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Currents in metric spaces

by

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Introduction

The development of intrinsic theories for area-minimization problems was motivated in the 1950's by the difficulty to prove, by parametric methods, existence for the Plateau problem for surfaces in Euclidean spaces of dimension higher than two. After the pioneering work of R. Caccioppoli [12] and E. De Giorgi [18], [19] on sets with finite perimeter, W. H. Fleming and H. Federer developed in [24] the theory of currents, which leads to existence results for the Plateau problem for oriented surfaces of any dimension and codimension. It is now clear that the interest of this theory, which includes in some sense the theory of Sobolev and BV-functions, goes much beyond the area-minimization problems that were its initial motivation: as an example one can consider the recent papers [3], [8], [27], [28], [29], [35], [41], [42], to quote just a few examples.

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The aim of this paper is to develop an extension of the Federer–Fleming theory to spaces without a differentiable structure, and virtually to any complete metric space; as a by-product we also show that actually the classical theory of currents depends very little on the differentiable structure of the ambient space, at least if one takes into account only normal or rectifiable currents, the classes of currents which are typically of interest in variational problems. The starting point of our research has been a very short paper of De Giorgi [20]: amazingly, he was able to formulate a generalized Plateau problem in any metric space E using (necessarily) only the metric structure; having done so, he raised some natural questions about the existence of solutions of the generalized Plateau problem in metric or in Banach and Hilbert spaces.

The basic idea of De Giorgi has been to replace the duality with differential forms with the duality with $(k+1)$-tuples $(f_0, f_1, ..., f_k)$, where k is the dimension, f_i are Lipschitz functions in E, and f_0 is also bounded; he called metric functionals all functions T defined on the space of these $(k+1)$-tuples which are linear with respect to f_0. We point out that the formal approach of De Giorgi has a strong analogy with the recent work of J. Cheeger [13] on differentiability of Lipschitz functions on metric measure spaces: indeed, also in this paper locally finitely many Lipschitz functions f_i play the role of the coordinate functions $x_1, ..., x_n$ in the Euclidean space $\mathbf{R}^n$. The basic operations of boundary $T \mapsto \partial T$, pushforward $T \mapsto \varphi_{\#} T$ and restriction $T \mapsto T \llcorner \omega$ can be defined in a natural way in the class of metric functionals; moreover, the mass, denoted by $\|T\|$, is simply defined as the least measure μ satisfying

$$|T(f_0, f_1, ..., f_k)| \leqslant \prod_{i=1}^{k} \mathrm{Lip}(f_i) \int_E |f_0| \, d\mu$$

for all $(k+1)$-tuples $(f_0, f_1, ..., f_k)$, where $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f. We also denote by $\mathbf{M}(T) = \|T\|(E)$ the total mass of T. Notice that in this setting it is natural to assume that the ambient metric space is complete, because $\mathrm{Lip}(E) \sim \mathrm{Lip}(\widehat{E})$ whenever E is a metric space and $\widehat{E}$ is the completion of E.

In order to single out in the general class of metric functionals the currents, we have considered all metric functionals with finite mass satisfying three independent axioms:

(1) linearity in all the arguments;

(2) continuity with respect to pointwise convergence in the last k arguments with uniform Lipschitz bounds;

(3) locality.

The latter axiom, saying that $T(f_0, f_1, ..., f_k) = 0$ if f_i is constant on a neighbourhood of $\{f_0 \neq 0\}$ for some $i \geqslant 1$, is necessary to impose, in a weak sense, a dependence on the derivatives of the f_i's, rather than a dependence on the f_i's themselves. Although

df has no pointwise meaning for a Lipschitz function in a general metric space E (but see [7], [13]), when dealing with currents we can denote the $(k+1)$-tuples by the formal expression $f_0\, df_1 \wedge ... \wedge df_k$, to keep in mind the analogy with differential forms; this notation is justified by the fact that, quite surprisingly, our axioms imply the usual product and chain rules of calculus:

$$T(f_0\, df_1 \wedge ... \wedge df_k) + T(f_1\, df_0 \wedge ... \wedge df_k) = T(1\, d(f_0 f_1) \wedge ... \wedge df_k),$$
$$T(f_0\, d\psi_1(f) \wedge ... \wedge d\psi_k(f)) = T(f_0 \det(\nabla\psi(f))\, df_1 \wedge ... \wedge df_k).$$

In particular, any current is alternating in $f = (f_1, ..., f_k)$.

A basic example of a k-dimensional current in $\mathbf{R}^k$ is

$$[\![g]\!](f_0\, df_1 \wedge ... \wedge df_k) := \int_{\mathbf{R}^k} g f_0 \det(\nabla f)\, dx$$

for any $g \in L^1(\mathbf{R}^k)$; in this case, by the Hadamard inequality, the mass is $|g|\mathcal{L}^k$. By the properties mentioned above, any k-dimensional current in $\mathbf{R}^k$ whose mass is absolutely continuous with respect to $\mathcal{L}^k$ is representable in this way. The general validity of this absolute-continuity property is still an open problem: we are able to prove it either for normal currents or in the cases $k=1$, $k=2$, using a deep result of D. Preiss [53], whose extension to more than two variables seems to be problematic.

In the Euclidean theory an important class of currents, in connection with the Plateau problem, is the class of rectifiable currents. This class can be defined also in our setting as

$$\mathcal{R}_k(E) := \{T : \|T\| \ll \mathcal{H}^k \text{ and is concentrated on a countably } \mathcal{H}^k\text{-rectifiable set}\}$$

or, equivalently, as the Banach subspace generated by Lipschitz images of Euclidean k-dimensional currents $[\![g]\!]$ in $\mathbf{R}^k$. In the same vein, the class $\mathcal{I}_k(E)$ of integer-rectifiable currents is defined by the property that $\varphi_\#(T \llcorner A)$ has integer multiplicity in $\mathbf{R}^k$ (i.e. is representable as $[\![g]\!]$ for some integer-valued g) for any Borel set $A \subset E$ and any $\varphi \in \mathrm{Lip}(E, \mathbf{R}^k)$; this class is also generated by Lipschitz images of Euclidean k-dimensional currents $[\![g]\!]$ in $\mathbf{R}^k$ with integer multiplicity.

One of the main results of our paper is that the closure theorem and the boundary-rectifiability theorem for integer-rectifiable currents hold in any complete metric space E; this result was quite surprising for us, since all the existing proofs in the case $E = \mathbf{R}^m$ heavily use the homogeneous structure of the Euclidean space and the Besicovitch derivation theorem; none of these tools is available in a general metric space (see for instance the counterexample in [17]). Our result proves that closure and boundary rectifiability are general phenomena; additional assumptions on E are required only when one looks for the analogues of the isoperimetric inequality and of the deformation theorem in this context.

If E is the dual of a separable Banach space (this assumption is not really restrictive, up to an isometric embedding) we also prove that any rectifiable current T can be represented, as in the Euclidean case, by a triplet $[\![M,\theta,\tau]\!]$ where M is a countably $\mathcal{H}^k$-rectifiable set, $\theta>0$ is the multiplicity function and τ, a unit k-vector field, is an orientation of the approximate tangent space to M (defined in [7]); indeed, we have

$$T(f_0\,df_1\wedge\ldots\wedge df_k)=\int_M \theta f_0\langle \textstyle\bigwedge_k d^M f,\tau\rangle\, d\mathcal{H}^k$$

where $\bigwedge_k d^M f$ is the k-covector field induced by the tangential differential on M of $f=(f_1,\ldots,f_k)$, which does exist in a pointwise sense. The only relevant difference with the Euclidean case appears in the formula for the mass. Indeed, in [38] the second author proved that for any countably $\mathcal{H}^k$-rectifiable set in a metric space the distance locally behaves as a k-dimensional norm (depending on the point, in general); we prove that $\|T\|=\theta\lambda\mathcal{H}^k\llcorner M$, where λ, called area factor, takes into account the local norm of M and is equal to 1 if the norm is induced by an inner product. We also prove that λ can always be estimated from below with $k^{-k/2}$, and from above with $2^k/\omega_k$; hence the mass is always comparable with the Hausdorff measure with multiplicities.

If the ambient metric space E is compact, our closure theorem leads, together with the lower semicontinuity property of the map $T\mapsto\mathbf{M}(T)$, to an existence theorem for the (generalized) Plateau problem

$$\min\{\mathbf{M}(T): T\in\mathcal{I}_k(E),\ \partial T=S\} \tag{1}$$

proposed by De Giorgi in [20]. The generality of this result, however, is, at least in part, compensated by the fact that even though S satisfies the necessary conditions $\partial S=0$ and $S\in\mathcal{I}_{k-1}(E)$, the class of admissible currents T in (1) could in principle be empty. A remarkable example of a metric space for which this phenomenon occurs is the 3-dimensional Heisenberg group H_3: we proved in [7] that this group, whose Hausdorff dimension is four, is purely k-unrectifiable for $k=2,3,4$, i.e.

$$\mathcal{H}^k(\varphi(A))=0 \quad \text{for all } A\subset\mathbf{R}^k \text{ Borel, } \varphi\in\mathrm{Lip}(A,H_3).$$

This, together with the absolute-continuity property, implies that the spaces $\mathcal{R}_k(H_3)$ reduce to $\{0\}$ for $k=2,3,4$; hence there is no admissible T in (1) if $S\neq 0$. Since a lot of analysis can be carried on in the Heisenberg group (Sobolev spaces, Rademacher theorem, elliptic regularity theory, Poincaré inequalities, quasiconformal maps, see [34] as a reference book), it would be very interesting to adapt some parts of our theory to the Heisenberg and to other geometries. In this connection, we recall the important recent work by B. Franchi, R. Serapioni and F. Serra Cassano [25], [26] on sets with finite perimeter and rectifiability (in an intrinsic sense) in the Heisenberg group. Related results, in doubling (or Ahlfors-regular) metric measure spaces are given in [6] and [47].

Other interesting directions of research that we do not pursue here are the extension of the theory to currents with coefficients in a general group, a class of currents recently studied by B. White in [62] in the Euclidean case, and the connection between bounds on the curvature of the space, in the sense of Alexandrov, and the validity of a deformation theorem. In this connection, we would like to mention the parametric approach to the Plateau problem for 2-dimensional surfaces pursued in [49], and the fact that our theory applies well to CBA metric spaces (i.e. the ones whose curvature in the Alexandrov sense is bounded from above) which are Ahlfors-regular of dimension k since, according to a recent work of B. Kleiner (see [39, Theorem B]), these spaces are locally bi-Lipschitz-parametrizable with Euclidean open sets.

With the aim to give an answer to the existence problems raised in [20], we have also studied some situations in which certainly there are plenty of rectifiable currents; for instance if E is a Banach space the cone construction shows that the class of admissible currents T in (1) is not empty, at least if S has bounded support. Assuming also that $\operatorname{spt} S$ is compact, we have proved that problem (1) has a solution (and that any solution has compact support) in a general class of Banach spaces, not necessarily finite-dimensional, which includes all l^p-spaces and Hilbert spaces. An amusing aspect of our proof of this result is that it relies in an essential way on the validity of the closure theorem in a general metric space. Indeed, our strategy (close to the Gromov existence theorem of "minimal fillings" in [32]) is the following: first, using the Ekeland–Bishop–Phelps principle, we are able to find a minimizing sequence (T_h) with the property that T_h minimizes the perturbed problem

$$T \mapsto \mathbf{M}(T) + \frac{1}{h}\mathbf{M}(T - T_h)$$

in the class $\{T : \partial T = S\}$. Using isoperimetric inequalities (that we are able to prove in some classes of Banach spaces, see Appendix B), we obtain that the supports of the T_h are equi-bounded and equi-compact. Now we use the Gromov compactness theorem (see [31]) to embed isometrically (a subsequence of) $\operatorname{spt} T_h$ in an abstract compact metric space X; denoting by i_h the embeddings, we apply the closure and compactness theorems for currents in X to obtain $S \in \mathcal{I}_k(X)$, limit of a subsequence of $i_{h\#} T_h$. Then a solution of (1) is given by $j_\# S$, where $j : \operatorname{spt} S \to E$ is the limit, in a suitable sense, of a subsequence of $(i_h)^{-1}$. We are able to circumvent this argument, working directly in the original space E, only if E has a Hilbert structure.

Our paper is organized as follows. In §1 we summarize the main notation and recall some basic facts on Hausdorff measures and measure theory. §2 contains essentially the basic definitions of [20] concerning the class of metric functionals, while in §3 we specialize to currents, and §4 and §5 deal with the main objects of our investigation, respectively the rectifiable and the normal currents. As in the classical theory of Federer–Fleming the basic operations of localization and slicing can be naturally defined in the class of normal currents. Using an equi-continuity property typical of normal currents we also obtain a compactness theorem.

In order to tackle the Plateau problem in duals of separable Banach spaces we study in §6 a notion of weak* convergence for currents; the main technical ingredient in the analysis of this convergence is an extension theorem for Lipschitz and w*-continuous functions $f: A \to \mathbf{R}$. If A is w*-compact we prove the existence of a Lipschitz and w*-continuous extension (a more general result has been independently proved by E. Matoušková in [43]). The reading of this section can be skipped by those who are mainly interested in the metric proof of closure and boundary-rectifiability theorems.

§7 collects some informations about metric-space-valued BV-maps $u: \mathbf{R}^k \to S$; this class of functions has been introduced by the first author in [4] in connection with the study of the Γ-limit as $\varepsilon \downarrow 0$ of the functionals

$$F_\varepsilon(u) := \int_{\mathbf{R}^k} \left[\varepsilon |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right] dx$$

with $W: \mathbf{R}^m \to [0, \infty)$ continuous (in this case S is a suitable quotient space of $\{W=0\}$ with the metric induced by $2\sqrt{W}$). We extend slightly the results of [4], dropping in particular the requirement that the target metric space is compact, and we prove a Lusin-type approximation theorem by Lipschitz functions for this class of maps.

§8 is devoted to the proof of the closure theorem and of the boundary-rectifiability theorem. The basic ingredient of the proof is the observation, due in the Euclidean context to R. Jerrard, that the slicing operator

$$\mathbf{R}^k \ni x \mapsto \langle T, \pi, x \rangle$$

provides a BV-map with values in the metric space S of 0-dimensional currents endowed with the flat norm whenever T is normal and $f \in \mathrm{Lip}(E, \mathbf{R}^k)$. Using the Lipschitz approximation theorem of the previous section, these remarks lead to a rectifiability criterion for currents involving only the 0-dimensional slices of the current. Once this rectifiability criterion is established, the closure theorem easily follows by a simple induction on the dimension. A similar induction argument proves the boundary-rectifiability theorem.

We also prove rectifiability criteria based on slices or projections: in particular, we show that a normal k-dimensional current T is integer-rectifiable if and only if $\varphi_{\#}T$ is integer-rectifiable in $\mathbf{R}^{k+1}$ for any Lipschitz function $\varphi: E \to \mathbf{R}^{k+1}$; this result, new even in the Euclidean case $E=\mathbf{R}^m$, is remarkable because no a priori assumption on the dimension of the support of T is made.

In §9 we recover, in duals of separable Banach spaces, the canonical representation of a rectifiable current by the integration over an oriented set with multiplicities. As a by-product, we are able to compare the mass of a rectifiable current with the restriction of $\mathcal{H}^k$ to its measure-theoretic support; the representation formula for the mass we obtain can be easily extended to the general metric case using an isometric embedding of the support of the current into l_∞. The results of this section basically depend on the area formula and the metric generalizations of the Rademacher theorem developed in previous papers [38], [7] of ours; we recall without proof all the results we need from those papers.

§10 is devoted to the cone construction and to the above-mentioned existence results for the Plateau problem in Banach spaces.

In Appendix A we compare our currents with the Federer–Fleming ones in the Euclidean case $E=\mathbf{R}^m$, and in Appendix B we prove in some Banach spaces the validity of isoperimetric inequalities, adapting to our case an argument of M. Gromov [32]. Finally, in Appendix C we discuss the problem of the lower semicontinuity of the Hausdorff measure, pointing out the connections with some long-standing open problems in the theory of Minkowski spaces.

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1. Notation and preliminary results

In this paper E stands for a complete metric space, whose open balls with center x and radius r are denoted by $B_r(x)$; $\mathcal{B}(E)$ is its Borel σ-algebra and $\mathcal{B}^\infty(E)$ is the algebra of bounded Borel functions on E.

We denote by $\mathcal{M}(E)$ the collection of finite Borel measures in E, i.e. σ-additive set functions $\mu: \mathcal{B}(E) \to [0,\infty)$; we say that $\mu \in \mathcal{M}(E)$ is *concentrated* on a Borel set B if $\mu(E \setminus B)=0$. The supremum and the infimum of a family $\{\mu_i\}_{i\in I} \subset \mathcal{M}(E)$ are respectively given by

$$\bigvee_{i\in I}\mu_i(B):=\sup\left\{\sum_{i\in J}\mu_i(B_i):B_i \text{ pairwise disjoint}, B=\bigcup_{i\in J}B_i\right\},\tag{1.1}$$

$$\bigwedge_{i\in I}\mu_i(B):=\inf\left\{\sum_{i\in J}\mu_i(B_i):B_i \text{ pairwise disjoint}, B=\bigcup_{i\in J}B_i\right\},\tag{1.2}$$

where J runs among all countable subsets of I and $B_i\in\mathcal{B}(E)$. It is easy to check that the infimum is a finite Borel measure and that the supremum is σ-additive in $\mathcal{B}(E)$.

Let (X,d) be a metric space and let k be an integer; the (outer) Hausdorff k-dimensional measure of $B\subset X$, denoted by $\mathcal{H}^k(B)$, is defined by

$$\mathcal{H}^k(B):=\lim_{\delta\downarrow 0}\frac{\omega_k}{2^k}\inf\left\{\sum_{i=0}^{\infty}[\operatorname{diam}(B_i)]^k:B\subset\bigcup_{i=0}^{\infty}B_i,\operatorname{diam}(B_i)<\delta\right\}$$

where ω_k is the Lebesgue measure of the unit ball of $\mathbf{R}^k$, and $\omega_0=1$. Since $\mathcal{H}^k_X(B)=\mathcal{H}^k_Y(B)$ whenever $B\subset X$ and X isometrically embeds in Y, our notation for the Hausdorff measure does not emphasize the ambient space. We recall (see for instance [38, Lemma 6 (i)]) that if X is a k-dimensional vector space and B_1 is its unit ball, then $\mathcal{H}^k(B_1)$ is a dimensional constant independent of the norm of X and equal, in particular, to ω_k. The Lebesgue measure in $\mathbf{R}^k$ will be denoted by $\mathcal{L}^k$.

The *upper and lower k-dimensional densities* of a finite Borel measure μ at x are respectively defined by

$$\Theta^*_k(\mu,x):=\limsup_{\varrho\downarrow 0}\frac{\mu(B_\varrho(x))}{\omega_k\varrho^k},\quad \Theta_{*k}(\mu,x):=\liminf_{\varrho\downarrow 0}\frac{\mu(B_\varrho(x))}{\omega_k\varrho^k}.$$

We recall that the implications

$$\Theta^*_k(\mu,x)\geqslant t \quad\text{for all } x\in B \quad\Rightarrow\quad \mu\geqslant t\mathcal{H}^k\llcorner B,\tag{1.3}$$

$$\Theta^*_k(\mu,x)\leqslant t \quad\text{for all } x\in B \quad\Rightarrow\quad \mu\llcorner B\leqslant 2^k t\mathcal{H}^k\llcorner B\tag{1.4}$$

hold in any metric space X whenever $t\in(0,\infty)$ and $B\in\mathcal{B}(X)$ (see [23, 2.10.19]).

