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Stability of embeddings for pseudoconcave surfaces and their boundaries

by

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1. Introduction

Let M denote a compact, strictly pseudoconvex, 3-dimensional CR-manifold. Such a structure is induced on a strictly pseudoconvex, real hypersurface in a complex surface, or as the boundary of a 2-dimensional Stein space. In the latter case we say that the CR-manifold is fillable or embeddable. It is a fundamental fact that many 3-dimensional, strongly pseudoconvex CR-manifolds cannot be realized as the boundary of any compact complex space. The CR-structure on M can be described as a subbundle $T^{0,1}M$ of the

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complexified tangent bundle, with fiber dimension 1. For each $p \in M$ we require that

$$T_p^{0,1}M \cap T_p^{1,0}M = \{0\},$$

where $T_p^{1,0}M = \overline{T_p^{0,1}M}$. There is a real two-plane field $H \subset TM$ such that

$$H \otimes \mathbf{C} = T^{0,1}M \oplus T^{1,0}M.$$

The plane field H is a contact field if and only if the CR-structure defining it is strictly pseudoconvex. All of the CR-structures with a given underlying plane field are, up to orientation deformations of one another. A given 3-manifold has infinitely many inequivalent contact structures. Recent work of Eliashberg, Kronheimer and Mrowka supports the view that only finitely many of these contact structures support any embeddable CR-structures, see [El], [KM]. In this paper we assume that $(M, T^{0,1}M)$ is an embeddable CR-manifold and restrict our attention to deformations of this CR-structure. It is now well known that the generic deformation of the CR-structure is not embeddable. We would like to understand the subset of deformations defining embeddable CR-structures. Presently so little is known about this set that it would be useful to have answers to the following essentially topological questions:

(1) Is the set of small embeddable deformations closed in the $\mathcal{C}^\infty$-topology?

(2) Is the set of small embeddable deformations (locally) connected or path-connected?

(3) Is the set of small embeddable deformations contained in an infinite-codimensional submanifold of the set of all deformations?

For strictly pseudoconvex, compact hypersurfaces in $\mathbf{C}^2$ results of Bland–Duchamp, Burns–Epstein and Lempert provide affirmative answers to questions (1) and (2), see [Bl], [BlD], [BuE], [Le1], [Le2] and [Ep2]. Various of these results together with the techniques of [EH1] also provide an affirmative answer to question (3) for the case of strictly pseudoconvex hypersurfaces in $\mathbf{C}^2$.

Lempert introduced a method for studying these questions in [Le1]–[Le3]. The basic idea is to compactify the problem by realizing an embeddable CR-manifold M as the boundary of a strictly *pseudoconcave* manifold X_-, containing a smooth, compact curve Z. As the normal bundle of Z is positive we say that Z is *positively embedded.* If X_+ denotes the normal Stein space bounded by M then $X = X_+ \amalg_M X_-$ is a compact complex space. As follows from the remarks above, the problem of extending a deformation of the CR-structure on M to an integrable complex structure on X_+ is very delicate. On the other hand, as follows from a theorem of Kiremidjian, extension to the pseudoconcave side is much simpler, see [Ki]. The formal obstruction is the cohomology group $H_c^2(X_-; \Theta)$, which is always finite-dimensional. The obstruction to finding an extension which preserves the dth formal neighborhood of Z is the cohomology group $H_c^2(X_-;\Theta\otimes[-dZ])$, which is again finite-dimensional, see [EH1]. Lempert's strategy is to extend an embeddable deformation of M to X_- in such a way that the divisor remains. Then one constructs elements of $H^0(X_-;[Z])$ with respect to the deformed complex structure by extending sections of $H^0(Z;N_Z)$. Finally one proves that $H^0(X_-;[Z])$ is stable under this process.

This approach has been applied successfully by Lempert and Li to the case that $Z\simeq\mathbf{P}^1$. For example, Lempert showed that if M is a strictly pseudoconvex hypersurface in $\mathbf{C}^2$ then the set of small embeddable deformations of the CR-structure on M is closed in the C^∞-topology. Indeed, in [Le2] it is shown that strictly pseudoconvex hypersurfaces $M\subset\mathbf{C}^2$ have a very strong stability property: any small, embeddable perturbation of the CR-structure can be realized as a small perturbation of the given embedding. In all previously known cases, the closedness of the set of embeddable deformations is established by constructing an embedding which is stable under such deformations. In these cases it can be shown that the entire algebra of CR-functions is stable under small, embeddable deformations. It is known from examples, however, that this is often not the case, see [CL], [Ep2].

This paper has three parts. In the first part we extend Lempert's methods to many cases where Z is not a rational curve. Lempert has recently shown that any embeddable, strictly pseudoconvex CR-structure on a 3-manifold can be realized as a separating hypersurface in a projective variety with an ample divisor contained in the pseudoconcave piece, see [Le3]. A similar result was announced in [Bog]. Under cohomological hypotheses on X_- which ensure that small deformations of the CR-structure on bX_- extend to X_-, in such a way that the divisor persists, we establish the closedness of the set of small embeddable deformations. This is accomplished without obtaining a stable embedding. In the second part we show that embeddable deformations of the CR-structure on M with extensions to X_-, vanishing to order 3 along Z, share many of the stability properties of hypersurfaces in $\mathbf{C}^2$. We also consider the consequences of higher-order vanishing and the deformations of the defining equations. In the final part we consider examples.

Assume that X_- is a pseudoconcave surface which contains a positively embedded, smooth, compact curve Z. Let $H^0(X_-;[dZ])$ denote the space of sections of the line bundle defined by the divisor dZ which are holomorphic in X_-. Restricting an element of $H^0(X_-;[dZ])$ to Z defines a holomorphic section of N_Z^d. A time-honored method for constructing sections of $H^0(X_-;[dZ])$ is to reverse this process by extending elements of $H^0(Z;N_Z^d)$. The Riemann–Roch theorem implies that

$$\sum_{j=0}^{d}[\dim H^0(Z;N_Z^j)-\dim H^1(Z;N_Z^j)]=\tfrac{1}{2}d(d+1)k+(1-g)(d+1), \tag{1.1}$$

where $k=\deg(N_Z)$ and g is the genus of Z. In §2 we show that if bX_- is embeddable then for large enough d we have the estimates

$$\sum_{j=0}^{d}[\dim H^0(Z;N_Z^j)-\dim H^1(Z;N_Z^j)]\leqslant H^0(X_-;[dZ])\leqslant\sum_{j=0}^{d}\dim H^0(Z;N_Z^j). \tag{1.2}$$

Let $\{X_{-j}\}$ denote X_- with a sequence of complex structures, each one containing a smooth, positively embedded curve Z_j. We suppose that each bX_{-j} is embeddable and that the sequence of pairs $\{(X_{-j},Z_j)\}$ converges in the $\mathcal{C}^\infty$-topology to (X_{-0},Z_0). Using (1.1), (1.2) and standard semicontinuity results for dimensions of cohomology groups we obtain that the limiting structure satisfies

$$\dim H^0(X_{-0};[dZ_0])\geqslant\tfrac{1}{2}d(d+1)k+(1-g)(d+1).$$

Even if X_- is embeddable, the ring

$$\mathcal{A}(X_-,[Z])=\bigcup_{d>0}H^0(X_-;[dZ])$$

may not contain sufficiently many sections to define an embedding. In light of this we have introduced two weakened notions of embeddability for pseudoconcave manifolds which contain a compact, positively embedded, smooth divisor: *weak embeddability* and *almost embeddability.* Essentially, the pair (X_-,Z) is weakly embeddable if the lower bounds in (1.2) are satisfied. If $(X_-,[Z])$ is weakly embeddable then for sufficiently large d the sections in $H^0(X_-;[dZ])$ define a holomorphic map of X_- into projective space which is an embedding of a neighborhood of Z. It seems difficult to show directly that these maps are embeddings on the complements of proper analytic subsets. This led us to introduce the subclass "almost embeddable concave structures", which enjoy this property.

This is an important subclass, for if bX_- is embeddable then (X_-,Z) is almost embeddable. One of our main results is that the converse is also true: If (X_-,Z) is almost embeddable then bX_- is embeddable, see Theorem 5.1. This in turn implies that X_- is itself embeddable in projective space. We prove this without assuming that the ring $\mathcal{A}(X_-,[Z])$ separates points on X_-, thus refining earlier work of Andreotti and Tomassini, see [AT]. In their work, the embeddability of a pseudoconcave surface X_- is proved under the assumption that there exists a holomorphic line bundle $F\to X_-$ such that the ring $\mathcal{A}(X_-;F)$ separates points on X_- and defines local coordinates at every point. For the case of a pseudoconcave surface with $F=[Z]$ we obtain the embeddability of X_- under the weaker assumptions:

(1) $\mathcal{A}(X_-, [Z])$ embeds some neighborhood of Z, and

(2) there is a proper analytic subset $G \subset X_- \setminus Z$ such that $\mathcal{A}(X_-, [Z])$ separates the points of $X_- \setminus G$.

Our second principal result is the fact that almost embeddability is closed under convergence in the $\mathcal{C}^\infty$-topology, see Theorem 6.2. Using this fact in tandem with Kiremidjian's extension theorem we are able to show that the set of small, embeddable perturbations of the CR-structure on bX_- is closed in the $\mathcal{C}^\infty$-topology for many new classes of CR-manifolds, see Theorem 6.1. Included among these are examples where the full algebra of CR-functions is not stable under small embeddable deformations.

In §4 we apply the results in §§2 and 3 to study the problem of projective fillability for surface germs containing a smooth curve with positive normal bundle. This is a germ analogue of the problem of embedding CR-manifolds. In this context we obtain a weak version of a conjecture of Lempert: even though the set of surface germs containing a given curve with a given positive normal bundle is infinite-dimensional, the set of projectively fillable germs is, in a reasonable sense, finite-dimensional. Each fillable germ has a representative as an open subset of a variety belonging to a finite-dimensional family of varieties. This generalizes a rigidity result for embeddings of $\mathbf{P}^1$ with normal bundle $\mathcal{O}(1)$ proved in [MR1].

In the second part of the paper we consider a compact, 2-dimensional subvariety of $X \subset \mathbf{P}^n$ which contains a smooth, strictly pseudoconvex, separating hypersurface $M \hookrightarrow X$. Let $X_\pm$ denote the components of $X \setminus M$. Now we assume that $Z \subset X_-$ is a smooth hyperplane section. As remarked above the obstruction to extending a small deformation of the CR-structure on M to an integrable almost complex structure on X_- vanishing to order d along Z is the finite-dimensional cohomology group $H^2_c(X_-; \Theta \otimes [-dZ])$. We show that embeddable deformations of M having extensions to X_- which vanish to order 3 along Z share many of the stability results of hypersurfaces in $\mathbf{C}^2$. We also consider the behavior of the defining equations for X under deformations of the CR-structure on M which extend to X_- vanishing to various orders along Z. A notable result in this part is Theorem 8.1: For any separating hypersurface M in a projective surface X there is a finite-codimensional space $\mathcal{E}_{D+2}$ of the deformations of the CR-structure on M, such that any *embeddable* deformation in $\mathcal{E}_{D+2}$ is "a wiggle". Our results on the deformations of the defining equations are especially useful if X is a cone. In this case we can show that the deformed CR-structure embeds as a hypersurface in a fiber of an analytic deformation space of X.

In the final part the results in the paper are applied to a variety of examples. If the curve "at infinity" is rational then it is possible to prove much more precise results. This case was considered by Lempert and Li, see [Le1], [Le2], [Li]. In §9 we refine their results. For the case of Hirzebruch surfaces we prove the natural analogue of Lempert's stability theorem for hypersurfaces in $\mathbf{C}^2$: A small embeddable deformation of the CR-structure on a strictly pseudoconvex, separating hypersurface in a Hirzebruch surface can be embedded as a hypersurface in a fiber of the versal deformation of the Hirzebruch surface. As a consequence, the set of small embeddable deformations is path-connected. In the last section we analyze various examples where Z is a non-rational curve.

The spirit of the methods used in this paper is essentially algebro-geometric. We do not know of any examples of surfaces with positive Kodaira dimension which satisfy the cohomological hypotheses used in this paper, (6.1) and (6.2). In a forthcoming paper we will present a more analytic approach to these issues. This involves a detailed analysis of the $\bar{\partial}$-operator on singular varieties and their (possibly singular) subdomains. We expect these methods to be more flexible allowing us to extend the range of our results. In particular, we hope that the analytic methods will allow us to treat certain surfaces of general type.

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I. Closedness of the set of embeddable deformations

2. The dimension of $H^0(X;[dZ])$

Let X denote a projective variety with Z a smooth holomorphic hypersurface embedded into $X\setminus\text{sing}(X)$. We say that Z is *positively embedded* if N_Z, the normal bundle of Z, is a positive line bundle over Z. Extending earlier work of Grauert, Griffiths showed in [Gr] that this implies that Z has a basis of neighborhoods in X with smooth, strictly pseudoconcave boundaries. We use the notation $[Z]$ to denote the holomorphic line bundle defined on X by the divisor Z. The bundle $[Z]$ is ample if for sufficiently large d the holomorphic sections of $[dZ]$ define an embedding of X in projective space. In order for $[Z]$ to be ample it is necessary that Z be positively embedded, though in general this is not sufficient.

Frequently we also let $[Z]$ denote the sheaf of germs of holomorphic sections of this line bundle. In this section we study $\dim H^0(X;[dZ])$ for $d>0$. If $\dim X=2$ and Z is a curve of genus g with $\deg N_Z=k$ then we show that for $d>1+(2g-2)/k$ we have

$$\dim H^0(X;[dZ]) = \tfrac{1}{2}d(d+1)k+(1-g)(d+1)+m_{X,Z}(d) \tag{2.1}$$

where $m_{X,Z}(d)$ is a non-negative, non-increasing function of d. Define the function

$$M(g,k,d) = \tfrac{1}{2}d(d+1)k+(1-g)(d+1). \tag{2.2}$$

By the Riemann–Roch theorem,

$$M(g,k,d) = \sum_{j=0}^{d}[\dim H^0(Z;N_Z^j)-\dim H^1(Z;N_Z^j)].$$

The main technical result needed for the proof of (2.1) is a slight elaboration of the Pardon–Stern extension of Kodaira's vanishing theorem.

PROPOSITION 2.1 (Pardon–Stern). *Let V be an n-dimensional normal, projective variety, and suppose that $Z\subset V\backslash\operatorname{sing}(V)$ is an ample divisor (i.e. the line bundle $[Z]$ is ample). Then the L^2-cohomology groups satisfy*

$$H_D^{0,q}(V\backslash\operatorname{sing}V;[-dZ])=0 \quad \text{for } q<n,\ d>0.$$

Remark. The subscript D on the cohomology group indicates that it is defined with respect to the Dirichlet boundary condition along $\operatorname{sing}V$. These groups are defined in [PS]. The basic vanishing result, from which the Pardon–Stern theorem is derived, was proved by Grauert and Riemenschneider, see [GR]. The notation in this argument follows that used in [PS]. For our applications, one could replace this result with an argument using a resolution of the singularities of V and the extension of Kodaira's vanishing theorem to seminegative bundles, found for example in [BPV] or [SS].

Proof. This result follows from Proposition 1.3 and Theorem C in [PS]: Because $Z\subset V\backslash\operatorname{sing}(V)$ the arguments given in [PS] to study cohomology with respect to the structure sheaf apply *mutatis mutandis* to cohomology with coefficients in $[dZ]$. In particular, as the Hodge star operator defined on $[dZ]$-valued forms acts fiberwise, one easily establishes that

$$\mathcal{H}_D^{0,n-q}(V\backslash\operatorname{sing}(V);[-dZ]) \simeq (\mathcal{H}_N^{n,q}(V\backslash\operatorname{sing}(V);[dZ]))'. \tag{2.3}$$

Hence the proposition follows from Theorem C.

In this section we make use of some well-known exact sequences of sheaves. Let $\mathcal{O}_X$ denote the structure sheaf of X, and $\mathcal{I}_Z$ the ideal sheaf of Z. Then for each $l\geqslant 1$ we have the exact sequences

$$0\to\mathcal{I}_Z^l\to\mathcal{O}_X\to\mathcal{O}_X/\mathcal{I}_Z^l\to 0, \tag{2.4}$$

$$0\to\mathcal{I}_Z^l/\mathcal{I}_Z^{l+1}\to\mathcal{O}_X/\mathcal{I}_Z^{l+1}\to\mathcal{O}_X/\mathcal{I}_Z^l\to 0. \tag{2.5}$$

Since the sheaf of germs of sections of $[Z]$ is locally free and $\mathcal{I}_Z$ is isomorphic to the sheaf of germs of sections of $[-Z]$, we can tensor (2.4) and (2.5) with $[dZ]$ to obtain

$$0\to[(d-l)Z]\to[dZ]\to\mathcal{O}_X/\mathcal{I}_Z^l\otimes[dZ]\to 0, \tag{2.6}$$

$$0\to N_Z^{d-l}\to\mathcal{O}_X/\mathcal{I}_Z^{l+1}\otimes[dZ]\to\mathcal{O}_X/\mathcal{I}_Z^l\otimes[dZ]\to 0. \tag{2.7}$$

We refer to sections of $\mathcal{O}_X/\mathcal{I}_Z^l\otimes[dZ]$ as order-l sections of $[dZ]$ along Z.

Let $\{U_1, ..., U_M\}$ denote a covering of a neighborhood of Z by coordinate neighborhoods, and $U_0\subset\subset X\setminus Z$ an open set with smooth strictly pseudoconvex boundary which covers $X\setminus U_1\cup...\cup U_M$. Let $\sigma_{0i}\in\mathcal{O}(U_i)$, $i>0$, vanish simply along Z. Set $\sigma_{00}=1$. The 1-cocycle

$$g_{ij}=\frac{\sigma_{0i}}{\sigma_{0j}}$$

defines the line bundle $[Z]$. We denote by σ_0 the holomorphic section of $[Z]$ defined by $\{\sigma_{0i}\}$. A smooth section of $[kZ]$ is a collection of functions $f_i\in\mathcal{C}^\infty(U_i)$ such that

$$f_i|_{U_i\cap U_j}=g_{ij}^k f_j|_{U_i\cap U_j}.$$

An element of $H^0(\mathcal{O}/\mathcal{I}_Z^l\otimes[kZ])$ is given by $s_i\in\mathcal{O}(U_i)$, $i\geqslant 1$, such that

$$s_i|_{U_i\cap U_j}-g_{ij}^k s_j|_{U_i\cap U_j}\in\mathcal{I}_Z^l(U_i\cap U_j). \tag{2.8}$$

Because the sheaf associated to $\mathcal{C}^\infty(X;[kZ])$ is a fine sheaf it is easy to show that there exist functions $f_i\in\mathcal{C}^\infty(U_i)$ such that

(a) $S=(s_i-f_i)$ defines a smooth section of $[kZ]$,

(b) $f_i\in\mathcal{I}_Z^l(U_i)\mathcal{C}^\infty(U_i)$,

(c) $\bar\partial S=\sigma_0^l\beta$ where β is a smooth $[(k-l)Z]$-valued $(0,1)$-form with compact support in a small neighborhood of Z.

We call S a smooth representative of (s_i).

Using the long exact sequence in cohomology and the Pardon–Stern vanishing theorem we obtain

PROPOSITION 2.2. *Let X be a normal, compact complex space and $Z\subset X\setminus\operatorname{sing} X$ be a smooth, positively embedded divisor. Then for each $d>0$ the natural map*

$$H^0(X;[dZ])\to H^0(\mathcal{O}_X/\mathcal{I}_Z^{d+1}\otimes[dZ])$$

is an isomorphism.

Proof. We first suppose that $[Z]$ is an ample divisor and X is projective. The short exact sequence of sheaves in (2.6) with $l=d+1$ leads to the long exact sequence in cohomology

$$0\to H^0(X;[-Z])\to H^0(X;[dZ])\xrightarrow{r} H^0(\mathcal{O}_X/\mathcal{I}_Z^{d+1}\otimes[dZ])\xrightarrow{\delta} H^1(X;[-Z])\to\dots\ . \quad (2.9)$$

As $[-Z]<0$, the group $H^0(X;[-Z])$ vanishes. To complete the proof in this case we need to show that r is surjective. Let $s\in H^0(\mathcal{O}/\mathcal{I}_Z^{d+1}\otimes[dZ])$ and let S be a smooth representative with compact support in a neighborhood of Z disjoint from $\operatorname{sing}(X)$. Evidently

$$\omega=\frac{\bar\partial S}{\sigma_0^{d+1}}$$

is a smooth, closed section of $\Lambda^{0,1}\otimes[-Z]$ with compact support in $X\setminus\operatorname{sing}(X)$. From Proposition 2.1 it follows that there is an element $f\in\mathcal{C}^\infty(X\setminus\operatorname{sing}(X);[-Z])$ such that

$$\bar\partial f=\omega.$$

As X is a normal variety the section $S-\sigma_0^{d+1}f$ extends to define an element of $H^0(X;[dZ])$ which satisfies

$$r(S-\sigma_0^{d+1}f)=s.$$

It is not necessary for $Z\subset X$ to be ample or for X to be projective. We use the following lemma to reduce to the previous case.

LEMMA 2.1. *Suppose that X is a normal, compact complex space and $Z\subset X\setminus\operatorname{sing}(X)$ is a positively embedded smooth divisor. There exists a normal, compact projective variety V and a holomorphic map $\pi\colon X\to V$. Let $W=\pi(Z)$; it is an ample divisor and there is a neighborhood of Z which is biholomorphic to a neighborhood of W.*

Proof. As N_Z is positive it follows from [Gr] that there is a neighborhood U of Z and a smooth, strictly plurisubharmonic function p defined in $U\setminus\{Z\}$ such that $\lim_{x\to Z}p(x)=\infty$. For sufficiently large $c\in\mathbf{R}$ set $S_c=\{p^{-1}(c)\}$. If c is a large regular value of p then S_c is a compact, strictly pseudoconvex hypersurface which bounds a neighborhood of Z. Such a hypersurface separates X into two connected components.

Let D_c denote the component of $X\setminus S_c$ not containing Z. This domain is a compact complex space with strictly pseudoconvex boundary. Arguing as in [Le2] we see that Grauert's theorem implies that the compact varieties in D_c can be blown down. Therefore we obtain a normal, compact complex space and a holomorphic mapping

$$\pi: X \to V.$$

The complex space V has no exceptional varieties disjoint from $\pi(Z)$. Set $W=\pi(Z)$; the map π is a biholomorphism from a neighborhood of Z to a neighborhood of W. Applying Theorems 2.4 and 1.8 from [MR2] (both due to Grauert) we conclude that $[W]$ is an ample line bundle and V is algebraic.

Let V, W be as in the lemma. The argument preceding it applies to show that

$$H^0(V;[dW]) \to H^0(\mathcal{O}_V/\mathcal{I}_V^{d+1}\otimes[dW])$$

is an isomorphism. As π is biholomorphism from a neighborhood of Z to a neighborhood of W it is obvious that

$$\pi^*: H^0(\mathcal{O}_V/\mathcal{I}_V^{d+1}\otimes[dW]) \to H^0(\mathcal{O}_X/\mathcal{I}_X^{d+1}\otimes[dZ])$$

is an isomorphism. Since both X and V are normal, and $[Z]|_{X\setminus Z}$ and $[W]|_{V\setminus W}$ are trivial, it follows that

$$\pi^*: H^0(V;[dW]) \to H^0(X;[dZ])$$

is an isomorphism as well.

We now restrict to the case $\dim X=2$; set

$$l_0 = \left[\!\left[\frac{2g-2}{k} \right]\!\right] + 1.$$

The cohomology groups $H^1(Z;N_Z^j)$ vanish for $j>(2g-2)/k$. Using this observation and Proposition 2.2 we obtain the main result of this section:

THEOREM 2.1. *Let X be a normal, compact complex space of dimension 2, and $Z\subset X\setminus\operatorname{sing}(X)$ be a smooth compact curve of genus g. Suppose further that its normal bundle N_Z has degree $k>0$. Then for $d>l_0$ we have that*

$$\dim H^0(X;[dZ]) = \tfrac{1}{2}d(d+1)k+(d+1)(1-g)+m_{X,Z}(d). \tag{2.10}$$

The function $m_{X,Z}(d)$ is a non-increasing function of d, satisfying the bounds

$$0 \leqslant m_{X,Z}(d) \leqslant \sum_{l=0}^{l_0-1} \dim H^1(Z;N_Z^l). \tag{2.11}$$

Proof. For $0\leqslant l\leqslant d-1$, the short exact sequences of sheaves (2.7) lead to the following long exact sequences in cohomology:

$$\begin{aligned}&H^0(Z;N_Z^l)\xrightarrow{i_l}H^0(\mathcal{O}_X/\mathcal{I}_Z^{d-l+1}\otimes[dZ])\xrightarrow{r_l}H^0(\mathcal{O}/\mathcal{I}_Z^{d-l}\otimes[dZ])\\&\xrightarrow{\delta_l^d}H^1(Z;N_Z^l)\longrightarrow H^1(\mathcal{O}_X/\mathcal{I}_Z^{d-l+1}\otimes[dZ])\longrightarrow H^1(\mathcal{O}/\mathcal{I}_Z^{d-l}\otimes[dZ])\longrightarrow 0.\end{aligned}\tag{2.12}$$

As $\deg N_Z=k>0$,

$$H^1(Z;N_Z^l)=0\quad\text{if } l\geqslant l_0.$$

Using the exact sequence (2.12), we obtain the recursion formula

$$\dim H^0(\mathcal{O}/\mathcal{I}_Z^{d-l+1}\otimes[dZ])=\dim H^0(\mathcal{O}/\mathcal{I}_Z^{d-l}\otimes[dZ])+\dim H^0(Z;N_Z^l)-\dim\operatorname{range}\delta_l^d.\tag{2.13}$$

For $d>l_0$ we use (2.7) with $l=1$ to derive the formula

$$\dim H^0(\mathcal{O}/\mathcal{I}_Z^2\otimes[dZ])=\dim H^0(Z;N_Z^d)+\dim H^0(Z;N_Z^{d-1}).$$

Using this formula along with (2.13) in a recursive argument we obtain

$$\dim H^0(\mathcal{O}/\mathcal{I}_Z^{d-l_0+1}\otimes[dZ])=\sum_{l=l_0}^{d}\dim H^0(Z;N_Z^l).$$

Let $\Delta_l(d)=\dim H^1(Z;N_Z^l)-\dim\operatorname{range}\delta_l^d$ so that

$$0\leqslant\Delta_l(d)\leqslant\dim H^1(Z;N_Z^l).\tag{2.14}$$

Note that $\Delta_l(d)=0$ for all $l\geqslant l_0$. For $d>l_0$ we have the formula

$$\dim H^0(\mathcal{O}/\mathcal{I}_Z^{d+1}\otimes[dZ])=\sum_{l=0}^{d}\dim H^0(Z;N_Z^l)-\dim H^1(Z;N_Z^l)+m_{X,Z}(d),\tag{2.15}$$

where

$$m_{X,Z}(d)=\sum_{l=0}^{d}\Delta_l(d).$$

Formula (2.10) now follows from Proposition 2.2.

