it follows that $\tilde{r}_n(w) \in (-1,1)$ for $w \in [R_1, \varepsilon_n^{-1/\alpha}]$, and from (3.22) together with (4.1), it further follows that $\tilde{r}_n(w) \in [-1,0]$ for $w \in [\varepsilon_n^{-1/\alpha}, \infty]$. Hence, with (4.158) and (4.159), we have proved that

$$\tilde{r}_n(w) \in \mathrm{Cl}(K_\alpha) \quad \text{for } w \in \partial H_{(n)} \tag{4.160}$$

with $H_{(n)} := \{\varrho e^{i\pi} \mid R_{(n)} \leqslant \varrho \leqslant \infty, 0 \leqslant t \leqslant \pi\}$.

Since analytic functions are open mappings, it follows that (4.160) implies $\tilde{r}_n(w) \in \mathrm{Cl}(K_\alpha)$ for all $w \in H_{(n)}$, and since the function $\tilde{r}_n$ is of real type, it further follows that $\tilde{r}_n(w) \in \mathrm{Cl}(\overline{K}_\alpha)$ for $w \in \{\varrho e^{i\pi} \mid R_{(n)} \leqslant \varrho \leqslant \infty, -\pi \leqslant t \leqslant 0\}$. Because of $R_{(n)} \leqslant R$, the last two assertions imply that (4.138) holds true.

(iii) At last, we prove monotonicity (4.139). From (3.6) in Lemma 3 and definition (3.7) of the positive measure μ_n, we have the representation

$$|z^{-\alpha} e_n(z)| = \frac{w_n(z)}{z^\alpha q_n(z)^2} \int \frac{q_n(x)^2}{|x-z|}\, d\mu_n(x) = \frac{w_n(z)}{z^\alpha q_n(z)^2} I(z) \tag{4.161}$$

with the polynomial w_n defined in (3.1) and q_n being the denominator of r_n^*. Since $\mathrm{supp}(\mu_n) = \mathbf{R}_-$, it follows that

$$|zI(z)| = \int \frac{q_n(x)^2}{|x/z-1|}\, d\mu_n(x) \tag{4.162}$$

is a strictly monotonically increasing function for $z \in (1, \infty)$. Since all zeros of

$$\frac{w_n(z)}{z^{\alpha+1} q_n(z)^2} \tag{4.163}$$

are contained in $(0,1)$ and all poles in $\mathbf{R}_-$, function (4.163) is also strictly monotonically increasing for $z \in (1,\infty)$. The two monotonicities together with (4.161) prove (4.139). $\square$

5. Tools from potential theory

Several aspects in the proofs of Lemmas 6 and 7 were already in spirit of a potential-theoretic nature; this orientation will become more dominant in the last two sections of the present paper. In the present section we start with the introduction of some terminology related to potential theory. We continue with an important result, which will be stated in Proposition 1. It deals with the representation of the log-function by a Green potential. Fortunately, the underlying problem has already been studied in [19] and in [14, Chapter 8]. Further, some special potential-theoretic results of an auxiliary nature will be proved.

The (logarithmic) *potential* of a measure μ is denoted by $p(\mu;\cdot)$ and defined as

$$p(\mu;z):=\int \log\frac{1}{|z-x|}\,d\mu(x). \tag{5.1}$$

By $\operatorname{cap}(\cdot)$ we denote the (logarithmic) *capacity* (for a definition see [22, Appendix I] or [13, Chapter II]). For a domain $D\subseteq\overline{\mathbf{C}}$ we denote the *Green function* in D by $g_D(z,v)$, $z,v\in\overline{\mathbf{C}}$ (for a definition see [22, Appendix V] or [13, Chapter IV]). We assume that $g_D(\cdot,\cdot)$ is defined throughout $\overline{\mathbf{C}}\times\overline{\mathbf{C}}$. If $D=\mathbf{C}\setminus\mathbf{R}_-$, then it follows from (4.15) and (4.16) that

$$g_{\mathbf{C}\setminus\mathbf{R}_-}(z,v)=\log\frac{1}{|B(z,v)|}=\log\frac{|\sqrt{z}+\sqrt{v}\,|}{|\sqrt{z}-\sqrt{\bar v}\,|} \tag{5.2}$$

with $\sqrt{\cdot}$ denoting the principle branch. For the domain $D_0:=\{\operatorname{Re}(z)>0\}$ the Green function is given by

$$g_{D_0}(z,v)=\log\frac{|z+v|}{|z-\bar v|}. \tag{5.3}$$

It follows from (5.2) and (5.3) that for $v\in(0,\infty)$ the Green functions $g_{\widetilde D}(re^{it},v)$, $\widetilde D=\mathbf{C}\setminus\mathbf{R}_-$ or $\widetilde D=D_0$, are monotonically decreasing functions of $|t|$ with $|t|\in[0,\pi]$ for a given $r>0$. For an arbitrary domain $D\subseteq\overline{\mathbf{C}}$ and a measure μ we define the *Green potential* as

$$g(\mu,D;z):=\int g_D(z,x)\,d\mu(x). \tag{5.4}$$

A useful tool in potential-theoretical investigations is the technique of *balayage*. A definition for logarithmic potentials can be found in [22, Appendix VII], [13, Chapter IV] or [18, Chapter II.4]. In our investigation we use this technique for Green potentials. In order to avoid technical subtilities, we assume that all domains involved are regular (with respect to Dirichlet problems) (cf. [22, Appendix II] or [18, Chapter I.5]). Let $D\subseteq\overline{\mathbf{C}}$ be a regular domain with $\operatorname{cap}(\overline{\mathbf{C}}\setminus D)>0$, μ a positive measure carried by D, i.e., $\mu(D)=\|\mu\|$, and $G\subseteq D$ a regular subdomain. Then there exists a positive measure $\hat\mu$, called the *balayage measure*, such that

$$g(\hat\mu,D;z)=g(\mu,D;z)\quad\text{for all } z\in\overline{\mathbf{C}}\setminus G, \tag{5.5}$$

$\hat\mu$ is carried by $D\setminus G$, and we have

$$\|\mu\|-\mu(G)=\mu(D\setminus G)\leqslant\|\hat\mu\|\leqslant\|\mu\|. \tag{5.6}$$

By $\|\cdot\|$ we denote the total mass of a measure (or the total variation in case of a signed measure). We have $\operatorname{supp}(\hat{\mu})\subseteq\overline{(\operatorname{supp}(\mu)\backslash G\cup\partial G)\cap D}$ since $\hat{\mu}$ is carried by $D\backslash G$. The balayage technique for Green potentials can be seen as a special case of balayage for logarithmic potentials since Green potentials can be represented as the difference of two logarithmic potentials (cf. [22, Appendix V] or [18, Chapter II.4]). The inequalities in (5.6) are consequences of the possibility that parts of the measure μ are swept on pieces of ∂G that are contained in ∂D, and that the mass swept there becomes irrelevant for the Green potential $g(\hat{\mu}, D;\cdot)$. Since it is assumed that the balayage measure $\hat{\mu}$ is carried by $D\backslash G$, these parts of the swept-out measure μ are no longer part of the balayage measure $\hat{\mu}$.

Green potentials in the domain $\mathbf{C}\backslash\mathbf{R}_-$, which represent linear transformations $c+\alpha\log|\cdot|$ of the log-function on a given interval $[R,x]\subseteq\mathbf{R}_+$, will play a fundamental role in the proof of Theorem 1. These potentials are studied in the next proposition.

PROPOSITION 1. *Let $c\in\mathbf{R}$, $\alpha>0$, $R\geqslant 4e^{-c/\alpha}$ and $x>R$. Then there exists a positive measure $\nu=\nu_{x,R,c,\alpha}$ with* $\operatorname{supp}(\nu)=[R,x]$ *such that*

$$g(\nu,\mathbf{C}\backslash\mathbf{R}_-;z)=\int g_{\mathbf{C}\backslash\mathbf{R}_-}(z,x)\,d\nu(x)=c+\log|z|^\alpha \quad \textit{for all } z\in[R,x], \tag{5.7}$$

and for $x\to\infty$ we have

$$\lim_{x\to\infty}\left(\pi\sqrt{2\alpha\|\nu\|}-\alpha\log x\right)=c+\alpha\log 4. \tag{5.8}$$

Remark. Proposition 1 shows that for $x\to\infty$ the total mass $\|\nu\|$ of the measure ν tends to infinity. However, the limit (5.8) shows more; it gives a quantitative estimate for the growth.

The proof of Proposition 1 follows after the next theorem, which has already been proved in [19], and with a more transparent and shorter proof in [14, Theorems 8.3.2 and 8.3.3].

THEOREM 4 ([19, Theorem 2]). *For the domain $D_0=\{\operatorname{Re}(z)>0\}$ and for any $a\in(0,1)$ there exists a positive measure ν_a with* $\operatorname{supp}(\nu_a)=[a,b(a)]$, $a<b(a)<1$, *such that the Green potential*

$$g_a(z):=g(\nu_a,D_0;z)=\int\log\left|\frac{z+x}{z-x}\right|d\nu_a(x) \tag{5.9}$$

satisfies

$$\begin{aligned} g_a(z)&=\log\frac{1}{|z|} \quad \textit{for } z\in[a,b(a)],\\ g_a(z)&>\log\frac{1}{|z|} \quad \textit{for } z\in(b(a),\infty). \end{aligned} \tag{5.10}$$

We have $\|\nu_{a'}\|>\|\nu_a\|$ *and* $b'=b(a')<b=b(a)$ *for* $a'<a$,

$$\tfrac{1}{2}<b(a)<1 \quad \text{for } a\in(0,1), \tag{5.11}$$

$$\lim_{a\to 0+} b(a)=\tfrac{1}{2}, \tag{5.12}$$

$$\nu_a((a_0,b(a)])=\mathcal{O}(1) \quad \text{as } a\to 0+ \tag{5.13}$$

for any $0<a_0\leqslant\frac{1}{2}$, *and*

$$\lim_{a\to 0+} a\exp\bigl(\pi\sqrt{\|\nu_a\|}\,\bigr)=2. \tag{5.14}$$

The proof of Theorem 4 in [19] is based on a systematic study of the function

$$f_a(z):=\frac{1}{c_0}\int_0^z\frac{d\zeta}{\zeta}\int_0^\zeta\frac{(c_1^2+t^2)\,dt}{\sqrt{(1-t^2)(c_2^2-t^2)}},\quad z,\zeta\in D_0, \tag{5.15}$$

with the three constants $c_0, c_1>0$, $c_2>1$ determined by the three conditions

$$\begin{gathered}\int_0^\infty\frac{(c_1^2-t^2)\,dt}{\sqrt{(1+t^2)(c_2^2+t^2)}}=0,\\ \int_0^\infty\frac{(c_1^2+t^2)\,dt}{\sqrt{(1-t^2)(c_2^2-t^2)}}=c_0,\\ \int_{c_2}^\infty\frac{dx}{x}\int_0^x\frac{(c_1^2+t^2)\,dt}{\sqrt{(1-t^2)(c_2^2-t^2)}}=c_0\log\frac{1}{ac_2}.\end{gathered} \tag{5.16}$$

The proof of Theorem 4 demands delicate estimates of elliptical integrals and will not be repeated here.