Let X,Y be metric spaces; we say that $f:X\to Y$ is a *Lipschitz* function if

$$d_Y(f(x),f(y))\leqslant Md_X(x,y)\quad\text{for all } x,y\in X,$$

for some constant $M\in[0,\infty)$; the least constant with this property will be denoted by $\mathrm{Lip}(f)$, and the collection of Lipschitz functions will be denoted by $\mathrm{Lip}(X,Y)$ (Y will be omitted if $Y=\mathbf{R}$). Furthermore, we use the notation $\mathrm{Lip}_1(X,Y)$ for the collection of Lipschitz functions f with $\mathrm{Lip}(f)\leqslant 1$, and $\mathrm{Lip}_b(X)$ for the collection of bounded real-valued Lipschitz functions.

We will often use isometric embeddings of a metric space into l^∞ or, more generally, duals of separable Banach spaces. To this aim, the following definitions will be useful.

Definition 1.1 (weak separability). Let (E,d) be a metric space. We say that E is weakly separable if there exists a sequence $(\varphi_h)\subset \mathrm{Lip}_1(E)$ such that

$$d(x,y) = \sup_{h\in\mathbf{N}} |\varphi_h(x)-\varphi_h(y)| \quad \text{for all } x,y\in E.$$

A dual Banach space $Y=G^*$ is said to be w*-separable if G is separable.

Notice that, by a truncation argument, the definition of weak separability can also be given by requiring φ_h to be also bounded. The class of weakly separable metric spaces includes the separable ones (it suffices to take $\varphi_h(\cdot)=d(\cdot,x_h)$ with $(x_h)\subset E$ dense) and all w*-separable dual spaces. Any weakly separable space can be isometrically embedded in l^∞ by the map

$$j(x) := (\varphi_1(x)-\varphi_1(x_0), \varphi_2(x)-\varphi_2(x_0), \dots), \quad x\in E,$$

and since any subset of a weakly separable space is still weakly separable also the converse is true.

2. Metric functionals

In this section we define, following essentially the approach of [20], a general class of metric functionals, in which the basic operations of boundary, pushforward, restriction can be defined. Then, functionals with finite mass are introduced.

Definition 2.1. Let $k\geqslant 1$ be an integer. We denote by $\mathcal{D}^k(E)$ the set of all $(k+1)$-tuples $\omega=(f,\pi_1,\dots,\pi_k)$ of real-valued Lipschitz functions in E with the first function f in $\mathrm{Lip}_b(E)$. In the case $k=0$ we set $\mathcal{D}^0(E)=\mathrm{Lip}_b(E)$.

If X is a vector space and $T\colon X\to\mathbf{R}$, we say that T is *subadditive* if $|T(x+y)|\leqslant |T(x)|+|T(y)|$ whenever $x,y\in X$, and we say that T is positively 1-homogeneous if $|T(tx)|=t|T(x)|$ whenever $x\in X$ and $t\geqslant 0$.

Definition 2.2 (metric functionals). We call k-dimensional metric functional any function $T\colon\mathcal{D}^k(E)\to\mathbf{R}$ such that

$$(f,\pi_1,\dots,\pi_k)\mapsto T(f,\pi_1,\dots,\pi_k)$$

is subadditive and positively 1-homogeneous with respect to $f\in\mathrm{Lip}_b(E)$ and $\pi_1,\dots,\pi_k\in \mathrm{Lip}(E)$. We denote by $MF_k(E)$ the vector space of k-dimensional metric functionals.

We can now define an "exterior differential"

$$d\omega = d(f,\pi_1,\dots,\pi_k) := (1,f,\pi_1,\dots,\pi_k)$$

mapping $\mathcal{D}^k(E)$ into $\mathcal{D}^{k+1}(E)$ and, for $\varphi \in \mathrm{Lip}(E,F)$, a pullback operator

$$\varphi^{\#}\omega = \varphi^{\#}(f, \pi_1, \ldots, \pi_k) = (f\circ\varphi, \pi_1\circ\varphi, \ldots, \pi_k\circ\varphi)$$

mapping $\mathcal{D}^k(F)$ on $\mathcal{D}^k(E)$. These operations induce in a natural way a boundary operator and a pushforward map for metric functionals.

Definition 2.3 (boundary). Let $k\geqslant 1$ be an integer and let $T\in MF_k(E)$. The boundary of T, denoted by ∂T, is the $(k-1)$-dimensional metric functional in E defined by $\partial T(\omega)=T(d\omega)$ for any $\omega\in\mathcal{D}^{k-1}(E)$.

Definition 2.4 (pushforward). Let $\varphi: E\to F$ be a Lipschitz map and let $T\in MF_k(E)$. Then, we can define a k-dimensional metric functional in F, denoted by $\varphi_\# T$, setting $\varphi_\# T(\omega)=T(\varphi^\#\omega)$ for any $\omega\in\mathcal{D}^k(F)$.

We notice that, by construction, $\varphi_\#$ commutes with the boundary operator, i.e.

$$\varphi_\#(\partial T) = \partial(\varphi_\# T). \tag{2.1}$$

Definition 2.5 (restriction). Let $T\in MF_k(E)$ and let $\omega=(g,\tau_1,\ldots,\tau_m)\in\mathcal{D}^m(E)$, with $m\leqslant k$ ($\omega=g$ if $m=0$). We define a $(k-m)$-dimensional metric functional in E, denoted by $T\llcorner\omega$, setting

$$T\llcorner\omega(f,\pi_1,\ldots,\pi_{k-m}) := T(fg,\tau_1,\ldots,\tau_m,\pi_1,\ldots,\pi_{k-m}).$$

Definition 2.6 (mass). Let $T\in MF_k(E)$; we say that T has finite mass if there exists $\mu\in\mathcal{M}(E)$ such that

$$|T(f,\pi_1,\ldots,\pi_k)| \leqslant \prod_{i=1}^k \mathrm{Lip}(\pi_i)\int_E |f|\,d\mu \tag{2.2}$$

for any $(f,\pi_1,\ldots,\pi_k)\in\mathcal{D}^k(E)$, with the convention $\prod_i \mathrm{Lip}(\pi_i)=1$ if $k=0$.

The minimal measure μ satisfying (2.2) will be called the mass of T and will be denoted by $\|T\|$.

The mass is well defined because one can easily check, using the subadditivity of T with respect to the first variable, that if $\{\mu_i\}_{i\in I}\subset\mathcal{M}(E)$ satisfy (2.3) also their infimum satisfies the same condition. By the density of $\mathrm{Lip}_b(E)$ in $L^1(E,\|T\|)$, which contains $\mathcal{B}^\infty(E)$, any $T\in MF_k(E)$ with finite mass can be uniquely extended to a function on $\mathcal{B}^\infty(E)\times[\mathrm{Lip}(E)]^k$, still subadditive and positively 1-homogeneous in all variables and satisfying

$$|T(f,\pi_1,\ldots,\pi_k)| \leqslant \prod_{i=1}^k \mathrm{Lip}(\pi_i)\int_E |f|\,d\|T\| \tag{2.3}$$

for any $f \in \mathcal{B}^\infty(E)$, $\pi_1, ..., \pi_k \in \mathrm{Lip}(E)$. Since this extension is unique we will not introduce a distinguished notation for it.

Functionals with finite mass are well behaved under the pushforward map: in fact, if $T \in MF_k(E)$ the functional $\varphi_\# T$ has finite mass, satisfying

$$\|\varphi_\# T\| \leqslant [\mathrm{Lip}(\varphi)]^k \varphi_\# \|T\|. \tag{2.4}$$

If φ is an isometry it is easy to check, using (2.6) below, that equality holds in (2.4). It is also easy to check that the identity

$$\varphi_\# T(f, \pi_1, ..., \pi_k) = T(f \circ \varphi, \pi_1 \circ \varphi, ..., \pi_k \circ \varphi)$$

remains true if $f \in \mathcal{B}^\infty(E)$ and $\pi_i \in \mathrm{Lip}(E)$.

Functionals with finite mass are also well behaved with respect to the restriction operator: in fact, the definition of mass easily implies

$$\|T \llcorner \omega\| \leqslant \sup |g| \prod_{i=1}^m \mathrm{Lip}(\tau_i) \|T\| \quad \text{with } \omega = (g, \tau_1, ..., \tau_m). \tag{2.5}$$

For metric functionals with finite mass, the restriction operator $T \llcorner \omega$ can be defined even though $\omega = (g, \tau_1, ..., \tau_m)$ with $g \in \mathcal{B}^\infty(E)$, and still (2.5) holds; the restriction will be denoted by $T \llcorner A$ in the special case $m=0$ and $g = \chi_A$.

PROPOSITION 2.7 (characterization of mass). *Let $T \in MF_k(E)$. Then T has finite mass if and only if*

(a) *there exists a constant $M \in [0, \infty)$ such that*

$$\sum_{i=0}^\infty |T(f_i, \pi_1^i, ..., \pi_k^i)| \leqslant M$$

whenever $\sum_i |f_i| \leqslant 1$ and $\mathrm{Lip}(\pi_j^i) \leqslant 1$;

(b) *$f \mapsto T(f, \pi_1, ..., \pi_k)$ is continuous along equi-bounded monotone sequences, i.e. sequences (f_h) such that $(f_h(x))$ is monotone for any $x \in E$ and*

$$\sup\{|f_h(x)| : x \in E, h \in \mathbf{N}\} < \infty.$$

If these conditions hold, $\|T\|(E)$ is the least constant satisfying (a), *and $\|T\|(B)$ is representable for any $B \in \mathcal{B}(E)$ by*

$$\sup\left\{\sum_{i=0}^\infty |T(\chi_{B_i}, \pi_1^i, ..., \pi_k^i)|\right\}, \tag{2.6}$$

where the supremum runs among all Borel partitions (B_i) *of* B *and all* k*-tuples of* 1-*Lipschitz maps* π^i_j.

Proof. The necessity of conditions (a) and (b) follows by the standard properties of integrals. If conditions (a) and (b) hold, for given 1-Lipschitz maps $\pi_1, ..., \pi_k: E \to \mathbf{R}$, we set $\pi = (\pi_1, ..., \pi_k)$ and define

$$\mu_\pi(A) := \sup\{|T(f, \pi_1, ..., \pi_k)| : |f| \leqslant \chi_A\}$$

for any open set $A \subset E$ (with the convention $\mu_\pi(\varnothing) = 0$). We claim that

$$\mu_\pi(A) \leqslant \sum_{i=1}^{\infty} \mu_\pi(A_i) \quad \text{whenever } A \subset \bigcup_{i=1}^{\infty} A_i. \tag{2.7}$$

Indeed, set $\psi_i^N(x) = \min\{1, N \operatorname{dist}(x, E \setminus A_i)\}$ and define

$$\varphi_i^N := \frac{\psi_i^N}{\sum_1^N \psi_j^N + 1/N}, \quad g_N := \sum_{i=1}^{N} \varphi_i^N = \left(1 + \left(N \sum_{i=1}^{N} \psi_i^N\right)^{-1}\right)^{-1}.$$

Notice that $0 \leqslant g_N \leqslant 1$, g_N is nondecreasing with respect to N, and $g_N \uparrow 1$ for any $x \in \bigcup_i A_i$. Hence, for any $f \in \operatorname{Lip}_b(E)$ with $|f| \leqslant \chi_A$, condition (b) gives

$$|T(f, \pi_1, ..., \pi_k)| = \lim_{N \to \infty} \left| T\left(\sum_{i=1}^{N} f \varphi_i^N, \pi_1, ..., \pi_k\right)\right| \leqslant \sum_{i=1}^{\infty} \mu_\pi(A_i).$$

Since f is arbitrary, this proves (2.7).

We can canonically extend μ_π to $\mathcal{B}(E)$ setting

$$\mu_\pi(B) := \inf\left\{\sum_{i=1}^{\infty} \mu_\pi(A_i) : A \subset \bigcup_{i=1}^{\infty} A_i\right\} \quad \text{for all } B \in \mathcal{B}(E),$$

and it is easily checked that μ_π is countably subadditive and additive on distant sets. Therefore, Carathéodory's criterion (see for instance [23, 2.3.2 (9)]) gives that $\mu_\pi \in \mathcal{M}(E)$. We now check that

$$|T(f, \pi_1, ..., \pi_k)| \leqslant \int_E |f| \, d\mu_\pi \quad \text{for all } f \in \operatorname{Lip}_b(E). \tag{2.8}$$

Indeed, assuming with no loss of generality that $f \geqslant 0$, we set $f_t = \min\{f, t\}$ and notice that the subadditivity of T and the definition of μ_π give

$$\big||T(f_s, \pi_1, ..., \pi_k)| - |T(f_t, \pi_1, ..., \pi_k)|\big| \leqslant \mu_\pi(\{f > t\})(s - t) \quad \text{for all } s > t.$$

In particular, $t\mapsto|T(f_t,\pi_1,...,\pi_k)|$ is a Lipschitz function, whose modulus of derivative can be estimated with $\phi(t)=\mu_\pi(\{f>t\})$ at any continuity point of ϕ. By integration with respect to t we get

$$|T(f,\pi_1,...,\pi_k)|=\int_0^\infty \frac{d}{dt}|T(f_t,\pi_1,...,\pi_k)|\,dt\leqslant\int_0^\infty \mu_\pi(\{f>t\})\,dt=\int_E f\,d\mu_\pi.$$

By the homogeneity condition imposed on metric functionals, (2.8) implies that the measure $\mu^*=\bigvee_\pi\mu_\pi$ satisfies condition (2.2). Since obviously

$$\mu^*(E)=\sup\left\{\sum_{i=0}^\infty \mu_{\pi^i}(f_i):\sum_{i=0}^\infty |f_i|\leqslant 1,\,\mathrm{Lip}(\pi_j^i)\leqslant 1\right\}$$

we obtain that $\mu^*(E)\leqslant M$, and this proves that $\|T\|(E)\leqslant M$, i.e. that $\|T\|(E)$ is the least constant satisfying (a).

It is easy to check that the set function τ defined in (2.6) is less than any other measure μ satisfying (2.2). On the other hand, a direct verification shows that τ is finitely additive, and the inequality $\tau\leqslant\mu^*$ implies the σ-additivity of τ as well. The inequality

$$|T(\chi_B,\pi_1,...,\pi_k)|\leqslant\tau(B)\quad\text{for all } B\in\mathcal{B}(E),\ \pi_i\in\mathrm{Lip}_1(E)$$

gives $\mu_\pi\leqslant\tau$, whence $\mu^*\leqslant\tau$ and also τ satisfies (2.2). This proves that τ is the least measure satisfying (2.2). □

Definition 2.8 (support). Let $\mu\in\mathcal{M}(E)$; the support of μ, denoted by $\operatorname{spt}\mu$, is the closed set of all points $x\in E$ satisfying

$$\mu(B_\varrho(x))>0\quad\text{for all } \varrho>0.$$

If $F\in MF_k(E)$ has finite mass we set $\operatorname{spt}T:=\operatorname{spt}\|T\|$.

The measure μ is clearly supported on $\operatorname{spt}\mu$ if E is separable; more generally, this is true provided the cardinality of E is an Ulam number, see [23, 2.1.6]. If B is a Borel set, we also say that T is concentrated on B if the measure $\|T\|$ is concentrated on B.

In order to deal at the same time with separable and nonseparable spaces, we will assume in the following that the cardinality of *any* set E is an Ulam number; this is consistent with the standard ZFC set theory. Under this assumption, we can use the following well-known result, whose proof is included for completeness.

LEMMA 2.9. *Any measure $\mu\in\mathcal{M}(E)$ is concentrated on a σ-compact set.*

Proof. We first prove that $S=\operatorname{spt}\mu$ is separable. If this is not true we can find by Zorn's maximal principle $\varepsilon>0$ and an uncountable set $A\subset S$ such that $d(x,y)\geqslant\varepsilon$ for any $x,y\in A$ with $x\neq y$; since A is uncountable we can also find $\delta>0$ and an infinite set $B\subset A$ such that $\mu(B_{\varepsilon/2}(x))\geqslant\delta$ for any $x\in B$. As the family of open balls $\{B_{\varepsilon/2}(x)\}_{x\in B}$ is disjoint, this gives a contradiction.

Let $(x_n)\subset S$ be a dense sequence and define $L_{k,h}:=\bigcup_{n=0}^h B_{1/k}(x_n)$, for $k\geqslant1$ and $h\geqslant0$ integers. Given $\varepsilon>0$ and $k\geqslant1$, since μ is supported on S we can find an integer $h=h(k,\varepsilon)$ such that $\mu(L_{k,h})\geqslant\mu(E)-\varepsilon/2^k$. It is easy to check that

$$K:=\bigcap_{k=1}^{\infty}\overline{L_{k,h(k,\varepsilon)}}$$

is compact and $\mu(E\setminus K)\leqslant\varepsilon$. □

We point out, however, that Lemma 2.9 does not play an essential role in the paper: we could have as well developed the theory making in Definition 2.6 the a priori assumption that the mass $\|T\|$ of any metric functional T is concentrated on a σ-compact set (this assumption plays a role in Lemma 5.3, Theorem 5.6 and Theorem 4.3).

3. Currents

In this section we introduce a particular class of metric functionals with finite mass, characterized by three independent axioms of linearity, continuity and locality. We conjecture that in the Euclidean case these axioms characterize, for metric functionals with compact support, the flat currents with finite mass in the sense of Federer–Fleming; this problem, which is not relevant for the development of our theory, is discussed in Appendix A.

Definition 3.1 (currents). Let $k\geqslant0$ be an integer. The vector space $\mathbf{M}_k(E)$ of k-dimensional currents in E is the set of all k-dimensional metric functionals with finite mass satisfying:

(i) T is multilinear in $(f,\pi_1,...,\pi_k)$;

(ii) $\lim_{i\to\infty}T(f,\pi_1^i,...,\pi_k^i)=T(f,\pi_1,...,\pi_k)$ whenever $\pi_j^i\to\pi_j$ pointwise in E with $\mathrm{Lip}(\pi_j^i)\leqslant C$ for some constant C;

(iii) $T(f,\pi_1,...,\pi_k)=0$ if for some $i\in\{1,...,k\}$ the function π_i is constant on a neighbourhood of $\{f\neq0\}$.

The independence of the three axioms is shown by the following three metric functionals with finite mass:

$$T_1(f,\pi):=\left|\int_{\mathbf{R}}f\pi'e^{-t^2}\,dt\right|,$$

$$T_2(f,\pi_1,\pi_2):=\int_{\mathbf{R}^2}f\,\frac{\partial\pi_1}{\partial x}\,\frac{\partial\pi_2}{\partial y}\,e^{-x^2-y^2}\,dx\,dy,$$

$$T_3(f,\pi):=\int_{\mathbf{R}}f(t)(\pi(t+1)-\pi(t))e^{-t^2}\,dt.$$

In fact, T_1 fails to be linear in π, T_2 fails to be continuous (continuity fails at $\pi_1(x,y)=\pi_2(x,y)=x+y$, see the proof of the alternating property in Theorem 3.5), and T_3 fails to be local.