The estimate (2.11) follows from (2.14). All that remains is to show that $m_{X,Z}(d)$ is non-increasing. We use the long exact sequence in cohomology defined by the short exact sequence of sheaves (2.6) with $l=1$ to conclude that as $l_0<d$ we have the estimate

$$\dim H^1(X;[(d-1)Z])\geqslant\dim H^1(X;[dZ]),$$

and therefore

$$m_{X,Z}(d)-m_{X,Z}(d-1)=\dim H^1(X;[dZ])-\dim H^1(X;[(d-1)Z])\leqslant 0.$$

3. Weakened notions of embeddability for concave structures

In this section we define two weakened notions of embeddability for a pseudoconcave surface X_- with a smooth, positively embedded, compact curve $Z \subset X_-$. Let $H^0(X_-; [dZ])$ denote the space of sections of $[dZ]$ which are holomorphic on X_-. Because bX_- is strictly pseudoconcave the estimates in §3 of [Le2] imply that every element of $H^0(X_-; [dZ])$ has a smooth extension to $\overline{X}_-$.

Definition. Suppose that X_- is a compact, complex surface with smooth, strongly pseudoconcave boundary. If X_- contains a compact, smooth, holomorphic curve Z of genus g with positive normal bundle of degree k then the pair (X_-, Z) is a *weakly embeddable, concave structure* if

$$\dim H^0(X_-; [dZ]) \geqslant M(g,k,d) \quad \text{for } d > \frac{2g-2}{k}+1. \tag{3.1}$$

The function $M(g,k,d)$ is defined in (2.2). Note that $H^1(Z; N_Z^j)=0$ for $j>(2g-2)/k$.

We consider the relationship between embeddability of bX_- and weak embeddability of (X_-, Z):

PROPOSITION 3.1. *Suppose that X_- is a compact, smooth, complex surface with strictly pseudoconcave boundary which contains a smooth, positively embedded curve Z. If bX_- is embeddable then (X_-, Z) is weakly embeddable.*

Proof. Let X_+ denote the normal Stein space with boundary bX_-. The compact complex space $X = X_+ \amalg_{bX_-} X_-$ contains a positively embedded, smooth curve. We can apply Theorem 2.1 to conclude that

$$\dim H^0(X; [dZ]) \geqslant \tfrac{1}{2}d(d+1)\deg N_Z + (1-g)(d+1) \quad \text{for } d > \frac{2g-2}{\deg N_Z}+1. \tag{3.2}$$

Definition. A weakly embeddable, concave structure (X_-, Z) defines a set of positive integers, $\mathcal{G}(X_-, Z)$. The positive integer $d \in \mathcal{G}(X_-, Z)$ if $H^0(X_-; [dZ])$ is base-point-free and the holomorphic mapping φ_d into projective space which it defines has the following properties:

(3.3) the image of Z is the intersection of $\varphi_d(X_-)$ with a hyperplane,

(3.4) there is a neighborhood U_d of Z such that $\varphi_d|_{U_d}$ is an embedding,

(3.5) the set $\varphi_d(U_d)$ is disjoint from $\varphi_d(X_- \setminus U_d)$.

The following result is used many times in the sequel:

PROPOSITION 3.2. *Suppose that (X_-, Z) is a weakly embeddable, concave structure. If g is the genus of Z and k is the degree of N_Z then there is a positive integer $C(g,k)$ such that the cardinality of $\mathbf{N}\backslash\mathcal{G}(X_-, Z)$ is at most $C(g,k)$.*

Proof. As N_Z is positive there is an integer m_0 which depends only on g, k, such that N_Z^m is very ample for $m_0 \leqslant m$. We tensor the exact sequence (2.4) with $l=1$ by $[mZ]$ to obtain

$$0 \to [(m-1)Z] \xrightarrow{\times\sigma_0} [mZ] \to N_Z^m \to 0.$$

As before, σ_0 denotes a holomorphic section of $[Z]$ with divisor exactly Z. This gives the following long exact sequence in cohomology:

$$\begin{aligned} 0 &\to H^0(X_-; [(m-1)Z]) \longrightarrow H^0(X_-; [mZ]) \xrightarrow{r_m} H^0(Z; N_Z^m) \\ &\to H^1(X_-; [(m-1)Z]) \xrightarrow{j_m} H^1(X_-; [mZ]) \longrightarrow H^1(Z; N_Z^m) \to \dots . \end{aligned} \tag{3.6}$$

Observe that (3.6) implies that

$$\dim H^0(X_-; [mZ]) - \dim H^0(X_-; [(m-1)Z]) \leqslant \dim H^0(Z; N_Z^m) \quad \text{for } 1 \leqslant m. \tag{3.7}$$

This shows that

$$\dim H^0(X_-; [dZ]) \leqslant 1 + \sum_{m=1}^{d} H^0(Z; N_Z^m). \tag{3.8}$$

It is an easy consequence of the Riemann–Roch theorem and the theory of divisors on a Riemann surface that

$$\begin{aligned} &\dim H^0(Z; N_Z^l) = 1-g+kl \quad \text{for } l \geqslant \frac{2g-2}{k}+1, \\ &\dim H^0(Z; N_Z^l) \leqslant kl+1 \quad \text{for } 0 \leqslant l < \frac{2g-2}{k}+1. \end{aligned} \tag{3.9}$$

These inequalities imply that there is a function $E(g,k)$ such that

$$\sum_{j=0}^{d} H^0(Z; N_Z^j) \leqslant M(g,k,d) + E(g,k), \tag{3.10}$$

and therefore

$$H^0(X_-; [dZ]) \leqslant M(g,k,d) + E(g,k).$$

Since (X_-, Z) is weakly embeddable it follows that

$$\dim H^0(X_-; [dZ]) \geqslant M(g,k,d) \quad \text{for } d > \frac{2g-2}{k}+1. \tag{3.11}$$

LEMMA 3.1. *Suppose that (X_-, Z) is a weakly embeddable, concave structure with Z of genus g and N_Z of degree k. The map r_j in* (3.6) *fails to be surjective for at most $E(g,k)$ values of j.*

Proof. From (3.6) we deduce that

$$\dim[H^0(Z;N_Z^j)/r_jH^0(X_-;[jZ])] \leqslant \dim H^0(Z;N_Z^j) + \dim H^0(X_-;[(j-1)Z]) - \dim H^0(X_-;[jZ]).$$

If we sum this estimate from 0 to m we obtain

$$\sum_{j=0}^{m} \dim[H^0(Z;N_Z^j)/r_jH^0(X_-;[jZ])] \leqslant \sum_{j=0}^{m} \dim H^0(Z;N_Z^j) - \dim H^0(X_-;[mZ]).$$

Applying (3.10) and (3.11) it follows that for any $m>1+(2g-2)/k$ we have

$$\sum_{j=0}^{m} \dim[H^0(Z;N_Z^j)/r_jH^0(X_-;[jZ])] \leqslant E(g,k). \tag{3.12}$$

This completes the proof of the lemma.

Combining this result with the next lemma we complete the proof of the proposition.

LEMMA 3.2. *Suppose that for a $j>m_0$ the mappings r_j, r_{j+1} are surjective. Then a basis of sections for $H^0(X_-;[(j+1)Z])$ defines a holomorphic map of X_- into projective space, satisfying conditions* (3.3), (3.4) *and* (3.5).

Proof. Suppose that $j>m_0$ and r_j, r_{j+1} are surjective. As j is at least m_0 it follows that $H^0(Z;N_Z^j)$ and $H^0(Z;N_Z^{j+1})$ define embeddings of Z into projective space. For $l=j$ or $j+1$ let $\{\sigma_{lk}\}$ denote sections of $[lZ]$ such that

$$\{r_l(\sigma_{lk}) : k=1, ..., d_l\}$$

define bases of $H^0(Z;N_Z^l)$. Here $d_l = \dim H^0(Z;N_Z^l)$; let $m=d_j+d_{j+1}$. As σ_0 vanishes exactly on Z we can define a map $\varphi': X_- \to \mathbf{P}^m$ in homogeneous coordinates by

$$\varphi': p \mapsto [\sigma_0^{j+1}(p) : \sigma_0\sigma_{j1}(p) : ... : \sigma_0\sigma_{jd_j}(p) : \sigma_{(j+1)1}(p) : ... : \sigma_{(j+1)d_{j+1}}(p)].$$

From the choice of j it is clear that $\varphi'|_Z$ is an embedding and $d\varphi'(x)$ has rank 2 for all $x\in Z$. It follows from the implicit function theorem and the compactness of Z that φ' defines an embedding of a neighborhood U_0 of Z into $\mathbf{P}^m$.

If we augment the components of φ' to obtain a basis for $H^0(X_-;[(j+1)Z])$ then after relabeling we obtain that

$$\varphi=[\sigma_0^{j+1}:\varphi_1:\ldots:\varphi_n]$$

defines a holomorphic map of X_- into $\mathbf{P}^n$. As the divisor of σ_0 equals Z, the image of Z under φ is the intersection of the hyperplane $H=\{a_0=0\}$ with $\varphi(X_-)$. Since φ' is the composition of φ with a projection onto a linear subspace, $\varphi|_{U_0}$ is an embedding. The line bundle $[Z]$ is trivial in $X_-\setminus Z$, and therefore we can take $\sigma_0\equiv 1$ in $X_-\setminus U_0$. In terms of linear coordinates on the affine chart $\mathbf{P}^n\setminus H$, the other components of φ, $\{\varphi_i/\sigma_0^{j+1}(x):i=1,\ldots,n\}$, are then holomorphic functions on $X_-\setminus U_0$ which are smooth and in particular remain bounded as x approaches bX_-. Hence the image of $X_-\setminus U_0$ under φ is contained in a relatively compact subset of $\mathbf{P}^n\setminus H$. From this we conclude that Z has a neighborhood $U\subset U_0$ such that

$$\varphi(U)\cap\varphi(\overline{X}_-\setminus U_0)=\varnothing.$$

Since φ is one-to-one on U_0 this implies that

$$\varphi(U)\cap\varphi(\overline{X}_-\setminus U)=\varnothing.$$

PROPOSITION 3.3. *Let (X_-,Z) be a weakly embeddable, concave structure with Z of genus g and N_Z of degree k. For $d\in\mathcal{G}(X_-,Z)$ let φ denote the map into projective space defined by $H^0(X_-;[dZ])$. There is a 2-dimensional projective variety V such that $\varphi(X_-)\subset V$. The degree of V is at most kd^2, and* $\operatorname{sing}(V)$ *is a finite set.*

Proof. Let $\dim H^0(X_-;[dZ])=n+1$. We can find a basis for this cohomology group of the form $\{\sigma_0^d,\varphi_1,\ldots,\varphi_n\}$ where each φ_i vanishes to order at most $d-1$ along Z. Let U denote the neighborhood of Z in (3.4)–(3.5). Since Z is positively embedded we can find a smooth, strictly pseudoconcave hypersurface $M\subset U$, bounding a domain D, such that $Z\subset D\subset U$. Since $D\subset U$ its image $\varphi(D)$ is a strictly pseudoconcave submanifold of $\mathbf{P}^n$. As $\varphi(M)\subset\mathbf{P}^n\setminus H$ the Harvey–Lawson theorem implies that it bounds an irreducible analytic variety W, see [HvL]. This variety is of course contained in a compact subset of the affine chart $\mathbf{P}^n\setminus H$. Observe that $V=\varphi(D)\sqcup_{\varphi(M)}W$ is a compact subvariety of $\mathbf{P}^n$.

If $f\in\mathcal{I}_V$ then $\varphi^*(f)|_D=0$, and thus $\varphi^*(f)\equiv 0$ on X_-. Hence $\varphi(X_-)$ is a subset of V. As $W\cap H=\varnothing$ the intersection (with multiplicity) of V with H equals $\varphi(D)\cap H$. From the form of φ it is easy to see that the order of contact between H and V is at most d, and therefore we have the estimate

$$\deg V\leqslant kd^2. \tag{3.13}$$

As V is smooth in a neighborhood of $V\cap H$ it follows from the maximum principle that $\operatorname{sing}(V)$ is a finite set.

From Theorem 2.1, Proposition 3.2 and Proposition 3.3 it follows that the notion of weak embeddability is equivalent to a simple geometric property.

PROPOSITION 3.4. *The pair (X_-, Z) is a weakly embeddable, concave structure if and only if there exists a positive integer d and a neighborhood $U\supset Z$ such that $H^0(X_-;[dZ])|_U$ defines an embedding of U into projective space.*

If bX_- is embeddable then we can obtain more precise geometric information about the maps described above.

COROLLARY 3.1. *Let X_- be a compact manifold with strictly pseudoconcave boundary. Suppose that X_- contains a smooth, compact, positively embedded curve Z, and that bX_- is an embeddable CR-manifold. If $d\in\mathcal{G}(X_-, Z)$ and φ is the holomorphic map defined by $H^0(X_-;[dZ])$, then X_- contains a proper analytic subset E such that $E\cap Z=\varnothing$ and $\varphi|_{X_-\setminus E}$ is an embedding.*

Remark. We call a holomorphic map $\varphi: X_-\to\mathbf{P}^n$ which satisfies the conclusion of this proposition a generically one-to-one mapping of (X_-, Z) into $\mathbf{P}^n$.

Proof. Let X_+ be the normal Stein space with $M=bX_+=-bX_-$, and let $X=X_+\amalg_M X_-$. Let $\varphi: X_-\to\mathbf{P}^n$ be the holomorphic map defined by $H^0(X_-;[dZ])$, and $V\subset\mathbf{P}^n$ the irreducible projective surface containing $\varphi(X_-)$. The line bundle $[Z]$ is trivial in $X\setminus Z$, and therefore the components of the map φ can be represented by holomorphic functions on $X_-\setminus Z$ which have smooth extensions to bX_-. As X_+ is normal it follows from the theorem of Kohn and Rossi that the components of φ have holomorphic extensions to X_+, see [KR]. Thus we have a holomorphic map $\widetilde{\varphi}: X\to\mathbf{P}^n$. As before, the permanence of functional relations implies that the image of $\widetilde{\varphi}$ is contained in V.

Let $H\subset\mathbf{P}^n$ be the hyperplane with $H\cap V=\varphi(Z)$. As $\widetilde{\varphi}(X_+)$ is a relatively compact subset of $\mathbf{P}^n\setminus H$, it follows that there is a neighborhood U of Z such that $\widetilde{\varphi}|_U$ is an embedding and $\widetilde{\varphi}(U)\cap\widetilde{\varphi}(X\setminus U)=\varnothing$. Thus there is an open neighborhood $W\subset V$ of $V\cap H$ such that $\widetilde{\varphi}^{-1}(W)=U$ and each point in W has a unique preimage. Let $[X]$, $[V]$ denote the generators of $H_4(X;\mathbf{Z})$, $H_4(V;\mathbf{Z})$, respectively. As $\widetilde{\varphi}$ is continuous it follows that

$$\widetilde{\varphi}_*[X]=q[V]$$

for some integer q. As $\widetilde{\varphi}$ is holomorphic it follows that $q>0$. In fact $q=1$, for let $\chi\in\mathcal{C}_c^\infty(W)$ be a non-negative function with $\chi(x)>0$ at some point $x\in W$, and let ω be the Kähler form for $\mathbf{P}^n$. As $\operatorname{supp}\chi\subset W$ we have that

$$q\int_W \chi\omega^2=\langle q[V],\chi\omega^2\rangle=\langle [X],\widetilde{\varphi}^*(\chi\omega^2)\rangle=\int_U\widetilde{\varphi}^*(\chi\omega^2)=\int_W\chi\omega^2.$$

Since $\int_W \chi\omega^2 > 0$ this proves the claim.

Let $F = \{x \in X : \operatorname{rk} \widetilde{\varphi}_*(x) < 2\}$ and $S = \widetilde{\varphi}^{-1}(\operatorname{sing} V)$. These are analytic subsets of X. As both are disjoint from U they are proper analytic subsets disjoint from Z. We define the proper analytic subset

$$E' = F \cup S \cup \operatorname{sing} X.$$

The map $\widetilde{\varphi}$ is injective on $X \setminus E'$. Suppose not; then there exists a pair of distinct points $y_1, y_2 \in X \setminus E'$ with $\widetilde{\varphi}(y_1) = \widetilde{\varphi}(y_2)$. In light of the definition of E' there exist disjoint open sets $U_1, U_2 \subset X \setminus E'$ such that $y_i \in U_i$, $\widetilde{\varphi}|_{U_i}$ is a biholomorphism onto its image, and $\widetilde{\varphi}(U_1) = \widetilde{\varphi}(U_2) \subset V \setminus \operatorname{sing}(V)$. Repeating the above computation with a non-negative function $\chi \in \mathcal{C}_c^\infty(\widetilde{\varphi}(U_i))$ we easily obtain a contradiction to the fact that $q=1$. As $\operatorname{rk} \widetilde{\varphi}_*(x) = 2$ for $x \in X \setminus E'$, setting $E = X_- \cap E'$ completes the proof of the corollary.

This corollary indicates that there is an important subclass of weakly embeddable, concave structures which includes those with an embeddable boundary:

Definition. Let X_- be a smooth, strictly pseudoconcave manifold containing a positively embedded, smooth, compact curve Z. If there exists a holomorphic map $\varphi: X_- \to \mathbf{P}^n$ which embeds a neighborhood of Z and is generically one-to-one then we say that (X_-, Z) is an *almost embeddable, concave structure.*

We conclude this section with a result on the $\mathcal{C}^\infty$-closedness of the set of weakly embeddable, concave structures. Let X_- be a compact, complex manifold with pseudoconcave boundary. Let Ω_j, $j \geqslant 1$, denote a sequence of deformations of the complex structure on X_-, smooth up to the boundary, which converge in the $\mathcal{C}^\infty$-topology to a deformation Ω_0. For $j \geqslant 0$ we let X_{-j} denote the complex manifold with complex structure induced by the deformation tensor Ω_j. Suppose that for each $j \geqslant 0$ there is a smooth, compact, holomorphic curve $Z_j \subset\subset X_{-j}$ and that, as $j \to \infty$, Z_j converges in the $\mathcal{C}^\infty$-topology on submanifolds to $Z_0 \subset\subset X_-$.

PROPOSITION 3.5. *If the pairs $\{(X_{-j}, Z_j) : j > 0\}$ are weakly embeddable then the limit (X_{-0}, Z_0) is as well.*

Proof. From the hypotheses it is clear that the genus of Z_j and the degree of its normal bundle are constant after some finite value of j. Denote these common values by g and k respectively. Starting at this value of j, we can therefore find a sequence $\{\psi_j\} \subset \operatorname{Diff}_c(X_-)$ converging to the identity in the $\mathcal{C}^\infty$-topology so that $\psi_j(Z_j) = Z_0$. We let X'_{-j} denote X_- with complex structure defined by $\Omega'_j = \psi_j^*(\Omega_j)$. The pairs (X'_{-j}, Z_0) are again weakly embeddable. The sequence $\{\Omega'_j\}$ also converges to Ω_0 in the $\mathcal{C}^\infty$-topology. We let L_j denote the holomorphic line bundle defined by the divisor Z_0 with respect to the complex structure defined by Ω'_j. Let $\overline{D}'_{jd}$ denote the natural $\bar{\partial}$-operators acting on sections of L_j^d. We need to understand the behavior of the space of holomorphic sections of L_j^d, i.e. $\ker \bar{D}'_{jd}$ as $j\to\infty$. As a $\mathcal{C}^\infty$-complex line bundle, each L_j^d is isomorphic to L_0^d. By fixing a sequence of isomorphisms we are reduced to considering a single line bundle L_0^d and a sequence of differential operators acting on its sections.

In [Le2] it is shown that in the present situation there exists a sequence of $\mathcal{C}^\infty$-bundle isomorphisms

$$\Phi_{j1}: L_0 \to L_j$$

which converge to the identity in the $\mathcal{C}^\infty$-topology. These bundle isomorphisms induce isomorphisms $\Phi_{jd}: L_0^d \to L_j^d$ for every integer d. In terms of this common (non-holomorphic) trivialization, the $\bar{\partial}$-operator on $\mathcal{C}^\infty(X_-; L_j^d)$ is represented by $\bar{D}_{jd} = \Phi_{jd}\circ\bar{D}'_{jd}\circ\Phi_{jd}^{-1}$. Fixing L^2-structures on $\mathcal{C}^\infty(X_-; \Lambda_b^{0,j}\otimes L_0^d)$, $j=0,1$, we define the quadratic forms

$$Q_{jd}(s) = \int_{X_-} |\bar{D}_{jd}s|^2\,dV$$

on $\mathcal{C}^\infty(\bar{X}_-; L_0^d)$. Using these quadratic forms and the Friedrichs extension we define unbounded, self-adjoint operators $\mathcal{D}_{jd}$ acting on $L^2(X_-; L_0^d)$. Using the estimates in [Le2] it follows that each operator $\mathcal{D}_{jd}$ has a compact resolvent. In light of the definition of $\bar{D}_{jd}$, it is clear that for $0\leqslant j$ and $0<d$ we have the isomorphisms

$$H^0(X_-; L_j^d) \simeq \ker \mathcal{D}_{jd}. \tag{3.14}$$

For each $d>0$ the sequence of operators $\{\mathcal{D}_{jd}: j>0\}$ converges in the strong resolvent sense to $\mathcal{D}_{0d}$. From classical perturbation theory, see [Ka], we conclude that

$$\dim\ker \mathcal{D}_{0d} \geqslant \limsup_{j\to\infty} \dim\ker \mathcal{D}_{jd}. \tag{3.15}$$

Since (X_{-j}, Z_j) is weakly embeddable it follows that for sufficiently large j and each $d>(2g-2)/k+1$, we have the estimate

$$\dim\ker \mathcal{D}_{jd} \geqslant \tfrac{1}{2}d(d+1)k+(1-g)(d+1).$$

The conclusion of the proposition is immediate from (3.14), these estimates and (3.15).

Remarks. (1) Note that having

$$\dim H^0(X_-; [dZ]) > cd^2$$

does not imply embeddability. The Andreotti–Grauert–Rossi example, see [R] or [AS], is a 1-parameter family of CR-structures on S^3: $\{{}^\varepsilon T^{0,1}S^3: \varepsilon\in\mathbf{C}, |\varepsilon|<1\}$. The structure

${}^0T^{0,1}S^3$ is that induced on the unit sphere in $\mathbf{C}^2$. If $\varepsilon\neq 0$ then $(S^3, {}^\varepsilon T^{0,1}S^3)$ is not embeddable. These structures extend to $\mathbf{P}^2\setminus B_1$ in such a way that Z, the line at infinity, remains holomorphic. For the standard structure,

$$\dim H^0(X_-^0;[dZ])=\tfrac{1}{2}(d+1)(d+2).$$

For a non-trivial deformation ($\varepsilon\neq 0$) we have

$$\tfrac{1}{4}d^2\leqslant \dim H^0(X_-^\varepsilon;[dZ])\leqslant \tfrac{1}{4}(d+1)^2.$$

For any $d>0$, $\varepsilon\neq 0$, the space of sections $H^0(X_-^\varepsilon;[dZ])$ gives local coordinates but fails to separate points on $\overline{X}_-$.

(2) In §5 we show that the boundary of an almost embeddable, concave structure is embeddable. It is a question of principal interest whether a weakly embeddable, concave structure is almost embeddable. In general, we do not know the answer to this question. Corollary 3.1 shows that if an embeddable CR-manifold bounds a pseudoconcave domain X_-, with smooth, positively embedded curve $Z\subset\subset X_-$, then the concave structure (X_-, Z) is almost embeddable. In §6 we show that, in a certain sense, almost embeddability is closed under convergence in the $\mathcal{C}^\infty$-topology. This allows us to show, for many new classes of 3-dimensional CR-manifolds, that the set of small embeddable deformations of the CR-structure is closed in the $\mathcal{C}^\infty$-topology.

4. Finiteness results for projectively fillable germs

In the previous sections we have considered strictly pseudoconcave, complex surfaces which contain a positively embedded curve. Our principal motivation is to understand the behavior, under deformation, of the algebra of CR-functions on the boundary of the manifold. There are two reasonable "germ models" for this problem, a concave one and a convex one. The convex one is the study of deformations of isolated surface singularities, whereas the concave problem is the study of projective fillability for surface germs containing a given curve with a given positive normal bundle. In this paper we consider only the latter problem, which we now formulate more precisely.

Let Σ denote a Riemann surface and L a line bundle over Σ. A surface germ containing (Σ, L) is a complex manifold Y of dimension 2, with a smooth, holomorphic embedding $i:\Sigma\to Y$ such that the normal bundle $N_\Sigma\to i(\Sigma)$ induced by i is isomorphic to L. We let $\mathcal{S}(\Sigma, L)$ denote the set of surface germs containing (Σ, L). Two such germs $Y_1, Y_2\in\mathcal{S}(\Sigma, L)$ are equivalent if there is a germ of a biholomorphism Φ defined on a neighborhood of $i_1(\Sigma)\subset Y_1$ such that

$$\Phi\circ i_1=i_2.$$

Let $\mathfrak{S}(\Sigma, L)$ denote the set of equivalence classes in $\mathcal{S}(\Sigma, L)$ under this equivalence relation. We do not attempt at this point to define a topology or analytic structure on this space. We say that a surface germ Y is projectively fillable if it is equivalent to an open subset $U \subset V$, a projective variety. We denote this subset of $\mathfrak{S}(\Sigma, L)$ by $\mathfrak{E}(\Sigma, L)$.

For the case of interest to us, $L>0$, it is known that $\mathfrak{S}(\Sigma, L)$ is infinite-dimensional. Using results of Docquier and Grauert, Morrow and Rossi gave a parametrization for the subset of $\mathfrak{S}(\Sigma, L)$ consisting of structures which are holomorphic neighborhood retracts (HNRs) of Σ. This subset is shown to be infinite-dimensional. On the other hand, results in [Gr] show that this subset is also of infinite codimension in $\mathfrak{S}(\Sigma, L)$.

