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Topological entropy of free product automorphisms

by

N. P. BROWN, K. DYKEMA and D. SHLYAKHTENKO

Pennsylvania State University, State College, PA, U.S.A.

Texas A&M University, College Station, TX, U.S.A.

University of California, Los Angeles, CA, U.S.A.

1. Introduction

The study of non-commutative topological dynamical systems, in other words, automorphisms of C^*-algebras, goes back a long way. Among the most important invariants of such systems is entropy, first introduced in the operator algebra context by Connes and Størmer in [13]. (See [40] for an extensive survey of this subject.) In this paper, we deal with the non-commutative generalization of topological entropy, discovered by Voiculescu [44] for nuclear C^*-algebras and extended by Brown [7] to the exact case. Although this entropy has been computed in many examples, we are very far from a good understanding of its behavior. For example, to this day we do not know in full generality the exact value of the topological entropy $\mathrm{ht}(\alpha\otimes\beta)$ of a tensor product of two automorphisms. In general the inequality $\mathrm{ht}(\alpha\otimes\beta)\leqslant\mathrm{ht}(\alpha)+\mathrm{ht}(\beta)$ is known. Equality is only known to hold when both α and β satisfy a CNT variational principle.

In this paper, we address four general types of questions regarding topological entropy.

The first question concerns the behavior of topological entropy under free products. The precise question is the value of $\mathrm{ht}(\alpha*\beta)$ of the topological entropy of the (possibly amalgamated) reduced free product automorphism $\alpha*\beta$. The main theorem of this paper states that $\mathrm{ht}(\alpha*\beta)=\max(\mathrm{ht}(\alpha),\mathrm{ht}(\beta))$, if the free product is with amalgamation over a finite-dimensional C^*-algebra. One surprising feature of this result is that the answer is precise—this is to be contrasted with the situation for tensor products. Although the free product of C^*-algebras is more complex than the tensor product, it seems to behave more like the direct sum for the purposes of entropy, which is, curiously, close to its

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behavior for the purposes of K-theory. One direct consequence of our result is that the free shift on an arbitrary reduced free product $*_{i\in\mathbf{Z}}(A,\phi)$ has zero entropy.

Giving a bound on the topological entropy of a free product of two automorphisms can be considered a refinement of the result [19] that the reduced amalgamated free product of two exact C^*-algebras is exact. Indeed, exactness of a C^*-algebra B is the statement that $\mathrm{ht}(\mathrm{id}: B\to B)<+\infty$. The reader will find that our proof is related to the argument given in [20] for the exactness of reduced amalgamated free products.

The second general question concerns the Connes–Narnhofer–Thirring (CNT) variational principle. The Connes–Narnhofer–Thirring (CNT) entropy [12] is a generalization to non-commutative measure spaces of the classical Kolmogorov–Sinai entropy. An automorphism α is said to satisfy the CNT variational principle if its topological entropy is equal to the supremum of the CNT entropy computed with respect to all invariant states. Although this principle fails for general non-commutative dynamical systems, we show that if two automorphisms $\alpha\in\mathrm{Aut}(A)$ and $\beta\in\mathrm{Aut}(B)$ satisfy the CNT variational principle, then so does $\alpha*\beta$.

The third general question concerns embeddings of dynamical systems. Kirchberg has shown that any separable exact C^*-algebra can be embedded into the Cuntz algebra on two generators. We show that any nuclear C^*-dynamical system can be covariantly embedded into the Cuntz algebra $\mathcal{O}_\infty$ in an entropy-preserving way.

The last question concerns the possible values of entropy that can be attained by automorphisms of a given C^*-algebra A, i.e., the set $TE(A)=\mathrm{ht}(\mathrm{Aut}(A))$. This is clearly an invariant of A. For instance, $TE(A)=\{+\infty\}$ if A is not exact; $TE(A)=\{0\}$ if A is finite-dimensional. We show that $TE(\mathcal{O}_\infty)=[0,+\infty]$. Thanks to Kirchberg's absorption results, this implies that any separable purely infinite nuclear simple C^*-algebra A admits an automorphism with any given value of entropy; i.e., $TE(A)=[0,+\infty]$.

The main result of this paper, computing entropy of amalgamated free products of automorphisms, applies only to the case that amalgamation takes place over a finite-dimensional subalgebra. Our results are likely to extend to the case of amalgamation over an arbitrary C^*-subalgebra, if the following question can be answered in the affirmative:

Question 1.1. Let $0\hookrightarrow I\to E\xrightarrow{\pi}B\to 0$ be an exact sequence of C^*-algebras, which is split (i.e., there exists a $*$-homomorphism $\eta: B\to E$ so that $\pi\circ\eta=\mathrm{id}_B$). Let α be an automorphism of E so that $\alpha(I)=I$; denote the resulting automorphism of B by $\widehat{\alpha}$. Assume that $\eta\circ\widehat{\alpha}=\alpha\circ\eta$. Is it true that $\mathrm{ht}(\alpha)\leqslant\max(\mathrm{ht}(\widehat{\alpha}),\mathrm{ht}(\alpha|_I))$?

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2. Preliminaries

2.1. *A review of topological entropy.* Let A be an exact C^*-algebra and let $\omega \subset A$ be a finite subset. Fix a faithful representation $\pi: A \to B(H)$. Define the set $CPA(A)$ to be the set of triples (Φ, Ψ, X), where X is a finite-dimensional C^*-algebra, and $\Phi: A \to X$, $\Psi: X \to B(H)$ are contractive completely-positive maps. For $\varepsilon > 0$, define

$$\operatorname{rcp}(\omega, \varepsilon) = \inf\{\operatorname{rank}(X) : (\Phi, \Psi, X) \in CPA(A) \text{ and } \|\Psi \circ \Phi(a) - \pi(a)\| < \varepsilon \text{ for all } a \in \omega\}.$$

(This quantity is independent of the choice of π, cf. [7].) If $\alpha \in \operatorname{Aut}(A)$ is an automorphism, then its topological entropy $\operatorname{ht}(\alpha)$ is defined as

$$\sup_{\omega \subset A} \sup_{\varepsilon} \limsup_{n \to \infty} \frac{1}{n} \log \operatorname{rcp}(\omega \cup \ldots \cup \alpha^{n-1}(\omega), \varepsilon).$$

This definition, obtained in [7], gives the same quantity as Voiculescu's original definition of dynamical topological entropy [44] for nuclear C^*-algebras A. We summarize below some properties of $\operatorname{ht}(\alpha)$, which we will need in this paper. The proofs can be found in [7].

THEOREM. *Let $\alpha \in \operatorname{Aut}(A)$ be as above. Then:*

(1) ht *is monotone: if $B \subset A$ and $\alpha(B) \subset B$, then* $\operatorname{ht}(\alpha|_B) \leqslant \operatorname{ht}(\alpha)$.

(2) *If $A = \overline{\bigcup A_n}$, with A_n subalgebras, $A_n \subset A_{n+1}$ and $\alpha(A_n) \subset A_n$, then* $\operatorname{ht}(\alpha) = \lim_n \operatorname{ht}(\alpha|_{A_n})$.

(3) *If $\beta \in \operatorname{Aut}(B)$, let $\alpha \otimes \beta \in \operatorname{Aut}(A \otimes_{\min} B)$ be the tensor product automorphism. Then* $\operatorname{ht}(\alpha \otimes \beta) \leqslant \operatorname{ht}(\alpha) + \operatorname{ht}(\beta)$. *If A contains an α-invariant projection, then* $\operatorname{ht}(\alpha \otimes \beta) \geqslant \operatorname{ht}(\beta)$.

(4) *If $\beta \in \operatorname{Aut}(B)$, let $\alpha \oplus \beta \in \operatorname{Aut}(A \oplus B)$ be the direct sum automorphism. Then* $\operatorname{ht}(\alpha \oplus \beta) = \max(\operatorname{ht}(\alpha), \operatorname{ht}(\beta))$.

(5) *If $\beta \in \operatorname{Aut}(A)$ commutes with α, i.e., $\beta \circ \alpha = \alpha \circ \beta$, then α extends to the obvious automorphism $\bar{\alpha} \in \operatorname{Aut}(A \rtimes_\beta \mathbf{Z})$. Then* $\operatorname{ht}(\bar{\alpha}) = \operatorname{ht}(\alpha)$.

2.2. *Amalgamated free products.* Let D be a unital C^*-algebra. Recall that a *D-valued non-commutative probability space* is a C^*-algebra A, containing D as a unital subalgebra, and endowed with a conditional expectation $E: A \to D$.

Let now A_1 and A_2 be two unital C^*-algebras with a common unital subalgebra D and with conditional expectations $E_j: A_j \to D$, so that the GNS representation associated to each E_j is faithful. Then there exists a C^*-algebra A, generated by A_1 and A_2 (in such a way that the copies of D inside $A_j \subset A$ are identified), a conditional expectation $E: A \to D$, and such that $A_j \subset A$ are *free with respect to* E (see [45]), and E gives rise to a faithful GNS representation of A. The C^*-algebra A is the *reduced amalgamated free* product over D of (A_1, E_1) and (A_2, E_2), and is denoted

$$(A, E) = (A_1, E_1) *_D (A_2, E_2). \tag{2.1}$$

When the conditional expectations E_j are clear from context we may write $A = A_1 *_D A_2$. In the case that $D = \mathbf{C}1 \subset A_1, A_2$, then E_1, E_2 and E are states and A is the reduced free product of A_1 and A_2, denoted simply

$$(A, E) = (A_1, E_1) * (A_2, E_2). \tag{2.2}$$

Moreover, in the situation of a free product (2.1) or (2.2), one has conditional expectations $\Phi_1: A \to A_1$ and $\Phi_2: A \to A_2$ such that $E_j \circ \Phi_j = E$.

Definition 2.2.1. Let $(A, E: A \to D)$ be a D-valued non-commutative probability space. We say that $\alpha \in \mathrm{Aut}(A)$ is an automorphism of this space, if (i) $\alpha(D) = D$ and (ii) $\alpha \circ E = E \circ \alpha$.

Notice that in the case that $D = \mathbf{C}$, an automorphism of a D-probability space A is simply an automorphism of A, fixing the state $E: A \to D = \mathbf{C}$.

Assume now that $(A_1, E: A_1 \to D)$ and $(A_2, E_2: A_2 \to D)$ are D-probability spaces. Assume further that $\alpha_j \in \mathrm{Aut}(A_j)$ are D-space automorphisms, and $\alpha_1|_D = \alpha_2|_D$. Then there is a unique automorphism $\alpha_1 * \alpha_2: A_1 *_D A_2 \to A_1 *_D A_2$ satisfying

$$(\alpha_1 * \alpha_2)|_{A_j \subset A_1 *_D A_2} = \alpha_j, \quad j = 1, 2.$$

We will need the following theorem of Blanchard and Dykema.

THEOREM [4]. *Let D and $\widetilde{D}$ be unital C^*-algebras, let (A_1, E_1) and (A_2, E_2) be D-probability spaces, and let $(\tilde{A}_1, \widetilde{E}_1)$ and $(\tilde{A}_2, \widetilde{E}_2)$ be $\widetilde{D}$-probability spaces. Assume that the GNS representations associated to E_j and $\widetilde{E}_j$ are faithful* $(j = 1, 2)$. *Let*

$$(A, E) = (A_1, E_1) *_D (A_2, E_2),$$
$$(\tilde{A}, \widetilde{E}) = (\tilde{A}_1, \widetilde{E}_1) *_{\widetilde{D}} (\tilde{A}_2, \widetilde{E}_2).$$

Suppose that $\pi_D: D \to \widetilde{D}$ is a (*not necessarily unital*) *injective $*$-homomorphism and that there are injective $*$-homomorphisms $\pi_1: A_1 \to \tilde{A}_1$ and $\pi_2: A_2 \to \tilde{A}_2$ such that $\pi_D \circ E_j = \widetilde{E}_j \circ \pi_j$. Then there is an injective $*$-homomorphism $\pi: A \to \tilde{A}$ such that $\pi|_{A_j} = \pi_j$* $(j = 1, 2)$ *and $\widetilde{E} \circ \pi = E$.*

2.3. *K-theory for reduced free products.* Thanks to fundamental work of E. Germain on K-theory of free products, one has the following six-term exact sequence for free products of nuclear C^*-algebras.

THEOREM [22], [21], [23]. *Let A and B be unital nuclear C^*-algebras and $\phi \in S(A)$, $\psi \in S(B)$ be states with faithful GNS representations. Let $C=(A,\phi)*(B,\psi)$, and let $\iota_A : A \hookrightarrow C$ and $\iota_B : B \hookrightarrow C$ denote the canonical inclusions. Then there is an exact sequence*

$$\begin{tikzcd} \mathbf{Z} \arrow[r, "1\mapsto[1_A]\oplus-[1_B]"] & K_0(A)\oplus K_0(B) \arrow[r, "(\iota_A)_*+(\iota_B)_*"] & K_0(C) \arrow[d] \\ K_1(C) \arrow[u] & K_1(A)\oplus K_1(B) \arrow[l, "(\iota_A)_*+(\iota_B)_*"'] & 0. \arrow[l] \end{tikzcd}$$

Moreover, if both A and B satisfy the Universal Coefficient Theorem of Rosenberg and Schochet (see e.g. [3]) then so does C.

2.4. *Cuntz–Pimsner algebras.* Let A be a C^*-algebra, and H be a Hilbert bimodule over A. Assume that H is *full*, i.e., $\langle H,H\rangle_A$ is dense in A. Let

$$F(H)=A\oplus\bigoplus_{n\geqslant 1} H^{\otimes_A n},$$

and for $\xi\in H$, let

$$l(\xi): F(H)\to F(H),$$
$$l(\xi)\cdot \xi_1\otimes\ldots\otimes\xi_n=\xi\otimes\xi_1\otimes\ldots\otimes\xi_n.$$

The Cuntz–Pimsner C^*-algebra $E(H)$ (cf. [33]) is then defined as $C^*(l(\xi):\xi\in H)$. It is not hard to see that

$$l(\xi)^* l(\zeta)=\langle\xi,\zeta\rangle_A,$$

and hence $A=\overline{\langle H,H\rangle}\subset E(H)$, acting on the left of $F(H)$. The projection from $F(H)$ onto $A\subset F(H)$ gives rise to a conditional expectation $E: E(H)\to A$. We summarize some of the properties of $E(H)$ below; the proofs can be found in [33], [37] and [20].