In the following we will use the expressive notation

$$\omega = f\,d\pi = f\,d\pi_1\wedge\ldots\wedge d\pi_k$$

for the elements of $\mathcal{D}^k(E)$; since we will mostly deal with currents in the following, this notation is justified by the fact that any current is alternating in $(\pi_1,\ldots,\pi_k)$ (see (3.2) below).

An important example of a current in Euclidean space is the following.

Example 3.2. Any function $g\in L^1(\mathbf{R}^k)$ induces a top-dimensional current $[\![g]\!]\in\mathbf{M}_k(\mathbf{R}^k)$ defined by

$$[\![g]\!](f\,d\pi_1\wedge\ldots\wedge d\pi_k) := \int_{\mathbf{R}^k} gf\,d\pi_1\wedge\ldots\wedge d\pi_k = \int_{\mathbf{R}^k} gf\det(\nabla\pi)\,dx$$

for any $f\in\mathcal{B}^\infty(\mathbf{R}^k)$, $\pi_1,\ldots,\pi_k\in\mathrm{Lip}(\mathbf{R}^k)$. The definition is well posed because of the Rademacher theorem, which gives $\mathcal{L}^k$-almost everywhere a meaning to $\nabla\pi$. The metric functional $[\![g]\!]$ is continuous by the well-known w*-continuity properties of determinants in the Sobolev space $W^{1,\infty}$ (see for instance [16]); hence $[\![g]\!]$ is a current. It is not hard to prove that $\|[\![g]\!]\|=|g|\mathcal{L}^k$.

In the case $k=2$ the previous example is optimal, in the sense that a functional

$$T(f,\pi_1,\pi_2)=\int_{\mathbf{R}^2} f\det(\nabla\pi)\,d\mu,$$

defined for $f\in\mathcal{B}^\infty(\mathbf{R}^2)$ and $\pi_1,\pi_2\in W^{1,\infty}(\mathbf{R}^2)\cap C^1(\mathbf{R}^2)$, satisfies the continuity property only if μ is absolutely continuous with respect to $\mathcal{L}^2$. This is a consequence of the following result, recently proved by D. Preiss in [53]. The validity of the analogous result in dimension higher than two is still an open problem.

THEOREM 3.3 (Preiss). *Let $\mu\in\mathcal{M}(\mathbf{R}^2)$ and assume that μ is not absolutely continuous with respect to $\mathcal{L}^2$. Then there exists a sequence of continuously differentiable functions $g_h\in\mathrm{Lip}_1(\mathbf{R}^2,\mathbf{R}^2)$ converging pointwise to the identity, and such that*

$$\lim_{h\to\infty}\int_{\mathbf{R}^2}\det(\nabla g_h)\,d\mu<\mu(\mathbf{R}^2).$$

Notice that the 1-dimensional version of the Preiss theorem is easy to obtain: assuming with no loss of generality that μ is singular with respect to $\mathcal{L}^1$, it suffices to define

$$g_h(t):=t-\mathcal{L}^1(A_h\cap(-\infty,t))\quad\text{for all } t\in\mathbf{R},$$

where (A_h) is a sequence of open sets such that $\mathcal{L}^1(A_h)\to 0$, containing an $\mathcal{L}^1$-negligible set on which μ is concentrated.

It is easy to check that $\mathbf{M}_k(E)$, endowed with the norm $\mathbf{M}(T):=\|T\|(E)$, is a Banach space. Notice also that the pushforward map $T\mapsto\varphi_\# T$ and the restriction operator $T\mapsto T\llcorner\omega$ (for $\omega\in\mathcal{D}^k(E)$), defined on the larger class of metric functionals, map currents into currents. As regards the boundary operator, we can give the following definition.

Definition 3.4 (normal currents). Let $k\geqslant 1$ be an integer. We say that $T\in\mathbf{M}_k(E)$ is a normal current if also ∂T is a current, i.e. $\partial T\in\mathbf{M}_{k-1}(E)$. The class of normal currents in E will be denoted by $\mathbf{N}_k(E)$.

Notice that ∂T is always a metric functional satisfying conditions (i) and (ii) above; concerning condition (iii) it can be proved using the stronger locality property stated in Theorem 3.5 below. Hence T is normal if and only if ∂T has finite mass. It is not hard to see that also $\mathbf{N}_k(E)$, endowed with the norm

$$\mathbf{N}(T):=\|T\|(E)+\|\partial T\|(E),$$

is a Banach space.

Now we examine the properties of the canonical extension of a current to $\mathcal{B}^\infty(E)\times[\mathrm{Lip}(E)]^k$, proving also that the action of a current on $\mathcal{D}^k(E)$ satisfies the natural chain and product rules for derivatives. An additional consequence of our axioms is the alternating property in $\pi_1,...,\pi_k$.

THEOREM 3.5. *The extension of any $T\in\mathbf{M}_k(E)$ to $\mathcal{B}^\infty(E)\times[\mathrm{Lip}(E)]^k$ satisfies the following properties*:

(i) (*product and chain rules*) *T is multilinear in $(f,\pi_1,...,\pi_k)$,*

$$T(f\,d\pi_1\wedge...\wedge d\pi_k)+T(\pi_1\,df\wedge...\wedge d\pi_k)=T(1\,d(f\pi_1)\wedge...\wedge d\pi_k) \tag{3.1}$$

whenever $f,\pi_1\in\mathrm{Lip}_b(E)$, and

$$T(f\,d\psi_1(\pi)\wedge...\wedge d\psi_k(\pi))=T(f\det\nabla\psi(\pi)\,d\pi_1\wedge...\wedge d\pi_k) \tag{3.2}$$

whenever $\psi=(\psi_1,...,\psi_k)\in[C^1(\mathbf{R}^k)]^k$ and $\nabla\psi$ is bounded;

(ii) (*continuity*)

$$\lim_{i\to\infty}T(f^i,\pi_1^i,...,\pi_k^i)=T(f,\pi_1,...,\pi_k)$$

whenever $f^i-f\to 0$ in $L^1(E,\|T\|)$ and $\pi_j^i\to\pi_j$ pointwise in E, with $\mathrm{Lip}(\pi_j^i)\leqslant C$ for some constant C;

(iii) (*locality*) *$T(f,\pi_1,...,\pi_k)=0$ if $\{f\neq 0\}=\bigcup_i B_i$ with $B_i\in\mathcal{B}(E)$ and π_i constant on B_i.*

Proof. We prove locality first. Possibly replacing f by $f\chi_{B_i}$ we can assume that π_i is constant on $\{f\neq 0\}$ for some fixed integer i. Assuming with no loss of generality that $\pi_i=0$ on B_i and $\mathrm{Lip}(\pi_j)\leqslant 1$, let us assume by contradiction the existence of $C\subset\{f\neq 0\}$ closed and $\varepsilon>0$ such that $|T(\chi_C\, d\pi)|>\varepsilon$, and let $\delta>0$ be such that $\|T\|(C_\delta\setminus C)<\varepsilon$, where C_δ is the open δ-neighbourhood of C. We set

$$g_t(x):=\max\left\{0, 1-\frac{3}{t}\operatorname{dist}(x,C)\right\},\quad c_t(x):=\operatorname{sign}(x)\max\{0,|x|-t\},$$

and using the finiteness of mass and the continuity axiom we find $t_0\in(0,\delta)$ such that $|T(g_{t_0}\, d\pi)|>\varepsilon$ and $t_1\in(0,t_0)$ such that $|T(g_{t_0}\, d\tilde{\pi})|>\varepsilon$, with $\tilde{\pi}_j=\pi_j$ for $j\neq i$ and $\tilde{\pi}_i=c_{t_1}\circ\pi_i$. Since $\tilde{\pi}_i$ is 0 on C_{t_1} and $\operatorname{spt} g_{t_1}\subset C_{t_1/2}$, the locality axiom (iii) on currents gives $T(g_{t_1}\, d\tilde{\pi})=0$. On the other hand, since $\mathrm{Lip}(\tilde{\pi}_j)\leqslant 1$ we get

$$|T((g_{t_0}-g_{t_1})\, d\tilde{\pi})|\leqslant\int_E|g_{t_0}-g_{t_1}|\, d\|T\|\leqslant\|T\|(C_{t_0}\setminus C)<\varepsilon.$$

This proves that $|T(g_{t_0}\, d\tilde{\pi})|<\varepsilon$ and gives a contradiction.

The continuity property (ii) easily follows by the definition of mass and the continuity axiom (ii) in Definition 3.1.

Using locality and multilinearity we can easily obtain that

$$T(f\, d\pi_1\wedge d\pi_{i-1}\wedge d\psi(\pi_i)\wedge\ldots\wedge d\pi_k)=T(f\psi'(\pi_i)\, d\pi_1\wedge\ldots\wedge d\pi_k) \tag{3.3}$$

whenever $i\in\{1,\ldots,k\}$ and $\psi\in\mathrm{Lip}(\mathbf{R})\cap C^1(\mathbf{R})$; in fact, the proof can be achieved first for affine functions ψ, then for piecewise affine functions ψ, and then for Lipschitz and continuously differentiable functions ψ (see also the proof of (3.2), given below).

Now we prove that T is alternating in $\pi_1,\ldots,\pi_k$; to this aim, it suffices to show that T vanishes if two functions π_i are equal. Assume, to fix the ideas, that $\pi_i=\pi_j$ with $i<j$, and set $\pi_l^k=\pi_l$ if $l\notin\{i,j\}$ and

$$\pi_i^k:=\frac{1}{k}\varphi(k\pi_i),\quad \pi_j^k:=\frac{1}{k}\varphi\left(k\pi_j+\frac{1}{2}\right)$$

where φ is a smooth function in $\mathbf{R}$ such that $\varphi(t)=t$ on $\mathbf{Z}$, $\varphi'\geqslant 0$ is 1-periodic and $\varphi'\equiv 0$ in $\left[0,\frac{1}{2}\right]$. The functions π^k uniformly converge to π, have equi-bounded Lipschitz constants, and since

$$\varphi'(k\pi_i)\varphi'\left(k\pi_j+\tfrac{1}{2}\right)=\varphi'(k\pi_i)\varphi'\left(k\pi_i+\tfrac{1}{2}\right)\equiv 0$$

from (3.3) we obtain that $T(f\, d\pi^k)=0$. Then the continuity property gives $T(f\, d\pi)=0$.

We now prove (3.2). By the axiom (i) and the alternating property just proved, the property is true if ψ is a linear function; if all components of ψ are affine on a common triangulation $\mathcal{T}$ of $\mathbf{R}^k$, representing $\mathbf{R}^k$ as a disjoint union of (Borel) k-simplices Δ and using the locality property (iii) we find

$$\begin{aligned} T(f\,d\psi_1(\pi)\wedge\ldots\wedge d\psi_k(\pi)) &= \sum_{\Delta\in\mathcal{T}} T\llcorner\pi^{-1}(\Delta)(f\,d\psi_1(\pi)\wedge\ldots\wedge d\psi_k(\pi)) \\ &= \sum_{\Delta\in\mathcal{T}} T\llcorner\pi^{-1}(\Delta)(f\det\nabla\psi|_\Delta(\pi)\,d\pi_1\wedge\ldots\wedge d\pi_k) \\ &= T\Big(f\sum_{\Delta\in\mathcal{T}}\det\nabla\psi|_\Delta(\pi)\,\chi_{\pi^{-1}(\Delta)}\,d\pi_1\wedge\ldots\wedge d\pi_k\Big). \end{aligned}$$

In the general case, the proof follows by the continuity property, using piecewise affine approximations ψ_h strongly converging in $W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}^k,\mathbf{R}^k)$ to ψ.

Finally, we prove (3.1); possibly replacing T by $T\llcorner\omega$ with $\omega=d\pi_2\wedge\ldots\wedge d\pi_k$ we can also assume that $k=1$. Setting $S=(f,\pi_1)_\# T\in\mathbf{M}_1(\mathbf{R}^2)$, the identity reduces to

$$S(g_1\,dg_2)+S(g_2\,dg_1)=S(1\,d(g_1g_2)) \tag{3.4}$$

where $g_i\in\mathrm{Lip}_b(\mathbf{R}^2)$ are smooth and $g_1(x,y)=x$ and $g_2(x,y)=y$ in a square $Q\supset(f,\pi)(E)\supset\operatorname{spt}S$. Let $g=g_1g_2$ and let u_h be obtained by linear interpolation of g on a family of regular triangulations $\mathcal{T}_h$ of Q (i.e. such that the smallest angle in the triangulations is uniformly bounded from below). It can be proved (see for instance [15]) that (u_h) strongly converges to g in $W^{1,\infty}(Q)$ as $h\to\infty$, and hence we can represent $u_h(x,y)$ on each $\Delta\in\mathcal{T}_h$ as $a_h^\Delta x+b_h^\Delta y+c^\Delta$, with

$$\lim_{h\to\infty}\sup_{\Delta\in\mathcal{T}_h}\sup_{(x,y)\in\Delta}|g_2-a_h^\Delta|+|g_1-b_h^\Delta|=0.$$

Using the continuity, the locality and the finiteness of mass of S we conclude

$$\begin{aligned} S(1\,dg) &= \lim_{h\to\infty} S(1\,du_h)=\lim_{h\to\infty}\sum_{\Delta\in\mathcal{T}_h} S\llcorner\Delta(a_h^\Delta\,dx)+S\llcorner\Delta(b_h^\Delta\,dy) \\ &= \lim_{h\to\infty}\sum_{\Delta\in\mathcal{T}_h} S\llcorner\Delta(g_2\,dg_1)+S\llcorner\Delta(g_1\,dg_2)=S(g_2\,dg_1)+S(g_1\,dg_2). \qquad\square \end{aligned}$$

A simple consequence of (3.1) is the identity

$$\partial(T\llcorner f)=(\partial T)\llcorner f-T\llcorner df \tag{3.5}$$

for any $f\in\mathrm{Lip}_b(E)$. In particular, $T\llcorner f$ is normal whenever T is normal and $f\in\mathrm{Lip}_b(E)$.

The strengthened locality property stated in Theorem 3.5 has several consequences: first

$$T(f\,d\pi)=T(f'\,d\pi') \quad \text{whenever } f=f',\ \pi=\pi' \text{ on } \operatorname{spt} T, \tag{3.6}$$

and this property can be used to define $\varphi_\# T\in\mathbf{M}_k(F)$ even if $\varphi\in\operatorname{Lip}(\operatorname{spt} T, F)$; in fact, we set

$$\varphi_\# T(f,\pi_1,...,\pi_k):=T(\tilde f,\tilde\pi_1,...,\tilde\pi_k)$$

where $\tilde f\in\operatorname{Lip}_b(E)$ and $\tilde\pi_i\in\operatorname{Lip}(E)$ are extensions to E, with the same Lipschitz constant, of $f\circ\varphi$ and $\pi_i\circ\varphi$. The definition is well posed thanks to (3.6), and still (2.1) and (2.4) hold. The second consequence of the locality property and of the strengthened continuity property is that the (extended) restriction operator $T\mapsto T\llcorner f\,d\tau_1\wedge...\wedge d\tau_m$ maps k-currents into $(k-m)$-currents whenever $f\in\mathcal{B}^\infty(E)$ and $\tau_i\in\operatorname{Lip}(E)$.

Definition 3.6 (weak convergence of currents). We say that a sequence $(T_h)\subset\mathbf{M}_k(E)$ weakly converges to $T\in\mathbf{M}_k(E)$ if T_h pointwise converge to T as metric functionals, i.e.

$$\lim_{h\to\infty} T_h(f\,d\pi)=T(f\,d\pi) \quad \text{for all } f\in\operatorname{Lip}_b(E),\ \pi_i\in\operatorname{Lip}(E),\ i=1,...,k.$$

The mapping $T\mapsto\|T\|(A)$ is lower semicontinuous with respect to the weak convergence for any open set $A\subset E$, because Proposition 2.7 (applied to the restrictions to A) easily gives

$$\|T\|(A)=\sup\left\{\sum_{i=0}^{\infty}|T(f_i\,d\pi^i)|:\sum_{i=0}^{\infty}|f_i|\leqslant\chi_A,\ \sup_{i,j}\operatorname{Lip}(\pi_j^i)\leqslant 1\right\}. \tag{3.7}$$

Notice also that the existence of the pointwise limit for a sequence $(T_h)\subset\mathbf{M}_k(E)$ is not enough to guarantee the existence of a limit current T and hence the weak convergence to T. In fact, suitable equi-continuity assumptions are needed to ensure that condition (ii) in Definition 3.1 and condition (b) in Proposition 2.7 hold in the limit.

The following theorem provides a simple characterization of normal k-dimensional currents in $\mathbf{R}^k$.

THEOREM 3.7 (normal currents in $\mathbf{R}^k$). *For any $T\in\mathbf{N}_k(\mathbf{R}^k)$ there exists a unique $g\in\mathrm{BV}(\mathbf{R}^k)$ such that $T=[\![g]\!]$. Moreover, $\|\partial T\|=|Dg|$, where Dg is the derivative in the sense of distributions of g, and $|Dg|$ denotes its total variation.*

Proof. Let now $T\in\mathbf{N}_k(\mathbf{R}^k)$. We recall that any measure μ with finite total variation in $\mathbf{R}^k$ whose partial derivatives in the sense of distributions are (representable by) measures with finite total variation in $\mathbf{R}^k$ is induced by a function $g\in\mathrm{BV}(\mathbf{R}^k)$. In fact, setting $f_\varepsilon=\mu*\varrho_\varepsilon\in C^\infty(\mathbf{R}^k)$, this family is bounded in $\mathrm{BV}(\mathbf{R}^k)$, and the Rellich theorem for BV-functions (see for instance [30]) provides a sequence (f_{ε_i}) converging in $L^1_{\mathrm{loc}}(\mathbf{R}^k)$ to $g\in\mathrm{BV}(\mathbf{R}^k)$, with $\varepsilon_i\to 0$. Since $f_\varepsilon\mathcal{L}^k$ weakly converge to μ as $\varepsilon\downarrow 0$ we conclude that $\mu=g\mathcal{L}^k$.

Setting

$$\mu(f):=T(f\,dx_1\wedge\ldots\wedge dx_k),\quad f\in\mathcal{B}^\infty(\mathbf{R}^k),$$

we first prove that all directional derivatives of μ are representable by measures. This is a simple consequence of (3.2) and of the fact that T is normal: indeed, for any orthonormal basis $(e_1,\ldots,e_k)$ of $\mathbf{R}^k$ we have

$$\left|\int_{\mathbf{R}^k}\frac{\partial\phi}{\partial e_i}\,d\mu\right|=\left|T\left(\frac{\partial\phi}{\partial e_i}\,d\pi_1\wedge\ldots\wedge d\pi_k\right)\right|=|T(1\,d\phi\wedge d\hat{\pi}_i)|=|\partial T(\phi\,d\hat{\pi}_i)|\leqslant\int_{\mathbf{R}^k}|\phi|\,\|\partial T\|$$

for any $\phi\in C_c^\infty(\mathbf{R}^k)$, where π_i are the projections on the lines spanned by e_i, and $d\hat{\pi}_i=d\pi_1\wedge\ldots\wedge d\pi_{i-1}\wedge d\pi_{i+1}\wedge\ldots\wedge d\pi_k$. This implies that $|D_v\mu|\leqslant\|\partial T\|$ for any unit vector v, whence $\mu=g\mathcal{L}^k$ for some $g\in L^1(\mathbf{R}^k)$ and $|D\mu|\leqslant\|\partial T\|$.