Proof of Proposition 1. The proposition follows from Theorem 4 by choosing the constant $a=a_x$ in the theorem in an appropriate way for each x, and by transforming the domain of definition D_0 of g_a in (5.9) into $\mathbf{C}\backslash\mathbf{R}_-$. Finally, it is necessary to use balayage in order to make sure that $\operatorname{supp}(\nu)=[R,x]$. The appropriate choice for the parameter a in Theorem 4 is

$$a=a_x:=\sqrt{\frac{e^{-c/\alpha}}{x}}\,. \tag{5.17}$$

With this choice we define the function $\tilde{g}$ as

$$\begin{aligned}\tilde{g}(w)&:=2\alpha g_{a_x}\bigl(e^{-c/2\alpha}/\sqrt{w}\,\bigr)\\ &=2\alpha\int\log\left|\frac{\sqrt{w}+\sqrt{v}}{\sqrt{w}-\sqrt{v}}\right|d\hat{\nu}_{a_x}(v)=\int g_{\mathbf{C}\backslash\mathbf{R}_-}(w,v)\,d\tilde{\nu}(v).\end{aligned} \tag{5.18}$$

The second equality in (5.18) follows from (5.9), with $\hat{\nu}_a$ the image of the measure ν_a in (5.9) under the mapping $z\mapsto w=e^{-c/\alpha}z^{-2}$. The third equality in (5.18) follows from (5.2) and the definition $\tilde{\nu}:=2\alpha\hat{\nu}_a$. Thus, we have

$$\|\tilde{\nu}\| = 2\alpha\|\nu_{a_x}\|. \tag{5.19}$$

Under $z\mapsto w=e^{-c/\alpha}z^{-2}$ the interval $[a_x, b(a_x)]$ transforms into $[e^{-c/\alpha}b(a_x)^{-2}, x]$. From (5.11) and the assumptions made in the proposition we deduce that $\tilde{b}_x:=e^{-c/\alpha}b(a_x)^{-2}<4e^{-c/\alpha}\leqslant R$. Hence, we have

$$\operatorname{supp}(\tilde{\nu}) = [\tilde{b}_x, x] \supset [R, x]. \tag{5.20}$$

From (5.10) and the definition of $\tilde{g}$ in (5.18), we deduce that

$$\tilde{g}(w) := 2\alpha\log\big(e^{c/2\alpha}\sqrt{w}\,\big) = c+\alpha\log(w) \quad \text{for } w\in[R,x]. \tag{5.21}$$

It is immediate from (5.18) that $\tilde{g}$ is a Green potential defined by the positive measure $\tilde{\nu}$. However, the support $[\tilde{b}_x, x]$ is larger than $[R, x]$. Therefore we use balayage to remove the measure $\tilde{\nu}$ from the subinterval $[\tilde{b}_x, R)$. Let $\hat{\nu}$ be the balayage measure of the measure $\tilde{\nu}$ resulting from balayage out of the domain $\mathbf{C}\setminus(\mathbf{R}_-\cup[R,x])$. We then have $\operatorname{supp}(\hat{\nu})=[R,x]$, and from (5.5) we learn that (5.21) implies (5.7) if we take $\nu:=\hat{\nu}$. From (5.6) we deduce that

$$\|\tilde{\nu}\| - \tilde{\nu}([\tilde{b}_x, R)) \leqslant \|\hat{\nu}\| \leqslant \|\tilde{\nu}\|. \tag{5.22}$$

It only remains to prove the limit (5.8). The interval $[\tilde{b}_x, R)$ is the image of $\big[e^{-c/2\alpha}/\sqrt{R}, b(a_x)\big)$ under the mapping $z\mapsto w=e^{-c/\alpha}z^{-2}$. From (5.12) and (5.13) we deduce that

$$\lim_{x\to\infty} \tilde{\nu}([\tilde{b}_x, R)) = 2\alpha \lim_{x\to\infty} \nu_{a_x}([\tilde{b}_x, R)) < \infty. \tag{5.23}$$

From (5.14), (5.9), (5.17), (5.18), (5.22), and taking $\nu=\hat{\nu}$, we deduce that

$$\begin{aligned} 2 &= \lim_{x\to\infty} a_x \exp\big(\pi\sqrt{\|\nu_{a_x}\|}\,\big) = \lim_{x\to\infty} x^{-1/2}e^{-c/2\alpha}\exp\big(\pi\sqrt{\|\nu\|/2\alpha+\mathcal{O}(1)}\,\big) \\ &= \lim_{x\to\infty} x^{-1/2}e^{-c/2\alpha}\exp\big(\pi\sqrt{\|\nu\|/2\alpha}\,\big) \end{aligned} \tag{5.24}$$

since $\|\nu\|\to\infty$ as $x\to\infty$. By taking logarithms and multiplying by 2α it follows from (5.24) that

$$\lim_{x\to\infty}\big(\pi\sqrt{2\alpha\|\nu\|} - \alpha\log x\big) = c+2\alpha\log 2, \tag{5.25}$$

which proves (5.8). □

The section is closed by two technical lemmas.

LEMMA 9. (i) *For any* $0<\alpha<\frac{1}{2}$ *and any* $R>0$ *there exists a positive measure* λ_α *with* $\operatorname{supp}(\lambda_\alpha)=[R,\infty]$ *and* $\|\lambda_\alpha\|<\infty$ *such that the Green potential* $g_\alpha(z):=g(\lambda_\alpha,\mathbf{C}\backslash\mathbf{R}_-;z)$ *satisfies*

$$g_\alpha(z)=z^{-\alpha} \quad \text{for } z\in[R,\infty). \tag{5.26}$$

(ii) *Let the function* h_α *be harmonic in the domain* $D_R:=\mathbf{C}\backslash(\mathbf{R}_-\cup\{|z|\leqslant R\})$ *with* $0<\alpha<\frac{1}{2}$, $R>0$, *and assume that* h_α *has boundary values* $h_\alpha(z)=|z|^{-\alpha}$ *for* $z\in\partial D_R$. *Then there exists a constant* $c=c_\alpha$ *such that*

$$0<h_\alpha(z)<c|z|^{-\alpha} \quad \text{for } z\in D_R. \tag{5.27}$$

Proof. (i) For $0<\alpha<\frac{1}{2}$, we consider the function

$$\tilde g(z):=\begin{cases}\operatorname{Re}(e^{i(-0.5+\alpha)\pi}z^{-\alpha})/\sin\pi\alpha & \text{for } z\in\overline{H}_+,\\ \operatorname{Re}(e^{i(0.5-\alpha)\pi}z^{-\alpha})/\sin\pi\alpha & \text{for } z\in H_-,\end{cases} \tag{5.28}$$

define the positive measure $\tilde\lambda$ by

$$d\tilde\lambda(x):=\frac{\alpha}{\pi}(\cot\pi\alpha)x^{-(1+\alpha)}\,dx=-\frac{1}{\pi}\frac{\partial}{\partial y}\tilde g(x+iy)|_{y=+0}\,dx, \quad x\in(0,\infty), \tag{5.29}$$

and then show that $\tilde g=g(\tilde\lambda,\mathbf{C}\backslash\mathbf{R}_-;\cdot)$. Proving the representation of $\tilde g$ by the Green potential $g(\tilde\lambda,\mathbf{C}\backslash\mathbf{R}_-;\cdot)$ demands some care since $\tilde g$ is unbounded in $\mathbf{C}\backslash\mathbf{R}_-$, and the measure $\tilde\lambda$ has infinite mass. Both problems appear near the origin. We therefore consider the domains $D_{1/n}=\mathbf{C}\backslash(\mathbf{R}_-\cup\{|z|\leqslant 1/n\})$, $n\in\mathbf{N}$. Since the normal derivatives $\partial/\partial y$ of $\tilde g$ to both sides of $(0,\infty)$ are negative, we see that $\tilde g$ is superharmonic in $\mathbf{C}\backslash\mathbf{R}_-$. From the Riesz decomposition theorem (cf. [18, Theorem II.3.1]) we therefore know that $\tilde g$ can be represented as

$$\tilde g(z)=h_n(z)+\int_{D_{1/n}}g_{D_{1/n}}(z,v)\,d\tilde\lambda(v) \quad \text{for } z\in D_{1/n} \tag{5.30}$$

with h_n being the solution of the Dirichlet problem in the domain $D_{1/n}$ with boundary values $h_n=\tilde g$ on $\partial D_{1/n}$. Actually, the Riesz decomposition theorem only ascertains that there exists a positive measure defining the Green potential on the right-hand side of (5.30). However, using the representation of $\tilde\lambda$ given after the second equality in (5.29), it can be shown with the help of the Green formula that the defining measure in the Green potential in (5.30) has to be the measure $\tilde\lambda$ which has been defined in (5.29). A method for recovering the defining measure of a potential has been shown in detail in Theorem II.1.5 of [18] under conditions that are applicable in the present situation.

The function $\varphi(z):=\sqrt{z}$ maps the domain $\mathbf{C}\backslash\mathbf{R}_-$ onto $D_0:=\{\operatorname{Re}(z)>0\}$. Let $\hat\delta_{z,n}$ be the balayage measure of the Dirac measure δ_z, $z\in D_{1/n}$, out of $D_{1/n}$ (cf. [22, Appendix VII] or [18, Chapter II.4]). By considering measures on $\overline{D}_0$ that correspond to $\hat\delta_{z,n}$

and δ_z under the mapping φ, it is not too difficult to verify that there exists a constant $c_1<\infty$, which is independent of n, such that

$$\hat{\delta}_{z,n}(\{|z|\leqslant 1/n\})\leqslant\frac{c_1}{\sqrt{n}}\quad\text{for } n\geqslant n_0 \tag{5.31}$$

and $z\in D_{1/n_0}$ fixed. Since

$$\tilde{g}(z)\leqslant|z|^{-\alpha}\quad\text{for } z\in\mathbf{C}\setminus\mathbf{R}_-, \tag{5.32}$$

$\alpha<\frac{1}{2}$ and $\tilde{g}(z)=0$ for all $z\in\partial D_{1/n}\setminus\{|z|\leqslant 1/n\}$, it follows from standard tools of potential theory (the construction of a solution of a Dirichlet problem with the help of harmonic measures) that

$$\lim_{n\to\infty}h_n(z)=0\quad\text{locally uniformly for } z\in\mathbf{C}\setminus\mathbf{R}_-. \tag{5.33}$$

This proves that

$$\tilde{g}=g(\tilde{\lambda},\mathbf{C}\setminus\mathbf{R}_-;\cdot\,). \tag{5.34}$$

We note that from (5.2) we can deduce that there exists a constant $c_2<\infty$ such that $g_{\mathbf{C}\setminus\mathbf{R}_-}(z,v)\leqslant c_2\operatorname{Re}(\sqrt{v})$ for $v\in\mathbf{C}$, $|v|<r$, $r>0$ small, and $z\in\mathbf{C}\setminus\mathbf{R}_-$ fixed. This estimate together with (5.29) and $\alpha<\frac{1}{2}$ shows that the Green function $g_{\mathbf{C}\setminus\mathbf{R}_-}(z,\cdot\,)$ is $\tilde{\lambda}$-integrable, and therefore the Green potential $g(\tilde{\lambda},\mathbf{C}\setminus\mathbf{R}_-;\cdot\,)$ is well defined.

Let now $R>0$ be fixed and let λ_α be the measure that results from balayage of the Green potential $g(\tilde{\lambda},\mathbf{C}\setminus\mathbf{R}_-;\cdot\,)$ out of the domain $\mathbf{C}\setminus(\mathbf{R}_-\cup[R,\infty))\subset\mathbf{C}\setminus\mathbf{R}_-$. Then we have

$$g_\alpha(z):=g(\lambda_\alpha,\mathbf{C}\setminus\mathbf{R}_-;z)=\begin{cases} z^\alpha & \text{for } z\in[R,\infty),\\ 0 & \text{for } z\in(-\infty,0].\end{cases} \tag{5.35}$$

Since the balayage measure λ_α is carried by $\mathbf{C}\setminus(\mathbf{R}_-\cup[R,\infty))$ (cf. the introduction of the balayage technique in (5.5) and (5.6) for the special situation of a Green potential), we have $\operatorname{supp}(\lambda_\alpha)\subseteq[R,\infty)$. Thus, it only remains to show that λ_α is of finite mass.

Indeed, from the definition of balayage (cf. [22, Formula A.15]) we know that $g(\lambda_\alpha,\mathbf{C}\setminus\mathbf{R}_-;\cdot\,)$ can be presented as

$$g(\lambda_\alpha,\mathbf{C}\setminus\mathbf{R}_-;z)=g(\tilde{\lambda},\mathbf{C}\setminus\mathbf{R}_-;z)-\int_0^R g_{\mathbf{C}\setminus(\mathbf{R}_-\cup[R,\infty))}(z,v)\,d\tilde{\lambda}(v). \tag{5.36}$$

Since $\mathbf{C}\setminus(\mathbf{R}_-\cup[R,\infty))$ is a subdomain of $\mathbf{C}\setminus\mathbf{R}_-$, we have

$$g_{\mathbf{C}\setminus(\mathbf{R}_-\cup[R,\infty))}(z,v)\leqslant g_{\mathbf{C}\setminus\mathbf{R}_-}(z,v)$$

for all $v\in\overline{\mathbf{C}}$ and $z\in(0,R)$. Hence, from the $\tilde{\lambda}$-integrability of $g_{\mathbf{C}\setminus\mathbf{R}_-}(z,\cdot\,)$ we deduce the $\tilde{\lambda}$-integrability of $g_{\mathbf{C}\setminus(\mathbf{R}_-\cup[R,\infty))}(z,\cdot\,)$, which shows that the second term on the right-hand side of (5.36) is bounded in a neighborhood of $[R,\infty]$ seen as a subset of $\overline{\mathbf{C}}$, which implies that the difference $\lambda_\alpha - \tilde{\lambda}|_{[R,\infty]}$ is a measure of finite mass, and therefore λ_α is, like $\tilde{\lambda}|_{[R,\infty]}$, also of finite mass. Note that by a Möbius transform, a neighborhood of $[R,\infty]$ can always be mapped on a neighborhood of a finite interval.

