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Anderson localization for Schrödinger operators on $\mathbf{Z}^2$ with quasi-periodic potential

by

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1. Introduction

The study of spectral properties of the Schrödinger operator on $l^2(\mathbf{Z}^d)$

$$H = -\Delta + V, \tag{1.1}$$

where Δ is the discrete Laplacian on $\mathbf{Z}^d$ and V a potential, plays a central role in quantum mechanics. Starting with the seminal paper by P. Anderson [2], many works have been devoted to the study of families of operators with some kind of random potential. The best developed part of the theory deals with potentials given by identically distributed, independent random variables at different lattice sites. It is not our intention to present the long and rich history of this area. Rather, we merely would like to mention the fundamental work by Fröhlich and Spencer [17], which lead to a proof of localization in [16] in all dimensions for large disorder, see also Delyon–Lévy–Souillard [12] and Simon–Taylor–Wolff [24]. More recently, a simple proof of the Fröhlich–Spencer theorem was found by Aizenman and Molchanov [1], again for the case of i.i.d. potentials. A central open problem in the random case is to show that localization occurs for any disorder in two dimensions, whereas in three and higher dimensions it is believed that there is a.c. spectrum for small disorders. Basic references in this field that cover the history roughly up to 1991 are Figotin–Pastur [15] and Carmona–Lacroix [11]. Some of the more recent literature is cited in [19]. Another case that has attracted considerable attention are quasi-periodic potentials. In the one-dimensional case Sinai [25] and Fröhlich–Spencer–Wittwer [18] have shown that one has pure point spectrum and exponentially decaying eigenfunctions for large disorder provided the potential is cosine-like

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and the frequency is Diophantine. In this paper we show that for potentials V of the form

$$V(n_1,n_2)=\lambda v(\theta_1+n_1\omega_1,\theta_2+n_2\omega_2), \tag{1.2}$$

where v is a real-analytic function on $\mathbf{T}^2$ which is nonconstant on any horizontal or vertical line, and λ is large, Anderson localization takes place for every $(\theta_1,\theta_2)\in\mathbf{T}^2$ provided the frequency vector $\underline{\omega}$ is restricted suitably. More precisely, for every $\varepsilon>0$, any $\lambda\geqslant\lambda_0(\varepsilon,v)$ and any $\underline{\theta}\in\mathbf{T}^2$ there exists $\mathcal{F}_\varepsilon\subset\mathbf{T}^2$ depending on $\underline{\theta}$ and λ so that $\operatorname{mes}(\mathbf{T}^2\setminus\mathcal{F}_\varepsilon)<\varepsilon$ and such that for any $\underline{\omega}\in\mathcal{F}_\varepsilon$ the operator with potential (1.2) has pure point spectrum and exponentially decaying eigenfunctions, see Theorem 6.2 below. At a lecture at the Institute for Advanced Study, H. Eliasson [14] has announced that this result can be obtained by means of a perturbative technique similar to [13]. In this paper we show that one can use basically nonperturbative methods similar in spirit to those in Bourgain–Goldstein [7] and Bourgain–Goldstein–Schlag [8]. The requirement of large λ is needed to insure that a certain inductive assumption holds. As in the aforementioned works, semi-algebraic sets also play a crucial role in this paper. In fact, we apply various recent results from the theory of those sets which are collected in §7. Another aspect of our work is the use of subharmonic functions. This basically replaces the Weierstrass preparation theorem which usually appears in perturbative proofs.

Finally, we would like to mention Bourgain–Jitomirskaya [9], where the case of a strip in $\mathbf{Z}^2$ with quasi-periodic potentials on each horizontal line is treated. The methods there, however, do not directly apply here.

We now proceed to give a brief overview of the proof. Suppose that there is a basis $\{\psi_j\}_{j=1}^\infty$ of l^2-normalized, exponentially decaying eigenfunctions of $H_{\underline{\omega}}(\underline{\theta})$ for some $\underline{\omega}$. More precisely, suppose that for all large squares $\Lambda\subset\mathbf{Z}^2$ centered at the origin of side length N there is a basis $\{\psi_j\}_{j=1}^{N^2}$ of eigenfunctions of $H_{\underline{\omega}}(\underline{\theta})\restriction\Lambda$ with Dirichlet boundary conditions on $\partial\Lambda$ so that for every j there is n_j so that

$$|\psi_j(n)|\leqslant C\exp(-\gamma|n-n_j|)\quad\text{for all } n\in\mathbf{Z}^2.$$

Here $\gamma>0$ is some fixed constant. Then the Green's function

$$G_{\underline{\omega}}^\Lambda(\underline{\theta},E)(n,m):=[(H_{\underline{\omega}}(\underline{\theta})-E)\restriction\Lambda]^{-1}(n,m)=\sum_j\frac{\psi_j(n)\psi_j(m)}{E_j-E}$$

satisfies

$$|G_{\underline{\omega}}^\Lambda(\underline{\theta},E)(n,m)|\leqslant C\exp\bigl(-\tfrac14\gamma|n-m|\bigr)\quad\text{for every } n,m\in\Lambda,\ |n-m|\geqslant\tfrac12N,$$

provided $\|G_{\underline{\omega}}^{\Lambda}(\underline{\theta},E)\|\leqslant e^{N^b}$ where $b<1$ and N large. This suggests the following terminology: We call a Green's function $G_{\underline{\omega}}^{\Lambda}(\underline{\theta},E)$ *good* if

$$\|G_{\underline{\omega}}^{\Lambda}(\underline{\theta},E)\|\leqslant e^{N^b},$$
$$|G_{\underline{\omega}}^{\Lambda}(\underline{\theta},E)(n,m)|\leqslant C\exp\left(-\tfrac{1}{4}\gamma|n-m|\right)\quad\text{for every } n,m\in\Lambda,\ |n-m|\geqslant\tfrac{1}{2}N,$$

and *bad* otherwise. §§ 2, 3, 4 below are devoted to establishing *large deviation theorems* for the Green's functions. This means that we show that for a *fixed* energy E and suitably restricted $\underline{\omega}$ a given Green's function $G_{\underline{\omega}}^{\Lambda}(\underline{\theta},E)$ satisfies

$$\operatorname{mes}[\underline{\theta}\in\mathbf{T}^2: G_{\underline{\omega}}^{\Lambda}(\underline{\theta},E)\text{ is bad}]<e^{-(\operatorname{diam}\Lambda)^{\sigma}}\tag{1.3}$$

for some constant $\sigma>0$. This large deviation estimate is the first crucial ingredient in the proof, the second being the method of energy elimination via semi-algebraic sets, which is presented in §5. It is easy to see that for a fixed side length N_0 of Λ the estimate (1.3) holds provided $\lambda\geqslant\lambda_0(N_0)$ (λ is as in (1.2)). This is precisely the origin of our assumption of large λ, and nowhere else does one need large λ in the proof. For larger scales $N\geqslant N_0$, (1.3) is proved inductively. Thus assume that (1.3) is known for N and we want to prove it for $N_1=N^C$, where C is some large constant (it turns out that this is precisely the way in which the scales increase). Partition a square Λ of side length N_1 into smaller squares $\{\Lambda_j\}$ of length N, and mark each such small square as either good or bad, depending on whether or not $G_{\underline{\omega}}^{\Lambda_j}(\underline{\theta},E)$ is good or bad. Since shifts by integer vectors (n_1,n_2) on $\mathbf{Z}^2$ correspond to shifts by $(n_1\omega_1,n_2\omega_2)$ on $\mathbf{T}^2$, it follows that the number of bad cubes is bounded by

$$\#\{(n_1,n_2)\in[-N_1,N_1]^2:(n_1\omega_1,n_2\omega_2)\in\mathcal{B}_{N,\underline{\omega}}(E)\},\tag{1.4}$$

where $\mathcal{B}_{N,\underline{\omega}}(E):=\{\underline{\theta}\in\mathbf{T}^2:G_{\underline{\omega}}^{\Lambda_0}(\underline{\theta},E)\text{ is bad}\}$, Λ_0 being a square centered at zero of side length N. The entire proof hinges on nontrivial estimates for the cardinality in (1.4). More precisely, one needs to prove that there is some $\delta>0$ so that $(1.4)<N_1^{1-\delta}$ for most $\underline{\omega}$. This is relevant for several reasons. One being that the usual "multi-scale analysis", i.e., repeated applications of the resolvent identity, fails if there is a chain of bad squares connecting two points in Λ. Clearly, such a chain might exist if $(1.4)\asymp N_1$. On the other hand, the entire §2 is devoted to showing that a sublinear bound $N_1^{1-\delta}$ is sufficient in order to obtain the desired off-diagonal decay of the Green's function on scale N_1 provided the energy E is separated from the spectra of all submatrices of intermediate sizes, see Lemma 2.4 and in particular (2.8) for a precise statement. Another, perhaps more crucial reason is of an analytical nature as can be seen from Lemma 4.4. That lemma is the central analytical result in this paper. It shows how to use bounds for subharmonic functions in order to treat the typical "resonance" problems that appear when one tries

to invert large matrices. This is in contrast to the usual KAM-type approach that is based on the Weierstrass preparation theorem. More precisely, one splits the N_1-square Λ into

$$\Lambda = \Lambda_* \cup \Lambda_*^c,$$

where $\Lambda_* = \bigcup_j \Lambda_j$, the union being over all bad squares. If $(1.4) < N_1^{1-\delta}$, then $|\Lambda_*| \leqslant N_1^{1-\delta} N^2 \leqslant N_1^{1-\delta/2}$ provided C was chosen large enough (recall $N_1 = N^C$). This relatively small size of Λ_* allows one to treat the "resonant sites" as a "black box". In fact, it translates into a sublinear bound (in N_1) for the Riesz mass of the subharmonic function $\log|\det A(\underline{\theta})|$ that controls the invertibility of $(H_{\underline{\omega}}(\underline{\theta}) - E) \restriction \Lambda$, see (4.19) and Lemma 4.8.

All of §3 is devoted to establishing a sublinear bound on (1.4). This section is entirely arithmetic, being devoted to finding a large set of $\underline{\omega} \in \mathbf{T}^2$ that have the desired property. It turns out that this set can be characterized as being those $\underline{\omega} = (\omega_1, \omega_2)$ for which the lattice

$$\{(n_1\omega_1, n_2\omega_2) \ (\mathrm{mod}\ \mathbf{Z}^2) : |n_1|, |n_2| \leqslant N_1\} \tag{1.5}$$

does not contain too many small nontrivial triangles of too small area. This is carried out in Lemma 3.1. Lemma 3.3 is the central result of §3. It states that the set of $\underline{\omega}$ that was singled out in Lemma 3.1 has the property that no algebraic curve of relatively small degree has more than $N_1^{1-\delta}$ many points from (1.5) coming too close to it. It is essential to realize that the set of $\underline{\omega}$ that needs to be excluded for this purpose does not depend on the algebraic curve under consideration, but is defined a priori. The logic of the proof of Lemma 3.3 is that too many points close to the curve would force that curve to oscillate more than it can, given its small degree. The oscillations are due to the fact that the curve would need to pass close to the vertices of triangles with comparatively large areas.

Returning to the actual proof of localization, recall that by the Shnol–Simon theorem, [22] and [23], the spectrum of $H_{\underline{\omega}}(0) = -\Delta + \lambda v(n_1\omega_1, n_2\omega_2)$ is characterized as those numbers E for which a nonzero, polynomially bounded solution exists, i.e., there is a nonzero function ξ on $\mathbf{Z}^2$ satisfying

$$(H_{\underline{\omega}}(0) - E)\xi = 0 \quad \text{and} \quad |\xi(\underline{x})| \lesssim 1 + |\underline{x}|^{c_0} \ \text{ for all } \underline{x} \in \mathbf{Z}^2,$$

where $c_0 > 0$ is some constant. The goal is to show that ξ decays exponentially. The key to doing so is to show that "double resonances" occur with small probability. More precisely, given two disjoint squares Λ_0 and Λ_1 of sizes N_0 and N_1 respectively, one says that a "double resonance" occurs if both

$$\|G_{\underline{\omega}}^{\Lambda_0}(0, E)\| > e^{N_0^c} \quad \text{and} \quad G_{\underline{\omega}}^{\Lambda_1}(0, E) \ \text{is bad.} \tag{1.6}$$

Here N_0 will be much larger than N_1 (some power of it), and c is a small constant. The proof of localization easily reduces to showing that (this is the approach from [7]) such double resonances do not occur for any such Λ_0 centered at the origin and any Λ_1 that is at a distance between $\overline{N}$ and $2\overline{N}$ from Λ_0. Here $\overline{N}$ is very large compared to N_0. To achieve this property one needs to remove a certain bad set of $\underline{\omega}\in\mathbf{T}^2$ whose size is ultimately seen to be very small as a result of the large deviation estimate (1.3). However, this reduction to (1.3) is nontrivial, and requires the "elimination of the energy" which is accomplished as a result of complexity bounds on semi-algebraic sets. The main result in that direction is Proposition 5.1 in §5 whose meaning should become clear when compared to the goal of preventing (1.6) (recall that shifts in $\mathbf{Z}^2$ correspond to shifts on $\mathbf{T}^2$). The set $\mathfrak{F}_K$ is precisely the set of bad $\underline{\omega}$ that needs to be removed, whereas conditions (5.1) are guaranteed by the large deviation estimates. The details of this reduction can be found in §6. Finally, we would like to mention that results on semi-algebraic sets are collected in §7.

2. Exponential decay of the Green's function via the resolvent identity

In this section, we consider a general operator

$$H=-\Delta+V \quad \text{on } l^2(\mathbf{Z}^2),$$

where V is an arbitrary potential indexed by lattice points $(n_1,n_2)\in\mathbf{Z}^2$. For any subset $\Lambda\subset\mathbf{Z}^2$ the restriction operator on Λ will be denoted throughout this paper by R_Λ, and

$$H_\Lambda := R_\Lambda H R_\Lambda$$

is the restriction of H to Λ. If Λ is a square, for example, then H_Λ is the same as H on Λ with Dirichlet boundary conditions. The main purpose of this section is to establish exponential off-diagonal decay of the Green's function

$$G_\Lambda(E) := (H_\Lambda - E)^{-1}$$

for certain regions Λ that do not contain too many bad subregions of a smaller scale. Here *bad* simply means that the Green's function on the smaller region does not exhibit exponential decay. The precise meaning of "too many" and "region" is given in Definition 2.1 and Lemma 2.4 below.

Definition 2.1. The distance between the points $\underline{x}=(x_1,x_2)\in\mathbf{Z}^2$ and $\underline{y}=(y_1,y_2)\in\mathbf{Z}^2$ is defined as

$$|\underline{x}-\underline{y}| = \max(|x_1-y_1|,|x_2-y_2|).$$

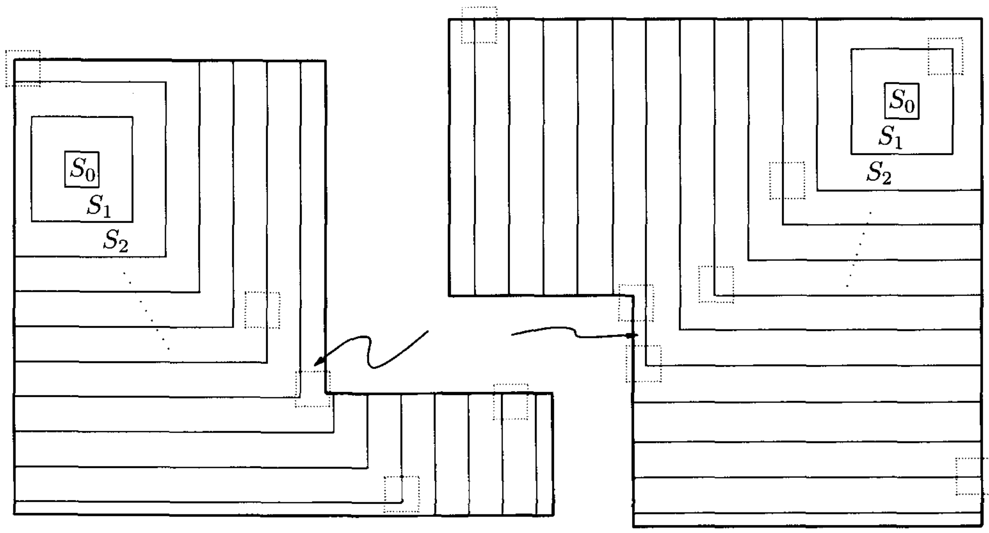

Fig. 1. Some examples of exhaustions of elementary regions.

The *M-square* centered at the point $\underline{x}=(x_1,x_2)\in\mathbf{Z}^2$ is the set

$$\begin{aligned} Q_M(\underline{x}) &:= \{\underline{y}\in\mathbf{Z}^2 : x_1-M\leqslant y_1\leqslant x_1+M,\ x_2-M\leqslant y_2\leqslant x_2+M\} \\ &= \{\underline{y}\in\mathbf{Z}^2 : |\underline{x}-\underline{y}|\leqslant M\}. \end{aligned} \tag{2.1}$$

An *elementary region* is defined to be a set Λ of the form

$$\Lambda := R\setminus(R+\underline{z}),$$

where $\underline{z}\in\mathbf{Z}^2$ is arbitrary and R is a rectangle

$$R = \{\underline{y}\in\mathbf{Z}^2 : x_1-M_1\leqslant y_1\leqslant x_1+M_1,\ x_2-M_2\leqslant y_2\leqslant x_2+M_2\}.$$

The *size* of Λ, denoted by $\sigma(\Lambda)$, is simply its diameter. The set of all elementary regions of size M will be denoted by $\mathcal{ER}(M)$. Elements of $\mathcal{ER}(M)$ are also referred to as M-regions.

The class of elementary regions consists of rectangles, L-shaped regions, and horizontal or vertical line segments. In what follows, we shall repeatedly apply the resolvent identity to the Green's functions $(H_{\Lambda_0}-E)^{-1}$ and $(H_{\Lambda_1}-E)^{-1}$, where $\Lambda_1\subset\Lambda_0$ are elementary regions. In fact, in the proof of the following lemma we shall establish exponential decay of the Green's function in some large region Λ_0, given suitable bounds on the Green's functions on smaller scales. This will require surrounding a given point in Λ_0

by a sequence of increasing regions inside Λ_0. More precisely, we consider *exhaustions* $\{S_j(\underline{x})\}_{j=0}^{l}$ *of* Λ_0 *of width* $2M$ *centered at* $\underline{x}$ defined inductively as follows:

$$\begin{aligned} S_0(\underline{x}) &:= Q_M(\underline{x})\cap\Lambda_0, \\ S_j(\underline{x}) &:= \bigcup_{\underline{y}\in S_{j-1}(\underline{x})} Q_{2M}(\underline{y})\cap\Lambda_0 \quad \text{for } 1\leqslant j\leqslant l, \end{aligned} \tag{2.2}$$

where l is maximal such that $S_l(\underline{x})\neq\Lambda_0$. Two examples of such exhaustions are given in Figure 1. It is clear that the sets S_j form an increasing sequence of elementary regions. Of particular importance to us are the "annuli" $A_j(\underline{x})=S_j(\underline{x})\setminus S_{j-1}(\underline{x})$, where $S_{-1}:=\varnothing$. With the possible exception of a single annulus, any $A_j(\underline{x})$ has the property that $Q_M(\underline{y})\cap A_j(\underline{x})$ is an elementary region for all $\underline{y}\in A_j(\underline{x})$. We have indicated this by means of the small dotted squares in Figure 1. Notice that in the left-hand region the square marked by an arrow does not lead to an elementary region. Thus, the aforementioned exceptional annulus is the one that contains the unique corner of Λ_0 that lies in the interior of the convex hull of Λ_0. See the annuli that are marked with arrows in Figure 1. Finally, we shall also need the fact that squares $Q_M(\underline{y}_1)$ and $Q_M(\underline{y}_2)$ with centers in nonadjacent annuli are disjoint (recall that the width of the annuli is $2M$).