By (3.2) we get

$$T(f\,d\pi_1\wedge\ldots\wedge d\pi_k)=\int_{\mathbf{R}^k}gf\det(\nabla\pi)\,dx$$

for any $f\in\mathcal{B}^\infty(\mathbf{R}^k)$ and any $\pi\in C^1(\mathbf{R}^k,\mathbf{R}^k)$ with $\nabla\pi$ bounded. Using the continuity property, a smoothing argument proves that the equality holds for all $\omega=f\,d\pi\in\mathcal{D}^k(\mathbf{R}^k)$; hence $T=[\![g]\!]$.

Finally, we prove that

$$|\partial T(f\,d\pi_1\wedge\ldots\wedge d\pi_{k-1})|\leqslant\prod_{i=1}^{k-1}\mathrm{Lip}(\pi_i)\int_{\mathbf{R}^k}|f|\,d|Dg|,\tag{3.8}$$

which implies that $\|\partial T\|\leqslant|Dg|$. By a simple smoothing and approximation argument we can assume that f and all functions π_i are smooth and that f has bounded support; denoting by H_π the $(k\times k)$-matrix having $Dg/|Dg|$ and $\nabla\pi_1,\ldots,\nabla\pi_{k-1}$ as rows we have

$$\begin{aligned}\partial T(f\,d\pi_1\wedge\ldots\wedge d\pi_{k-1})&=\int_{\mathbf{R}^k}g\,df\wedge d\pi_1\wedge\ldots\wedge d\pi_{k-1}\\&=\sum_{i=1}^k(-1)^i\int_{\mathbf{R}^k}f\det\left(\frac{\partial\pi}{\partial\hat{x}_i}\right)dD_ig\\&=\sum_{i=1}^k(-1)^i\int_{\mathbf{R}^k}f\frac{D_ig}{|Dg|}\det\left(\frac{\partial\pi}{\partial\hat{x}_i}\right)d|Dg|\\&=-\int_{\mathbf{R}^k}f\det(H_\pi)\,d|Dg|,\end{aligned}$$

whence (3.8) follows using the Hadamard inequality. □

The previous representation result can be easily extended to those k-dimensional currents in $\mathbf{R}^k$ whose mass is absolutely continuous with respect to $\mathcal{L}^k$. Except for $k=1,2$, we do not know whether *all* currents in $\mathbf{M}_k(\mathbf{R}^k)$ satisfy this absolute-continuity property. As the proof of Theorem 3.8 below shows, the validity of this statement is related to the extension of the Preiss theorem to any number of dimensions.

THEOREM 3.8. *A current $T\in\mathbf{M}_k(\mathbf{R}^k)$ is representable as $[\![g]\!]$ for some $g\in L^1(\mathbf{R}^k)$ if and only if $\|T\|\ll\mathcal{L}^k$. For $k=1,2$ the mass of any $T\in\mathbf{M}_k(\mathbf{R}^k)$ is absolutely continuous with respect to $\mathcal{L}^k$.*

Proof. The first part of the statement can be obtained from (3.2) arguing as in the final part of the proof of Theorem 3.7. In order to prove the absolute-continuity property, let us assume that $k=2$. Let

$$\mu(B):=T(\chi_B\,dx_1\wedge dx_2),\quad B\in\mathcal{B}(\mathbf{R}^2),$$

and let $\mu\llcorner A+\mu\llcorner(\mathbf{R}^2\setminus A)$ be the Hahn decomposition of μ. Since T is continuous, by applying Theorem 3.3 to the measures $\mu\llcorner A$ and $-\mu\llcorner(\mathbf{R}^2\setminus A)$, and using (3.2), we obtain that $\mu\ll\mathcal{L}^2$; hence $\mu=g\mathcal{L}^2$ for some $g\in L^1(\mathbf{R}^2)$. In the case $k=1$ the proof is analogous, by the remarks following Theorem 3.3. □

In the following theorem we prove, by a simple projection argument, the absolute-continuity property of normal currents in any metric space E.

THEOREM 3.9 (absolute continuity). *Let $T\in\mathbf{N}_k(E)$ and let $N\in\mathcal{B}(\mathbf{R}^k)$ be $\mathcal{L}^k$-negligible. Then*

$$\|T\llcorner d\pi\|(\pi^{-1}(N))=0\quad \textit{for all } \pi\in\mathrm{Lip}(E,\mathbf{R}^k). \tag{3.9}$$

Moreover, $\|T\|$ vanishes on Borel $\mathcal{H}^k$-negligible subsets of E.

Proof. Let $L=\pi^{-1}(N)$ and $f\in\mathrm{Lip}_b(E)$; since

$$(T\llcorner d\pi)(f\chi_L)=T\llcorner(f\,d\pi)(\chi_L)=\pi_\#(T\llcorner f)(\chi_N\,dx_1\wedge\ldots\wedge dx_k)$$

and $\pi_\#(T\llcorner f)\in\mathbf{N}_k(\mathbf{R}^k)$, from Theorem 3.7 we conclude that $T\llcorner d\pi(f\chi_L)=0$. Since f is arbitrary we obtain $\|T\llcorner d\pi\|(L)=0$.

If $L\in\mathcal{B}(E)$ is any $\mathcal{H}^k$-negligible set and $\pi\in\mathrm{Lip}(E,\mathbf{R}^k)$, taking into account that $\pi(L)$ (being $\mathcal{H}^k$-negligible) is contained in a Lebesgue-negligible Borel set N we obtain $\|T\llcorner d\pi\|(L)\leqslant\|T\llcorner d\pi\|(\pi^{-1}(N))=0$. From (2.6) we conclude that $\|T\|(L)=0$. □

4. Rectifiable currents

In this section we define the class of rectifiable currents. We first give an intrinsic definition and then, as in the classical theory, we compare it with a parametric one adopted, with minor variants, in [20].

We say that an $\mathcal{H}^k$-measurable set $S \subset E$ is *countably $\mathcal{H}^k$-rectifiable* if there exist sets $A_i \subset \mathbf{R}^k$ and Lipschitz functions $f_i: A_i \to E$ such that

$$\mathcal{H}^k\Big(S \setminus \bigcup_{i=0}^{\infty} f_i(A_i)\Big) = 0. \tag{4.1}$$

It is not hard to prove that any countably $\mathcal{H}^k$-rectifiable set is separable; by the completeness assumption on E the sets A_i can be required to be closed, or compact.

LEMMA 4.1. *Let $S \subset E$ be countably $\mathcal{H}^k$-rectifiable. Then there exist finitely or countably many compact sets $K_i \subset \mathbf{R}^k$ and bi-Lipschitz maps $f_i: K_i \to S$ such that $f_i(K_i)$ are pairwise disjoint and $\mathcal{H}^k(S \setminus \bigcup_i f_i(K_i)) = 0$.*

Proof. By Lemma 4 of [38] we can find compact sets $K_i \subset \mathbf{R}^k$ and bi-Lipschitz maps $f_i: K_i \to E$ such that $S \subset \bigcup_i f_i(K_i)$, up to $\mathcal{H}^k$-negligible sets. Then, setting $B_0 = K_0$ and

$$B_i := K_i \setminus f^{-1}\Big(S \cap \bigcup_{j<i} f_j(K_j)\Big) \in \mathcal{B}(\mathbf{R}^k) \quad \text{for all } i \geqslant 1,$$

we represent $\mathcal{H}^k$-almost all of S as the disjoint union of $f_i(B_i)$. For any $i \in \mathbf{N}$, representing $\mathcal{L}^k$-almost all of B_i by a disjoint union of compact sets the proof is achieved. □

Definition 4.2 (rectifiable currents). Let $k \geqslant 1$ be integer and let $T \in \mathbf{M}_k(E)$; we say that T is rectifiable if

(a) $\|T\|$ is concentrated on a countably $\mathcal{H}^k$-rectifiable set;

(b) $\|T\|$ vanishes on $\mathcal{H}^k$-negligible Borel sets.

We say that a rectifiable current T is integer-rectifiable if for any $\varphi \in \mathrm{Lip}(E, \mathbf{R}^k)$ and any open set $A \subset E$ we have $\varphi_\#(T \llcorner A) = [\![\theta]\!]$ for some $\theta \in L^1(\mathbf{R}^k, \mathbf{Z})$.

The collections of rectifiable and integer-rectifiable currents will be respectively denoted by $\mathcal{R}_k(E)$ and $\mathcal{I}_k(E)$. The space of integral currents $\mathbf{I}_k(E)$ is defined by

$$\mathbf{I}_k(E) := \mathcal{I}_k(E) \cap \mathbf{N}_k(E).$$

We have proved in the previous section that condition (b) holds if either $k = 1, 2$ or T is normal. We will also prove in Theorem 8.8 (i) that condition (a) can be weakened by requiring that T is concentrated on a Borel set, σ-finite with respect to $\mathcal{H}^{n-1}$, and that, for normal currents T, the integer rectifiability of all projections $\varphi_{\#}(T \llcorner A)$ implies the integer rectifiability of T.

In the case $k=0$ the definition above can be easily extended by requiring the existence of countably many points $x_h \in E$ and $\theta_h \in \mathbf{R}$ (or $\theta_h \in \mathbf{Z}$, in the integer case) such that

$$T(f) = \sum_h \theta_h f(x_h) \quad \text{for all } f \in \mathcal{B}^\infty(E).$$

It follows directly from the definition that $\mathcal{R}_k(E)$ and $\mathcal{I}_k(E)$ are Banach subspaces of $\mathbf{M}_k(E)$.

We will also use the following rectifiability criteria, based on Lipschitz projections, for 0-dimensional currents; the result will be extended to k-dimensional currents in Theorem 8.8.

THEOREM 4.3. *Let $S \in \mathbf{M}_0(E)$. Then*

(i) *$S \in \mathcal{I}_0(E)$ if and only if $S(\chi_A) \in \mathbf{Z}$ for any open set $A \subset E$;*

(ii) *$S \in \mathcal{I}_0(E)$ if and only if $\varphi_{\#} S \in \mathcal{I}_0(\mathbf{R})$ for any $\varphi \in \mathrm{Lip}(E)$;*

(iii) *if $E = \mathbf{R}^N$ for some N, then $S \in \mathcal{R}_0(E)$ if and only if $\varphi_{\#} S \in \mathcal{R}_0(\mathbf{R})$ for any $\varphi \in \mathrm{Lip}(E)$.*

Proof. (i) If $S(\chi_A)$ is integer for any open set A, we set

$$\Sigma := \{x \in E : \|S\|(B_\varrho(x)) \geqslant 1 \text{ for all } \varrho > 0\}$$

and notice that Σ is finite and that, by a continuity argument, $S \llcorner \Sigma \in \mathbf{I}_0(E)$. If $x \notin \Sigma$ we can find a ball B centered at x such that $\|S\|(B) < 1$; as $S(\chi_A)$ is an integer for any open set $A \subset B$, it follows that $S(\chi_A) = 0$, and hence $\|S\|(B) = 0$. A covering argument proves that $\|S\|(K) = 0$ for any compact set $K \subset E \setminus \Sigma$, and Lemma 2.9 implies that S is supported on Σ.

(ii) Let $A \subset E$ be an open set and let φ be the distance function from the complement of A. Since

$$S(\chi_A) = \varphi_{\#} S(\chi_{(0,\infty)}) \in \mathbf{Z}$$

the statement follows from (i).

(iii) The statement follows by Lemma 4.4 below. □

LEMMA 4.4. *Let μ be a signed measure in $\mathbf{R}^N$. Set $\mathcal{Q} = \mathbf{Q}^N \times (\mathbf{Q} \cap (0,\infty))^N$ and consider the countable family of Lipschitz maps*

$$f_{x,\lambda}(y) = \max_{i \leqslant N} \lambda_i |x_i - y_i|, \quad y \in \mathbf{R}^N,$$

where (x,λ) *runs through* $\mathcal{Q}$.

Then $\mu\in\mathcal{R}_0(\mathbf{R}^N)$ *if and only if* $f_{x,\lambda\#}\mu\in\mathcal{R}_0(\mathbf{R})$ *for all* $(x,\lambda)\in\mathcal{Q}$.

Proof. We can assume with no loss of generality that μ has no atom and denote by $\|\cdot\|_\infty$ the l_∞-norm in $\mathbf{R}^N$. Assume μ to be a counterexample to our conclusion and let $K\leqslant N$ be the smallest dimension of a coordinate-parallel subspace of $\mathbf{R}^N$ charged by $|\mu|$, i.e. K is the smallest integer such that there exist $x^0\in\mathbf{R}^N$, $I\subset\{1,...,N\}$ with cardinality $N-K$ such that $|\mu|(P_I(x^0))>0$, where

$$P_I(x^0):=\{x\in\mathbf{R}^N: x_i=x_i^0 \text{ for any } i\in I\}.$$

Since μ has no atom, $K>0$. Replacing μ by $-\mu$ if necessary, we find $\varepsilon>0$ and $x^1\in\mathbf{Q}^N$ such that

$$\mu(M)>3\varepsilon \quad \text{where } M:=P_I(x^0)\cap\{y:\|y-x^1\|_\infty<1\}.$$

Next we choose k sufficiently large such that

$$|\mu|(\widetilde{M})<\varepsilon \quad \text{with } \widetilde{M}:=\{y\in\mathbf{R}^N:\operatorname{dist}_\infty(y,M)\in(0,2/k)\}.$$

Modifying x^1 only in the ith coordinates for $i\in I$ we can, without changing M, in addition assume that $|(x^0-x^1)_i|<1/k$ for all $i\in I$. We define $\lambda\in(\mathbf{Q}\cap(0,\infty))^N$ by $\lambda_i=k$ if $i\in I$, and $\lambda_i=1$ otherwise. Observe that

$$M\subset f^{-1}_{x^1,\lambda}([0,1))\subset M\cup\widetilde{M}.$$

Let T be the countable set on which $\tilde{\mu}=f_{x^1,\lambda\#}\mu$ is concentrated. Due to our minimal choice of K we have $|\mu|(M\cap f^{-1}_{x,\lambda}(s))=0$ for any $s\in\mathbf{R}$; hence our choice of $\widetilde{M}$ gives

$$|\mu|(f^{-1}_{x^1,\lambda}(T\cap[0,1)))\leqslant|\mu|(f^{-1}_{x^1,\lambda}([0,1))\setminus M)<\varepsilon,$$

and we obtain that $|\tilde{\mu}|([0,1))<\varepsilon$. On the other hand,

$$\tilde{\mu}((0,1])=\mu(f^{-1}_{x^1,\lambda}([0,1)))\geqslant\mu(M)-|\mu|(\widetilde{M})\geqslant 2\varepsilon.$$

This contradiction finishes our proof. □

It is also possible to show that this kind of statement fails in any infinite-dimensional situation, for instance when E is L^2. In fact, it could be proved that given any sequence of Lipschitz functions on a Hilbert space, we can always find a continuous probability measure on it whose images under all these maps are purely atomic.

Now we show that rectifiable currents have a parametric representation, as sums of images of rectifiable Euclidean currents (see also [20]).

THEOREM 4.5 (parametric representation). *Let $T\in\mathbf{M}_k(E)$. Then, $T\in\mathcal{R}_k(E)$ (resp. $T\in\mathcal{I}_k(E)$) if and only if there exist a sequence of compact sets K_i, functions $\theta_i\in L^1(\mathbf{R}^k)$ (resp. $\theta_i\in L^1(\mathbf{R}^k,\mathbf{Z})$) with $\operatorname{spt}\theta_i\subset K_i$, and bi-Lipschitz maps $f_i: K_i\to E$, such that*

$$T=\sum_{i=0}^{\infty} f_{i\#}[\![\theta_i]\!] \quad \textit{and} \quad \sum_{i=0}^{\infty}\mathbf{M}(f_{i\#}[\![\theta_i]\!])=\mathbf{M}(T).$$

Moreover, if E is a Banach space, T can be approximated in mass by a sequence of normal currents.

Proof. One implication is trivial, since $f_{i\#}[\![\theta_i]\!]$ is rectifiable, being concentrated on $f_i(K_i)$ (the absolute-continuity property (b) is a consequence of the fact that f_i^{-1}: $f_i(K_i)\to K_i$ is a Lipschitz function), and $\mathcal{R}_k(E)$ is a Banach space. For the integer case, we notice that $T_i=f_{i\#}[\![\theta_i]\!]$ is integer-rectifiable if θ_i takes integer values, because for any $\varphi\in\operatorname{Lip}(E,\mathbf{R}^k)$ and any open set $A\subset E$, setting $h=\varphi\circ f_i: K_i\to\mathbf{R}^k$ and $A'=f_i^{-1}(A)$, we have

$$\varphi_\#(T_i\llcorner A)=h_\#([\![\theta_i]\!]\llcorner A')=\left[\!\!\left[\sum_{x\in h^{-1}(y)\cap A'}\theta_i(x)\operatorname{sign}(\det\nabla h(x))\right]\!\!\right]$$

as a simple consequence of the Euclidean area formula.

Conversely, let us assume that T is rectifiable, let S be a countably $\mathcal{H}^k$-rectifiable set on which $\|T\|$ is concentrated, and let K_i and f_i be given by Lemma 4.1. Let $g_i=f_i^{-1}\in\operatorname{Lip}(S_i,K_i)$, with $S_i=f_i(K_i)$, and set $R_i=g_{i\#}(T\llcorner S_i)$; since $\|R_i\|$ vanishes on $\mathcal{H}^k$-negligible sets, by Theorem 3.7 there exists an integrable function θ_i vanishing outside of K_i such that $R_i=[\![\theta_i]\!]$, with integer values if $T\in\mathcal{I}_k(E)$. Since $f_i\circ g_i(x)=x$ on S_i, the locality property (3.6) of currents implies

$$T\llcorner S_i=(f_i\circ g_i)_\#(T\llcorner S_i)=f_{i\#}R_i=f_{i\#}[\![\theta_i]\!].$$

Adding with respect to i, the desired representation of T follows. Finally, if E is a Banach space we can assume (see [37]) that f_i are Lipschitz functions defined on the whole of $\mathbf{R}^k$ and, by a rescaling argument, that $\operatorname{Lip}(f_i)\leqslant 1$; for $\varepsilon>0$ given, we can choose $\theta_i'\in\operatorname{BV}(\mathbf{R}^k)$ such that $\int_{\mathbf{R}^k}|\theta_i-\theta_i'|\,dx<\varepsilon 2^{-i}$ to obtain that the normal current $\widetilde{T}=\sum_i f_{i\#}[\![\theta_i']\!]$ satisfies $\mathbf{M}(T-\widetilde{T})<\varepsilon$. □

The following theorem provides a canonical (and minimal) set S_T on which a rectifiable current T is concentrated.