(ii) Let the function $\tilde{h}$ be defined by

$$\tilde{h}(z) := \frac{1}{\cos \pi\alpha} \operatorname{Re}(z^{-\alpha}) \quad \text{for } z \in D_R. \tag{5.37}$$

The function $\tilde{h}$ is harmonic in D_R and has boundary values

$$\begin{aligned} \tilde{h}(z) &= |z|^{-\alpha} && \text{for } z \in (-\infty, 0] \pm i0, \\ \tilde{h}(z) &\geqslant R^{-\alpha} && \text{for } |z| = R. \end{aligned} \tag{5.38}$$

Comparing the boundary values of $\tilde{h}$ with those of h_α on ∂D_R, we see that

$$\tilde{h}(z) - h_\alpha(z) \geqslant 0 \quad \text{for all } z \in \bar{D}_R. \tag{5.39}$$

If we choose $c = 1/\cos \pi\alpha$ we deduce from (5.37) and (5.39) that

$$h_\alpha(z) \leqslant \tilde{h}(z) \leqslant c|z|^{-\alpha} \quad \text{for all } z \in D_R, \tag{5.40}$$

which proves (5.27). □

LEMMA 10. *Set, as in Lemma 9,* $D_R := \mathbf{C} \setminus (\mathbf{R}_- \cup \{|z| \leqslant R\})$, $R > 0$.

(i) *Let the function h be harmonic in the domain D_R with boundary values $h(z) = 0$ for $z \in (-\infty, -R) \pm i0$, $h(z) = 1$ for $|z| = R$, $z \neq -R$, and let h be bounded in a neighborhood of infinity. Then for every $r > R$ there exists a constant $c = c_{R,r}$ such that*

$$0 \leqslant h(z) \leqslant c \operatorname{Re}(1/\sqrt{z}) \quad \textit{for all } z \in D_r. \tag{5.41}$$

(ii) *For $z_0 \in (R, \infty)$ and $r > z_0$ there exist two constants $c_1 = c_{1,z_0,r} > 0$ and $c_2 = c_{2,z_0,r} < \infty$ such that*

$$c_1 \operatorname{Re}(1/\sqrt{v}) \leqslant g_{D_R}(z_0, v) \leqslant c_2 \operatorname{Re}(1/\sqrt{v}) \quad \textit{for all } v \in D_r. \tag{5.42}$$

(iii) *For $z_0 \in (R, \infty)$ and $r > R$ there exists a constant $c = c_{z_0,R,r}$ such that*

$$g_{\mathbf{C}\setminus\mathbf{R}_-}(z_0, x) \leqslant c\, g_{D_R}(z_0, x) \quad \textit{for all } x \in [r, \infty). \tag{5.43}$$

Proof. (i) We use the function H introduced in (4.65), and define the function h by

$$h(z) := \frac{2}{\pi} \log \frac{1}{|H(R/z)|}. \tag{5.44}$$

Then it follows from (4.66) and (4.67) that h possesses the required boundary values, and (5.41) follows from a geometric consideration of the function h.

(ii) Let $\varphi: D_R \to D_0 = \{\mathrm{Re}(w) > 0\}$ be the Riemann mapping function with $\varphi(R+1) = 1$ and $\varphi(R) = \infty$. Near infinity we then have the development

$$\varphi(z) = \frac{c_0}{\sqrt{z}} + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \to \infty,\ z \in D_R, \tag{5.45}$$

with $c_0 > 0$. From (5.45) and the concrete form (5.3) of $g_{D_0}(z, v)$ the inequalities (5.42) follow.

(iii) For $x \in [r_0, \infty)$, $r_0 > z_0$, the estimate (5.43) is an immediate consequence of the lower estimate in (5.42) and the concrete definition of $g_{\mathbf{C}\setminus\mathbf{R}_-}(z_0, v)$ in (5.2), from which we see that there exists a constant $c_3 < \infty$ such that $g_{\mathbf{C}\setminus\mathbf{R}_-}(z_0, v) \leqslant c_3 \mathrm{Re}(1/\sqrt{|v|})$ for $v \in D_{r_0}$. For the interval $[r, r_0]$ the estimate (5.43) is rather immediate. □

6. Auxiliary functions II

In the present section we introduce and study a quadratic transformation of the function r_n. The function r_n is a rational transform of the error function e_n, and it has been investigated in detail in §4. The final form of the quadratic transformation is the function Ψ_n, which will be defined via two intermediate functions R_n and Φ_n. We define R_n as

$$R_n(w) := \frac{4w^{2\alpha} - 1}{w^\alpha} r_n(\varepsilon_n^{1/\alpha} w) - \frac{1}{w^\alpha}, \quad n \in \mathbf{N}, \tag{6.1}$$

with r_n defined in (4.1). Comparing (4.1) with (6.1) shows that implicitly in (6.1) the independent variable w of transformation (4.49) has been used. Based on R_n we define

$$\Phi_n(w) := \frac{1}{8w^\alpha}\left(R_n(w) + \sqrt{R_n(w)^2 - 4}\right), \quad n \in \mathbf{N}, \tag{6.2}$$

where the sign of the root is chosen so that $R_n(w)$ and the square root $\sqrt{R_n(w)^2 - 4}$ have the same sign for $w \in \mathbf{R}_+$ near infinity. A Möbius transform ψ is defined by

$$\psi(z) := \frac{z}{\sin \pi\alpha + i(\cos \pi\alpha) z}. \tag{6.3}$$

It is immediate that $\psi: K_\alpha \to \mathbf{D} = \{|w|<1\}$ is a bijective map of the disc K_α, introduced in (4.8) in Lemma 5, onto the unit disc $\mathbf{D}$. Finally, the function Ψ_n is defined as

$$\Psi_n(w) := \begin{cases} \psi\circ\Phi_n(w) & \text{for } w\in \overline{H}_+ := \{\operatorname{Im}(w)\geqslant 0\}, \\ \overline{\psi\circ\Phi_n(w)} & \text{for } w\in H_- := \{\operatorname{Im}(w)<0\}. \end{cases} \tag{6.4}$$

In the next three lemmas relevant properties of the functions R_n, Φ_n, Ψ_n will be proved. Each of these lemmas deals with one of the three functions. The last lemma (Lemma 13) deals with Ψ_n, and it contains all information that is relevant for the proof of Theorem 1. The two earlier lemmas are only of intermediate interest, like the functions R_n and Φ_n themselves.

LEMMA 11. *As in* (2.10) *and* (2.8) *of Lemma 2, we denote by* ε_n *the minimal error* $E_{n+1+[\alpha],n}(f_\alpha,[0,1])$, *by* η_{nj}, $j=0,\ldots,2n+2+[\alpha]$, *the extreme points of the error function* e_n, *and by* $\tilde\eta_{nj} := \eta_{nj}\varepsilon_n^{-1/\alpha}$ *the transformed extreme points. For the function* R_n *defined in* (6.1), *we have*

$$-2\leqslant R_n(w)\leqslant 2 \quad \textit{for } w\in[2^{-1/\alpha}, \varepsilon_n^{-1/\alpha}], \tag{6.5}$$

$$R_n(\tilde\eta_{nj}) = 2(-1)^{j+[\alpha]+1} \quad \textit{for } j=1,\ldots,2n+2+[\alpha]. \tag{6.6}$$

For $R\geqslant 1$ *sufficiently large, we further have*

$$|R_n(w)|>2 \quad \textit{for all } |w| = \tilde\eta_{n,2k-1+[\alpha]},\ w\notin \mathbf{R}_+, \tag{6.7}$$

$k\in\{1,\ldots,n+1\}$, *and* $\tilde\eta_{n,2k-1+[\alpha]}\geqslant R$,

$$R_n(w)\notin[-2,2] \quad \textit{for all } w\in \mathbf{C}\setminus(\mathbf{R}_-\cup\{|w|\leqslant R\}\cup[0,\varepsilon_n^{-1/\alpha}]), \tag{6.8}$$

$$R_n(w) < -2 \quad \textit{for } w\in(\varepsilon_n^{-1/\alpha},\infty), \tag{6.9}$$

the function $w^{-\alpha}R_n(w)$ *is strictly monotonically decreasing for* $w\in(\varepsilon_n^{-1/\alpha},\infty)$, *the function* R_n *is analytic in* $D_R := \mathbf{C}\setminus(\mathbf{R}_-\cup\{|w|\leqslant R\})$, *and it has analytic continuations across the interval* $(-\infty,-R)$ *from both sides.*

Proof. We use the same notation as used in the proofs of Lemmas 7 and 8. The independent variable w of R_n is connected with the original variable z via transformation (4.49). It follows from (4.2) and (4.3) that

$$f_1(w) := \frac{-1}{2w^\alpha+1} \leqslant \tilde r_n(w) \leqslant \frac{-1}{2w^\alpha-1} =: f_2(w) \tag{6.10}$$

for $w\in[2^{-1/\alpha},\varepsilon_n^{-1/\alpha}]$. If we substitute the functions f_k, $k=1,2$, instead of $\tilde r_n(w)=r_n(\varepsilon_n^{1/\alpha}w)$ into definition (6.1) of R_n, then this yields the upper and the lower bounds

$$\begin{aligned}F_k(w):=\frac{4w^{2\alpha}-1}{w^\alpha}f_k(w)-\frac{1}{w^\alpha}&=-\left(\frac{f_k(w)}{f_1(w)f_2(w)}+1\right)w^{-\alpha}\\&=-((-1)^{k+1}2w^\alpha-1+1)w^{-\alpha}=(-1)^k2,\quad k=1,2,\end{aligned}\tag{6.11}$$

for R_n and $w\in[2^{-1/\alpha},\varepsilon_n^{-1/\alpha}]$. From (6.10) and (6.11), we then deduce the bounds (6.5).

From (4.52) we know that at each transformed extreme point $\tilde\eta_{nj}$, $j=1,...,2n+2+[\alpha]$, equality holds in one of the two inequalities in (6.10), and therefore also in (6.5). From (4.52) together with transformation (4.49) and the inequalities in (6.10) and (6.5), we then deduce the identities (6.6).

From the monotonicity (4.137) proved in Lemma 8, we know that there exists $R\geqslant 0$ such that $|\tilde r_n|$ is monotonically increasing and decreasing on the two half-circles $\{re^{it}\mid t\in[0,\pi]\}$ and $\{re^{it}\mid t\in[-\pi,0]\}$, respectively, for any $r\geqslant R$. For the factor in front of $\tilde r_n$ in definition (6.1) of R_n, we have the lower estimate

$$\left|\frac{4w^{2\alpha}-1}{w^\alpha}\right|\geqslant\frac{4|w|^{2\alpha}-1}{|w|^\alpha}\quad\text{for } w\in D_{2^{-1/\alpha}}\supset D_1.\tag{6.12}$$

Hence, it follows from (6.1) and (4.137) in Lemma 8 that the lower estimate (6.7) holds on circles passing through transformed extreme points $\tilde\eta_{nj}$ with $R_n(\tilde\eta_{nj})=2$. Indeed, it follows from (6.12), (6.1) and the monotonicity proved in (4.137) that for $|w|=\tilde\eta_{nj}$ we have

$$|R_n(w)|\geqslant\frac{4|w|^{2\alpha}-1}{|w|^\alpha}|r_n(\varepsilon_n^{1/\alpha}w)|-\frac{1}{|w|^\alpha}\geqslant\frac{4\tilde\eta_{nj}^{2\alpha}-1}{\tilde\eta_{nj}^\alpha}r_n(\eta_{nj})-\frac{1}{\tilde\eta_{nj}^\alpha}=R_n(\tilde\eta_{nj})=2.\tag{6.13}$$

From (6.6) we know that $R_n(\tilde\eta_{nj})=2$ holds for $j=2k-1+[\alpha]$, $k\in\{1,...,n+1\}$. In (6.7), the index $k\in\{1,...,n+1\}$ has to be chosen so large that $R=\tilde\eta_{n,2k-1+[\alpha]}$ is as large as required in Lemma 8. It follows from (4.38) in Lemma 7, part (vi) of Lemma 7, and the interlacing (2.16), that such a choice of $k\in\mathbf{N}$ is always possible if $n\in\mathbf{N}$ is sufficiently large.