The following lemma is a standard fact that will be used repeatedly.

LEMMA 2.2. *Suppose that* $\Lambda\subset\mathbf{Z}^2$ *is an arbitrary set with the following property: for every* $\underline{x}\in\mathbf{Z}^2$ *there is a subset* $W(\underline{x})\subset\Lambda$ *with* $\underline{x}\in W(\underline{x})$, $\operatorname{diam}(W(\underline{x}))\leqslant N$, *and such that the Green's function* $G_{W(\underline{x})}(E)$ *satisfies for certain* $t, N, A>0$

$$\|G_{W(\underline{x})}(E)\| < A, \tag{2.3}$$

$$|G_{W(\underline{x})}(E)(\underline{x},\underline{y})| < e^{-tN} \quad \textit{for all } \underline{y}\in\partial_* W(\underline{x}). \tag{2.4}$$

Here $\partial_* W(\underline{x})$ *is the* interior boundary *of* $W(\underline{x})$ *relative to* Λ *given by*

$$\partial_* W(\underline{x}) := \{\underline{y}\in W(\underline{x}) : \textit{there exists } \underline{z}\in\Lambda\setminus W(\underline{x}) \textit{ with } |\underline{z}-\underline{y}|=1\}. \tag{2.5}$$

Then

$$\|G_\Lambda(E)\| < 2N^2 A$$

provided $4N^2e^{-tN}\leqslant\frac{1}{2}$.

Proof. Let $\varepsilon>0$ be arbitrary. By the resolvent identity

$$G_\Lambda(E+i\varepsilon)(\underline{x},\underline{y}) = G_{W(\underline{x})}(E+i\varepsilon)(\underline{x},\underline{y}) + \sum_{\substack{\underline{z}\in W(\underline{x}) \\ \underline{z}'\in\Lambda\setminus W(\underline{x}) \\ |\underline{z}-\underline{z}'|=1}} G_{W(\underline{x})}(E+i\varepsilon)(\underline{x},\underline{z})\, G_\Lambda(E+i\varepsilon)(\underline{z}',\underline{y}).$$

Summing over $\underline{y}\in\Lambda$ yields

$$\sup_{\underline{x}\in\Lambda}\sum_{\underline{y}\in\Lambda}|G_\Lambda(E+i\varepsilon)(\underline{x},\underline{y})|\leqslant\sup_{\underline{x}\in\Lambda}\sum_{\underline{y}\in W(\underline{x})}\|G_{W(\underline{x})}(E+i\varepsilon)\| \\ +\sup_{\underline{x}\in\Lambda}\sum_{\substack{\underline{z}\in W(\underline{x})\\ \underline{z}'\in\Lambda\setminus W(\underline{x})\\ |\underline{z}-\underline{z}'|=1}}|G_{W(\underline{x})}(E+i\varepsilon)(\underline{x},\underline{z})|\sup_{\underline{w}\in\Lambda}\sum_{\underline{y}\in\Lambda}|G_\Lambda(E+i\varepsilon)(\underline{w},\underline{y})|.$$

In view of (2.3) and (2.4) one obtains

$$\sup_{\underline{x}\in\Lambda}\sum_{\underline{y}\in\Lambda}|G_\Lambda(E+i0)(\underline{x},\underline{y})|\leqslant N^2A+4N^2e^{-tN}\sup_{\underline{w}\in\Lambda}\sum_{\underline{y}\in\Lambda}|G_\Lambda(E+i0)(\underline{w},\underline{y})|. \tag{2.6}$$

By self-adjointness, the left-hand side of (2.6) is an upper bound on $G_\Lambda(E)$. Hence the lemma follows from Schur's lemma. □

The following lemma is the main result of this section. First we introduce some useful notation.

Definition 2.3. For any positive numbers a,b the notation $a\lesssim b$ means $Ca\leqslant b$ for some constant $C>0$. By $a\ll b$ we mean that the constant C is very large. If both $a\lesssim b$ and $a\gtrsim b$, then we write $a\asymp b$. The various constants will be defined by the context in which they arise. Finally, $N^{\alpha-}$ means $N^{\alpha-\varepsilon}$ with some small $\varepsilon>0$ (the precise meaning of "small" can again be derived from the context).

LEMMA 2.4. *Suppose that M,N are positive integers such that for some $0<\tau<1$*

$$N^\tau\leqslant M\leqslant 2N^\tau. \tag{2.7}$$

Let $\Lambda_0\in\mathcal{ER}(N)$ be an elementary region of size N with the property that for all $\Lambda\subset\Lambda_0$, $\Lambda\in\mathcal{ER}(L)$ with $M\leqslant L\leqslant N$, the Green's function $G_\Lambda(E):=(H_\Lambda-E)^{-1}$ of Λ at energy E satisfies

$$\|G_\Lambda(E)\|\leqslant e^{L^b} \tag{2.8}$$

for some fixed $0<b<1$. We say that $\Lambda\in\mathcal{ER}(L)$, $\Lambda\subset\Lambda_0$ is good, *if in addition to (2.8) the Green's function exhibits the off-diagonal decay*

$$|G_\Lambda(E)(\underline{x},\underline{y})|\leqslant e^{-\gamma|\underline{x}-\underline{y}|}\quad \text{for all } \underline{x},\underline{y}\in\Lambda,\ |\underline{x}-\underline{y}|>\tfrac14 L, \tag{2.9}$$

where $\gamma>0$ is fixed. Otherwise Λ is called bad. *Assume that for any family $\mathcal{F}$ of pairwise disjoint bad M'-regions in Λ_0 with $M+1\leqslant M'\leqslant 2M+1$,*

$$\#\mathcal{F}\leqslant N^b. \tag{2.10}$$

Under these assumptions one has

$$|G_{\Lambda_0}(E)(\underline{x},\underline{y})| \leqslant e^{-\gamma'|\underline{x}-\underline{y}|} \quad \textit{for all } \underline{x},\underline{y}\in\Lambda_0,\ |\underline{x}-\underline{y}|>\tfrac{1}{4}N, \tag{2.11}$$

where $\gamma'=\gamma-N^{-\delta}$ *and* $\delta=\delta(b,\tau)>0$, *provided* N *is sufficiently large, i.e.,* $N\geqslant N_0(b,\tau,\gamma)$.

Proof. Choose a constant $c>1$ so that both

$$cb<1 \quad \text{and} \quad c\tau\leqslant 1. \tag{2.12}$$

Define inductively scales $M_{j+1}=\lceil M_j^c\rceil$, $M_0=M$. Fix an elementary region $\Lambda_1\subset\Lambda_0$ of size M_1. For any $\underline{x}\in\Lambda_1$ consider the exhaustion $\{S_j(\underline{x})\}_{j=0}^{l}$ of Λ_1 of width $2M$, see (2.2). We say that the annulus $A_j=S_j(\underline{x})\setminus S_{j-1}(\underline{x})$ is *good*, if for any $\underline{y}\in A_j$ both the elementary regions

$$Q_M(\underline{y})\cap A_j \quad \text{and} \quad Q_M(\underline{y})\cap\Lambda_1 \tag{2.13}$$

satisfy (2.9). Otherwise the annulus is called *bad*. Recall that there is at most one annulus A_{j_0} for which $Q_M(\underline{y})\cap A_{j_0}$ is not an elementary region. In that case A_{j_0} is counted among the bad annuli. Moreover, it is clear that the size of $Q_M(\underline{y})\cap A_j$ is between $M+1$ and $2M+1$. Fix some small $\varkappa=\gamma^{-1}N^{-2\delta}$ which will be determined below. An elementary region $\Lambda_1\subset\Lambda_0$ of size M_1 is called *bad* provided for some $\underline{x}\in\Lambda_1$ the number of bad annuli $\{A_j\}$ exceeds

$$B_1 := \varkappa\frac{M_1}{M}. \tag{2.14}$$

M will be assumed large enough so that $B_1\geqslant 10$, say. Let $\mathcal{F}_1$ be an arbitrary family of pairwise disjoint bad M_1-regions contained in Λ_0. If $\Lambda_1\in\mathcal{F}_1$, then by construction there are at least $\frac{1}{2}B_1$ many pairwise disjoint bad M-regions contained in Λ_1 (squares Q_M with centers in nonadjacent annuli are disjoint). Consequently, there are at least

$$\tfrac{1}{2}B_1\cdot\#\mathcal{F}_1$$

many pairwise disjoint bad M-regions in Λ_0. By assumption (2.10), this implies that

$$\#\mathcal{F}_1 \leqslant \frac{2N^b}{\varkappa M_1/M} \tag{2.15}$$

for any such family $\mathcal{F}_1$.

Suppose that $\Lambda_1\subset\Lambda_0$ is a *good* M_1-region and fix any pair $\underline{x},\underline{y}\in\Lambda_1$ with $|\underline{x}-\underline{y}|>\frac{1}{4}M_1$. Consider the exhaustion $\{S_j(\underline{x})\}$ of Λ_1 of width $2M$ centered at $\underline{x}$ as in (2.2). By assumption, there are no more than B_1 bad annuli in this exhaustion. Let $A_j(\underline{x}), A_{j+1}(\underline{x}), \dots, A_{j+s}(\underline{x})$ be adjacent good annuli and define

$$U=\bigcup_{i=j}^{j+s} A_i(\underline{x}).$$

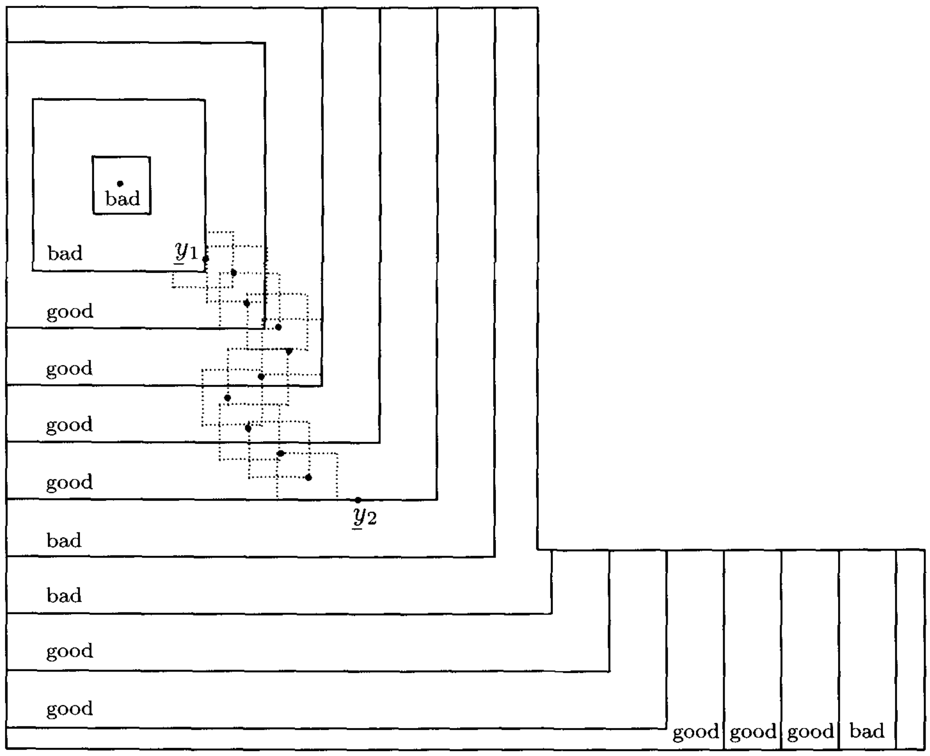

Fig. 2. Applying the resolvent identity to adjacent good annuli.

First, we estimate $\|G_U(E)\|$. Since U is in general not an elementary region, one cannot invoke (2.8). Instead, one uses that for each $\underline{y}\in U$

$$W(\underline{y}) := Q_M(\underline{y})\cap U \tag{2.16}$$

satisfies (2.9). This follows from the definition of good annuli, see (2.13), since if $\underline{y}\in A_j$ either

$$W(\underline{y}) = Q_M(\underline{y})\cap A_j \quad \text{or} \quad W(\underline{y}) = Q_M(\underline{y})\cap \Lambda_1.$$

By Lemma 2.2 with $N=2M+1$, $t=\frac{1}{4}\gamma$, $A=e^{(2M+1)^b}$,

$$\|G_U(E)\| \leqslant 2(2M+1)^2 e^{(2M+1)^b} \tag{2.17}$$

for large M. Next we turn to exponential off-diagonal decay of $G_U(E)$. More precisely, choose two points $\underline{y}_1\in\partial_* S_{j-1}(\underline{x})$ and $\underline{y}_2\in\partial_* S_{j+s}(\underline{x})$, see Figure 2. Here $\partial_* S_{-1}(\underline{x}):=\{\underline{x}\}$ and

$$\partial_* S_j(\underline{x}) := \{\underline{y}\in S_j(\underline{x}) : \text{there exists } \underline{z}\in\Lambda_1\backslash S_j(\underline{x}) \text{ with } |\underline{y}-\underline{z}|=1\}$$

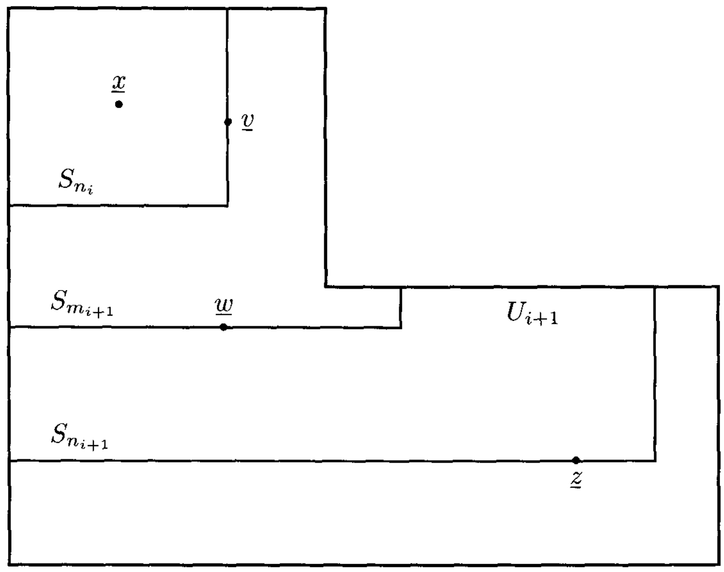

Fig. 3. Passing from S_{n_i} to $S_{n_{i+1}}$.

for $j\geqslant 0$. In Figure 3 the interior boundaries $\partial_* S_j$ are given by the thin ⅃-shaped curves inside Λ_1. By construction,

$$|\underline{y}_1-\underline{y}_2|\geqslant 2M(s+1).$$

Applying the resolvent identity $t=2(s+1)$ times therefore yields (with $G_U(E)=G_U$ for simplicity)

$$\begin{aligned} G_U(\underline{y}_1,\underline{y}_2) = & \sum_{\substack{\underline{z}_1\in W(\underline{y}_1)\\ \underline{z}_1'\in U\setminus W(\underline{y}_1)}} \sum_{\substack{\underline{z}_2\in W(\underline{z}_1')\\ \underline{z}_2'\in U\setminus W(\underline{z}_1')}} \cdots \sum_{\substack{\underline{z}_t\in W(\underline{z}_{t-1})\\ \underline{z}_t'\in U\setminus W(\underline{z}_{t-1})}} G_{W(\underline{y}_1)}(\underline{y}_1,\underline{z}_1) \\ & \times G_{W(\underline{z}_1')}(\underline{z}_1',\underline{z}_2)\cdot\ldots\cdot G_{W(\underline{z}_{t-1}')}(\underline{z}_{t-1}',\underline{z}_t)\,G_U(\underline{z}_t',\underline{y}_2), \end{aligned} \tag{2.18}$$

where it is understood that $|\underline{z}_i-\underline{z}_i'|=1$. A possible chain of regions $W(\underline{y}_1), W(\underline{z}_1'), \ldots$ starting at $\underline{y}_1$ is shown in Figure 2. Consequently, (2.18), (2.9) and (2.17) imply that

$$\begin{aligned} |G_U(E)(\underline{y}_1,\underline{y}_2)| &\leqslant 2(2M+1)^2(16M_1)^{2(s+1)}e^{(2M+1)^b}e^{-\gamma|\underline{y}_1-\underline{y}_2|} \\ &\leqslant (40M_1)^{2(s+2)}e^{(2M+1)^b}e^{-\gamma|\underline{y}_1-\underline{y}_2|}. \end{aligned} \tag{2.19}$$

Our next goal is to obtain exponential off-diagonal decay of $G_{\Lambda_1}(E)$ from (2.19). Recall that there is the exhaustion

$$S_0(\underline{x})\subsetneq S_1(\underline{x})\subsetneq \ldots \subsetneq S_k(\underline{x})\subsetneq \Lambda_1.$$