THEOREM 4.6. *Let $T\in\mathcal{R}_k(E)$ and set*

$$S_T:=\{x\in E:\Theta_{*k}(\|T\|,x)>0\}. \tag{4.2}$$

Then S_T is countably $\mathcal{H}^k$-rectifiable, and $\|T\|$ is concentrated on S_T; moreover, any Borel set S on which $\|T\|$ is concentrated contains S_T, up to $\mathcal{H}^k$-negligible sets.

Proof. Let S be a countably $\mathcal{H}^k$-rectifiable set on which $\|T\|$ is concentrated; by the Radon–Nikodym theorem we can find a nonnegative function $\theta \in L^1(\mathcal{H}^k \llcorner S)$ such that $\|T\| = \theta \mathcal{H}^k \llcorner S$. By Theorem 5.4 of [7] we obtain that $\Theta_k(\|T\|, x) = \theta(x)$ for $\mathcal{H}^k$-a.e. $x \in S$, while (1.3) gives $\Theta_k(\|T\|, x) = 0$ for $\mathcal{H}^k$-a.e. $x \in E \setminus S$. This proves that $S_T = S \cap \{\theta > 0\}$, up to $\mathcal{H}^k$-negligible sets, and since $\|T\|$ is concentrated on $S \cap \{\theta > 0\}$ the proof is achieved. □

Definition 4.7 (size of a rectifiable current). The *size* of $T \in \mathcal{R}_k(E)$ is defined by

$$\mathbf{S}(T) := \mathcal{H}^k(S_T)$$

where S_T is the set described in Theorem 4.6.

5. Normal currents

In this section we study more closely the class of normal currents; together with rectifiable currents, this is one of the main objects of our investigation, in connection with the isoperimetric inequalities and the general Plateau problem. We start with a useful equi-continuity property which leads, under suitable compactness assumptions on the supports, to a compactness theorem in $\mathbf{N}_k(E)$.

PROPOSITION 5.1 (equi-continuity of normal currents). *Let $T \in \mathbf{N}_k(E)$. Then the estimate*

$$|T(f\,d\pi) - T(f\,d\pi')| \leqslant \sum_{i=1}^{k} \int_E |f|\,|\pi_i - \pi_i'|\,d\|\partial T\| + \mathrm{Lip}(f) \int_{\mathrm{spt}\, f} |\pi_i - \pi_i'|\,d\|T\| \tag{5.1}$$

holds whenever $f, \pi_i, \pi_i' \in \mathrm{Lip}(E)$ and $\mathrm{Lip}(\pi_i) \leqslant 1$, $\mathrm{Lip}(\pi_i') \leqslant 1$.

Proof. Assume first that f, π_i and π_i' are bounded. We set $d\pi_0 = d\pi_2 \wedge ... \wedge d\pi_k$ and, using the definition of ∂T, we find

$$\begin{aligned} &T(f\,d\pi_1 \wedge d\pi_0) - T(f\,d\pi_1' \wedge d\pi_0) \\ &\qquad = T(1\,d(f\pi_1) \wedge d\pi_0) - T(1\,d(f\pi_1') \wedge d\pi_0) - T(\pi_1\,df \wedge d\pi_0) + T(\pi_1'\,df \wedge d\pi_0) \\ &\qquad = \partial T(f\pi_1\,d\pi_0) - \partial T(f\pi_1'\,d\pi_0) - T(\pi_1\,df \wedge d\pi_0) + T(\pi_1'\,df \wedge d\pi_0); \end{aligned}$$

hence using the locality property, $|T(f\,d\pi_1 \wedge d\pi_0) - T(f\,d\pi_1' \wedge d\pi_0)|$ can be estimated with

$$\int_E |f|\,|\pi_1 - \pi_1'|\,d\|\partial T\| + \mathrm{Lip}(f) \int_{\mathrm{spt}\, f} |\pi_1 - \pi_1'|\,d\|T\|.$$

Repeating $k-1$ more times this argument the proof is achieved. In the general case the inequality (5.1) is achieved by a truncation argument, using the continuity axiom. □

THEOREM 5.2 (compactness). *Let $(T_h)\subset \mathbf{N}_k(E)$ be a bounded sequence, and assume that for any integer $p\geqslant 1$ there exists a compact set $K_p\subset E$ such that*

$$\|T_h\|(E\setminus K_p)+\|\partial T_h\|(E\setminus K_p)<\frac{1}{p}\quad \textit{for all } h\in\mathbf{N}.$$

Then, there exists a subsequence $(T_{h(n)})$ converging to a current $T\in\mathbf{N}_k(E)$ satisfying

$$\|T\|\Big(E\setminus\bigcup_{p=1}^{\infty}K_p\Big)+\|\partial T\|\Big(E\setminus\bigcup_{p=1}^{\infty}K_p\Big)=0.$$

Proof. Possibly extracting a subsequence, we can assume the existence of measures $\mu,\nu\in\mathcal{M}(E)$ such that

$$\lim_{h\to\infty}\int_E f\,d\|T_h\|=\int_E f\,d\mu,\quad \lim_{h\to\infty}\int_E f\,d\|\partial T_h\|=\int_E f\,d\nu$$

for any bounded continuous function f in E. It is also easy to see that $(\mu+\nu)(E\setminus K_p)\leqslant 1/p$, and hence $\mu+\nu$ is concentrated on $\bigcup_p K_p$.

Step 1. We will first prove that (T_h) has a pointwise converging subsequence $(T_{h(n)})$; to this aim, by a diagonal argument, we need only to show for any integer $q\geqslant 1$ the existence of a subsequence $(h(n))$ such that

$$\limsup_{n,m\to\infty}|T_{h(n)}(f\,d\pi)-T_{h(m)}(f\,d\pi)|\leqslant\frac{3}{q}$$

whenever $f\,d\pi\in\mathcal{D}^k(E)$ with $|f|\leqslant q$, $\mathrm{Lip}(f)\leqslant 1$ and $\mathrm{Lip}(\pi_i)\leqslant 1$. To this aim, we choose $g\in\mathrm{Lip}(E)$ with bounded support such that

$$\sup_{h\in\mathbf{N}}\mathbf{N}(T_h-T_h\llcorner g)<\frac{1}{q^2}$$

(it suffices to take $g\colon E\to[0,1]$ with $\mathrm{Lip}(g)\leqslant 1$ and $g=1$ in K_{2q^2}), and prove the existence of a subsequence $h(n)$ such that $T_{h(n)}\llcorner g(f\,d\pi)$ converges whenever $f\,d\pi\in\mathcal{D}^k(E)$ with $\mathrm{Lip}(f)\leqslant 1$ and $\mathrm{Lip}(\pi_i)\leqslant 1$.

Endowing $Z=\mathrm{Lip}_1(\bigcup_p K_p)$ with a separable metric inducing uniform convergence on any compact set K_p, we can find a countable dense set $D\subset Z$ and a subsequence $(h(n))$ such that $T_{h(n)}\llcorner g(f\,d\pi)$ converge whenever $f,\pi_1,...,\pi_k$ belong to D. Now we claim that $T_{h(n)}\llcorner g(f\,d\pi)$ converge for $f,\pi_1,...,\pi_k\in\mathrm{Lip}_1(E)$; in fact, for any $\tilde f,\tilde\pi_1,...,\tilde\pi_k\in D$ we can use (5.1) to obtain

$$
\begin{aligned}
\limsup_{n,n'\to\infty} & |T_{h(n)} \llcorner g(f\,d\pi) - T_{h(n')} \llcorner g(f\,d\pi)| \\
& \leqslant 2 \limsup_{h\to\infty} |T_h(f\,d\pi) - T_h(\tilde f\,d\tilde\pi)| \\
& \leqslant \limsup_{h\to\infty} \sum_{i=1}^k \int_E (|f|+1)|\pi_i - \tilde\pi_i|\,d[\|\partial(T_h \llcorner g)\| + \|T \llcorner g\|] \\
& \quad + \int_E |f-\tilde f|\,d\|T_h \llcorner g\| \\
& \leqslant \sum_{i=1}^k \int_{\operatorname{spt} g} (|f|+1)|\pi_i - \tilde\pi_i|\,d\mu \\
& \quad + \int_E (|f|+1)|g|\,|\pi_i - \tilde\pi_i|\,d\nu + \int_E |f-\tilde f|\,|g|\,d\mu.
\end{aligned}
$$

Since $\tilde f$ and $\tilde\pi_i$ are arbitrary, this proves the convergence of $T_{h(n)} \llcorner g(f\,d\pi)$.

Step 2. Since $T_{h(n)}(\omega)$ converge to $T(\omega)$ for any $\omega \in \mathcal{D}^k(E)$, T satisfies conditions (i) and (iii) stated in Definition 3.1. Passing to the limit as $n\to\infty$ in the definition of mass we obtain that both T and ∂T have finite mass, and that $\|T\| \leqslant \mu$, $\|\partial T\| \leqslant \nu$. In order to check the continuity property (ii) in Definition 3.1 we can assume, by the finiteness of mass, that f has bounded support; under this assumption, passing to the limit as $h\to\infty$ in (5.1) we get

$$
|T(f\,d\pi) - T(f\,d\pi')| \leqslant \sum_{i=1}^k \int_E |f|\,|\pi_i - \pi_i'|\,d\mu + \operatorname{Lip}(f) \int_{\operatorname{spt} f} |\pi_i - \pi_i'|\,d\nu,
$$

whenever $\operatorname{Lip}(\pi_i) \leqslant 1$ and $\operatorname{Lip}(\pi_i') \leqslant 1$. This estimate trivially implies the continuity property. □

A simple consequence of the compactness theorem, of (3.5) and of (3.1) is the following localization lemma; in (5.2) we estimate the extra boundary created by the localization.

LEMMA 5.3 (localization). *Let $\varphi \in \operatorname{Lip}(E)$ and let $T \in \mathbf{N}_k(E)$. Then, $T \llcorner \{\varphi > t\} \in \mathbf{N}_k(E)$ and*

$$
\|\partial(T \llcorner \{\varphi > t\})\|(\{\varphi = t\}) \leqslant \frac{d}{d\tau} \|T \llcorner d\varphi\|(\{\varphi \leqslant \tau\})\Big|_{\tau=t} \tag{5.2}
$$

for $\mathcal{L}^1$-a.e. $t \in \mathbf{R}$. Moreover, if S is any σ-compact set on which T and ∂T are concentrated, $T \llcorner \{\varphi > t\}$ and its boundary are concentrated on S for $\mathcal{L}^1$-a.e. $t \in \mathbf{R}$.

Proof. Let $\mu=\|T\|+\|\partial T\|$, let (K_p) be a sequence of pairwise disjoint compact sets whose union covers μ-almost all of E, and set

$$g(t):=\mu(\{\varphi\leqslant t\}),\quad g_p(t):=\mu(K_p\cap\{\varphi\leqslant t\}).$$

We denote by L the set of all $t\in\mathbf{R}$ such that $g'(t)=\sum_p g'_p(t)$ is finite and the derivative in (5.2) exists; these conditions are fulfilled $\mathcal{L}^1$-almost everywhere in $\mathbf{R}$, and hence L has full measure in $\mathbf{R}$.

Let $t\in L$, let $\varepsilon_h\downarrow 0$ and set

$$f_h(s)=\begin{cases}0 & \text{for } s\leqslant t,\\ 1 & \text{for } s\geqslant t+\varepsilon_h,\\ (s-t)/\varepsilon_h & \text{for } s\in[t,t+\varepsilon_h];\end{cases}$$

by (3.5) and the locality property we obtain that the currents $T\llcorner f_h\circ\varphi$ satisfy

$$\partial(T\llcorner f_h\circ\varphi)=\partial T\llcorner f_h\circ\varphi-R_h \tag{5.3}$$

with $R_h=\varepsilon_h^{-1}T\llcorner\chi_{\{t<\varphi<t+\varepsilon_h\}}\,d\varphi$. By (3.5) and locality again we get

$$\partial R_h=\partial(\partial T\llcorner f_h\circ\varphi)=-\frac{1}{\varepsilon_h}\partial T\llcorner\chi_{\{t<\varphi<t+\varepsilon_h\}}\,d\varphi.$$

It is easy to see that our choice of t implies that the sequence (R_h) satisfies the assumptions of Theorem 5.2. Hence, possibly extracting a subsequence, we can assume that (R_h) converges as $h\to\infty$ to some $R\in\mathbf{N}_{k-1}(E)$ such that $\|R\|$ and $\|\partial R\|$ are concentrated on $\bigcup_p K_p$.

Since $\partial T\llcorner f_h(\varphi)$ converge to $\partial T\llcorner\{\varphi>t\}$, passing to the limit as $h\to\infty$ in (5.3) we obtain

$$\partial(T\llcorner\{\varphi>t\})=\partial T\llcorner\{\varphi>t\}-R,$$

and hence $\|\partial(T\llcorner\{\varphi>t\})\|(\{\varphi=t\})\leqslant\mathbf{M}(R)$. Finally, the lower semicontinuity of mass gives

$$\mathbf{M}(R)\leqslant\liminf_{h\to\infty}\mathbf{M}(R_h)\leqslant\frac{d}{d\tau}\|T\llcorner d\varphi\|(\{\varphi\leqslant\tau\})\Big|_{\tau=t}. \qquad\square$$

In the proof of the uniqueness part of the slicing theorem we need the following technical lemma, which allows us to represent the mass as a supremum of a countable family of measures.

LEMMA 5.4. *Let $S\subset E$ be a σ-compact set. Then, there exists a countable set $D\subset \mathrm{Lip}_1(E)\cap\mathrm{Lip}_b(E)$ such that*

$$\|T\| = \bigvee\{\|T\llcorner d\pi\| : \pi_1, ..., \pi_k \in D\} \tag{5.4}$$

whenever T is concentrated on S.

Proof. Let $X=\mathrm{Lip}_b(E)\cap\mathrm{Lip}_1(E)$ and let $S=\bigcup_h K_h$ with $K_h\subset E$ compact. The proof of Proposition 2.7 and a truncation argument based on the continuity axiom give

$$\|T\| = \bigvee\{\|T\llcorner d\pi\| : \pi_1, ..., \pi_k \in X\} \tag{5.5}$$

for any $T\in\mathbf{M}_k(E)$. Let $D_h\subset X$ be a countable set with the property that any $q\in X$ can be approximated by a sequence $q^i\subset D_h$ with $\sup|q^i|$ equi-bounded and q^i uniformly converging to q on K_h. Taking into account (5.5), the proof will be achieved with $D=\bigcup_h D_h$ if we show that

$$\|T\llcorner d\pi\|\llcorner K_h \leqslant \bigvee\{\|T\llcorner dq\| : q_1, ..., q_k \in D_h\} \quad \text{for all } \pi_1, ..., \pi_k \in X. \tag{5.6}$$

Let $f\in\mathcal{B}^\infty(E)$ vanish outside of K_h, and let $\pi_j^i\in D_h$ converge as $i\to\infty$ to π_j as above (i.e. uniformly on K_h with $\sup_h|\pi_j^i|$ equi-bounded). Then, the functions

$$\tilde\pi_j^i(x) := \min_{y\in K_h} \pi_j^i(y)+d(x,y) \in \mathrm{Lip}_1(E)$$

coincide with π_j^i on K_h and pointwise converge to $\tilde\pi_j(x)=\min_{K_h}\pi_j(y)+d(x,y)$. Using the locality property and the continuity axiom we get

$$T(f\,d\pi) = T(f\,d\tilde\pi) = \lim_{i\to\infty} T(f\,d\tilde\pi^i) = \lim_{i\to\infty} T(f\,d\pi^i) \leqslant \int_E |f|\,d\mu_h$$

where μ_h is the right-hand side in (5.6). Since f is arbitrary this proves (5.6). □

In an analogous way we can prove the existence of a countable dense class of open sets.

LEMMA 5.5. *Let $S\subset E$ be a σ-compact set. There exists a countable collection $\mathcal{A}$ of open subsets of E with the following property: for any open set $A\subset E$ there exists a sequence $(A_i)\subset\mathcal{A}$ such that*

$$\lim_{i\to\infty}\chi_{A_i} = \chi_A \quad \textit{in } L^1(\mu) \textit{ for any } \mu\in\mathcal{M}(E) \textit{ concentrated on } S.$$

Proof. Let $S=\bigcup_h K_h$, with K_h compact and increasing, let D be constructed as in the previous lemma, and let us define

$$\mathcal{A} := \left\{\left\{\pi > \tfrac{1}{2}\right\} : \pi\in D\right\}.$$

The characteristic function of any open set $A\subset E$ can be approximated by an increasing sequence $(g_i)\subset \mathrm{Lip}(E)$, with $g_i\geqslant 0$. For any $i\geqslant 1$ we can find $f_i\in D$ such that $|f_i-g_i|<1/i$ on K_i. By the dominated convergence theorem, the characteristic functions of $\{f_i>\frac{1}{2}\}$ converge in $L^1(\mu)$ to the characteristic function of A whenever μ is concentrated on S. $\square$

The following slicing theorem plays a fundamental role in our paper; it allows to represent the restriction of a k-dimensional normal current T as an integral of $(k-m)$-dimensional ones. This is the basic ingredient in many proofs by induction on the dimension of the current.

We denote by $\langle T,\pi,x\rangle$ the sliced currents, $\pi\colon E\to\mathbf{R}^m$ being the slicing map, and characterize them by the property

$$\int_{\mathbf{R}^m}\langle T,\pi,x\rangle\,\psi(x)\,dx=T\llcorner(\psi\circ\pi)\,d\pi\quad\text{for all } \psi\in C_c(\mathbf{R}^k).\tag{5.7}$$

We emphasize that the current-valued map $x\mapsto\langle T,\pi,x\rangle$ will be measurable in the following weak sense: whenever $g\,d\tau\in\mathcal{D}^{k-m}(E)$, the real-valued map

$$x\mapsto\langle T,\pi,x\rangle(g\,d\tau)$$

is $\mathcal{L}^m$-measurable in $\mathbf{R}^m$. This weak measurability property is necessary to give a sense to (5.7) and suffices for our purposes. An analogous remark applies to $x\mapsto\|\langle T,\pi,x\rangle\|$.