THEOREM. *Let H and A be as above. Then:*

(1) *If $K\subset H$ is a Hilbert A-subbimodule which is full, then $E(K)$ is canonically isomorphic to $C^*(l(\xi): F(H)\to F(H)$ such that $\xi\in K)\subset E(H)$.*

(2) *Let H' be another full Hilbert A-bimodule, then $E(H\oplus H')=E(H)*_A E(H')$, where the reduced free product is taken with respect to the canonical conditional expectations from $E(H)$ (resp., $E(H')$) onto A.*

(3) *Assume that* $\bar{\alpha}: H \to H$ *is a linear map so that for some* $\alpha \in \mathrm{Aut}(A)$,

$$\bar{\alpha}(a_1 \cdot \xi \cdot a_2) = \alpha(a_1) \cdot \bar{\alpha}(\xi) \cdot \alpha(a_2), \quad a_1, a_2 \in A,\ \xi \in H, \tag{2.3}$$

$$\langle \bar{\alpha}(\xi_1), \bar{\alpha}(\xi_2) \rangle_A = \alpha(\langle \xi_1, \xi_2 \rangle_A), \quad \xi_1, \xi_2 \in H. \tag{2.4}$$

Then there is a unique automorphism $E(\bar{\alpha})$ *of the* A*-probability space* $(E(H), E\colon E(H) \to A)$ *so that* $E(\bar{\alpha})|_{A \subset E(H)} = \alpha$ *and* $E(\bar{\alpha})(l(\xi)) = l(\bar{\alpha}(\xi))$. *The automorphism* $E(\bar{\alpha})$ *is called the Bogolyubov automorphism associated to* $\bar{\alpha}$.

2.5. *Some examples of Bogolyubov automorphisms.* In the course of proving the main theorem of the paper, we will encounter a particular class of Bogolyubov automorphisms of $E(H)$, which we will presently describe. Let D be a C^*-algebra and $(A, E\colon A \to D)$ be a D-probability space. Let

$$K^o = A \otimes A$$

(algebraic tensor product), and endow K^o with the A-valued inner product given by

$$\langle a \otimes b, a' \otimes b' \rangle_A = b^* E(a^* a') b', \quad a, a', b, b' \in A.$$

Denote by K_D the Hilbert A-bimodule obtained from K^o after separation and completion. Another description of K_D is as the internal tensor product $K_D = L^2(A, E) \otimes_D A$. Notice that any automorphism α of the D-probability space A extends to a linear map $\bar{\alpha}\colon K_D \to K_D$ satisfying equations (2.3) and (2.4).

THEOREM [35]. $E(K_D) \cong (A, E) *_D (D \otimes \mathcal{T}, \mathrm{id} \otimes \psi)$, *where* $\mathcal{T}$ *denotes the Toeplitz algebra* (*the algebra generated by the unilateral shift on* $l^2(\mathbf{N})$), *and* $\psi\colon \mathcal{T} \to \mathbf{C}$ *is the vacuum state* (*corresponding to the vector* $\delta_1 \in l^2(\mathbf{N})$). *Moreover,* $E(\bar{\alpha})$ *corresponds in this isomorphism to* $\alpha * ((\alpha|_D) \otimes \mathrm{id})$.

Let $(A, E\colon A \to D)$ be a D-probability space, and let α be an automorphism of this D-probability space. Assume that D is finite-dimensional. Let ϕ be an $\alpha|_D$-invariant faithful trace on D. Consider the Hilbert (A, A)-bimodule

$$H = L^2(A, \phi \circ E) \otimes_{\mathbf{C}} A,$$

together with the vector $\xi = 1 \otimes 1 \in H$ and the inner product

$$\langle a_1 \otimes a_2, b_1 \otimes b_2 \rangle = \langle a_1, b_1 \rangle_{L^2(A, \phi \circ E)} a_2^* b_2.$$

We will henceforth also write ϕ to mean $\phi \circ E$. Let $U\colon H \to H$ be given by

$$U(x \otimes a) = V(x) \otimes \alpha(a),$$

where $V\colon L^2(A, \phi) \to L^2(A, \phi)$ is the unitary induced by $\alpha\colon A \to A$. Then $UaU^* = \alpha(a)$ where a and $\alpha(a)$ act on the left of H.

LEMMA 2.5.1. *There exists a vector $\zeta \in H$, with the following properties:*
(1) $d\zeta = \zeta d$ *for all* $d \in D$,
(2) $U\zeta = \zeta$,
(3) $\langle \zeta, a\zeta \rangle = E(a)$ *for all* $a \in A$.

Proof. Let $U(D)$ denote the unitary group of D, endowed with Haar measure μ. Let

$$\zeta' = \int_{u \in U(D)} u\xi u^* \, d\mu(u).$$

For each $w \in U(D)$, we get

$$w\zeta' = w \int_{u \in U(D)} u\xi u^* \, d\mu(u) = \left(\int_{uw \in U(D)} wu\xi uw^* \, d\mu(u) \right) w = \zeta' w,$$

so $d\zeta' = \zeta' d$ for all $d \in D$. Furthermore,

$$\begin{aligned} U\zeta' = \int_{u \in U(D)} U(u \otimes u^*) \, d\mu(u) &= \int_{u \in U(D)} \alpha(u) \cdot 1 \otimes \alpha(u^*) \, d\mu(u) \\ &= \int_{u \in U(D)} \alpha(u) \xi \alpha(u^*) \, d\mu(u) = \zeta'. \end{aligned}$$

Lastly, for $a \in A$, set

$$\begin{aligned} \Phi(a) = \langle \zeta', a\zeta' \rangle &= \int_{u,v \in U(D)} \langle u \otimes u^*, av \otimes v^* \rangle \, d\mu(u) \, d\mu(v) \\ &= \int_{u,v} u\phi(u^*av)v^* \, d\mu(u) \, d\mu(v) \\ &= \int_{u,v} \phi(vu^*a)uv^* \, d\mu(u) \, d\mu(v) \\ &= \int_{w = uv^*, v} \phi(wa)w^* \, d\mu(w) \, d\mu(v) \\ &= \int_{w} \phi(wa)w^* \, d\mu(w). \end{aligned}$$

Assume now that $E(a) = 0$, i.e., $\phi(da) = 0$ for all $d \in D$. Then

$$\Phi(a) = \int_w \phi(wa)w^* \, d\mu(w) = 0.$$

Since whenever $v \in U(D)$,

$$\Phi(v) = \int_w \phi(wv)w^* \, d\mu(w) = \int_{u = wv} \phi(u)vu^* \, d\mu(u) = v\Phi(1),$$

and any $d\in D$ is a linear combination of unitaries from $U(D)$, we get that

$$\Phi(d)=d\Phi(1)\quad\text{for all } d\in D.$$

Since for any $a\in A$, $a=(a-E(a))+E(a)$ and $E(a)\in d$, we get that

$$\Phi(a)=\Phi(a-E(a))+\Phi(E(a))=E(a)\Phi(1).$$

We also have $d\Phi(1)=\Phi(d)=\langle\zeta',d\zeta'\rangle=\langle\zeta',\zeta'd\rangle=\Phi(1)d$. Hence $\Phi(1)$ is in the center of D; moreover, $\Phi(1)=\langle\zeta',\zeta'\rangle\geqslant 0$ and $\alpha(\Phi(1))=\alpha(\langle\zeta',\zeta'\rangle)=\langle U\zeta',U\zeta'\rangle=\langle\zeta',\zeta'\rangle=\Phi(1)$. We claim that $\Phi(1)$ is invertible. Writing $D=\bigoplus M_{n_k}$ as a direct sum of matrix algebras, we find that

$$\Phi(1)=\int_{u\in U(D)}\phi(u)u^*\,d\mu(u)=\bigoplus\Phi_k,$$

where

$$\Phi_k=\alpha_k\cdot 1_{M_{n_k}}\cdot\int_{u\in U(n_k)}\mathrm{Tr}(u)u^*\,d\mu(u),$$

and $\alpha_k\geqslant 0$ is related to the value of ϕ on the minimal projection of D corresponding to the kth matrix summand in the direct sum decomposition of D. To show that $\Phi(1)$ is invertible, it is sufficient to show that $c_k=\int_{u\in U(k)}\mathrm{Tr}(u)u^*\,d\mu(u)$ is a strictly positive scalar for all $k\geqslant 1$. Repeating the argument above with D replaced by M_k, we find that c_k is in the center of M_k and is non-negative. Furthermore,

$$\mathrm{Tr}(c_k)=\int_{u\in U(k)}\mathrm{Tr}(u)\,\mathrm{Tr}(u^*)\,d\mu(u)=\int_{u\in U(k)}|\mathrm{Tr}(u)|^2\,d\mu(u)>0,$$

since the subset of $u\in M_k$ with $\mathrm{Tr}(u)\neq 0$ has non-zero measure.

Now let $\zeta=\Phi(1)^{-1/2}\zeta'$. Then $U\zeta=U\Phi(1)^{-1/2}U^*U\zeta'=\Phi(1)^{-1/2}\zeta'=\zeta$; for all $d\in D$, $d\zeta=d\Phi(1)^{-1/2}\zeta'=\Phi(1)^{-1/2}d\zeta'=\Phi(1)^{-1/2}\zeta'd=\zeta d$; and for all $a\in A$,

$$\begin{aligned}\langle\zeta,a\zeta\rangle=\langle\Phi(1)^{-1/2}\zeta',a\Phi(1)^{-1/2}\zeta'\rangle&=\langle\zeta',\Phi(1)^{-1/2}a\Phi(1)^{-1/2}\zeta'\rangle\\&=\Phi(\Phi(1)^{-1/2}a\Phi(1)^{-1/2})\\&=\Phi(1)^{-1/2}E(a)\Phi(1)^{-1/2}\Phi(1)\\&=E(a),\end{aligned}$$

as desired. □

COROLLARY 2.5.2. *Let α be an automorphism of a D-probability space $(A,E\colon A\to D)$ with $\dim(D)<\infty$. Assume that $E\colon A\to D$ gives rise to a faithful GNS representation. Let ϕ be an α-invariant trace on D. Consider the automorphism $\beta=\alpha*\mathrm{id}$ on $B=(A,\phi\circ E)*\mathcal{T}$, where $\mathcal{T}$ is the Toeplitz algebra with its vacuum state. Consider the automorphism $\gamma=\alpha*(\alpha|_D\otimes\mathrm{id})$ on $C=(A,E)*_D(D\otimes\mathcal{T})$. Then there exists a covariant embedding of (C,γ) into (B,β).*

Proof. Let H be as above. Consider the Cuntz–Pimsner C^*-algebra $E(H)$, generated by A and the operators $l(h)$ for $h \in H$. Then

$$B = E(H) = C^*(A, l(h) : h \in H) = C^*(A, l(\xi)).$$

The automorphism β is identified with the Bogolyubov automorphism $E(U)$: indeed we have $\beta(a) = \alpha(a)$, $a \in A$, and $E(U)(l(a \otimes b)) = l(U(a \otimes b)) = l(\alpha(a) \otimes \alpha(b)) = \alpha(a) l(\xi) \alpha(b) = \beta(a l(\xi) b)$. Choose ζ as in Lemma 2.5.1. Let $L = l(\zeta)$. Then $L^* a L = \langle \zeta, a\zeta \rangle = E(a)$. Let $K = \overline{A \zeta A} \subset H$. It follows that $C^*(A, L) = C^*(aLb : a, b \in A) = C^*(l(h) : h \in K) = E(K)$. It is easily seen that the map $a \zeta b \mapsto a \otimes b$ defines an isomorphism of K with the A-bimodule K_D defined in §2.5. Hence by the results of that section, $C^*(A, L) \cong C$. Moreover, since

$$E(U)(a) = \alpha(a), \quad a \in A,$$
$$E(U)(L) = l(U(\zeta)) = l(\zeta) = L,$$

we get that $E(U)|_{C^*(A,L)} = \gamma$. □

3. Topological entropy in certain extensions

The main result of this section gives an estimate of topological entropy in certain extensions. We begin with a lemma which gives a way of estimating the δ-rank of finite sets in an extension.

Consider a short exact sequence $0 \to I \to E \xrightarrow{\pi} B \to 0$ where E is a unital exact C^*-algebra, and assume that there exists a unital completely positive splitting $\varrho: B \to E$ (i.e., $\pi \circ \varrho = \mathrm{id}_B$).

Lemma 3.1. *Let I, E, B, π and ϱ be as above. Given $\varepsilon > 0$ there exists a $\delta > 0$ with the following property: if $\omega = \{\varrho(b_1) + x_1, ..., \varrho(b_s) + x_s\} \subset E$ is a finite set containing the unit of E and such that $b_i \in B$, $x_i \in I$ and $\|b_i\|, \|x_i\| \leqslant 1$, $1 \leqslant i \leqslant s$, and if $0 \leqslant e \leqslant 1_E$ is an element of I such that $\|[e, \varrho(b_i)]\|, \|x_i - e x_i\|, \|x_i - x_i e\| < \delta$ for $1 \leqslant i \leqslant s$, then*

$$\mathrm{rcp}_E(\omega, 30\varepsilon) \leqslant \mathrm{rcp}_I(e^{1/n} \omega e^{1/n}, \varepsilon) + \mathrm{rcp}_B(\pi(\omega), \varepsilon),$$

for any $n > 1/\varepsilon$.

Proof. Fix $\varepsilon > 0$. By the lemma in [1, p. 332] we can find a $\tilde{\delta} > 0$ such that for every pair of elements f, g in the unit ball of E such that $f \geqslant 0$ we have the implication

$$\|[f, g]\| < \tilde{\delta} \;\Rightarrow\; \|[f^{1/2}, g]\| < \varepsilon.$$

Let $\delta=\min(\tilde{\delta}, \frac{1}{4}\varepsilon^2)$.