Here k is chosen so that $\underline{y}\notin S_k(\underline{x})$, but $\underline{y}\in S_{k+1}(\underline{x})$. Let

$$n_0=-1\leqslant m_1<n_1<m_2<n_2<\ldots<m_g<n_g\leqslant k$$

be such that *all* annuli between S_{m_i} and S_{n_i} are good, whereas all annuli between S_{n_i} and $S_{m_{i+1}}$ are bad. Moreover, g is maximal with this property. If $n_g<k$, we set $m_{g+1}=k$. Define

$$U_i=S_{n_i}\backslash S_{m_i}\quad \text{for } 1\leqslant i\leqslant g.$$

Using the resolvent identity, we shall now inductively obtain estimates of the form

$$|G_{S_{n_i}}(E)(\underline{x},\underline{z})|\leqslant B_i\, e^{-\gamma|\underline{x}-\underline{z}|}\quad \text{for all } \underline{z}\in\partial_* S_{n_i}, \tag{2.20}$$

with certain constants B_i. Consider the case $i=1$. If $m_1=0$, then $S_{n_1}=U_1$, and thus

$$|G_{S_{n_1}}(E)(\underline{x},\underline{z})|\leqslant (40M_1)^{2(n_1-m_1+1)}e^{(2M+1)^b}e^{-\gamma|\underline{x}-\underline{z}|} \tag{2.21}$$

by (2.19). If $m_1>0$, then by (2.8) and (2.19)

$$|G_{S_{n_1}}(E)(\underline{x},\underline{z})|\leqslant \sum_{\substack{\underline{w}\in S_{n_1}\backslash U_1\\ \underline{w}'\in U_1\\ |\underline{w}-\underline{w}'|=1}} |G_{S_{n_1}}(E)(\underline{x},\underline{w})|\,|G_{U_1}(E)(\underline{w}',\underline{z})| \tag{2.22}$$

$$\leqslant 16M_1(40M_1)^{2(n_1-m_1+1)}e^{M_1^b}e^{(2M+1)^b+2\gamma(m_1+1)M}e^{-\gamma|\underline{x}-\underline{z}|}. \tag{2.23}$$

In view of (2.21) and (2.22), the estimate stated in (2.20) for $i=1$ therefore holds with

$$B_1=16M_1(40M_1)^{2(n_1-m_1+1)}e^{2M_1^b+2\gamma(m_1-n_0)M}. \tag{2.24}$$

To pass from S_{n_i} to $S_{n_{i+1}}$ one argues as follows. Fix any $\underline{z}\in\partial_* S_{n_{i+1}}$.

$$\begin{aligned}
&|G_{S_{n_{i+1}}}(E)(\underline{x},\underline{z})|\\
&\quad\leqslant \sum_{\substack{\underline{w}\in S_{n_{i+1}}\backslash U_{i+1}\\ \underline{w}'\in U_{i+1}}} |G_{S_{n_{i+1}}}(E)(\underline{x},\underline{w})|\,|G_{U_{i+1}}(E)(\underline{w}',\underline{z})|\\
&\quad\leqslant \sum_{\substack{\underline{w}\in S_{n_{i+1}}\backslash U_{i+1}\\ \underline{w}'\in U_{i+1}}}\ \sum_{\substack{\underline{v}\in S_{n_{i+1}}\backslash S_{n_i}\\ \underline{v}'\in S_{n_i}}} |G_{S_{n_i}}(E)(\underline{x},\underline{v}')|\,|G_{S_{n_{i+1}}}(E)(\underline{v},\underline{w})|\,|G_{U_{i+1}}(E)(\underline{w}',\underline{z})|\\
&\quad\leqslant \sum_{\substack{\underline{w}\in S_{n_{i+1}}\backslash U_{i+1}\\ \underline{w}'\in U_{i+1}}}\ \sum_{\substack{\underline{v}\in S_{n_{i+1}}\backslash S_{n_i}\\ \underline{v}'\in S_{n_i}}} B_i e^{-\gamma|\underline{x}-\underline{v}'|}e^{M_1^b}e^{-\gamma|\underline{w}'-\underline{z}|}(40M_1)^{2(n_{i+1}-m_{i+1}+1)}e^{(2M+1)^b} &(2.25)\\
&\quad\leqslant B_i(16M_1)^2(40M_1)^{2(n_{i+1}-m_{i+1}+1)}e^{2M_1^b+2M\gamma(m_{i+1}-n_i)}e^{-\gamma|\underline{x}-\underline{z}|}, &(2.26)
\end{aligned}$$

where it is again understood that $|\underline{w}-\underline{w}'|=|\underline{v}-\underline{v}'|=1$. To pass from (2.25) to (2.26) one uses that

$$|\underline{x}-\underline{z}| \leqslant |\underline{x}-\underline{v}'|+|\underline{w}'-\underline{z}|+2M(m_{i+1}-n_i),$$

see Figure 3. By means of (2.26) and (2.24) one obtains the following expression for B_g:

$$B_g := (16M_1)^{2g-1}(40M_1)^{2\sum_1^g (n_i-m_i+1)} \exp\Big(2gM_1^b+2M\gamma\sum_{i=0}^{g-1}(m_{i+1}-n_i)\Big). \tag{2.27}$$

By definition,

$$\sum_{i=0}^{g-1}(m_{i+1}-n_i) \leqslant B_1 \quad \text{and} \quad 2g \leqslant \sum_{i=1}^{g}(n_i-m_i+1) \leqslant \frac{M_1}{M}.$$

Recalling (2.14), this shows that (2.27) reduces to

$$\log B_g \lesssim \gamma\varkappa M_1+M_1M^{cb-1} \tag{2.28}$$

provided N (and thus M) is large. Inserting this into (2.26) one obtains

$$|G_{S_{n_g}}(E)(\underline{x},\underline{z})| \leqslant \exp[-\gamma|\underline{x}-\underline{z}|(1-C\varkappa-C\gamma^{-1}M^{cb-1})] \tag{2.29}$$

for all $\underline{z}\in\partial_* S_{n_g}(\underline{x})$. By maximality of g, one has $|\underline{x}-\underline{z}|\geqslant|\underline{x}-\underline{y}|-2B_1M$ for all such $\underline{z}$. Hence a final application of the resolvent identity allows one to deduce the desired bound for $G_{\Lambda_1}(E)$ from (2.29), i.e.,

$$|G_{\Lambda_1}(E)(\underline{x},\underline{y})| \leqslant e^{-\gamma_1|\underline{x}-\underline{y}|}, \tag{2.30}$$

where

$$\gamma_1 = \gamma(1-C\varkappa-C\gamma^{-1}M^{cb-1}) \tag{2.31}$$

with some absolute constant C.

This process can be repeated to pass from scale M_1 to scale M_2, and so on. More precisely, we call an M_2-region $\Lambda_1\subset\Lambda_0$ *bad* if there is some exhaustion of Λ_1 by annuli of thickness $2M_1$ for which the number of bad annuli exceeds

$$B_2 := \varkappa\frac{M_2}{M_1}, \tag{2.32}$$

with the same $\varkappa$ as above. An annulus A is called *bad* if it contains some point $\underline{y}$ for which one of the two M_1-regions

$$Q_{M_1}(\underline{y})\cap A \quad \text{and} \quad Q_{M_1}(\underline{y})\cap\Lambda_1$$

does not satisfy (2.30), cf. (2.13). For the same reason as before, any family $\mathcal{F}_2$ of pairwise disjoint bad M_2-regions satisfies

$$\#\mathcal{F}_2 \leqslant \left(\frac{2}{\varkappa}\right)^2 \frac{N^b}{M_2/M}, \tag{2.33}$$

cf. (2.15). Moreover, if $\Lambda_1 \subset \Lambda_0$ is a good M_2-region, then the same arguments involving the resolvent identity that lead to (2.30) show that one has the off-diagonal decay

$$|G_{\Lambda_1}(\underline{x},\underline{y})| \leqslant \exp(-\gamma_2|\underline{x}-\underline{y}|) \quad \text{for any } \underline{x},\underline{y}\in\Lambda_1,\ |\underline{x}-\underline{y}| > \tfrac{1}{4}M_2,$$

where

$$\gamma_2 := \gamma(1-C\varkappa-C\gamma^{-1}M^{bc-1})(1-C\varkappa-C\gamma_1{}^{-1}M^{bc-1}).$$

Continuing inductively, the lemma follows provided one reaches a scale $M_s \leqslant N$ for which there are no bad M_s-regions. In analogy to (2.33), (2.15), any family $\mathcal{F}_s$ of pairwise disjoint bad M_s-regions satisfies

$$\#\mathcal{F}_s \leqslant \left(\frac{2}{\varkappa}\right)^s \frac{N^b}{M_s/M}.$$

Ignoring the difference between M_s and M^{c^s} (which is justified for large N), one therefore needs to ensure the existence of a positive integer s for which

$$\left(\frac{2}{\varkappa}\right)^s \frac{N^b}{M^{c^s-1}} < 1 \quad \text{and} \quad M^{c^s} \leqslant N.$$

Since $M \asymp N^\tau$ and $\varkappa = \gamma^{-1}N^{-2\delta}$, this can be done for any $N \geqslant N_0(b,\tau,\gamma)$ provided

$$c\tau \leqslant 1 \quad \text{and} \quad c\left(b+2\frac{\log(1/\tau)}{\log c}\delta\right) < 1.$$

In view of (2.12) this holds for small $\delta>0$, as claimed. Thus (2.11) has been established with

$$\gamma' = \gamma \prod_{j=0}^{s-1} (1-C\varkappa-C\gamma_j^{-1}N^{-\tau(1-bc)}),$$

where $\gamma_0=\gamma$. Since $\varkappa=\gamma^{-1}N^{-2\delta}$ and $s \leqslant \log(1/\tau)/\log c$, for sufficiently large N and small $\delta>0$ one has

$$\gamma' \geqslant \gamma(1-N^{-\delta}),$$

and the lemma follows. □

3. An arithmetic condition on the frequency vector

This section deals exclusively with the two-dimensional dynamics given by the frequency vector $\underline{\omega}=(\omega_1,\omega_2)\in\mathbf{T}^2$. The main result here is Lemma 3.3, which states that for an algebraic curve $\Gamma\subset\mathbf{T}^2$ of degree B, the number of points $(n_1\omega_1, n_2\omega_2) \pmod{\mathbf{Z}^2}$ with $1\leqslant|n_1|,|n_2|\leqslant N$, falling into an η-neighborhood Γ^η of Γ, is no larger than $N^{1-\delta_0}$. This requires a relation between the numbers η, B, N, and, most importantly, a suitable condition on $\underline{\omega}$. That condition turns out to be of the form $\underline{\omega}\in\Omega_N$, where $\operatorname{mes}(\mathbf{T}^2\setminus\Omega_N)<N^{-\varepsilon}$, $\varepsilon>0$ a small positive constant. It is essential that the set Ω_N is determined by purely arithmetic considerations that do not depend on the curve Γ, see Lemma 3.1 below. In order to understand the conclusion of the following lemma, it might be helpful to recall the following simple fact: Let n,m be positive integers, and suppose $1>\delta>0$. Then

$$\operatorname{mes}[\theta\in\mathbf{T}: \|\theta m\|<\delta,\ \|\theta n\|<\delta] \asymp \delta^2+\frac{\delta\gcd(m,n)}{m+n},$$

where $\|\cdot\|$ denotes the distance to the nearest integer. This implies that the fractional parts of θm and θn, considered as random variables, are strongly dependent if and only if $\gcd(m,n)$ is large relative to $m+n$.

LEMMA 3.1. *Let N be a positive integer. There exists a set $\Omega_N\subset[0,1]^2$ so that*

$$\operatorname{mes}([0,1]^2\setminus\Omega_N)<N^{0-}$$

and such that any $\underline{\omega}=(\omega_1,\omega_2)\in\Omega_N$ has the following property:

Let q_1,q_1',q_2,q_2' be nonzero integers bounded in absolute value by N, and suppose that the numbers

$$\begin{cases} \theta_1\equiv q_1\omega_1 \pmod 1,\\ \theta_1'\equiv q_1'\omega_1 \pmod 1,\\ \theta_2\equiv q_2\omega_2 \pmod 1,\\ \theta_2'\equiv q_2'\omega_2 \pmod 1\end{cases}$$

satisfy

$$|\theta_i|,|\theta_i'|<N^{-1+\delta_1},\quad i=1,2, \tag{3.3}$$

and

$$-N^{-3+\delta_2}<\begin{vmatrix}\theta_1 & \theta_1'\\ \theta_2 & \theta_2'\end{vmatrix}<N^{-3+\delta_2}, \tag{3.4}$$

with $\delta_1,\delta_2>0$ sufficiently small. Then

$$\gcd(q_1,q_1')>N^{1-11\delta_1},$$
$$\gcd(q_2,q_2')>N^{1-11\delta_1}.$$

Proof. Partition $\mathbf{T}^2$ into squares I of size $1/N^2$, and restrict $\underline{\omega}=(\omega_1,\omega_2)$ to one such square I. From (3.3),

$$|q_i\omega_i-m_i|<N^{-1+\delta_1} \quad \text{and} \quad |q_i'\omega_i-m_i'|<N^{-1+\delta_1}$$

for some $m_i,m_i'\in\mathbf{Z}$. We may clearly assume (by suitable restriction of ω_i) that

$$|q_i|,|q_i'|>N^{1-\delta_1-}. \tag{3.6}$$

Thus

$$\left|\omega_i-\frac{m_i}{q_i}\right|<N^{-2(1-\delta_1)+} \quad \text{and} \quad \left|\omega_i-\frac{m_i'}{q_i'}\right|<N^{-2(1-\delta_1)+}. \tag{3.7}$$

Since ω_i is restricted to an interval of size $1/N^2$, the number of pairs (q,m) bounded by N so that $|\omega_i-m/q|<N^{-2(1-\delta_1)+}$ for some ω_i in that interval is at most $N^{2\delta_1+}$. Fix then q_i,q_i',m_i,m_i' and consider the relative measure of $\underline{\omega}\in I$ such that (3.4) holds, i.e.,

$$-N^{-3+\delta_2}<\begin{vmatrix} q_1\omega_1-m_1 & q_2\omega_2-m_2 \\ q_1'\omega_1-m_1' & q_2'\omega_2-m_2' \end{vmatrix}<N^{-3+\delta_2}. \tag{3.8}$$

Writing $\omega_i=\omega_{i,0}+\varkappa_i$ with $|\varkappa_i|<1/N^2$, (3.8) is of the form

$$|(q_1q_2'-q_1'q_2)\varkappa_1\varkappa_2+\alpha_1\varkappa_1+\alpha_2\varkappa_2+\beta|<N^{-3+\delta_2}. \tag{3.9}$$

Assume

$$|q_1q_2'-q_1'q_2|\geqslant N^{1+\delta_2+10\delta_1}.$$

Then (3.9) defines a $(\varkappa_1,\varkappa_2)$-set of measure at most

$$\frac{N^{-3+\delta_2+}}{|q_1q_2'-q_1'q_2|}\leqslant N^{-4-10\delta_1+}.$$

The relative measure in I is therefore less than $N^{-10\delta_1+}$, and summing over all possible choices of q_i,m_i,q_i',m_i', $i=1,2$, gives the bound $N^{-10\delta_1}N^{8\delta_1+}<N^{-\delta_1}$. It thus remains to consider the case where

$$|q_1q_2'-q_1'q_2|<N^{1+\delta_2+10\delta_1}. \tag{3.12}$$

We need to estimate the measure of those $\underline{\omega}=(\omega_1,\omega_2)\in\mathbf{T}^2$ for which there are $q_i,q_i'\in\mathbf{Z}\cap[-N,N]$ such that $\|q_i\omega_i\|<N^{-1+\delta_1}$, $\|q_i'\omega_i\|<N^{-1+\delta_1}$, (3.12) holds and

$$\min_{i=1,2}|\gcd(q_i,q_i')|<R. \tag{3.13}$$

Write

$$\gcd(q_i,q_i')=r_i, \quad q_i=r_iQ_i, \quad q_i'=r_iQ_i', \quad \text{for } i=1,2.$$

Fixing r_1 and r_2, (3.12) becomes

$$|Q_1 Q_2' - Q_1' Q_2| < \frac{1}{|r_1 r_2|} N^{1+\delta_2+10\delta_1}. \tag{3.15}$$

Estimate the measure of ω_i so that

$$\|q_i \omega_i\| < N^{-1+\delta_1}, \quad \|q_i' \omega_i\| < N^{-1+\delta_1},$$

for given q_i, q_i', $\gcd(q_i, q_i') = r_i$. From (3.7), for some $m_i, m_i' \in \mathbf{Z}$,

$$\begin{aligned} \left| \frac{m_i}{q_i} - \frac{m_i'}{q_i'} \right| &< N^{-2(1-\delta_1)+}, \\ |m_i q_i' - m_i' q_i| &< N^{2\delta_1+}, \\ |m_i Q_i' - m_i' Q_i| &< \frac{1}{|r_i|} N^{2\delta_1+}. \end{aligned} \tag{3.17}$$

Since $\gcd(Q_i, Q_i') = 1$, the number of possible (m_i, m_i') in (3.17) is at most

$$\left(1 + \frac{1}{|r_i|} N^{2\delta_1+}\right) |r_i| \leqslant |r_i| + N^{2\delta_1+}.$$

Since $|\omega_i - m_i/q_i| < N^{-2(1-\delta_1)+}$, the ω_i-measure estimate is

$$(|r_i| + N^{2\delta_1+}) N^{-2+2\delta_1} \leqslant N^{-2+4\delta_1+} |r_i|. \tag{3.18}$$

Distinguish the cases

$$|r_1 r_2| < N^{1+\delta_2+10\delta_1}, \tag{3.19}$$

$$|r_1 r_2| \geqslant N^{1+\delta_2+10\delta_1}. \tag{3.20}$$

Assume $|r_1| \geqslant |r_2|$. Observe that

$$\frac{N^{1-\delta_1-}}{|r_i|} \leqslant |Q_i|, |Q_i'| < \frac{N}{|r_i|}$$

by (3.6). If (3.19) holds, then the number of Q_i, Q_i' satisfying (3.15) is at most

$$\left(\frac{N}{|r_1|}\right)^2 \frac{N^{1+\delta_2+10\delta_1}}{|r_1 r_2|} N^{\delta_1+} \frac{|r_1|}{|r_2|} < \frac{N^{3+\delta_2+12\delta_1}}{(r_1 r_2)^2}. \tag{3.22}$$

In view of (3.18) and (3.22) the corresponding $\underline{\omega}$-contribution is of measure less than

$$\sum_{\substack{r_1, r_2 \\ |r_1 r_2| < N^{1+\delta_2+10\delta_1}}} \frac{N^{3+\delta_2+12\delta_1}}{(r_1 r_2)^2} N^{-4+8\delta_1} |r_1 r_2| < N^{-1+\delta_2+20\delta_1+}.$$

For the contribution of (3.20) we obtain (recalling (3.13))

$$\sum_{\substack{r_1,r_2\\ |r_1|\wedge|r_2|\leqslant|r_2|<R}} \left(\frac{N}{r_1}\right)^2 N^{\delta_1+\frac{|r_1|}{|r_2|}} N^{-4+8\delta_1}|r_1r_2| < N^{-2+10\delta_1}\sum_{\substack{r_1,r_2\\ |r_2|<R}} 1 < RN^{-1+10\delta_1}. \tag{3.24}$$

Taking $R=N^{1-11\delta_1}$, the measure contribution by (3.24) is less than $N^{-\delta_1}$. Thus, under previous restrictions of $\underline{\omega}$, necessarily $(q_i, q_i')>N^{1-11\delta_1}$, proving Lemma 3.1. □

Remark 3.2. It is clear that the set Ω_N is basically stable under perturbations of order N^{-4}. More precisely, one can replace Ω_N with the set

$$\widetilde{\Omega}_N := \bigcup_{i:Q_i\cap\Omega_N\neq\varnothing} Q_i, \tag{3.25}$$

where the union runs over a partition of $\mathbf{T}^2$ of cubes of side length N^{-4}. This point is not an essential one, but will be useful in §6 below.