The problem which we consider in this section is the "size" of $\mathfrak{E}(\Sigma, L)$. We show that it injects into a subset of a finite-dimensional space. In [MR1] it is shown that $\mathfrak{E}(\mathbf{P}^1, \mathcal{O}(1))$ consists of a single structure: the standard embedding of $\mathbf{P}^1$ as a linear subspace of $\mathbf{P}^2$.

The following theorem is a consequence of the results in §2 and §3:

THEOREM 4.1. *Let g denote the genus of Σ, and $0<k$ the degree of $L \to \Sigma$. There are integers $N(g,k)$ and $D(g,k)$ such that each class of germs in $\mathfrak{E}(\Sigma, L)$ contains a representative which is an open subset of a projective surface S, embedded in $\mathbf{P}^N$ for an $N \leqslant N(g,k)$ of degree at most $D(g,k)$. The curve Σ is realized as the intersection of S with a hyperplane.*

Remark. The sense in which $\mathfrak{E}(\Sigma, L)$ is finite-dimensional is given by this theorem. The proof is a refinement of the proof of the Nakai–Moishezon criterion for the ampleness of a divisor, see [Ht, Chapter V, §1]. The extra bookkeeping is done in Theorem 2.1.

Proof. Let Y be a representative of a class in $\mathfrak{E}(\Sigma, L)$. Without loss of generality we can consider Y to be an open subset in a smooth projective surface X. Let Z denote the embedded image of Σ with normal bundle $N_Z \simeq L$. As N_Z is positive, a small neighborhood X_- of Z can be found with a smooth, strictly pseudoconcave boundary. From Theorem 2.1 it follows that (X_-, Z) is a weakly embeddable, concave structure. Proposition 3.2 implies that there exists a $d \in \mathcal{G}(X_-, Z)$ such that $d < C(g,k)+1$. The conclusions of the theorem now follow from (3.10) and Proposition 3.3.

Remarks. (1) As shown in Proposition 3.3, the image of the map defined by $H^0(X_-; [dZ])$ provides a representative of a fillable germ as an open set in a projective surface with at worst finitely many isolated singularities.

(2) The theorem implies that every class in $\mathfrak{E}(\Sigma, L)$ has a representative with $i(\Sigma)$ the intersection of a hyperplane in $\mathbf{P}^n$, $n \leqslant N(g,k)$, with a surface in $\mathbf{P}^n$ of degree at most $D(g,k)$. Let $\mathcal{C}(d,n)$ denote the surfaces in $\mathbf{P}^n$ of degree at most d. This is a

finite-dimensional space. We see that there is an injective map of $\mathfrak{E}(\Sigma, L)$ into the finite-dimensional space

$$\mathcal{C}(D(g,k), N(g,k)) \times \mathbf{P}^{*N(g,k)}. \tag{4.1}$$

We let $\mathfrak{e}(g,k)$ denote the finite-dimensional image of this correspondence. This is a sense in which the set of projectively fillable surface germs containing (Σ, L) with $0<L$ is finite-dimensional. This is a weak version of Lempert's conjecture on the existence of a universally stable embedding: every fillable germ with the given data can be embedded into a finite-dimensional family of varieties. The set $\mathfrak{e}(g,k)$ may be quite complicated, e.g. it may have infinitely many connected components. Note that the moduli space (4.1) only depends on the topological data (g,k). Hence if we allow the complex structures on Σ and L to vary then we obtain an injective map into the same space.

(3) The results in this section have a direct bearing on question (3) of the introduction. If $H_c^2(X_-; \Theta\otimes[-Z])=0$ then we can extend a small deformation of the CR-structure on M to a complex structure on X_- so that Z remains a positively embedded holomorphic curve. Indeed, using results in [EH1] we obtain an analytic map from small deformations of the CR-structure on M to deformations of the complex structure on X_- which extend the deformation of the CR-structure and vanish on Z. Let $\mathrm{Def}(M, \bar\partial_b)$ denote the small deformations of the CR-structure on M. Composing this extension map with the projection to $\mathfrak{S}(Z, N_Z)$ defines a map

$$\mathfrak{g}\colon \mathrm{Def}(M, \bar\partial_b) \to \mathfrak{S}(Z, N_Z).$$

If a small deformation ω is embeddable then $\mathfrak{g}(\omega)$ is fillable. It would be quite interesting to know if the converse is also true.(1) We conclude from Theorem 4.1 that the set of embeddable deformations is, at least *formally* contained in a subvariety of infinite codimension in the set of all deformations. To prove a theorem to this effect one would of course need to define appropriate topologies and analytic structures on $\mathfrak{S}(Z, N_Z)$ and $\mathrm{Def}(M, \bar\partial_b)$ in which to analyze the properties of this map. We hope to return to this in a latter publication. A related result for the case of the 3-sphere appears in [BlD].

5. Almost embeddability implies embeddability

Let X_- be a complex surface with smooth, strictly pseudoconcave boundary M. Assume that there is a smooth holomorphic curve $Z \subset\subset X_-$ of genus g, with positive normal bundle N_Z of degree k. In §3 we showed that if M is embeddable then there are a projective variety $V \subset \mathbf{P}^n$ and a holomorphic mapping $\varphi\colon X_- \to V$ which enjoy several

(1) *Note added in proof.* A recent result of Bruno de Oliveira shows that the converse is false.

special properties. Notably, φ is an embedding in a neighborhood of Z, and there is a proper analytic subset $E\subset X_-\backslash Z$ such that $\varphi|_{X_-\backslash E}$ is an embedding. In this section we show that an almost embeddable, pseudoconcave surface is actually embeddable.

A section $\omega\in\mathcal{C}^\infty(M;\mathrm{Hom}(T^{0,1}M,T^{1,0}M))$ with $\|\omega\|_{L^\infty}<1$ defines a deformation of the CR-structure with the same underlying contact field. For each $p\in M$ we set

$$^\omega T_p^{0,1}M=\{\bar{Z}+\omega(\bar{Z}):\bar{Z}\in T_p^{0,1}M\}.$$

Note that ω does not have to satisfy an integrability condition. If

$$\Omega\in\mathcal{C}^\infty(\bar{X}_-;\mathrm{Hom}(T^{0,1}X_-,T^{1,0}X_-))$$

satisfies $\|\Omega\|_{L^\infty}<1$ and the integrability condition

$$\bar{\partial}\Omega-\tfrac{1}{2}[\Omega,\Omega]=0$$

then

$$^\Omega T_x^{0,1}X_-=\{\bar{Z}+\Omega(\bar{Z}):\bar{Z}\in T_x^{0,1}X_-\},\quad x\in X_-,$$

defines an integrable almost complex structure on X_-. If

$$\Omega_b=\Omega|_{T^{0,1}M}=\omega$$

then we say that Ω is an extension of ω to X_-.

We now establish that if (X_-,Z) is an almost embeddable, concave structure then bX_- is an embeddable CR-manifold. The first observation is that we do not need to actually embed bX_- itself, it suffices to embed a sequence of hypersurfaces tending to bX_-. The proof of this statement uses the relative index introduced in [Ep2]. For the convenience of the reader we recall the basic definitions. Let $(M,\bar{\partial}_b)$ denote an embeddable, strictly pseudoconvex, 3-dimensional CR-manifold, and $\mathcal{S}$ an orthogonal projection onto $\ker\bar{\partial}_b$. If ω denotes an embeddable deformation of the CR-structure then it is shown in [Ep2] that

$$\mathcal{S}:\ker\bar{\partial}_b^\omega\to\ker\bar{\partial}_b$$

is a Fredholm operator, and the relative index $\mathrm{Ind}(\bar{\partial}_b,\bar{\partial}_b^\omega)$ is defined as the Fredholm index of this operator.

Choose a non-negative function $p_0\in\mathcal{C}^\infty(\bar{X}_-)$ which vanishes on bX_- but such that dp_0 is non-vanishing on this set. We can further arrange that p_0 is strictly plurisubharmonic in a neighborhood U_0 of bX_-. For $\varepsilon>0$ we let

$$X_\varepsilon=p_0^{-1}((\varepsilon,\infty)).$$

LEMMA 5.1. *Suppose that for a decreasing sequence $\{\varepsilon_n\}$ tending to zero the CR-manifolds $\{bX_{\varepsilon_n}\}$ are embeddable. Then bX_- is embeddable.*

Proof of Lemma 5.1. For each $0<\tau\ll\varepsilon_0$, we can find a smooth family of contact transformations

$$\psi_t: bX_t \to bX_\tau, \quad t\in[0,\tau],$$

with $\psi_\tau=\mathrm{Id}$. Using these contact transformations we pull back the CR-structures $T^{0,1}bX_t$ to obtain a smooth family of CR-structures on bX_τ. Let ω_t denote the deformation tensors. By selecting τ sufficiently small we can ensure that

$$\|\omega_t\|_{L^\infty} < \tfrac{1}{4} \quad \text{for } t\in[0,\tau]. \tag{5.1}$$

Observe that for any $0<s<\tau$ the CR-manifold bX_s is embeddable: We choose an n so that $\varepsilon_n<s$. As bX_{ε_n} is embeddable there is a pseudoconvex, complex manifold X_n^+ with $bX_n^+=-bX_{\varepsilon_n}$. By gluing X_n^+ to the collar $\{x:\varepsilon_n\leqslant p_0(x)\leqslant\tau\}$ we obtain a complex manifold which contains bX_s as a hypersurface bounding a compact, pseudoconvex domain. Using this construction and Corollary 3.1 in [Ep2] it follows that for $0<s<\tau$ we have

$$\mathrm{Ind}(bX_\tau, bX_s)=0.$$

As $\omega_0=\lim_{s\to 0}\omega_s$ in the $\mathcal{C}^\infty$-topology it follows from (5.1) and Theorem E in [Ep2] that

$$\mathrm{Ind}(bX_\tau, bX_-)\geqslant 0.$$

We can therefore apply Theorem A from [Ep2] to conclude that bX_- is embeddable.

Remark. This gluing argument implies that

$$\mathrm{Ind}(bX_-, bX_s)=0 \quad \text{for } s<\tau.$$

For sufficiently small s we can therefore embed bX_s as a small deformation of a given embedding of bX_-.

THEOREM 5.1. *Suppose that (X_-, Z) is an almost embeddable, strongly pseudoconcave structure. Then bX_- is embeddable.*

Proof of Theorem 5.1. Let $\psi: X_-\to\mathbf{P}^N$ be a generically one-to-one map of the concave structure (X_-, Z) and let G denote the minimal, proper analytic subset such that $\psi|_{X_-\setminus G}$ is an embedding. If there is an exhaustion function p_0 vanishing on bX_- and plurisubharmonic on $X_-\setminus Z$, then, as $G\cap Z=\varnothing$, it follows that $\dim G=0$. In this circumstance we can easily use the lemma to show that bX_- is embeddable. Set $M_t=\{p_0^{-1}(t)\}$.

Since 0 is not a critical value for p_0 there is an $\varepsilon>0$ such that p_0 has no critical values in $[0,\varepsilon]$. The generically one-to-one map ψ is an embedding on the complement of a discrete set. Thus we can find a decreasing sequence $\{t_n\}$ such that $\lim t_n=0$. For each n we have:

(1) $t_n<\varepsilon$, and hence dp is non-vanishing on M_{t_n},

(2) $\psi|_{M_{t_n}}$ is an embedding.

In this case the theorem follows from Lemma 5.1.

Now we treat the general case. Our proof makes extensive usage of monoidal transformations and the resolution of singularities. The data we have is X_-, a pseudoconcave manifold with a positively embedded smooth divisor Z. There is an irreducible projective variety $W\subset \mathbf{P}^N$ such that $\psi(X_-)\subset W$. *A priori* we do not know that W is a normal variety. If $q\colon \widetilde{W}\to W$ is the normalization of W then W and $\widetilde{W}$ are bimeromorphic and biholomorphic in the complement of a finite set. They are moreover biholomorphic in a neighborhood of Z. Let ψ' denote the composition $q^{-1}\circ\psi$. This is a meromorphic map which is therefore defined in the complement of a set $I\subset X_-\setminus Z$ of codimension at least 2, see [Gu]. As $\widetilde{W}\setminus\psi'(Z)$ is an affine variety it follows that we can find coordinates for $\widetilde{W}\setminus\psi'(Z)$ in which the components of ψ' are uniformly bounded in a neighborhood of I. By Riemann's extension theorem, ψ' has a holomorphic extension to X_-. This shows that there is no loss in generality in assuming that ψ maps X_- into a normal subvariety W of $\mathbf{P}^n$.

Let $\pi\colon \widehat{W}\to W$ be a resolution of the singularities of W.

LEMMA 5.2. *The mapping ψ factors through $\widehat{W}$ on the complement of a discrete set. That is, there is a discrete set $I\subset X_-$ and a holomorphic map $\hat{\psi}\colon X_-\setminus I\to\widehat{W}$ such that*

$$\psi|_{X_-\setminus I}=\pi\circ\hat{\psi}.$$

Remark. We use the notation $f\colon A \dashrightarrow B$ for meromorphic maps.

Proof of Lemma 5.2. The proof is essentially immediate: $\pi^{-1}\colon W\to\widehat{W}$ is a meromorphic map, and it is biholomorphic from $W\setminus\operatorname{sing}(W)$ to its image. We set $\hat{\psi}=\pi^{-1}\circ\psi$; this map is a biholomorphism from $X_-\setminus G$ to $\pi^{-1}(W\setminus\operatorname{sing}(W))$. More importantly it is a meromorphic mapping from X_- to $\widehat{W}$. This implies that the map extends to the complement of the indeterminacy locus I. As I is a subvariety of codimension at least 2, it is a discrete set of points.

Henceforth we let $\pi\colon\widehat{W}\to W$ denote a resolution of the singularities of W, and $\hat{\psi}$ the lift of ψ. The map $\hat{\psi}\colon X_- \dashrightarrow \widehat{W}$ is holomorphic on the complement of a discrete set and an embedding on the complement of a proper subvariety $\widehat{G}$. To complete our argument we need to show that by finitely many blow-ups of $\widehat{W}$ we can obtain a map which is defined on the complement of a discrete set and globally injective. To that end we first treat the local problem: Let (z,w) denote coordinates for $\mathbf{C}^2$ and $D, D' \subset \mathbf{C}^2$ be neighborhoods of $(0,0)$. We call a holomorphic map $f: D \to D'$ a germ of a blow-down if

(1) $f(0,w)=(0,0)$,

(2) f is injective on $D \setminus \{z=0\}$.

THEOREM 5.2. *Suppose that $f: D \to D'$ is a germ of a blow-down. Then there are local coordinates (ζ, ξ) on a neighborhood of $(0,0)$ such that in these coordinates the map is either*

$$f(\zeta,\xi)=(\zeta,\zeta^k\xi), \quad k\in\mathbf{N}, \tag{5.2}$$

or

$$f(\zeta,\xi)=(\zeta^j,\zeta^{k_1}(\alpha_1+\zeta^{k_2}(\alpha_2+\ldots+\zeta^{k_p}(\alpha_p+\xi)\ldots))), \quad \alpha_i\in\mathbf{C},\ k_i\in\mathbf{N},\ i=1,\ldots,p. \tag{5.3}$$

As a consequence of the theorem we obtain

COROLLARY 5.1. *If $f: D \to D'$ is a germ of a blow-down, then there is a finite sequence of point blow-ups*

$$D'=D_0' \xleftarrow{\pi_1} D_1' \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_m} D_m'$$

and lifted maps $f_i: D \to D_i'$ so that

$$f=\pi_1\circ\ldots\circ\pi_i\circ f_i$$

and $f_m: D \to D_m'$ is a germ of a biholomorphism.

These results are proved in [EH2]. Combining this corollary with Lemma 5.2 we obtain

PROPOSITION 5.1. *If $\psi: X_- \to W$ is a generically one-to-one mapping of the concave structure (X_-, Z), then there exists a desingularization $\tilde{\pi}: \widetilde{W} \to W$ and a meromorphic map $\tilde{\psi}: X_- \dashrightarrow \widetilde{W}$ which is a biholomorphism on the complement of a discrete set and such that*

$$\psi=\tilde{\pi}\circ\tilde{\psi}.$$

Proof of Proposition 5.1. We let $\pi: \widehat{W} \to W$ and $\hat{\psi}$ be respectively the desingularization of W and the meromorphic lift of ψ obtained above. Let I denote the indeterminacy locus of $\hat{\psi}$, and E the union of 1-dimensional components of $\widehat{G}$. As $\widehat{W}$ is smooth and $\hat{\psi}$ is generically one-to-one it is clear that $\operatorname{rk} d\hat{\psi}(x)<2$ for $x\in E\setminus I$. We show that at smooth points of $E\setminus I$ the map $\hat{\psi}$ is a germ of a blow-down. This follows easily from the next proposition.

PROPOSITION 5.2. *Let $U, V \subset \mathbf{C}^2$ be neighborhoods of* $(0,0)$ *and let* $f: U \to V$ *be a holomorphic map. Suppose that the set* $K = \{z : \operatorname{rk} df(z) < 2\}$ *is a smooth proper submanifold passing through* $(0,0)$. *If* $\operatorname{rk} d(f|_K)(0,0) = 1$ *then* f *is a non-trivially ramified cover in a neighborhood of* $(0,0)$.

Proof of Proposition 5.2. As $\operatorname{rk} df$ is generically 2 we can find local coordinates (z_1, z_2) near to 0, and (w_1, w_2) near to $f(0)$, so that $K = \{z_2 = 0\}$,

$$f(z_1, z_2) = (g(z_1, z_2), z_2^j h(z_1, z_2))$$

with $j > 1$ and

$$\frac{\partial g}{\partial z_1}(0,0) \neq 0, \quad h(0,0) \neq 0.$$

For each ξ sufficiently close to zero there is a unique $z_1(\xi)$ close to 0 with

$$g(z_1(\xi), 0) = \xi.$$

We can solve for $z_1(z_2; \xi)$ such that

$$z_1(0; \xi) = z_1(\xi) \quad \text{and} \quad g(z_1(z_2; \xi), z_2) = \xi.$$

On the other hand, for $\eta \neq 0$ sufficiently close to zero the equation

$$z_2^j h(z_1(z_2; \xi), z_2) = \eta$$

has $j > 1$ distinct solutions. Thus f is a j-sheeted ramified covering map in a neighborhood of 0.

Let $\{E_1, ..., E_m\}$ be the irreducible components of E. As $\hat{\psi}$ is generically one-to-one, the proposition shows that $\operatorname{rk}(d\hat{\psi}|_{E_j}) = 0$ for $j = 1, ..., m$. Hence each component is mapped to a point by $\hat{\psi}$, and on each, the Jacobian determinant has a positive, generic order of vanishing. We use the following iterative procedure to desingularize the map $\hat{\psi}$: Begin with E_1 and blow up the point $\hat{\psi}(E_1)$, obtaining a space $\pi_1: \widehat{W}_1 \to \widehat{W}$ and a lifted map $\hat{\psi}_1$. As the indeterminacy locus of $\hat{\psi}_1$ is discrete, the map $\hat{\psi}_1$ is either locally biholomorphic in a neighborhood of E_1 or $\hat{\psi}_1(E_1)$ is a point. In the former case we turn our attention to $\hat{\psi}_1(E_2)$, in the latter case we simply blow up $\hat{\psi}_1(E_1)$. Proceeding recursively in this fashion we obtain a sequence of monoidal transformations $\pi_i: \widehat{W}_i \to \widehat{W}_{i-1}$ and lifted meromorphic maps $\hat{\psi}_i: X_- \dashrightarrow \widehat{W}_i$ which satisfy

$$\hat{\psi}_{i-1} = \pi_i \circ \hat{\psi}_i.$$

As the sum of the generic vanishing orders of the Jacobians of the $\hat{\psi}_i$ along the components of E is strictly decreasing, after finitely many such steps we obtain a space $\tilde{\pi}: \widetilde{W} \to \widehat{W}$ and a lifted map $\tilde{\psi}: X_- \dashrightarrow \widetilde{W}$ with $\hat{\psi} = \tilde{\pi} \circ \tilde{\psi}$. The map $\tilde{\psi}$ is defined, holomorphic and has non-vanishing Jacobian on the complement of a discrete set $I_{\tilde{\psi}}$. It is evident that $\tilde{\psi}|_{X_- \setminus I_{\tilde{\psi}}}$ is a biholomorphism onto its range, which is an open subset of $\widetilde{W}$.

Since the set $I_{\tilde{\psi}}$ is discrete, for almost all ε we obtain an embedding

$$\tilde{\psi}: bX_\varepsilon \to S_\varepsilon \subset \widetilde{W}.$$

By a finite sequence of monoidal transformations of X_ε we can resolve the indeterminacy of $\tilde{\psi}$ at the finite set of points $I_{\tilde{\psi}} \cap X_\varepsilon$. Denote the blown-up space by $\widehat{X}_\varepsilon$, and the lifted map by $\tilde{\psi}_\varepsilon$. For such generic ε, the image $\tilde{\psi}_\varepsilon(\widehat{X}_\varepsilon)$ is a domain in $\widetilde{W}$ with smooth, pseudoconcave boundary S_ε. Therefore $\widetilde{W}_{+\varepsilon} = \widetilde{W} \setminus \tilde{\varphi}_\varepsilon(\widehat{X}_\varepsilon)$ is a compact domain with a smooth, strictly pseudoconvex boundary, CR-equivalent to bX_ε. This completes the proofs of Proposition 5.1 and Theorem 5.1.

The following corollary is a simple consequence of Theorem 5.1 and classical results of Chow and Kodaira, see [CK] and [Kd1].

COROLLARY 5.2. *If (X_-, Z) is an almost embeddable concave structure, then X_- is embeddable in projective space.*

Proof. From Theorem 5.1 it follows that $-bX_-$ is the boundary of a strictly pseudoconvex, normal Stein space X_+. Therefore $X = X_- \amalg_{bX_-} X_+$ is a compact complex space with a positively embedded divisor $Z \subset X \setminus \operatorname{sing}(X)$. Let $\widehat{X}$ be the minimal resolution of X. As X_- is smooth it follows that $\widehat{X}$ contains an open subset $\widehat{X}_-$, biholomorphic to X_-. It therefore also contains a positively embedded holomorphic curve $\widehat{Z}$. As $\widehat{Z} \cdot \widehat{Z} \geqslant 1$ it follows from Theorem 3.3 in Part I of [Kd1] that $\widehat{X}$ is an algebraic surface and per force embeddable in projective space. Thus $X_- \simeq \widehat{X}_-$ also embeds in projective space.

Remark. Consider the following example: Let (V, p) be a compact, projective surface with a unique singular point at p. Let $Z \subset V \setminus \{p\}$ denote a smooth hyperplane section and $\pi: \widehat{V} \to V$ a resolution of the singularity of V. Let $E \subset \widehat{V}$ be the exceptional divisor which, for simplicity, we suppose is a smooth, compact curve. Let $\{B_1, ..., B_m\}$ be pairwise disjoint disks contained in E. As $E \setminus \bigcup B_i$ is a Stein space, Siu's theorem implies that it has a Stein neighborhood basis $\{U_i\}$, see [Si]. For each i we can suppose that $-bU_i$ is the smooth, strictly pseudoconcave boundary of a domain $X_i \subset \widehat{V}$, and that $Z \cap U_i = \varnothing$. The domains X_i are almost embeddable: a basis φ of sections of $H^0(\widehat{V}; [Z])$ defines a generically one-to-one map. Evidently $\varphi(E)$ is a single point. There does not exist a projective embedding of X_i such that $[dZ]$ is linearly equivalent to a hyperplane section

$Z'=H\cap X_i$. If this were possible then a basis φ' of sections for $H^0(X_i;[Z'])$ would extend to define a projective immersion of $\widehat{V}$ where the compact curve E would be mapped non-trivially into an affine chart. This is because $E\cap X_i$ is a union of open disks and $\varphi'|_{E\cap X_i}$ would be an embedding. This shows that in order to embed bX_- it is sometimes necessary to enlarge the algebra of CR-functions beyond those which arise as restrictions of meromorphic functions with polar divisor Z.

6. Limits of embeddable deformations

We now establish that if certain cohomological conditions are satisfied then the set of small embeddable perturbations of the CR-structure on the boundary of an embeddable, pseudoconcave surface is closed in the $\mathcal{C}^\infty$-topology.

THEOREM 6.1. *Suppose that X_- is a complex surface with an embeddable, smooth, strictly pseudoconcave boundary such that there exists a smooth, positively embedded, holomorphic curve $Z\subset\subset X_-$ of genus at least 1. If either of the hypotheses*

$$H^2_c(X_-;\Theta\otimes[-Z])=0 \tag{6.1}$$

or

$$H^2_c(X_-;\Theta)=0 \quad \textit{and} \quad H^1(Z;N_Z)=0 \tag{6.2}$$

is satisfied, then the set of small embeddable deformations of the CR-structure on bX_- is closed in the $\mathcal{C}^\infty$-topology.

Remark. If the genus of Z is zero then the complications encountered here do not arise. This case, which has already been treated in [Le1], [Le2] and [Li], is considered in §9.

This theorem has the following corollary:

COROLLARY 6.1. *Suppose that $\overline{X}_-$ is a strictly pseudoconcave manifold with boundary which is embeddable in projective space and contains a positively embedded, smooth, compact curve Z. If (6.1) holds, then the set of small deformations of the complex structure on X_- which are smooth up to bX_-, vanish on Z and embed into projective space is closed in the $\mathcal{C}^\infty$-topology. If (6.2) holds, then the set of small deformations of the complex structure on X_- which are smooth up to bX_- and embed into projective space is closed in the $\mathcal{C}^\infty$-topology.*

Proof of Corollary 6.1. Suppose that $\{\Omega_j\}$ is a sequence of integrable, embeddable deformations of the complex structure on X_- which are smooth up to bX_- and satisfy the hypotheses of the corollary. Assume that this sequence converges in the $\mathcal{C}^\infty$-topology to Ω_0. Let $\{\omega_j : j\geqslant 0\}$ be the deformations of the CR-structure on bX_- induced by these deformations of the complex structure on X_-. The sequence $\{\omega_j : j>0\}$ converges to ω_0 in the $\mathcal{C}^\infty$-topology. Using Andreotti's theorem (see [A]) and the resolution of singularities, it follows, for each $j>0$, that $(bX_-, {}^{\omega_j}T^{0,1}bX_-)$ bounds a compact complex manifold. It follows from Kohn's theorem (see [Ko]) that each is therefore embeddable. If the ω_j are sufficiently small then Theorem 6.1 implies that $(bX_-, {}^{\omega_0}T^{0,1}bX_-)$ is embeddable as well. Let X'_+ denote the normal Stein space with this boundary, and X'_- the manifold X_- with the complex structure defined by Ω_0. If the deformations $\{\Omega_j\}$ are sufficiently small, then under either hypothesis X'_- contains a positively embedded, compact curve (see Step 1 below). Let $\widehat{X}$ be the minimal resolution of the singularities of the compact complex space $X'=X'_+\amalg X'_-$. Since X'_- is smooth, $\widehat{X}$ contains an open subset biholomorphic to X'_-. Hence $\widehat{X}$ contains a positively embedded, compact curve, and so satisfies the hypotheses of Theorem 3.3 in Part I of [Kd1]. It is therefore an algebraic surface which embeds into projective space.