From the limits (4.38) and the limit (4.46) in Lemma 7, it follows that for a given $R\geqslant 1$ only a finite number of transformed extreme points $\tilde\eta_{nj}$ can be contained in the interval $[0,R]$, and we can choose $j_n\in\mathbf{N}$ so that $\tilde\eta_{n,j_n}\leqslant R<\tilde\eta_{n,j_n+1}$. In the sequel we shall exclude the j_n+1 first extreme points $\tilde\eta_{nj}$, $j=0,...,j_n$, from our considerations. Note that the number j_n depends on $R\geqslant 1$, but it follows from (4.46) in Lemma 7 that there exists $j_0\in\mathbf{N}$ such that $j_n\leqslant j_0$ for all $n\in\mathbf{N}$. From the alternation property (6.6), we conclude that the function R_n has a zero $\tilde{x}_{nj}$ between two adjacent transformed extreme points $\tilde{\eta}_{n,j-1}$ and $\tilde{\eta}_{nj}$, i.e., we have

$$\tilde{\eta}_{n,j-1} < \tilde{x}_{nj} < \tilde{\eta}_{nj} \quad \text{for } j = j_n+1, \ldots, 2n+2+[\alpha]. \tag{6.14}$$

The zeros $\tilde{x}_{nj}$ are in general different from the zeros $\tilde{z}_{nj}$, which have been studied in Lemmas 5 and 7, but each pair $\{\tilde{x}_{nj}, \tilde{z}_{nj}\}$ always lies in the open interval $(\tilde{\eta}_{n,j-1}, \tilde{\eta}_{nj})$ for $j = j_n+1, \ldots, 2n+2+[\alpha]$.

Next, we show that relation (6.8) holds true for $R \geqslant 1$ sufficiently large. We choose $n \in \mathbf{N}$ fixed, and set

$$D := D_R = \mathbf{C} \backslash (\mathbf{R}_- \cup \{|w| \leqslant R\}) \quad \text{with } R := \tilde{\eta}_{n,2k-1+[\alpha]} \tag{6.15}$$

and $2k-1+[\alpha] = j_n$. We have already earlier mentioned that it follows from (4.46) in Lemma 7 that $R \geqslant 1$ can be made arbitrarily large if $k \in \mathbf{N}$ is chosen sufficiently large.

In a first step we show that the function R_n has in D exactly $2(n+1-k)+1$ zeros, which all lie in the open interval $(R, \varepsilon_n^{-1/\alpha})$. Indeed, from (6.6) and (6.15) we know that $R_n(R) = 2$. From Lemma 5 together with (6.15), we further know that $\tilde{r}_n$ has no other zeros in D than the $2(n+1-k)+1$ zeros $\tilde{z}_{nj}$, $j = 2k+[\alpha], \ldots, 2n+2+[\alpha]$. It is immediate that the function $(4w^{2\alpha}-1)w^{-\alpha}\tilde{r}_n(w)$ has exactly the same zeros in D. From (4.4) in Lemma 5 we conclude that the zeros $\tilde{z}_{nj}$ interlace with the $2(n+2-k)$ transformed extreme points $\tilde{\eta}_{nj}$, $j = 2k-1+[\alpha], \ldots, 2n+2+[\alpha]$, i.e., each zero $\tilde{z}_{nj}$ is lying in the open interval $(\tilde{\eta}_{n,j+1}, \tilde{\eta}_{nj})$, $j = 2k+[\alpha], \ldots, 2n+2+[\alpha]$.

From (6.7) together with (4.9) in Lemma 5 in combination with (6.12), we deduce that

$$|R_n(w)| > 2 \quad \text{for all } w \in \partial D \backslash \{R\} \tag{6.16}$$

and $R \geqslant 1$ sufficiently large. Hence, we have

$$\begin{aligned} \frac{1}{|w|^\alpha} < 1 < \left|2 - \frac{1}{|w|^\alpha}\right| \leqslant \left| |R_n(w)| - \frac{1}{|w|^\alpha} \right| \leqslant \left| R_n(w) + \frac{1}{w^\alpha} \right| \\ = \left| \frac{4w^{2\alpha}-1}{w^\alpha} \right| |\tilde{r}_n(w)| \quad \text{for } w \in \partial D \end{aligned} \tag{6.17}$$

and $R \geqslant 1$ sufficiently large. By Rouché's theorem we therefore deduce from (6.16) and (6.17) that R_n has only the $2(n+1-k)+1$ zeros $\tilde{x}_{nj}$, $j = 2k+[\alpha], \ldots, 2n+2+[\alpha]$, in D which are the ones that have been listed in (6.14) with k and j_n chosen so that $2k-1+[\alpha] = j_n$.

Let us now assume that $a \in (-2, 2)$ is arbitrary. From (6.16) we deduce

$$|a| < |R_n(w)| \quad \text{for all } w \in \partial D. \tag{6.18}$$

Hence, it follows again from Rouché's theorem that the two functions R_n and R_n-a have the same number of $2(n+1-k)+1$ zeros in D. From (6.6) we deduce that R_n-a has these $2(n+1-k)+1$ zeros in the interval $(R, \varepsilon_n^{-1/\alpha}]$. Hence, it follows that $R_n(w)\neq a$ for all $w\in D\setminus[0, \varepsilon_n^{-1/\alpha}]$, which proves (6.8) for the open interval $(-2, 2)$.

That the conclusion holds also true for the two limiting cases $a=2$ and $a=-2$ follows from the detailed investigation of the zeros of the function R_n-a in the domain D that just has been done for $a\in(-2, 2)$. Indeed, the zeros of the functions R_n-a depend continuously on a. If R_n-2 or R_n+2 would have a zero at a point $w_0\in D\setminus[0, \varepsilon_n^{-1/\alpha}]$, then in every neighborhood of w_0 there should be a zero of R_n-a with $a\in(-2, 2)$. However, this possibility has already been excluded. Hence, (6.8) is completely proved.

It follows from (6.6) that the largest transformed extreme point $\tilde{\eta}_{n,2n+2+[\alpha]}=\varepsilon_n^{-1/\alpha}$ is a zero of the function R_n+2, and a zero counting shows that this zero has to be simple. The function R_n-a has exactly one zero in the open interval $(\tilde{\eta}_{n,2n+1+[\alpha]}, \varepsilon_n^{-1/\alpha})$ for $a\in(-2, 2)$, and this zero converges to $\tilde{\eta}_{n,2n+2+[\alpha]}=\varepsilon_n^{-1/\alpha}$ as $a\to-2+0$. From this it follows that (6.9) has to hold true for all $w\in(\varepsilon_n^{-1/\alpha}, \infty)$ since R_n-a has no zero in the interval $(\varepsilon_n^{-1/\alpha}, \infty)$.

It remains to prove that the function $w^{-\alpha}R_n(w)$ is strictly decreasing for $w\in(\varepsilon_n^{-1/\alpha}, \infty)$. Inserting the identity $\tilde{r}_n(w)=\tilde{e}_n(w)/(2w^{\alpha}-\tilde{e}_n(w))$ in (6.1) yields after some simplifications that

$$\begin{aligned} R_n(w) &= 4w^{\alpha}\tilde{r}_n(w)-(1+\tilde{r}_n(w))w^{-\alpha} \\ &= -4w^{\alpha}\frac{\tilde{e}_n(w)-\frac{1}{2}w^{-\alpha}}{\tilde{e}_n(w)-2w^{\alpha}} = -4w^{\alpha}\frac{w^{-\alpha}\tilde{e}_n(w)-\frac{1}{2}w^{-2\alpha}}{w^{-\alpha}\tilde{e}_n(w)-2}. \end{aligned} \tag{6.19}$$

Since $\tilde{e}_n(w)=\varepsilon_n^{-1}e_n(\varepsilon_n^{1/\alpha}w)$, it follows from (4.137) in Lemma 8 that $0>w^{-\alpha}\tilde{e}_n(w)>(w')^{-\alpha}\tilde{e}_n(w')$ for $\varepsilon_n^{-1/\alpha}<w<w'$. The monotonicity of $w^{-\alpha}R_n(w)$ then follows from (6.19). □

LEMMA 12. *Let* $\mathbf{D}(R)$ *denote the disc* $\{|w|\leqslant R\}$, *and let further* D_R *and* $D_{R,n}$ *be the domains* $D_R:=\mathbf{C}\setminus(\mathbf{R}_-\cup\overline{\mathbf{D}(R)})$ *and* $D_{R,n}:=\mathbf{C}\setminus(\mathbf{R}_-\cup\overline{\mathbf{D}(R)}\cup[0, \varepsilon_n^{-1/\alpha}])$ *with* $R\geqslant1$ *and* $n\in\mathbf{N}$. *As in* (2.10) *and* (2.8) *of Lemma* 2 (*and also in Lemma* 11), *we denote the minimal error* $E_{n+1+[\alpha],n}(f_\alpha, [0, 1])$ *by* ε_n, *the extreme points of the error function* e_n *by* η_{nj}, $j=0, ..., 2n+2+[\alpha]$, *and the transformed extreme points by* $\tilde{\eta}_{nj}=\eta_{nj}\varepsilon_n^{-1/\alpha}$. *For the function* Φ_n *defined in* (6.2), *for* $R\geqslant1$, *and for* $n_0\in\mathbf{N}$ *sufficiently large, the following assertions hold true*:

(i) *We have*

$$|\Phi_n(w)| = \tfrac{1}{4}|w|^{-\alpha} \quad \textit{for } w\in[2^{-1/\alpha}, \varepsilon_n^{-1/\alpha}]\pm i0,\ n\geqslant n_0, \tag{6.20}$$

and the function $|\Phi_n(w)|$ *is monotonically increasing in* $(\varepsilon_n^{-1/\alpha}, \infty)$.

(ii) *The function* Φ_n *is analytic and different from zero in* $D_{R,n}$. *It has analytic continuations across the intervals* $(2^{-1/\alpha}, \varepsilon_n^{-1/\alpha})$ *and* $(-\infty, 0)$ *from both sides, there exists a constant* $c<\infty$ *such that for* $n\geqslant n_0$ *we have*

$$\begin{aligned} \operatorname{dist}(\Phi_n(w), \operatorname{Cl}(K_\alpha)) &\leqslant c|w|^{-\alpha} \quad \textit{for } w\in \overline{H}_+\backslash \mathbf{D}(R), \\ \operatorname{dist}(\Phi_n(w), \operatorname{Cl}(\overline{K}_\alpha)) &\leqslant c|w|^{-\alpha} \quad \textit{for } w\in \overline{H}_-\backslash \mathbf{D}(R), \end{aligned} \tag{6.21}$$

with $\operatorname{Cl}(\cdot)$ *denoting the closure, and*

$$\begin{aligned} \operatorname{dist}(\Phi_n(w), \partial K_\alpha) &\leqslant c|w|^{-\alpha} \quad \textit{for } w\in(-\infty,-R)+i0, \\ \operatorname{dist}(\Phi_n(w), \partial \overline{K}_\alpha) &\leqslant c|w|^{-\alpha} \quad \textit{for } w\in(-\infty,-R)-i0. \end{aligned} \tag{6.22}$$

(iii) *The constant* $c<\infty$ *in* (ii) *can be chosen so that*

$$|\Phi_n(w)|\geqslant \frac{1}{c} \quad \textit{for } |w|=R,\ n\geqslant n_0. \tag{6.23}$$

(iv) *For* $\log|\Phi_n|$ *we have the representation*

$$\log|\Phi_n(w)| = \varphi_n(w) - \int g_D(z,x)\,d\tilde{\mu}_n(x) \quad \textit{for } w\in D_R \tag{6.24}$$

with a positive measure $\tilde{\mu}_n$ *on* $[R, \varepsilon_n^{-1/\alpha}]$ *that is defined by*

$$d\tilde{\mu}_n(x) = \frac{R_n'(x)\,dx}{\pi\sqrt{4-R_n^2(x)}} \quad \textit{for } x\in[R, \varepsilon_n^{-1/\alpha}], \tag{6.25}$$

and the function φ_n *in* (6.24) *is the solution of a Dirichlet problem, i.e., it is harmonic in the domain* D_R *and has boundary values*

$$\varphi_n(w) := \log|\Phi_n(w)| \quad \textit{for } w\in\partial D_R. \tag{6.26}$$

(v) *On the intervals* $[\tilde{\eta}_{nj}, \tilde{\eta}_{n,j+1}]$ *between consecutive extreme points* $\tilde{\eta}_{nj}$ *and* $\tilde{\eta}_{n,j+1}$, *we have*

$$\tilde{\mu}_n([\tilde{\eta}_{nj}, \tilde{\eta}_{n,j+1}]) = 1 \quad \textit{if } \tilde{\eta}_{nj}\geqslant R, \tag{6.27}$$

and consequently

$$\tilde{\mu}_n([R, \varepsilon_n^{-1/\alpha}]) = 2n + \mathcal{O}(1) \quad \textit{as } n\to\infty. \tag{6.28}$$

Proof. (i) Since we know from (6.8) in Lemma 11 that $R_n(w)^2-4\neq 0$ for all $w\in D_{R,n}$, it follows that Φ_n is analytic in $D_{R,n}$ for $R\geqslant 1$ and $n\in\mathbf{N}$ sufficiently large. Since from (6.5) and (6.6) we know that $R_n(w)^2-4$ has double zeros at $\tilde{\eta}_{nj}$, $j=1,\ldots,2n+1+[\alpha]$, the function Φ_n has analytic continuations across the interval $(2^{-1/\alpha}, \varepsilon_n^{-1/\alpha})$ from both sides.