So assume that $\omega=\{\varrho(b_1)+x_1, ..., \varrho(b_s)+x_s\}\subset E$ is a finite set such that $b_i\in B$, $x_i\in I$ and $\|b_i\|, \|x_i\|\leqslant 1$, $1\leqslant i\leqslant s$, and $0\leqslant e\leqslant 1_E$ is an element of I such that $\|[e, \varrho(b_i)]\|$, $\|x_i-ex_i\|, \|x_i-x_ie\|<\delta$ for $1\leqslant i\leqslant s$. By our choice of δ we then have $\|[e^{1/2}, \varrho(b_i)]\|$, $\|[(1_E-e)^{1/2}, \varrho(b_i)]\|<\varepsilon$ for all $1\leqslant i\leqslant s$.

We also claim that $\|[e^{1/2}, x_i]\|<\varepsilon$ for all $1\leqslant i\leqslant s$. To see this we first note that since $e\leqslant e^{1/2}$ we have that $1_E-e\geqslant 1_E-e^{1/2}$. Thus

$$\|e^{1/2}x-x\|^2=\|x^*(1_E-e^{1/2})^2x\|\leqslant\|x^*(1_E-e^{1/2})x\|\leqslant\|x^*(1_E-e)x\|\leqslant\|x-ex\|$$

for all $x\in I$ with $\|x\|\leqslant 1$. Similarly, $\|xe^{1/2}-x\|\leqslant\|x-xe\|$ and hence

$$\|[e^{1/2}, x_i]\|\leqslant\|e^{1/2}x_i-x_i\|+\|x_ie^{1/2}-x_i\|\leqslant\|ex_i-x_i\|^{1/2}+\|x_ie-x_i\|^{1/2}<\varepsilon.$$

Since $0\leqslant e\leqslant 1_E$ some routine functional calculus shows that $\|e^{1/2+1/n}-e^{1/2}\|\leqslant 2/n$ and $\|e^{1+2/n}-e\|\leqslant 2/n$. Combining these inequalities, the inequalities in the previous paragraphs and a standard interpolation argument we get

$$\begin{aligned}\|[e^{1/2+1/n}, \varrho(b_i)]\| &< 4/n+\varepsilon,\\ \|[e^{1/2+1/n}, x_i]\| &< 4/n+\varepsilon,\\ \|\varrho(b_i)e^{1+2/n}-\varrho(b_i)e\| &\leqslant 2/n,\\ \|x_ie^{1+2/n}-x_ie\| &\leqslant 2/n.\end{aligned}$$

These four inequalities will be needed at the end of the proof.

Assume that $E\subset B(H)$ and $B\subset B(K)$. By Arveson's extension theorem we may assume that $\varrho: B\to E\subset B(H)$ is defined on all of $B(K)$ (and takes values in $B(H)$). Now choose $(\tilde{\phi}_1, \tilde{\phi}_2, D_1)\in CPA(B)$ such that $\|\tilde{\phi}_2\circ\tilde{\phi}_1(b_i)-b_i\|<\varepsilon$, $1\leqslant i\leqslant s$, and $\operatorname{rank}(D_1)=\operatorname{rcp}_B(\pi(\omega), \varepsilon)$. Using the techniques in the proof of Proposition 1.4 in [7] we may replace the (not necessarily unital) maps $\tilde{\phi}_1$ and $\tilde{\phi}_2$ with unital maps $\phi_1: B\to D_1$ and $\phi_2: D_1\to B(K)$ such that $\|\phi_2\circ\phi_1(b_i)-b_i\|<14\varepsilon$. (Here we use the facts that $1_B\in\pi(\omega)$ and $\|b_i\|\leqslant 1$, $1\leqslant i\leqslant s$.)

Similarly, choose $(\psi_1, \psi_2, D_2)\in CPA(I)$ such that

$$\|\psi_2\circ\psi_1(e^{1/n}(b_i+x_i)e^{1/n})-e^{1/n}(b_i+x_i)e^{1/n}\|<\varepsilon,$$

$1\leqslant i\leqslant s$, and $\operatorname{rank}(D_2)=\operatorname{rcp}_I(e^{1/n}\omega e^{1/n}, \varepsilon)$. By Arveson's extension theorem we may assume that ψ_1 is defined on all of $B(H)$.

Define $\chi_1: E\to D_1\oplus D_2$ by

$$\chi_1(y)=\phi_1(\pi(y))\oplus\psi_1(e^{1/n}ye^{1/n})$$

for all $y \in E$, and $\chi_2\colon D_1 \oplus D_2 \to B(H)$ by

$$\chi_2(S \oplus T) = (1_E - e)^{1/2} \varrho(\phi_2(S))(1_E - e)^{1/2} + e^{1/2} \psi_2(T) e^{1/2}$$

for all $S \in D_1$, $T \in D_2$. Since we have arranged that ϕ_2 is unital (and ϱ is unital by assumption) we see that $\chi_2(1_{D_1} \oplus 1_{D_2}) = 1_E - e + e^{1/2} \psi_2(1_{D_2}) e^{1/2} = 1_E - (e^{1/2}(1_E - \psi_2(1_{D_2}))e^{1/2})$. Since ψ_2 is a contractive completely positive map, this shows that $\chi_2(1_{D_1} \oplus 1_{D_2})$ is a positive operator of norm less than or equal to one. Since it is clear that χ_2 is a completely positive map this, in turn, implies that χ_2 is also a contractive map (see [31, Proposition 3.5]).

Hence $(\chi_1, \chi_2, D_1 \oplus D_2) \in CPA(E)$ and

$$\operatorname{rank}(D_1 \oplus D_2) = \operatorname{rcp}_I(e^{1/n} \omega e^{1/n}, \varepsilon) + \operatorname{rcp}_B(\{\pi(\omega)\}, \varepsilon).$$

Thus we only have to check that $\|\chi_2 \circ \chi_1(\varrho(b_i) + x_i) - \varrho(b_i) - x_i\| < 31\varepsilon$ for $1 \leqslant i \leqslant s$. But, letting $y = \varrho(b_i) + x_i$ we have

$$\begin{aligned}
\|\chi_2 \circ \chi_1(y) - y\| &= \|(1_E - e)^{1/2} \varrho(\phi_2(\phi_1(b_i)))(1_E - e)^{1/2} \\
&\qquad + e^{1/2} \psi_2 \circ \psi_1(e^{1/n}(\varrho(b_i) + x_i)e^{1/n}) e^{1/2} - \varrho(b_i) - x_i\| \\
&\leqslant \|(1_E - e)^{1/2} \varrho(\phi_2(\phi_1(b_i)))(1_E - e)^{1/2} - (1_E - e)^{1/2} \varrho(b_i)(1_E - e)^{1/2}\| \\
&\qquad + \|e^{1/2} \psi_2 \circ \psi_1(e^{1/n}(\varrho(b_i) + x_i)e^{1/n}) e^{1/2} \\
&\qquad\quad - e^{1/2}(e^{1/n}(\varrho(b_i) + x_i)e^{1/n}) e^{1/2}\| \\
&\qquad + \|(1_E - e)^{1/2} \varrho(b_i)(1_E - e)^{1/2} + e^{(n+2)/2n}(\varrho(b_i) + x_i) e^{(n+2)/2n} \\
&\qquad\quad - \varrho(b_i) - x_i\| \\
&\leqslant 15\varepsilon + \|[(1_E - e)^{1/2}, \varrho(b_i)]\| + \|[e^{(n+2)/2n}, \varrho(b_i)]\| + \|[e^{(n+2)/2n}, x_i]\| \\
&\qquad + \|\varrho(b_i)(1_E - e) + (\varrho(b_i) + x_i)e^{1+2/n} - \varrho(b_i) - x_i\| \\
&\leqslant 18\varepsilon + 8/n + \|\varrho(b_i)e^{1+2/n} - \varrho(b_i)e\| + \|x_i e^{1+2/n} - x_i e\| \\
&\leqslant 18\varepsilon + 12/n.
\end{aligned}$$

Hence for any $n > 1/\varepsilon$ we have the desired inequality. □

Remark 3.2. The previous lemma is easily generalized to arbitrary extensions, though a precise formulation is somewhat awkward (and does not appear to be useful for entropy calculations). The idea is that if E is a unital exact C^*-algebra then the quotient map $\pi\colon E \to B$ is always locally liftable (cf. [25, Proposition 7.2]). Hence the δ-rank of any finite subset of E can be estimated in terms of finite subsets of I and B.

Though it will not be needed in what follows, it seems appropriate to point out the following application.

PROPOSITION 3.3. *With I, E, B, π and ϱ as above, let $\alpha\in\mathrm{Aut}(E)$ be an automorphism such that $\alpha(I)=I$, and let $\widehat{\alpha}\in\mathrm{Aut}(B)$ be the induced automorphism. Assume that $\varrho\circ\widehat{\alpha}=\alpha\circ\varrho$ and there exists an approximate unit $\{e_\lambda\}\subset I$ such that $\alpha(e_\lambda)=e_\lambda$ for all λ (which happens, for example, if there exists a strictly positive element $h\in I$ such that $\alpha(h)=h$). Then*

$$\mathrm{ht}(\alpha)=\max(\mathrm{ht}(\alpha|_I),\mathrm{ht}(\widehat{\alpha})).$$

Proof. By [7, Proposition 2.10] it suffices to show the inequality

$$\mathrm{ht}(\alpha)\leqslant\max(\mathrm{ht}(\alpha|_I),\mathrm{ht}(\widehat{\alpha})).$$

Let $\varepsilon>0$ be given and $\omega=\{\varrho(b_1)+x_1,...,\varrho(b_s)+x_s\}\subset E$ be any finite set containing the unit of E and such that $\|b_i\|,\|x_i\|\leqslant 1$, $1\leqslant i\leqslant s$. Let $\{e_\lambda\}\subset I$ be an approximate unit such that $\alpha(e_\lambda)=e_\lambda$ for all λ. Since we can manufacture a *quasicentral* approximate unit out of the convex hull of $\{e_\lambda\}$ [1], we may further assume that $\{e_\lambda\}$ is quasicentral in E (and still fixed by α).

Choose $\delta>0$ according to the previous lemma and take λ large enough so that $\|[e_\lambda,\varrho(b_i)]\|,\|x_i-e_\lambda x_i\|,\|x_i-x_ie_\lambda\|<\delta$ for $1\leqslant i\leqslant s$. Since $\alpha^j(e_\lambda)=e_\lambda$ for all $j\in\mathbf{N}$ it is clear that $\|[e_\lambda,\alpha^j(\varrho(b_i))]\|,\|\alpha^j(x_i)-e_\lambda\alpha^j(x_i)\|,\|\alpha^j(x_i)-\alpha^j(x_i)e_\lambda\|<\delta$ for $1\leqslant i\leqslant s$ and all $j\in\mathbf{N}$. Hence letting $\omega_I=e_\lambda^{1/k}\omega e_\lambda^{1/k}\subset I$ for some $k>1/\varepsilon$, the previous lemma implies that $\mathrm{rcp}(\omega\cup...\cup\alpha^n(\omega),30\varepsilon)$ is bounded above by

$$\mathrm{rcp}_I(\omega_I\cup...\cup\alpha^{n-1}(\omega_I),\varepsilon)+\mathrm{rcp}_B(\pi(\omega)\cup...\cup\widehat{\alpha}^{n-1}(\pi(\omega)),\varepsilon),$$

which is bounded above by

$$2\max(\mathrm{rcp}_I(\omega_I\cup...\cup\alpha^{n-1}(\omega_I),\varepsilon),\mathrm{rcp}_B(\pi(\omega)\cup...\cup\widehat{\alpha}^{n-1}(\pi(\omega)),\varepsilon)).$$

This inequality implies the result. □

The next lemma is inspired by §5 in [43].

LEMMA 3.4. *Let $F\subset B(H)$ be a finite set of self-adjoint contractive operators on a Hilbert space H. Let P be a projection in $B(H)$, of rank $l<\infty$. Then for any $\delta>0$, there exists a positive finite-rank contraction $X\in B(H)$ so that*

(1) $XP=PX=P$,

(2) $\|[X,T]\|<\delta$ *for all* $T\in F$,

(3) *the rank of X is no bigger than* $l\cdot(|F|+1)^{2/\delta+1}$.

Proof. Denote by $K_1\subset H$ the range of P. Define recursively

$$K_n=\mathrm{span}\{K_{n-1}\cup\bigcup_{T\in F}TK_{n-1}\}.$$

Let $q=|F|$ be the cardinality of F. Then K_n has dimension at most $q+1$ times the dimension of K_{n-1}, so that $\dim K_n \leqslant l\cdot(q+1)^n$.

Let P_n be the orthogonal projection onto K_n; then P_n are clearly an increasing sequence, and $P_i P = PP_i = P = P_1$ for all i. Let

$$X_n = \frac{1}{n}(P_1 + \ldots + P_n).$$

Then $X_n P = PX_n = (PP_1 + \ldots + PP_n)/n = P$ for all n. Note that the rank of X_n is the same as that of P_n, which is bounded by $l\cdot(q+1)^n$.