Proof of Theorem 6.1. The proof has two main steps:

(1) Let $\{\omega_j\}$ be a sequence of small embeddable deformations of the CR-structure on bX_- which converges in the $\mathcal{C}^\infty$-topology to ω_0. Using the cohomological hypotheses we show that each deformation of the CR-structure ω_j can be extended to an integrable deformation Ω_j of the complex structure on X_-. Let X_{-j} denote X_- with the complex structure defined by Ω_j. We further establish that Ω_j can be chosen so that there is a smooth, compact, holomorphic curve $Z_j\subset X_{-j}$. Finally we show that Ω_j converges to Ω_0 and Z_j to Z_0 in appropriate topologies.

(2) After passing to a subsequence, we study the limits of the generically one-to-one maps and projective varieties $\varphi_j: X_{-j}\to V_j$. This allows us to show that there is a generically one-to-one, holomorphic map $\psi: X_{-0}\to \mathbf{P}^m$, and therefore that (X_{-0}, Z_0) is almost embeddable. From Theorem 5.1 this suffices to conclude that bX_{-0} is embeddable.

Step 1. If (6.1) holds then we can apply the extended Kiremidjian theorem, see [Le2], [EH1], to obtain $\{\Omega_j : j\geqslant 0\}$, integrable extensions of $\{\omega_j : j\geqslant 0\}$ which vanish along Z. That is,

$$\Omega_j=\sigma_0\Xi_j,$$

where $\sigma_0\in H^0(X_-;[Z])$ with divisor equal to Z, and the $\{\Xi_j\}$ are smooth tensors. We can also assume that the sequence $\{\Omega_j : j\geqslant 1\}$ converges in the $\mathcal{C}^\infty$-topology to Ω_0 as $j\to\infty$.

If (6.2) holds then we proceed a little differently. We apply Kiremidjian's theorem to obtain integrable extensions $\{\Omega'_j : j\geqslant 0\}$ of $\{\omega_j : j\geqslant 0\}$, converging to Ω'_0 as $j\to\infty$. If the deformation tensor Ω_0' is sufficiently small, then, as $H^1(Z;N_Z)=0$, we can apply Kodaira's stability theorem (see [Kd2]) to locate an Ω_0'-holomorphic curve Z_0, a small deformation of Z such that

$$\deg N_{Z_0}=\deg N_Z. \tag{6.3}$$

As Ω_j' converges to Ω_0' in the $\mathcal{C}^\infty$-topology we can, for sufficiently large j, locate Ω_j'-holomorphic curves Z_j' which are small deformations of Z_0 and converge to it in the $\mathcal{C}^\infty$-topology on submanifolds. We henceforth omit the finitely many terms of the sequence for which such a curve does not exist, and relabel the remaining terms beginning with $j=1$.

Select diffeomorphisms $\{\psi_j\in\mathrm{Diff}_c(X_-):j\geqslant 1\}$ which reduce to the identity in a neighborhood of bX_- and carry Z_j' onto Z_0. The size of ψ_j in the $\mathcal{C}^\infty$-topology on $\mathrm{Diff}_c(X_-)$ can be controlled by the $\mathcal{C}^\infty$-seminorms measuring the distance between Z_j' and Z_0, and therefore by the $\mathcal{C}^\infty$-seminorms of $\omega_j-\omega_0$. We let Ω_j be the pullback of Ω_j' via ψ_j. This is again an integrable extension of ω_j, with Z_0 a holomorphic curve. For $j\geqslant 0$ we let X_{-j} denote the manifold X_-, with the complex structure defined by the deformation tensor Ω_j, and $Z_j=\psi_j(Z_j')$ denote the smooth divisor with respect to this complex structure.

Step 2. We now consider the behavior of the generically one-to-one maps

$$\varphi_j: X_{-j}\to V_j$$

obtained in Corollary 3.1. Our goal is to prove the following result:

THEOREM 6.2. *There exists a $d>0$ such that $H^0(X_{-0};[dZ_0])$ is base-point-free and defines a generically one-to-one map of (X_{-0},Z_0) into projective space.*

Remark. In a subsequent paper we will give an analytic proof of this statement. It follows from the estimates

$$\dim H_D^{0,1}(V_j,[dZ_j])=O(1),$$

uniform in d and j. This proof is also rather involved.

Proof of Theorem 6.2. As the proof of this theorem is quite long we begin with a short outline. Because the manifolds $\{X_{-j}\}$ can be compactified we can use Corollary 3.1 to obtain a generically one-to-one map of each (X_{-j},Z_j) into a projective space such that the images are contained in irreducible subvarieties of uniformly bounded degrees. Using standard compactness results we show that the bases of sections defining these maps have convergent subsequences. Two problems arise in the limit:

(1) The limiting collection of sections might have base points, and so fail to give a globally defined map of X_{-0} into projective space. In general, the limiting map is defined in the complement of a finite subset of Z_0.

(2) It is not evident that the limiting map is generically one-to-one.

To handle these problems we consider the image varieties as defining holomorphic currents. As the degrees are uniformly bounded, this sequence of currents necessarily has convergent subsequences. This allows us to use the approximating maps to control the global properties of the limiting map, and thereby to prove the proposition.

The hypotheses of the theorem imply that for each $j>0$ there is a normal Stein space X_{+j}, with $bX_{+j}=-bX_{-j}$ as CR-manifolds. Let $X_j=X_{+j}\amalg_{bX_{+j}}X_{-j}$. These are compact complex spaces each with a positively embedded, smooth curve Z_j. Of course, with our normalizations the Z_j are all represented by the same "physical curve" Z_0, though the complex structure in general depends on j. Let $H^0(X_{-j};[dZ_j])$, $j\geqslant 0$, be the holomorphic sections of the line bundle defined by the divisor $[dZ_j]$. Let g denote the genus of Z_0 and k the degree of the normal bundle. From Proposition 3.5, (3.10) and (3.11) it follows that, for $j\geqslant 0$ and $d>1+(2g-2)/k$, we have the estimates

$$M(g,k,d)\leqslant \dim H^0(X_{-j};[dZ_j])\leqslant M(g,k,d)+E(g,k). \tag{6.4}$$

For each $j>0$ the complex space X_{+j} is normal. As the line bundles $[dZ_j]$ are trivial in a neighborhood of X_{+j}, it follows easily that every element of $H^0(X_{-j};[dZ_j])$ extends holomorphically to define an element of $H^0(X_j;[dZ_j])$.

Recall that $\mathcal{G}(X_{-j},Z_j)$ is the set of integers such that $H^0(X_{-j};[dZ_j])$ defines a holomorphic map of X_{-j} into a projective space which is an embedding of a neighborhood of Z_j. From Proposition 3.1 we know that

$$|\mathbf{N}\backslash\mathcal{G}(X_{-j},Z_j)|\leqslant C(g,k). \tag{6.5}$$

For $j>0$ it follows from the Corollary 3.1 that these maps are generically one-to-one on X_{-j}. The proof of this statement required an embedding of X_{-j} as an open subset of a compact complex space. To obtain a generically one-to-one map for the limiting structure we need to choose d judiciously. First we state a lemma in algebraic geometry whose proof is deferred to the end of this section.

LEMMA 6.1. *Suppose that Σ is a Riemann surface of genus g, and $L\to\Sigma$ a holomorphic line bundle of degree $k>0$. For each non-negative integer l there is an integer $d(g,k,l)$ such that, if $d\geqslant d(g,k,l)$ and $S\subset H^0(\Sigma;L^d)$ is a linear subspace of codimension l, then there is a finite subset $\{x_1,\dots,x_r\}\subset\Sigma$, containing the base points of S. Moreover, S defines an embedding of $\Sigma\backslash\{x_1,\dots,x_r\}$ into $\mathbf{P}S^*$.*

Existence of a limiting map. We need to consider more carefully the sequence of vector spaces $\{H^0(X_{-j};[dZ_j])\}$. We are in precisely the situation considered in the proof of Proposition 3.5. Let

$$\Phi_{jd}\colon[dZ_0]\to[dZ_j]$$

denote, as before, bundle isomorphisms and $\overline{D}_{jd}$ the induced representation of the $\bar{\partial}$-operator on sections of $[dZ_j]$ as an operator on sections of $[dZ_0]$. As before we have the identifications

$$H^0(X_{-j};[dZ_j]) = \Phi_{j,d} \ker \overline{D}_{jd} \quad \text{for } j,d\geqslant 0. \tag{6.6}$$

We now make use of the estimates given in §3 of [Le2]. Lempert considers a complex manifold with boundary Y, such that the Levi form of bY has at least one negative eigenvalue at each point. Let L be a holomorphic line bundle on Y with $\bar{\partial}$-operator $\overline{D}$, acting on $\mathcal{C}^\infty(Y;L)$. Lempert shows that there are constants C_s, C_s' such that for $u\in \mathcal{C}^\infty(\overline{Y};L)$ we have the estimates

$$\begin{aligned} |u|_s &\leqslant C_s(|\overline{D}u|_{s-1}+|u|_0), && s=1,2,\dots, \\ |u|_s &\leqslant C_s'|u|_0 \quad \text{for } u\perp H^0(Y;L), && s=2,3,\dots. \end{aligned} \tag{6.7}$$

Here $\{|\cdot|_s : s\in[0,\infty)\}$ is a family of seminorms defining the standard topology on $\mathcal{C}^\infty(\overline{Y};L)$. The constants $\{C_s\}$ are geometric in nature, obtained by localizing to coordinate patches where the line bundle L is *smoothly* trivialized. Thus if we have a family of complex structures on Y and L with uniformly bounded geometries on $\overline{Y}$, then we have uniform bounds on these constants. From this it follows that for each pair (s,d) there is a constant $\widehat{C}_{s,d}$ such that each $u\in H^0(X_{-j};[dZ_j])$, $j\geqslant 0$, satisfies

$$|\Phi_{jd}^{-1}u|_s \leqslant \widetilde{C}_{s,d}|u|_s \leqslant \widetilde{C}_{s,d}C_{s,d}|u|_0 \leqslant \widehat{C}_{s,d}|\Phi_{jd}^{-1}u|_0. \tag{6.8}$$

Let $r_{j,d}$ denote the restriction mappings

$$r_{j,d}: H^0(X_{-j};[dZ_j]) \to H^0(Z_j;N_{Z_j}^d).$$

In Lemma 3.1 we showed that there is an integer $E(g,k)$ such that $r_{j,d}$ may fail to be surjective for at most $E(g,k)$ values of d. From this observation it follows that we can find arbitrarily large values of $d>d(g,k,E(g,k))$, where $d(g,k,l)$ is defined in Lemma 6.1, such that

(6.9) the maps $r_{0,d}$ and $r_{0,d+1}$ are surjective,

(6.10) there exists an infinite subsequence $\{j_1,j_2,\dots\}$ such that $r_{j_k,d}$ and $r_{j_k,d+1}$ are surjective for all $k>0$.

To see this let d_0 be chosen so that $r_{0,d}$ is surjective for $d\geqslant d_0$. Suppose that for a $d\geqslant d_0$ there does not exist a subsequence satisfying (6.10). Then there is an integer J_d such that for $j>J_d$ at least one of the maps $r_{j,d}, r_{j,d+1}$ is not surjective. It is clear that this can happen at most $2E(g,k)$ times. Let d_1 be henceforth fixed so that $d_1\geqslant \max\{d_0, d(g,k,E(g,k))\}$ and (6.10) holds. To simplify our notation we relabel this subsequence $\{X_{-1},X_{-2},\dots\}$.

Fix a volume form on X_- and a Hermitian metric on the fibers of $[Z_0]$, thereby obtaining metrics on $[dZ_0]$ for all $d>0$ as well as inner products on $\mathcal{C}^\infty(X_-;[dZ_0])$. For each $j>0$ let σ_{j0} denote a section of $H^0(X_{-j};[Z_j])$ with divisor equal to Z_j. If we normalize so that each $\Phi_{j1}^{-1}(\sigma_{j0})$ has norm 1 then it follows from the estimates (6.8) and the Rellich compactness theorem (see [Ka]) that there is a subsequence converging to σ_{00} in the $\mathcal{C}^\infty$-topology. Here σ_{00} is a holomorphic section of $[Z_0]$ which vanishes simply along the divisor. We continue to denote this subsequence by $\{\sigma_{j0}\}$.

For $0<j$ we construct ordered bases for $H^0(X_{-j};[dZ_j])$, $0<d\leqslant d_1+1$, in the following manner: Fix a basis for $H^0(X_{-j};[Z_j])$ of the form

$$e_{j1}=(\sigma_{j0},f_{j11},\dots,f_{j1n_{1j}}).$$

Inductively define

$$e_{jm}=(\sigma_{j0}e_{j(m-1)},f_{jm1},\dots,f_{jmn_{mj}}),$$

where $\{r_{j,m}(f_{jmi}):i=1,\dots,n_{mj}\}$ is a basis for range $r_{j,m}$. Finally for each pair j,d let $\Psi_{j,d}$ denote the result of applying the Gram–Schmidt process to the ordered basis $\Phi_{jd}^{-1}(e_{jd})$.

It follows from the pigeon hole principle and Lemma 3.1 that we can select a subsequence $\{j_i\}$ so that $N_d=\dim H^0(X_{-j_i};[dZ_{j_i}])$ is constant for $0<d\leqslant d_1+1$. Again using the estimates (6.8) and Rellich compactness we can further assume that the subsequence $\{\Psi_{j_i,d}\}$ converges to $\Psi_{0,d}$ in the $\mathcal{C}^\infty$-topology on $\overline{X}_-$, for $0<d\leqslant d_1+1$. Here $\Psi_{0,d}$ is a linearly independent collection of holomorphic sections of $[dZ_0]$; though of course it may fail to span $H^0(X_{-0};[dZ_0])$. Let $W_d\subset H^0(X_{-0};[dZ_0])$ denote the subspace spanned by $\Psi_{0,d}$, for $0<d\leqslant d_1+1$. As $\Psi_{j_i,d}$ are orthonormal bases it follows easily that $\dim W_d=N_d$ for $0<d\leqslant d_1+1$. Let $N=N_{d_1+1}$.

From our choices of subsequence and d_1 it is apparent that r_{j_i,d_1} and r_{j_i,d_1+1} are surjective for all i. Corollary 3.1 shows that for $i>0$ the maps into $\mathbf{P}^N$ defined by

$$H^0(X_{-j_i};[(d_1+1)Z_{j_i}]),$$

which we denote by φ_{j_i,d_1+1}, are generically one-to-one and embed a neighborhood of Z_{j_i}. The image of φ_{j_i,d_1+1} is contained in an irreducible projective variety V_i. From Proposition 3.3 it follows that $\deg V_i\leqslant D$ for some fixed integer D. There is a fixed hyperplane H such that

$$\varphi_{j_i,d_1+1}(Z_{j_i})=V_i\cap H.$$

Since the sections σ_{j0}, $j\geqslant 0$, vanish only along Z, the uniform convergence of the bases $\{\Psi_{j_i,d_1+1}\}$ and the maximum principle imply that there is a neighborhood R of Z and a neighborhood S of H so that

$$\varphi_{j_i,d_1+1}(X_{j_i}\setminus R)\subset\mathbf{P}^N\setminus S\quad\text{for all } i. \tag{6.11}$$

Basic structure of the limiting map. We have not been able to show that $r_{0,d}|_{W_d}$ are surjective for $d=d_1, d_1+1$. It follows easily from (3.12), however, that for $d=d_1, d_1+1$ we have the estimates

$$\dim[H^0(Z_0; N^d_{Z_0})/r_{0,d}(W_d)] \leqslant E(g,k).$$

From our choice of d_1 and Lemma 6.1 it follows that there is a finite set $E_0=\{x_1, ..., x_r\}$ containing the base points of $r_{0,d}(W_d)$, $d=d_1$ and d_1+1, such that spaces of sections $r_{0,d_1}(W_{d_1})$ and $r_{0,d_1+1}(W_{d_1+1})$ define embeddings of $Z_0\setminus E_0$ into projective spaces. As the first component of Ψ_{0,d_1+1} is a constant multiple of $\sigma_{00}^{d_1+1}$, it follows that the base locus of the subspace W_{d_1+1} is contained in Z_0, and therefore in the finite set E_0. Thus Ψ_{0,d_1+1} defines a holomorphic map $\varphi_{0,d_1+1}: X_{-0}\setminus E_0 \to \mathbf{P}^N$ and a meromorphic map from X_{-0} to $\mathbf{P}^N$. Note that as Z_0 is 1-dimensional the singularities of $\varphi_{0,d_1+1}|_{Z_0}$ are removable. The image of the extended map is simply the closure of $\varphi_{0,d_1+1}(Z_0\setminus E_0)$. Since $r_{0,d_1}(W_{d_1})$ is base-point-free on $Z_0\setminus E_0$, it follows that the rank of $d\varphi_{0,d_1+1}(x)$ equals 2 for every $x\in Z_0\setminus E_0$. Fix a Riemannian metric on X_- and let $B_\delta(x)$ denote the metric ball of radius δ centered at x, $S_\delta(x)=bB_\delta(x)$ and

$$X^\delta_- = X_- \setminus \bigcup_{i=1}^{r} B_\delta(x_i).$$

We can remove the indeterminacy of the meromorphic map φ_{0,d_1+1} by blowing up, successively, a sequence of points to obtain a complex manifold $\pi: X'_{-0} \to X_{-0}$ and a globally defined map $\varphi'_{0,d_1+1}: X'_{-0} \to \mathbf{P}^N$. Let $E_1 \subset \pi^{-1}(E_0)$ denote the exceptional divisor. Then we have that

$$\pi: X'_{-0}\setminus E_1 \to X_{-0}$$

is a biholomorphism onto its image, and

$$\varphi_{0,d_1+1}\circ\pi|_{X'_{-0}\setminus E_1} = \varphi'_{0,d_1+1}|_{X'_{-0}\setminus E_1}.$$

Let $Z'_0 = \overline{\pi^{-1}(Z_0\setminus E_0)}$; it is biholomorphic to Z_0.

The exceptional divisor E_1 is a finite collection of $\mathbf{P}^1$'s. As the genus of Z'_0 is greater than 0 it follows from the Riemann–Hurwitz theorem that

$$I = \varphi'_{0,d_1+1}(E_1) \cap \varphi'_{0,d_1+1}(Z'_0) \tag{6.12}$$

is a finite set of points. Note that as $\operatorname{rk} d\varphi'_{0,d_1+1}(x)=2$ on an open set, the set

$$E'_2 = \{x : \operatorname{rk} \varphi'_{0,d_1+1}(x) < 2\}$$

is a proper subvariety of dimension at most 1. The analogous statement is therefore true for the meromorphic map φ_{0,d_1+1}.

Structure of the limiting variety. Consider the sequence of varieties $\{V_i\}$ as defining a sequence of positive currents $\{[V_i]\}$ on $\mathbf{P}^N$. As $\deg V_i \leqslant D$ for all i, it follows that the masses of the $[V_i]$ are uniformly bounded. A result of Bishop implies that $\{[V_i]\}$ has a weakly convergent subsequence $\{[V_{i_k}]\}$, and that the limiting current C is a holomorphic 2-chain. In fact,

$$\lim_{k\to\infty} [V_{i_k}] = C = \sum_{j=1}^{m} n_j [Y_j], \tag{6.13}$$

where the Y_j are distinct, purely 2-dimensional, irreducible subvarieties of $\mathbf{P}^N$, and the n_j are positive integers, see [Bi] and [Hv]. The image $\varphi_{0,d_1+1}(X_{-0}\setminus E_0)$ is contained in a single irreducible component of C. If it were not, then one could easily construct a non-trivial meromorphic function on X_{-0} which vanished on an open set. Call this component Y_1. We again relabel so that the subsequence $\{[V_{i_k}]\}$ is simply $\{V_i\}$ with corresponding relabeling of the bases, maps and pseudoconcave manifolds.

We need to have a somewhat more precise picture of the convergence of the sequence $\{[V_i]\}$. The exceptional divisor $E_1 \subset X'_-$ is a union of $\mathbf{P}^1$'s, and in particular it has real codimension 2. This implies that for each $\varepsilon>0$ there exists an $r(\varepsilon)>0$ so that if $r<r(\varepsilon)$ then the 3-dimensional Hausdorff measure of

$$T_{0,r} = \varphi_{0,d_1+1}\big(\bigcup S_r(x_l)\big) = \varphi'_{0,d_1+1}\circ\pi^{-1}\big(\bigcup S_r(x_l)\big)$$

satisfies

$$\mathcal{H}_3(T_{0,r}) < \varepsilon.$$

$\mathcal{H}_3$ denotes 3-dimensional Hausdorff measure. The (relabeled) bases $\{\Psi_{i,d}\}$ converge in the $\mathcal{C}^\infty$-topology on $\overline{X}_-$ to $\Psi_{0,d}$. This easily implies that the maps into projective space defined by these bases converge locally uniformly in the $\mathcal{C}^\infty$-topology on $X_-\setminus E_0$. From this we conclude that, for a fixed $r<r(\varepsilon)$ and a sufficiently large j, we have the estimate

$$\mathcal{H}_3(T_{j,r}) < 2\varepsilon \quad \text{where } T_{j,r} = \varphi_{j,d_1+1}\big(\bigcup S_r(x_l)\big). \tag{6.14}$$

This implies that we can choose a subsequence $\{i_k\}$ and a sequence $r_k\to 0$ such that if we set

$$V_{i_k}^+ = \varphi_{i_k,d_1+1}\big(X_{i_k}\setminus \bigcup B_{r_k}(x_l)\big), \quad V_{i_k}^- = \varphi_{i_k,d_1+1}\big(\bigcup B_{r_k}(x_l)\big)$$

then

$$[V_{i_k}] = V_{i_k}^+ + V_{i_k}^- \quad \text{and} \quad d[V_{i_k}^+] = -d[V_{i_k}^-] = T_k \tag{6.15}$$

and

$$\lim_{k\to\infty} \mathcal{H}_3(T_k) = 0.$$

By choosing a further subsequence of $\{[V_{i_k}]\}$ (which we continue to denote by $\{[V_{i_k}]\}$) we can assume that $\{[V_{i_k}^{\pm}]\}$ and $\{\pm d[V_{i_k}^{\pm}]\}$ converge weakly to currents $C^{\pm}$ and D respectively. Of course,

$$dC^{+}=-dC^{-}=D, \quad C=C^{+}+C^{-} \tag{6.16}$$

and $C^{\pm}$ are integer multiplicity currents with support on the holomorphic varieties $\{Y_i\}$. As $\mathcal{H}_3(D)=0$ it follows from the support theorem of Federer that $d[C^{\pm}]=0$. It now follows from the structure theorem of Harvey and Shiffman for closed, positive currents that

$$C^{\pm}=\sum_{i=1}^{m} n_i^{\pm}[Y_i]. \tag{6.17}$$

From (6.13) and (6.16) it follows that $\{n_i^{\pm}\}$ are non-negative integers with

$$n_i=n_i^{+}+n_i^{-}, \quad i=1,...,m,$$

see [Hv].

The limiting map is unramified. Let $w\in\varphi'_{0,d_1+1}(Z'_0\setminus\pi^{-1}(E_0))\setminus I$, see (6.12). There is a unique point $x\in Z_0\setminus E_0$ with

$$\varphi_{0,d_1+1}(x)=\varphi'_{0,d_1+1}(x)=w.$$

We claim that the point x has a neighborhood $U_x\subset X'_{-0}\setminus E_1$ such that

$$(\varphi'_{0,d_1+1})^{-1}(\varphi'_{0,d_1+1}(U_x))=U_x. \tag{6.18}$$

The rank of $d\varphi'_{0,d_1+1}(x)$ is two, and therefore φ'_{0,d_1+1} is one-to-one in a neighborhood of x. If U_x did not exist then we could find sequences $\{p_n\}$, $\{q_n\}$ such that

$$\varphi'_{0,d_1+1}(p_n)=\varphi'_{0,d_1+1}(q_n), \quad \lim_{n\to\infty} p_n=x,$$

but x is not a limit point of $\{q_n\}$. As $\varphi'_{0,d_1+1}(q_n)$ converges to a point on H it follows that the sequence $\{q_n\}$ remains in a compact neighborhood of $Z'_0\cup E_1$, and that its limit points must lie on this set. Let q be a limit point of this sequence. Evidently $\varphi'_{0,d_1+1}(q)=\varphi'_{0,d_1+1}(x)=w$. Since $\varphi'_{0,d_1+1}|_{Z_0\setminus E_0}$ is an embedding, the point q must belong to E_1. This would, however, imply that $w\in I$, which is a contradiction. This shows that a neighborhood U_x satisfying (6.18) exists.