Throughout this paper semi-algebraic sets play a crucial role. We refer the reader to §7 for the definitions as well as some basic properties of semi-algebraic sets.

The idea behind Lemma 3.3 is as follows: If too many points $(n_1\omega_1, n_2\omega_2)$ fall very close to an algebraic curve Γ, then there would have to be many small triangles with vertices close to Γ. Here "small" means both small sides and small area. This, however, is excluded by Lemma 3.1.

LEMMA 3.3. *Let $\mathcal{A}\subset[0,1]^2$ be a semi-algebraic set of degree at most B, see Definition* 7.1. *Assume further that*

$$\operatorname{mes}(\mathcal{A}_{\theta_1})<\eta, \quad \operatorname{mes}(\mathcal{A}_{\theta_2})<\eta, \quad \textit{for all } (\theta_1,\theta_2)\in\mathbf{T}^2, \tag{3.26}$$

where $\mathcal{A}_{\theta_i}$ denotes a section of $\mathcal{A}$. Let

$$\log B \ll \log N \ll \log\frac{1}{\eta}. \tag{3.27}$$

Then, for $\underline{\omega}\in\Omega_N$ introduced in Lemma 3.1 *with*

$$\operatorname{mes}([0,1]^2\setminus\Omega_N)<N^{0-},$$

one has that

$$\#\{(n_1,n_2)\in\mathbf{Z}^2 : |n_1|\vee|n_2|<N,\ (n_1\omega_1,n_2\omega_2)\in\mathcal{A}\ (\operatorname{mod}\mathbf{Z}^2)\}<N^{1-\delta_0}$$

with some absolute constant $\delta_0>0$.

Proof. In view of Definition 7.1 there are polynomials $\{P_i\}_{i=1}^s$ with $\deg(P_i)\leqslant d$ so that

$$\partial\mathcal{A}\subset\bigcup_{i=1}^{s}\{[0,1]^2:P_i=0\}.$$

Let $\Gamma_i=\{[0,1]^2:P_i=0\}$. Unless the algebraic curve Γ_i contains the vertical segment $\theta_1=\text{const}$, it can intersect it in at most $\deg(P_i)\leqslant d$ many points. Therefore, the sections $\mathcal{A}_{\theta_1}$, $\mathcal{A}_{\theta_2}$ are unions of at most $sd=B$ many intervals of total measure less than η, see (3.26). By (3.27) we may assume that each of these intervals contains at most one element $n\omega_i$ (mod 1). Hence

$$\sup_{\theta_2}\#\{n_1\in\mathbf{Z}:|n_1|<N,\ n_1\omega_1\in\mathcal{A}_{\theta_2}\ (\text{mod }1)\}\leqslant B, \tag{3.29}$$

$$\sup_{\theta_1}\#\{n_2\in\mathbf{Z}:|n_2|<N,\ n_2\omega_2\in\mathcal{A}_{\theta_1}\ (\text{mod }1)\}\leqslant B. \tag{3.30}$$

Since $\operatorname{mes}(\mathcal{A})<\eta$, one has $\operatorname{dist}((\theta_1,\theta_2),\partial\mathcal{A})<\eta^{1/2}$ for each $(\theta_1,\theta_2)\in\mathcal{A}$. Fix one of the $\Gamma=\Gamma_i$ from above and assume that

$$\#\{(n_1\omega_1,n_2\omega_2)\in\mathcal{A}'\ (\text{mod }\mathbf{Z}^2):|n_1|<N,\ |n_2|<N\}>N^{1-\varrho}, \tag{3.31}$$

where

$$\mathcal{A}'\equiv\mathcal{A}\cap\{\underline{\xi}\in[0,1]^2:\operatorname{dist}(\underline{\xi},\Gamma)<\eta^{1/2}\}.$$

Since P_i has at most B irreducible factors, for at least one of them (3.31) remains true (with $\mathcal{A}'$ being defined in terms of the respective factor, and with $N^{1-\varrho-}$ instead of $N^{1-\varrho}$). In what follows we can therefore assume that P_i is irreducible. Thus, by Bezout's theorem,

$$\#[P_i=0,\ |\partial_{\theta_1}P_i|=|\partial_{\theta_2}P_i|]<2B^2$$

(if $\partial_{\theta_1}P_i\pm\partial_{\theta_2}P_i$ vanishes identically, then Γ is a line). One can therefore restrict $\mathcal{A}'$ to a piece of Γ where $|\partial_{\theta_1}P_i|<|\partial_{\theta_2}P_i|$, say, so that (3.31) remains true (again with $N^{1-\varrho-}$). Observe that we have reduced ourselves to the case where $\mathcal{A}'$ is a $\sqrt{\eta}$-strip around the graph of an analytic function

$$\theta_2=\Theta(\theta_1)\quad\text{satisfying }|\Theta'|\leqslant 1. \tag{3.33}$$

Moreover, the function Θ is defined over an interval of size greater than $N^{-\varrho-}$. Now let $\varepsilon_1:=N^{-1+\varrho_1}$ with some $\varrho_1>\varrho$ to be specified. Clearly, $\mathcal{A}'$ is covered by $\lesssim\varepsilon_1^{-1}$ many ε_1-disks D_α. Furthermore, for any disk D_α one has

$$\#\{(n_1\omega_1,n_2\omega_2)\in\mathcal{A}'\cap D_\alpha\ (\text{mod }\mathbf{Z}^2):|n_1|<N,\ |n_2|<N\}\lesssim N^{1+}\varepsilon_1\lesssim N^{\varrho_1+}.$$

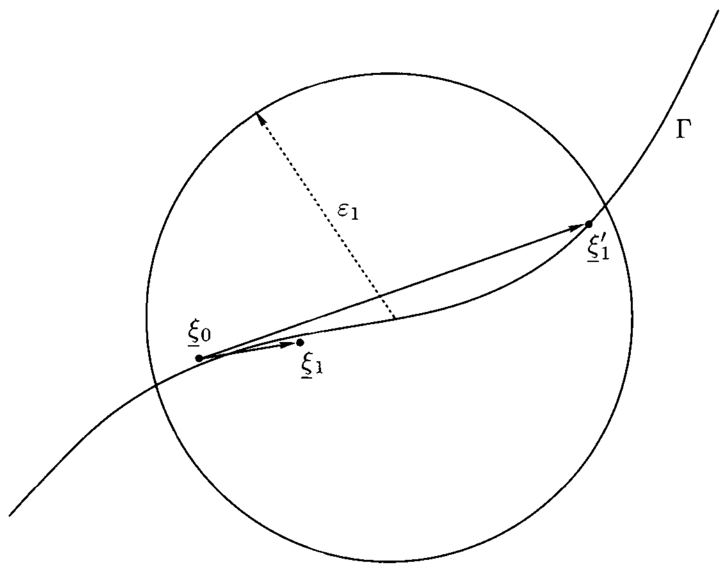

Fig. 4. The triangle $\underline{\xi}_0, \underline{\xi}_1, \underline{\xi}_1'$.

Thus there are at least $(\varepsilon_1 N^{\varrho+})^{-1}$ many disks D_α so that

$$\#\{(n_1\omega_1, n_2\omega_2) \in \mathcal{A}' \cap D_\alpha \pmod{\mathbf{Z}^2} : |n_1| < N, |n_2| < N\} \gtrsim \frac{N^{1-\varrho-}}{\varepsilon_1^{-1}} \gtrsim N^{\varrho_1 - \varrho-}. \tag{3.35}$$

Finally, we claim that the majority of the disks D_α have the property that for any choice of distinct points $\underline{\xi}_0, \underline{\xi}_1, \underline{\xi}_1'$ in $\{(n_1\omega_1, n_2\omega_2) \in \mathcal{A}' \cap D_\alpha \pmod{\mathbf{Z}^2} : |n_1| < N, |n_2| < N\}$, one has

$$\mathrm{angle}([\underline{\xi}_0, \underline{\xi}_1], [\underline{\xi}_0, \underline{\xi}_1']) \lesssim B^2 \varepsilon_1 N^\varrho, \tag{3.36}$$

see Figure 4. Suppose that this fails. Then there are at least $M \gtrsim \varepsilon_1^{-1} N^{-\varrho}$ many disks D_α which contain triples $\underline{\xi}_0, \underline{\xi}_1, \underline{\xi}_1'$ as above so that (3.36) is violated. It is not hard to see that on any such disk D_α the unit vector $\nabla P_i / |\nabla P_i|$ covers an interval on S^1 of size at least $\varphi \asymp B^2 \varepsilon_1 N^\varrho$, cf. [7, §13]. Consequently, there exists some $\zeta \in S^1$ so that $\nabla P_i / |\nabla P_i|$ attains ζ at least $M\varphi$ many times. Equivalently,

$$\#\{P_i = 0, \zeta^\perp \cdot \nabla P_i = 0\} \geqslant \lfloor M\varphi \rfloor.$$

By Bezout's theorem, the left-hand side is no larger than B^2, and the claim follows ($\zeta^\perp \cdot \nabla P_i$ cannot vanish identically, as then $\nabla P_i / |\nabla P_i|$ would be constant). Alternatively, one can use Theorem 7.4 to write Γ as the graph of no more than B^C many piecewise analytic functions with a second derivative bound of the form $B^C N^{2\varrho}$, which immediately leads to (3.36) with a bound $B^C \varepsilon_1 N^{2\varrho}$.

Now choose any D_α such that (3.35) and (3.36) hold. By means of (3.29) (or (3.30)), one may fix

$$\underline{\xi}_0, \underline{\xi}_1 \in \{(m_1\omega_1, m_2\omega_2) \in \mathcal{A}' \cap D_\alpha \ (\text{mod } \mathbf{Z}^2) : |m_1| < N, |m_2| < N\}$$

such that

$$|\underline{\xi}_0 - \underline{\xi}_1| \lesssim \frac{B\varepsilon_1}{N^{\varrho_1 - \varrho -}} \tag{3.37}$$

and $\underline{\xi}_0 - \underline{\xi}_1$ is not parallel to either one of the coordinate axes. Let

$$\underline{\xi}_1' \in \{(n_1\omega_1, n_2\omega_2) \in \mathcal{A}' \cap D_\alpha \ (\text{mod } \mathbf{Z}^2) : |n_1| < N, |n_2| < N\}$$

so that $\underline{\xi}_0 - \underline{\xi}_1'$ is also not parallel to a coordinate axis. This can be rewritten in the form

$$\underline{\xi}_1 - \underline{\xi}_0 = (\theta_1, \theta_2) \equiv (q_1\omega_1, q_2\omega_2) \quad (\text{mod } 1),$$
$$\underline{\xi}_1' - \underline{\xi}_0 = (\theta_1', \theta_2') \equiv (q_1'\omega_1, q_2'\omega_2) \quad (\text{mod } 1),$$

with, see (3.37),

$$|\theta_1| + |\theta_2| < \frac{B\varepsilon_1}{N^{\varrho_1 - \varrho -}} < N^{-1+\varrho +} \quad \text{and} \quad |\theta_1'| + |\theta_2'| < \varepsilon_1 = N^{-1+\varrho_1}. \tag{3.38}$$

Moreover, in view of (3.36),

$$\text{Area triangle}(\underline{\xi}_0, \underline{\xi}_1, \underline{\xi}_1') \asymp \text{abs} \begin{vmatrix} \theta_1 & \theta_2 \\ \theta_1' & \theta_2' \end{vmatrix} \lesssim \varepsilon_1 N^{-1+\varrho +} B^2 \varepsilon_1 N^{\varrho} \lesssim N^{-3+2\varrho_1 + 2\varrho +}.$$

Apply Lemma 3.1 with $\delta_1 = \varrho_1$, $\delta_2 = 2\varrho_1 + 2\varrho$. By construction, $q_i, q_i' = \mathbf{Z} \backslash \{0\}$. Since $\underline{\omega} \in \Omega_N$, it follows that

$$\gcd(q_1, q_1') > N^{1 - 11\varrho_1} \quad \text{and} \quad \gcd(q_2, q_2') > N^{1 - 11\varrho_1}.$$

Write

$$q_1 = r_1 Q, \quad q_1' = r_1' Q, \quad \text{with } Q > N^{1 - 11\varrho_1}, \gcd(r_1, r_1') = 1. \tag{3.39}$$

Hence, $|r_1| + |r_1'| < N^{11\varrho_1}$. Take $k_1, k_1' \in \mathbf{Z}$, $|k_1|, |k_1'| < N^{11\varrho_1}$ so that $r_1 k_1 + r_1' k_1' = 1$. Hence

$$\|Q\omega_1\| < |k_1| \, \|q_1\omega_1\| + |k_1'| \, \|q_1'\omega_1\| < 2N^{11\varrho_1} N^{-1+\varrho_1} = 2N^{-1+12\varrho_1},$$

and therefore, by (3.38),

$$N^{-1+\varrho +} > |\theta_1| = \|q_1\omega_1\| = |r_1| \, \|Q\omega_1\|,$$
$$\|Q\omega_1\| < \frac{N^{-1+\varrho +}}{|r_1|} \leqslant N^{-1+\varrho +}, \tag{3.40}$$
$$Q > N^{1-\varrho -}.$$

Fixing $\underline{\xi}_0, \underline{\xi}_1$ as above and considering a variable point

$$\underline{\xi}_1' \in \{(n_1\omega_1, n_2\omega_2) \in \mathcal{A}' \cap D_\alpha \ (\mathrm{mod}\ \mathbf{Z}^2) : |n_1| < N, \ |n_2| < N\},$$

(3.39), (3.40) imply that $q_1' = r_1' Q$, where $Q > N^{1-\varrho-}$ is a divisor of q_1. Since q_1 has at most N^{0+} divisors and $|r_1'| < N^{\varrho+}$, this limits the number of q_1''s to $N^{\varrho+}$. Therefore, recalling (3.35),

$$N^{\varrho_1 - \varrho} < \#\{(n_1\omega_1, n_2\omega_2) \in \mathcal{A}' \cap D_\alpha \ (\mathrm{mod}\ \mathbf{Z}^2) : |n_1| < N, \ |n_2| < N\} < N^{2\varrho+}.$$

Letting $\varrho_1 = 4\varrho$, ϱ small enough, a contradiction follows. This finally leads to the bounds

$$\#\{(n_1\omega_1, n_2\omega_2) \in \mathcal{A}' : |n_1| < N, \ |n_2| < N\} \lesssim N^{1-\varrho},$$
$$\#\{(n_1\omega_1, n_2\omega_2) \in \mathcal{A} : |n_1| < N, \ |n_2| < N\} \lesssim B^C N^{1-\varrho},$$

for some $\varrho > 0$, and the lemma follows. □

Remark 3.4. It is natural to ask to what extent the previous lemma depends on the fact that $\mathcal{A}$ is semi-algebraic. Does it hold, for example, if $\mathcal{A}$ is the diffeomorphic image of a semi-algebraic set? It is easy to see that the answer is affirmative for diffeomorphisms that act in each variable separately, i.e., $\Phi(\theta_1, \theta_2) = (f_1(\theta_1), f_2(\theta_2))$ so that $C^{-1} < |f_i'| < C$ and $|f_i''| < C$. Indeed, the only properties that directly depend on Γ are (3.29), (3.30), (3.33) and (3.36), which are preserved under such diffeomorphisms. In the applications below one deals with sets $\mathcal{A}$ defined by trigonometric polynomials on $\mathbf{T}^2$ rather than polynomials. Covering the torus $\mathbf{T}^2$ by coordinate charts, one obtains diffeomorphisms of the form $\Phi(\theta_1, \theta_2) = (\sin\theta_1, \sin\theta_2)$, say, with θ_1, θ_2 small. Hence Lemma 3.3 still applies to this case.

4. A large deviation theorem for the Green's functions

In this section we consider Hamiltonians of the form

$$H(\underline{\theta}) = -\Delta + \lambda V(\underline{\theta}), \tag{4.1}$$

where $V(\underline{\theta})(n_1, n_2) = v(\theta_1 + n_1\omega_1, \theta_2 + n_2\omega_2)$ and $\lambda \geqslant 1$ is a large parameter. In order to emphasize the dependence of H on $\underline{\omega}$, we sometimes write $H_{\underline{\omega}}$. The real-analytic function $v: \mathbf{T}^2 \to \mathbf{R}$ is assumed to be nondegenerate in the sense that

$$\theta_1 \mapsto v(\theta_1, \theta_2) \quad \text{and} \quad \theta_2 \mapsto v(\theta_1, \theta_2) \tag{4.2}$$

are nonconstant functions for any choice of the other variable. It is a well-known fact that this implies that for all $\delta>0$

$$\begin{aligned}\sup_{\theta_2\in\mathbf{T},E} \operatorname{mes}[\theta_1\in\mathbf{T}:|v(\theta_1,\theta_2)-E|<\delta]\leqslant C\delta^a,\\ \sup_{\theta_1\in\mathbf{T},E} \operatorname{mes}[\theta_2\in\mathbf{T}:|v(\theta_1,\theta_2)-E|<\delta]\leqslant C\delta^a,\end{aligned}\tag{4.3}$$

where $C,a>0$ are constants depending only on v. See, for example, the last section of [19]. For any $\gamma>0$ and $0<b<1$ let

$$\begin{aligned}\mathcal{G}^{\gamma,b}(\Lambda,E):=\{\underline{\theta}\in\mathbf{T}^2:\|G_\Lambda(\underline{\theta},E)\|<\lambda^{-1}e^{\sigma(\Lambda)^b},\ |G_\Lambda(\underline{\theta},E)(\underline{x},\underline{y})|<e^{-\gamma|\underline{x}-\underline{y}|}\\ \text{for all } \underline{x},\underline{y}\in\Lambda,\ |\underline{x}-\underline{y}|>\tfrac{1}{4}\sigma(\Lambda)\},\end{aligned}\tag{4.4}$$

$$\mathcal{B}^{\gamma,b}(\Lambda,E):=\mathbf{T}^2\backslash\mathcal{G}^{\gamma,b}(\Lambda,E),\tag{4.5}$$

Λ being an elementary region. The main purpose of this section is to show that the measure of $\mathcal{B}^{\gamma,b}(\Lambda,E)$ is sub-exponentially small in $\sigma(\Lambda)$, provided $\underline{\omega}\in\Omega$, where

$$\Omega:=\liminf_{N\text{ dyadic}}\Omega_N,\tag{4.6}$$

Ω_N being the set from the previous section. Notice that $\operatorname{mes}(\mathbf{T}^2\backslash\Omega)=0$. This will be done inductively, with the first step being given by the following lemma.