Set $Q_n = P_n - P_{n-1}$, $Q_1 = P_1$. Since $TK_n \subset K_{n+1}$ and thus $TP_n = P_{n+1}TP_n$, if $m-n>2$ then we have $Q_m T Q_n = Q_m T P_n Q_n = Q_m P_{n+1} T P_n Q_n = 0$ for all $T\in F$. Since T is self-adjoint, also $Q_n T Q_m = 0$ if $m-n>2$. Hence $Q_n T Q_m = 0$ if $|n-m|>2$. Let Z be the orthocomplement of $\overline{\bigcup K_n}$ in H. Then in the decomposition $H = \bigoplus Q_n H \oplus Z$, each $T\in F$ has the form

$$\begin{pmatrix} Q_1TQ_1 & Q_1TQ_2 & 0 & & \\ Q_2TQ_1 & Q_2TQ_2 & Q_2TQ_3 & 0 & \\ 0 & Q_3TQ_2 & Q_3TQ_3 & Q_3TQ_4 & \ddots \\ & 0 & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & 0 \end{pmatrix},$$

i.e., it is a "block tri-diagonal" matrix. Hence using the convention $Q_0 = 0$ we have the identities

$$Q_n T = \sum_{j=-1,0,1} Q_n T Q_{n+j}, \quad TQ_n = \sum_{j=-1,0,1} Q_{n+j} T Q_n.$$

Now

$$X_n = Q_1 + \left(1 - \frac{1}{n}\right) Q_2 + \ldots + \frac{1}{n} Q_n = \sum_{i=1}^{n} \frac{n-i+1}{n} Q_i,$$

so that for any $T\in F$,

$$X_n T = \sum_{i=1}^{n} \frac{n-i+1}{n} Q_i T = \sum_{j=-1,0,1} \sum_{i=1}^{n} \frac{n-i+1}{n} Q_i T Q_{i+j}.$$

Similarly,

$$TX_n = \sum_{i=1}^{n} \frac{n-i+1}{n} TQ_i = \sum_{j=-1,0,1} \sum_{i=1}^{n} \frac{n-i+1}{n} Q_{i+j} T Q_i$$

$$= \sum_{j=-1,0,1} \sum_{i=1}^{n} \frac{n-i+1-j}{n} Q_i T Q_{i+j}.$$

Hence

$$\begin{aligned}\|X_nT-T_nX\| &= \left\| \sum_{j=-1,0,1} \sum_{i=1}^{n} \frac{(n-i+1)-(n-i+1-j)}{n} Q_iTQ_{i+j} \right\| \\ &= \left\| \sum_{j=-1,0,1} \sum_{i=1}^{n} \frac{j}{n} Q_iTQ_{i+j} \right\| \\ &\leqslant \sum_{j=-1,0,1} \frac{|j|}{n} \left\| \sum_i Q_iTQ_{i+j} \right\| \\ &\leqslant \sum_{j=-1,1} \frac{|j|}{n} = \frac{2}{n}.\end{aligned}$$

The last inequality is due to the fact that for a fixed j, the operator $\sum Q_iTQ_{i+j}$ is a block-diagonal operator, with the blocks having orthogonal ranges, so that $\|\sum Q_iTQ_{i+j}\| \leqslant \max\{\|Q_iTQ_{i+j}\|\} \leqslant 1$.

Choose the smallest integer n with $n>2/\delta$, and set $X=X_n$. Then $\|[X,T]\| \leqslant 2/n<\delta$; by construction, $XP=PX=P$ and the rank of X is bounded by $l(q+1)^n \leqslant l(q+1)^{2/\delta+1}$. $\square$

We finally come to the main result of this section.

THEOREM 3.5. *Let $A, B\subset B(H)$ be unital exact C^*-algebras such that*

$$B\cap\mathcal{K}(H)=\{0\} \quad \textit{and} \quad C=B\otimes 1+\mathcal{K}(H)\otimes A\subset B(H\otimes H).$$

For any two unitaries $V, W\in B(H)$ such that $\operatorname{Ad} V(B)=B$ and $\operatorname{Ad} W(A)=A$, the unitary $U=V\otimes W$ has the property that $\operatorname{Ad} U(C)=C$ and

$$\operatorname{ht}(\operatorname{Ad} U|_C) \leqslant \max(\operatorname{ht}(\operatorname{Ad} V|_B), \operatorname{ht}(\operatorname{Ad} W|_A)).$$

Proof. We have a short exact sequence

$$0\to\mathcal{K}(H)\otimes A\to C \xrightarrow{\pi} B\to 0 \tag{3.1}$$

with splitting $\varrho: B\to C$ given by $\varrho(b)=b\otimes 1$.

Let $k_1, ..., k_N\in\mathcal{K}(H)$, $a_1, ..., a_N\in A$ and $b_1, ..., b_N\in B$ be self-adjoint elements, each of norm at most 1, so that each k_i has finite rank. Let L be the sum of the ranks of $k_1, ..., k_N$. Let

$$\omega=\{k_i\otimes a_i+b_i\otimes 1: 1\leqslant i\leqslant N\}.$$

Fixing $\varepsilon>0$ and a positive integer n, let

$$\begin{aligned}\omega(n) &= \omega\cup(\operatorname{Ad} U)(\omega)\cup...\cup(\operatorname{Ad} U)^{n-1}(\omega) \\ &= \{1\}\cup\{V^rk_iV^{-r}\otimes W^ra_iW^{-r}+V^rb_iV^{-r}\otimes 1: 1\leqslant i\leqslant N,\, 0\leqslant r\leqslant n-1\}.\end{aligned}$$

Then the sum of the ranks of

$$k_1, ..., k_N, ..., V^{n-1}k_1V^{-(n-1)}, ..., V^{n-1}k_NV^{-(n-1)}$$

is at most nL, so there exists a projection P of rank nL so that $PV^rk_jV^{-r}=V^rk_jV^{-r}P=V^rk_jV^{-r}$ for all $j=1, ..., N$ and $r=0, ..., n-1$.

Consider the collection

$$F=\{b_1, ..., b_N\}\cup...\cup\{V^{n-1}b_1V^{-(n-1)}, ..., V^{n-1}b_NV^{-(n-1)}\}$$

of at most nN self-adjoint operators on $B(H)$. Let δ be as in Lemma 3.1 for the given value of ε and the short exact sequence (3.1). By applying Lemma 3.4 with these choices of δ, P, F, and $l=nL$, we find a positive, finite-rank $X\in\mathcal{K}(H)$ so that

(1) $XP=PX=P$, and hence $XV^rk_jV^{-r}=XPV^rk_jV^{-r}=PV^rk_jV^{-r}=V^rk_jV^{-r}=V^rk_jV^{-r}X$ for all $1\leqslant j\leqslant N$ and $0\leqslant r\leqslant n-1$,

(2) $\|[X, V^rb_jV^{-r}]\|\leqslant\delta$ for all $1\leqslant j\leqslant N$ and $0\leqslant r\leqslant n-1$,

(3) the rank of X is at most $nL\cdot(nN+1)^{2/\delta+1}$.

Let $e=X\otimes 1$. Then e satisfies the hypotheses of Lemma 3.1. Indeed,

$$\|[e, V^rb_iV^{-r}\otimes 1]\|=\|[X, V^rb_iV^{-r}]\|\leqslant\delta,\quad 1\leqslant i\leqslant N,\ 0\leqslant r\leqslant n-1,$$

and

$$eV^rk_iV^{-r}\otimes W^ra_iW^{-r}-V^rk_iV^{-r}\otimes W^ra_iW^{-r}=(XV^rk_iV^{-r}-V^rk_iV^{-r})\otimes W^ra_iW^{-r}=0$$

if $1\leqslant i\leqslant N$ and $0\leqslant r\leqslant n-1$, etc. Hence by Lemma 3.1, as long as $m>\varepsilon^{-1}$ we have

$$\mathrm{rcp}_C(\omega(n), 30\varepsilon)\leqslant\mathrm{rcp}_{\mathcal{K}(H)\otimes A}(e^{1/m}\omega(n)e^{1/m}, \varepsilon)+\mathrm{rcp}_B(\pi(\omega(n)), \varepsilon).$$

Note that

$$\begin{aligned}e^{1/m}\omega(n)e^{1/m}=\{X^{2/m}\otimes 1\}\cup\{&X^{1/m}V^rk_iV^{-r}X^{1/m}\otimes W^ra_iW^{-r}\\&+X^{1/m}V^rb_iV^{-r}X^{1/m}\otimes 1:1\leqslant i\leqslant N,\ 0\leqslant r\leqslant n-1\}.\end{aligned}$$

Hence setting

$$\begin{aligned}\omega_A(n)&=\{1\}\cup\{W^ra_iW^{-r}:1\leqslant i\leqslant N,\ 0\leqslant r\leqslant n-1\},\\ \omega_B(n)&=\{1\}\cup\{V^rb_iV^{-r}:1\leqslant i\leqslant N,\ 0\leqslant r\leqslant n-1\}\end{aligned}$$

we have

$$\mathrm{rcp}_{\mathcal{K}(H)\otimes A}(e^{1/n}\omega(n)e^{1/n}, \varepsilon)\leqslant\mathrm{rcp}_A\big(\omega_A(n), \tfrac{1}{2}\varepsilon\big)\cdot\mathrm{rank}(X).$$

On the other hand,

$$\mathrm{rcp}_B(\pi(\omega(n)),\varepsilon)=\mathrm{rcp}_B(\omega_B(n),\varepsilon).$$

Since the rank of X is at most $nL(nN+1)^{\delta/2+1}$ it follows that

$$\begin{aligned}\mathrm{rcp}(\omega(n),30\varepsilon)&\leqslant \mathrm{rcp}_A\big(\omega_A(n),\tfrac{1}{2}\varepsilon\big)nL(nN+1)^{2/\delta+1}+\mathrm{rcp}_B(\omega_B(n),\varepsilon)\\&\leqslant 2(nL+1)(nN+1)^{2/\delta+1}\max\big(\mathrm{rcp}_A(\omega_A(n),\varepsilon),\mathrm{rcp}_B(\omega_B(n),\varepsilon)\big),\end{aligned}$$

so that

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log\mathrm{rcp}(\omega(n),30\varepsilon)\\&\qquad\leqslant\max\Big(\limsup_{n\to\infty}\frac{1}{n}\log\mathrm{rcp}(\omega_A(n),\tfrac{1}{2}\varepsilon),\limsup_{n\to\infty}\frac{1}{n}\log\mathrm{rcp}(\omega_B(n),\varepsilon)\Big)\\&\qquad\leqslant\max(\mathrm{ht}(\mathrm{Ad}\,W|_A),\mathrm{ht}(\mathrm{Ad}\,V|_B)).\end{aligned}$$

Since C can be written as the closure of the linear span of elements of the form appearing in ω, the statement of the theorem follows. □

We shall record the following corollary, which will be the basis for entropy computations in this paper.

COROLLARY 3.6. *Let A be a C^*-algebra, $\alpha\in\mathrm{Aut}(A)$, $\pi: A\to B(H)$ be a faithful representation of A, and let $U\in U(H)$ be such that $\pi(\alpha(a))=U\pi(a)U^*$. Assume that $\pi(A)\cap K(H)=\{0\}$. Let $A_0=A$, $\alpha_0=\alpha$, $\pi_0=\pi$, $H_0=H$ and $U_0=U$. Recursively construct C^*-algebras A_n, $\alpha_n\in\mathrm{Aut}(A_n)$, $\pi_n: A_n\to B(H_n)$ and $U_n\in U(H_n)$ by setting*

$$\begin{aligned}H_n&=H_{n-1}\otimes H,\\A_n&=K(H_{n-1})\otimes\pi(A)+\pi_{n-1}(A_{n-1})\otimes\mathrm{Id}_H,\\\pi_n&=\textit{obvious representation on } H_n,\\U_n&=U_{n-1}\otimes U,\\\alpha_n&=\mathrm{Ad}\,U_n.\end{aligned}$$

View $A_{n-1}\subset A_n$ as $A_{n-1}\cong\pi_{n-1}(A_{n-1})\otimes\mathrm{Id}_H$. Let

$$A_\infty=\overline{\bigcup_n A_n},\quad \alpha_\infty=\varinjlim\alpha_n.$$

Then:

(i) $\mathrm{ht}(\alpha_\infty)=\mathrm{ht}(\alpha)$.

(ii) *If γ is an injective endomorphism of A_∞ so that $\gamma\circ\alpha_\infty=\alpha_\infty\circ\gamma$, denote by $\bar{\alpha}$ the obvious extension of α_∞ to $A_\infty\rtimes_\gamma\mathbf{N}$. Then $\mathrm{ht}(\bar{\alpha})=\mathrm{ht}(\alpha)$.*

Proof. Statement (ii) follows from statement (i) and the results of [20]. Hence it is sufficient to prove (i); for that one only needs to prove that $\mathrm{ht}(\alpha_n) \leqslant \mathrm{ht}(\alpha)$, in view of the behavior of entropy with respect to inductive limits. We now proceed by induction on n. Since $\alpha_0 = \alpha$, the statement is true for $n=0$. Applying Theorem 3.5 to $A_n = K(H_{n-1}) \otimes \pi(A) + \pi_{n-1}(A_{n-1}) \otimes \mathrm{Id}_H$ gives $\mathrm{ht}(\alpha_n) \leqslant \max(\mathrm{ht}(\alpha_{n-1}), \mathrm{ht}(\alpha))$, which is equal to $\mathrm{ht}(\alpha)$ by the induction hypothesis. □

4. Free products with the Toeplitz algebra

The main technical result of this section (which will be used to prove the more general result about entropy of amalgamated free products of automorphisms) is the following theorem.

THEOREM 4.1. *Let α be an automorphism of a D-probability space $(A, E: D \to A)$. Assume that D is finite-dimensional. Assume that the GNS representation associated to E is faithful. Let $\mathcal{T} \subset B(l^2)$ be the Toeplitz algebra generated by the unilateral shift $l(\delta_n) = \delta_{n+1}$ $(n \geqslant 1)$, and ψ be the vector state on $\mathcal{T}$ associated to $\delta_1 \in l^2$. Consider on the algebra $(A, E) *_D (D \otimes \mathcal{T}, \mathrm{id}_D \otimes \psi)$ the automorphism $\alpha * (\alpha|_D \otimes \mathrm{id})$. Then*

$$\mathrm{ht}(\alpha * (\alpha|_D \otimes \mathrm{id})) = \mathrm{ht}(\alpha).$$

Because of Corollary 2.5.2 and monotonicity of ht, it is sufficient to prove Theorem 4.1 in the particular case that $D = \mathbf{C}$. For convenience, we shall restate this particular case as

PROPOSITION 4.2. *Let A be a unital C^*-algebra and $\phi: A \to \mathbf{C}$ a state with a faithful GNS representation. Let $\alpha \in \mathrm{Aut}(A)$ be an automorphism so that $\phi \circ \alpha = \phi$. Consider the algebra $(A, \phi) * (\mathcal{T}, \psi)$. Then $\mathrm{ht}(\alpha * \mathrm{id}) = \mathrm{ht}(\alpha)$.*

Proof. Let $H = L^2(A, \phi)$ be the GNS Hilbert space associated to A, and let $1 \in H$ be the cyclic vector associated to ϕ. Let $\mathcal{O}_2$ be the Cuntz algebra on two generators [14]. Without loss of generality, by replacing A with $A \otimes \mathcal{O}_2$ and α with $\alpha \otimes \mathrm{id}$ we may assume that the GNS representation $\pi: A \to B(H)$ satisfies $\pi(A) \cap K(H) = \{0\}$. Let $U: H \to H$ be the unitary induced on $H = L^2(A, \phi)$ by α. We shall covariantly identify $((A, \phi) * (\mathcal{T}, \psi), \alpha * \mathrm{id})$ as the crossed product by a certain endomorphism γ of the algebra A_∞ described in Corollary 3.6, taken with the automorphism $\bar{\alpha}$. By Corollary 3.6, we then have $\mathrm{ht}(\alpha) = \mathrm{ht}(\bar{\alpha})$, which, in view of our identification, is the same as $\mathrm{ht}(\alpha * \mathrm{id})$, hence proving the proposition. The remainder of the proof is essentially a special case of the techniques used in [20], where more general Cuntz–Pimsner algebras were shown to have a crossed product structure.