Recalling the definition of the neighborhood R from (6.11), choose a point $x\in Z_0\setminus E_0$ and let $U_x\subset R$ be a neighborhood which satisfies (6.18). By shrinking U_x we can assume that $\operatorname{rk} d\varphi_{0,d_1+1}(y)=2$ for all $y\in U_x$, and that φ_{0,d_1+1} embeds U_x. Now choose

$$w\in \varphi_{0,d_1+1}(U_x)\setminus I\cup Y_2\cup...\cup Y_m.$$

This is surely possible as $\varphi_{0,d_1+1}(U_x)$ is an open subset of Y_1, and $Y_1 \cap Y_i$, $i>1$, are proper analytic subvarieties. Let $Q_w \subset \varphi_{0,d_1+1}(U_x)$ be a neighborhood of w such that $Q_w \cap Y_i = \varnothing$ for $i>1$, and choose an open subset $Q'_w \subset S$ of $\mathbf{P}^N$ (see (6.11)) such that

$$Q'_w \cap Y_1 = Q_w \quad \text{and} \quad Q'_w \cap Y_i = \varnothing \quad \text{for } i>1. \tag{6.19}$$

Now choose a non-negative function $\chi \in \mathcal{C}_c^\infty(Q'_w)$ such that $\chi(w)=1$. Let η denote the canonical Kähler form on $\mathbf{P}^N$. It follows from (6.13) and (6.19) that

$$n_1[Y_1](\chi\eta^2) = \lim_{k\to\infty} [V_{i_k}](\chi\eta^2). \tag{6.20}$$

Since φ_{i_k,d_1+1} is generically one-to-one it follows that

$$[V_{i_k}](\chi\eta^2) = \int_{X_{i_k}} \varphi^*_{i_k,d_1+1}(\chi\eta^2).$$

As $U_x \subset R$ it follows that in fact

$$[V_{i_k}](\chi\eta^2) = \int_{X_{-i_k}} \varphi^*_{i_k,d_1+1}(\chi\eta^2). \tag{6.21}$$

For any sufficiently small $\delta>0$ we have that $U_x \subset\subset X^\delta_-$, and therefore

$$C^+(\chi\eta^2) = \lim_{k\to\infty} \int_{X^\delta_-} \varphi^*_{i_k,d_1+1}(\chi\eta^2) = \int_{X^\delta_-} \varphi^*_{0,d_1+1}(\chi\eta^2) = [Y_1](\chi\eta^2). \tag{6.22}$$

The last equality follows from (6.18). As $[Y_1](\chi\eta^2)>0$ equations (6.20), (6.21) and (6.22) imply that

$$n_1^+ = 1 \quad \text{and} \quad n_1^- = n_1 - 1. \tag{6.23}$$

Now suppose that we could find disjoint open sets $U_1, U_2 \subset X_- \setminus Z_0$ such that

$$\varphi_{0,d_1+1}(U_1) = \varphi_{0,d_1+1}(U_2).$$

Since $\operatorname{rk} d\varphi_{0,d_1+1} = 2$ in the complement of a proper analytic subset we could choose disjoint open sets U_1, U_2 such that

(1) $\varphi_{0,d_1+1}|_{U_i}$, $i=1,2$, are embeddings,
(2) $Q = \varphi_{0,d_1+1}(U_1) = \varphi_{0,d_1+1}(U_2)$,
(3) for $i>1$, $Q \cap Y_i = \varnothing$.

Let $Q'\subset\mathbf{P}^N$ be an open set such that $Q'\cap Y_1=Q$ and $Q'\cap Y_i=\varnothing$ for $i>1$. As before we select a non-negative function $\chi\in\mathcal{C}_c^\infty(Q')$, with $\chi(w)>0$ for some $w\in Q$. Choose a $\delta>0$ small enough so that $U_1\cup U_2\subset\subset X_-^\delta$. Then we have that

$$\begin{aligned} n_1[Y_1](\chi\eta^2) = \lim_{k\to\infty}[V_{i_k}](\chi\eta^2) &= \lim_{k\to\infty}\int_{X_-^\delta}\varphi^*_{i_k,d_1+1}(\chi\eta^2)+C^-(\chi\eta^2) \\ &\geqslant 2[Y_1](\chi\eta^2)+(n_1-1)[Y_1](\chi\eta^2). \end{aligned} \tag{6.24}$$

The last inequality follows from (6.23) and our choice of open sets U_1, U_2. As $[Y_1](\chi\eta^2)>0$ this contradiction proves that such subsets do not exist.

Construction of the generically one-to-one map. We now augment the basis Ψ_{0,d_1+1} to obtain a basis Ψ_{0,d_1+1} for $H^0(X_{-0};[(d_1+1)Z_0])$. In light of (6.9) this basis is base-point-free, and therefore defines a map

$$\psi: X_{-0}\to\mathbf{P}^M,$$

where $M=\dim H^0(X_{-0};[(d_1+1)Z_0])-1$. It is clear from (6.9) and Lemma 3.2 that ψ embeds a neighborhood of Z_0. There is a linear projection $p:\mathbf{P}^M\to\mathbf{P}^N$ such that

$$\varphi_{0,d_1+1}=p\circ\psi.$$

This implies that there do not exist disjoint open subsets $U_1, U_2\subset X_-$ with $\psi(U_1)=\psi(U_2)$.

Let $W\subset\mathbf{P}^M$ be the irreducible subvariety obtained in Proposition 3.3 such that $\psi(X_-)\subset W$ and $\operatorname{sing}(W)$ is a finite set. To complete the proof of Theorem 6.2 we need to show that ψ is generically one-to-one.

Let $E_2=\{x\in X_{-0}:\operatorname{rk} d\psi(x)<2\}$. This set is a proper subvariety, and therefore $E_2=E_2^0\cup E_2^1$ where E_2^i is the union of i-dimensional components of E_2. The subset E_2^0 is discrete.

PROPOSITION 6.2. *If E_2^1 is the union of 1-dimensional components of E_2 then $\psi(E_2^1)$ is a finite set of points.*

Proof of Proposition 6.2. First we show that $\operatorname{rk} d(\psi|_{E_2^1})_*(p)=0$ for $p\in E_2^1$. Suppose that this were not the case at some point in E_2^1. As $\operatorname{sing}(W)$ is a finite set we could therefore find a smooth point $x\in E_2^1$ such that $\psi(x)$ is a smooth point of W and

$$\operatorname{rk}(d(\psi|_{E_2^1}))(x)=1.$$

We apply Proposition 5.2 to conclude that ψ is a non-trivially ramified cover in a neighborhood of x. This would imply that there were disjoint open sets U_1, U_2 with

$\psi(U_1)=\psi(U_2)$. Each connected component of E_2^1 is therefore mapped to a single point by ψ, and each connected component is a union of irreducible components of E_2^1. There is a neighborhood U_0 of bX_- in which a defining function p_0 for bX_- is strictly plurisubharmonic. The maximum principle implies that no irreducible component of E_2^1 can be contained in U_0. It is clear that $\operatorname{sing}(E_2^1)\cap\{X_-\setminus U_0\}$ is a finite set, and therefore $E_2^1\cap\{X_-\setminus U_0\}$ has finitely many irreducible components. As every irreducible component of E_2^1 intersects $X_-\setminus U_0$, this implies that E_2^1 has finitely many irreducible components. This completes the proof of Proposition 6.2.

LEMMA 6.3. *There is a discrete set of points $E_3\subset X_-$ such that, if $G=E_2\cup E_3$, then $\psi|_{X_-\setminus G}$ is an embedding.*

Proof of Lemma 6.2. Suppose that there exist $x\neq y\in X_-\setminus E_2$ with $\psi(x)=\psi(y)$. From the definition of E_2 it follows that $\operatorname{rk} d\psi(x)=\operatorname{rk} d\psi(y)=2$. The image point $\psi(x)=\psi(y)\in\operatorname{sing}(W)$. Otherwise we could find disjoint open sets U_x, U_y with $\psi(U_x)=\psi(U_y)$. If there exists $x\in X_-\setminus E_2$ and $y\in E_2$ such that $\psi(x)=\psi(y)$, then again $\psi(x)\in\operatorname{sing}(W)$. If it were not, then there would again exist disjoint neighborhoods U_x, U_y of x, y, respectively, such that $\psi(U_x)\cap\psi(U_y)$ has non-empty interior. In either case the germ $(W,\psi(x))$ must be reducible and the disjoint neighborhoods U_x, U_y must have images lying in different components of the germ. As $\operatorname{sing}(W)$ is finite the different components must intersect in a point. This shows that for some set $G\subset E_2\cup\psi^{-1}(\operatorname{sing}(W))$ the restriction $\psi|_{X_-\setminus G}$ is an embedding. The lemma follows from the finiteness of the set $\operatorname{sing}(W)$.

Thus the map ψ is generically one-to-one. To complete the proof of Theorem 6.2, and thereby the proof of Theorem 6.1, all that remains is the proof of Lemma 6.1.

Proof of Lemma 6.1. To prove the lemma we use the following more precise statement:

SUBLEMMA 6.1. *Let Σ be a Riemann surface of genus g, and $L\to\Sigma$ a holomorphic line bundle of degree $k>0$. Let $N+1=\dim H^0(\Sigma;L^d)$, let φ be the canonical map*

$$\varphi\colon\Sigma\to\mathbf{P}H^0(\Sigma;L^d)^*\simeq\mathbf{P}^N,$$

and $\pi\colon\mathbf{P}^N\to\mathbf{P}^{N-l}$ be a linear projection. There are at most l pairs of distinct points $(a_i,b_i)\in\Sigma$ such that $\pi\circ\varphi(a_i)=\pi\circ\varphi(b_i)$, provided

$$dk-2(l+1)\geqslant 2g-1. \tag{6.25}$$

Remark. We would like to thank Ron Donagi for the statement and proof of this sublemma. It is a classical general position argument which makes extensive usage of the language and notions of projective geometry, see [GH].

Proof of the sublemma. We prove the sublemma by induction on l. Recall how a linear projection $\pi: \mathbf{P}^N \to \mathbf{P}^{N-l}$ is defined: given a subspace $Q \subset \mathbf{P}^N$ of dimension $l-1$ we obtain a holomorphic map $\pi_Q: \mathbf{P}^N \setminus Q \to \mathbf{P}^{N-l}$ by mapping a point $p \in \mathbf{P}^N \setminus Q$ into the unique l-dimensional subspace of $\mathbf{P}^N$ spanned by p and Q. More concretely one can select an $(N-l)$-dimensional linear subspace R disjoint from Q. Then $\pi_Q(p)$ is the intersection of the linear subspace spanned by p and Q with R.

Since $L>0$ there is a divisor D_L so that the line bundle $[D_L]$ is linearly equivalent to L. Let Φ be a fixed basis for $H^0(\Sigma; L^d) \simeq H^0(\Sigma; [dD_L])$. We begin the induction with $l=1$. Choose a point $q \in \mathbf{P}^N$. First suppose that $q=\varphi(c_1)$ for a $c_1 \in \Sigma$. There cannot exist two points $a_1 \neq b_1 \in \Sigma \setminus \varphi^{-1}(q)$ with

$$\pi_q \circ \varphi(a_1) = \pi_q \circ \varphi(b_1). \tag{6.26}$$

Let $h \in \mathbf{P}^{N*}$ be a linear functional such that

$$h \cdot \Phi(a_1) = h \cdot \Phi(b_1) = 0.$$

If (6.26) holds then q is collinear with $\varphi(a_1)$ and $\varphi(b_1)$, and therefore we would also have that

$$h \cdot q = h \cdot \Phi(c_1) = 0.$$

This would imply that

$$H^0(\Sigma; [dD_L \setminus \{a_1, b_1\}]) = H^0(\Sigma; [dD_L \setminus \{a_1, b_1, c_1\}]). \tag{6.27}$$

If $kd-3 \geqslant 2g-1$ then (6.27) contradicts the Riemann–Roch theorem. If $q \notin \varphi(\Sigma)$ then there cannot exist two distinct pairs (a_i, b_i), $i=1,2$, such that

$$\pi_q \circ \varphi(a_i) = \pi_q \circ \varphi(b_i), \quad i=1,2. \tag{6.28}$$

As $\varphi(a_1), \varphi(b_1)$ and q are collinear, if $h \in \mathbf{P}^{N*}$ is chosen so that $h \cdot \Phi(a_1) = h \cdot \Phi(b_1) = 0$ then $h \cdot q = 0$ as well. As $\varphi(a_2)$, $\varphi(b_2)$ and q are again collinear, if $h \cdot \Phi(a_2)$ also vanishes then $h \cdot \Phi(b_2) = 0$ as well. This would entail

$$H^0(\Sigma; [dD_L \setminus \{a_1, b_1, a_2\}]) = H^0(\Sigma; [dD_L \setminus \{a_1, b_1, a_2, b_2\}]).$$

If $kd-4 \geqslant 2g-1$ this again contradicts the Riemann–Roch theorem. This completes the case $l=1$.

Suppose that the result is proved for $\{1, 2, ..., j\}$ and that $dk-2(j+2) \geqslant 2g-1$. Let Q be a j-dimensional subspace of $\mathbf{P}^N$, and let π_Q be the linear projection to $\mathbf{P}^{N-(j+1)}$

it defines. For convenience we select an $[N-(j+1)]$-dimensional subspace R disjoint from Q and $\varphi(\Sigma)$ so that $\pi_Q\colon \mathbf{P}^N\setminus Q\to R$. To shorten the exposition we only present the details of the induction step for the generic case $Q\cap\varphi(\Sigma)=\varnothing$. As in the case $l=1$, the non-generic cases are even more restrictive.

Suppose that there are $j+2$ distinct pairs $\{(a_i,b_i)\}$ such that

$$\pi_Q\circ\varphi(a_i)=\pi_Q\circ\varphi(b_i)=r_i\in R,\quad i=1,\ldots,j+2.$$

Let q_i denote the intersection of the line through $\varphi(a_i)$ and r_i with Q, and q_i' the intersection of the line through $\varphi(b_i)$ and r_i with Q. This collinearity entails the linear relations

$$\alpha_{i1}r_i+\beta_{i1}\varphi(a_i)=q_i,\quad \alpha_{i2}r_i+\beta_{i2}\varphi(b_i)=q_i'.$$

Here and in the sequel we tacitly identify points in $\mathbf{P}^N$ with points on the lines in $\mathbf{C}^{N+1}\setminus\{0\}$ which define them. For homogeneous relations of this type this practice should cause no confusion. As both $Q\cap R$ and $Q\cap\varphi(\Sigma)$ are empty, none of these coefficients can vanish. These relations imply that

$$q_i''=\alpha_{i2}q_i-\alpha_{i1}q_i'=\alpha_{i2}\beta_{i1}\varphi(a_i)-\alpha_{i1}\beta_{i2}\varphi(b_i). \tag{6.29}$$

As the map φ separates points it follows that $q_i''\neq0$, $i=1,\ldots,j+1$. We claim that the points $\{q_1'',\ldots,q_{j+1}''\}$ are linearly independent. If not, we would have a linear relation

$$\sum_{i=1}^{j+1}c_iq_i''=0, \tag{6.30}$$

with some coefficient non-zero. We can relabel so that $c_1\neq0$. Combining (6.29) and (6.30) we would deduce that

$$\alpha_{12}\beta_{11}\varphi(a_1)-\alpha_{11}\beta_{12}\varphi(b_1)=-\sum_{i=2}^{j+1}\frac{c_i}{c_1}\alpha_{i2}\beta_{i1}\varphi(a_i)-\alpha_{i1}\beta_{i2}\varphi(b_i). \tag{6.31}$$

This would imply that

$$H^0(\Sigma;[dD_L-\{a_1,a_2,b_2,\ldots,a_{j+1},b_{j+1}\}])=H^0(\Sigma;[dD_L-\{a_1,b_1,a_2,b_2,\ldots,a_{j+1},b_{j+1}\}]).$$

As $kd-2(j+1)\geqslant 2g-1$ this contradicts the Riemann–Roch theorem. Since $\dim Q=j$ it follows that the points $\{q_1'',\ldots,q_{j+1}''\}$ span the subspace. If $h\in\mathbf{P}^{N*}$ is chosen so that

$$h\cdot\Phi(a_i)=h\cdot\Phi(b_i)=0\quad\text{for } i=1,\ldots,j+1, \tag{6.32}$$

then (6.29) implies that $h\cdot q_i''=0$, $i=1,\dots,j+1$, and therefore, as $\{q_1'',\dots,q_{j+1}''\}$ span Q, we have that

$$h\cdot q=0 \quad \text{for all } q\in Q.$$

In particular, $h\cdot q_{j+2}=h\cdot q_{j+2}'=0$. If $h\cdot\Phi(a_{j+2})=0$ as well, then $h\cdot r_{j+2}=0$, and therefore as $\varphi(b_{j+2})$, q_{j+2}' and r_{j+2} are collinear we see that $h\cdot\Phi(b_{j+2})=0$. This would imply that

$$\begin{aligned} &H^0(\Sigma;[dD_L-\{a_1,b_1,\dots,a_{j+1},b_{j+1},a_{j+2}\}]) \\ &\qquad\qquad = H^0(\Sigma;[dD_L-\{a_1,b_1,\dots,a_{j+1},b_{j+1},a_{j+2},b_{j+2}\}]). \end{aligned}$$

As $dk-2(j+2)\geqslant 2g-1$ this contradicts the Riemann–Roch theorem and completes the proof of the sublemma in the generic case. The non-generic cases are left to the reader.

Using a similar argument one easily shows that a subspace $S\subset H^0(\Sigma;L^d)$ of codimension l, where $kd-2(l+1)\geqslant 2g-1$ has a finite number of base points $B=\{s_1,\dots,s_m\}$. Thus the subspace defines a map $\varphi'\colon\Sigma\setminus B\to\mathbf{P}S^*$. The lemma now follows immediately as the sublemma shows that at most a finite number of points in $\varphi'(\Sigma\setminus B)$ can have two or more preimages. As $d\varphi'$ is not identically zero it vanishes at finitely many points. This completes the proof of the lemma.

II. Stability results for deformations with extensions vanishing to high order along Z

7. Extending sections of the normal bundle

In a previous paper, [EH1], we considered the problem of extending deformations of the CR-structure on bX_- as integrable, almost complex structures on X_-. From Lempert's work it is clear that it is often useful to have an extension for which the deformation tensor vanishes to a given order on a smooth, compact curve $Z\subset X_-$. Suppose that ω is a deformation of the CR-structure on M, and Ω is an integrable deformation of the complex structure on X_-, that satisfies

(1) $\Omega_b=\Omega|_{T^{0,1}M}=\omega$,
(2) $\Xi=\sigma_0^{-d}\Omega$ has a smooth extension across Z,
(3) $\bar\partial\Omega-\frac{1}{2}[\Omega,\Omega]=0$,

then we say that ω has an integrable extension (to X_-) vanishing to order d (along Z). Let $\mathcal{E}_d(M,X_-,Z)$ denote the set of deformations of the CR-structure with such an extension. In [EH1] the following theorem is proved:

THEOREM EH1. *Let X_- be a smooth, strictly pseudoconcave surface containing a smooth, positively embedded holomorphic curve Z. We further suppose that the CR-structure induced on bX_- is embeddable. For any $0\leqslant d$ the set $\mathcal{E}_d(M, X_-, Z)$ contains an analytic subvariety of the set of all deformations of the CR-structure of codimension at most* $2\dim H_c^2(X_-;\Theta\otimes[-dZ])$.

Remarks. (1) Using a slight variant of the proof of Theorem EH1 one can improve the codimension estimate to the more natural bound

$$\operatorname{codim}\mathcal{E}_d(M, X_-, Z)\leqslant \dim H_c^2(X_-;\Theta\otimes[-dZ]),$$

see [EH3]. For applications of the improved results see §10.

(2) Kiremidjian proved a version of Theorem EH1 under the assumption that the formal obstruction to extending the deformation tensor $H_c^2(X_-;\Theta)$ vanishes. Using the Nash–Moser theorem he obtains extensions in $\mathcal{C}^\infty(\overline{X}_-)$ which *do not* depend analytically on the boundary data. In [EH1] we obtain extensions of finite differentiability which depend analytically, in appropriately defined Hilbert spaces, on the boundary data.

In Part I we considered circumstances where small deformations of the CR-structure on bX_- could always be extended to X_- in such a way that a smooth, positively embedded divisor Z also deforms in a controlled manner. Unless $Z\simeq\mathbf{P}^1$, 2 is the minimal order of vanishing, which implies that N_Z, the holomorphic normal bundle, remains fixed under the deformation of the ambient complex structure. We now consider the consequences of having extensions of deformations which vanish to order 3 or more along the divisor. It must be stressed that we know of only one class of examples with $H_c^2(X_-;\Theta\otimes[-3Z])=0$, neighborhoods of $\mathbf{P}^1$ in $\mathbf{P}^2$. As follows from the extension result above, however, the codimension of $\mathcal{E}_3(M, X_-, Z)$ is always finite, and one hopes that the collection of embeddable structures with such an extension is also of "finite codimension". In this and the following section we show that embeddable structures in $\mathcal{E}_3(M, X_-, Z)$ have stability properties analogous to those of hypersurfaces in $\mathbf{C}^2$.

Suppose that ω defines a deformation of the CR-structure on M which has an extension to an integrable, almost complex structure Ω on X_-, vanishing to order $d>0$ along Z. We denote X_- with this complex structure by X'_-. When confusion might arise we continue to denote objects connected with the deformed complex structure with a $'$, e.g. $\bar\partial'$. We denote the line bundle defined by the divisor Z with respect to the deformed complex structure by $[Z']$. Suppose that $(M, {}^{\omega}T^{0,1}M)$ is embeddable and hence the boundary of a compact Stein space X'_+.

To proceed with our analysis it is useful to have a more explicit description of the complex structure on the line bundle $[Z']$. To that end we construct local $\bar\partial'$-holomorphic defining functions for the hypersurface Z'. Fix a cover of a neighborhood of $Z\subset X_-$ by open balls $\{U_1, ..., U_Q\}$ such that in each U_i we have $\bar\partial$-holomorphic coordinates (z_i, w_i) with

$$Z\cap U_i = \{z_i = 0\}.$$

We further suppose that each U_i contains the unit ball B_i in the (z_i, w_i)-coordinates, and that these balls also define a cover of a neighborhood of Z. Let U_0 denote an open set in X_- disjoint from Z such that U_0 along with the balls of radius 1 is a cover of X_-. If we set

$$\sigma_{0i} = \begin{cases} z_i, & 1\leqslant i\leqslant Q, \\ 1, & i=0, \end{cases}$$

then $\sigma_0 = \{\sigma_{0i}\}$ is a holomorphic 0-cochain with $(\sigma_0) = [Z]$.

The basic facts we need are contained in the following technical lemma. Here we let $(z, w_1, ..., w_{n-1})$ denote linear coordinates on $\mathbf{C}^n$ with $H = \{z=0\}$.

LEMMA 7.1. *Let $\Omega\in\mathcal{C}^\infty(\overline{B}_1; \Theta\otimes\Lambda^{0,1})$ define an integrable deformation of the complex structure on the unit ball B_1 in $\mathbf{C}^n$, such that $\Omega = z^d\Xi$ for some $d>0$. Here Ξ is smooth in $\overline{B}_1$ of sufficiently small $\mathcal{C}^{n+2}$-norm. There exists a function $z'\in\mathcal{C}^\infty(\overline{B}_1)$, holomorphic with respect to the deformed complex structure, such that $z' = hz$ where*

$$h = 1+O(|z|^{d-1}).$$

The map $\Xi\to h-1$ is continuous from $\mathcal{C}^\infty(\overline{B}_1; \Theta\otimes\Lambda^{0,1}\otimes[-dH])$ to $\mathcal{C}^\infty(\overline{B}_1)$.

This is an extension of Lemma 4.1 in [Le2]. The proof can be found in [O].

We apply the lemma to obtain local $\bar\partial'$-holomorphic defining functions for Z'. For each $i>0$ let $z_i'\in\mathcal{O}(B_i')$ denote the function defined in Lemma 7.1, and set

$$\sigma'_{0i}|_{B_i} = \begin{cases} z_i', & i>0, \\ 1, & i=0. \end{cases} \tag{7.1}$$

The $\bar\partial'$-holomorphic cochain σ_0' satisfies

$$(\sigma_0') = [Z'].$$

Transition functions for $[Z]$ and $[Z']$ are given by

$$g_{ij} = \frac{z_i}{z_j} \quad \text{and} \quad g'_{ij} = \frac{z_i'}{z_j'}$$

respectively.

We let

$$h_i = \frac{z_i'}{z_i}. \tag{7.2}$$

The smooth 0-cochain $h=\{h_i\}$ defines an isomorphism between the smooth line bundles $[Z]$ and $[Z']$. If s is a smooth section of $[Z]$ we denote the corresponding section of $[Z']$ by $h\cdot s$. From the lemma it follows that $1-h_i$ is estimated on B_i by bounds on $\Xi|_{B_i}$, and moreover

$$h_i = 1+O(|z_i|^{d-1}), \quad i>0. \tag{7.3}$$

Of course, $h_0=1$.

We now consider the extension of sections of N_Z to sections of $[Z']$.

LEMMA 7.2. *Let Ω be an integrable deformation of the complex structure on X_- vanishing to order $d>1$ along Z. If τ is a section of $H^0(Z;N_Z)$ which has a holomorphic extension as a section of $[Z]$, then τ has a smooth extension to $\overline{X}_-$ as a section $\hat{s}$ of $[Z']$ which satisfies*

$$\bar{\partial}'\hat{s} = \sigma_0^{d-1}\alpha, \tag{7.4}$$

where α is a smooth $[(2-d)Z]$-valued 1-form.

Proof. For $s\in H^0(X_-;[Z])$ we let $\hat{s}=h\cdot s$ be a smooth section of $[Z']$. If Ω vanishes to order $d>1$ along Z then $\hat{s}|_Z=s|_Z$, and it follows from (7.3) that

$$\bar{\partial}'\hat{s} = (\sigma_0')^{d-1}\alpha,$$

where α is a smooth $[(2-d)Z']$-valued 1-form.

Suppose that X_-' is a compact domain in a projective surface V'. We augment the set U_0 by adding to it $V'\backslash X_-'$. The line bundle $[Z']$ is extended to all of V' by simply extending σ_{00}' to be 1 on the expanded U_0. We extend $\hat{s}$ as a smooth function across bX_- and cut it off smoothly to obtain something compactly supported in $V'\backslash\operatorname{sing}(V')$. Denote by $\tilde{s}$ this extension of τ to V' as a smooth section of $[Z']$.

In order to find a holomorphic extension we need to solve

$$\begin{cases} \bar{\partial}'t = \bar{\partial}'\tilde{s}, \\ t|_Z = 0. \end{cases}$$

It would suffice to solve

$$\bar{\partial}'t_0 = \frac{\bar{\partial}'\tilde{s}}{\sigma_0'} \tag{7.5}$$

and set $s'=\tilde{s}-\sigma_0' t_0$. In general, (7.5) is solvable only if $H_D^{0,1}(V'\setminus\operatorname{sing} V')=0$. Unless $Z\simeq\mathbf{P}^1$ our information about this cohomology group is rather limited, so we instead consider the equation

$$\bar{\partial}' t_1 = \frac{\bar{\partial}'\tilde{s}}{(\sigma_0')^2}. \tag{7.6}$$

Here we see the utility of working with perturbations of the complex structure on X_- that vanish to order 3 along Z.

In the proof of the following theorem we use the anisotropic Lempert–Sobolev norms defined in §3 of [Le2]. We refer the reader to the cited work for their definitions and main properties, see also (6.7). As in §6 we denote these norms by $|\cdot|_s$.

THEOREM 7.1. *Let Ω be an integrable deformation of the complex structure on X_- which vanishes to order $d\geqslant 3$ along Z, such that X'_- is a subdomain in a normal, projective variety V'. If $\Xi=\sigma_0^{-d}\Omega$ is sufficiently small and $\tau\in H^0(Z;N_Z)$ has a holomorphic extension s as a section of $[Z]$, then it has a $\bar{\partial}'$-holomorphic extension s' as a section of $[Z']$ which satisfies*

$$s_i'-h_i s_i = O(\sigma_{0i}^{d-1}), \quad i=1,\ldots,Q.$$

Additionally the Lempert–Sobolev norms of the difference $s'-h\cdot s$ are estimated on $\overline{X}_-$ by bounds on Ξ.