Only at the last transformed extreme point $\tilde{\eta}_{n,2n+2+[\alpha]}=\varepsilon_n^{-1/\alpha}$, the function $R_n(w)^2-4$ has a simple zero, and consequently the function Φ_n has an algebraic singularity there.

From (6.5) we deduce that

$$\left|R_n(w)+\sqrt{R_n(w)^2-4}\,\right|^2=R_n(w)^2+(4-R_n(w)^2)=4 \tag{6.29}$$

for $w\in[2^{-1/\alpha},\varepsilon_n^{-1/\alpha}]$, which proves (6.20). From (6.9) and the assumption made after (6.2) with respect to the sign of the square root in the definition of Φ_n, we conclude that both functions R_n and Φ_n are negative on the interval $(\varepsilon_n^{-1/\alpha},\infty)$. From (6.2) we therefore get

$$|\Phi_n(w)|=\tfrac{1}{8}\left(|w^{-\alpha}R_n(w)|+\sqrt{w^{-2\alpha}R_n(w)^2-4w^{-2\alpha}}\,\right) \tag{6.30}$$

for $w\in(\varepsilon_n^{-1/\alpha},\infty)$. Since it has been proved in Lemma 11 that $|w^{-\alpha}R_n(w)|$ is strictly monotonically increasing for $w\in(\varepsilon_n^{-1/\alpha},\infty)$, the monotonicity of $|\Phi_n(w)|$ on $(\varepsilon_n^{-1/\alpha},\infty)$ follows from (6.30).

(ii) First, we derive estimates for $|\Phi_n|$ that hold throughout D_R for $R\geqslant 1$ and $n\in\mathbf{N}$ sufficiently large. These estimates will be derived in the upper half-plane H_+; the corresponding results in the lower half-plane H_- then are a consequence of the symmetry of the function $|\Phi_n|$ with respect to $\mathbf{R}$.

Let the two sets D_1 and D_2 be defined as $D_1:=\{w\in H_+\mid |R_n(w)|>2\}$ and $D_2:=H_+\setminus D_1$. It follows immediately from (6.2) that on the set D_2 we have

$$|\Phi_n(w)|\leqslant\frac{1}{8|w|^\alpha}\left(2+\sqrt{8}\,\right)\leqslant\frac{5}{8|w|^\alpha}\quad\text{for } w\in D_2. \tag{6.31}$$

To prove a corresponding estimate on D_1 turns out to be more involved. Because of (4.13) in Lemma 6 the function R_n is analytic in D_R for $R\geqslant 1$ sufficiently large. The set D_1 is open in H_+. From (4.9) in Lemma 5 together with (6.1) it follows that $|R_n(x\pm i0)|>2$ for $x\in(-\infty,-R]$ if $R\geqslant 1$ is sufficiently large. Taking into consideration (6.7) in Lemma 11, it follows that the domain $D_1\setminus\overline{\mathbf{D}(R)}$ is contained in a single component of D_1 if $R\geqslant 1$ is chosen as in (6.7) and sufficiently large. From the assumption made after (6.2) with respect to the sign of the square root in the definition of Φ_n, we then deduce that

$$\left|\arg\left[\frac{\sqrt{R_n(w)^2-4}}{R_n(w)}\right]\right|<\frac{\pi}{4}\quad\text{and}\quad\left|\frac{\sqrt{R_n(w)^2-4}}{R_n(w)}\right|\leqslant\sqrt{2} \tag{6.32}$$

for $w\in D_1\setminus\overline{\mathbf{D}(R)}$. It is immediate that for $z\in\mathbf{D}(1)$, we have $\left|{}_+\sqrt{1-z^2}-1\right|\leqslant|z|^2$. From (6.2), it then follows that

$$\begin{aligned}\left|\Phi_n(w)-\frac{R_n(w)}{4w^\alpha}\right|&=\frac{|R_n(w)|}{8|w|^\alpha}\left|\sqrt{1-4/R_n(w)^2}-1\right|\\&\leqslant\frac{4}{8|w|^\alpha|R_n(w)|}=\frac{1}{8|w|^\alpha}\quad\text{for } w\in D_1\setminus\overline{\mathbf{D}(R)}.\end{aligned} \tag{6.33}$$

From (4.138) in Lemma 8 we know that there exists $c_1<\infty$ such that $|\tilde{r}_n(w)|=|r_n(\varepsilon_n^{1/\alpha}w)|\leqslant c_1$ for $w\in\overline{D}_R$ and $R\geqslant 1$ sufficiently large. From (6.1) and (6.33) we therefore conclude that there exist $R\geqslant 1$ and $c_2<\infty$ such that

$$\begin{aligned}|\Phi_n(w)-\tilde{r}_n(w)| &\leqslant \left|\Phi_n(w)-\frac{R_n(w)}{4w^\alpha}\right|+\left|\frac{R_n(w)}{4w^\alpha}-\tilde{r}_n(w)\right| \\ &\leqslant \frac{1}{8|w|^\alpha}+\frac{|\tilde{r}_n(w)|+1}{4|w|^{2\alpha}}\leqslant \frac{c_2}{|w|^\alpha} \quad \text{for } w\in D_1\backslash\overline{\mathbf{D}(R)}.\end{aligned} \tag{6.34}$$

Since $0\in K_\alpha$, and since from (4.138) in Lemma 8 we know that $\tilde{r}_n(w)\in \mathrm{Cl}(K_\alpha)$ for $w\in D_R\cap H_+$ and $R\geqslant 1$ sufficiently large, it follows from (6.31) and (6.34) that there exists $c_3<\infty$ such that

$$\operatorname{dist}(\Phi_n(w),\mathrm{Cl}(K_\alpha))\leqslant \frac{c_3}{|w|^\alpha} \quad \text{for } w\in D_R\cap H_+, \tag{6.35}$$

where R is the same constant as that used in (6.34), and the constant c_3 is the maximum of $\frac{5}{8}$ and c_2. (The closure of the open disc K_α has been denoted by $\mathrm{Cl}(K_\alpha)$ since the notation $\overline{K}_\alpha$ has already been used to denote conjugation.) Since Φ_n is a function of real type, a conjugated result of (6.35) holds in H_-, which then proves (6.21).

Above, we have seen that $(-\infty,-R)\subset D_1$ for $R\geqslant 1$ sufficiently large. From (6.34) and relation (4.9) in Lemma 5, estimate (6.22) therefore follows. The constants c and R are the same as those in (6.34).

(iii) Choose $k_1,k_2\in\{1,\dots,n+1\}$ so that

$$R_{(1,n)}:=\tilde{\eta}_{n,2k_1-1+[\alpha]}\leqslant R<R_{(2,n)}:=\tilde{\eta}_{n,2k_2-1+[\alpha]}$$

for all $n\geqslant n_0$, $n\in N$. Because of the limits (4.77), the limits (4.38) in Lemma 7, and the interlacing (2.16), such a choice is always possible. Because of (6.7) in Lemma 11, the half-circles $\{w\in H_+ \mid |w|=R_{(j,n)}\}$, $j=1,2$, are contained in D_1. For $k_1,k_2\in\mathbf{N}$ and $n\in N$ sufficiently large, from (6.2) and (6.32) we deduce that

$$|\Phi_n(w)|=\frac{|R_n(w)|}{8|w|^\alpha}\left|1+\sqrt{1-4/R_n(w)^2}\,\right|\geqslant \frac{|R_n(w)|}{8|w|^\alpha}\geqslant\frac{1}{4|w|^\alpha} \tag{6.36}$$

for $|w|=R_{(1,n)}$, and also for $|w|=R_{(2,n)}$. Since Φ_n is analytic and different from zero in the half-annulus $\{w\in H_+ \mid R_{(1,n)}<|w|<R_{(2,n)}\}$, it follows from (6.36), (6.20), (6.22) and $R_{(1,n)}$ sufficiently large that $|\Phi_n(w)|\geqslant 1/4R_{(2,n)}$ for $w\in H_+$, $R_{(1,n)}<|w|<R_{(2,n)}$. This conclusion proves (6.23).

(iv) Next, we prove (6.24). From definition (6.2) and identity (6.20) we deduce that

$$\begin{aligned}\frac{\partial}{\partial y}\log|\Phi_n(w)| &= \frac{\partial}{\partial y}\log|\Phi_n(w)8w^\alpha| = i\frac{\partial}{\partial w}\log(\Phi_n(w)8w^\alpha)\\ &= i\frac{\left(R_n+\sqrt{R_n^2-4}\right)'}{R_n+\sqrt{R_n^2-4}}(w) = \frac{i}{R_n+\sqrt{R_n^2-4}}\left(R_n'+\frac{R_nR_n'}{\sqrt{R_n^2-4}}\right)(w)\\ &= \frac{iR_n'(w)}{\sqrt{R_n(w)^2-4}} = \frac{R_n'(w)}{\sqrt{4-R_n(w)^2}}\end{aligned} \tag{6.37}$$

for $w=x+iy\in[2^{-1/\alpha},\varepsilon_n^{-1/\alpha}]+i0$. From (6.5) and (6.6) in Lemma 11 we deduce that at each transformed extreme point $\tilde{\eta}_{nj}$ in the open interval $(2^{-1/\alpha},\varepsilon_n^{-1/\alpha})$ both functions R_n' and $\sqrt{4-R_n^2}$ have a simple zero. These two zeros cancel out in the quotient in the last term of (6.37). Consequently, this quotient has no sign changes in the interval $(2^{-1/\alpha},\varepsilon_n^{-1/\alpha})$. From (6.9) in Lemma 11 and the assumptions made with respect to the sign of the square root in (6.2), it follows that the last term of (6.37) is positive imaginary for $w\in(\varepsilon_n^{-1/\alpha},\varepsilon_n^{-1/\alpha}+\delta)$, $\delta>0$ small. Since $4-R_n^2$ has a simple zero at $\varepsilon_n^{-1/\alpha}=\tilde{\eta}_{n,2n+2+[\alpha]}$, it follows that the last term of (6.37) is positive on $(2^{-1/\alpha},\varepsilon_n^{-1/\alpha})+i0$. Hence, the measure $\tilde{\mu}_n$ defined by

$$d\tilde{\mu}_n(x) := \frac{R_n'(x)\,dx}{\pi\sqrt{4-R_n(x)^2}},\quad x\in[R,\varepsilon_n^{-1/\alpha}], \tag{6.38}$$

is positive. The constant $R\geqslant 1$ in (6.37) has to be chosen large enough so that Φ_n is analytic and different from zero in D_R, and $n\in\mathbf{N}$ has to be so large that $\varepsilon_n^{-1/\alpha}>R$.

Let the function φ_n in (6.24) be the solution of the Dirichlet problem in D_R with boundary values (6.26). The representation (6.24) follows from the Riesz representation theorem (cf. [18, Theorem II.3.1]) together with (6.37) and (6.38) in the same way, as this theorem has been applied for the proof of representation (5.34) in the proof of Lemma 9. The place of (5.29) is now taken by (6.37). Again, the Riesz decomposition theorem only ascertains that there exists a representation of the form (6.24); the more specific assertion that the measure $\tilde{\mu}_n$ in (6.24) is given by (6.38) can be shown with the help of the Green formula. Details of this method for recovering the defining measure from a potential has been proved in Theorem II.1.5 of [18] under conditions that are applicable in the present situation.

(v) For two adjacent transformed extreme points $\tilde{\eta}_{nj}$, $\tilde{\eta}_{n,j+1}\in[R,\varepsilon_n^{-1/\alpha}]$, we have

$$\tilde{\mu}_n([\tilde{\eta}_{nj},\tilde{\eta}_{n,j+1}]) = \frac{1}{\pi}\int_{\tilde{\eta}_{nj}}^{\tilde{\eta}_{n,j+1}}\frac{R_n'(x)\,dx}{\sqrt{4-R_n(x)^2}} = \frac{1}{\pi}\int_{-2}^{2}\frac{dt}{\sqrt{4-t^2}} = 1, \tag{6.39}$$

which proves (6.27). In the last equality in (6.39) we have applied the substitution $x\mapsto t:=R_n(x)$ and have used (6.6). Note that between two adjacent transformed extreme points the function R_n is monotonic. Since the interval $[2^{-1/\alpha}, \varepsilon_n^{-1/\alpha}]$ contains $2n+2+[\alpha]$ transformed extreme points $\tilde{\eta}_{nj}$, the estimate (6.28) follows from (4.46) in Lemma 7 for any $R \geqslant 1$. □

LEMMA 13. *Let the domain* $D_R := \mathbf{C} \backslash (\mathbf{R}_- \cup \overline{\mathbf{D}(R)})$ *and the minimal error* ε_n *be defined as before in Lemma* 12, *let further* $n_0 \in \mathbf{N}$ *be chosen so that* $\varepsilon_{n_0}^{-1/\alpha} > R$. *Then the following assertions hold true for the function* Ψ_n *defined in* (6.4):

(i) *For* $R \geqslant 1$ *sufficiently large, there exists a constant* $c < \infty$ *such that*

$$|\log |\Psi_n(w)|| \leqslant c|w|^{-\alpha} \quad \text{for } w \in [-\infty, -R] \pm i0, \tag{6.40}$$

$$|\log(|4w^\alpha \sin(\pi\alpha)\Psi_n(w)|)| \leqslant c|w|^{-\alpha} \quad \text{for } w \in [R, \varepsilon_n^{-1/\alpha}] \pm i0, \tag{6.41}$$

$$|\log |\Psi_n(w)|| \leqslant c \quad \text{for } |w| = R, \tag{6.42}$$

and $n \geqslant n_0$.