THEOREM 5.6 (slicing theorem). *Let $T\in\mathbf{N}_k(E)$, let L be a σ-compact set on which T and ∂T are concentrated, and let $\pi\in\mathrm{Lip}(E,\mathbf{R}^m)$, with $m\leqslant k$.*

(i) *There exist currents $\langle T,\pi,x\rangle\in\mathbf{N}_{k-m}(E)$ such that*

$$\langle T,\pi,x\rangle \text{ and } \partial\langle T,\pi,x\rangle \text{ are concentrated on } L\cap\pi^{-1}(x),\tag{5.8}$$

$$\int_{\mathbf{R}^m}\|\langle T,\pi,x\rangle\|\,dx=\|T\llcorner d\pi\|\tag{5.9}$$

and (5.7) *holds.*

(ii) *If L' is a σ-compact set, and if $T^x\in\mathbf{M}_{k-m}(E)$ are concentrated on L', satisfy* (5.7) *and $x\mapsto\mathbf{M}(T^x)$ is integrable on $\mathbf{R}^k$, then $T^x=\langle T,\pi,x\rangle$ for $\mathcal{L}^m$-a.e. $x\in\mathbf{R}^m$.*

(iii) *If $m=1$, there exists an $\mathcal{L}^1$-negligible set $N\subset\mathbf{R}$ such that*

$$\langle T,\pi,x\rangle=\lim_{y\downarrow x}T\llcorner\frac{\chi_{\{x<\pi<y\}}}{y-x}\,d\pi=(\partial T)\llcorner\{\pi>x\}-\partial(T\llcorner\{\pi>x\})$$

for any $x\in\mathbf{R}\setminus N$. Moreover, $\mathbf{M}(\langle T,\pi,x\rangle)\leqslant\mathrm{Lip}(\pi)\mathbf{M}(T\llcorner\{\pi\leqslant x\})'$ for $\mathcal{L}^1$-a.e. x, and

$$\int_{-\infty}^{\infty}\mathbf{N}(\langle T,\pi,x\rangle)\,dx\leqslant\mathrm{Lip}(\pi)\mathbf{N}(T).\tag{5.10}$$

Proof. Step 1. In the case $m=1$ we take statement (iii) as a definition. The proof of the localization lemma shows that

$$S_x := (\partial T)\llcorner\{\pi > x\} - \partial(T\llcorner\{\pi > x\}) = \lim_{y\downarrow x}\frac{1}{y-x}T\llcorner\chi_{\{x<\pi<y\}}\,d\pi \tag{5.11}$$

for $\mathcal{L}^1$-a.e. x; hence $\operatorname{spt} S_x\subset L\cap\pi^{-1}(x)$ and

$$\mathbf{M}(S_x\llcorner\omega)\leqslant\frac{d}{dt}\|(T\llcorner d\pi)\llcorner\omega\|(\{\pi>t\})\Big|_{t=x}\quad\text{for }\mathcal{L}^1\text{-a.e. }x\in\mathbf{R}$$

whenever $\omega\in\mathcal{D}^p(E)$, $0\leqslant p\leqslant k-1$. By integrating with respect to x we obtain

$$\int_{\mathbf{R}}^{*}\mathbf{M}(S_x\llcorner\omega)\,dx\leqslant\mathbf{M}((T\llcorner d\pi)\llcorner\omega) \tag{5.12}$$

where $\int^*$ denotes the upper integral (we will use also the lower integral $\int_*$ later on).

Now we check (5.7): any function $\psi\in C_c(\mathbf{R})$ can be written as the difference of two bounded functions $\psi_1,\psi_2\in C(\mathbf{R})$ with $\psi_i\geqslant 1$. Setting $\gamma_i(t)=\int_0^t\psi_i(\tau)\,d\tau$, for $i=1,2$ and $\omega\in\mathcal{D}^{k-1}(E)$ we compute

$$\begin{aligned}\int_0^\infty S_x(\omega)\psi_i(x)\,dx &= \int_0^\infty \partial T\llcorner\{\pi>x\}(\omega)\psi_i(x)\,dx-\int_0^\infty\partial(T\llcorner\{\pi>x\})(\omega)\psi_i(x)\,dx\\ &=\int_0^\infty\partial T\llcorner\{\gamma_i\circ\pi>t\}(\omega)\,dt-\int_0^\infty T\llcorner\{\gamma_i\circ\pi>t\}(d\omega)\,dt\\ &=\partial T(\gamma_i^+\circ\pi\omega)-T(\gamma_i^+\circ\pi\,d\omega).\end{aligned}$$

Analogously, using the identity $S_x=\partial(T\llcorner\{\pi\leqslant x\})-\partial T\llcorner\{\pi\leqslant x\}$ we get

$$\int_{-\infty}^0 S_x(\omega)\psi_i(x)\,dx=-\partial T(\gamma_i^-\circ\pi\omega)+T(\gamma_i^-\circ\pi\,d\omega).$$

Hence, setting $\omega=f\,dp$, we obtain

$$\begin{aligned}\int_{\mathbf{R}} S_x(f\,dp)\psi_i(x)\,dx&=\partial T(\gamma_i\circ\pi f\,dp)-T(\gamma_i\circ\pi\,df\wedge dp)\\ &=T(f\,d(\gamma_i\circ\pi)\wedge dp)=T(f\gamma_i'\circ\pi\,d\pi\wedge dp)=T\llcorner\psi_i\circ\pi\,d\pi(f\,dp).\end{aligned}$$

Since $\psi=\psi_1-\psi_2$ this proves (5.7).

By (5.7) we get

$$T\llcorner d\pi(g\,d\tau)=\int_{\mathbf{R}}S_x(g\,d\tau)\,dx\leqslant\prod_{i=1}^{k-1}\operatorname{Lip}(\tau_i)\int_{*\mathbf{R}}\|S_x\|(|g|)\,dx$$

whenever $g\,d\tau\in\mathcal{D}^{k-1}(E)$. The representation formula for the mass and the superadditivity of the lower integral give

$$\|T\llcorner d\pi\|(|g|)\leqslant\int_{*\mathbf{R}}\|S_x\|(|g|)\,dx\quad\text{for all }g\in L^1(E,\|T\llcorner d\pi\|).$$

This, together with (5.12) with $\omega=|g|$, gives the weak measurability of $x\mapsto\|S_x\|$ and (5.9).

To complete the proof of statement (iii) we use the identity

$$\partial\langle T,\pi,x\rangle=-\langle\partial T,\pi,x\rangle,\tag{5.13}$$

and apply (5.9) to the slices of T and ∂T to recover (5.10).

Step 2. In this step we complete the existence of currents $\langle T,\pi,x\rangle$ satisfying (i) by induction with respect to m. Assuming the statement true for some $m\in[1,k-1]$, let us prove it for $m+1$. Let $\pi=(\pi_1,\tilde\pi)$, with $\tilde\pi\in\mathrm{Lip}(E,\mathbf{R}^{m-1})$, and set $x=(y,t)$ and

$$T_t:=\langle T,\pi_1,t\rangle,\quad T_x:=\langle T_t,\tilde\pi,y\rangle.$$

By the induction assumption and (5.12) with $\omega=d\tilde\pi$ we get

$$\int_{\mathbf{R}}^{*}\int_{\mathbf{R}^{m-1}}\mathbf{M}(T_x)\,dy\,dt=\int_{\mathbf{R}}^{*}\mathbf{M}(T_t\llcorner d\tilde\pi)\,dt\leqslant\mathbf{M}(T\llcorner d\pi).\tag{5.14}$$

By applying twice (5.7) we get

$$\int_{\mathbf{R}^m}T_x\psi_1(y)\,\psi_2(t)\,dy\,dt=\int_{\mathbf{R}}T_t\llcorner\psi_1(\tilde\pi)\,d\tilde\pi\,\psi_2(t)\,dt=T\llcorner\psi_1(\tilde\pi)\,\psi_2(\pi_1)\,d\pi$$

whenever $\psi_1\in C_c(\mathbf{R}^{m-1})$ and $\psi_2\in C_c(\mathbf{R})$; then, a simple approximation argument proves that T_x satisfy (5.7). Finally, the equality (5.9) can be deduced from (5.7) and (5.14) arguing as in Step 1.

Step 3. Now we prove the uniqueness of $\langle T,\pi,x\rangle$; let $f\,dp\in\mathcal{D}^{k-m}(E)$ be fixed; denoting by (ϱ_ε) a family of mollifiers, by (5.7) we get

$$T^x(f\,d\pi)=\lim_{\varepsilon\downarrow 0}T(f\varrho_\varepsilon\circ\pi\,d\pi\wedge dp)\quad\text{for }\mathcal{L}^m\text{-a.e. }x\in\mathbf{R}^m.$$

This shows that, for given ω, $T^x(\omega)$ is uniquely determined by (5.7) for $\mathcal{L}^m$-a.e. $x\in\mathbf{R}^m$. Let D be given by Lemma 5.4 with $S=L\cup L'$, and let $N\subset\mathbf{R}^m$ be an $\mathcal{L}^m$-negligible Borel set such that $T^x(f\,d\pi)=\langle T,\pi,x\rangle(f\,d\pi)$ whenever $\pi_i\in D$ and $x\in\mathbf{R}^m\setminus N$. By applying (5.4) to $T^x-\langle T,\pi,x\rangle$ we conclude that $T^x=\langle T,\pi,x\rangle$ for any $x\in\mathbf{R}^m\setminus N$. $\square$

Now we consider the case of (integer-)rectifiable currents, proving that the slicing operator is well defined and preserves the (integer) rectifiability. Our proof of these facts uses only the metric structure of the space; in w*-separable dual spaces a more precise result will be proved in Theorem 9.7 using the coarea formula of [7].

THEOREM 5.7 (slices of rectifiable currents). *Let* $T\in\mathcal{R}_k(E)$ *(resp.* $T\in\mathcal{I}_k(E)$*) and let* $\pi\in\mathrm{Lip}(E,\mathbf{R}^m)$*, with* $1\leqslant m\leqslant k$*. Then there exist currents* $\langle T,\pi,x\rangle\in\mathcal{R}_{k-m}(E)$ *(resp.* $\langle T,\pi,x\rangle\in\mathcal{I}_{k-m}(E)$*) concentrated on* $S_T\cap\pi^{-1}(x)$ *and satisfying* (5.7), (5.9),

$$\langle T\llcorner A,\pi,x\rangle=\langle T,\pi,x\rangle\llcorner A \quad \textit{for all } A\in\mathcal{B}(E) \tag{5.15}$$

for $\mathcal{L}^m$*-a.e.* $x\in\mathbf{R}^m$ *and*

$$\int_{\mathbf{R}^m}\mathbf{S}(\langle T,\pi,x\rangle)\,dx\leqslant c(k,m)\prod_{i=1}^m\mathrm{Lip}(\pi_i)\,\mathbf{S}(T). \tag{5.16}$$

Moreover, if $T^x\in\mathbf{M}_{k-m}(E)$ *are concentrated on* $L\cap\pi^{-1}(x)$ *for some* σ*-compact set* L*, satisfy* (5.7) *and* $\int_{\mathbf{R}^k}\mathbf{M}(T^x)\,dx<\infty$*, then* $T^x=\langle T,\pi,x\rangle$ *for* $\mathcal{L}^m$*-a.e.* $x\in\mathbf{R}^m$*.*

Proof. We construct the slices of the current first under the additional assumption that E is a Banach space. Under this assumption, Theorem 4.5 implies that we can write T as a mass-converging series of normal currents T_h; by applying (5.9) to T_h we get

$$\int_{\mathbf{R}^m}\sum_{h=0}^{\infty}\langle T_h,\pi,x\rangle\,dx\leqslant\prod_{i=1}^m\mathrm{Lip}(\pi_i)\sum_{h=0}^{\infty}\mathbf{M}(T_h)=\prod_{i=1}^m\mathrm{Lip}(\pi_i)\,\mathbf{M}(T),$$

and hence $\sum_h\langle T_h,\pi,x\rangle$ converges in $\mathbf{M}_{k-m}(E)$ for $\mathcal{L}^m$-a.e. $x\in\mathbf{R}^m$. Denoting by $\langle T,\pi,x\rangle$ the sum, obviously (5.7), (5.9) and condition (b) in Definition 4.2 follow by a limiting argument. Since $\langle T_h,\pi,x\rangle$ are concentrated on $\pi^{-1}(x)$, the same is true for $\langle T,\pi,x\rangle$. In the general case, we can assume by Lemma 2.9 that $F=\operatorname{spt}T$ is separable; we choose an isometry j embedding F into l_∞ and define

$$\langle T,\pi,t\rangle:=j_{\#}^{-1}\langle j_{\#}T,\tilde\pi,t\rangle \quad \text{for all } t\in\mathbf{R}$$

where $\tilde\pi$ is a Lipschitz extension to l_∞ of $\pi\circ j^{-1}:j(F)\to\mathbf{R}$. It is easy to check that (5.7) and (5.9) still hold, and that $\langle T,\pi,t\rangle$ are concentrated on $\pi^{-1}(x)$. Moreover, since (5.9) gives

$$\int_{\mathbf{R}^m}\|\langle T,\pi,x\rangle\|(E\setminus S_T)\,dx\leqslant\prod_{i=1}^m\mathrm{Lip}(\pi_i)\,\|T\|(E\setminus S_T)=0$$

we obtain that $\langle T,\pi,x\rangle$ is concentrated on S_T for $\mathcal{L}^m$-a.e. $x\in\mathbf{R}^m$. Using this property, the inequality (see Theorem 2.10.25 of [23])

$$\int_{\mathbf{R}^m}\mathcal{H}^{k-m}(S_T\cap\pi^{-1}(x))\,dx\leqslant c(k,m)\prod_{i=1}^m\mathrm{Lip}(\pi_i)\,\mathcal{H}^k(S_T)$$

and Theorem 4.6 imply (5.16).

The uniqueness of $\langle T, \pi, x\rangle$ can be proved arguing as in Theorem 5.6 (ii). The uniqueness property easily implies the validity for $\mathcal{L}^m$-a.e. $x\in\mathbf{R}^m$ of the identity

$$\langle T\llcorner A, \pi, x\rangle = \langle T, \pi, x\rangle\llcorner A$$

for any $A\in\mathcal{B}(E)$ fixed. Let $\mathcal{A}$ be given by Lemma 5.5 and let $N\subset\mathbf{R}^m$ be an $\mathcal{L}^m$-negligible set such that the identity above holds for any $A\in\mathcal{A}$ and any $x\in\mathbf{R}^m\setminus N$. By Lemma 5.5 we infer that the identity holds for any open set $A\subset E$ and any $x\in\mathbf{R}^m\setminus N$, whence (5.15) follows.

Finally, we show that $\langle T, \pi, x\rangle\in\mathcal{I}_{k-m}(E)$ for $\mathcal{L}^m$-a.e. $x\in\mathbf{R}^m$ if $T\in\mathcal{I}_k(E)$. The proof relies on the well-known fact that this property is true in the Euclidean case, as a consequence of the Euclidean coarea formula; see also Theorem 9.7, where this property is proved in a much more general setting. By Theorem 4.5 we can assume with no loss of generality that $T=f_\#[\![\theta]\!]$ for some integer-valued $\theta\in L^1(\mathbf{R}^k)$ vanishing outside of a compact set K, and that $f\colon K\to E$ is bi-Lipschitz. Then, it is easy to check that

$$T^x := f_\#\langle[\![\theta]\!], \pi\circ f, x\rangle$$

are concentrated on $f(K)\cap\pi^{-1}(x)$, satisfy (5.7), and that $\int_{\mathbf{R}^m}\mathbf{M}(T^x)\,dx<\infty$. Hence

$$\langle T, \pi, x\rangle = T^x\in\mathcal{I}_{k-m}(E) \quad \text{for } \mathcal{L}^m\text{-a.e. } x\in\mathbf{R}^m. \qquad \square$$

We conclude this section with two technical lemmas about slices, which will be used in §8. The first one shows that the slicing operator, when iterated, produces lower-dimensional slices of the original current; the second one shows that in some sense the slicing operator and the projection operator commute if the slicing and projection maps are properly chosen.

LEMMA 5.8 (iterated slices). *Let $T\in\mathcal{R}_k(E)\cup\mathbf{N}_k(E)$, $1\leqslant m<k$, $\pi\in\mathrm{Lip}(E,\mathbf{R}^m)$ and $T_t=\langle T, \pi, t\rangle$. Then, for any $n\in[1, k-m]$ and any $\varphi\in\mathrm{Lip}(E,\mathbf{R}^n)$ we have*

$$\langle T, (\pi,\varphi), (t,y)\rangle = \langle T_t, \varphi, y\rangle \quad \textit{for } \mathcal{L}^{m+n}\textit{-a.e. } (t,y)\in\mathbf{R}^{m+n}.$$

Proof. The proof easily follows by the characterization of slices based on (5.7). $\square$

LEMMA 5.9 (slices of projections and projections of slices). *Let $m\in[1,k]$, $n>m$, $S\in\mathcal{R}_k(E)$, $\varphi\in\mathrm{Lip}(E,\mathbf{R}^{n-m})$ and $\pi\in\mathrm{Lip}(E,\mathbf{R}^m)$. Then*

$$q_\#\langle(\varphi,\pi)_\# S, p, t\rangle = \varphi_\#\langle S, \pi, t\rangle \quad \textit{for } \mathcal{L}^m\textit{-a.e. } t\in\mathbf{R}^m,$$

where $p: \mathbf{R}^n \to \mathbf{R}^m$ *and* $q: \mathbf{R}^n \to \mathbf{R}^{n-m}$ *are respectively the projections on the last* m *coordinates and on the first* $n-m$ *coordinates.*

Proof. Set $\phi=(\varphi,\pi)$, let $f\,d\tau\in\mathcal{D}^{k-m}(\mathbf{R}^{n-m})$ and let $g\in C_c^\infty(\mathbf{R}^m)$ be fixed. By the same argument used in the proof of Theorem 5.6 (ii) we need only to prove that

$$\int_{\mathbf{R}^m} g(x)\, q_\# \langle \phi_\# S, p, t\rangle (f\,d\tau)\,dx = \int_{\mathbf{R}^m} g(x)\,\varphi_\#\langle S,\pi,t\rangle(f\,d\tau)\,dx. \tag{5.17}$$

Using (5.7) we obtain that the right-hand side in (5.17) is equal to

$$\int_{\mathbf{R}^m} g(x)\langle S,\pi,x\rangle(f\circ\varphi\, d(\tau\circ\varphi))\,dx = S(f\circ\varphi\cdot g\circ\pi\, d\pi\wedge d(\tau\circ\varphi)).$$

On the other hand, a similar argument implies that the left-hand side is equal to

$$\begin{aligned}\int_{\mathbf{R}^m} g(x)\langle\phi_\# S,p,x\rangle(f\circ q\, d(\tau\circ q))\,dx &= \phi_\# S(f\circ q\cdot g\circ p\, dp\wedge d(\tau\circ q))\\ &= S(f\circ\varphi\cdot g\circ\pi\, d\pi\wedge d(\tau\circ\varphi))\end{aligned}$$

because $q\circ\phi=\varphi$ and $p\circ\phi=\pi$. □

We conclude this section by noticing that in the special case when $k=m$ and $\pi=\varphi$ an analogous formula holds with p equal to the identity map, i.e.

$$\langle\varphi_\# S,p,x\rangle = \varphi_\#\langle S,\varphi,x\rangle \quad \text{for } \mathcal{L}^k\text{-a.e. } x\in\mathbf{R}^k. \tag{5.18}$$

6. Compactness in Banach spaces

In the compactness theorem for normal currents seen in the previous section, the existence of a given compact set K containing all the supports of T_h is too strong for some applications. This is the main motivation for the introduction of a *weak** convergence for normal currents in dual Banach spaces, which provides a more general compactness property, proved in Theorem 6.6.

Definition 6.1 (weak* convergence). Let Y be a w*-separable dual space. We say that a sequence $(T_h)\subset\mathbf{M}_k(Y)$ w*-converges to $T\in\mathbf{M}_k(Y)$, and we write $T_h\rightharpoonup T$, if $T_h(f\,d\pi)$ converge to $T(f\,d\pi)$ for any $f\,d\pi\in\mathcal{D}^k(Y)$ with f and π_i Lipschitz and w*-continuous.