Proof. With $\tilde{s}$ defined as above it follows that

$$\bar{\partial}'\tilde{s}_i = \sigma_{0i}^{d-1}\beta_i.$$

Here $\{\beta_i\}$ are smooth 1-forms which are estimated on $\overline{X}_-$ by bounds on Ξ. From the lemma it follows that

$$\frac{\bar{\partial}'\tilde{s}_i}{(\sigma_{0i}')^{d-1}} = \frac{\beta_i}{h_i^{d-1}}$$

is smooth and obviously $\bar{\partial}'$-closed. As $1-h_i$ is estimated on B_i by bounds on Ξ it follows that this ratio also satisfies such bounds on $\overline{X}_-$.

Thus we have a smooth, closed $(0,1)$-form with values in $[(2-d)Z']$ and compact support in $V'\setminus\operatorname{sing}(V')$. As $d>2$, Proposition 2.1 implies that we can solve

$$\bar{\partial}' u = \frac{\bar{\partial}'\tilde{s}}{(\sigma_0')^{d-1}}.$$

By standard regularity results, u is smooth on $V'\setminus\operatorname{sing}(V')$. Setting $s'=\tilde{s}-(\sigma_0')^{d-1}u$ we obtain a holomorphic section of $[Z']$ which has the desired behavior in a neighborhood of Z. As V' is normal and the line bundle $[Z']$ is trivial in a neighborhood of $\operatorname{sing}(V')$, this section has a holomorphic extension to all of V'.

All that remains is to establish the bounds on s' in $\overline{X}_-$. To that end we use the estimates (6.7), originally from §3 of [Le2]. As noted before, the constants $\{C_s\}$ are uniformly bounded under small deformations of Y and L. To complete the proof we need to show that for a negative line bundle L the constants $\{C'_s\}$ are also uniformly bounded under small deformations. For the case $Y=X_-$, $L=[(2-d)Z]$, it is clear that $H^0(Y;L)=0$. Indeed, we can find a strictly pseudoconcave neighborhood U of Z such that $L|_U$ is strictly negative. If Y', L' denote small enough deformations of the complex structures on Y, L, then $U\subset Y'$ is again strictly pseudoconcave, and $L'|_U$ is strictly negative. This implies that $H^0(Y';L')=0$ for sufficiently small deformations of the complex structure. This in turn implies that the constants $\{C'_s\}$ are uniformly bounded.

If not, then we could find a sequence of deformations (Y_n, L_n) converging to (Y, L) in the $\mathcal{C}^\infty$-topology, and an $l\geqslant 2$ for which there exists a sequence $\{u_n\}$ so that $u_n\in \mathcal{C}^\infty(Y_n;L_n)$ and

$$|u_n|_l=1, \quad |u_n|_l\geqslant n|\overline{D}_n u_n|_{l-1}. \tag{7.7}$$

On the other hand, we have that

$$|u_n|_l\leqslant C_l(|\overline{D}_n u_n|_{l-1}+|u_n|_0) \tag{7.8}$$

with a constant independent of n. Let Φ_n denote $\mathcal{C}^\infty$-isomorphisms of the bundles $L_n\to L$, converging to the identity as $n\to\infty$. From (7.7) and the Rellich compactness theorem we conclude that $\{\Phi_n u_n\}$ has a subsequence converging to $v\in H^0(Y;L)$ in the $(l-1)$-norm. The second estimate, (7.8), implies that $|v|_0\geqslant C_l^{-1}$, and therefore $v\neq 0$. This contradicts the fact that $H^0(Y;L)=0$.

In the case at hand, $Y=X'_-$, $L=(2-d)Z'$, we have solved the equation

$$\overline{D}u=\frac{\bar\partial'\tilde{s}}{(\sigma'_0)^{d-1}}.$$

The right-hand side is estimated on $\overline{X}_-$ by bounds on Ξ. Observe that $s'-h\cdot s=(\sigma'_0)^{d-1}u$. As we can assume that the constants $\{C'_s\}$ in (6.7) are uniformly bounded, it follows that u itself is also bounded on X_- by Ξ. This observation coupled with the bounds on σ'_0 given in Lemma 7.1 completes the proof of the theorem.

Using Lemma 2.1 along with Theorem 7.1 we obtain the corollary.

COROLLARY 7.1. *If Ω is an integrable deformation on X_- which vanishes to order $d\geqslant 3$ along Z such that Ω induces an embeddable CR-structure on bX_-, and $\sigma_0^{-d}\Omega$ is sufficiently small, then every section $\tau\in H^0(Z;N_Z)$ which has a holomorphic extension σ*

as a section of $[Z]$ *also has a holomorphic extension* σ' *as a section of* $[Z']$ *satisfying the estimate*

$$h\sigma-\sigma'=O((\sigma_0')^{d-1})$$

along Z.

Let M be a separating, strictly pseudoconvex hypersurface in a projective surface X. Let $X_{\pm}$ denote the components of $X\setminus M$. Assume that $X_-\subset X\setminus\operatorname{sing}(X)$ and that $Z\subset X_-$ is a smooth, positively embedded holomorphic curve.

PROPOSITION 7.1. *Suppose that* $H^0(X;[Z])\to H^0(Z;N_Z)$ *is surjective and* ω *is a sufficiently small deformation of the CR-structure on* M *with an integrable extension to* X_- *vanishing to order* $d\geqslant 3$ *along* Z. *Let* X_+' *denote the normal Stein space bounded by* $(M,{}^{\omega}T^{0,1}M)$, *and set* $X'=X_+'\amalg_M X_-'$. *Then*

$$\dim H^0(X';[Z'])=\dim H^0(X;[Z]).$$

Proof. Let $N=\dim H^0(Z;N_Z)$; from Corollary 7.1 it follows that there exist sections $\{\sigma_1',...,\sigma_N'\}\in H^0(X';[Z'])$ such that $\{\sigma_1'|_Z,...,\sigma_N'|_Z\}$ is a basis for $H^0(Z;N_Z)$, and therefore

$$\dim H^0(X';[Z'])\geqslant N+1=\dim H^0(X;[Z]).$$

It follows from (3.7) that

$$\dim H^0(X';[Z'])\leqslant\dim H^0(Z;N_Z)+1$$

as well, thus completing the proof of the proposition.

From this point, Lempert's argument establishing the stability of the embedding of M into $\mathbf{C}^N$,

$$p\to\left(\frac{\sigma_1}{\sigma_0}\bigg|_p,...,\frac{\sigma_N}{\sigma_0}\bigg|_p\right),$$

defined by a basis of sections $\{\sigma_0,...,\sigma_N\}$ for $H^0(X;[Z])$, is a consequence of (6.7). Using his argument one easily proves

PROPOSITION 7.2. *Suppose that* $H^0(X;[Z])\to H^0(Z;N_Z)$ *is surjective, and* $[Z]\to X$ *is a very ample line bundle. Let* ω *be a sufficiently small deformation of the CR-structure on* M *with an integrable extension to* X_- *vanishing to order* $d\geqslant 3$ *along* Z. *Let* X_+' *denote the normal Stein space bounded by* $(M,{}^{\omega}T^{0,1}M)$, *and set* $X'=X_+'\amalg_M X_-'$. *Then for each* $l>0$ *and* $\varepsilon>0$ *there exist a* $k>0$ *and a* $\delta>0$ *such that, if* $\|\omega\|_{C^k(M)}<\delta$, *then there are holomorphic sections* $\{\sigma_0',...,\sigma_N'\}$ *of* $[Z']$ *which satisfy the estimates*

$$\left\|\frac{\sigma_i'}{\sigma_0'}-\frac{\sigma_i}{\sigma_0}\right\|_{C^l(M)}<\varepsilon\quad\textit{for } i=1,...,N.$$

With additional hypotheses we see that $\dim H^0(X;[dZ])=\dim H^0(X';[dZ'])$ for all positive values of d.

COROLLARY 7.2. *There exists an integer* $m(k,g)$ *such that if the maps*

$$\mathrm{Sym}^m H^0(Z;N_Z)\to H^0(Z;N_Z^m) \text{ are surjective for } 0<m\leqslant m(k,g) \tag{7.9}$$

then, under the hypotheses of Proposition 7.1, *we have*

$$\dim H^0(X;[dZ])=\dim H^0(X';[dZ']) \quad \text{for } d>0.$$

Proof. If $k>2g-2$ then it follows from Castelnuovo's bound that the maps

$$\mathrm{Sym}^m H^0(Z;N_Z)\to H^0(Z;N_Z^m)$$

are surjective for all $m>0$, see [GH]. If $k<2g-2$ then let $l=[\![(2g-2)/k]\!]+2$; by the same reasoning the maps

$$\mathrm{Sym}^m H^0(Z;N_Z^l)\to H^0(Z;N_Z^{lm})$$

are surjective for all $m>0$. Let $j>2l$ and choose $0<n$ so that

$$(n+1)l<j\leqslant(n+2)l.$$

With this choice of n we can find two sections $s_1,s_2\in H^0(Z;N_Z^{j-nl})$ without common zeros. Hence the following sequence of sheaves is exact:

$$0\to N_Z^{nl}\oplus N_Z^{nl}\xrightarrow{f}N_Z^j\to 0$$

where $f(t_1,t_2)=s_1t_1+s_2t_2$. The long exact sequence in cohomology implies that

$$s_1H^0(Z;N_Z^{nl})+s_2H^0(Z;N_Z^{nl})=H^0(Z;N_Z^j).$$

As $j-nl<2l$ this implies that there is an $m(k,g)$ such that for any $m>0$ the group $H^0(Z;N_Z^m)$ is generated by monomials of weighted degree m in

$$\bigoplus_{j=0}^{m(k,g)}H^0(Z;N_Z^j).$$

The hypotheses of the theorem therefore imply that the following sequences are exact:

$$0\to H^0(X;[(d-1)Z])\to H^0(X;[dZ])\xrightarrow{r}H^0(Z;N_Z^d)\to 0,\quad 1\leqslant d,$$
$$0\to H^0(X';[(d-1)Z'])\to H^0(X';[dZ'])\xrightarrow{r}H^0(Z;N_Z^d)\to 0,\quad 1\leqslant d.$$

Thus

$$\dim H^0(X;[dZ]) = \dim H^0(X';[dZ']) = \sum_{j=0}^{d} H^0(Z;N_Z^j)$$

for $1 \leqslant d$.

Remarks. (1) Note that the hypotheses of the corollary imply that $\dim H^0(X;[dZ])$ achieves the maximum possible value for all $0<d$.

(2) It seems likely that the conclusion of the corollary does not require an additional hypothesis as strong as (7.9). Using once again Lempert's estimate one can show that for a *fixed* d and sufficiently small ω, sections in $H^0(X;[dZ])$ are well approximated on X_- by sections in $H^0(X';[dZ'])$. *A priori* the necessary smallness of ω depends on d. The results in [Ep2], however, suggest that, at least when $\bigcup_{d>0} H^0(X;[dZ])|_M$ is dense in the CR-functions on M, there should be a uniform estimate that works for all d.

In the case that $X \setminus Z$ is a Stein manifold, the stability property obtained in the previous corollary can be strengthened.

COROLLARY 7.3. *Suppose that (X,Z) satisfies the hypotheses of Proposition 7.2, and that $X\setminus Z$ is a Stein manifold. Then any sufficiently small deformation of the CR-structure on M with an integrable extension to X_-, vanishing to order $d \geqslant 3$ on M, can be realized by a small deformation of the embedding of M into X itself.*

Remark. Colloquially one says that all sufficiently small embeddable deformations of the CR-structure on M are obtained by "wiggling" M within X. In this case the results in [Ep2] do indeed show that $\ker \bar\partial_b$ is uniformly well approximated by elements of $\ker \bar\partial_b'$, and vice versa.

Proof. Let $Y \subset \mathbf{C}^n$ denote a proper embedding of $X\setminus Z$ obtained from a basis of sections of $[Z]$. It follows from a theorem of Docquier and Grauert that there is a neighborhood U of Y and a holomorphic retraction

$$R: U \to Y,$$

see [DG]. For a sufficiently small, embeddable deformation, the image of the perturbed embedding, defined in Proposition 7.2, is transverse to the fibers of R. Hence we can compose this embedding with R to obtain an embedding of the deformed CR-manifold into Y itself. As Y is biholomorphic to $X\setminus Z$ this proves the corollary.

8. Deformation of the defining equations

We now let $X \subset \mathbf{P}^N$ be an algebraic surface with Z a smooth hyperplane section and $M \subset X$ a separating, smooth, strictly pseudoconvex hypersurface. Let $X_\pm$ be the components of $X \setminus M$; assume that X_- is smooth and that $Z \subset\subset X_-$. We have seen that an embeddable deformation of the CR-structure on M which has an extension to X_- vanishing to sufficiently high order along Z can be embedded as a small deformation of M in $\mathbf{P}^N$. In this section we show, under additional hypotheses, that the deformed embedding lies in an algebraic variety which is a deformation of X.

We let $(\zeta_0 : \ldots : \zeta_N)$ denote homogeneous coordinates for $\mathbf{P}^N$. Let H denote the hyperplane $\{\zeta_0 = 0\}$, and suppose that X is embedded into $\mathbf{P}^N$ with

$$Z = X \cap H$$

a *transverse* intersection. We can also think of the functions $\{\zeta_i\}$ as a basis for $H^0(\mathbf{P}^N; [H])$. With this interpretation we denote the restrictions of $\{\zeta_j : j = 0, \ldots, N\}$ to X by $\{\sigma_j\}$, and the restrictions of $\{\sigma_j : j = 1, \ldots, N\}$ to Z by $\{\tau_j\}$.

We begin with a crude algebraic stability result:

PROPOSITION 8.1. *Let ω denote a sufficiently small, embeddable deformation of the CR-structure on M, having an extension to X_- vanishing to order $d \geqslant 3$ along Z. The deformed CR-manifold $(M, {}^{\omega}T^{0,1}M)$ can be embedded into a projective variety $\widehat{X}$ which satisfies*

$$\deg(\widehat{X}) = \deg(X).$$

Proof. Let Ω denote the integrable deformation extending ω which vanishes to order $d \geqslant 3$ along Z. We let $X'_\pm$ denote the normal, complex spaces bounded by $(M, {}^{\omega}T^{0,1}M)$, and set

$$X' = X'_+ \amalg_M X'_-.$$

Let σ'_0 denote the section of $[Z']$ introduced in (7.1). Recall that $h = \{h_i\}$, defined in (7.2), is a smooth 0-cochain defining an isomorphism between $[Z]$ and $[Z']$. Using Proposition 7.2 it follows that for each $l > 0$ and $\varepsilon > 0$ there exist a $k > 0$ and a $\delta > 0$ such that if $\|\omega\|_{C^k(M)} < \delta$ then there are holomorphic sections $\{\sigma'_0, \ldots, \sigma'_N\}$ of $[Z']$ such that

$$\begin{gathered} \sigma'_i|_Z = \tau_i, \\ h_i \sigma_i - \sigma'_i = O(|\sigma'_0|^{d-1}), \\ \|h^{-1}\sigma'_i - \sigma_i\|_{C^l(X_-)} < \varepsilon \quad \text{for } i = 0, \ldots, N. \end{gathered}$$

From the definition of h it follows that

$$\sigma'_0 = h\sigma_0.$$

Thus the embedding of X'_- into $\mathbf{P}^N$ defined by $\{\sigma'_0, ..., \sigma'_N\}$ meets the embedding of X along $Z=X\cap H$. On the other hand, by taking ω sufficiently small we obtain the estimates above for $l=2$ and an $0<\varepsilon\ll 1$. This gives an embedding of bX'_- which is a small deformation of bX_-. As bX'_- is contained in the affine chart $\mathbf{P}^N\setminus H$, we can apply the Harvey–Lawson construction to obtain an irreducible analytic variety $\widehat{X}_+$ with $b\widehat{X}_+=-bX'_-$.

As $\widehat{X}_+$ is contained in a compact subset of the affine chart $\mathbf{P}^N\setminus H$, we see that

$$H\cap \widehat{X}_+=\varnothing.$$

Thus $\widehat{X}=\widehat{X}_+\amalg_{bX'_-} X'_-$ is a subvariety of $\mathbf{P}^N$ which meets H along Z. Since X intersects H transversely we have the formula

$$\deg X = \#(X\cap P)$$

for $P\subset H$ a generic $(N-2)$-dimensional subspace. As transversality is an open condition, under the smallness hypotheses of the proposition we also have that

$$\deg \widehat{X} = \#(\widehat{X}\cap P).$$

As $\widehat{X}\cap H=X\cap H$ the proposition follows immediately.

Suppose that the ideal of the subvariety $X\subset \mathbf{P}^N$ is generated by homogeneous polynomials $\{p_1, ..., p_l\}$ of degrees $\{m_1, ..., m_l\}$. The sections $\{\sigma_i\}$ satisfy the relations

$$p_j(\sigma_0, ..., \sigma_N)=0, \quad j=1, ..., l. \tag{8.1}$$

If Ω vanishes to order $d\geqslant 3$ along Z then the holomorphic sections $\{\sigma'_i\}$ of $[Z']$ satisfy

$$h\cdot\sigma_i-\sigma'_i=O(|\sigma_0|^{d-1}). \tag{8.2}$$

As a consequence of (7.3) it follows that, in local coordinates near to a point on Z, we have the estimate

$$\sigma_i-\sigma'_i=O(|\sigma_0|^{d-1})=O(|\sigma'_0|^{d-1}). \tag{8.3}$$

Using the homogeneity of p_j it follows that, along Z,

$$p_j(\sigma')=O(|\sigma_0|^{d-1}),$$

and therefore by the Riemann removable singularities theorem it follows that

$$\xi_j=\frac{p_j(\sigma')}{(\sigma'_0)^{d-1}}$$

extends to define a holomorphic section of $[(m_j-d+1)Z']$.

Let

$$D=\max\{m_1, ..., m_l\}.$$

The discussion in the previous paragraph establishes the following strong stability result:

THEOREM 8.1. *If ω is a sufficiently small, embeddable perturbation of the CR-structure on M which has an integrable extension to X_- vanishing to order $d \geqslant D+2$ along Z, then $(M, {}^{\omega}T^{0,1}M)$ is realizable as a small perturbation of bX_+ within X.*

Proof. Under these hypotheses the bundles $[(m_j - d+1)Z']$ are negative, and therefore the holomorphic sections $\{\xi_j\}$ must vanish identically. As the polynomials $\{p_j\}$ generate the ideal of X it follows that the image of the map $x \mapsto (\sigma_0'(x) : ... : \sigma_N'(x))$ is contained in X. That the image of M under this embedding is a small perturbation of bX_+ follows from Proposition 7.2.

This result has an interpretation in terms of the relative index between the reference CR-structure and that defined by ω, see [Ep2]. Let $(M, \bar{\partial}_b)$ denote an embeddable, strictly pseudoconvex, 3-dimensional CR-manifold, $\mathcal{S}$ an orthogonal projection onto $\ker \bar{\partial}_b$, and ω an embeddable deformation of the CR-structure on M. If ω arises from a wiggle of M then it is shown in [Ep2] that the relative index vanishes, see §5. Let $\mathcal{S}^{\omega}$ denote the orthogonal projection onto $\ker \bar{\partial}_b^{\omega}$. If $\operatorname{Ind}(\bar{\partial}_b, \bar{\partial}_b^{\omega}) = 0$ then

$$\|\mathcal{S} - \mathcal{S}^{\omega}\| = O(|\omega|). \tag{8.4}$$

Here $\|\cdot\|$ denotes the operator norm relative to the H^s-topology for some fixed s, and $|\cdot|$ is an appropriate $\mathcal{C}^k$-norm. Thus we see that if $\operatorname{Ind}(\bar{\partial}_b, \bar{\partial}_b^{\omega}) = 0$ and $|\omega|$ is small then the entire algebra $\ker \bar{\partial}_b$ is uniformly well approximated in the H^s-topology by $\ker \bar{\partial}_b^{\omega}$. In [EH1] it is shown that, for any $0 \leqslant d$, $\mathcal{E}_d(M, X_-, Z)$ contains a finite-codimensional subvariety of the set of all deformations.

COROLLARY 8.1. *If ω belongs to the intersection of $\mathcal{E}_{D+2}$ with the set of embeddable CR-structures, then*

$$\operatorname{Ind}(\bar{\partial}_b, \bar{\partial}_b^{\omega}) = 0.$$

Proof. This is an immediate consequence of Theorem 8.1 and the fact, proved in [Ep2], that the relative index vanishes for wiggles.

Remark. This corollary will perhaps be an important step in the understanding of the set of embeddable CR-structures in the general case. As $\operatorname{codim} \mathcal{E}_{D+2}$ is finite it gives strong support for the conjecture that $\operatorname{Ind}(\bar{\partial}_b, \bar{\partial}_b^{\omega})$ assumes only finitely many values among small embeddable deformations. In [Ep2] it is shown that this conjecture implies that the set of embeddable structures is locally closed in the $\mathcal{C}^{\infty}$-topology.

We now consider the stability properties of deformations of the CR-structure on M which have extensions to X_- vanishing along Z to order $0 < d < D+2$. We make an additional assumption on X and its embedding into projective space:

$$H^0(\mathbf{P}^N; [H]) \to H^0(X; [Z]) \text{ and } H^0(X; [Z]) \to H^0(Z; N_Z) \text{ are surjective.} \tag{8.5}$$

As above we let $\{\zeta_0, ..., \zeta_N\}$ denote homogeneous coordinates for $\mathbf{P}^N$ with $H=\{\zeta_0=0\}$. We may consider $\{\zeta_i\}$ as a basis of sections for $H^0(\mathbf{P}^N; [H])$, and, with this interpretation, set

$$\sigma_i = \zeta_i|_X, \quad i=0, ..., N,$$
$$\tau_i = \sigma_i|_Z, \quad i=1, ..., N.$$

Our assumptions on X and Z imply that $\{\sigma_i : i=0, ..., N\}$ is a basis for $H^0(X; [Z])$, and $\{\tau_i : i=1, ..., N\}$ is a basis for $H^0(Z; N_Z)$.

We also require that the normal bundle satisfy the condition:

$$\operatorname{Sym}^m H^0(Z; N_Z) \to H^0(Z; N_Z^m) \text{ is surjective for } 2 \leqslant m. \tag{8.6}$$

As noted in the proof of Corollary 7.2, there is a constant $m(k,g)$ such that it suffices to assume this surjectivity for $m \leqslant m(k,g)$. The conditions in (8.6) hold for $Z=\mathbf{P}^1$, any hypersurface in $\mathbf{P}^3$, the projective completion of the total space of the canonical bundle of a non-hyperelliptic Riemann surface. As follows from the Castelnuovo bound, this is the case for the compactification of any line bundle with degree sufficiently large compared to the genus, see [GH]. We can now establish a stronger algebraic stability statement.

PROPOSITION 8.2. *Suppose that X is a projective variety containing a smooth, very ample divisor Z, satisfying* (8.5) *and* (8.6). *Let ω be a sufficiently small, embeddable deformation of the CR-structure on M which has an extension to X_- vanishing on Z satisfying*:

(a) *the normal bundle of Z' is isomorphic to the normal bundle of Z,*

(b) *the map $H^0(X'_-; [Z']) \to H^0(Z'; N_{Z'})$ is surjective.*

Then the CR-manifold $(bX_-, {}^{\omega}T^{0,1}bX_-)$ has an embedding into a projective variety $\widehat{X}$. For each j there exist homogeneous polynomials $\{p_{j1}, ..., p_{jm_j}\}$ in the variables $(\zeta_1, ..., \zeta_N)$ such that the homogeneous polynomials

$$P_j(\zeta) = p_j(\zeta) - \sum_{k=1}^{m_j} p_{jk}(\zeta)\zeta_0^k, \quad j=1, ..., m,$$

belong to the ideal of $\widehat{X}$.

Proof. Let $\{\sigma'_0, ..., \sigma'_N\}$ denote sections of $[Z']$ which satisfy

$$(\sigma'_0) = Z', \quad \sigma'_i|_{Z'} = \tau_i, \quad i=1, ..., N. \tag{8.7}$$

As the sections $\{\tau_i\}$ satisfy the relations $p_j(\tau)=0$, $j=1, ..., l$, it follows that

$$\xi_{j0} = \frac{p_j(\sigma')}{\sigma'_0}$$

are holomorphic sections of $[(m_j-1)Z']$. Our assumptions (8.5) and (8.6) imply that there exist homogeneous polynomials $\{p_{j1}\}$ of degrees $\{m_j-1\}$ such that

$$(p_j(\sigma')/\sigma_0')|_{Z'} = p_{j1}(\tau), \quad j=1,\dots,l.$$

Note that these polynomials are functions of N variables, i.e. they do not depend on ζ_0. We can repeat this argument with

$$\xi_{j1} = \frac{p_j(\sigma')-\sigma_0' p_{j1}(\sigma')}{(\sigma_0')^2},$$

which are holomorphic sections of $[(m_j-2)Z']$. Arguing recursively we obtain a sequence of homogeneous polynomials

$$\{p_{jk}(\zeta_1,\dots,\zeta_N): k=1,\dots,m_j\},$$

where $\deg p_{jk}=m_j-k$, which satisfy

$$p_j(\sigma')-\sum_{k=1}^{m_j}(\sigma_0')^k p_{jk}(\sigma')=0. \tag{8.8}$$

Setting

$$P_j(\zeta)=p_j(\zeta)-\sum_{k=1}^{m_j} p_{jk}(\zeta)\zeta_0^k$$

completes the proof of the proposition.

Remark. If ω has an extension vanishing to order $d\geqslant 3$ along Z then conditions (a) and (b) are automatically satisfied. Moreover one can arrange that the polynomials p_{jk}, $k=1,\dots,d-2$, vanish identically. If $Z\simeq\mathbf{P}^1$ then such high-order vanishing is not needed.

Now we suppose that $X\setminus Z$ is a normal cone in $\mathbf{C}^N$. This implies that the polynomials $\{p_j(\zeta)\}$ are independent of ζ_0. Using the simple structure of the deformed polynomials we can show that the variety $\widehat{X}$ is a fiber in a deformation space of X.