LEMMA 4.1. *Let v be as above and fix any $0<b<1$. Then with $\gamma=\frac{1}{2}\log\lambda$,*

$$\sup_{\theta_i,E}\operatorname{mes}(\mathcal{B}^{\gamma,b}_{\theta_i}(\Lambda,E))\leqslant C\exp(-c\sigma(\Lambda)^b)\quad \textit{for } i=1,2,$$

for any $\Lambda\in\mathcal{ER}(N)$ provided $\lambda\geqslant\lambda_0(N,b,v)$, $N\geqslant N_0(b,v)$. Here c,C are constants depending only on v, and λ_0 grows sub-exponentially in N.

Proof. By definition (4.1),

$$(H_\Lambda-E)^{-1}=(\lambda V_\Lambda-E-\Delta_\Lambda)^{-1}=(I-(\lambda V_\Lambda-E)^{-1}\Delta_\Lambda)^{-1}(\lambda V_\Lambda-E)^{-1}.\tag{4.7}$$

It suffices to consider the case where θ_2 is the fixed variable. Since

$$\|(V_\Lambda-E/\lambda)^{-1}\|=\max_{\underline{x}\in\Lambda}|v(\theta_1+x_1\omega_1,\theta_2+x_2\omega_2)-E/\lambda|^{-1},$$

it follows that outside the set

$$\{\theta_1\in\mathbf{T}:\min_{\underline{x}\in\Lambda}|v(\theta_1+x_1\omega_1,\theta_2+x_2\omega_2)-E/\lambda|\leqslant\delta\}\tag{4.8}$$

one has the bounds

$$|(H_\Lambda(\underline{\theta})-E)^{-1}(\underline{x},\underline{y})|\leqslant \sum_{l\geqslant|\underline{x}-\underline{y}|}(\lambda^{-1}\delta^{-1})^{l+1}4^l\leqslant(1-4\lambda^{-1}\delta^{-1})^{-1}(4\lambda^{-1}\delta^{-1})^{|\underline{x}-\underline{y}|+1},$$

$$\|(H_\Lambda(\underline{\theta})-E)^{-1}\|\leqslant 2\delta^{-1}\lambda^{-1}.$$

The factor 4^l arises as upper bound on the number of nearest neighbor walks joining $\underline{x}$ to $\underline{y}$. Since the measure of the set in (4.8) is controlled by (4.3), the lemma follows by choosing $\delta=2\exp(-\sigma(\Lambda)^b)$ and $\lambda\geqslant\delta^{-4}$. □

For the meaning of semi-algebraic in the following lemma, see Definition 7.1.

LEMMA 4.2. *Let* $v(\theta_1,\theta_2)=\sum_{k,l=-D}^{D}a_{k,l}e(k\theta_1+l\theta_2)$ *be a real-valued trigonometric polynomial of degree* D *on* $\mathbf{T}^2$. *There is some absolute constant* C_0 *so that for any choice of* $\Lambda\subset\mathbf{Z}^2$ *the set* $\mathcal{B}^{\gamma,b}(\Lambda,E)\subset[0,1]^2$ *is semi-algebraic of degree no more than* $B=C_0D\sigma(\Lambda)^6$.

Proof. The conditions in the definition of the sets (4.4) and (4.5) can be rewritten in terms of determinants by means of Cramer's rule as in [7]. This shows that there exist polynomials $P_j(x_1,y_1,x_2,y_2)$, $1\leqslant j\leqslant s=\sigma(\Lambda)^4$, so that

$$\mathcal{G}^{\gamma,b}(\Lambda,E)=\bigcap_{j=1}^{s}\{\underline{\theta}\in\mathbf{T}^2: P_j(\sin\theta_1,\cos\theta_1,\sin\theta_2,\cos\theta_2)>0\} \tag{4.9}$$

and such that $\max_j\deg(P_j)\lesssim D\sigma(\Lambda)^2$. By Definition 7.1, $\mathcal{B}^{\gamma,b}(\Lambda,E)=\mathbf{T}^2\setminus\mathcal{G}^{\gamma,b}(\Lambda,E)$ is a closed semi-algebraic set of degree at most $\lesssim D\sigma(\Lambda)^6$. One now views $\mathbf{T}^2$ as a subset of $\mathbf{R}^4$ given by

$$x_1^2+y_1^2=1 \quad\text{and}\quad x_2^2+y_2^2=1.$$

In order to pass to the square $[0,1]^2$, one covers $\mathbf{T}^2$ by finitely many coordinate charts (16 suffice). More precisely, suppose that $y_1>1/\sqrt{2}$ and $-1/\sqrt{2}<x_1<1/\sqrt{2}$. Then one can write $y_1=\sqrt{1-x_1^2}$. Inserting this into an inequality of the form $P(x_1,y_1,x_2,y_2)\geqslant0$ one obtains that

$$Q_1(x_1,x_2,y_2)+\sqrt{1-x_1^2}\,Q_2(x_1,x_2,y_2)\geqslant0 \quad\text{and}\quad -\frac{1}{\sqrt{2}}<x_1<\frac{1}{\sqrt{2}},$$

where Q_1 and Q_2 are polynomials. Denote this set by $\mathcal{S}$. Suppressing x_2, y_2 for simplicity, one has

$$\begin{aligned}\mathcal{S}=&\Big\{Q_1(x_1)\geqslant0,\,Q_2(x_1)\geqslant0,\,-\frac{1}{\sqrt{2}}<x_1<\frac{1}{\sqrt{2}}\Big\}\\ &\cap\Big\{Q_1(x_1)<0,\,Q_2(x_1)\geqslant0,\,(1-x_1^2)Q_2^2(x_1)\geqslant Q_1^2(x_1),\,-\frac{1}{\sqrt{2}}<x_1<\frac{1}{\sqrt{2}}\Big\}\\ &\cap\Big\{Q_1(x_1)\geqslant0,\,Q_2(x_1)<0,\,(1-x_1^2)Q_2^2(x_1)\leqslant Q_1^2(x_1),\,-\frac{1}{\sqrt{2}}<x_1<\frac{1}{\sqrt{2}}\Big\}.\end{aligned}$$

Repeating this procedure in the variables x_2, y_2 one clearly obtains semi-algebraic sets in x_1 and x_2, say, and the degree has increased at most by some fixed factor. □

Remark 4.3. (1) Observe that in the previous proof $\mathcal{B}^{\gamma,b}(\Lambda, E)$ was shown to be semi-algebraic in the variables $\sin\theta_1$ and $\sin\theta_2$, say. In view of Remark 3.4 this distinction is irrelevant for our purposes.

(2) Since we choose v to be a real-analytic function, Lemma 4.2 does not apply directly. This, however, can be circumvented systematically by truncation. More precisely, given M, there is a trigonometric polynomial $P_M = P_M(\underline{\theta})$ of degree $\lesssim M$ so that

$$\|v - P_M\|_\infty < e^{-M}.$$

This follows from the fact that the Fourier coefficients of v decay exponentially. Hence, if

$$\|(-\Delta + \lambda V(\theta) - E)_\Lambda^{-1}\| < \lambda^{-1} e^{M^b},$$

as in the definition of (4.4), then also

$$\|(-\Delta + \lambda P_M(\theta) - E)_\Lambda^{-1}\| < 2\lambda^{-1} e^{M^b}$$

provided $M = \sigma(\Lambda)$ is large enough. A similar statement holds for the exponential decay. Strictly speaking, one should therefore replace v by P_M in the definitions (4.4) and (4.5) with $\sigma(\Lambda) = M$. In view of Lemma 4.2 these new sets are semi-algebraic of degree at most $C_0\sigma(\Lambda)^7$. For the sake of simplicity, however, we do not distinguish between v and P_M.

In this section, it is convenient for us to work with squares

$$Q_M(\underline{x}) := \{\underline{y} \in \mathbf{Z}^2 : x_1 - M \leqslant y_1 < x_1 + M,\ x_2 - M \leqslant y_2 < x_2 + M\} \tag{4.10}$$

rather than those defined in (2.1). This is relevant in connection with Figure 6, as will be explained in the following proof.

LEMMA 4.4. *Let $\delta_0 > 0$ be as in Lemma 3.3 and suppose that b, ϱ, γ are fixed positive numbers so that*

$$0 < b, \varrho < 1 \quad \textit{and} \quad b + \delta_0 > 1 + 3\varrho. \tag{4.11}$$

Let $N_0 \leqslant N_1$ be positive integers satisfying

$$\overline{N}_0(\gamma, b, \varrho) \leqslant 100 N_0 \leqslant N_1^\varrho$$

with some large constant $\overline{N}_0$ depending only on γ, b and ϱ. Assume that for any $N_0 \leqslant M \leqslant N_1$ and any $\Lambda \in \mathcal{ER}(M)$,

$$\sup_{\theta_i, E} \operatorname{mes}(\mathcal{B}_{\theta_i}^{\gamma,b}(\Lambda, E)) \leqslant \exp(-\sigma(\Lambda)^\varrho) \quad \textit{for } i = 1, 2. \tag{4.12}$$

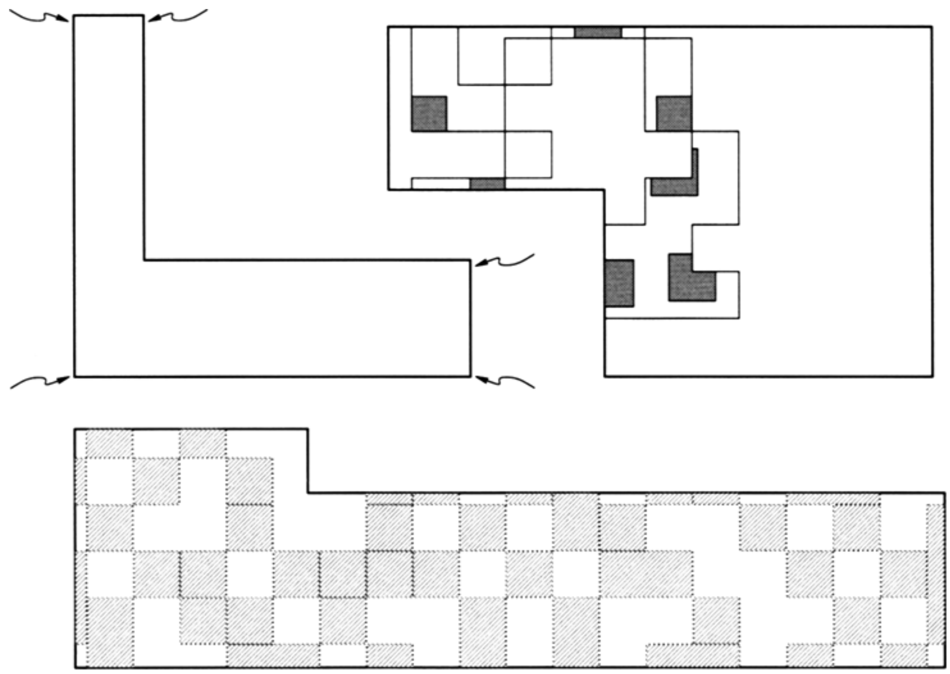

Fig. 5. Examples of partitions of Λ_0 and regions $W(\underline{x})$.

Assume moreover that

$$\underline{\omega} \in \bigcap_{\substack{N_0^{C_1} \leqslant N \leqslant N_1^{\varrho C_1} \\ N \text{ dyadic}}} \Omega_N,$$

where Ω_N is as in Lemma 3.1 and $C_1(b,\varrho) \gg 1/\varrho$ is a large constant depending only on b and ϱ. Then for all $\Lambda \in \mathcal{ER}(N)$,

$$\sup_{\theta_1, E} \operatorname{mes}[\theta_2 \in \mathbf{T} : \|G_\Lambda(\underline{\theta}, E)\| > e^{N^b}] < e^{-N^{3\varrho}},$$

provided $N_0^{C_1} \leqslant N \leqslant N_1^{\varrho C_1}$, and similarly with θ_1 and θ_2 interchanged.

Proof. Choose M_0 with $N_0 \leqslant M_0 \leqslant N_1$, and let N be given by $M_0 = \lceil N^{\varepsilon_0} \rceil$ where $\varepsilon_0 > 0$ is a small number that will be specified below (C_1 will be chosen to be ε_0^{-1}). Partition $\mathbf{Z}^2$ into squares $\{Q_\alpha\}$ where each Q_α is of the form $Q_{M_0}(\underline{x})$ and $\underline{x}$ belongs to the sublattice $2M_0\mathbf{Z}^2$. Let

$$\Lambda_0 = \bigcup_\alpha \Lambda_\alpha, \quad \text{where each } \Lambda_\alpha = Q_\alpha \cap \Lambda_0,$$

be the resulting partition of Λ_0. The union here runs only over all nonempty Λ_α. For each such α, with the possible exception of at most five values of α, one has that $\Lambda_\alpha \in$

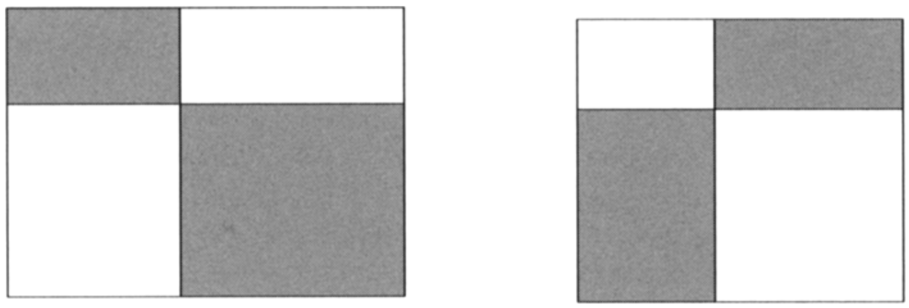

Fig. 6. Two good regions Λ_α and Λ_β meeting at one point.

$\mathcal{ER}(M')$ where $M_0 \leqslant M' \leqslant 2M_0$. These exceptional α are given by the corners of Λ_0, see Figure 5 where they are marked with arrows (observe that there Λ_α might have very small diameter). Let

$$\mathcal{A} := \bigcup_{M_0 \leqslant M \leqslant 2M_0} \bigcup_{\substack{\Lambda \in \mathcal{ER}(M) \\ \Lambda \subset [-M,M]^2}} \mathcal{B}^{\gamma,b}(\Lambda, E).$$

By Lemma 4.2, $\mathcal{A}$ is the semi-algebraic sets of degree at most $C_0 M_0^{14}$ (see Remark 4.3), and by hypothesis (4.12),

$$\max_{i=1,2} \sup_{\theta_i} \operatorname{mes}(\mathcal{A}_{\theta_i}) \lesssim M_0^7 \exp(-M_0^\varrho). \tag{4.13}$$

Fix some $\underline{\theta} \in \mathbf{T}^2$. By Lemma 3.3, and our choice of $\underline{\omega}$,

$$\#\{(n_1,n_2) \in [-N,N]^2 : (\theta_1 + n_1\omega_1, \theta_2 + n_2\omega_2) \in \mathcal{A} \ (\operatorname{mod} \mathbf{Z}^2)\} \lesssim N^{1-\delta_0}. \tag{4.14}$$

Here ε_0 needs to be chosen small enough, and then $\overline{N}_0$ large enough, such that condition (3.27) is fulfilled with $B = M_0^{14}$ and η equal to the right-hand side of (4.13). We say that Λ_α is *good* if

$$(\theta_1 + n_1\omega_1, \theta_2 + n_2\omega_2) \notin \mathcal{A} \quad (\operatorname{mod} \mathbf{Z}^2) \quad \text{for all } (n_1,n_2) \in \Lambda_\alpha.$$

Define the bad set $\Lambda_* \subset \Lambda_0$ as

$$\Lambda_* := \bigcup_{\alpha \text{ bad}} \Lambda_\alpha. \tag{4.15}$$

In view of (4.14),

$$\#\Lambda_* \lesssim M_0^2 N^{1-\delta_0}. \tag{4.16}$$

In addition, the at most six regions intersecting the corners are counted among the bad set. An example of a possible bad set is given in the lower region in Figure 5 (the shaded regions are supposed to be the good ones). Now consider the good set $\Lambda_*^c := \Lambda_0 \backslash \Lambda_*$

and fix any $\underline{x}\in\Lambda_\alpha\subset\Lambda_*^c$. It will follow from Lemma 2.2 that the norm of the Green's function $G_{\Lambda_*^c}(E)$ is not too large. In order to define the regions $W(\underline{x})$ appearing in that lemma, one needs to distinguish several cases. Evidently, $Q_{M_0}(\underline{x})$ intersects no more than three other Λ_β, $\beta\neq\alpha$. In case these regions are as in Figure 6, one lets $W(\underline{x}):=Q_{M_0}(\underline{x})\cap\Lambda_\alpha$. Otherwise, set $W(\underline{x}):=Q_{M_0}(\underline{x})\cap\Lambda_*^c$. A selection of such $W(\underline{x})$ is shown in the right-hand region of Figure 5. It is easy to see that each $W(\underline{x})$ is an elementary region with $M_0\leqslant\sigma(W(\underline{x}))\leqslant 2M_0$ satisfying $\operatorname{dist}(\underline{x},\partial_* W(\underline{x}))\geqslant M_0-1$ (here ∂_* stands for the interior boundary relative to Λ_*^c, cf. (2.5)). We want to call the reader's attention to an important detail in connection with the situation shown in Figure 6. Since we are working with squares defined by (4.10), the point $\underline{x}_0$ at which the two regions Λ_α and Λ_β meet belongs to at most one of them (in the left-hand situation of Figure 6 this point does not belong to either, in the right-hand situation it belongs to the upper shaded square). Moreover, if it belongs to Λ_α, then it has no immediate neighbors in Λ_β. For this reason the interior boundary of $W(\underline{x}_0)$ belongs entirely to Λ_α. Hence Lemma 2.2 with $N=2M_0$, $A=\lambda^{-1}e^{(2M_0)^b}$ and $t=\frac{1}{2}\gamma$ yields

$$\|G_{\Lambda_*^c}(\underline{\theta},E)\|\lesssim\lambda^{-1}M_0^2e^{(2M_0)^b}, \tag{4.17}$$

if $M_0^2e^{-\gamma M_0/2}\ll 1$. This bound is basically preserved inside a polydisk $B(\underline{\theta},e^{-M_0})\subset\mathbf{C}^2$. Indeed, by the standard Neumann series argument and (4.17),

$$\|G_{\Lambda_*^c}(\underline{\theta}',E)\|\leqslant\|[I-\lambda(V_{\underline{\theta}}-V_{\underline{\theta}'})G_{\Lambda_*^c}(\underline{\theta},E)]^{-1}\|\,\|G_{\Lambda_*^c}(\underline{\theta},E)\|\leqslant 2\|G_{\Lambda_*^c}(\underline{\theta},E)\|, \tag{4.18}$$

provided $|\underline{\theta}'-\underline{\theta}|<e^{-M_0}$. Define a matrix-valued analytic function $A(\underline{\theta}')$ on $B(\underline{\theta},e^{-M_0})$ as

$$A(\underline{\theta}')=R_{\Lambda_*}H(\underline{\theta}')R_{\Lambda_*}-R_{\Lambda_*}H(\underline{\theta}')R_{\Lambda_*^c}G_{\Lambda_*^c}(\underline{\theta}',E)R_{\Lambda_*^c}H(\underline{\theta}')R_{\Lambda_*}. \tag{4.19}$$