The uniqueness of the w*-limit follows by a Lipschitz extension theorem: if A is w*-compact and f is w*-continuous, we can extend f preserving both the Lipschitz constant and the w*-continuity.

THEOREM 6.2. *Let Y be a w^*-separable dual space, let $A\subset Y$ be w^*-compact and let $f\colon A\to\mathbf{R}$ be Lipschitz and w^*-continuous. Then, there exists a uniformly w^*-continuous map $\tilde f\colon Y\to\mathbf{R}$ such that $\tilde f|_A=f$, $\sup|\tilde f|=\sup|f|$ and $\mathrm{Lip}(\tilde f)=\mathrm{Lip}(f)$.*

Proof. Of course, we can assume $f(A)\subset[0,1]$. Using compactness (and metrizability) of the w*-topology on any bounded subset of Y we find a sequence $\{U_n\}_{n\geqslant 0}$ of w*-neighbourhoods of zero such that

$$|f(x)-f(y)|\leqslant 2^{-n}+\mathrm{Lip}(f)\,\mathrm{dist}_{\|\cdot\|}(x-y,U_n)\quad\text{if } x,y\in A,\ n\geqslant 0. \tag{6.1}$$

Clearly, we can also modify this sequence (gradually replacing the U_n by smaller sets if necessary) in a way that additionally

$$U_0=Y\quad\text{and}\quad U_{n+1}+U_{n+1}\subset U_n\quad\text{for all } n\geqslant 0. \tag{6.2}$$

For $x\in Y$ we define

$$d_1(x):=\inf\{2^{-n}:x\in U_n\},\qquad d_2(x):=\min\{2d_1(x),\mathrm{Lip}(f)\|x\|\}.$$

Due to (6.2) we have $d_1(x+y)\leqslant 2\max(d_1(x),d_1(y))$ for any pair of points x,y. This implies by induction with respect to n that $d_1(\sum_1^n x_i)\leqslant 2d_1(x_n)$ provided $d_1(x_1)<d_1(x_2)<\ldots<d_1(x_n)$. We prove also by induction in n that $d_1(\sum_1^n x_i)\leqslant 2\sum_1^n d_1(x_i)$ for any $x_1,\ldots,x_n\in Y$. Indeed, if all values $d_1(x_i)$ are different, then this is a consequence of what was just said. But if $d_1(x_{n-1})=d_1(x_n)$ then the estimate $d_1(x_{n-1})+d_1(x_n)\geqslant d_1(x_{n-1}+x_n)$ shows that the claimed inequality follows from the induction assumption

$$d_1\biggl(\sum_1^{n-2}x_i+(x_{n-1}+x_n)\biggr)\leqslant 2\sum_1^{n-2}d_1(x_i)+2d_1(x_{n-1}+x_n).$$

Now we put for any $x\in Y$

$$d(x):=\inf\biggl\{\sum_{i=1}^n d_2(x):x=\sum_{i=1}^n x_i\biggr\}.$$

We note that

$$|f(x)-f(y)|\leqslant d(x-y)\quad\text{whenever } x,y\in A. \tag{6.3}$$

To see this take an arbitrary representation $x-y=\sum_1^n z_i$. We define S to be the set of those indices i such that $d_2(z_i)=2d_1(z_i)$, and put $z=\sum_{i\in S}z_i$, $\bar z=x-y-z$. Then

$$\mathrm{Lip}(f)\|\bar z\|\leqslant\sum_{i\notin S}\mathrm{Lip}(f)\|z_i\|=\sum_{i\notin S}d_2(z_i).$$

Moreover, $\sum_{i\in S} d_2(z_i)=2\sum_{i\in S} d_1(z_i)\geqslant d_1(z)$. Since $|f(x)-f(y)|\leqslant d_1(z)+\mathrm{Lip}(f)\|\bar z\|$ due to (6.1), we just established (6.3).

Finally, we define our function $\tilde f$ by

$$\tilde f(x) := \inf_{y\in A} f(y)+d(x-y).$$

Since obviously $|d(x-y)-d(\bar x-y)|\leqslant d_2(x-\bar x)$ for any $x,\bar x,y$, we see that $\tilde f(x)-\tilde f(\bar x)\leqslant d_2(x-\bar x)\leqslant \mathrm{Lip}(f)\|x-\bar x\|$. Hence $\mathrm{Lip}(\tilde f)=\mathrm{Lip}(f)$, and due to the w*-continuity of d_1 at zero the function $\tilde f$ is a uniformly w*-continuous one. Moreover, the condition (6.3) ensures that $\tilde f(x)=f(x)$ for each $x\in A$. The function $\min\{f(x),1\}$ satisfies all stated conditions. □

In the following proposition we state some basic properties of the w*-convergence.

PROPOSITION 6.3 (properties of w*-convergence). *Let Y be a w^*-separable dual space and let $(T_h)\subset \mathbf{M}_k(Y)$ be a bounded sequence. Then*

(i) *the w^*-limit is unique*;

(ii) *$T_h \rightharpoonup T$ implies $\mathbf{M}(T)\leqslant\liminf_h \mathbf{M}(T_h)$*;

(iii) *w^*-convergence is equivalent to weak convergence if all currents T_h are supported on a compact set S.*

Proof. (i) The uniqueness of the limit obviously follows from (ii).

To prove (ii) we fix 1-Lipschitz functions π^i_j in E and functions $f_i\in\mathrm{Lip}(E)$ with $\sum|f_i|\leqslant 1$, for $i=1,\dots,p$. By (3.7) we need only to show that

$$\sum_{i=1}^p T(f_i\,d\pi^i)\leqslant \liminf_{h\to\infty}\mathbf{M}(T_h).$$

Let $\varepsilon>0$ and let $K_\varepsilon\subset Y$ be a compact set such that $\|T\|(Y\setminus K_\varepsilon)+\|\partial T\|(Y\setminus K_\varepsilon)<\varepsilon$; since the restrictions of f_i and π^i to K_ε are w*-continuous we can find by Theorem 6.2 w*-continuous extensions $f_{i\varepsilon},\pi^i_{j\varepsilon}$ of $f_i|_{K_\varepsilon},\pi^i_j|_{K_\varepsilon}$. As the condition $\sum_i|f_{i\varepsilon}|\leqslant 1$ need not be satisfied, we define $\hat f_{i\varepsilon}=q_i(f_{1\varepsilon},\dots,f_{p\varepsilon})$, where $q\colon\mathbf{R}^p\to\mathbf{R}^p$ is the orthogonal projection on the convex set $\sum_i|z_i|\leqslant 1$. The convergence of T_h to T implies

$$\sum_{i=1}^p T(\hat f_{\varepsilon i}\,d\pi^i_\varepsilon)=\lim_{h\to\infty}\sum_{i=1}^p T_h(\hat f_{\varepsilon i}\,d\pi^i_\varepsilon)\leqslant\liminf_{h\to\infty}\mathbf{M}(T_h).$$

Since $\hat f_{\varepsilon i}=f_{\varepsilon i}=f_i$ on K_ε, by letting $\varepsilon\downarrow 0$ the inequality follows.

(iii) The equivalence follows by Theorem 6.2 and the locality property (3.6). □

Another link between w*-convergence and weak convergence is given by the following lemma.

LEMMA 6.4. *Let X be a compact metric space, let $C_h \subset X$ and $j_h \in \mathrm{Lip}_1(C_h, Y)$ with*

$$\sup\{\|j_h(x)\| : x \in C_h,\ h \in \mathbf{N}\} < \infty.$$

Let us assume that (C_h) converge to C in the sense of Kuratowski and that $j: C \to Y$ satisfies

$$x_{h(k)} \in C_{h(k)} \to x \quad \Rightarrow \quad \mathrm{w}^*\text{-}\lim_{k\to\infty} j_{h(k)}(x_{h(k)}) = j(x). \tag{6.4}$$

Then, $j \in \mathrm{Lip}_1(C, Y)$ and $S_h \to S$ implies that $j_{h\#}S_h \rightharpoonup j_\# S$ for any bounded sequence $(S_h) \subset \mathbf{N}_k(X)$ with $\operatorname{spt} S_h \subset C_h$.

Proof. The w*-lower semicontinuity of the norm implies $j \in \mathrm{Lip}_1(X, Y)$ and clearly $\operatorname{spt} S \subset C$. Let $f: Y \to \mathbf{R}$ be any w*-continuous Lipschitz map; we claim that for any Lipschitz extension $\tilde f$ of $f \circ j$ we have $\sup_{C_h} |f \circ j_h - \tilde f| \to 0$; in fact, assuming by contradiction that $|f \circ j_h(x_h) - \tilde f(x_h)| \geqslant \varepsilon$ for some $\varepsilon > 0$ and $x_h \in C_h$, we can assume that a subsequence $(x_{h(k)})$ converges to $x \in C$, and hence that $\tilde f(x_{h(k)})$ converge to $\tilde f(x) = f \circ j(x)$; on the other hand, $j_{h(k)}(x_{h(k)})$ w*-converge to $j(x)$, hence $f \circ j_{h(k)}(x_{h(k)})$ converge to $f \circ j(x)$, and a contradiction is found.

Let now $f\, d\pi \in \mathcal{D}^k(Y)$ with f and π_i Lipschitz and w*-continuous, and let $\tilde f, \tilde\pi_i$ be Lipschitz extensions of $f \circ j$, $\pi_i \circ j$ respectively with $\tilde f$ bounded; notice that

$$j_{h\#}S_h(f\, d\pi) - j_\# S(f\, d\pi) = [S_h(f \circ j_h\, d(\pi \circ j_h)) - S_h(\tilde f\, d\tilde\pi)] + [S_h(\tilde f\, d\tilde\pi) - S(\tilde f\, d\tilde\pi)].$$

The equi-continuity of normal currents and the uniform convergence to 0 of $f \circ j_h - \tilde f$ and $\pi_i \circ j_h - \tilde\pi_i$ on C_h imply that the quantity in the first square bracket tends to 0; the second one is also infinitesimal by the weak convergence of S_h to S. □

Definition 6.5 (equi-compactness). A sequence of compact metric spaces (X_h) is called *equi-compact* if for any $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that any space X_h can be covered by at most N balls with radius ε.

Using the equi-compactness assumption and the Gromov–Hausdorff convergence of metric spaces (see [31]), Theorem 5.2 can be generalized as follows.

THEOREM 6.6 (weak* compactness). *Let Y be a w*-separable dual space, let $(T_h) \subset \mathbf{N}_k(Y)$ be a bounded sequence, and assume that for any $\varepsilon > 0$ there exists $R > 0$ such that $K_h = \bar B_R(0) \cap \operatorname{spt} T_h$ are equi-compact and*

$$\sup_{h\in\mathbf{N}} \|T_h\|(Y \setminus K_h) + \|\partial T_h\|(Y \setminus K_h) < \varepsilon.$$

Then, there exists a subsequence $(T_{h(k)})$ w-converging to some $T \in \mathbf{N}_k(Y)$. Moreover, T has compact support if $\operatorname{spt} T_h$ are equi-bounded.*

Proof. Assume first that $\operatorname{spt} T_h$ are equi-bounded and put $K_h = \operatorname{spt} T_h$; since K_h are equi-compact, by Gromov's embedding theorem [31], possibly extracting a subsequence (not relabelled), we can find a compact metric space X and isometric immersions $i_h: K_h \to X$. By our extra assumption on K_h the maps $j_h = i_h^{-1}$ are equi-bounded in $i_h(K_h)$, and we denote by B a closed ball in Y containing all sets $j_h(X)$. Let d_w be a metric inducing in B the w*-topology; since $Y = (B, d_w)$ is compact, possibly extracting a subsequence we can assume the existence of a compact set $C \subset X$ and of $j: C \to B$ such that $C_h = i_h(K_h)$ converge to C in the sense of Kuratowski and (6.4) holds (for instance, this can be proved by taking a Kuratowski limit of a subsequence of the graphs of j_h in $X \times B$). By Theorem 5.2 we can also assume that the currents $S_h = i_{h\#} T_h$ weakly converge as $h \to \infty$ to some current S. By Lemma 6.4 we conclude that $T_h = j_{h\#} S_h$ w*-converge to $T = j_\# S$.

If the supports are not equi-bounded, the proof can be achieved by a standard diagonal argument if we show the existence, for any $\varepsilon > 0$, of a sequence $\widetilde{T}_h$ still satisfying the assumptions of the theorem, with $\operatorname{spt} \widetilde{T}_h$ equi-bounded and $\mathbf{M}(T_h - \widetilde{T}_h)(Y) < \varepsilon$. These currents can be easily obtained setting $\widetilde{T}_h = T_h \llcorner B_{R_h}(0)$, where $R_h \in (R, R+1)$ are chosen in such a way that $\mathbf{M}(\partial \widetilde{T}_h)(Y)$ are equi-bounded. This choice can be done using the localization lemma with $\varphi(x) = \|x\|$. $\square$

7. Metric-space-valued BV-functions

In this section we introduce a class of BV-maps $u: \mathbf{R}^k \to S$, where S is a metric space. We follow essentially the approach developed by L. Ambrosio in [4] but, unlike [4], we will not make any compactness assumption on S, assuming only that S is weakly separable. If $S = \mathbf{M}_0(E)$ we use a Lipschitz approximation theorem for metric-valued BV-maps to prove in Theorem 7.4 the rectifiability of the collection of all atoms of $u(x)$, as x varies in (almost all of) $\mathbf{R}^k$.

Let (S, d) be a weakly separable metric space and let $\mathcal{F} \subset \operatorname{Lip}_b(S)$ be a countable family such that

$$d(x, y) = \sup_{\varphi \in \mathcal{F}} |\varphi(x) - \varphi(y)| \quad \text{for all } x, y \in S. \tag{7.1}$$

Definition 7.1 (functions of metric bounded variation). We say that a function $u: \mathbf{R}^k \to S$ is a function of metric bounded variation, and we write $u \in \operatorname{MBV}(\mathbf{R}^k, S)$, if $\varphi \circ u \in \operatorname{BV}_{\mathrm{loc}}(\mathbf{R}^k)$ for any $\varphi \in \mathcal{F}$ and

$$\|Du\| := \bigvee_{\varphi \in \mathcal{F}} |D(\varphi \circ u)| < \infty.$$

Notice that in the definition above we implicitly make the assumption that $\varphi\circ u$ is Lebesgue-measurable for any $\varphi\in\mathrm{Lip}_1(S)$; since S is a metric space, this condition is easily seen to be equivalent to measurability of u between $\mathbf{R}^k$, endowed with the σ-algebra of Lebesgue-measurable sets, and S, endowed with the Borel σ-algebra. Notice also that, even in the Euclidean case $S=\mathbf{R}^m$, the space MBV is strictly larger than BV, because not even the local integrability of u is required, and is related to the class of generalized functions with bounded variation studied in [22], [55].

The class $\mathrm{MBV}(\mathbf{R}^k,S)$ and $\|Du\|$ are independent of the choice of $\mathcal{F}$; this is a direct consequence of the following lemma. It is also easy to check that $u\in\mathrm{MBV}(\mathbf{R}^k,\mathbf{R})$ if $u\in\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^k,\mathbf{R})$ and $|Du|(\mathbf{R}^k)<\infty$, and in this case $\|Du\|=|Du|$.

LEMMA 7.2. *Let $\mathcal{F}\subset\mathrm{Lip}_b(S)$ be as in (7.1), and let $u\in\mathrm{MBV}(\mathbf{R}^k,S)$ and $\psi\in\mathrm{Lip}_1(S)\cap\mathrm{Lip}_b(S)$. Then $\psi\circ u\in\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^k)$ and*

$$|D(\psi\circ u)|\leqslant\bigvee_{\varphi\in\mathcal{F}}|D(\varphi\circ u)|.$$

In particular, $\|Du\|=\bigvee\{|D(\varphi\circ u)|:\varphi\in\mathrm{Lip}_1(S)\cap\mathrm{Lip}_b(S)\}$.

Proof. Let us first assume $k=1$. Let $A\subset\mathbf{R}$ be an open interval and let $v:A\to\mathbf{R}$ be a bounded function. We denote by L_v the Lebesgue set of v and put $|Dv|(A)=+\infty$ if $v\notin\mathrm{BV}_{\mathrm{loc}}(A)$. It can be easily proved that

$$|Dv|(A)=\sup\left\{\sum_{i=1}^{p-1}|v(t_{i+1})-v(t_i)|:t_1<\ldots<t_p,\,t_i\in A\setminus N\right\}$$

whenever $\mathcal{L}^1(N)=0$ and $N\supset A\setminus L_v$. Choosing

$$N:=(A\setminus L_{\psi\circ u})\cup\bigcup_{\varphi\in\mathcal{F}}[(A\setminus L_{\varphi\circ u})\cup\{t\in A:|D(\varphi\circ u)|(\{t\})>0\}]$$

we get

$$|\psi\circ u(t_{i+1})-\psi\circ u(t_i)|\leqslant\sup_{\varphi\in\mathcal{F}}|\varphi\circ u(t_{i+1})-\varphi\circ u(t_i)|\leqslant\|Du\|((t_{i+1},t_i))$$

whenever $t_i,t_{i+1}\in A\setminus N$. Adding with respect to i and taking the supremum, we obtain that $|D(\psi\circ u)|(A)$ can be estimated with $\|Du\|(A)$. By approximation the same inequality remains true if A is an open set or a Borel set.

In the case $k>1$ the proof follows by the 1-dimensional case recalling the following facts (see [23, 4.5.9(27) and 4.5.9(28)] or [4]): first

$$|Dv|=\bigvee_{\nu\in\mathbf{S}^{k-1}}|D_\nu v|\quad\text{for all }v\in\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^k)\tag{7.2}$$

and the directional total variations $|D_\nu v|$ can be represented as integrals of variations on lines, namely

$$|D_\nu v| = \int_{\pi_\nu} V_u(x,\nu)\, d\mathcal{H}^{k-1}(x) \quad \text{for all } \nu \in \mathbf{S}^{k-1},$$

where π_ν is the hyperplane orthogonal to ν, $u(x,\nu)(t) = u(x+t\nu)$ and

$$V_u(x,\nu)(B) := |Du(x,\nu)|(\{t : x+t\nu \in B\}) \quad \text{for all } B \in \mathcal{B}(\mathbf{R}^k).$$

Hence, for $\nu \in \mathbf{S}^{k-1}$ fixed and $v = \psi \circ u$, using (1.8) of [4] to commute the supremum with the integral we get

$$\begin{aligned} |D_\nu v| &= \int_{\pi_\nu} V_v(x,\nu)\, d\mathcal{H}^{k-1}(x) \leqslant \int_{\pi_\nu} \bigvee_{\varphi \in \mathcal{F}} V_{\varphi\circ u}(x,\nu)\, d\mathcal{H}^{k-1}(x) \\ &= \bigvee_{\varphi \in \mathcal{F}} \int_{\pi_\nu} V_{\varphi\circ u}(x,\nu)\, d\mathcal{H}^{k-1}(x) = \bigvee_{\varphi \in \mathcal{F}} |D(\varphi\circ u)| \leqslant \|Du\|. \end{aligned}$$

Since ν is arbitrary the inequality $|Dv| \leqslant \|Du\|$ follows by (7.2). □

Given $u \in \mathrm{MBV}(\mathbf{R}^k, S)$, we denote by MDu the maximal function of $\|Du\|$, namely

$$MDu(x) := \sup_{\varrho>0} \frac{\|Du\|(B_\varrho(x))}{\omega_k \varrho^k}.$$

By the Besicovitch covering theorem, $\mathcal{L}^k(\{MDu > \lambda\})$ can be easily estimated from above with a dimensional constant times $\|Du\|(\mathbf{R}^k)/\lambda$; hence $MDu(x)$ is finite for $\mathcal{L}^k$-a.e. x. The following lemma provides a Lipschitz property of MBV-functions (reversing the roles of $\mathbf{R}^k$ and S, an analogous property can be used to define Sobolev functions on a metric space, see [33], [34]).