THEOREM 8.2. *If, under the hypotheses of Proposition* 8.2, $V_0\cap\mathbf{P}^N\setminus Z$ *is a normal cone, then there exists a flat family of projective varieties,* $\pi:\mathcal{V}\to\mathbf{C}$, *such that* $\widehat{X}$ *is a fiber.*

Proof. We define a family of polynomials, depending upon $t\in\mathbf{C}$,

$$P_j^t(\zeta)=p_j(\tilde{\zeta})-\sum_{k=1}^{m_j} t^k p_{jk}(\tilde{\zeta})\zeta_0^k, \quad j=1,\dots,l,\ \tilde{\zeta}=(\zeta_1,\dots,\zeta_N),$$

and a holomorphic action of $\mathbf{C}^*$ on $\mathbf{P}^N$ by

$$K_t(\zeta_0:\zeta_1:\ldots:\zeta_N)=(\zeta_0:t\zeta_1:\ldots:t\zeta_N).$$

Observe that for $t\neq 0$ the equation $P_j(\zeta)=P_j^1(\zeta)=0$ is satisfied if and only if $P_j^t(K_t(\zeta))=0$. Define the analytic space

$$\mathcal{V}\subset\mathbf{C}\times\mathbf{P}^N=\{(t,\zeta):P_j^t(\zeta)=0,\ j=1,\ldots,l\}.$$

Let π denote the natural projection $\pi:\mathcal{V}\to\mathbf{C}$ and $V_t=\pi^{-1}(t)$.

If $ts\neq 0$ then the map $M_{t/s}$ induces a biholomorphic equivalence between V_s and V_t. As $\widehat{X}\subset V_1$ it is apparent that for each $t\in\mathbf{C}$, V_t contains a 2-dimensional subvariety of $\mathbf{C}\times\mathbf{P}^N$. Observe that as subvarieties of $\mathbf{P}^N$ we have, for all t, that

$$V_t\cap\{\zeta_0=0\}=Z. \tag{8.9}$$

In light of (8.9), if we can show that the germ of V_t at each point $x\in Z$ is irreducible then it follows that V_t is globally irreducible. For each $x\in Z$ we can select a neighborhood $U_x\subset\mathbf{P}^N$ and a subset $\{j_1,\ldots,j_{N-2}\}\subset\{1,\ldots,m\}$ such that $\{dp_{j_1}(\zeta),\ldots,dp_{j_{N-2}}(\zeta)\}$ are linearly independent in U_x. For sufficiently small t and a possibly smaller neighborhood U'_x of x, the differentials $\{dP^t_{j_1}(\zeta),\ldots,dP^t_{j_{N-2}}(\zeta)\}$ are linearly independent in U'_x. The implicit function theorem then implies that

$$\{P^t_{j_1}(\zeta)=\ldots=P^t_{j_{N-2}}(\zeta)=0\}\cap U'_x$$

is a smooth submanifold. As $K_t(V_1)\cap U'_x$ is contained in this manifold and is itself a 2-dimensional manifold they must coincide. This shows that, for small enough t, the germs of V_t at points along Z are irreducible and smooth. As K_t fixes $\{\zeta_0=0\}$ and is a biholomorphism, this proves the desired statement for all $t\in\mathbf{C}\setminus\{0\}$.

We have now established that $\mathcal{V}$ is an analytic space with irreducible fibers. All that remains is to show that the map π is flat. Arguing as above, this is essentially immediate near smooth points of V_0. We use the method introduced in [BlE] to establish the flatness near to the singular point of V_0.

Let y be the singular point of V_0 and let $\{z_1,\ldots,z_N\}$ denote affine coordinates in a neighborhood of y with $y=\{z=0\}$. In virtue of the Artin approximation theorem, to establish the flatness of π at y it suffices to show that if $r=(r_1,\ldots,r_m)$ is a relation satisfied by the generators p of $\mathcal{I}_{V_0}$, that is,

$$r\cdot p=\sum_{j=1}^{m}r_ip_i\equiv 0,$$

then there is a formal series R^t, with polynomial m-vector coefficients

$$R^t \sim r + \sum_{k=1}^{\infty} t^k r_k, \tag{8.10}$$

such that

$$R^t \cdot P^t \equiv O(t^q) \quad \text{for every } q. \tag{8.11}$$

For $\varrho > 0$ let

$$A_\varrho = \{\varrho < |z| < 2\varrho\}.$$

For sufficiently small $0 < \varrho$, $A_\varrho \cap V_0$ is a smooth manifold, and therefore we can find a smooth family of diffeomorphisms $\{F_t : |t| < \eta\}$ such that

$$F_0 = \mathrm{Id} \quad \text{and} \quad F_t(A_\varrho \cap V_0) \subset V_t. \tag{8.12}$$

This implies that $F_t^* P_j^t \equiv 0$ for $j = 1, ..., m$. Since the coordinate functions of r are polynomials, as are the generators $\{p_1, ..., p_m\}$ of the ideal $\mathcal{I}_{V_0}$, we can apply the proof of Theorem 7.2 in [BlE] to obtain a series as in (8.10) satisfying (8.11). This completes the proof of the theorem.

Remark. In this circumstance we have embedded the hypersurface M, with the deformed CR-structure, as a hypersurface in a fiber of an analytic deformation space of X.

Appendix: The exact obstruction to extending sections of the normal bundle

As before, $M \subset X$ is a strictly pseudoconvex, separating hypersurface, $X_\pm$ are the components of $X \setminus M$, and $Z \subset X_-$ is a smooth, compact, holomorphic curve. In this appendix we analyze precisely when $H^0(X; [Z]) \to H^0(Z; N_Z)$ is surjective. To answer this question we use (2.7) with $l = 1, d = 1$ to obtain the long exact sequence in cohomology

$$H^0(Z; \mathcal{O}_Z) \to H^0(\mathcal{O}_X / \mathcal{I}_Z^2 \otimes [Z]) \xrightarrow{r} H^0(Z; N_Z) \xrightarrow{\delta} H^1(Z; \mathcal{O}_Z) \to \dots . \tag{A.1}$$

By Proposition 2.2 the issue is to decide if a section $s \in H^0(Z; N_Z)$ is in the kernel of δ. Of course, if $Z \simeq \mathbf{P}^1$ then the group $H^1(Z; \mathcal{O}_Z) = 0$, and thus all sections of the normal bundle extend to holomorphic sections of $[Z]$. In this appendix we assume that $Z \not\simeq \mathbf{P}^1$.

Suppose that ω defines a deformation of the CR-structure on M which has an extension to an integrable almost complex structure Ω on X_-, vanishing to order 2 along Z. Denote X_- with this complex structure by X'_-. As before, we denote objects connected with the deformed complex structure with a $'$, e.g. Z'. Suppose that $(M, {}^{\omega}T^{0,1}M)$ is the boundary of a compact Stein space X'_+. From Lemma 2.1 we conclude that the compact complex space

$$X' = X'_- \amalg_M X'_+$$

is a modification of a projective variety. As the deformation tensor vanishes to order 2 along Z, the complex structure on Z and its normal bundle are unchanged. Thus we have the exact sequence

$$H^0(Z;\mathcal{O}_Z) \to H^0(\mathcal{O}_{X'}/\mathcal{I}_Z^2 \otimes [Z']) \xrightarrow{r'} H^0(Z;N_Z) \xrightarrow{\delta'} H^1(Z;\mathcal{O}_Z) \to \dots . \tag{A.2}$$

Fix a cover of a neighborhood of $Z \subset X_-$ by open balls $\{U_1, \dots, U_Q\}$ such that in each U_i we have $\bar\partial$-holomorphic coordinates (z_i, w_i) with

$$Z \cap U_i = \{z_i = 0\}.$$

We further suppose that each U_i contains the unit ball in the (z_i, w_i)-coordinates, and that these balls also define a cover of a neighborhood of Z. Let U_0 denote an open set in X_-, disjoint from Z, such that U_0 along with the balls of radius 1 is a cover of X_-. If we set

$$\sigma_{0i} = \begin{cases} z_i, & 1 \leqslant i \leqslant Q, \\ 1, & i = 0, \end{cases}$$

then $\sigma_0 = \{\sigma_{0i}\}$ is a holomorphic 0-cochain with $(\sigma_0) = [Z]$.

Let $\{z'_i\}$ be as defined in (7.1) noting that

$$\frac{z'_i}{z_i} = 1 + O(|z_i|) \quad \text{and} \quad \bar\partial \frac{z'_i}{z_i} = z_i \alpha_i \tag{A.3}$$

for a smooth 1-form α_i. Transition functions for $[Z]$ and $[Z']$ are given by

$$g_{ij} = \frac{z_i}{z_j} \quad \text{and} \quad g'_{ij} = \frac{z'_i}{z'_j}$$

respectively. From (A.3) it follows that

$$z'_i = z_i + v_i z_i^2 + O(|z_i|^3), \tag{A.4}$$

and therefore

$$\bar\partial g'_{ij} = O(|\sigma_0|^2) \quad \text{and} \quad \bar\partial z'_i = O(|\sigma_0|^2). \tag{A.5}$$

We have an expansion for g'_{ij}:

$$g'_{ij} = g_{ij}(1 + z_i(v_i - g_{ji} v_j) + O(|z_i|^2)). \tag{A.6}$$

From (A.5) and (A.6) it follows that

$$\bar{\partial}(v_i - g_{ji} v_j)|_Z = 0. \tag{A.7}$$

Invariantly $(v_i - g_{ji} v_j)|_Z$ is a Čech 1-cocycle with values in N_Z^*. We define a cohomology class $\beta_2 \in H^1(Z; N_Z^*)$ as the cohomology class of the Čech cocycle

$$\beta_2|_{U_i \cap U_j \cap Z} = g_{ij} v_i - v_j|_{U_i \cap U_j \cap Z}. \tag{A.8}$$

PROPOSITION A.1. *If Ω is the deformation tensor of an integrable almost complex structure which vanishes to order 2 along Z, then the connecting homomorphism in* (A.2) *for the deformed structure is given by*

$$\delta'(s) = \delta(s) - \beta_2(s), \quad s \in H^0(Z; N_Z). \tag{A.9}$$

Proof. Let $W_i = Z \cap U_i$. A section s of N_Z is represented by a collection of functions $s_i \in \mathcal{O}_Z(W_i)$ which satisfy

$$s_i|_{W_i \cap W_j} = g_{ij} s_j|_{W_i \cap W_j}. \tag{A.10}$$

Let $\{\tilde{s}_i\}$ denote $\bar{\partial}$-holomorphic extensions of $\{s_i\}$ to neighborhoods of $\{W_i\}$. The connecting homomorphism is defined as a Čech 1-cocycle by

$$\delta(s) = \frac{\tilde{s}_i - g_{ij} \tilde{s}_j}{\sigma_{0i}}. \tag{A.11}$$

To compute δ' we find $\bar{\partial}'$-holomorphic extensions of $\{s_i\}$, $\{s_i'\}$ which satisfy

$$s_i' = \tilde{s}_i + O((z_i')^2). \tag{A.12}$$

To accomplish this we need to solve the equation

$$\bar{\partial}' u_i = \frac{\bar{\partial}' \tilde{s}_i}{(z_i')^2}.$$

That this is possible follows easily from the fact that $\bar{\partial}' \tilde{s}_i = z_i^2 \alpha$, where α is a smooth $\bar{\partial}'$-closed $(0,1)$-form. Formula (A.9) now follows easily from (A.4), (A.6), (A.11) and (A.12).

LEMMA A.1. *If X satisfies:*

$$H^0(X; [Z]) \to H^0(Z; N_Z) \to 0 \textit{ is exact,} \tag{A.13}$$

and Ω is the deformation tensor for an integrable almost complex structure which vanishes to order 2 along Z, then $H^0(X_-'; [Z']) \to H^0(Z; N_Z)$ is surjective if and only if the map from $H^0(Z, N_Z)$ to $H^1(Z, \mathcal{O}_Z)$ defined by $\beta_2(\Omega)$ is trivial.

Proof. The hypothesis implies that the map r in (A.1) is also surjective, and therefore, for the reference structure the connecting homomorphism δ is the zero map. Thus

$\delta'=0$ if and only if the map defined by β_2 is trivial, and so r' in (A.2) is surjective if and only if this map is trivial. The proposition now follows from Proposition 2.2.

Remark. If the genus of Z is at least 1 then one can easily show that $H^1(Z;N_Z^*)\simeq H^0(Z;\varkappa\otimes N_Z)'=[H^0(Z;\varkappa)\otimes H^0(Z;N_Z)]'$. Thus the map defined by β_2 is trivial if and only if $\beta_2=0$ in $H^1(Z,N_Z^*)$.

III. Examples

9. The case $Z=\mathbf{P}^1$

In this section we consider a projective surface X with a smooth, ample divisor $Z\simeq\mathbf{P}^1$. Castelnuovo's condition implies that X is a rational surface. As $H^1(X;\mathcal{O}_X)=0$ it follows from (2.6) with $d=l=1$ that

$$0\to H^0(X;\mathcal{O}_X)\to H^0(X;[Z])\to H^0(Z;N_Z)\to 0$$

is exact, and therefore

$$\dim H^0(X;[Z])=\dim H^0(Z;N_Z)+1. \tag{9.1}$$

Suppose that ω is an embeddable deformation of the CR-structure on bX_- which has an integrable extension Ω to X_-. As $Z\simeq\mathbf{P}^1$ it follows that

$$H^1(Z;N_Z)=0. \tag{9.2}$$

Thus, provided that Ω is sufficiently small we can apply the stability theorem of Kodaira to conclude that there exists a smooth rational curve Z' which is holomorphic with respect to the deformed complex structure and a small deformation of Z. As observed in Step 1 of the proof of Theorem 6.1 we can actually assume that $Z=Z'$. Let X'_- denote X_- with this complex structure, X'_+ the normal Stein space bounded by $(bX_-,{}^{\omega}T^{0,1}bX_-)$, and $X'=X'_+\amalg_{bX_-}X'_-$.

As holomorphic line bundles over $\mathbf{P}^1$ are classified by their degree,

$$N_{Z'}\simeq N_Z. \tag{9.3}$$

As X' is also rational it follows as above that

$$\dim H^0(X';[Z])=\dim H^0(Z;N_Z)+1. \tag{9.4}$$

Indeed, we can easily show that

$$\dim H^0(X;[kZ])=\dim H^0(X';[kZ'])\quad \text{for } k>0. \tag{9.5}$$

Using the cohomology sequence derived from the sheaf exact sequence

$$0 \to [-Z] \to [kZ] \to \mathcal{O}_X/\mathcal{I}_Z^{k+1} \otimes [kZ] \to 0$$

we conclude that

$$H^0(X;[kZ]) \simeq H^0(\mathcal{O}_X/\mathcal{I}_Z^{k+1} \otimes [kZ]). \tag{9.6}$$

Using the cohomology sequence derived from

$$0 \to N_Z^{k-j} \to \mathcal{O}_X/\mathcal{I}_Z^{j+1} \otimes [kZ] \to \mathcal{O}_X/\mathcal{I}_Z^{j} \otimes [kZ] \to 0$$

in an inductive argument, and the fact that $H^1(Z;N_Z^j)=0$ for all $j>0$, we conclude that

$$\dim H^0(\mathcal{O}_X/\mathcal{I}_Z^{k+1} \otimes [kZ]) = \sum_{j=0}^{k} \dim H^0(Z;N_Z^j). \tag{9.7}$$

From Proposition 2.2, (9.6) and (9.7) we easily derive (9.5).

Let $\{\sigma_0, ..., \sigma_N\}$ denote a basis of sections for $H^0(X;[Z])$ with $(\sigma_0)=Z$.

THEOREM 9.1. *Suppose that M is a smooth, strictly pseudoconvex, separating hypersurface in a rational surface X. Let $X_\pm$ be the components of $X\setminus M$, and suppose that X_- contains a very ample, smooth rational curve Z. If $H_c^2(X_-;\Theta)=0$ and ω is a sufficiently small, embeddable deformation of the CR-structure on M, then there exists a basis $\{\sigma_0', ..., \sigma_N'\}$ for $H^0(X';[Z'])$ such that the differences*

$$\frac{\sigma_i}{\sigma_0} - \frac{\sigma_i'}{\sigma_0'}, \quad i=1,...,N, \tag{9.8}$$

are bounded on M by the size of ω. The set of embeddable deformations of the CR-structure on M is closed in the $\mathcal{C}^\infty$-topology.

Proof. As $H_c^2(X_-;\Theta)=0$ we can extend ω as an integrable deformation of the complex structure on X_- as described above. As $\dim H^0(X;[Z])=H^0(X';[Z'])$ the argument of Lempert used in the proof of Proposition 7.2 applies in this case as well to establish the existence of the basis $\{\sigma_0', ..., \sigma_N'\}$ for $H^0(X';[Z'])$ satisfying the conclusions of the theorem.

The last statement follows as in Lempert's work: let $\{\omega_n\}$ be a sequence of small embeddable deformations converging to ω in the $\mathcal{C}^\infty$-topology. For each n the theorem provides a CR-embedding ψ_n of $(M, {}^{\omega_n}T^{0,1}M)$ into $\mathbf{C}^N$ with uniform estimates on the coordinate functions. We can therefore select a convergent subsequence of embeddings $\{\psi_{n_j}\}$ which converge to a CR-embedding of $(M, {}^{\omega}T^{0,1}M)$.

Theorem 9.1 implies that the affine embedding of M defined by $\{\sigma_i/\sigma_0\}$ is stable under small embeddable deformations of the CR-structure on M. Arguing as in the proof of Corollary 7.3 we easily establish

COROLLARY 9.1. *If in addition to the hypotheses of Theorem* 9.1, X *is assumed to be smooth and reduced, then every sufficiently small, embeddable deformation of the CR-structure on* M *can be realized as a small deformation of* M *within* X.

We now consider the special case of Hirzebruch surfaces. Let $\mathcal{O}(k)$ denote the unique holomorphic line bundle of degree k over $\mathbf{P}^1$, and S_k denote the rational ruled surface obtained as the projectivization of the Whitney sum $\mathcal{O}(k)\oplus\mathcal{O}(0)$, i.e.

$$S_k=\mathbf{P}[\mathcal{O}(k)\oplus\mathcal{O}(0)].$$

If $k>0$ then the boundary of a strictly pseudoconcave neighborhood of the 0-section in S_k satisfies the hypotheses of Theorem 9.1. Consequently the CR-structure on such a boundary has a stable embedding. Let S_{k0} denote the space obtained by blowing down the exceptional curve in S_k. The space $S_{k0}\backslash\mathbf{P}^1$ embeds into $\mathbf{C}^{k+1}$ as a cone with an isolated normal singularity. This space has non-trivial smooth deformations. These are of course just the affine bundles associated with $\mathcal{O}(k)$. These deformations are naturally parametrized by $H^1(\mathbf{P}^1;\mathcal{O}(-k))$. We denote the versal deformation by $\mathcal{V}_k$, see [MR1], [Pi], [BlE]. Let $X_\pm$ denote the components of $S_{k0}\backslash M$, and $Z\subset X_-$ denote the 0-section of $\mathcal{O}(k)$. In §10 (a) it is shown that $H^2_c(X_-;\Theta\otimes[-Z])=0$.

THEOREM 9.2. *Let* M *be a strongly pseudoconvex, separating hypersurface embedded into* $S_{k0}\backslash\{0\}$. *Any sufficiently small, embeddable deformation* ω *of the CR-structure on* M *embeds as a hypersurface in a fiber of the versal deformation of the singular space* S_{k0}.

Proof. The deformation ω has an extension to X_- which vanishes along Z. As S_{k0} is a cone we can apply Theorem 8.2 to construct a flat family of projective varieties $\pi:\mathcal{V}\to\mathbf{C}$ such that $\pi^{-1}(0)=S_{k0}$ and $(M,{}^{\omega}T^{0,1}M)$ is embedded in the fiber $\pi^{-1}(1)$. From the versality of $\mathcal{V}_k$ it follows that if ω is sufficiently small then $\pi^{-1}(\bar{D}_1)$ has a holomorphic embedding into this space, and therefore the deformed CR-structure embeds into a fiber of $\mathcal{V}_k$.

Using special features of the affine bundles we can say even more about the topology of the space of embeddable deformations. If X' denotes a fiber of $\mathcal{V}_k$ with $Z'\subset X'$ the curve at infinity, then it is shown in [MR1] that

$$\mathcal{O}_X/\mathcal{I}_Z^2\simeq\mathcal{O}_{X'}/\mathcal{I}_{Z'}^2. \tag{9.9}$$

Let $\{\sigma_{0i}\}$ be a 0-cochain defining Z. A consequence of (9.9) is

LEMMA 9.1. *There exists a* $\bar\partial'$*-holomorphic* 0*-cochain* $\{\sigma'_{0i}\}$ *with* $(\sigma'_{0i})=Z'$ *such that*

$$\bar\partial'\frac{\sigma_{0i}}{\sigma'_{0i}}=\sigma'_{0i}\alpha_i \tag{9.10}$$

for a 0*-cochain of smooth* $(0,1)$*-forms* $\{\alpha_i\}$.

To prove this lemma we use the "tubular neighborhood theorem in the holomorphic category". This gives a convenient way to study the deformation tensor describing the complex structure on a neighborhood of Z'.

LEMMA 9.2. *Let Y be a compact, complex submanifold of a complex manifold X with holomorphic normal bundle NY. Let $i: NY \to T^{1,0}X|_Y$ be a complex linear inclusion. There exists a neighborhood U of the 0-section in NY, a neighborhood V of Y in X, and a diffeomorphism $\psi: U \to V$ such that*

(1) $\psi|_Y = \mathrm{Id}$, *where we identify Y with the 0-section of NY,*

(2) *for each $y \in Y$ the restriction $\psi|_{N_y Y \cap V}$ is holomorphic, that is, ψ is fiber-holomorphic,*

(3) *restricting ψ_* to the vertical tangent space along the 0-section induces the inclusion $i: NY \to T^{1,0}X|_Y$.*

The result is a simple consequence of Lemma 4.1 in [Ku].

Proof of Lemma 9.1. The normal bundle sequence of $Z' \hookrightarrow X'$ splits, and therefore there is a holomorphic bundle map

$$i: NZ' \to T^{1,0}X'|_{Z'}.$$

Let ψ be a normal fibration as in Lemma 9.2 with ψ_* inducing i along the 0-section. Using ψ we pull back the complex structure from a neighborhood of Z' in X' to a neighborhood of the 0-section in NZ'. Let Ω denote the deformation tensor for this deformed complex structure on NZ'. We introduce local coordinates (z, w) in a neighborhood of a point p in the 0-section so that $\{w=0\}$ is the 0-section and $\{z=c\}$ are the fibers of the normal bundle. A computation in these coordinates shows that

$$\Omega = w(a(z,\bar{z},w)\partial_z + b(z,\bar{z},w)\partial_w) \otimes d\bar{z},$$

where, as indicated, a and b depend holomorphically on w.

Since the inclusion i is holomorphic it follows that ∂_w is a $\bar{\partial}_b^{\Omega}$-holomorphic vector field along the 0-section. This is equivalent to the condition

$$\pi_{\Omega}^{1,0}[\partial_w, \partial_{\bar{z}} + wa\partial_z + wb\partial_w]|_{w=0} = 0.$$

This in turn implies that $a(z,\bar{z},0) = b(z,\bar{z},0) = 0$. Thus Ω is of the form

$$\Omega = w^2 \Xi,$$

for Ξ a smooth tensor. The assertion of the lemma follows from Lemma 7.1.

We use this to show that the set of small embeddable deformations is path-connected. Let ω denote a small embeddable deformation of the CR-structure on M. From Theorem 9.1 it follows that $(M, {}^{\omega}T^{0,1}M)$ is either a wiggle within S_{k0} or embeds as a hypersurface in a smooth fiber of $\mathcal{V}_k$. In the former case we can obviously isotope the deformed embedding through smooth, pseudoconvex hypersurfaces in S_{k0} back to the original embedding of M. Henceforth we assume that the deformed structure embeds into a smooth fiber X' of $\mathcal{V}_k$.

We can apply Lemma 9.1 to conclude that the 0-cochain $\{h_i\}$, which defines the smooth bundle isomorphism between $[Z]$ and $[Z']$, satisfies

$$h_i|_Z = 1, \quad \bar{\partial}' h_i = \sigma'_{0i}\alpha_i. \tag{9.11}$$

Let $\{\sigma_j\}$ denote a basis of sections of $H^0(S_k; [Z])$ with

$$\tau_j = \sigma_j|_Z, \quad j = 1, ..., k+1.$$

We set $s_j = h\sigma_j$, obtaining a $\mathcal{C}^\infty$-section of $[Z']$ which extends τ_j. From (9.11) and the fact that the deformation tensor vanishes on Z it follows that

$$\beta_j = \frac{\bar{\partial}' s_j}{\sigma'_0}$$

is a smooth, closed $(0,1)$-form. The size of β_j is controlled by estimates on the deformation tensor. As X' is a smooth rational surface it follows that $H^{0,1}(X') = 0$. Thus we can solve the equation

$$\bar{\partial}' u_j = \beta_j,$$

where again the size of u_j is controlled by estimates on the deformation tensor. We set $\sigma'_j = s_j - \sigma'_0 u_j$ to obtain a holomorphic section of $[Z']$ which restricts to τ_j on Z'.

The space $S_{k0} \backslash Z$ has an embedding into $\mathbf{C}^{k+1}$ such that its ideal is generated by the (2×2)-minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & \dots & x_k \\ x_2 & x_3 & \dots & x_{k+1} \end{pmatrix}.$$

We let $(\sigma_0, \sigma_1, ..., \sigma_{k+1})$ denote a basis of sections of $[Z]$ such that

$$x_j = \frac{\sigma_j}{\sigma_0}.$$

Let (a, b, c, d) be indices such that

$$x_a x_b - x_c x_d$$

belongs to the ideal of S_{k0}. Then

$$((\sigma_a'\sigma_b'-\sigma_c'\sigma_d')/\sigma_0')|_{Z'}=\tau_d u_c+\tau_c u_d-\tau_a u_b-\tau_b u_a. \tag{9.12}$$

This is a holomorphic section of N_Z, and therefore there are constants $\{\gamma_e^{abcd}\}$ such that the right-hand side of (9.12) equals

$$\sum \gamma_e^{abcd}\tau_e. \tag{9.13}$$

As $H^0(Z;N_Z)$ is a finite-dimensional vector space it is apparent that the coefficients in (9.13) can be estimated by the supremum norm of $\tau_d u_c+\tau_c u_d-\tau_a u_b-\tau_b u_a$ and therefore by size of the deformation tensor. There is one further constant γ_{abcd} such that

$$x_a x_b - x_c x_d - \sum \gamma_e^{abcd} x_e - \gamma_{abcd}$$

is a generator for the ideal defining X'. Evidently this constant is also estimated by the size of the deformation tensor.