In view of (4.18),

$$\log|\det A(\underline{\theta}')|\lesssim M_0\#\Lambda_*\lesssim M_0^3N^{1-\delta_0}. \tag{4.20}$$

Furthermore, Lemma 4.8 and (4.18) imply that

$$\|A(\underline{\theta}')^{-1}\|\lesssim\|G_{\Lambda_0}(\underline{\theta}',E)\|\lesssim e^{2M_0}\|A(\underline{\theta}')^{-1}\|. \tag{4.21}$$

Fix the variable $\theta_1'=\theta_1$ and let $|\theta_2-\theta_2'|<e^{-M_0}$. Introduce a new scale M_1 so that $M_1^\varrho=\lceil 10M_0\rceil$. For each $\underline{x}\in\Lambda_0$ define an elementary region $\widetilde{W}(\underline{x}):=Q_{M_1}(\underline{x})\cap\Lambda_0$. Applying (4.12) at scale M_1 yields a set $\Theta\subset\mathbf{T}$ of measure

$$\operatorname{mes}(\Theta)\lesssim N^2e^{-M_1^\varrho}\lesssim e^{-M_1^\varrho/2} \tag{4.22}$$

and so that for any $y\in\mathbf{T}\setminus\Theta$ the Green's function $G_{\widetilde{W}(\underline{x})}(\theta_1,y,E)$ satisfies the conditions of Lemma 2.2 for all $\underline{x}\in\Lambda_0$. Lemma 2.2 with $N=2M_1$, $A=e^{(2M_1)^b}$ and $t=\frac{1}{2}\gamma$ therefore implies that

$$\|G_{\Lambda_0}(\theta_1,y)\|\lesssim e^{M_1}\quad\text{for all } y\in\mathbf{T}\setminus\Theta. \tag{4.23}$$

By our choice of M_1 there is some y_0 for which (4.23) holds and so that $|y_0-\theta_2|<\frac{1}{20}e^{-M_0}$. In view of (4.21) this implies that

$$\begin{aligned}\|A(\theta_1,y_0)^{-1}\| &\lesssim e^{M_1},\\ |\det A(\theta_1,y_0)| &\gtrsim e^{-M_1|\Lambda_*|},\\ \log|\det A(\theta_1,y_0)| &\gtrsim -M_1|\Lambda_*|\gtrsim -M_1M_0^2N^{1-\delta_0},\end{aligned} \tag{4.24}$$

see (4.15). Recalling (4.20), there is the universal upper bound

$$\log|\det A(\theta_1,z)|\lesssim M_0^3N^{1-\delta_0}\quad\text{for all } |z-y_0|\leqslant\tfrac{1}{2}e^{-M_0}. \tag{4.25}$$

Define the function

$$F(w):=\det A\big(\theta_1,y_0+\tfrac{1}{4}e^{-M_0}w\big)\quad\text{where } |w|\leqslant 2. \tag{4.26}$$

Since A is analytic, $\log|F|$ is a subharmonic function on the disk $\mathbf{D}_2:=[|w|\leqslant 2]$ satisfying

$$\log|F(w)|\lesssim M_0^3N^{1-\delta_0},\quad \log|F(0)|\gtrsim -M_1M_0^2N^{1-\delta_0}.$$

For any $0<r<2$ the submean value property of $\log|F|$ implies that

$$\frac{1}{2\pi}\int_0^{2\pi}\log|F(re^{i\phi})|\,d\phi\gtrsim -M_1M_0^2N^{1-\delta_0},$$

which in turn leads to the L^1-bounds

$$\int|\log|F(re^{i\phi})||\,d\phi=2\int\log^+|F(re^{i\phi})|\,d\phi-\int\log|F(re^{i\phi})|\,d\phi\lesssim M_1M_0^2N^{1-\delta_0}, \tag{4.27}$$

$$\int_{[r<2]}|\log|F(re^{i\phi})||\,r\,dr\,d\phi\lesssim M_1M_0^2N^{1-\delta_0}. \tag{4.28}$$

As a subharmonic function, $\log|F|$ has a unique Riesz representation on $\mathbf{D}=[|w|<1]$, see Levin [21, §7.2]:

$$\log|F(w)|=\int_{\mathbf{D}}\log|w-w'|\,d\mu(w')+h(w). \tag{4.29}$$

$\mu=(1/2\pi)\Delta\log|F|\geqslant 0$ is a measure on $\mathbf{D}$ of mass bounded by, see (4.28),

$$2\pi\mu(\mathbf{D})=\int_{\mathbf{D}}\Delta\log|F|<\left|\int\log|F|\Delta\zeta\right|\lesssim\|\log|F|\|_{L^1(\mathbf{D}_2)}\lesssim M_1M_0^2N^{1-\delta_0}, \tag{4.30}$$

where $\zeta\geqslant 0$ is a smooth bump function, $\operatorname{supp}(\zeta)\subset \mathbf{D}_2$, $\zeta=1$ on $\mathbf{D}$. Furthermore, the harmonic function h on $\mathbf{D}$ is given by

$$h(re^{i\phi})=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}\log|F(e^{i\theta})|\,d\theta-\int_{\mathbf{D}}\log|1-w\bar{w}'|\,d\mu(w').$$

In particular,

$$\sup_{|w|\leqslant 1/2}|h(w)|\lesssim\int|\log|F(e^{i\theta})||\,d\theta+\mu(\mathbf{D})\lesssim M_1M_0^2N^{1-\delta_0}. \tag{4.31}$$

Combining the bounds on μ and h with (4.29) yields

$$\begin{gathered}\|\log|F(t)|\|_{\mathrm{BMO}[-1/2,1/2]}\lesssim M_1M_0^2N^{1-\delta_0},\\ \operatorname{mes}\big[t\in\big[-\tfrac12,\tfrac12\big]:|\log|F(t)|-\langle\log|F|\rangle|>N^{1-\delta_0+\tau}\big]\lesssim\exp\biggl(-c\frac{N^\tau}{M_1M_0^2}\biggr)\end{gathered} \tag{4.32}$$

for any $\tau>0$, where $\langle\log|F|\rangle$ denotes the mean on $\big[-\frac12,\frac12\big]$. The constant c is an absolute one provided by the John–Nirenberg inequality. Estimates (4.30), (4.31) and the representation (4.29) imply

$$|\langle\log|F|\rangle|\lesssim M_1M_0^2N^{1-\delta_0}.$$

Recalling the definition (4.26) of F, (4.32) therefore implies the bound

$$\operatorname{mes}[y\in I:\log|\det A(\theta_1,y)|\lesssim -M_1M_0^2N^{1-\delta_0+\tau}]\lesssim e^{-M_0}\exp\biggl(-c\frac{N^\tau}{M_1M_0^2}\biggr),$$

where $I=\big(\theta_2-\frac12e^{-M_0},\theta_2+\frac12e^{-M_0}\big)$ (this estimate could also have been obtained via Cartan's theorem, see [21, §11.2]). If y is not in the set on the left-hand side, then

$$\|A(\theta_1,y)^{-1}\|\lesssim e^{C|\Lambda_*|}e^{CM_1M_0^2N^{1-\delta_0+\tau}}.$$

Combining this with (4.21) and covering $\mathbf{T}$ by intervals I of size e^{-M_0} yields

$$\|G_{\Lambda_0}(\theta_1,\theta_2,E)\|\lesssim e^{2M_0}e^{CM_1M_0^2N^{1-\delta_0+\tau}} \tag{4.33}$$

for all $y\in\mathbf{T}\setminus\mathcal{S}_{\theta_1}$ where

$$\operatorname{mes}(\mathcal{S}_{\theta_1})\lesssim\exp\biggl(-c\frac{N^\tau}{M_1M_0^2}\biggr). \tag{4.34}$$

Since $M_0\asymp N^{\varepsilon_0}$ and $M_1\asymp N^{\varepsilon_0/\varrho}$, this proves the lemma provided

$$N^b\gtrsim N^{1-\delta_0+3\varepsilon_0/\varrho+\tau},\quad N^{3\varrho}\lesssim N^{\tau-3\varepsilon_0/\varrho}.$$

Choosing $\tau=3\varrho+3\varepsilon_0/\varrho+$, there exists $\varepsilon_0>0$ so small that

$$b>1-\delta_0+3\varrho+\frac{6\varepsilon_0}{\varrho},$$

see (4.11). Letting $C_1=\varepsilon_0^{-1}$, this finishes the proof. □

The following corollary combines the previous lemma with Lemma 2.4 in order to obtain exponential off-diagonal decay.

COROLLARY 4.5. *Suppose that all the assumptions of Lemma* 4.4 *are valid. Furthermore, let* $N_1=\lfloor N_0^{C_1}\rfloor$ *where* C_1 *is the constant from Lemma* 4.4. *Then for all* $\Lambda\in\mathcal{ER}(N)$,

$$\sup_{\theta_i,E}\operatorname{mes}(\mathcal{B}_{\theta_i}^{\gamma',b}(\Lambda,E))<\exp(-N^{\varrho})$$

for any $N_1\leqslant N\leqslant N_1^2$, *where* $\gamma'=\gamma-N^{-\delta}$, $\delta=\delta(b,\gamma)>0$.

Proof. Recall that $C_1\gg 1/\varrho$, so that it is possible to satisfy $100N_0\leqslant N_1^{\varrho}\leqslant N_0^{\varrho C_1}$, as required. Fix some $N\in[N_1,N_1^2]$ and $\Lambda_0\in\mathcal{ER}(N)$. Let $M_0=N_1^{\varepsilon}$ and define

$$\mathcal{A}:=\bigcup_{M_0+1\leqslant L\leqslant 2M_0+1}\ \bigcup_{\Lambda_1\in\mathcal{ER}(L)}\mathcal{B}^{\gamma,b}(\Lambda,E).$$

By Lemma 4.2 and Remark 4.3, $\mathcal{A}$ is semi-algebraic of degree at most $C_0M_0^{14}=N_1^{14\varepsilon}$. Choose ε small enough so that conditions (3.27) hold with $B\asymp M_0^{14}$. On the other hand, we also require that $M_0\geqslant N_0$ so that (4.12) is satisfied at scale M_0. This can be done provided C_1 is chosen large enough (in fact, inspection of the proof of Lemma 4.4 shows that there $\varepsilon_0=C_1^{-1}$ was chosen sufficiently small to verify (3.27), so that one may set $\varepsilon=\varepsilon_0$). Hence, for N_0 large, we may apply Lemma 3.3 to conclude that for any choice of $\underline{\theta}\in\mathbf{T}^2$

$$\#\{(n_1,n_2)\in[-N,N]^2:(\theta_1+n_1\omega_1,\theta_2+n_2\omega_2)\in\mathcal{A}\ (\mathrm{mod}\ \mathbf{Z}^2)\}\lesssim N^{1-\delta_0}. \tag{4.35}$$

Now suppose that $\Lambda_1\in\mathcal{ER}(M')$ has the following property, where $N_1+1\leqslant M'\leqslant 2N_1+1$: for every $\underline{x}\in\Lambda_1$ the Green's function $G_{W(\underline{x})}(\underline{\theta},E)$ of the elementary region $W(\underline{x}):=Q_{M_0}(\underline{x})\cap\Lambda_1$ satisfies

$$|G_{W(\underline{x})}(\underline{\theta},E)(\underline{x},\underline{y})|\leqslant e^{-\gamma|\underline{x}-\underline{y}|}\quad\text{for every }\underline{y}\in\partial_*W(\underline{x}). \tag{4.36}$$

Here the interior boundary ∂_* is defined relative to Λ_1, see (2.5). A standard application of the resolvent identity then shows that

$$|G_{\Lambda_1}(\underline{\theta},E)(\underline{x},\underline{y})|\leqslant e^{-\gamma|\underline{x}-\underline{y}|+CM_0}\quad\text{for every }\underline{x},\underline{y}\in\Lambda_1,\ |\underline{x}-\underline{y}|>\tfrac{1}{4}M'.$$

On the other hand, suppose that $\mathcal{F}_1$ is a family of pairwise disjoint elementary regions $\Lambda_1\in\mathcal{ER}(M')$ with M' as above, so that for each Λ_1 there is at least one $\underline{x}\in\Lambda_1$ violating (4.36). In view of (4.35), this implies that

$$\#\mathcal{F}_1 < N^{1-\delta_0}. \tag{4.37}$$

Now fix θ_1. By Lemma 4.4, the θ_2-set where some $\Lambda_1\subset\Lambda_0$ with $\Lambda_1\in\mathcal{ER}(L)$, $N_1\leqslant L\leqslant N_1^2$, violates

$$\|G_{\Lambda_1}(\theta_1,\theta_2,E)\| < e^{L^b} \tag{4.38}$$

is no larger than $N_1^8 e^{-N_1^{3\varrho}} < e^{-N^\varrho}$. For θ_2 off this set Lemma 2.4 with $M=N_1$, $\tau\geqslant\frac{1}{2}$, implies that, see (4.37) and (4.38),

$$|G_{\Lambda_0}(\underline{\theta},E)(\underline{x},\underline{y})| \leqslant e^{-\gamma'|\underline{x}-\underline{y}|} \quad \text{for every } \underline{x},\underline{y}\in\Lambda_1,\ |\underline{x}-\underline{y}|>\tfrac{1}{4}N,$$

with $\gamma'=\gamma-N^{-\delta}$, as claimed. □

The following proposition is the main result of this section. It follows from Lemma 4.1 and Corollary 4.5 by means of induction.

PROPOSITION 4.6. *Let v be a real-analytic function satisfying* (4.2). *Let $\underline{\omega}\in\Omega$, see* (4.6). *Then for sufficiently large $\lambda\geqslant\lambda_0(v,\underline{\omega})$ and all $N\geqslant N_0(v,\underline{\omega})$ there is the estimate*

$$\sup_{\theta_i,E} \operatorname{mes}(\mathcal{B}_{\theta_i}^{\gamma,b}(\Lambda,E)) \leqslant \exp(-\sigma(\Lambda)^\varrho) \quad \text{for } i=1,2, \tag{4.39}$$

for any $\Lambda\in\mathcal{ER}(N)$ with $\gamma=\frac{1}{4}\log\lambda$ and some constants $0<b,\varrho<1$.

Proof. Fix positive numbers b,ϱ satisfying (4.11). Choose N_0 sufficiently large so that both Lemma 4.1 holds and $\underline{\omega}\in\bigcap_{N\geqslant N_0}\Omega_N$. Let C_1 be the constant from Lemma 4.4 and let λ_0 be so large that Lemma 4.1 holds in the whole range $[N_0,N_0^{C_1}]$ with $\gamma=\gamma_0:=\frac{1}{2}\log\lambda$. In view of Lemma 4.4 and Corollary 4.5, the bound (4.39) holds for all $N\in[N_1,N_1^2]$ with $\gamma=\gamma_1:=\gamma_0-N_1^{-\delta}$. One can now continue inductively applying Lemma 4.4 and Corollary 4.5 to cover the interval $[N_2,N_2^2]$ where $N_2=N_1^2$ and $\gamma=\gamma_2:=\gamma_1-N_2^{-\delta}$, etc. It is evident that always $\gamma>\frac{1}{2}\gamma_0$ if N_0 is large enough. □

Remark 4.7. It is clear that for (4.39) to hold up to scale N and $\lambda\geqslant\lambda_0(N_0)$ where λ_0 is some sufficiently large number, it suffices to assume that

$$\underline{\omega}\in\bigcap_{\substack{N_0\leqslant M\leqslant N\\ N \text{ dyadic}}}\Omega_M.$$

Furthermore, for any $\varepsilon>0$ one can choose N_0 such that

$$\operatorname{mes}\Big(\mathbf{T}^2\setminus\bigcap_{\substack{N_0\leqslant M\\ M\text{ dyadic}}}\Omega_M\Big)<\varepsilon.$$

These properties will be relevant for the proof of localization in §6.

The following lemma is a standard fact that was used above. The simple proof is included for the reader's convenience.

LEMMA 4.8. *Let M be the matrix*

$$M=\begin{bmatrix} B & U\\ U^t & C\end{bmatrix},$$

where B is an invertible $(n\times n)$-matrix, U is an $(n\times k)$-matrix, and C is a $(k\times k)$-matrix. Let $A=C-U^tB^{-1}U$. Then M is invertible if and only if A is invertible, and

$$\|A^{-1}\|\lesssim\|M^{-1}\|\lesssim(1+\|B^{-1}\|)^2(1+\|A^{-1}\|),$$

where the constants only depend on $\|U\|$.

Proof. This follows from the identity

$$\begin{bmatrix} I & 0\\ -U^tB^{-1} & I\end{bmatrix}\begin{bmatrix} B & U\\ U^t & C\end{bmatrix}\begin{bmatrix} B^{-1} & 0\\ 0 & I\end{bmatrix}=\begin{bmatrix} I & U\\ 0 & C-U^tB^{-1}U\end{bmatrix}.$$ □

5. The elimination of the energy and semi-algebraic sets in $\mathbf{T}^4$

The final step to get localization is to establish the following key result on semi-algebraic sets needed in the energy elimination argument. The significance of Proposition 5.1 in connection with the elimination of the energy will become clear in the proof of localization given in the following section.

PROPOSITION 5.1. *Let $\mathcal{A}\subset\mathbf{T}^4$ be a semi-algebraic set of elements*

$$(\underline{\omega},\underline{\theta})=(\omega_1,\omega_2,\theta_1,\theta_2)$$

so that

$$\deg\mathcal{A}<B$$

and

$$\operatorname{mes}_{\mathbf{T}}(\mathcal{A}_{\omega_1,\omega_2,\theta_1})<\eta,\quad \operatorname{mes}_{\mathbf{T}}(\mathcal{A}_{\omega_1,\omega_2,\theta_2})<\eta\quad \textit{for all sections.}\tag{5.1}$$

Let

$$\log B \ll \log K \ll \log \frac{1}{\eta}. \tag{5.2}$$

Then there exists a set $\mathfrak{F}_K$ *of measure*

$$\operatorname{mes} \mathfrak{F}_K < K^{-1/100} \tag{5.3}$$

and such that for any $\underline{\omega}=(\omega_1,\omega_2)\notin \mathfrak{F}_K \subset \mathbf{T}^2$ *we may ensure that*

$$\{(\underline{\omega}, n_1\omega_1, n_2\omega_2) \pmod{\mathbf{Z}^4} : |n_1|\vee|n_2| \asymp K\} \cap \mathcal{A} = \varnothing. \tag{5.4}$$

The one-dimensional version of this result, which is given in the following lemma, was established by Bourgain and Goldstein [7].