Consider now the Hilbert space

$$F = L^2(A,\phi)\oplus \bigoplus_{n\geqslant 2} L^2(A,\phi)^{\otimes n}$$

and the representation $\varrho: A\to B(H)$ given by $\varrho=\pi\oplus(\bigoplus_{n\geqslant 2}\pi\otimes 1)$. Consider the isometry $l: F\to F$ defined by

$$l(\xi) = 1\otimes\xi,$$

where 1 denotes the image of the unit of A in $L^2(A,\phi)$, and $\xi\in F$. Denote by $U: L^2(A,\phi)\to L^2(A,\phi)$ the unitary implementing α. Denote by V the unitary $U\oplus\bigoplus_n U^{\otimes n}$ acting on the Hilbert space F. One sees that $\mathrm{Ad}_V(l)=l$ and $\mathrm{Ad}_V(\varrho(a))=\varrho(\alpha(a))$.

CLAIM 4.3. $(C^*(\varrho(A),l),\mathrm{Ad}_V)\cong((A,\phi)*(\mathcal{T},\psi),\alpha*\mathrm{id})$.

Proof. This actually follows from §2.5, since $C^*(\varrho(A),l)$ is isomorphic to the Cuntz–Pimsner algebra associated to the (A,A)-bimodule $L^2(A,\phi)\otimes A$. For the reader's convenience, we give a proof.

Let θ be the vector state on $B(F)$ associated to the vector $1\in L^2(A,\phi)\oplus 0\subset F$. Then $l^*1=0$, and one can easily verify that (i) $\theta(a_0 l \dots l a_n l^* a_{n+1}\dots l^* a_{n+m})=0$ for all $a_j\in\varrho(A)$, $n,m\geqslant 0$, $n+m>0$; and (ii) $l^*\varrho(a)l=\phi(a)$ for all $a\in A$. It follows from [35] that A and $C^*(l)$ are free in $(B(F),\theta)$. Since $C^*(l)$ is (obviously) isomorphic to $\mathcal{T}$, $\theta|_{C^*(l)}=\psi$, and since ϱ is injective and $\theta|_{\varrho(A)}=\phi$, the claim is proved. □

From now on, write $C=C^*(A,l)$.

Denote by C_n the closed linear span

$$C_n = \overline{\mathrm{span}}\{a_0 l a_1 \dots l a_m l^* a_{m+1} l^* \dots a_{2m} : m\leqslant n,\ a_0,\dots,a_{2m}\in\varrho(A)\}$$

(each monomial above has exactly m terms equal to l and the same number of terms equal to l^*). Note that because of the relation $l^*\varrho(a)l=\phi(a)$, $a\in A$, each C_n is a C^*-subalgebra of C. The action of $w=a_0 l a_1\dots l a_n l^* a_{n+1} l^*\dots a_{2n}\in C_k$ on a vector $\xi=\xi_1\otimes\dots\otimes\xi_r\in H$ can be described as

$$w\cdot\xi = \begin{cases} 0 & \text{if } n\geqslant r,\\ (a_0\otimes a_1\otimes\dots\otimes(a_n\cdot\xi_{n+1})\otimes\dots\otimes\xi_r)\prod_{j=1}^n\langle\xi_j, a_{2n-j+1}^*\rangle & \text{otherwise}\end{cases}$$

(here we identify $a_j\in\varrho(A)$ with $a_j\cdot 1\in L^2(A,\phi)\subset H$).

Denote by C_∞ the C^*-algebra $\overline{\bigcup_{n\geqslant 0} C_n}$. Then $lC_nl^*\subset C_{n+1}$, and hence $\gamma=l\cdot l^*$ determines an endomorphism of C_∞. It can be easily seen that C is isomorphic to the crossed product of C_∞ by this endomorphism (cf. the discussion after Proposition 1.2 in [20] for the definition). Moreover, Ad_V leaves C_∞ invariant and commutes with the endomorphism $l\cdot l^*$ ([20, Proposition 1.5]).

It remains to show that $(C_n, \mathrm{Ad}_V|_{C_n}) \cong (A_n, \alpha_n)$, where A_n and α_n are as in Corollary 3.6.

We proceed by induction. In the case that $n=0$, $C_0=A$, $\mathrm{Ad}_V|_{C_0}=\alpha$.

Let

$$F_n = L^2(A,\phi) \oplus \bigoplus_{k=2}^{n+1} L^2(A,\phi)^{\otimes k} \subset F.$$

Then F_n is invariant under the action of C_n on F. Consider the isomorphism W,

$$\begin{aligned} W: L^2(A,\phi) \oplus \bigoplus_{k\geqslant 2} L^2(A,\phi)^{\otimes k} \\ \to \Big(L^2(A,\phi) \oplus \bigoplus_{k=2}^{n+1} L^2(A,\phi)^{\otimes k}\Big) \otimes \Big(\mathbf{C} \oplus \bigoplus_{m\geqslant 1} L^2(A,\phi)^{\otimes m(n+1)}\Big) = F_n \otimes K_n. \end{aligned}$$

For each $a \in C_n$, $W^* a W: F_n \otimes K_n \to F_n \otimes K_n$ has the form $a|_{F_n} \otimes \mathrm{Id}_{K_n}$. It follows that the representation of C_n on F_n is faithful.

Denote by $B_n \subset C_n$ the ideal

$$B_n = \mathrm{span}\{a_0 l a_1 \dots l a_n l^* a_{n+1} l^* \dots a_{2n} : a_0, \dots, a_{2n} \in \varrho(A)\}.$$

Write

$$F_n = L^2(A,\phi) \oplus \bigoplus_{k=2}^{n+1} L^2(A,\phi)^{\otimes k} = F_{n-1} \oplus L^2(A,\phi)^{\otimes(n+1)}.$$

Then $B_n \cdot F_{n-1}=0$; hence the representation of B_n obtained by restricting its action to the space $L^2(A,\phi)^{\otimes(n+1)}$ is faithful. The action of $a_0 l a_1 \dots l a_n l^* a_{n+1} l^* \dots a_{2n} \in B_n$ on $\xi_1 \otimes \dots \otimes \xi_{n+1} \in L^2(A,\phi)^{\otimes(n+1)}$ can be explicitly written as

$$\begin{aligned} a_0 l a_1 \dots l a_n l^* a_{n+1} l^* \dots a_{2n} \cdot \xi_1 \otimes \dots \otimes \xi_n \otimes \xi_{n+1} \\ = \langle \xi_1, a_{2n}^* \rangle \dots \langle \xi_n, a_{n+1}^* \rangle a_0 \otimes \dots \otimes a_{n-1} \otimes (a_{n+1} \cdot \xi_{n+1}). \end{aligned}$$

Denote by $\theta(a_0, \dots, a_{n-1}, a_{n+1}, \dots, a_{2n}) \in B(L^2(A,\phi)^{\otimes n})$ the compact operator given by

$$\theta(a_0, \dots, a_{n-1}^*, a_{n+1}, \dots, a_{2n}) \xi_1 \otimes \dots \otimes \xi_n = \langle \xi_1, a_{2n}^* \rangle \dots \langle \xi_n, a_{n+1}^* \rangle a_0 \otimes \dots \otimes a_{n-1}.$$

Then the map

$$\begin{aligned} a_0 l a_1 \dots l a_n l^* a_{n+1} l^* \dots a_{2n} \mapsto \theta(a_0, \dots, a_{n-1}^*, a_{n+1}, \dots, a_{2n}) \otimes a_n \\ \in K(L^2(A,\phi)^{\otimes n}) \otimes A \subset B(L^2(A,\phi)^{\otimes n} \otimes L^2(A,\phi)) \end{aligned}$$

is a C^*-algebra isomorphism of B_n with $K(L^2(A,\phi)^{\otimes n}) \otimes A$.

For all $m \leqslant n+1$, the subspaces $L^2(A,\phi)^{\otimes m} \subset F_n$ are invariant under the action of C_n. Denote by ϱ_n the representation of C_n obtained by restricting its action on H to the space $L^2(A,\phi)^{\otimes(n+1)}$. The image $\varrho_n(A_{n-1})$ lies inside $B(L^2(A,\phi)^{\otimes(n-1)})\otimes A\otimes \mathrm{Id}_{L^2(A,\phi)}$. By our assumption, $\pi(A)\cap K(L^2(A,\phi))=\{0\}$. This implies that

$$\varrho_n(C_n)\cap\varrho_n(B_n)=\varrho_n(C_n)\cap K(L^2(A,\phi)^{\otimes n})\otimes A=\{0\}.$$

We claim that ϱ_n is faithful. Indeed, assume that it is not, and that some $0\neq a\in C_n$ is annihilated by ϱ. Write $a=a_{n-1}+b$, where $a_{n-1}\in C_{n-1}$ and $b\in B$. Since $\varrho_n(a)=\varrho_n(a_{n-1})+\varrho_n(b)=0$ and $\varrho_n(B_n)\cap\varrho_n(C_{n-1})=\{0\}$, it then follows that $\varrho_n(a_{n-1})=0$ and $\varrho_n(b)=0$. But for $a_{n-1}\in C_{n-1}$, $\varrho(a_{n-1})=\varrho_{n-1}(a_{n-1})\otimes\mathrm{Id}_{L^2(A,\phi)}$. Proceeding inductively, we see that ϱ_n is faithful.

We have therefore proved that

$$C_n=\varrho_{n-1}(A_{n-1})\otimes\mathrm{Id}_{L^2(A,\phi)}+K(L^2(A,\phi)^{\otimes(n-1)})\otimes\pi(A),$$

which means that $C_n\cong A_n$. Since $\mathrm{Ad}\,V_n|_{C_n}$ is given by $\mathrm{Ad}_{U^{\otimes(n-1)}\otimes U}$, it follows that the dynamical system $(C_n,\mathrm{Ad}\,V|_{C_n})\cong(A_n,\alpha_n)$. □

COROLLARY 4.4. *Let α be an automorphism of a non-commutative probability space $(A,E\colon A\to D)$, and assume that D is finite-dimensional. Assume that the GNS representation with respect to E is faithful. Let $\bar\alpha$ be the Bogolyubov automorphism of the Cuntz–Pimsner C^*-algebra described in §2.5. Then $\mathrm{ht}(\bar\alpha)=\mathrm{ht}(\alpha)$.*

It would be interesting to find a formula for the entropy of more general Bogolyubov automorphisms of Cuntz–Pimsner algebras. For example, we believe that the corollary above should hold for more general D.

5. Entropy for free products of automorphisms

We have now almost arrived at the main result of the paper (Theorem 5.7), calculating entropy of the free product of two automorphisms. First, we shall prove several technical lemmas which reduce the general case to the case of a free product with the Toeplitz algebra as in Theorem 4.1.

LEMMA 5.1. *Let $D\subset B\subset\mathcal{B}$ be C^*-subalgebras, and let $U\in\mathcal{B}$ be a unitary. Let E be a conditional expectation from $\mathcal{B}$ onto D. Assume that*

(i) *$C^*(D,U)$ and B are free with respect to E,*

(ii) *$[U,D]=0$ and $E(U)=E(U^*)=0$.*

Then the algebras B and UBU^* are free with respect to E. If in addition $E(U^k)=0$ for all $k\neq0$, then the algebras $\{U^kBU^{-k}\}_{k\in\mathbf{Z}}$ are free with respect to E.

Proof. Since U is free from B and commutes with D, we see that for all $b\in B$, $E(UbU^*)=E(U(b-E(b))U^*)+E(UE(b)U^*)=0+E(E(b)UU^*)=E(b)$. Now let $b_j\in B$ be such that $E(b_j)=0$. Then

$$E(b_0Ub_1U^*b_2Ub_3U^*\ldots)=0$$

because U and B are free with respect to E. But then B and UBU^* are free with respect to E, since for all j, $Ub_{2j+1}U^*\in UBU^*\cap\ker E$, and any element in $UBU^*\cap\ker E$ has the form UbU^* for some $b\in B\cap\ker E$.

The proof that $\{U^kBU^{-k}\}_k$ are free with respect to E proceeds along similar lines. $\square$

The following lemma is included for completeness. We will use the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) in the sequel.

LEMMA 5.2. *Let $D\subseteq B\subseteq A$ be unital inclusions of unital C^*-algebras, and suppose that there are conditional expectations $E_B^A: A\to B$ and $E_D^B: B\to D$. Let $E_D^A=E_D^B\circ E_B^A$: $A\to D$. Consider the following statements*:

(i) *The GNS representations associated to E_B^A and E_D^B are faithful.*

(ii) *The GNS representation associated to E_D^A is faithful.*

(iii) *The GNS representation associated to E_B^A is faithful.*

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *None of the reverse implications hold in general.*

Proof. We will denote by $a\to\hat{a}$ the defining map from A onto a dense subspace of $L^2(A,E_D^A)$, and similarly for the other L^2-spaces. We will also need the isomorphism $\pi: L^2(A,E_D^A)\to L^2(A,E_B^A)\otimes_B L^2(B,E_D^B)$ given by $\pi(\hat{a})=\hat{a}\otimes\hat{1}$.