(ii) *For* $R \geqslant 1$ *sufficiently large, we have the representation*

$$\log |\Psi_n(w)| = \psi_n(w) - \int g_{D_R}(z, x)\, d\mu_n(x) \quad \text{for } w \in D_R \tag{6.43}$$

and μ_n *a measure of finite mass defined on* $[R, \infty]$. *On* $[R, \varepsilon_n^{-1/\alpha}]$ *the measure* μ_n *is very similar to the positive measure* $\tilde{\mu}_n$ *in representation* (6.24) *of Lemma* 12. *With the same constant* c *as used in* (6.40), (6.41) *and* (6.42), *we have*

$$|\mu_n([R, x]) - \tilde{\mu}_n([R, x])| \leqslant c x^{-\alpha} \quad \text{for all } x \in [R, \varepsilon_n^{-1/\alpha}], \tag{6.44}$$

and further we have

$$\mu_n([\tilde{x}_{n,j-1}, \tilde{x}_{nj}]) = \tilde{\mu}_n([\tilde{x}_{n,j-1}, \tilde{x}_{nj}]) = 1 \tag{6.45}$$

for $j = j_1, \ldots, 2n+2+[\alpha]$, *where* $\tilde{x}_{nj}$, $j = 1, \ldots, 2n+2+[\alpha]$, *are the zeros of the function* R_n, *which have been studied in Lemma* 11, *and the index* j_1 *is determined by the condition* $\tilde{x}_{n,j_1-1} \leqslant R < \tilde{x}_{nj_1}$. *On* $[\varepsilon_n^{-1/\alpha}, \infty]$ *we have the estimate*

$$\left\| \mu_n |_{[\varepsilon_n^{-1/\alpha}, \infty)} \right\| \leqslant \tfrac{1}{2} \quad \text{for } n \geqslant n_0(R). \tag{6.46}$$

The function ψ_n *in* (6.43) *is harmonic in the domain* D_R *and has boundary values*

$$\psi_n(w) = \log |\Psi_n(w)| \quad \text{for } w \in \partial D_R. \tag{6.47}$$

Proof. The proof of the lemma will be carried out in the upper half-plane $\overline{H}_+ = \{w \in \overline{\mathbf{C}} \mid \operatorname{Im}(w) \geqslant 0\}$. The transfer to the lower half-plane $\overline{H}_-$ is immediate. Since the Möbius transform ψ defined in (6.3) is analytic in a neighborhood of the closed disc $\mathrm{Cl}(K_\alpha)$, it follows from (6.21) in Lemma 12 that Ψ_n is analytic in $\overline{H}_+ \backslash (\overline{\mathbf{D}(R)} \cup \{\varepsilon_n^{-1/\alpha}\})$ for $n \in \mathbf{N}$ and $R \geqslant 1$ sufficiently large. Note that Φ_n and therefore also Ψ_n has analytic continuations across the three subintervals of $\mathbf{R} \backslash \{0, 2^{-1/\alpha}, \varepsilon_n^{-1/\alpha}\}$.

(i) The Möbius transform ψ is a bijective map of the circle ∂K_α onto the unit circle $\partial \mathbf{D}$. Hence, from estimate (6.22) in Lemma 12, it follows that there exists a constant $c > 0$ such that for $n \in \mathbf{N}$ and $R \geqslant 1$ sufficiently large, we have

$$||\Psi_n(w)| - 1| \leqslant c|w|^{-\alpha} \quad \text{for } w \in (-\infty, -R] + i0. \tag{6.48}$$

Since the same considerations can be repeated on $\overline{H}_-$, estimate (6.40) follows from (6.48).

At the origin the Möbius transform ψ has the development

$$\psi(w) = \frac{w}{\sin \pi\alpha + i(\cos \pi\alpha) w} = \frac{1}{\sin \pi\alpha} w - i \frac{\cos \pi\alpha}{\sin^2 \pi\alpha} w^2 - \frac{\cos^2 \pi\alpha}{\sin^3 \pi\alpha} w^3 + \dots . \tag{6.49}$$

From definition (6.4) it therefore follows that

$$\Psi_n(w) = \frac{1}{\sin \pi\alpha} \Phi_n(w) + \mathcal{O}(\Phi_n(w)^2) \quad \text{as } \Phi_n(w) \to 0. \tag{6.50}$$

Using identity (6.20) of Lemma 12 together with (6.49), we deduce that there exists a constant $c > 0$ such that for $n \in \mathbf{N}$ and $R \geqslant 1$ sufficiently large, we have

$$||\Psi_n(w) 4 w^\alpha \sin \pi\alpha| - 1| \leqslant c|w|^{-\alpha} \quad \text{for } w \in [R, \varepsilon_n^{-1/\alpha}] + i0. \tag{6.51}$$

The same inequality holds for $w \in [R, \varepsilon_n^{-1/\alpha}] - i0$ since $|\Psi_n|$ is symmetric with respect to $\mathbf{R}$. Hence, estimate (6.41) follows from (6.51).

From estimate (6.21) and (6.23) in Lemma 12, we deduce that there exists a constant $c_1 < \infty$ such that for $n \in \mathbf{N}$ and $R \geqslant 1$ sufficiently large we have

$$\frac{1}{c_1} \leqslant |\Psi_n(w)| \leqslant c_1 \quad \text{for } |w| = R. \tag{6.52}$$

The estimate (6.42) follows from (6.52). Note that in (6.42) the radius $R \geqslant 1$ is fixed.

(ii) Repeating the analysis that has been done for proving representation (6.24) in part (iv) of the proof of Lemma 12, or the derivation of representation (5.34) in the proof of Lemma 9, we again apply the Riesz representation theorem (cf. [18, Theorem II.3.1]) for a proof of representation (6.43). Using, as before, Theorem II.1.5 from [18], where a technique for recovering the defining measure from a potential has been described in detail, we see that the measure μ_n in representation (6.43) is given by

$$d\mu_n(x) = \frac{1}{\pi}\frac{\partial}{\partial y}\log|\Psi_n(x+iy)|\,dx \quad \text{for } x\in[R,\infty),\ y=+0. \tag{6.53}$$

The harmonic function ψ_n in (6.43) is the solution of the Dirichlet problem in D_R with boundary values given by (6.47).

From the Cauchy–Riemann differential equations we know that

$$\frac{\partial}{\partial y}\log|\Psi_n(x+iy)| = -\frac{\partial}{\partial x}\arg\Psi_n(x+iy), \quad w=x+iy\in D_R\cap H_+. \tag{6.54}$$

In the simply-connected domain $D_R\cap H_+$, the functions $\arg\Psi_n$ and $\arg\Phi_n$ are well defined if we fix their value at one point. From (6.54), (6.53), (6.38) and (6.37), we deduce that

$$\begin{aligned}\mu_n([R,x]) &= \arg\Psi_n(R+i0)-\arg\Psi_n(x+i0),\\ \tilde{\mu}_n([R,x]) &= \arg\Phi_n(R+i0)-\arg\Phi_n(x+i0) \quad \text{for } x\in[R,\varepsilon_n^{-1/\alpha}].\end{aligned} \tag{6.55}$$

From the definition of the Möbius transform ψ in (6.3), it follows that

$$\arg\Psi_n(w) = \arg\Phi_n(w)-\arg(\sin\pi\alpha+i(\cos\pi\alpha)\Phi_n(w)) \tag{6.56}$$

for $w\in D_R\cap H_+$. From (6.55), (6.56) and identity (6.20) in Lemma 12, it then further follows that

$$|\mu_n([R,x])-\tilde{\mu}_n([R,x])| \leqslant \sin^{-1}((\cot\pi\alpha)\tfrac{1}{4}x^{-\alpha}) \quad \text{for } x\in[R,\varepsilon_n^{-1/\alpha}], \tag{6.57}$$

which proves (6.44). From (6.2) and the definition of the points $\tilde{x}_{nj}$, $j=1,\ldots,2n+2+[\alpha]$, as the zeros of the function R_n, it follows that $\Phi_n(\tilde{x}_{nj})\in i\mathbf{R}$ for $j=1,\ldots,2n+2+[\alpha]$, and therefore the identities (6.45) are a consequence of (6.55) and (6.56).

The Möbius transform ψ maps the interval $[-1,1]\subseteq \mathrm{Cl}(K_\alpha)$ onto the semi-circle

$$C_\alpha:\ \psi(t) = \frac{t\sin(\pi\alpha)-it^2\cos(\pi\alpha)}{\sin^2(\pi\alpha)+t^2\cos^2(\pi\alpha)} \quad \text{with } -1\leqslant t\leqslant 1. \tag{6.58}$$

By $\{\alpha\}$ we denote $\mathrm{dist}(\alpha,\mathbf{N})$. If $\alpha\in\mathbf{N}+\frac{1}{2}$, then we have $\psi(w)=(-1)^{[\alpha]}w$.

From part (i) of Lemma 12 together with (6.9) and the definitions (6.1), (6.2) and (6.4), we know that $\Phi_n(x)$ is monotonically decreasing from $-\frac{1}{4}\varepsilon_n$ to -1 if x runs through the interval $[\varepsilon_n^{-1/\alpha},\infty)$ from $\varepsilon_n^{-1/\alpha}$ to $+\infty$. Consequently, $\arg\Psi_n(x)$ varies monotonically if x runs through the interval $[\varepsilon_n^{-1/\alpha},\infty)$. The maximal span of this variation is $\pi\{\alpha\}<\frac{1}{2}\pi$. Hence, estimate (6.46) follows from (6.53) together with (6.54). □

7. Proof of Theorem 1

The proof of Theorem 1 is based on a comparison of the function $\log|\Psi_n|$ studied in the last section with a special Green potential of the type introduced in Proposition 1 of §5. Representation (6.43) of $\log|\Psi_n|$ in Lemma 13 contains a Green potential with defining measure μ_n, its total mass $\|\mu_n\|$ being approximately $2n$. The comparison with the potential from Proposition 1 will allow us to derive an asymptotic estimate of ε_n that is precise enough to prove the limit (1.1) in Theorem 1.

In Lemma 1 it has been shown that there exists some freedom in choosing the numerator degree m_n of the approximant r_n^*. Instead of considering identical numerator and denominator degrees $m_n=n$, as has been done in Theorem 1, it turns out that the choice of numerator degrees $m_n=n+1+[\alpha]$ is more favorable, and this has indeed been the degree chosen in (2.4) and used throughout §§ 3, 4 and 6.

The two domains $\mathbf{C}\setminus\mathbf{R}_-$ and $D_R:=\mathbf{C}\setminus(\mathbf{R}_-\cup\overline{\mathbf{D}(R)})$ will frequently be used, where $R>0$ is a fixed number chosen large enough so that all the conclusions of Lemmas 13, 9 and 10 hold true. In Proposition 1 of §5 the existence of a Green potential $g(\nu,\mathbf{C}\setminus\mathbf{R}_-;\cdot)$ with special properties has been established. In in this proposition we choose for the constants c and x the values

$$c:=\log(4|\sin\pi\alpha|) \quad\text{and}\quad x_n:=\varepsilon_n^{-1/\alpha} \quad\text{for } n\in\mathbf{N}. \tag{7.1}$$

We assume that $R>4e^{-c/\alpha}$ and $n\in\mathbf{N}$ so large that $\varepsilon_n^{-1/\alpha}>R$. With the special choice (7.1) the defining measure of the Green potential in (5.7) of Proposition 1 is denoted by ν_n. It is a positive measure supported on the interval $[R,x_n]=[R,\varepsilon_n^{-1/\alpha}]$. Relation (5.7) in Proposition 1 then has the form

$$g(\nu_n,\mathbf{C}\setminus\mathbf{R}_-;w)=\int g_{\mathbf{C}\setminus\mathbf{R}_-}(w,u)\,d\nu_n(u)=\log(4|\sin\pi\alpha|)+\alpha\log w \tag{7.2}$$

for $w\in[R,\varepsilon_n^{-1/\alpha}]$, and from (5.8) in Proposition 1, we know that the total mass $\|\nu_n\|$ of the measure ν_n in (7.2) satisfies the relation

$$\lim_{n\to\infty}\left(\pi\sqrt{2\alpha\|\nu_n\|}+\log\varepsilon_n\right)=\log(4|\sin\pi\alpha|)+\alpha\log 4. \tag{7.3}$$

Note that because of $\varepsilon_n\to0$ as $n\to\infty$, we have $x_n\to\infty$ as $n\to\infty$.