LEMMA 5.2. *Let* $\mathcal{A}_1 \subset \mathbf{T}^2$ *be semi-algebraic,* $\deg \mathcal{A}_1 < B$, $\operatorname{mes} \mathcal{A}_1 < \eta$ *so that* (5.2) *holds. Then*

$$\int_{\mathbf{T}_{\mathrm{D.C.}_b}} \sum_{n\asymp K} \chi_{\mathcal{A}_1}(\omega, n\omega)\, d\omega < \frac{B^{C_1}}{K} + K^{C_2} \operatorname{mes} \mathcal{A}_1. \tag{5.5}$$

Here $\mathbf{T}_{\mathrm{D.C.}_b}$ *refers to those points satisfying a Diophantine condition with parameter* b. *The constant* C_2 *depends on* b.

For the detailed proof we refer the reader to [7], see Lemma 6.1. The origin of the two terms on the right-hand side of (5.5) is easy to explain. In fact, observe that any horizontal line can intersect $\mathcal{A}_1$ in at most B^C many intervals. This follows from the fact that any such section $\mathcal{A}_{\theta_2}$ is again semi-algebraic of degree at most B, and therefore consists of no more than B^C intervals, see Definition 7.1. The first term in (5.5) arises if the set $\mathcal{A}_1$ consists of very thin neighborhoods of lines of slope $\asymp K$. Since no horizontal line can intersect $\mathcal{A}_1$ in more than B^C many intervals, there can be no more than B^C of these neighborhoods. Since each of them projects onto the ω-axis as an interval of size $\asymp 1/K$, one obtains a contribution of the form B^C/K. This is sketched in Figure 7 by means of the steep lines on the left-hand side. On the other hand, if $\mathcal{A}_1$ contains a horizontal strip of width η (see Figure 7), then the contribution to the sum in (5.5) is equal to $K\eta$. Observe that the first term in (5.5), which is usually the larger one since $\operatorname{mes} \mathcal{A}_1$ is very small, derives from the "almost vertical" pieces of $\mathcal{A}_1$. This intuition also applies to the two-dimensional version stated in Proposition 5.1. More precisely, let us suppose that $\mathcal{A}$ is contained in a small neighborhood of a zero-set $[P=0]$ where $P(\omega_1,\omega_2,\theta_1,\theta_2)$ is a polynomial of degree B. Then we need to control the number of times the hypersurface $[P=0]$ is close to being vertical, i.e., where $|\nabla_{\underline{\theta}} P| \leqslant \delta |\nabla_{\underline{\omega}} P|$ on $[P=0]$ with some small $\delta>0$. Lemma 5.3 is the required tool for this purpose.

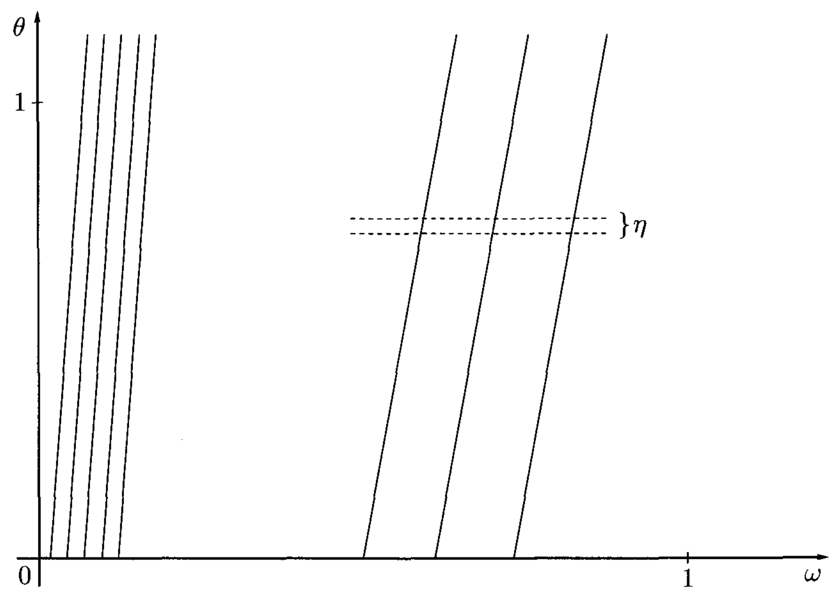

Fig. 7. The geometry of Lemma 5.2.

LEMMA 5.3. *Let $P(\omega_1, \omega_2, \theta_1, \theta_2)$ be a polynomial of degree B and $\delta > 0$. Define*

$$\mathfrak{V} := \left\{ (\underline{\omega}, \underline{\theta}) \in \mathbf{T}^4 : \max_{i=1,2} |\partial_{\theta_i} P| \leqslant \delta \max_{i=1,2} |\partial_{\omega_i} P| \right\} \cap [P=0].$$

Then

$$\operatorname{mes}_{\mathbf{T}^2}(\operatorname{Proj}_{\underline{\omega}}(\mathfrak{V})) < B^C \delta.$$

Proof. Let $\mathfrak{V} = \mathfrak{V}_1 \cup \mathfrak{V}_2$ where

$$\mathfrak{V}_j = \left\{ (\underline{\omega}, \underline{\theta}) \in \mathbf{T}^4 : \max_{i=1,2} |\partial_{\theta_i} P| \leqslant \delta |\partial_{\omega_j} P| \right\} \cap [P=0]$$

(this set is still semi-algebraic of degree $\lesssim B$). Take $j=1$ and consider the section $\mathfrak{V}_{1,\omega_2} \subset \mathbf{T}^3$. Fix ω_2 and write

$$\mathfrak{V}_{1,\omega_2} = \bigcup_{\alpha < B^C} \mathfrak{V}_{1,\omega_2}^{\alpha}$$

with $\mathfrak{V}_{1,\omega_2}^{\alpha}$ connected. We need to estimate the measure of the intervals

$$I_\alpha = [a_\alpha, b_\alpha] = \operatorname{Proj}_{\omega_1}(\mathfrak{V}_{1,\omega_2}^{\alpha}) \subset \mathbf{T}.$$

There is a piecewise analytic curve $\gamma(s) = (\omega_1(s), \theta_1(s), \theta_2(s))$, $s \in [0,1]$, such that

$$\begin{gathered} \gamma(s) \in \mathfrak{V}_{1,\omega_2}^{\alpha}, \\ \omega_1(0) = a_\alpha, \quad \omega_1(1) = b_\alpha, \\ |\dot{\gamma}(s)| < B^C. \end{gathered} \tag{5.6}$$

The existence of such a curve with a power-like bound (5.6) in the degree follows from Yomdin's quantitative triangulation Theorem 7.4 in §7. Since

$$P(\omega_1(s),\omega_2,\theta_1(s),\theta_2(s))=0$$

it follows that

$$\partial_{\omega_1}P(\gamma(s))\cdot\dot\omega_1(s)+\partial_{\theta_1}P\cdot\dot\theta_1(s)+\partial_{\theta_2}P\cdot\dot\theta_2(s)=0,$$

$$|\dot\omega_1(s)|<\left\{\max_{\mathfrak{V}_1}\frac{|\partial_{\theta_1}P|+|\partial_{\theta_2}P|}{|\partial_{\omega_1}P|}\right\}[|\dot\theta_1|+|\dot\theta_2|]<B^C\delta,$$

from (5.6) and the definition of $\mathfrak{V}_1$. Thus

$$|I_\alpha|=b_\alpha-a_\alpha=\omega_1(1)-\omega_1(0)\leqslant\int_0^1|\dot\omega_1|<B^C\delta.$$

Hence

$$\mathrm{mes}_{\mathbf{T}}[\mathrm{Proj}_{\omega_1}(\mathfrak{V}_{1,\omega_2})]<B^C\delta,$$

$$\mathrm{mes}_{\mathbf{T}^2}(\mathrm{Proj}_{\underline{\omega}}(\mathfrak{V}_1))=\int_{\mathbf{T}}\mathrm{mes}_{\mathbf{T}}[\mathrm{Proj}_{\omega_1}(\mathfrak{V}_{1,\omega_2})]\,d\omega_2<B^C\delta,$$

and the lemma follows. □

Proof of Proposition 5.1. Lemma 5.2 allows us to control the contribution to (5.4) by those pairs $(n_1,n_2)\in\mathbf{Z}_+^2$ satisfying $\min(n_1,n_2)<K^{1/2}$. In fact,

$$\begin{aligned}\iint&\sum_{\substack{0<n_1<K^{1/2}\\ n_2\asymp K}}\chi_{\mathcal{A}}(\omega_1,\omega_2,n_1\omega_1,n_2\omega_2)\,d\omega_1\,d\omega_2\\ &\overset{\text{by (5.5)}}{<}\sum_{0<n_1<K^{1/2}}\int d\omega_1\left[\frac{B^C}{K}+K^C\int\chi_{\mathcal{A}}(\omega_1,\omega_2,n_1\omega_1,\theta_2)\,d\omega_2\,d\theta_2\right]\\ &<B^CK^{-1/2}+K^{C+1/2}\sup_{\omega_1,\omega_2,\theta_1}(\mathrm{mes}\,\mathcal{A}_{\omega_1,\omega_2,\theta_1})\\ &\overset{\text{by (5.1)}}{<}B^CK^{-1/2}+K^{C+1/2}\eta<K^{-1/3}.\end{aligned}$$

We may therefore assume that $\min(n_1,n_2)>K^{1/2}$. Since $\mathrm{mes}\,\mathcal{A}<\eta$ and since $\log B\ll\log K$, it suffices to consider the case where $\mathcal{A}$ is contained in an $\eta^{1/2}$-neighborhood of a zero-set $[P=0]$, $P(\omega_1,\omega_2,\theta_1,\theta_2)$ being a polynomial of degree less than B^C. Take

$$\delta=K^{-1/10}. \tag{5.7}$$

Let $\mathfrak{V}$ be as in Lemma 5.3. Thus $\mathfrak{V}'=\mathrm{Proj}_{\underline{\omega}}(\mathfrak{V})$ is a semi-algebraic set in $\mathbf{T}^2$ of degree bounded by B^C, see Theorem 7.2. Since $\operatorname{mes}\mathfrak{V}'<B^C\delta$, $\mathfrak{V}'$ lies within a $\delta_1=B^C\delta^{1/2}$-neighborhood of $\partial\mathfrak{V}'$ and may therefore be covered by $B^C\delta_1^{-1}$ disks of radius δ_1. The total measure in $\underline{\omega}$-space of the union of those δ_1-disks is thus at most

$$B^C\delta_1<B^CK^{-1/20}. \tag{5.8}$$

Restrict next $\underline{\omega}$ to a δ_1-disk Q outside $\mathfrak{V}'$. Fix $K^{1/2}<n_1,n_2<K$. We estimate (everything being understood mod $\mathbf{Z}^2$)

$$\operatorname{mes}[\underline{\omega}\in Q:(\omega_1,\omega_2,n_1\omega_1,n_2\omega_2)\in\mathcal{A}]\leqslant\operatorname{mes}[\underline{\omega}\in Q:\operatorname{dist}((\omega_1,\omega_2,n_1\omega_1,n_2\omega_2),[P=0])<\eta].$$

Since $Q\cap\mathfrak{V}'=\varnothing$,

$$\max_i|\partial_{\theta_i}P|>\delta\max_i|\partial_{\omega_i}P| \tag{5.9}$$

in each point of $[P=0]\cap(Q\times\mathbf{T}^2)$. Let

$$\begin{aligned}[P=0]\cap(Q\times\mathbf{T}^2)&=\mathfrak{S}_1\cup\mathfrak{S}_2,\\ \mathfrak{S}_1&=[P=0]\cap(Q\times\mathbf{T}^2)\cap[|\partial_{\theta_1}P|\geqslant|\partial_{\theta_2}P|],\\ \mathfrak{S}_2&=[P=0]\cap(Q\times\mathbf{T}^2)\cap[|\partial_{\theta_1}P|<|\partial_{\theta_2}P|].\end{aligned} \tag{5.10}$$

It suffices to estimate

$$\operatorname{mes}[\underline{\omega}\in Q:\operatorname{dist}((\underline{\omega},n_1\omega_1,n_2\omega_2),\mathfrak{S}_1)<\eta^{1/2}], \tag{5.11}$$

where $n_i\omega_i$ is understood mod 1. Restrict ω_1 to an interval of size $1/n_1$, say $[0,1/n_1]$ (by translation). Consider the segment

$$S=[(\omega_1,\omega_{2,0},n_1\omega_1,\theta_{2,0}):\omega_1\in[0,1/n_1]]\subset\mathbf{T}^4, \tag{5.12}$$

and denote by Γ the intersection of $\mathfrak{S}_1$ with a $\sqrt{\eta}$-tube around S, see Figure 8. Then one has

$$\Gamma=\mathfrak{S}_1\cap[\operatorname{dist}((\underline{\omega},\underline{\theta}),S)<\eta^{1/2}]=\bigcup_{\alpha<B^C}\Gamma_\alpha,$$

where Γ_α are the connected components. Thus

$$\begin{aligned}&\operatorname{mes}[\omega_1\in[0,1/n_1]:\operatorname{dist}((\omega_1,\omega_{2,0},n_1\omega_1,\theta_{2,0}),\mathfrak{S}_1)<\eta^{1/2}]\\ &\quad=\operatorname{mes}[\omega_1\in[0,1/n_1]:\operatorname{dist}((\omega_1,\omega_{2,0},n_1\omega_1,\theta_{2,0}),\Gamma)<\eta^{1/2}]\\ &\quad\leqslant\sum_\alpha\operatorname{mes}[\omega_1\in[0,1/n_1]:\operatorname{dist}((\omega_1,\omega_{2,0},n_1\omega_1,\theta_{2,0}),\Gamma_\alpha)<\eta^{1/2}]\\ &\quad\leqslant\sum_\alpha[|\operatorname{Proj}_{\omega_1}(\Gamma_\alpha)|+2\eta^{1/2}].\end{aligned} \tag{5.13}$$

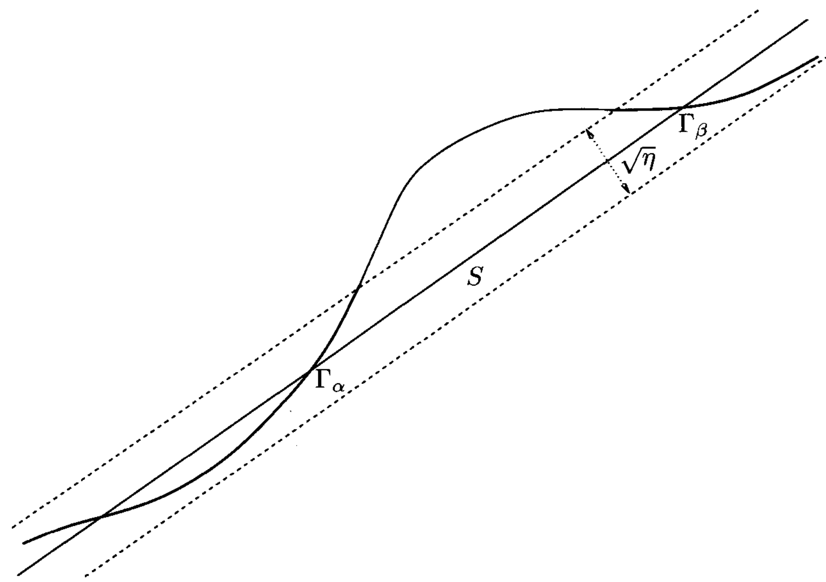

Fig. 8. The components of Γ in a $\sqrt{\eta}$-neighborhood of S.

Fix α and denote

$$I=\mathrm{Proj}_{\omega_1}(\Gamma_\alpha), \tag{5.14}$$

which is an interval. Let I' be an $\eta^{1/2}$-neighborhood of I. Thus if $(\underline{\omega},\underline{\theta})\in\Gamma_\alpha$, then

$$\mathrm{dist}((\underline{\omega},\underline{\theta}), S\cap[\omega_1\in I'])<\eta^{1/2}$$

and, from the definition of S,

$$\Gamma_\alpha\subset I'\times[|\omega_2-\omega_{2,0}|<\eta^{1/2}]\times n_1 I'\times[|\theta_2-\theta_{2,0}|<\eta^{1/2}]. \tag{5.15}$$

Denote

$$\varrho=|I'|>\eta^{1/2}. \tag{5.16}$$

The right-hand side of (5.15) is a $(\varrho\times\eta^{1/2}\times n_1\varrho\times\eta^{1/2})$-box that may be rescaled to the unit cube by a $(1/\varrho\times 1/\eta^{1/2}\times 1/n_1\varrho\times 1/\eta^{1/2})$-dilation. Since Γ_α is semi-algebraic of degree less than B^C, for given points $P_0, P_1\in\Gamma_\alpha$, we may again by means of Theorem 7.4 obtain a curve $\gamma(s)=(\omega_1(s),\omega_2(s),\theta_1(s),\theta_2(s))$ in Γ_α such that

$$\gamma(0)=P_0,\quad \gamma(1)=P_1,\quad \frac{|\dot\omega_1|}{\varrho}+\frac{|\dot\omega_2|}{\eta^{1/2}}+\frac{|\dot\theta_1|}{n_1\varrho}+\frac{|\dot\theta_2|}{\eta^{1/2}}<B^C.$$

Since $\gamma(s)\in\mathfrak{S}_1\subset[P=0]$,

$$\partial_{\omega_1}P(\gamma(s))\cdot\dot\omega_1(s)+\partial_{\omega_2}P\cdot\dot\omega_2+\partial_{\theta_1}P\cdot\dot\theta_1+\partial_{\theta_2}P\cdot\dot\theta_2=0. \tag{5.17}$$

From (5.9) and (5.10),

$$|(\partial_{\theta_1}P)(\gamma(s))|>\delta[|\partial_{\omega_1}P|+|\partial_{\omega_2}P|]\vee|\partial_{\theta_2}P|, \tag{5.18}$$

and hence from (5.17) and (5.18),

$$|\dot\theta_1|<\frac{|\partial_{\omega_1}P|+|\partial_{\omega_2}P|}{|\partial_{\theta_1}P|}(|\dot\omega_1|+|\dot\omega_2|)+\frac{|\partial_{\theta_2}P|}{|\partial_{\theta_1}P|}|\dot\theta_2|<\frac{1}{\delta}(\varrho+\sqrt{\eta})B^C+\sqrt{\eta}\,B^C\lesssim\frac{\varrho}{\delta}B^C,$$

$$|\theta_1(1)-\theta_1(0)|\lesssim\frac{\varrho}{\delta}B^C.$$

It follows that

$$\operatorname{diam}(\operatorname{Proj}_{\theta_1}(\Gamma_\alpha))<\frac{\varrho B^C}{\delta}. \tag{5.19}$$