(i) $\Rightarrow$ (ii): Let $a\in A$. Choose $a_1\in A$ so that $\widehat{aa_1}\in L^2(A,E_B^A)\backslash\{0\}$, i.e.,

$$E_B^A(a_1^*a^*aa_1)\neq 0.$$

There exists an element $b_1\in B$ such that

$$(E_B^A(a_1^*a^*aa_1))^{1/2}b_1\in L^2(B,E_D^B)\backslash\{0\},$$

i.e.,

$$0\neq E_D^B(b_1^*E_B^A(a_1^*a^*aa_1)b_1)=E_D^A(b_1^*a_1^*a^*aa_1b_1).$$

Thus $\widehat{aa_1b_1}\in L^2(A,E_D^A)\backslash\{0\}$.

(ii) $\Rightarrow$ (iii): Given $a\in A$ take $a_1\in A$ so that $E_D^A(a_1^*a^*aa_1)\neq 0$. Hence $E_B^A(a_1^*a^*aa_1)\neq 0$.

A counterexample to (ii) $\Rightarrow$ (i) is provided by taking $D=\mathbf{C}$, $B=\mathbf{C}\oplus\mathbf{C}$, $A=M_2(\mathbf{C})$, $E_B^A((c_{ij})_{1\leqslant i,j\leqslant 2})=c_{11}\oplus c_{22}$ and $E_D^B(z_1\oplus z_2)=z_1$.

A counterexample to (iii) $\Rightarrow$ (ii) is provided by taking D, B and E_D^B as in the above example, $A=M_2(\mathbf{C})\oplus M_2(\mathbf{C})$ and $E_B^A((c_{ij})_{1\leqslant i,j\leqslant 2}\oplus(d_{ij})_{1\leqslant i,j\leqslant 2})=c_{11}\oplus d_{11}$. $\square$

LEMMA 5.3. *Let $E_i: A_i \to D$, $i=1,2$, be D-probability spaces, with automorphisms α_i. Assume that the GNS representations associated to E_i are faithful, and that $\alpha_1|_D = \alpha_2|_D$. Let $E_1 \oplus E_2: A_1 \oplus A_2 \to D \oplus D$ be the obvious conditional expectation. Consider the algebra $M_2(D)$ of (2×2)-matrices over D, and view $D \oplus D \subset M_2(D)$ as diagonal matrices. Let $F: M_2(D) \to D \oplus D$ be the conditional expectation*

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \oplus d.$$

Then there exists an isomorphism

$$\phi: (A_1 \oplus A_2, E_1 \oplus E_2) *_{D\oplus D} (M_2(D), F) \cong ((A_1, E_1) *_D (A_2, E_2)) \otimes M_2(\mathbf{C})$$

so that ϕ intertwines the automorphisms $(\alpha_1 \oplus \alpha_2) * (\mathrm{id} \otimes \alpha_1|_D)$ and $(\alpha_1 * \alpha_2) \otimes \mathrm{id}$.

Proof. Consider in $M_2(D) \subseteq (A_1 \oplus A_2, E_1 \oplus E_2) *_{D\oplus D} (M_2(D), F)$ the unitary

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider the subalgebras

$$A_1' = \{a \oplus 0 : a \in A_1\},$$
$$A_2' = \{w(0 \oplus a)w^* : a \in A_2\}.$$

Since w is free from $A_1 \oplus A_2$ with amalgamation over $D \oplus D$, it is easily seen from the definition of freeness and Lemma 5.1 that the algebras A_1' and A_2' are also free with amalgamation over $D \oplus 0$. Furthermore, the restriction of $(E_1 \oplus E_2) * F$ to A_i' is E_i, and hence the GNS representation associated to each E_i is faithful. It follows from the embedding result (see §2.2) that A_1' and A_2' together generate the reduced free product $A_1 *_D A_2$. It is easily seen that the algebra $(A_1 \oplus A_2) *_{D\oplus D} (M_2(D))$ is generated by A_1', A_2' and $C^*(1_D \oplus 1_D, w) \cong M_2(\mathbf{C})$, and is isomorphic to $(A_1 *_D A_2) \otimes M_2(\mathbf{C})$ via the map

$$\phi: A_1' \ni a_1 \oplus 0 \mapsto \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \in A_1 \otimes M_2(\mathbf{C}), \quad a_1 \in A_1,$$

$$\phi: A_2' \ni w(0 \oplus a_2)w^* \mapsto \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} \in A_2 \otimes M_2(\mathbf{C}), \quad a_2 \in A_2,$$

$$\phi(w) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is clear that ϕ intertwines the automorphisms $(\alpha_1 \oplus \alpha_2) * \mathrm{id}$ and $(\alpha_1 * \alpha_2) \otimes 1$. □

In the following lemma, $\mathcal{O}_2$ will be the Cuntz algebra [14] generated by isometries S_1 and S_2 such that $S_1 S_1^* + S_2 S_2^* = 1$, and σ will denote the state on $\mathcal{O}_2$ satisfying

$$\sigma(S_{i_1} \dots S_{i_p} S_{j_q}^* \dots S_{j_1}^*) = \begin{cases} 2^{-p} & \text{if } p=q \text{ and } i_1 = j_1, \dots, i_p = j_p, \\ 0 & \text{otherwise.} \end{cases}$$

Note that σ is faithful.

LEMMA 5.4. *Let σ be the state on $\mathcal{O}_2$ as above. Let C be a commutative C^*-algebra having a faithful state ϱ and a unitary $u\in C$ such that $\varrho(u)=0$. Consider the algebra $B=((\mathbf{C}\oplus\mathbf{C})\otimes\mathcal{O}_2,\theta\otimes\sigma)*(C,\varrho)$, where $\theta(a\oplus b)=\frac{1}{2}(a+b)$. Then B is simple and purely infinite. Moreover, denoting by $E=E^B_{(\mathbf{C}\oplus\mathbf{C})\otimes\mathcal{O}_2}$ the conditional expectation from B onto $(\mathbf{C}\oplus\mathbf{C})\otimes\mathcal{O}_2$ arising from the free product construction, there exists a subalgebra $M\subset B$ so that*

(i) *$M\cong M_2(\mathbf{C})$ in such a way that the element $(a\oplus b)\otimes 1$ corresponds to the diagonal matrix*

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

(ii) *$E(M)\subset(\mathbf{C}\oplus\mathbf{C})\otimes 1$.*

Proof. Since $\mathcal{O}_2$ has trivial K-theory, it follows from Germain's exact sequence for free products (see §2.3) that $K_0(B)$ is zero. Once B is known to be simple and purely infinite, it will follow from Cuntz's fundamental results [15] that there exists a partial isometry $w\in B$ so that $ww^*=(1\oplus 0)\otimes 1$ and $w^*w=(0\oplus 1)\otimes 1$. Set $M=C^*(w)$. Then $M\cong M_2(\mathbf{C})$, in such a way that w corresponds to the partial isometry

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and therefore (i) is satisfied. To see that (ii) is satisfied, note that M is the linear span of w, w^*, ww^*, w^*w; hence it is sufficient to check that $E(w)$, $E(w^*w)$, $E(ww^*)$ and $E(w^*)=E(w)^*$ all lie in $(\mathbf{C}\oplus\mathbf{C})\otimes 1$. This is clearly true of $E(w^*w)$ and $E(ww^*)$, since ww^* and w^*w lie in $(\mathbf{C}\oplus\mathbf{C})\otimes 1$. Furthermore, since $(ww^*)E(w)=E(ww^*w)=E(w)w^*w)$, and the projections ww^* and w^*w are orthogonal, it follows that $ww^*E(w)ww^*$ and $w^*wE(w)w^*w$ are both zero. Since ww^* and w^*w lie in the center of $(\mathbf{C}\oplus\mathbf{C})\otimes\mathcal{O}_2$ and $ww^*+w^*w=1$, it follows that $E(w)=0$, and hence $E(w^*)=0$, so that (ii) is satisfied as well.

Note that $B\cong(\mathcal{O}_2\oplus\mathcal{O}_2,\sigma')*(C,\varrho)$, where $\sigma'=\frac{1}{2}(\sigma+\sigma)$. We shall now apply Theorem 3.1 of [18] to show that B is simple and purely infinite; the following five observations show that the hypotheses of this theorem are satisfied:

(1) σ' and ϱ are faithful states.

(2) Let $v=S_2S_2S_1^*\oplus 0\in\mathcal{O}_2\oplus\mathcal{O}_2$; then v is a partial isometry belonging to the spectral subspace of the state $\sigma'*\varrho$ corresponding to 2, i.e., $F(xv)=2F(vx)$ for all $x\in B$.

(3) Let $q_1=v^*v$ and $q_2=vv^*$; then $q_1\perp q_2$ and we have $F(q_1)=\frac{1}{4}$, $F(q_2)=\frac{1}{8}$.

(4) Consider

$$B_1=C^*(q_1,q_2,u)\subseteq B,$$
$$B_2=C^*(uq_1u^*,q_2)\subseteq B_1.$$

By Lemma 5.1, uq_1u^* and q_2 are free with respect to F, so we may write B_2 as the free product of two copies of $\mathbf{C}\oplus\mathbf{C}$,

$$B_2 \cong \Big(\underset{3/4}{\mathbf{C}} \oplus \overset{uq_1u^*}{\underset{1/4}{\mathbf{C}}} \Big) * \Big(\underset{7/8}{\mathbf{C}} \oplus \overset{q_2}{\underset{1/8}{\mathbf{C}}} \Big),$$

one containing minimal projection uq_qu^* which has trace $\frac{1}{4}$, and the other containing minimal projection q_2 which has trace $\frac{1}{8}$. It is then well known (cf. Proposition 2.7 of [16]) that

$$B_2 \cong \mathbf{C}\oplus C([0,1], M_2(\mathbf{C}))\oplus\mathbf{C}$$

with

$$uq_1u^* \sim 1\oplus\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\oplus 0,$$

$$q_2 \sim 0\oplus\begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}\oplus 0.$$

Therefore, q_2 is equivalent in B_1 to a subprojection of q_1, and $q_2B_1q_2$ contains the diffuse Abelian subalgebra $q_2B_2q_2$.

(5) The centralizer of σ' contains a diffuse Abelian subalgebra; hence by Proposition 3.2 of [16], B is simple, and thus q_1+q_2 is full in B.

The above facts allow us to apply Theorem 3.1 of [18], and we conclude that B is simple and purely infinite. □

LEMMA 5.5. *Let W be a unital C^*-algebra and let $E_W^X: X\to W$ and $E_W^Y: Y\to W$ be W-probability spaces, with automorphisms α_X and α_Y, respectively, which agree on W. Assume that $W\subset Z\subset X$ is an α_X-invariant subalgebra, and let E_W^Z be the restriction of E_W^X to Z. Assume that there exists an E_W^X-preserving α_X-invariant conditional expectation $E_Z^X: X\to Z$, and assume that the GNS representations associated to E_W^X, E_W^Z, E_Z^X and E_W^Y are faithful. Then there exists an isomorphism of the reduced free products*

$$X *_Z (Z *_W Y) \cong X *_W Y$$

which intertwines the automorphisms $\alpha_X * (\alpha_X|_Z * \alpha_Y)$ and $\alpha_X * \alpha_Y$.

Proof. Consider the free product conditional expectation $E_Z: X *_Z (Z *_W Y) \to Z$. The GNS representation associated to E_Z is faithful, by definition. Let $E_W = E_W^Z \circ E_Z$. By Lemma 5.2, E_W also gives rise to a faithful GNS representation. By Lemma 2.6 in [36], X and Y are free with respect to E_W. It follows from the assumptions and the embedding result (see §2.2) that the C^*-algebra generated by X and Y in the GNS representation of $X *_Z (Z *_W Y)$ associated to E_W is isomorphic to $X *_W Z$. The desired isomorphism follows. □

LEMMA 5.6. *Let $E_i: A_i \to D$, $i=1,2$, be D-probability spaces, with automorphisms α_i. Assume that the GNS representations associated to E_i are faithful, and that $\alpha_1|_D = \alpha_2|_D$.*

Let $E: A_1 \oplus A_2 \to D = \{d \oplus d : d \in D\} \subset A_1 \oplus A_2$ be given by

$$E(a_1 \oplus a_2) = \tfrac{1}{2}([E_1(a_1) + E_2(a_2)] \oplus [E_1(a_1) + E_2(a_2)]).$$

Let $\sigma: \mathcal{O}_2 \to \mathbf{C}$ be as in Lemma 5.4. Let C be any C^*-algebra with a state $\hat{\varrho}$, giving rise to a faithful GNS representation, and containing a unitary u, such that $\varrho = \hat{\varrho}|_{C^*(u)}$ is faithful and $\varrho(u) = 0$.

Then there exists an embedding

$$\lambda: (A_1, E_1) *_D (A_2, E_2) \to B = ((A_1 \oplus A_2) \otimes \mathcal{O}_2, E \otimes \sigma) *_D (C \otimes_{\min} D, \hat{\varrho} \otimes \mathrm{id})$$

so that λ intertwines $(\alpha_1 * \alpha_2)$ and $((\alpha_1 \oplus \alpha_2) \otimes \mathrm{id}) * (\mathrm{id} \otimes \alpha_1|_D)$.

Proof. Using the embedding result (see §2.2), we can reduce to the case $C = C^*(u)$ and $\hat{\varrho} = \varrho$, by replacing B with the algebra generated by $(A_1 \oplus A_2) \otimes \mathcal{O}_2$ and $C^*(u) \otimes D \subset C \otimes_{\min} D$.

We have the following sequence of covariant inclusions, justified below:

$$\begin{aligned} A_1 *_D A_2 &\hookrightarrow M_2 \otimes (A_1 *_D A_2) \\ &\cong M_2 \otimes D *_{D \oplus D} (A_1 \oplus A_2) \quad \text{(by Lemma 5.3)} \\ &\overset{(a)}{\hookrightarrow} ([\mathbf{C}^2 \otimes \mathcal{O}_2) * C] \otimes D) *_{\mathbf{C}^2 \otimes \mathcal{O}_2 \otimes D} [(A_1 \oplus A_2) \otimes \mathcal{O}_2] \\ &\cong ((\mathbf{C}^2 \otimes \mathcal{O}_2 \otimes D) *_D (C \otimes D)) *_{\mathbf{C}^2 \otimes \mathcal{O}_2 \otimes D} [(A_1 \oplus A_2) \otimes \mathcal{O}_2] \\ &\overset{(b)}{\cong} C \otimes D *_D [(A_1 \oplus A_2) \otimes \mathcal{O}_2]. \end{aligned}$$

Inclusion (a) is implied by Lemma 5.4, together with the embedding result (see §2.2).