We consider the sum

$$d_n(w):=\log|\Psi_n(w)|+g(\nu_n,\mathbf{C}\setminus\mathbf{R}_-;w) \quad\text{for } w\in\mathbf{C}\setminus\mathbf{R}_-,\ n\in\mathbf{N}, \tag{7.4}$$

and derive estimates for the functions d_n on ∂D_R. Since $g(\nu_n,\mathbf{C}\setminus\mathbf{R}_-;w)=0$ for all $w\in(-\infty,0)$, we deduce from (7.4) and (6.40) in Lemma 13 that there exists $c<\infty$ such that

$$|d_n(w)|\leqslant c|w|^{-\alpha} \quad\text{for } w\in(-\infty,-R). \tag{7.5}$$

From (7.4), (7.2) and (6.41) in Lemma 13, we deduce that

$$|d_n(w)| \leqslant c|w|^{-\alpha} \quad \text{for } w \in [R, \varepsilon_n^{-1/\alpha}]. \tag{7.6}$$

The Green function $g_{\mathbf{C}\setminus\mathbf{R}_-}(\cdot, u)$, $u \in \mathbf{R}_+$, is monotonic on circles around the origin, as has been discussed after (5.3) and follows directly from (5.2). Since ν_n is a positive measure, it follows from the monotonicity that the Green potential $g(\nu_n, \mathbf{C}\setminus\mathbf{R}_-; re^{it})$ is a monotonically decreasing function of $|t|$ for $|t| \in [0, \pi]$ and $r > 0$. We deduce from (6.42) in Lemma 13 and the monotonicity that the constant c in (7.5) and (7.6) can be chosen so that

$$|d_n(w)| \leqslant cR^{-\alpha} \quad \text{for } |w| = R. \tag{7.7}$$

Putting the estimates (7.5), (7.6) and (7.7) together, we see that $|d_n(w)| \leqslant c|w|^{-\alpha}$ for $w \in \partial D_R$.

The Green potential in representation (6.43) of Lemma 13 for $\log|\Psi_n|$ and the Green potential $g(\nu_n, \mathbf{C}\setminus\mathbf{R}_-; \cdot)$ in (7.2) are defined in two different domains D_R and $\mathbf{C}\setminus\mathbf{R}_-$. We will develop estimates for a comparison of both types of Green potentials. With representation (6.43) in Lemma 13, we rewrite (7.4) as

$$\begin{aligned} d_n(w) &= \psi_n(w) - \int g_{D_R}(w, u)\, d\mu_n(u) + \int g_{\mathbf{C}\setminus\mathbf{R}_-}(w, u)\, d\nu_n(u) \\ &= \psi_n(w) + \delta_n(w) + \int g_{\mathbf{C}\setminus\mathbf{R}_-}(w, u)\, d(\nu_n - \mu_n)(u) \end{aligned} \tag{7.8}$$

with

$$\delta_n(w) := \int [g_{\mathbf{C}\setminus\mathbf{R}_-}(w, u) - g_{D_R}(w, u)]\, d\mu_n(u). \tag{7.9}$$

LEMMA 14. *Both functions ψ_n and δ_n in (7.8) are harmonic in D_R, and there exists a constant $c < \infty$ such that for $R > 0$ sufficiently large we have*

$$|\psi_n(w)| \leqslant cw^{-\alpha_0} \qquad \textit{for } w \geqslant R \tag{7.10}$$

and

$$|\delta_n(w)| \leqslant c\sqrt{1/w} \quad \textit{for } w \geqslant R, \tag{7.11}$$

with $\alpha_0 := \min\left(\alpha, \tfrac{1}{2} - \varepsilon\right)$ for a given $\varepsilon > 0$.

Proof. From the definition of ψ_n in (6.47) of Lemma 13 and the estimates (6.40) and (6.42) in Lemma 13, we know that there exists a constant $c_1 < \infty$ such that

$$|\psi_n(w)| \leqslant c_1 |w|^{-\alpha} \quad \text{for } w \in \partial D_R \text{ and } n \in \mathbf{N}. \tag{7.12}$$

By choosing, if necessary, a new constant c_1, we can assume that (7.12) holds also true with the exponent $-\alpha_0$ instead of $-\alpha$. From part (ii) of Lemma 9 we know that there exists a function h_{α_0} which is harmonic and bounded in D_R, and from (7.12) and (5.27) in Lemma 9 we deduce that

$$|\psi_n(w)| \leqslant c_1 h_{\alpha_0}(w) \leqslant c_2 |w|^{-\alpha_0} \quad \text{for } w \in D_R, \tag{7.13}$$

where $c_2 < \infty$ is an appropriately chosen constant. Assertion (7.10) follows from (7.13).

Next, we come to the proof of (7.11). We shall show further down that for $R_3 > R$ there exists $c_3 < \infty$ such that

$$|g(\mu_n|_{(R,\infty)}, D_R; w)| \leqslant c_3 \quad \text{for } R \leqslant |w| \leqslant R_3 \text{ and all } n \in \mathbf{N}. \tag{7.14}$$

The difference $d_R(\cdot, u) := g_{\mathbf{C}\setminus\mathbf{R}_-}(\cdot, u) - g_{D_R}(\cdot, u)$ is harmonic in D_R and $d_R(\cdot, u) = g_{\mathbf{C}\setminus\mathbf{R}_-}(\cdot, u)$ on ∂D_R for all $u \in D_R$. Hence, it follows from (7.9) and (7.14) that

$$\begin{aligned} |\delta_n(w)| &\leqslant c_3 \quad \text{for } |w| = R, \\ |\delta_n(w)| &= 0 \quad\ \ \text{for } w \in [-\infty, -R], \end{aligned} \tag{7.15}$$

for all $n \in \mathbf{N}$. Since δ_n is harmonic in D_R, it follows from (7.15) and part (i) of Lemma 10 that for $R_4 > R$ there exists $c_4 < \infty$ such that

$$|\delta_n(w)| \leqslant c_4 \operatorname{Re}\big(\sqrt{1/w}\big) \quad \text{for } |w| \geqslant R_4,\ n \in \mathbf{N}. \tag{7.16}$$

Estimate (7.11) then follows from (7.14) and (7.16).

It remains to verify (7.14). For this purpose we investigate the behavior of the measure μ_n as $n \to \infty$. From (4.43) in Lemma 7 together with part (iv) of the same lemma, we know that the sequence of functions $\tilde{r}_n = r_n(\varepsilon_n^{1/\alpha}\cdot)$, $n \in \mathbf{N}$, converges locally uniformly in D_R to a function $\tilde{r}$ that is analytic in D_R if $R > 0$ is chosen so large that $\tilde{b}_1, \dots, \tilde{b}_{[\alpha]} \in \mathbf{D}(R)$. With definition (6.1) it then further follows that also the sequence R_n, $n \in \mathbf{N}$, converges to the function $\widetilde{R}(w) := w^{-\alpha}(4w^{2\alpha} - 1)\tilde{r}(w) - w^{-\alpha}$ locally uniformly for $w \in D_R$. From the definitions (6.2) and (6.4) it follows that the two sequences Φ_n and Ψ_n, $n \in \mathbf{N}$, converge to functions Φ and Ψ, respectively, uniformly on a neighborhood of every compact subset of $[R, \infty) \pm i0$ as $n \to \infty$. The limit functions Φ and Ψ have analytic continuations across $[R, \infty)$ from both sides, but these continuations define different branches. All functions involved are analytic and different from zero in a neighborhood of compact subsets of $[R, \infty)$.

From representation (6.53) of the measure μ_n and the convergence of the functions Ψ_n, $n \in \mathbf{N}$, we deduce that the density functions

$$\frac{d\mu_n(x)}{dx} = \frac{1}{\pi} \frac{\partial}{\partial y} \log |\Psi_n(x + i0)|, \quad x \in [R, \infty), \tag{7.17}$$

of the measures μ_n converge uniformly on a neighborhood of every compact subset of $[R,\infty)$ to the density function

$$\frac{d\mu(x)}{dx}=\frac{1}{\pi}\frac{\partial}{\partial y}\log|\Psi(x+i0)|, \quad x\in[R,\infty), \tag{7.18}$$

of the limit measure μ of the sequence of measures $\{\mu_n\}$. As a uniform limit, the function Ψ and its derivatives are bounded on compact subsets of $[R,\infty)$. From (7.18) we therefore know that the limit measure μ has a bounded density function on $[R,\infty)$. Consequently, for any $R_5>R$ there exists $c_5<\infty$ such that

$$\begin{aligned}0&\leqslant|g(\mu_n|_{(R,R_5)},\mathbf{C}\backslash\mathbf{R}_-;w)|\leqslant g(|\mu_n||_{(R,R_5)},\mathbf{C}\backslash\mathbf{R}_-;w)\\&=\int_R^{R_5}g_{\mathbf{C}\backslash\mathbf{R}_-}(w,u)\,d|\mu_n|(u)\leqslant c_5 \quad \text{for } w\in\mathbf{C} \text{ and } n\in\mathbf{N}.\end{aligned} \tag{7.19}$$

In the proof of Lemma 7, and there especially in the proof of (4.81), it has been shown that the sequence of products $\prod_{j=1}^{2n+2+[\alpha]}B(\cdot,\tilde{z}_{nj})$, $n\in\mathbf{N}$, converges to the infinite product (4.79) locally uniformly in $\mathbf{C}$ as $n\to\infty$, and the infinite product (4.79) is not identically zero. With the same arguments as applied after (4.79), we conclude that there exist $c_6<\infty$ and $n_6\in\mathbf{N}$ such that

$$\sum_{j=2+[\alpha]}^{2n+2+[\alpha]}\frac{1}{\sqrt{\tilde{z}_{nj}}}\leqslant c_6 \quad \text{for all } n\geqslant n_6, \tag{7.20}$$

where the $\tilde{z}_{nj}$, $j=1,\ldots,2n+2+[\alpha]$, are the zeros that the function $\tilde{r}_n$ has in $\mathbf{C}\backslash\mathbf{R}_-$.

From (7.20) and the estimates (6.44), the equalities (6.45), and estimate (6.46), it follows that for any $R_7>R$ there exist constants $c_7,c_8<\infty$ and $n_7\in\mathbf{N}$ such that the estimate

$$\begin{aligned}|g(\mu_n|_{(R_7,\infty)},\mathbf{C}\backslash\mathbf{R}_-;w)|&\leqslant\int_{R_7}^{\infty}g_{\mathbf{C}\backslash\mathbf{R}_-}(w,u)\,d|\mu_n|(u)\\&\leqslant\sum_{j=j_7}^{2n+1+[\alpha]}(1+c\tilde{x}_{nj}^{-\alpha})g_{\mathbf{C}\backslash\mathbf{R}_-}(w,\tilde{x}_{nj})\\&\quad+(1.5+c\tilde{x}_{n,2n+2+[\alpha]}^{-\alpha})g_{\mathbf{C}\backslash\mathbf{R}_-}(w,\tilde{x}_{n,2n+2+[\alpha]}^{-\alpha})\\&\leqslant c_7\sum_{j=j_7-1}^{2n+1+[\alpha]}g_{\mathbf{C}\backslash\mathbf{R}_-}(w,\tilde{z}_{nj})\leqslant c_8\end{aligned} \tag{7.21}$$

holds true for $|w|=R$ and $n\geqslant n_7$, where the $\tilde{x}_{nj}$, $j=j_7,\ldots,2n+2+[\alpha]$, are the zeros of the function R_n in D_{R_7}, the index j_7 is determined by the condition $\tilde{x}_{n,j_7-1}\leqslant R_7<\tilde{x}_{nj_7}$, and the $\tilde{z}_{nj}$, $j=j_7-1,\ldots,2n+1+[\alpha]$, are zeros of the function $\tilde{r}_n$.