Thus we see that the deformed CR-manifold embeds as a hypersurface in a surface whose defining equations are perturbations of the defining equations for S_{k0}. The coefficients of the linear and constant terms are estimated by the size of the deformation tensor. Let M' denote the embedded image of the deformed CR-manifold in X'. Let $\pi:\mathcal{V}\to\mathbf{C}$ denote the 1-parameter deformation space containing X' constructed in the proof of Theorem 8.2. Lemma 9.2 implies that there is a foliation of a fibered neighborhood U of $M\hookrightarrow S_{k0}$ by holomorphic disks, transverse to the fibers of $\mathcal{V}$. If the deformation of the CR-structure is sufficiently small then $\pi^{-1}(1)$ lies in the neighborhood U. Let t denote the deformation parameter. The foliation by disks defines an analytic 1-parameter family of diffeomorphisms F_t between M and hypersurfaces K_t, lying in $\pi^{-1}(t)$. Again, if ω is sufficiently small then $F_1(M)$ is a smooth, strictly pseudoconvex hypersurface in X' isotopic to M' through strictly pseudoconvex hypersurfaces. We can then deform $F_1(M)$ back to M through $F_t(M)$, $t\in[0,1]$. If we resolve the singularity at $(0,0)$ in $\mathcal{V}$ then all the fibers of resolved space satisfy

$$H^{0,1}(X_t)=H^{2,1}(X_t)=0.$$

We can therefore apply Theorem C in [Ep2] to conclude:

THEOREM 9.3. *The set of small embeddable deformations of the CR-structure on a smooth, separating, strictly pseudoconvex hypersurface $M\subset S_{k0}$ is locally path-connected. Each sufficiently small, embeddable deformation has relative index zero.*

If we instead begin with a smooth, separating, strictly pseudoconvex hypersurface in an affine bundle L_ξ over $\mathbf{P}^1$, then it is easy to show that all sufficiently small deformations are wiggles. By semicontinuity it follows that $H_c^2(X_-;\Theta\otimes[-Z])=0$, so every small deformation has an extension to X_- retaining the holomorphic $\mathbf{P}^1$. Thus we can extend all sections in $H^0(Z; N_Z)$ as holomorphic sections of $[Z']$. Using Lempert's argument we can show that a small embeddable deformation has an embedding near to a given embedding of L_ξ. As $L_\xi \backslash Z$ is a Stein manifold, there is a holomorphic retraction of a neighborhood of $L_\xi \backslash Z$ onto $L_\xi \backslash Z$. If the deformation is sufficiently small then the deformed embedding is transverse to the fibers of the retraction, and so we can compose them to obtain an embedding of the deformed CR-structure into L_ξ.

THEOREM 9.4. *Let L_ξ be an affine bundle over $\mathbf{P}^1$, and M a smooth, separating, strictly pseudoconvex hypersurface in L_ξ. All sufficiently small, embeddable deformations of the CR-structure on M can be realized as small deformations of M within L_ξ.*

Remarks. (1) Theorem 9.1 was previously obtained by Hua-Lun Li. Among other things, he proved

THEOREM 3.4.3 ([Li]). *If M is a CR-manifold which bounds a strictly pseudoconvex, open neighborhood of the 0-section in the line bundle $\mathcal{O}(-m)$ over $\mathbf{P}^1$, then there exists a stable embedding of M into $\mathbf{C}^m$.*

(2) Let $(M, T^{0,1}M)$ be an embeddable 3-dimensional CR-manifold with a stable embedding

$$\varphi: M \hookrightarrow \mathbf{C}^N,$$

where $N \geqslant 3$. If ω defines a small, embeddable deformation of $T^{0,1}M$, then there is a $\bar{\partial}_b^\omega$-CR-embedding

$$\varphi_\omega: M \hookrightarrow \mathbf{C}^N$$

so that $\|\varphi - \varphi_\omega\| = O(|\omega|)$. This does not imply the local path connectedness of the space of small embeddable deformations. To obtain the path connectedness one needs an isotopy from a deformed embedding to the reference embedding through maximally complex submanifolds. If the codimension is greater than 1 then this is difficult to accomplish because maximal complexity is non-generic.

10. Some examples and problems

In the examples below we compute $H_c^2(X_-; \Theta \otimes [-jZ])$ under various circumstances. The size of the pseudoconcave neighborhood X_- of Z which we employ is not very important. It follows from results in [Gr] that this cohomology group, or rather its Serre dual $H^0(X_-; \Omega^1 \otimes \mathcal{K} \otimes [jZ])$, is determined by a fixed, finite-order formal neighborhood of Z in X_-.

(a) *Line bundles over curves.* Let $L\to\Sigma$ be a holomorphic line bundle of degree $k>0$ over a surface Σ of genus g. We denote by $\hat{L}$ the compactification of L obtained by adding the "curve at infinity". We let $\{z_\alpha, U_\alpha\}$ denote a cover of Σ by coordinate charts where

$$z_\alpha = f_{\alpha\beta}(z_\beta) \quad \text{in } U_\alpha \cap U_\beta,$$

and we let ξ_α denote a fiber variable for $L|_{U_\alpha}$. On the overlaps we have

$$\xi_\alpha = g_{\alpha\beta}(z_\beta)\xi_\beta, \quad g_{\alpha\beta}\in \mathcal{O}^*(U_\alpha\cap U_\beta).$$

Let p be a smooth function which vanishes along the 0-section Z of L and is given in a local coordinate chart by

$$p(z_\alpha,\xi_\alpha) = h_\alpha(z_\alpha)|\xi_\alpha|^2 + O(|\xi|^3).$$

We further assume that $-\log p$ is strictly plurisubharmonic in a deleted neighborhood of Z. Let X_- be a strictly pseudoconcave neighborhood of Z, with bX_- a level surface of p. We first compute $H^2_c(X_-;\Theta\otimes[-jZ])$. Next we consider the deformations which define the affine bundles associated to L.

To compute $H^2_c(X_-;\Theta\otimes[-jZ])$ we use Serre duality to identify this group as the dual of $H^0(X_-;\Omega^1\otimes\mathcal{K}\otimes[jZ])$. Here Ω^1 is the sheaf of holomorphic 1-forms, and $\mathcal{K}$ is the canonical sheaf. We compute the latter group by expanding sections into power series along Z. This has an invariant description in terms of the S^1-action induced on sections of $\Omega^1\otimes\mathcal{K}\otimes[jZ]$. Let s denote a section of $H^0(X_-;\Omega^1\otimes\mathcal{K}\otimes[jZ])$; in local coordinates we have

$$s = (a_\alpha(z_\alpha,\xi_\alpha)\,dz_\alpha + b_\alpha(z_\alpha,\xi_\alpha)\,d\xi_\alpha)\otimes dz_\alpha\wedge d\xi_\alpha\otimes e^j_\alpha.$$

Here e_α is a section locally trivializing L. We expand a_α, b_α in Taylor series along Z:

$$a_\alpha = \sum_{k=0}^{\infty} a_{\alpha k}(z_\alpha)\xi^k_\alpha, \quad b_\alpha = \sum_{k=0}^{\infty} b_{\alpha k}(z_\alpha)\xi^k_\alpha.$$

The coefficients of these series satisfy the transition relations

$$\begin{gathered} a_{\beta 0} = (f'_{\alpha\beta})^2 g^{1-j}_{\alpha\beta} a_{\alpha 0}, \\ \begin{pmatrix} a_{\beta(k+1)} \\ b_{\beta k}\end{pmatrix} = g^{k-j+2}_{\alpha\beta} f'_{\alpha\beta} \begin{pmatrix} f'_{\alpha\beta} & g'_{\alpha\beta}/g_{\alpha\beta} \\ 0 & 1\end{pmatrix}\begin{pmatrix} a_{\alpha(k+1)} \\ b_{\alpha k}\end{pmatrix}, \quad k=0,\dots . \end{gathered} \tag{10.1}$$

The transition relations appearing in the first equation in (10.1) define holomorphic sections of $\varkappa^2\otimes L^{\otimes(j-1)}$, where $\varkappa$ is the canonical bundle of Z. The matrix appearing in the second equation in (10.1) defines a rank-2 vector bundle $E\to Z$. The sheaf of sections $\mathcal{E}$ of E fits into a short exact sequence

$$0\to\mathcal{O}_Z\to\mathcal{E}\to\varkappa\to 0.$$

The factor in front of the matrix amounts to tensoring by the line bundle $\varkappa\otimes L^{j-k-2}$.

We begin with the observation that

$$\dim H_c^2(X_-;\Theta\otimes[-Z])\geqslant\dim H^0(Z;\varkappa^2).$$

The group on the right-hand side vanishes if and only if $Z\simeq\mathbf{P}^1$. In this case using the exact sequences considered below one easily shows that

$$\begin{aligned} H_c^2(X_-;\Theta\otimes[-3Z])&=0 \quad \text{if and only if } \deg L=1,\\ H_c^2(X_-;\Theta\otimes[-2Z])&=0 \quad \text{if and only if } \deg L=1,2,3,\\ H_c^2(X_-;\Theta\otimes[-Z])&=0 \quad \text{if } \deg L>0.\end{aligned}$$

We now turn our attention to line bundles over curves of genus at least 1. In these cases the only group which may in fact vanish is $H_c^2(X_-;\Theta)$. Let $H_c^2(X_-;\Theta)_{(k)}$, $k=-1,0,\ldots$, denote the various Taylor series components of these cohomology classes. We have that

$$H_c^2(X_-;\Theta)_{(-1)}\simeq[H^0(Z;\varkappa^2\otimes L^{-1})]'.$$

For $k\geqslant 0$ the other components fit into long exact sequences:

$$H^0(Z;\varkappa\otimes L^{-k-2})\to[H_c^2(X_-;\Theta)_{(k)}]'\to H^0(Z;\varkappa^2\otimes L^{-k-2})\to\ldots\,.$$

Evidently if $\deg L>2\deg\varkappa=4g-4$ then the group $H_c^2(X_-;\Theta)=0$. For $\deg\varkappa<\deg L\leqslant 2\deg\varkappa$ the size of this group depends in a subtle way on the holomorphic moduli of L. If $\deg L\leqslant\varkappa$ then $H_c^2(X_-;\Theta)\neq 0$.

From these considerations we conclude:

PROPOSITION 10.1. *Suppose that $L\to\Sigma$ is as above and $\deg L>2\deg\varkappa$. Let X_- be a strictly pseudoconcave neighborhood of the 0-section in L. The set of sufficiently small, embeddable perturbations of the CR-structure on bX_- is closed in the C^∞-topology.*

Proof. This is an application of Theorem 6.1. We have verified that $H_c^2(X_-;\Theta)=0$. The normal bundle to the 0-section is canonically isomorphic to L, and thus as $\deg L>2\deg\varkappa$ it follows that

$$H^1(Z;N_Z)=0.$$

We now consider a special family of deformations of the complex structure on the total space of L. These are the affine bundles. Let $\{\nu_{\alpha\beta}\}$ be a 1-cocycle with values in L^*. Using this 1-cocycle we can define a new complex structure on L using the transition relations

$$\xi_\alpha = \frac{g_{\alpha\beta}\xi_\beta}{1-\nu_{\alpha\beta}\xi_\beta}.$$

In terms of coordinates on the dual bundle $\eta_\alpha = \xi_\alpha^{-1}$ this becomes

$$\eta_\beta = g_{\alpha\beta}\eta_\alpha + \nu_{\alpha\beta}.$$

We let $\hat{L}_\nu$ denote the compact manifold defined by these relations. The effect of such a deformation is to eliminate the holomorphic representative of the "curve at infinity". Thus $\hat{L}\backslash\{Z\}$ is a complex manifold with a non-trivial exceptional locus, whereas the deformation $\hat{L}_\nu\backslash\{Z\}$ is a Stein manifold. Let M be an S^1-invariant, strictly pseudoconcave hypersurface bounding a neighborhood X_- of Z. The deformation of the complex structure on L defined by ν can be identified with a deformation of the CR-structure on M with an integrable extension to X_-. We use $X_{\nu-}$ to denote the pseudoconcave domain with the deformed complex structure.

The class $\nu \in H^1(Z; N_Z^*)$ can be identified with the class β_2 defined in the appendix to §8. A little care is required as the tensor defining the complex structure on L_ν as a deformation of the complex structure on L does not vanish to order 2 along Z. As shown in [MR1], however, there is a canonical identification

$$\mathcal{O}_L/\mathcal{I}_Z^2 \simeq \mathcal{O}_{L_\nu}/\mathcal{I}_{Z_\nu}^2, \tag{10.2}$$

so the argument from the appendix can be carried through in this case as well, see also Lemma 9.1. It is evident that

$$H^0(X_-; [Z]) \to H^0(Z; N_Z)$$

is surjective. It follows from Proposition A.1 that the obstruction to extending a section $s \in H^0(Z; N_Z)$ to an element of $H^0(X_{\nu-}; [Z_\nu])$ is exactly

$$\nu(s) \in H^1(Z; \mathcal{O}_Z).$$

If $g \geqslant 1$ then the map $\nu: H^0(Z, N_Z) \to H^1(Z, \mathcal{O}_Z)$ is non-trivial if $\nu \neq 0$ in $H^1(Z, N_Z^*)$. For such an element one can show that

$$\operatorname{Ind}(bX_-, bX_{\nu-}) < 0.$$

These examples illustrate two points: (1) The condition in (10.2) does not suffice to conclude that the algebra of CR-functions is stable under small deformations. Indeed, these are the first examples where the entire algebra of CR-functions is *not* stable under all small embeddable deformations, and yet the set of such deformations is known to be closed in the $\mathcal{C}^\infty$-topology. (2) In [Ep1] it is shown that under deformations of the CR-structure on circle bundles with "non-negative" Fourier coefficients the whole algebra of CR-functions is stable. These examples show that there are embeddable deformations not satisfying the non-negativity hypothesis, but that the algebra of CR-functions is not stable under such a deformation.

(b) *Neighborhoods of curves in* $\mathbf{P}^2$. Let $Z\subset\mathbf{P}^2$ be a smoothly embedded curve of degree d. The classical formula

$$g=\tfrac{1}{2}(d-1)(d-2)$$

gives the genus of Z in terms of its degree. The normal bundle is $\mathcal{O}(d)|_Z$. By Serre duality we obtain that

$$\dim H^1(Z;N_Z)=\dim H^0(Z;\mathcal{O}(-3))=0.$$

Note that the degree of N_Z is $d^2>2g-2$. Let X_- be a neighborhood of Z with strictly pseudoconvex boundary. As $\mathbf{P}^2\setminus\{X_-\}$ is a Stein manifold with boundary bX_-, it follows that bX_- is an embeddable CR-manifold. Using Serre duality, we identify

$$H_c^2(X_-;\Theta\otimes[-jZ])\simeq[H^0(X_-;\Omega^1\otimes\mathcal{K}\otimes[jZ])]'. \tag{10.3}$$

From Hartogs's extension theorem for sections of a holomorphic vector bundle we conclude that

$$H^0(X_-;\Omega^1\otimes\mathcal{K}\otimes[jZ])\simeq H^0(\mathbf{P}^2;\Omega^1\otimes\mathcal{K}_{\mathbf{P}^2}\otimes[jZ]).$$

Let $[H]$ denote the hyperplane section bundle. As is well known, $\mathcal{K}_{\mathbf{P}^2}\simeq[-3H]$, and therefore the group we must compute is

$$H^0(\mathbf{P}^2;\Omega^1\otimes[(dj-3)H]).$$

A calculation shows that

$$\dim H^0(\mathbf{P}^2;\Omega^1\otimes[kH])=\begin{cases}0 & \text{if } k<2,\\ \tfrac{1}{2}(k+2)(k+1)(k-1) & \text{if } k\geqslant 2,\end{cases} \tag{10.4}$$

see [Bot]. Combining (10.3) and (10.4) we obtain

$$\begin{aligned}
H^2_c(X_-;\Theta\otimes[-4Z]) &= 0 && \text{if and only if } d=1,\\
H^2_c(X_-;\Theta\otimes[-2Z]) &= 0 && \text{if and only if } d=1,2,\\
H^2_c(X_-;\Theta\otimes[-Z]) &= 0 && \text{if and only if } d=1,2,3,4,\\
\dim H^2_c(X_-;\Theta\otimes[-Z]) &= (d-2)(d-4) && \text{if } d\geqslant 5,\\
H^2_c(X_-;\Theta) &= 0 && \text{for } d>0.
\end{aligned} \tag{10.5}$$

Note that $(d-2)(d-4)$ is the codimension of the planar deformations in the space of all deformations of the complex structure on Z.

The case $d=1$ is simply that of a domain in $\mathbf{C}^2$, which was treated in [Le2]. If $d=2$ then $Z\simeq\mathbf{P}^1$, and the analysis presented in §9 applies to show that every sufficiently small, embeddable perturbation can be obtained by wiggling bX_- in $\mathbf{P}^2$. This is true because the normal Stein space bounded by bX_- is $\mathbf{P}^2\backslash\{X_-\}$, which is a smooth manifold. From Theorem 6.1 we obtain

PROPOSITION 10.2. *If X_- is a smoothly bounded, strictly pseudoconcave neighborhood of a curve in $\mathbf{P}^2$ of degree $d>2$, then the set of sufficiently small, embeddable perturbations of the CR-structure on bX_- is closed in the C^∞-topology.*

Problem 10.1. Is the embedding of bX_- in $\mathbf{P}^2$ stable for any $d>2$?

Problem 10.2. Let X_- be a strictly pseudoconcave domain in $\mathbf{P}^2$. Is the set of sufficiently small, embeddable perturbations of the CR-structure on bX_- closed in the C^∞-topology?

Remark. There exist smoothly bounded, strictly pseudoconcave domains in $\mathbf{P}^2$ which do not contain any compact, holomorphic curves, see [F].

(c) *Quadric hypersurfaces.* The quadric hypersurfaces in $\mathbf{P}^3$ are classified by the rank of the quadratic form defining them. If we require the surface to be connected and irreducible then there are only two examples:

$$Q_0=\{[\zeta]:\zeta_1^2+\zeta_2^2+\zeta_3^2=0\},\quad Q_1=\{[\zeta]:\zeta_0^2+\zeta_1^2+\zeta_2^2+\zeta_3^2=0\}.$$

In either case there is a rational curve Z "at infinity" with normal bundle of degree 2.

The cone Q_0 is the compactified total space of the line bundle of degree 2 over $\mathbf{P}^1$ with the exceptional locus blown down. This is the same as the space denoted as S_{20} in §9. Let X_- denote a neighborhood of $Z\subset Q_0$ with strictly pseudoconcave boundary. In Theorem 9.3 it is shown that any sufficiently small, embeddable perturbation of the

CR-structure on bX_- is realizable as a hypersurface in a fiber of the total space of the versal deformation of Q_0. That space can be described quite simply in this case:

$$\mathcal{V}=\{(t,[\zeta]):\zeta_0^2+\zeta_1^2+\zeta_2^2+t\zeta_3^2=0\}.$$

Now let $X_-\subset Q_1$ be a neighborhood of Z with strictly pseudoconcave boundary. Using the semicontinuity of $\dim H_c^2(Y_t;\Theta\otimes[-Z])$ for $\{Y_t\}$ a smooth family of strictly pseudoconcave domains in the fibers of $\mathcal{V}$, it follows that $H_c^2(X_-,\Theta\otimes[-Z])=0$. We can therefore apply Theorem 9.1 to conclude:

PROPOSITION 10.3. *If $X_-\subset Q_1$ is a smoothly bounded, strictly pseudoconcave neighborhood of Z, then every sufficiently small, embeddable perturbation of the CR-structure on bX_- is realizable as a small wiggle of bX_- within Q_1.*

(d) *Cubic hypersurfaces.* Let $X\subset\mathbf{P}^3$ be a cubic surface, not necessarily smooth. Let $Z=X\cap\mathbf{P}^2$ be a smooth hyperplane section and X_- a smoothly bounded, strictly pseudoconcave neighborhood of Z. A computation shows that both

$$H_c^2(X_-;\Theta\otimes[-2Z])=0 \quad\text{and}\quad H^1(Z;N_Z)=0.$$

Thus we can apply Theorem 6.1 using either cohomological hypothesis to conclude:

PROPOSITION 10.4. *Let $X_-\subset X$, as above, with X a cubic surface in $\mathbf{P}^3$. Then the set of sufficiently small, embeddable perturbations of the CR-structure on bX_- is closed in the C^∞-topology.*

(e) *Quartic hypersurfaces.* Now we suppose that $X\subset\mathbf{P}^3$ is a quartic surface, not necessarily smooth. Let $Z_d=X\cap Y_d$ be a smooth intersection in $\mathbf{P}^3$ of the quartic X with a hypersurface Y_d of degree d. Let X_- be a smoothly bounded, strictly pseudoconcave neighborhood of Z_d. Computations show in this case that, for all $d\geqslant 1$, $\dim H^1(Z_d,N_{Z_d})=1$. On the other hand, $\dim H_c^2(X_-,\Theta\otimes[-Z_d])$ is equal to the codimension of the set of deformations of Z_d extendible to deformations of X in the space of all deformations of the complex structure on Z_d. If $d=1$ then $H_c^2(X_-,\Theta\otimes[-Z_1])=0$, and we can apply Theorem 6.1 to conclude that the set of small embeddable perturbations of the CR-structure on bX_- is closed in the C^∞-topology.

If $d\geqslant 2$ then

$$H_c^2(X_-,\Theta\otimes[-Z_d])\neq 0 \quad\text{and}\quad H^1(Z_d,N_{Z_d})\neq 0.$$

Hence we cannot directly apply Theorem 6.1. Nevertheless with the help of the information above and the precise version of Theorem EH1, we obtain

PROPOSITION 10.5. *Let X_- be a smooth domain with strictly pseudoconcave boundary in a quartic surface in $\mathbf{P}^3$. If X_- contains a smooth divisor $Z_d = X \cap Y_d$, with Y_d a surface of degree $1 \leqslant d$, then the set of small embeddable perturbations of the CR-structure on bX_- is closed in the $\mathcal{C}^\infty$-topology.*

By modifying the construction of Catlin–Lempert, see [CL], one can obtain an example of a singular quartic hypersurface such that the embedding of bX_- into $\mathbf{P}^3$ is not stable. In this example we again have a case where the algebra of CR-functions is not stable under all small embeddable deformations, but the set of such deformations is closed in the $\mathcal{C}^\infty$-topology.

Note that a quartic surface X is a K3-surface. The deformations of the complex structure on X are parametrized by a 20-dimensional complex space T^{20}. The algebraic deformations depend on only 19 parameters. In fact, T^{20} has a countable collection of 19-dimensional proper subvarieties $\{\mathcal{A}_n\}$ which parametrize the algebraic K3-surfaces. The index n is the minimal degree of a curve defining a very ample divisor on a K3-surface with complex structures parametrized by $\mathcal{A}_n$. The union of the $\{\mathcal{A}_n\}$ is dense in $\mathcal{S}$, see [GH]. The closedness problem for sequences of complex structures on projective varieties analogous to that which we have been considering for CR-manifolds has in the present case a negative solution: Choose a sequence $\Omega_n \in \mathcal{A}_n$ which converge to a point $\Omega_\infty \in T^{20} \setminus \bigcup \mathcal{A}_n$. Let X_n denote X with complex structure defined by Ω_n. Observe that X_∞ is not a projective variety.

As the pseudoconcave manifold X_- with $bX_- = M$ is highly non-unique, it is not immediately apparent what bearing this example has on the stability problem for embeddable CR-structures. Let X be a smooth algebraic K3-surface. We can construct the versal deformation space $\pi: \mathcal{V} \to T^{20}$ for the complex structure on X. We identify X with $\pi^{-1}(0)$. Let $M \hookrightarrow X$ be a smooth, strictly pseudoconvex, separating hypersurface. We can find a hypersurface germ $\mathcal{M} \subset \mathcal{V}$ such that

$$M = \mathcal{M} \cap \pi^{-1}(0),$$

and so that

$$M_t = \mathcal{M} \cap \pi^{-1}(t)$$

is separating and strictly pseudoconvex for sufficiently small $t \in T^{20}$. For the generic t the surface $\pi^{-1}(t)$ is not algebraic, and so we have M_t embedded in a "non-embeddable", compact surface. On the other hand, M_t bounds a compact region in $\pi^{-1}(t)$, and is therefore itself embeddable. We can find a Stein neighborhood $U \subset X$ which contains M; this neighborhood has in turn a Stein neighborhood W in $\mathcal{V}$ on which there is a holomorphic retraction $R: W \to U$. For sufficiently small t the hypersurfaces M_t are transverse to the fibers of R, and therefore can be reembedded into X as wiggles of M.

(f) *Quintic hypersurfaces.* Let $X \subset \mathbf{P}^3$ be a quintic surface, not necessarily smooth, and let Z denote a smooth hyperplane section. As usual we take X_- to be a smooth, strictly pseudoconcave neighborhood of Z in X. A computation shows that

$$\begin{aligned} H_c^2(X_-;\Theta) &= 0, \\ \dim H_c^2(X_-;\Theta\otimes[-Z]) &= 6, \\ \dim H^1(Z;N_Z) &= 3. \end{aligned}$$

The results in this paper cannot be applied directly to study the structure of the embeddable perturbations on bX_-. With these computations and a more precise version of Theorem EH1 we can show that the set of embeddable structures lying in a codimension-3 subspace of the set of all deformations is closed in the $\mathcal{C}^\infty$-topology.

Problem 10.3. Is the set of all sufficiently small, embeddable deformations of the CR-structure on bX_- closed in the $\mathcal{C}^\infty$-topology?

The principal difference between quintic and quartic surfaces is that for a quintic surface not every element of $H^1(Z,N_Z)$ can be generated by global deformations of the complex structure on X. It follows from results of Kodaira, see [Kd1], that all deformations of the complex structures on hypersurfaces in $\mathbf{P}^3$ of degree different from 4 are algebraic.

Let M be a smooth, compact, strictly pseudoconvex, embeddable 3-manifold. In [Ep2] it was conjectured that among the sufficiently small, embeddable perturbations of the CR-structure on M only finitely many different relative indices actually arise. This in turn implies that the set of sufficiently small, embeddable perturbations is closed in the $\mathcal{C}^\infty$-topology. In Theorem 6.1 we have shown that the set of small embeddable perturbations is closed in the $\mathcal{C}^\infty$-topology without verifying this conjecture.

Problem 10.4. Suppose that M is as above and also satisfies the hypotheses of Theorem 6.1. Show that only finitely many different relative indices arise among the set of sufficiently small, embeddable perturbations.(2)

(2) *Note added in proof.* In recent work of the first author this question has been answered affirmatively.

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