Denote by J an $\eta^{1/2}$-neighborhood of $\operatorname{Proj}_{\theta_1}(\Gamma_\alpha)$. From (5.19), $|J|<\varrho B^C/\delta$. Clearly, if $(\omega_1,\omega_2,\theta_1,\theta_2)\in\Gamma_\alpha$, then ω_1 is at distance less than $\eta^{1/2}$ from J/n_1. Indeed, there is an element $(\omega_1',\omega_{2,0},n_1\omega_1',\omega_{2,0})\in S$ at distance less than $\eta^{1/2}$ from $(\underline{\omega},\underline{\theta})$. Thus

$$|\theta_1-n_1\omega_1'|<\eta^{1/2}\ \Rightarrow\ n_1\omega_1'\in J\ \Rightarrow\ \omega_1'\in\frac{J}{n_1}$$

and $|\omega_1-\omega_1'|<\eta^{1/2}$. Therefore,

$$|\operatorname{Proj}_{\omega_1}(\Gamma_\alpha)|<\frac{|J|}{n_1}+\eta^{1/2}<\frac{\varrho B^C}{\delta n_1}+\eta^{1/2}$$

and, recalling (5.14), (5.16), (5.7),

$$\varrho<\frac{\varrho B^C}{\delta n_1}+5\eta^{1/2}<B^C\frac{1}{K^{1/2-1/10}}\varrho+5\eta^{1/2},$$

which implies that $\varrho<6\eta^{1/2}$. In conclusion,

$$\begin{aligned}|\operatorname{Proj}_{\omega_1}(\Gamma_\alpha)|&<c\eta^{1/2},\\ (5.13)&<B^C\eta^{1/2},\\ (5.11)&<KB^C\eta^{1/2}.\end{aligned}$$

Summing over $K^{1/2}<n_1,n_2<K$ and the δ_1-disks disjoint from $\mathfrak{S}'$ gives the measure estimate

$$\delta_1^{-2}K^2KB^C\eta^{1/2}<B^CK^4\eta^{1/2}. \tag{5.20}$$

From (5.8), (5.20), the resulting bound on the bad $\underline{\omega}$-set is

$$B^C(K^{-1/20}+K^4\eta^{1/2})<K^{-1/21}$$

from assumption (5.2). This proves Proposition 5.1. □

6. The proof of localization

In this section we prove Theorem 6.2 below. The scheme of the proof is very similar to that in [7], and we refer the reader to [7] for further motivation and some details. See also [10] and [8], where related arguments are used. We first prove a lemma that shows that double resonances occur only with small probability. Throughout this section we let $\mathcal{B}^{\gamma,b}(\Lambda,E)$ be as in Proposition 4.6. Observe that this set depends on the frequency vector $\underline{\omega}$, although we do not explicitly indicate this in the notation. In this section we shall make use of the sets $\widetilde{\Omega}_N$ introduced in Remark 3.2, see (3.25). Observe that those sets are semi-algebraic of degree at most $4N^8$ (each small square is described by 4 lines, and there are at most N^8 squares in total).

LEMMA 6.1. *For any pair of positive integers $N,\overline{N}$ define the set*

$$\begin{aligned}\mathfrak{D}(N,\overline{N}):=\{(\underline{\omega},\underline{\theta})\in\mathbf{T}^4: \text{for some } E\in\mathbf{R},\ \|(H_{\underline{\omega}}(0)\restriction_{Q_{\overline{N}}}-E)^{-1}\|>e^{N^2},\\ \text{and } \underline{\theta}\in\mathcal{B}^{\gamma,b}(\Lambda,E) \text{ for some } \Lambda\in\mathcal{ER}(N)\}.\end{aligned}\tag{6.1}$$

Then for any constant $C_2\geqslant 1$, all sufficiently large integers N_0 and N, and $\lambda\geqslant\lambda_0(N_0)$, the set

$$\mathcal{A}_N:=\bigcup_{\overline{N}\asymp N^{C_2}}\mathfrak{D}(N,\overline{N})\cap\bigcap_{\substack{N_0<M<N^{C_2}\\ M \text{ dyadic}}}(\widetilde{\Omega}_M\times\mathbf{T}^2)$$

satisfies the requirements of Proposition 5.1 *with* $K\asymp\exp((\log N)^2)$, $\log B\asymp\log N$ *and* $\log(1/\eta)\asymp N^{\varrho}$.

Proof. Suppose $(\underline{\omega},\underline{\theta})\in\mathcal{A}_N$. Then for some choice of $\overline{N}\asymp N^{C_2}$, and some eigenvalue E_j of $H_{\underline{\omega}}(0)\restriction_{Q_{\overline{N}}}$,

$$\underline{\theta}\in\mathcal{B}^{\gamma,b}(\Lambda,E_j)\quad\text{for some }\Lambda\in\mathcal{ER}(N).\tag{6.2}$$

More precisely, with E as in (6.1), E_j was chosen such that $|E-E_j|<e^{-N^2}$. This is possible because

$$\|(H_{\underline{\omega}}(0)\restriction_{Q_{\overline{N}}}-E)^{-1}\|^{-1}=\operatorname{dist}(E,\operatorname{sp}(H_{\underline{\omega}}(0)\restriction_{Q_{\overline{N}}}))$$

by self-adjointness. Since the e^{-N^2}-perturbation basically preserves the conditions in the definition of the bad set, see (4.5), one arrives at (6.2). By the restriction on $\underline{\omega}$ the measure of the set in (6.2) is at most $\exp(-N^{\varrho})$, cf. Remark 4.7. Summing over the possible choices of E_j and Λ, one derives that

$$\operatorname{mes}\mathcal{A}_N\lesssim N^{C_2}N^6e^{-N^{\varrho}}=:\eta,$$

so that $\log K \ll \log(1/\eta)$ for large N, as claimed. To verify the semi-algebraic condition in Proposition 5.1, we apply the projection Theorem 7.2 below to the sets $\mathcal{A}_N = \mathrm{Proj}_{\mathbf{T}^4}(\tilde{\mathcal{A}}_N)$, $\tilde{\mathcal{A}}_N \subset \mathbf{R}^5$, where

$$\begin{aligned}\tilde{\mathcal{A}}_N &:= \bigcup_{\overline{N} \asymp N^{C_2}} \widetilde{\mathfrak{D}}(N, \overline{N}) \cap \bigcap_{\substack{N_0 < M < N^{C_2} \\ M \text{ dyadic}}} (\widetilde{\Omega}_M \times \mathbf{T}^2 \times \mathbf{R}), \\ \widetilde{\mathfrak{D}}(N, \overline{N}) &:= \{(\underline{\omega}, \underline{\theta}, E) \in \mathbf{T}^4 \times \mathbf{R} : \|(H_{\underline{\omega}}(0) \restriction_{Q_{\overline{N}}} - E)^{-1}\| > e^{N^2} \\ &\qquad \text{and } \underline{\theta} \in \mathcal{B}^{\gamma, b}(\Lambda, E) \text{ for some } \Lambda \in \mathcal{ER}(N)\}.\end{aligned} \tag{6.3}$$

It therefore suffices to show that $\tilde{\mathcal{A}}_N$ is semi-algebraic of degree at most N^C. Since $\widetilde{\Omega}_N$ is semi-algebraic of degree at most $4N^8$, this will follow if $\widetilde{\mathfrak{D}}(N, \overline{N}) \subset \mathbf{R}^5$ is semi-algebraic of degree N^C. As in [7], this is done by expressing the Green's function appearing in (6.3) and (4.5) in terms of determinants via Cramer's rule, and using the Hilbert–Schmidt norm instead of the operator norm. This requires approximating the analytic function v by polynomials, see Remark 4.3. □

THEOREM 6.2. *Let v be a real-analytic potential satisfying* (4.2). *Given $\underline{\theta} \in \mathbf{T}^2$, any $\varepsilon > 0$ and any $\lambda \geqslant \lambda_0(\varepsilon, v)$ there is a set $\mathcal{F}_\varepsilon = \mathcal{F}_\varepsilon(\underline{\theta}, \lambda) \subset \mathbf{T}^2$ so that*

$$\mathrm{mes}(\mathbf{T}^2 \setminus \mathcal{F}_\varepsilon) < \varepsilon$$

and such that for any $\underline{\omega} \in \mathcal{F}_\varepsilon$ the operator (4.1) *displays Anderson localization, i.e., the spectrum is pure point and the eigenfunctions decay exponentially.*

Proof. Without loss of generality, we shall let $\underline{\theta} = 0$. Given $\varepsilon > 0$, choose N_0 large enough so that

$$\mathrm{mes}\Big(\mathbf{T}^2 \setminus \bigcap_{\substack{N_0 \leqslant N \\ N \text{ dyadic}}} \widetilde{\Omega}_N\Big) < \tfrac{1}{2}\varepsilon.$$

Applying Proposition 5.1 with dyadic $N > N_0$, $\mathcal{A}_N$ as in Lemma 6.1, and $K = K(N) \asymp \exp((\log N)^2)$, one obtains a set $\mathfrak{F}_K$. We shall prove the theorem for all

$$\underline{\omega} \in \bigcap_{\substack{N_0 \leqslant N \\ N \text{ dyadic}}} (\widetilde{\Omega}_N \setminus \mathfrak{F}_{K(N)}).$$

For N_0 large this will remove only ε in measure. Now fix such an $\underline{\omega}$. By the Shnol–Simon theorem [22], [23] it suffices to prove that generalized eigenfunctions decay exponentially. More precisely, let ψ be a nonzero function on $\mathbf{Z}^2$ satisfying

$$(H_{\underline{\omega}}(0) - E)\psi = 0 \quad \text{and} \quad |\psi(\underline{x})| \lesssim 1 + |\underline{x}|^{c_0} \text{ for all } \underline{x} \in \mathbf{Z}^2, \tag{6.4}$$

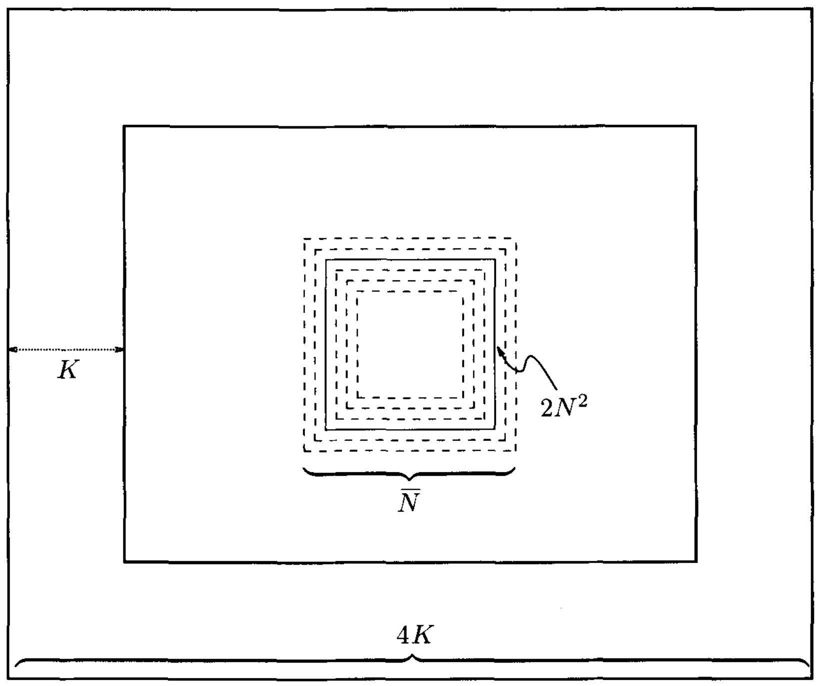

Fig. 9. The squares and annuli in the proof of localization.

where E is arbitrary and $c_0>0$ is some constant. Pick a large integer N and set $\overline{N}=N^{C_2}$, C_2 as in Lemma 6.1. First we claim that there is some square $Q_M(0)$ centered at zero so that

$$\|(H_{\underline{\omega}}(0)\restriction_{Q_M}-E)^{-1}\|>e^{N^2}\quad\text{for some } M\asymp\overline{N}. \tag{6.5}$$

This follows from the fact that there is an annulus A at distance $\asymp M$ around the origin of thickness $2N^2$ (see the dashed squares in Figure 9), so that for every $\underline{x}\in A$ one has

$$(x_1\omega_1,x_2\omega_2)\in\mathcal{G}^{\gamma,b}(Q_{N^2}(0),E)\quad(\operatorname{mod}\mathbf{Z}^2).$$

The existence of this annulus is proved by means of Proposition 4.6 and the same semi-algebraic considerations as above. More precisely, the number of $\underline{x}$ in a square $Q_{\overline{N}}(0)$ for which the Green's function of a square of size N^2 centered at $\underline{x}$ is bad, is at most $\overline{N}^{1-\delta_0}$ by Lemma 3.3 and Proposition 4.6. Since $\overline{N}/N^2>\overline{N}^{1-\delta_0}$ for large C_2, there has to be some annulus of thickness N^2 which is free of bad points, as desired. In Figure 9 we have indicated this good annulus by means of a solid line surrounded by dashed annuli. It follows from (6.4) that for any cube Q

$$(H_{\underline{\omega}}(0)\restriction_Q-E)\psi=\xi, \tag{6.6}$$

where $\xi(\underline{y})=\psi(\underline{y})$ if $\underline{y}\in\mathbf{Z}^2\backslash Q$ and there is $\underline{z}\in Q$ with $|\underline{y}-\underline{z}|=1$. Otherwise $\xi=0$. Hence for any $\underline{x}\in\mathbf{Z}^2$,

$$\psi(\underline{x})=\sum_{\substack{\underline{y}\in Q_{N^2}(\underline{x})\\ \underline{y}'\in\mathbf{Z}^2\backslash Q_{N^2}(\underline{x})\\ |\underline{y}-\underline{y}'|=1}} G_{Q_{N^2}(\underline{x})}(0,E)(\underline{x},\underline{y})\,\psi(\underline{y}').$$

In particular, if $\underline{x}\in A$ where A is this good annulus, then

$$|\psi(\underline{x})|\lesssim e^{-N^2},$$

which implies (6.5), see (6.6) (we are assuming here, as we may, that $\gamma\gg1$). By our choice of $\underline{\omega}$, see (6.1), therefore

$$(x_1\omega_1,x_2\omega_2)\notin\mathcal{B}^{\gamma,b}(\Lambda,E)\quad(\mathrm{mod}\ \mathbf{Z}^2)$$

for every $\underline{x}$ such that $|x_1|\vee|x_2|\asymp K$ and any $\Lambda\in\mathcal{ER}(N)$. One now checks from Lemma 2.2 and the resolvent identity that the Green's function of the set $U:=Q_{2K}(0)\backslash Q_K(0)$ exhibits off-diagonal decay, i.e.,

$$|G_U(0,E)(\underline{x},\underline{y})|\leqslant\exp(-|\underline{x}-\underline{y}|+O(N^2))\quad\text{if } \underline{x},\underline{y}\in U. \tag{6.7}$$

In view of (6.4) one has again

$$(H_{\underline{\omega}}(0)\restriction_U-E)\psi=\tilde{\xi},$$

where $\tilde{\xi}$ is supported on points in $\mathbf{Z}^2\backslash U$ at distance one from ∂U. Let $\underline{x}\in U$ such that $\mathrm{dist}(\underline{x},\partial U)\asymp K$. By the polynomial growth of ψ and (6.7) one finally obtains that

$$|\psi(\underline{x})|<e^{-|\underline{x}|/2}$$

provided N and thus K are large. □

7. Semi-algebraic sets

The purpose of this section, which should be regarded as an appendix, is to introduce semi-algebraic sets and to present those results from the literature that are used in this paper.

Definition 7.1. A set $\mathcal{S}\subset\mathbf{R}^n$ is called *semi-algebraic* if it is a finite union of sets defined by a finite number of polynomial equalities and inequalities. More precisely, let

$\mathcal{P}=\{P_1,...,P_s\}\subset\mathbf{R}[X_1,...,X_n]$ be a family of real polynomials whose degrees are bounded by d. A (closed) semi-algebraic set $\mathcal{S}$ is given by an expression

$$\mathcal{S}=\bigcup_j\bigcap_{l\in\mathcal{L}_j}\{\mathbf{R}^n:P_l\,s_{jl}\,0\},\tag{7.1}$$

where $\mathcal{L}_j\subset\{1,...,s\}$ and $s_{jl}\in\{\geqslant,\leqslant,=\}$ are arbitrary. We say that $\mathcal{S}$ has degree at most sd and its degree is the infimum of sd over all representations as in (7.1).

The projection of a semi-algebraic set of $\mathbf{R}^{k+l}$ onto $\mathbf{R}^k$ is semi-algebraic. This is known as the *Tarski–Seidenberg* principle, see Bochnak, Coste and Roy [6]. The currently best quantitative version of this principle is due to Basu, Pollak and Roy [5], [4]. For the history of such effective Tarski–Seidenberg results we refer the reader to those papers.

THEOREM 7.2. *Let $\mathcal{S}\subset\mathbf{R}^n$ be semi-algebraic defined in terms of s polynomials of degree at most d as in* (7.1). *Then there exists a semi-algebraic description of its projection onto $\mathbf{R}^{n-1}$ by a formula involving at most $s^{2n}d^{O(n)}$ polynomials of degree at most $d^{O(n)}$. In particular, if $\mathcal{S}$ has degree B, then any projection of $\mathcal{S}$ has degree at most B^C, $C=C(n)$.*

Proof. This is a special case of the main theorem in [5]. □

Another fundamental result on semi-algebraic sets is the following bound on the sum of the Betti numbers by Milnor, Oleinik and Petrovsky, and Thom. Strictly speaking, their result only applies to *basic* semi-algebraic sets, which are given purely by intersections without unions. The general case as in Definition 7.1 above was settled by Basu [3].

THEOREM 7.3. *Let $\mathcal{S}\subset\mathbf{R}^n$ be as in* (7.1). *Then the sum of all Betti numbers of $\mathcal{S}$ is bounded by $s^n(O(d))^n$. In particular, the number of connected components of $\mathcal{S}$ does not exceed $s^n(O(d))^n$.*

Proof. This is a special case of Theorem 1 in [3]. □

Another result that we shall need is the following triangulation theorem of Yomdin [26], later refined by Yomdin and Gromov [20]. We basically reproduce the statement of that result from [20], see p. 239.

THEOREM 7.4. *For any positive integers r,n there exists a constant $C=C(n,r)$ with the following property: Any semi-algebraic set $\mathcal{S}\subset[0,1]^n\subset\mathbf{R}^n$ can be triangulated into $N\lesssim(\deg\mathcal{S}+1)^C$ simplices, where for every closed k-simplex $\Delta\subset\mathcal{S}$ there exists a homeomorphism h_Δ of the regular simplex $\Delta^k\subset\mathbf{R}^k$ with unit edge length onto Δ such that h_Δ is real analytic in the interior of each face of Δ. Furthermore, $\|D_rh_\Delta\|\leqslant1$ for all Δ.*

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