Isomorphism (b) is implied by Lemma 5.5. □

THEOREM 5.7. *Let D be a finite-dimensional C^*-algebra, and let α_j be an automorphism of a D-probability space $(A_j, E_j: A_j \to D)$ ($j \in J$, J a finite or countably infinite set). Assume that the GNS representations associated to E_j ($j \in J$) are faithful. Assume that $\alpha_j|_D = \alpha_i|_D$ for all $i, j \in J$. Let $(A, E) = *_D((A_j, E_j), j \in J)$ and let $*_j \alpha_j$ denote the free product automorphism of A. Then*

$$\mathrm{ht}(*_j \alpha_j) = \sup_{j \in J}(\mathrm{ht}(\alpha_j)). \tag{5.1}$$

Proof. Because of the embedding result (see §2.2) and the behavior of entropy with respect to inductive limits, it suffices to prove the statement when $J = \{1, 2\}$ is a set with two elements.

By monotonicity of ht, the inequality $\geqslant$ in (5.1) is clear. Let $\mathcal{T}$ be the Toeplitz algebra generated by the unilateral shift l, and denote by $\hat{\varrho}$ the vacuum expectation on $\mathcal{T}$. Voiculescu [42] showed that l^*+l is a semicircular element with its spectrum an interval. Hence by functional calculus $C=C^*(l^*+l)$ contains a unitary u such that $\hat{\varrho}(u^k)=0$ for all $k\neq 0$. It is not difficult to see that C contains no non-zero compact operator, and thus $\varrho=\hat{\varrho}|_C$ is faithful. By Lemma 5.6 and Lemma 5.3, the amalgamated free product dynamical system $(A_1 *_D A_2, \alpha_1 * \alpha_2)$ can be covariantly embedded into (B, β), where

$$(B, F) = ((A_1 \oplus A_2) \otimes \mathcal{O}_2, \widetilde{E} \otimes \sigma) *_D (\mathcal{T} \otimes D, \hat{\varrho} \otimes \mathrm{id}_D)$$

and where

$$\beta = ((\alpha_1 \oplus \alpha_2) \otimes \mathrm{id}_{\mathcal{O}_2}) * (\mathrm{id}_{\mathcal{T}} \otimes (\alpha_1|_D)).$$

So by monotonicity again, $\mathrm{ht}(\alpha_1 * \alpha_2) \leqslant \mathrm{ht}(\beta)$. Using Theorem 4.1 we have

$$\mathrm{ht}(\beta) = \mathrm{ht}((\alpha_1 \oplus \alpha_2) \otimes \mathrm{id}_{\mathcal{O}_2}) = \mathrm{ht}(\alpha_1 \oplus \alpha_2) = \max(\mathrm{ht}(\alpha_1), \mathrm{ht}(\alpha_2)). \qquad \square$$

Remark 5.8. (1) The only reason that the hypotheses of Theorem 5.7 require finite dimensionality of D is because the proof appeals to Theorem 4.1. If one could prove a version of Theorem 4.1 for more general types of D, then Theorem 5.7 would be valid for those more general algebras D, with the same proof.

(2) Theorem 5.7 also holds for amalgamated free products of injective *endomorphisms* of non-commutative probability spaces. This can be seen by realizing an injective endomorphism as the restriction of an automorphism (see e.g. [20]), and then utilizing our result for automorphisms.

(3) One can actually prove a partial version of Theorem 5.7 with amalgamation taking place over an AF algebra. Assume that $E_1: A_1 \to D$ and $E_2: A_2 \to D$ are D-probability spaces, with automorphisms α_1, α_2, which restrict to the same automorphism on D. Assume that $D=\overline{\bigcup_i D^{(i)}}$, with $D^{(i)} \subset D^{(i+1)}$ finite-dimensional. Assume further that

$$A_j = \overline{\bigcup_i A_j^{(i)}}, \quad A_j^{(i)} \subset A_j^{(i+1)},$$

and that $E_j(A_j^{(i)})=D^{(i)}$ and $D^{(i)} \subset A_j^{(i)}$, $j=1,2$. Let $\alpha_1 * \alpha_2$ denote the free product automorphism on $A_1 *_D A_2$. Then $\mathrm{ht}(\alpha_1 * \alpha_2)=\max(\mathrm{ht}(\alpha_1), \mathrm{ht}(\alpha_2))$. This is due to the fact that under these assumptions, $A_1 *_D A_2$ is the direct limit of $A_1^{(i)} *_{D^{(i)}} A_2^{(i)}$.

6. Applications

6.1. *Shifts on infinite free products and twisted free permutations.* Let A be a unital C^*-algebra, D a finite-dimensional unital subalgebra, and let $E: A \to D$ be a conditional expectation with a faithful GNS representation. Let I be a non-empty set, and for every $i \in I$ let (A_i, E_i) be a copy of (A, E). Let

$$(B, F) = (*_D)_{i \in I}(A_i, E_i)$$

be the reduced amalgamated free product. Let $\sigma: I \to I$ be a bijection. The *free permutation* arising from σ is the automorphism β of B that permutes the copies of A inside B by sending A_i to $A_{\sigma(i)}$. (The *free shift* is the free permutation arising from the shift on $I = \mathbf{Z}$.) The following theorem is a generalization of results of Størmer [38], Brown and Choda [8] and Dykema [17], and a partial generalization of other results of Størmer [39]. It answers affirmatively in the case of amalgamation over a finite-dimensional C^*-algebra Question 11 of [17]:

THEOREM 6.1.1. *If A is an exact unital C^*-algebra and D is finite-dimensional then for every free permutation β of $B = (*_D)_{i \in I} A$ one has* $\operatorname{ht}(\beta) = 0$.

Proof. Consider the reduced group C^*-algebra $C_r^*(\mathbf{F}_{|I|})$ of the non-Abelian free group on $|I|$ generators, where $|I|$ is the cardinality of I. Let $(u_i)_{i \in I}$ be the unitary generators of $C_r^*(\mathbf{F}_{|I|})$ corresponding to the free generators of $\mathbf{F}_{|I|}$. Let $\sigma_* \in \operatorname{Aut}(C_r^*(\mathbf{F}_{|I|}))$ be the automorphism such that $\sigma_*(u_i) = u_{\sigma(i)}$. Then from [8] and [17], $\operatorname{ht}(\sigma_*) = 0$. Let

$$(\widetilde{B}, \widetilde{F}) = (C_r^*(\mathbf{F}_{|I|}) \otimes D, \tau \otimes \mathrm{id}_D) *_D (A, E),$$

where τ is the canonical tracial state on $C_r^*(\mathbf{F}_{|I|})$, and let

$$\tilde{\beta} = (\sigma_* \otimes \mathrm{id}_D) * \mathrm{id}_A \in \operatorname{Aut}(\widetilde{B}).$$

Then by Theorem 5.7, $\operatorname{ht}(\tilde{\beta}) = \operatorname{ht}(\sigma_*) = 0$. Let $B' = C^*(\bigcup_{i \in I} u_i A u_i^*) \subseteq \widetilde{B}$. Then $\widetilde{F}(u_i a u_i^*) = E(a)$ for all $a \in A$ and $i \in I$, and, by Lemma 5.1, the family $(u_i A u_i^*)_{i \in I}$ is free with respect to $\widetilde{F}$. It is not difficult to see, though somewhat tedious to write down in detail, that the inclusion representation of B' on $L^2(\widetilde{B}, \widetilde{F})$ is a multiple of the GNS representation, ϱ, of B' on $L^2(B', \widetilde{F}|_{B'})$. Indeed, one chooses vectors $\xi_i \in L^2(\widetilde{B}, \widetilde{F})$ such that $\bigcup_i B' \xi_i$ spans a dense subset of $L^2(\widetilde{B}, \widetilde{F})$ and so that the representation of B' on $B' \xi_i$ is equivalent to ϱ for all i. Hence there is an isomorphism $\pi: B \to B'$ sending the copy of A_i in B to $u_i A u_i^*$. We see that the automorphisms β and $\tilde{\beta}|_{B'}$ are conjugate via π. Hence, by monotonicity of ht we have $\operatorname{ht}(\beta) \leqslant \operatorname{ht}(\tilde{\beta}) = 0$. □

Remark 6.1.2. Just as in Remark 5.8 (3), it is possible to weaken the hypothesis of the preceding theorem by requiring that $(A, D, E: A \to D)$ is an inductive limit of $(A^{(i)}, D^{(i)}, E_i)$, with $D^{(i)}$ finite-dimensional.

Definition 6.1.3. Let D be a unital C^*-algebra, let I be a set, and for every $i \in I$ let (A_i, E_i) be a D-probability space such that the GNS representation associated to E_i is faithful. Let

$$(A, E) = (*_D)_{i \in I}(A_i, E_i)$$

be the reduced amalgamated free product. Let $\sigma: I \to I$ be a bijection such that for every $i \in I$ there is an isomorphism $\alpha_i: A_i \to A_{\sigma(i)}$ such that $\alpha_i(D) = D$ and $E_{\sigma(i)} \circ \alpha_i = E_i$. Assume that $\alpha_i|_D$ is the same for all i. Then there is an automorphism α of A sending the copy of A_i in A onto the copy of $A_{\sigma(i)}$ in A via α_i. An automorphism α arising in this way is called a *twisted free permutation* of A.

The next theorem generalizes Theorem 6.1.1 and also some results of [17].

THEOREM 6.1.4. *Suppose that D is finite-dimensional and $\alpha \in \mathrm{Aut}(A)$ is a twisted free permutation as described above. If the permutation σ of I has no cycles then $\mathrm{ht}(\alpha) = 0$. Otherwise, whenever c is a cycle of σ let $l(c)$ be the length of the cycle, let $i \in I$ be one of the elements moved by the cycle, and let $\beta_c \in \mathrm{Aut}(A_i)$ be the restriction of $\alpha^{l(c)}$ to the copy of A_i in A. (Note that β_c depends on i only up to conjugation.) Then*

$$\mathrm{ht}(\alpha) = \sup_c \frac{\mathrm{ht}(\beta_c)}{l(c)}, \tag{6.1}$$

where the supremum is over all cycles c of σ.

Proof. If σ has no cycles then α is (conjugate to) a free permutation, so $\mathrm{ht}(\alpha) = 0$ by Theorem 6.1.1. In general, by making a cycle decomposition of σ and using Theorem 5.7, we see that in order to prove (6.1) we may without loss of generality assume that σ itself is a cyclic permutation, $\sigma = c$, of a finite set I. However, then $\alpha^{l(c)} = *_{i \in I} \gamma_i$, where each $\gamma_i \in \mathrm{Aut}(A_i)$ is conjugate to β_c. Hence again applying Theorem 5.7 we have $\mathrm{ht}(\alpha) = \mathrm{ht}(\alpha^{|I|})/|I| = \mathrm{ht}(\beta_c)/l(c)$. $\square$

6.2. *The CNT variational principle.* In classical ergodic theory an important result connecting topological and measurable entropy is the variational principle.

Definition 6.2.1. Let (A, α) be a unital exact C^*-dynamical system. We say that (A, α) satisfies a CNT variational principle if

$$\mathrm{ht}(\alpha) = \sup_\phi h_\phi(\alpha),$$

where the supremum is taken over all α-invariant states on A, and $h_\phi(\alpha)$ denotes the CNT entropy of α with respect to ϕ (cf. [12]).

By [17, Proposition 9] we always have the inequality

$$\mathrm{ht}(\alpha) \geqslant \sup_\phi h_\phi(\alpha).$$

Not every C^*-dynamical system satisfies a CNT variational principle. In [29] an example of a highly non-asymptotically Abelian system was given for which $\mathrm{ht}(\alpha) \geqslant \frac{1}{2}\log 2$ while $h_\phi(\alpha)=0$ for the unique α-invariant (tracial) state. But, since $\mathrm{ht}(\cdot)$ agrees with classical topological entropy and $h_\phi(\cdot)$ agrees with classical Kolmogorov–Sinai entropy when A is Abelian, the classical variational principle says that if A is unital and Abelian then (A,α) always satisfies a CNT variational principle. The list of examples of noncommutative dynamical systems which satisfy a CNT variational principle is also rapidly growing (cf. [44, 4.7], [10, 4.6, 4.7], [7, 3.6, 3.7], [5], [34], [30]). Moreover, the class of C^*-dynamical systems which satisfy a CNT variational principle is closed under taking (minimal) tensor products (cf. [41, Lemma 3.4]) and crossed products by $\mathbf{Z}$ (cf. [7, Theorem 3.5]). Unfortunately it is *not* closed under taking quotients or subalgebras (even those with a conditional expectation onto them—simply take a direct sum of something Abelian with large entropy and the example from [29]) and it is not yet known what happens in extensions. However, we now show that it is also closed under taking reduced free products. The next theorem also gives lots of examples of non-asymptotically Abelian dynamical systems for which the CNT variational principle holds.

THEOREM 6.2.2. *Let $E_j: A_j \to D$, $j=1,2$, be non-commutative probability spaces with automorphisms α_1 and α_2. Assume that D is finite-dimensional, that $\alpha_1|_D=\alpha_2|_D$, and that E_j give rise to faithful GNS representations. If both (A_1,α_1) and (A_2,α_2) satisfy the CNT variational principle, then so does $(A_1 *_D A_2, \alpha_1 * \alpha_2)$.*

Proof. Assume without loss of generality that $\mathrm{ht}(\alpha_1) \geqslant \mathrm{ht}(\beta_1)$. Given $\varepsilon>0$ we can find an α_1-invariant state $\gamma \in S(A_1)$ such that $\mathrm{ht}(\alpha_1) \leqslant h_\gamma(\alpha_1)+\varepsilon$.