Indeed, the Green function $g_{\mathbf{C}\setminus\mathbf{R}_-}(w,u)$ is monotonically decreasing for $u\in[R,\infty)$ and $|w|=R$ fixed, which implies the second inequality in (7.21). Note that it follows from (6.44) and (6.45) in Lemma 13 that $|\mu_n|([\tilde{x}_{n,j-1},\tilde{x}_{nj}])\leqslant 1+c\tilde{x}_{n,j-1}^{-\alpha}$ for $j=j_7,...,2n+1+[\alpha]$, and from (6.44), (6.45) and (6.46) that $|\mu_n|([\tilde{x}_{n,2n+2+[\alpha]},\infty))\leqslant 1.5+c\tilde{x}_{n,2n+2+[\alpha]}^{-\alpha}$. The third inequality in (7.21) is a consequence of the inequalities $\tilde{z}_{n,j-1}<\tilde{x}_{nj}$, $j=2,...,2n+2+[\alpha]$, which follow from (6.14) and the discussion after (6.14). From the explicit definition of $g_{\mathbf{C}\setminus\mathbf{R}_-}(w,u)$ in (5.2), it is immediate that there exists $c_9<\infty$ such that

$$0\leqslant g_{\mathbf{C}\setminus\mathbf{R}_-}(w,u)\leqslant c_9\frac{1}{\sqrt{u}}\quad\text{for } |w|=R \text{ and } u\in(R,\infty). \tag{7.22}$$

The last inequality in (7.21) follows from (7.22) and (7.20). From (7.19) and (7.21), then inequality (7.14) follows. □

After the completion of the proof of Lemma 14, we come back to the main stream of the proof of Theorem 1. From (7.8), (7.10), (7.11) and (7.6) we deduce that there exists a constant $c<\infty$ such that

$$\left|\int g_{\mathbf{C}\setminus\mathbf{R}_-}(w,u)\,d(\nu_n-\mu_n)(u)\right|\leqslant c|w|^{-\alpha_0}\quad\text{for } w\in[R,\varepsilon_n^{-1/\alpha}] \tag{7.23}$$

with $\alpha_0:=\min(\alpha,\frac{1}{2}-\varepsilon)$, $\varepsilon>0$. From the estimate (7.23) we shall deduce a relation between the two masses $\|\mu_n\|$ and $\|\nu_n\|$ of the measures μ_n and ν_n, respectively. Let $\hat{\mu}_n$ denote the measure resulting from balayage of the measure μ_n out of the domain $\mathbf{C}\setminus(\mathbf{R}_-\cup[R,\varepsilon_n^{-1/\alpha}])$. From inequality (6.46) in Lemma 13 together with (5.5) and (5.6), we deduce that

$$|\,\|\mu_n\|-\|\hat{\mu}_n\|\,|<\tfrac{1}{2}. \tag{7.24}$$

Using the measure λ_{α_0} and the Green potential $g_{\alpha_0}(z)=g(\lambda_{\alpha_0},\mathbf{C}\setminus\mathbf{R}_-;z)$ introduced before (5.26) in Lemma 9, it follows from (7.23) and identity (5.26) in Lemma 9 that

$$d_{1,n}(w):=\int g_{\mathbf{C}\setminus\mathbf{R}_-}(w,u)\,d(\nu_n-\hat{\mu}_n-c\lambda_{\alpha_0})(u)\leqslant 0 \tag{7.25}$$

for $w\in[R,\varepsilon_n^{-1/\alpha}]$. Since the measure λ_{α_0} is positive, the function $d_{1,n}$ is subharmonic in $\mathbf{C}\setminus(\mathbf{R}_-\cup[R,\varepsilon_n^{-1/\alpha}])$. We have $d_{1,n}(w)=0$ for $w\in\mathbf{R}_-$. Hence, from (7.25) it follows that $d_{1,n}(w)\leqslant 0$ for $w\in\mathbf{C}\setminus\mathbf{R}_-$, which implies that $(\partial/\partial y)d_{1,n}(x\pm i0)\leqslant 0$ for $x\in\mathbf{R}_-$. From the Gauss theorem (cf. [18, Theorem II.1.1]) applied to $\mathbf{C}\setminus\mathbf{R}_-$ it then follows that

$$(\nu_n-\hat{\mu}_n-c\lambda_{\alpha_0})([R,\infty))\leqslant 0, \tag{7.26}$$

which implies with (7.24) that

$$\|\nu_n\| \leqslant \|\mu_n\| + \tfrac{1}{2} + c\|\lambda_{\alpha_0}\|. \tag{7.27}$$

Complementary to (7.25), we consider

$$d_{2,n}(w) := \int g_D(w,u)\, d(\nu_n - \hat{\mu}_n + c\lambda_{\alpha_0})(u) \geqslant 0 \quad \text{for } w \in [R, \varepsilon_n^{-1/\alpha}], \tag{7.28}$$

where the estimate again follows from (7.23) and (5.26) in Lemma 9. From (7.28), it then follows that

$$(\nu_n - \hat{\mu}_n + c\lambda_{\alpha_0})([R,\infty)) \geqslant 0, \tag{7.29}$$

which implies that

$$\|\nu_n\| \geqslant \|\mu_n\| - \tfrac{1}{2} - c\|\lambda_{\alpha_0}\|. \tag{7.30}$$

From Lemma 9 we know that $\|\lambda_{\alpha_0}\| < \infty$. Hence, we deduce from the relations (6.44) and (6.45) in Lemma 13, the relations (6.27) and (6.28) in Lemma 12, (7.27) and (7.30) that

$$\|\nu_n\| = 2n + \mathcal{O}(1) \quad \text{as } n \to \infty. \tag{7.31}$$

From (7.3) (or from (5.8) in Proposition 1) we know that (7.31) implies that

$$\lim_{n\to\infty} \left(\pi\sqrt{2\alpha(2n+\mathcal{O}(1))} + \log \varepsilon_n\right) = \log(4|\sin \pi\alpha|) + \alpha \log 4, \tag{7.32}$$

where $\mathcal{O}(1)$ denotes the Landau symbol from (7.31). From (7.32) we deduce that

$$\lim_{n\to\infty} \left(2\pi\sqrt{\alpha n} + \log \varepsilon_n\right) = \log|\sin \pi\alpha| + (1+\alpha)\log 4 \tag{7.33}$$

or

$$\lim_{n\to\infty} \varepsilon_n e^{2\pi\sqrt{\alpha n}} = 4^{1+\alpha}|\sin \pi\alpha|, \tag{7.34}$$

which proves (1.2) in Theorem 1. With this last conclusion the purpose of the paper is completed.

Acknowledgement. We feel deeply indebted to one of the referees who has suggested a large number of corrections and improvements to the original manuscript. These suggestions have led to greater transparency at many places and more than that to the elimination of some substantial faults.

References

[1] ANDERSSON, J.-E., Rational approximation to functions like x^α in integral norms. *Anal. Math.*, 14 (1988), 11–25.

[2] BERNSTEIN, S., Sur meilleure approximation de $|x|$ par des polynomes de degrés donnés. *Acta Math.*, 37 (1914), 1–57.

[3] — Sur la meilleure approximation de $|x|^p$ par des polynômes de degrés très élevés. *Izv. Akad. Nauk SSSR Ser. Mat.*, 2 (1938), 169–180 (Russian); 181–190 (French); *Collected Works*, Vol. II, pp. 262–272. Izdat. Akad. Nauk SSSR, Moscow, 1960 (Russian).

[4] BULANOV, A. P., Asymptotics for the least deviation of $|x|$ from rational functions. *Mat. Sb. (N.S.)*, 76 (118) (1968), 288–303; English translation in *Math. USSR-Sb.*, 5 (1968), 275–290.

[5] — The approximation of $x^{1/3}$ by rational functions. *Vesci Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk*, 1968:2 (1968), 47–56 (Russian).

[6] FREUD, G. & SZABADOS, J., Rational approximation to x^α. *Acta Math. Acad. Sci. Hungar.*, 18 (1967), 393–399.

[7] GANELIUS, T., Rational approximation to x^α on $[0,1]$. *Anal. Math.*, 5 (1979), 19–33.

[8] GONCHAR, A. A., On the speed of rational approximation of continuous functions with characteristic singularities. *Mat. Sb. (N.S.)*, 73 (115) (1967), 630–638; English translation in *Math. USSR-Sb.*, 2 (1967), 561–568.

[9] — Rational approximation of the function x^α, in *Constructive Theory of Functions* (Varna, 1970), pp. 51–53. Izdat. Bolgar. Akad. Nauk, Sofia, 1972 (Russian).

[10] — The rate of rational approximation and the property of univalence of an analytic function in the neighborhood of an isolated singular point. *Mat. Sb. (N.S.)*, 94 (136) (1974), 265–282 (Russian); English translation in *Math. USSR-Sb.*, 23 (1974), 254–270.

[11] GONCHAR, A. A. & LOPES, L. G., Markov's theorem for multipoint Padé approximants. *Mat. Sb. (N.S.)*, 105 (147) (1978), 512–524 (Russian); English translation in *Math. USSR-Sb.*, 34 (1978), 449–459.

[12] KARLIN, S. & STUDDEN, W. J., *Tchebycheff Systems: With Applications in Analysis and Statistics.* Pure Appl. Math., 15. Interscience, Wiley, New York–London–Sydney, 1966.

[13] LANDKOF, N. S., *Foundations of Modern Potential Theory.* Grundlehren Math. Wiss., 180. Springer-Verlag, New York-Heidelberg, 1972.

[14] LORENTZ, G. G., GOLITSCHEK, M. VON & MARKOVOZ, Y., *Constructive Approximation. Advanced Problems.* Grundlehren Math. Wiss., 304. Springer-Verlag, Berlin, 1996.

[15] MEINARDUS, G., *Approximation of Functions: Theory and Numerical Methods.* Springer Tracts Nat. Philos., 13. Springer-Verlag, New York, 1967.

[16] NEWMAN, D. J., Rational approximation to $|x|$. *Michigan Math. J.*, 11 (1964), 11–14.

[17] RIVLIN, T. J., *An Introduction to the Approximation of Functions.* Blaisdell, Waltham, MA–Toronto, ON–London, 1969.

[18] SAFF, E. B. & TOTIK, V., *Logarithmic Potentials with External Fields.* Grundlehren Math. Wiss., 316. Springer-Verlag, Berlin, 1997.

[19] STAHL, H., Best uniform rational approximation of $|x|$ on $[-1,1]$. *Mat. Sb.*, 183 (1992), 85–118 (Russian); English translation in *Russian Acad. Sci. Sb. Math.*, 76 (1993), 461–487.

[20] — Best uniform rational approximation of x^α on $[0,1]$. *Bull. Amer. Math. Soc. (N.S.)*, 28 (1993), 116–122.

[21] — Poles and zeros of best rational approximants of $|x|$. *Constr. Approx.*, 10 (1994), 469–522.

[22] STAHL, H. & TOTIK, T., *General Orthogonal Polynomials.* Encyclopedia Math. Appl., 43. Cambridge Univ. Press, Cambridge, 1992.

[23] SZEGŐ, G., *Orthogonal Polynomials*, 4th edition. Amer. Math. Soc. Colloq. Publ., 23. Amer. Math. Soc., Providence, RI, 1975.

[24] TZIMBALARIO, J., Rational approximation to x^α. *J. Approx. Theory*, 16 (1976), 187–193.

[25] VARGA, R. S., *Scientific Computation on Mathematical Problems and Conjectures*. CBMS-NSF Regional Conf. Ser. in Appl. Math., 60. SIAM, Philadelphia, PA, 1990.

[26] VARGA, R. S. & CARPENTER, A. J., On the Bernstein conjecture in approximation theory. *Constr. Approx.*, 1 (1985), 333–348.

[27] — On the Bernstein conjecture in approximation theory. *Mat. Sb. (N.S.)*, 129 (171) (1986), 535–548 (Russian); English translation in *Math. USSR-Sb.*, 57 (1987), 547–560.

[28] — Some numerical results on best uniform rational approximation of x^α on $[0,1]$. *Numer. Algorithms*, 2 (1992), 171–185.

[29] VARGA, R. S., RUTTAN, A. & CARPENTER, A. J., Numerical results on best uniform rational approximation of the function $|x|$ on the interval $[-1,1]$. *Mat. Sb.*, 182 (1991), 1523–1541 (Russian); English translation in *Math. USSR-Sb.*, 74 (1993), 271–290.

[30] VYACHESLAVOV, N. S., The approximation of $|x|$ by rational functions. *Mat. Zametki*, 16 (1974), 163–171 (Russian); English translation in *Math. Notes*, 16 (1974), 680–685.

[31] — The uniform approximation of $|x|$ by rational functions. *Dokl. Akad. Nauk SSSR*, 220 (1975), 512–515 (Russian); English translation in *Soviet Math. Dokl.*, 16 (1975), 100–104.

[32] — Approximation of x^α by rational functions. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44 (1980), 92–109 (Russian); English translation in *Math. USSR-Izv.*, 16 (1981), 83–101.

HERBERT R. STAHL
TFH-Berlin/FB II
Luxemburgerstrasse 10
DE-13353 Berlin
Germany
stahl@tfh-berlin.de

Received January 22, 2001