Let $E: A_1 * A_2 \to A_1$ be the canonical conditional expectation and define a state $\tilde{\gamma} = \gamma \circ E \in S(A_1 *_D A_2)$. Then one checks that $\tilde{\gamma}\circ(\alpha_1 * \alpha_2)=\tilde{\gamma}$ and $\tilde{\gamma}\circ E=\tilde{\gamma}$. But under these conditions, CNT entropy is also monotone (cf. [12, III.6]), and so $\mathrm{ht}(\alpha_1 * \alpha_2)=\mathrm{ht}(\alpha_1) \leqslant h_\gamma(\alpha_1)+\varepsilon \leqslant h_{\tilde{\gamma}}(\alpha_1 * \alpha_2)+\varepsilon$. Since ε was arbitrary, we are done. □

A particularly interesting class of dynamical systems for which one would like to have a CNT variational principle is those arising from automorphisms of discrete groups. For any discrete group G let $C_r^*(G)$ be the reduced group C^*-algebra and τ_G the canonical trace on $C_r^*(G)$. G is called exact if $C_r^*(G)$ is exact. If $\gamma: G \to G$ is a group automorphism then there is an induced automorphism $\widehat{\gamma}\in\mathrm{Aut}(C_r^*(G))$ such that $\tau_G\circ\widehat{\gamma}=\tau_G$. If G is Abelian then a classical theorem of Berg [2] implies that $\mathrm{ht}(\widehat{\gamma})=h_{\tau_G}(\widehat{\gamma})$.

THEOREM 6.2.3. *Let H be a finite group and let G_1 and G_2 be discrete exact groups having H as a common subgroup. Suppose that $\gamma_i\in\mathrm{Aut}(G_i)$ $(i=1,2)$ are automorphisms preserving H such that $\gamma_1|_H=\gamma_2|_H$. Let $\gamma_1*\gamma_2$ denote the resulting automorphism of the free product $G_1*_H G_2$ with amalgamation over H. If $\mathrm{ht}(\widehat{\gamma}_i)=h_{\tau_{G_i}}(\widehat{\gamma}_i)$ $(i=1,2)$, then*

$$\mathrm{ht}(\widehat{\gamma_1*\gamma_2})=h_{\tau_{G_1*_H G_2}}(\widehat{\gamma_1*\gamma_2}).$$

COROLLARY 6.2.4. *If G_1,G_2 are discrete exact groups with automorphisms $\gamma_i\in\mathrm{Aut}(G_i)$ $(i=1,2)$, and if $\mathrm{ht}(\widehat{\gamma}_i)=h_{\tau_{G_i}}(\widehat{\gamma}_i)$ $(i=1,2)$, then $\mathrm{ht}(\widehat{\gamma}_1*\widehat{\gamma}_2)=h_{\tau_{G_1*G_2}}(\widehat{\gamma}_1*\widehat{\gamma}_2)$.*

It would be interesting to know whether or not the above theorem or its corollary can be extended to the dual entropy defined in [9] as well.

6.3. *Entropy-preserving embeddings.* Kirchberg first proved that every separable exact C^*-algebra is isomorphic to a subalgebra of the Cuntz algebra on two generators (cf. [27]). In [7, Remark 2.3] it was asked whether or not one can always find a unital embedding $\varrho: A\rtimes_\alpha\mathbf{Z}\hookrightarrow\mathcal{O}_2$ such that $\mathrm{ht}_{\mathcal{O}_2}(\mathrm{Ad}\,\varrho(u))=\mathrm{ht}(\alpha)$, where $u\in A\rtimes_\alpha\mathbf{Z}$ is the implementing unitary. We now solve this problem affirmatively in the case that A is nuclear and there exists an α-invariant state $\phi\in S(A)$ with faithful GNS representation. We also show that if (A,α) is *any* nuclear C^*-dynamical system then there always exists an entropy-preserving covariant embedding into the Cuntz algebra on infinitely many generators. Since many C^*-algebras are stable under tensor products with $\mathcal{O}_\infty$ we thus get entropy-preserving embeddings into a very large class of C^*-algebras. It follows that the topological entropy invariant of all such algebras is $[0,\infty]$.

We begin with a simple proposition.

PROPOSITION 6.3.1. *Assume that A is a unital exact C^*-algebra, $\alpha\in\mathrm{Aut}(A)$ and there exists an α-invariant state $\phi\in S(A)$ with faithful GNS representation. Let B be unital and exact, let $\psi\in S(B)$ have faithful GNS representation, let $E: A\rtimes_\alpha\mathbf{Z}\to A$ be the canonical conditional expectation and let $u\in A\rtimes_\alpha\mathbf{Z}$ be the implementing unitary. Regarding u as a unitary in $(A\rtimes_\alpha\mathbf{Z},\phi\circ E)*(B,\psi)$ we have $\mathrm{ht}(\mathrm{Ad}\,u)=\mathrm{ht}(\alpha)$.*

Proof. Since $E: A\rtimes_\alpha\mathbf{Z}\to A$ is faithful, by Lemma 5.2 the GNS representation of $\phi\circ E$ is faithful.

Consider the C^*-algebra $(A,\phi)*(*_{\mathbf{Z}}(B,\psi))\rtimes_{\alpha*S}\mathbf{Z}$, where $S: *_{\mathbf{Z}}(B,\psi)\to *_{\mathbf{Z}}(B,\psi)$ is the free shift, and let $v\in(A,\phi)*(*_{\mathbf{Z}}(B,\psi))\rtimes_{\alpha*S}\mathbf{Z}$ be the implementing unitary. It is fairly easy to see (cf. the proof of [11, Claims 2 and 4]) that there exists a $*$-isomorphism

$$\varrho: (A\rtimes_\alpha\mathbf{Z},\phi\circ E)*(B,\psi)\to(A,\phi)*((*_{\mathbf{Z}}(B,\psi))\rtimes_{\alpha*S}\mathbf{Z})$$

such that $\varrho(u)=v$. Hence $\mathrm{ht}(\mathrm{Ad}\,u)=\mathrm{ht}(\mathrm{Ad}\,v)$. But [7, Theorem 3.5], Theorem 6.1.1 above and our main result imply that $\mathrm{ht}(\mathrm{Ad}\,v)=\mathrm{ht}(\alpha*S)=\mathrm{ht}(\alpha)$. □

THEOREM 6.3.2. *Let A be a unital, separable, nuclear C^*-algebra and $\alpha\in\mathrm{Aut}(A)$ be an automorphism such that there exists an α-invariant state $\phi\in S(A)$ with faithful GNS representation. If $\mathcal{O}_2$ denotes the Cuntz algebra on two generators then there exists a unital embedding $\varrho: A\rtimes_\alpha \mathbf{Z}\hookrightarrow\mathcal{O}_2$ such that $\mathrm{ht}_{\mathcal{O}_2}(\mathrm{Ad}\,\varrho(u))=\mathrm{ht}(\alpha)$, where $u\in A\rtimes_\alpha\mathbf{Z}$ is the implementing unitary.*

Proof. Replacing (A,α) with $(A\otimes C(\mathbf{T}),\alpha\otimes\mathrm{id}_{C(\mathbf{T})})$, if necessary, we may assume that there exists a Haar unitary in the centralizer of ϕ. In this setting, [16, Proposition 3.2] ensures that $(A\rtimes_\alpha\mathbf{Z},\phi\circ E)*(\mathcal{T},v)$ is a simple C^*-algebra, where $\mathcal{T}$ is the Toeplitz algebra and v is the vacuum state. Moreover, this free product is nuclear since it is isomorphic to a Cuntz–Pimsner algebra over $A\rtimes_\alpha\mathbf{Z}$. Thus Kirchberg's absorption theorem for $\mathcal{O}_2$ (i.e., $B\otimes\mathcal{O}_2\cong\mathcal{O}_2$ for any simple, separable, unital, nuclear C^*-algebra B, cf. [27]) together with the previous proposition implies the result. □

We now turn to covariant entropy-preserving embeddings into $\mathcal{O}_\infty$.

THEOREM 6.3.3. *Let A be any separable, nuclear C^*-algebra and $\alpha\in\mathrm{Aut}(A)$ be an automorphism. If $\mathcal{O}_\infty$ denotes the Cuntz algebra on infinitely many generators then there exists an automorphism $\beta\in\mathrm{Aut}(\mathcal{O}_\infty)$ and a non-unital $*$-monomorphism $\pi: A\to\mathcal{O}_\infty$ such that $\beta\circ\pi=\pi\circ\alpha$ and $\mathrm{ht}(\beta)=\mathrm{ht}(\alpha)$.*

Proof. By adding a unit to A and replacing A by its crossed product by α, we may assume that A is unital and α is inner. We may also assume that there exists an α-invariant state $\phi\in S(A)$ with faithful GNS representation. Indeed, it follows from Theorem 3.5 that the covariant embedding $(A,\alpha)\hookrightarrow(A\oplus A+\mathcal{K}(H\oplus H),\mathrm{Ad}(u\oplus 1_A))$, where we regard $A\subset B(H)$, and $u\in A$ is a unitary which implements α, is entropy-preserving. Now any vector state arising from the second copy of H will be $\mathrm{Ad}(u\oplus 1_A)$-invariant and have faithful GNS representation since it is a cyclic vector.

So, we assume that A is unital, α is inner and $\phi\in S(A)$ is α-invariant with faithful GNS representation. Consider the free product

$$B=((A\otimes C[0,1]\otimes\mathcal{O}_2\oplus C[0,1],\eta))*(\mathcal{T},v).$$

Here η is the average of Lebesgue measure on the second copy of $C[0,1]$ and the tensor product of ϕ, Lebesgue measure and an arbitrary faithful state on $\mathcal{O}_2$, and v is the vacuum state on the Toeplitz algebra $\mathcal{T}$. Since there is a Haar unitary in the centralizer of η (coming from the copies of $C[0,1]$), [16, Proposition 3.2] implies that B is simple. Moreover, B is nuclear being isomorphic to a Cuntz–Pimsner algebra over $A\otimes C[0,1]\otimes\mathcal{O}_2\oplus C[0,1]$ (cf. [20], [24]). Since $\mathcal{O}_2$ is KK-equivalent to zero, $A\otimes C[0,1]\otimes\mathcal{O}_2\oplus C[0,1]$ satisfies the Universal Coefficient Theorem. Since $\mathcal{T}$ is Type I it also satisfies the UCT, and hence, by a result of Germain [23] so does B. Moreover, it follows from Germain's six-term exact sequence for K-theory (see §2.3) that B has the K-theory of $\mathcal{O}_\infty$. Hence $B\otimes\mathcal{O}_\infty$ is a simple, unital, purely infinite, nuclear C^*-algebra which satisfies the UCT and has the K-theory of $\mathcal{O}_\infty$. So, by the classification results of Kirchberg [26] and Phillips [32], $B\otimes\mathcal{O}_\infty\cong\mathcal{O}_\infty$. By our main result we have that the automorphism $\gamma=(\alpha\otimes\mathrm{id}_{C[0,1]}\otimes\mathrm{id}_{\mathcal{O}_2}\oplus\mathrm{id}_{C[0,1]})*\mathrm{id}_{\mathcal{T}}\in\mathrm{Aut}(B)$ has the same entropy as α. Hence defining $\beta=\gamma\otimes\mathrm{id}_{\mathcal{O}_\infty}\in\mathrm{Aut}(B\otimes\mathcal{O}_\infty)=\mathrm{Aut}(\mathcal{O}_\infty)$ we get the desired entropy-preserving covariant embedding. □

Subquestion 6.3.4. Let $u\in A$ be a unitary and regard $\mathrm{Ad}\,u$ as an automorphism of $(A,\phi)*(B,\psi)$. As in Proposition 6.3.1, is it true that $\mathrm{ht}_{A*B}(\mathrm{Ad}\,u)=\mathrm{ht}(\mathrm{Ad}\,u|_A)$? Clearly, an affirmative answer to this question would imply that the automorphism in Theorem 6.3.3 can be taken to be inner.

6.4. *Possible values of entropy of all automorphisms of a C^*-algebra.*

Definition 6.4.1. If A is an exact C^*-algebra then put

$$TE(A)=\{t\in[0,\infty]:\text{there exists }\alpha\in\mathrm{Aut}(A)\text{ such that }\mathrm{ht}(\alpha)=t\}.$$

For A unital, we can also consider the set

$$TE_{\mathrm{Inn}}(A)=\{t\in[0,\infty]:\text{there exists a unitary }u\in A\text{ such that }\mathrm{ht}(\mathrm{Ad}\,u)=t\}.$$

The results of §6.3 imply, in particular, that $TE(\mathcal{O}_\infty)=[0,\infty]$ and $TE_{\mathrm{Inn}}(\mathcal{O}_2)=[0,\infty]$, since there are automorphisms of nuclear (in fact, Abelian) algebras with any prescribed entropy. (See [6] for some nice examples.) Clearly, if B contains a projection, then $TE(B\otimes A)\supset TE(A)$. If A and B are both unital, we have

$$TE_{\mathrm{Inn}}(A\otimes B)\supset TE_{\mathrm{Inn}}(A)\cup TE_{\mathrm{Inn}}(B).$$

If B is any nuclear simple separable purely infinite C^*-algebra then Kirchberg has shown that $B\otimes\mathcal{O}_\infty\cong B$ (see [27]). Recently Kirchberg and Rørdam introduced a class of non-simple purely infinite C^*-algebras (cf. [28]). Though it is not yet known whether all of their nuclear purely infinite C^*-algebras will absorb $\mathcal{O}_\infty$, many examples are known.

THEOREM 6.4.2. *Let B be any exact C^*-algebra which contains a projection. Then $TE(B\otimes\mathcal{O}_\infty)=[0,\infty]$. If B is unital then $TE_{\mathrm{Inn}}(B\otimes\mathcal{O}_2)=[0,\infty]$.*

In particular, if B is nuclear, simple, separable and purely infinite then $TE(B)=[0,\infty]$.

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N. P. Brown
Department of Mathematics
Pennsylvania State University
State College, PA 16802
U.S.A.
nbrown@math.psu.edu

K. Dykema
Department of Mathematics
Texas A&M University
College Station, TX 77843-3368
U.S.A.
kjd@tamu.edu

D. Shlyakhtenko
Department of Mathematics
University of California
Los Angeles, CA 90095
U.S.A.
shlyakht@math.ucla.edu

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