LEMMA 7.3. *Let (S,d) be a weakly separable metric space. Then, for any $u \in \mathrm{MBV}(\mathbf{R}^k, S)$ there exists an $\mathcal{L}^k$-negligible set $N \subset \mathbf{R}^k$ such that*

$$d(u(x), u(y)) \leqslant c[MDu(x) + MDu(y)]|x-y| \quad \textit{for all } x, y \in \mathbf{R}^k \setminus N$$

with c depending only on k.

Proof. Any function $w \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^k)$ satisfies

$$|w(x) - w(y)| \leqslant c(k)[MDw(x) + MDw(y)]|x-y| \quad \text{for all } x, y \in L_w,$$

where L_w is the set of Lebesgue points of w; this is a simple consequence of the estimate

$$\frac{1}{\omega_k \varrho^k} \int_{B_\varrho(x)} \frac{|w(z) - w(x)|}{|z-x|}\, dz \leqslant \int_0^1 \frac{|Dw|(B_{t\varrho}(x))}{\omega_k (t\varrho)^k}\, dt \leqslant MDw(x)$$

for any ball $B_\varrho(x)\subset\mathbf{R}^k$ centered at some point $x\in L_w$ (see for instance (2.5) and Theorem 2.3 of [5]). Taking into account (7.1) and the inequality $MDu\geqslant MD(\varphi\circ u)$, the statement follows with $N=\mathbf{R}^k\setminus\bigcap_{\varphi\in\mathcal{F}}L_{\varphi\circ u}$. □

In the following we endow $\mathrm{Lip}_b(E)$ with the flat norm $\mathbf{F}(\phi)=\sup|\phi|+\mathrm{Lip}(\phi)$, and, by duality, we endow the space $\mathbf{M}_0(E)$ with the flat norm

$$\mathbf{F}(T):=\sup\{T(\phi):\phi\in\mathrm{Lip}_b(E),\,\mathbf{F}(\phi)\leqslant 1\}.$$

If E is a weakly separable metric space it is not hard to see that $\mathbf{M}_0(E)$ is still weakly separable. In fact, assuming $E=l^\infty$ (up to an isometric embedding of $\mathbf{M}_0(E)$ into $\mathbf{M}_0(l^\infty)$), by Theorem 6.2 and Lemma 2.9 we see that

$$\begin{aligned}\mathbf{F}(T)&=\sup\{T(\phi):\phi\in\mathrm{Lip}^*(E)\cap\mathcal{B}^\infty(E),\,\mathbf{F}(\phi)\leqslant 1\}\\&=\sup\{T(\phi):n\geqslant 1,\,\phi\in\mathcal{L}_n(E),\,\mathbf{F}(\phi)\leqslant 1\},\end{aligned}$$

where $\mathrm{Lip}^*(E)$ is the vector subspace of w*-continuous functions in $\mathrm{Lip}(E)$, and $\mathcal{L}_n(E)$ is the subspace consisting of all functions depending only on the first n coordinates of x; since all the sets $\{\phi\in\mathcal{L}_n(E):\mathbf{F}(\phi)\leqslant 1\}$ are separable, when endowed with the topology of uniform convergence on bounded sets, a countable subfamily is easily achieved.

THEOREM 7.4 (rectifiability criterion). *Let E be a weakly separable metric space, let $S=\mathbf{M}_0(E)$ be endowed with the flat norm and let $T\in\mathrm{MBV}(\mathbf{R}^k,S)$. Then, there exists an $\mathcal{L}^k$-negligible set $N\subset\mathbf{R}^k$ such that*

$$\mathcal{R}_K:=\bigcup_{z\in\mathbf{R}^k\setminus N}\{x\in K:\|T(z)\|(\{x\})>0\}$$

is contained in a countably $\mathcal{H}^k$-rectifiable set for any compact set $K\subset E$.

Proof. Let $N_1\subset\mathbf{R}^k$ be given by Lemma 7.3 with $S=\mathbf{M}_0(E)$, $N=N_1\cup\{MDT=\infty\}$, $K\subset E$ compact and $\varepsilon,\delta>0$. For simplicity we use the notation T_z for $T(z)$, while $T_z(\phi)$ will stand for $\int_E\phi\,dT_z$.

We define $Z_{\varepsilon,\delta}$ as the collection of points $z\in\mathbf{R}^k\setminus N$ such that $MDT(z)<1/2\varepsilon$ and

$$\|T_z\|(\{x\})\geqslant\varepsilon\quad\Rightarrow\quad\|T_z\|(B_{3\delta}(x)\setminus\{x\})\leqslant\tfrac{1}{3}\varepsilon$$

for any $x\in K$. Setting $\mathcal{R}_{\varepsilon,\delta}=\{x\in K:\|T_z\|(\{x\})\geqslant\varepsilon$ for some $z\in Z_{\varepsilon,\delta}\}$, we notice that $\mathcal{R}_K=\bigcup_{\varepsilon,\delta>0}\mathcal{R}_{\varepsilon,\delta}$; hence it suffices to prove that $\mathcal{R}_{\varepsilon,\delta}$ is contained in a countably $\mathcal{H}^k$-rectifiable set.

Denoting by B any subset of $\mathcal{R}_{\varepsilon,\delta}$ with diameter less than δ, we now check that

$$d(x,x')\leqslant\frac{3c(k)(\delta+1)}{\varepsilon^2}|z-z'|\tag{7.3}$$

whenever $x, x' \in B$, $\|T_z\|(\{x\}) \geqslant \varepsilon$ and $\|T_{z'}\|(\{x'\}) \geqslant \varepsilon$ for some $z, z' \in Z_{\varepsilon,\delta}$. In fact, setting $d = d(x, x') \leqslant \delta$, we can define a function $\phi(y)$ equal to $d(y, x)$ in $B_d(x)$, equal to 0 in $E \setminus B_{2\delta}(x)$ with $\sup|\phi| = d$, $\mathrm{Lip}(\phi) \leqslant 1$; since

$$|T_z(\phi)| \leqslant \tfrac{1}{3}\varepsilon d, \quad |T_{z'}(\phi)| \geqslant \varepsilon d - \tfrac{1}{3}\varepsilon d,$$

we get

$$\tfrac{1}{3}\varepsilon d(x, x') \leqslant |T_{z'}(\phi) - T_z(\phi)| \leqslant \frac{c(k)(\delta+1)}{\varepsilon}|z - z'|.$$

By (7.3) it follows that for any $z \in Z_{\varepsilon,\delta}$ there exists at most one $x = f(z) \in B$ such that $\|T_z\|(\{x\}) \geqslant \varepsilon$; moreover, denoting by D the domain of f, the map $f: D \to B$ is Lipschitz and onto, and hence B is contained in the countably $\mathcal{H}^k$-rectifiable set $f(\bar{D})$. A covering argument proves that $\mathcal{R}_{\varepsilon,\delta}$ is contained in a countably $\mathcal{H}^k$-rectifiable set. □

Actually, it could be proved that, for a suitable choice of N, the set $\mathcal{R}_K$ is *universally measurable* in E, i.e., for any $\mu \in \mathcal{M}(E)$ it belongs to the completion of $\mathcal{B}(E)$ with respect to μ. The proof follows by the projection theorem (see [23, 2.2.12]), checking first that the set

$$\mathcal{R}'_K := \{(z, x) \in (\mathbf{R}^k \setminus N) \times K : \|T_z\|(\{x\}) > 0\}$$

belongs to $\mathcal{B}(\mathbf{R}^k) \otimes \mathcal{B}(E)$, and then noticing that $\mathcal{R}_K$ is the projection of $\mathcal{R}'_K$ on E. Since the projection theorem is a quite sophisticated measure-theoretic result, we preferred to state Theorem 7.4 in a weaker form, which is actually largely sufficient for our purposes.

8. Closure and boundary-rectifiability theorems

In this section we prove the classical closure and boundary-rectifiability theorems for integral currents, proved in the Euclidean case by H. Federer and W. H. Fleming in [24] (see also [58], [61]). Actually, we prove a more general closure property for rectifiable currents with equi-bounded masses and sizes, proved in the Euclidean case by F. J. Almgren in [1] using multivalued function theory. We also provide new characterizations of integer-rectifiable currents based on the Lipschitz projections.

The basic ingredient of our proofs is the following theorem, which allows us to deduce rectifiability of a k-current from the rectifiability of its 0-dimensional slices (for Euclidean currents in general coefficient groups, a similar result has been obtained by B. White in [62]). The proof is based on Theorem 7.4, the slicing theorem and the key observation, due to R. Jerrard in the Euclidean context (see [36]), that $x \mapsto \langle T, \pi, x\rangle$ is a BV-map whenever $T \in \mathbf{N}_k(E)$ and $\pi \in \mathrm{Lip}(E, \mathbf{R}^k)$.

THEOREM 8.1 (rectifiability and rectifiability of slices). *Let* $T\in\mathbf{N}_k(E)$. *Then* $T\in\mathcal{R}_k(E)$ *if and only if*

$$\text{for any } \pi\in\mathrm{Lip}(E,\mathbf{R}^k),\quad \langle T,\pi,x\rangle\in\mathcal{R}_0(E)\quad \text{for } \mathcal{L}^k\text{-a.e. } x\in\mathbf{R}^k. \tag{8.1}$$

Moreover, $T\in\mathbf{I}_k(E)$ *if and only if* (8.1) *holds with* $\mathbf{I}_0(E)$ *in place of* $\mathcal{R}_0(E)$.

Proof. Let $\pi\in\mathrm{Lip}(E,\mathbf{R}^k)$ with $\mathrm{Lip}(\pi_i)\leqslant 1$; we will first prove that for any $T\in\mathbf{N}_k(E)$ the map $x\mapsto T_x=\langle T,\pi,x\rangle$ belongs to $\mathrm{MBV}(\mathbf{R}^k,S)$, where S as in Theorem 7.4 is $\mathbf{M}_0(E)$ endowed with the flat norm. Let $\psi\in C_0^1(\mathbf{R}^k)$ and $\phi\in\mathrm{Lip}_b(E)$ with $\mathbf{F}(\phi)\leqslant 1$; using (3.2) we compute

$$\begin{aligned}(-1)^{i-1}\int_{\mathbf{R}^k}T_x(\phi)\frac{\partial\psi}{\partial x_i}(x)\,dx&=(-1)^{i-1}T\llcorner d\pi\left(\phi\frac{\partial\psi}{\partial x_i}\circ\pi\right)=T(\phi\, d(\psi\circ\pi)\wedge d\hat{\pi}_i)\\&=\partial T(\phi(\psi\circ\pi)\,d\hat{\pi}_i)-T(\psi\circ\pi\, d\phi\wedge d\hat{\pi}_i)\\&\leqslant\|\partial T\|(\psi\circ\pi)+\|T\|(\psi\circ\pi),\end{aligned}$$

where

$$d\hat{\pi}_i=d\pi_1\wedge\ldots\wedge d\pi_{i-1}\wedge d\pi_{i+1}\wedge\ldots\wedge d\pi_k.$$

Since ψ is arbitrary, this proves that $x\mapsto T_x(\phi)$ belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^k)$ and

$$|DT_x(\phi)|\leqslant k\pi_{\#}\|T\|+k\pi_{\#}\|\partial T\|.$$

Since ϕ is arbitrary, this proves that $T_x\in\mathrm{MBV}(\mathbf{R}^k,S)$.

Now we consider the rectifiable case. By Theorem 5.7, the rectifiability of T implies the generic rectifiability of T_x. Conversely, let L be a σ-compact set on which $\|T\|$ is concentrated; by Theorem 7.4 there exists an $\mathcal{L}^k$-negligible set $N\subset\mathbf{R}^k$ such that

$$\bigcup_{x\in\mathbf{R}^k\setminus N}\{y\in L:\|T_x\|(\{y\})>0\}$$

is contained in a countably $\mathcal{H}^k$-rectifiable set $\mathcal{R}_\pi$. Now, if $T_x\in\mathcal{R}_0(E)$ for $\mathcal{L}^k$-a.e. x, by (5.9) we infer

$$\|T\llcorner d\pi\|(E\setminus\mathcal{R}_\pi)=\|T\llcorner d\pi\|(L\setminus\mathcal{R}_\pi)=\int_{\mathbf{R}^k}\|T_x\|(L\setminus\mathcal{R}_\pi)\,dx=0.$$

Hence, $T\llcorner d\pi$ is concentrated on a countably $\mathcal{H}^k$-rectifiable set for any $\pi\in\mathrm{Lip}(E,\mathbf{R}^k)$. By Lemma 5.4 this implies the same for T, and hence T is rectifiable.

Finally, we consider the integer-rectifiable case. The proof is straightforward in the special case when $E=\mathbf{R}^k$ and $p=\pi: E\to\mathbf{R}^k$ is the identity map (in this case, representing T as $[\![\theta]\!]$, $\langle T,\pi,x\rangle$ is the Dirac delta at x with multiplicity $\theta(x)$ for $\mathcal{L}^k$-a.e. $x\in\mathbf{R}^k$).

In the general case, one implication follows by Theorem 5.7. Conversely, let us assume that the slices of T are generically integer-rectifiable. For $A\in\mathcal{B}(E)$ and $\varphi\in \mathrm{Lip}(E,\mathbf{R}^k)$ given, from (5.18) and (5.15) we infer

$$\langle\varphi_\#(T\llcorner A),p,x\rangle=\varphi_\#\langle T\llcorner A,\varphi,x\rangle=\varphi_\#(\langle T,\varphi,x\rangle\llcorner A)\in\mathcal{I}_0(\mathbf{R}^k)$$

for $\mathcal{L}^k$-a.e. $x\in\mathbf{R}^k$, whence $\varphi_\#(T\llcorner A)\in\mathcal{I}_k(\mathbf{R}^k)$. □

Remark 8.2. Analogously, if E is a w*-separable dual space we can say that $T\in \mathcal{R}_k(E)$ (resp. $T\in\mathcal{I}_k(E)$) if

$$\langle T,\pi,x\rangle\in\mathcal{R}_0(E)\ \ (\text{resp. } \mathcal{I}_0(E))\quad \text{for } \mathcal{L}^k\text{-a.e. } x\in\mathbf{R}^k$$

for any w*-continuous map $\pi\in\mathrm{Lip}(E,\mathbf{R}^k)$. In fact, this condition implies that $T\llcorner d\pi$ is concentrated on a countably $\mathcal{H}^k$-rectifiable set for any such π, and Lemma 5.4 together with Theorem 6.2 imply the existence of a sequence of w*-continuous Lipschitz functions $\pi^i: E\to\mathbf{R}^k$ such that

$$\|T\|=\bigvee_{i\in\mathbf{N}}\|T\llcorner d\pi^i\|.$$

We also notice that in the Euclidean case $E=\mathbf{R}^n$ it suffices to consider the canonical linear projection and correspondingly the slices along the coordinate axes (in fact, our notion of mass is comparable with the Federer–Fleming one, see Appendix A).

The following technical proposition will be used in the proof, by induction on the dimension, of the closure theorem.

PROPOSITION 8.3. *Let $(T_h)\subset\mathbf{N}_k(E)$ be a bounded sequence weakly converging to $T\in\mathbf{N}_k(E)$ and let $\pi\in\mathrm{Lip}(E)$. Then, for $\mathcal{L}^1$-a.e. $t\in\mathbf{R}$ there exists a subsequence $(h(n))$ such that $(\langle T_{h(n)},\pi,t\rangle)$ is bounded in $\mathbf{N}_{k-1}(E)$ and*

$$\lim_{n\to\infty}\langle T_{h(n)},\pi,t\rangle=\langle T,\pi,t\rangle.$$

In addition, if $T_h\in\mathcal{R}_k(E)$ and $\mathbf{S}(T_h)$ are equi-bounded, the subsequence $(h(n))$ can be chosen in such a way that $\mathbf{S}(\langle T_{h(n)},\pi,t\rangle)$ are equi-bounded.

Proof. We first prove the existence of a subsequence $h(n)$ such that $\langle T_{h(n)},\pi,t\rangle$ converge to $\langle T,\pi,t\rangle$ for $\mathcal{L}^1$-a.e. $t\in\mathbf{R}$. Recalling Proposition 5.6 (iii), we need only to prove that

$$\lim_{n\to\infty}T_{h(n)}\llcorner\{\pi>t\}=T\llcorner\{\pi>t\},\quad \lim_{n\to\infty}\partial T_{h(n)}\llcorner\{\pi>t\}=\partial T\llcorner\{\pi>t\}\tag{8.2}$$

for $\mathcal{L}^1$-a.e. $t\in\mathbf{R}$. Let $\mu_h=\pi_\#(\|T_h\|+\|\partial T_h\|)$ and let $\mu_{h(n)}$ be a subsequence w*-converging to μ in $\mathbf{R}$. If t is not an atom of μ, noticing that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}[\|T_{h(n)}\|+\|\partial T_{h(n)}\|](\pi^{-1}([t-\delta,t+\delta])\leqslant\lim_{\delta\downarrow 0}\mu([t-\delta,t+\delta])=0,$$

and approximating $\chi_{\{\pi>t\}}$ by Lipschitz functions we obtain (8.2). As

$$\int_{\mathbf{R}}\liminf_{n\to\infty}\mathbf{N}(\langle T_{h(n)},\pi,t\rangle)\,dt\leqslant\liminf_{n\to\infty}\int_{\mathbf{R}}\mathbf{N}(\langle T_{h(n)},\pi,t\rangle)\,dt\leqslant\operatorname{Lip}(\pi)\sup_{h\in\mathbf{N}}\mathbf{N}(S_h)<\infty$$

we can also find for $\mathcal{L}^1$-a.e. $t\in\mathbf{R}$ a subsequence of $(\langle S_{h(n)},\pi,t\rangle)$ bounded in $\mathbf{N}_{k-1}(E)$. If the sequence $(\mathbf{S}(T_h))$ is bounded we can use (5.16) and a similar argument to obtain a subsequence with equi-bounded size. □

Remark 8.4. If E is a w*-separable dual space the same property holds, with a similar proof, if weak convergence is replaced by w*-convergence, provided π is w*-continuous.

Now we can prove the closure theorem for (integer-)rectifiable currents, assuming as in [1], the existence of suitable bounds on mass and size. Actually, we will prove in Theorem 9.5 that for rectifiable currents T whose multiplicity is bounded from below by $a>0$ (in particular, the integer-rectifiable currents) the bound on size follows by the bound on mass, since $\mathbf{S}(T)\leqslant k^{k/2}\mathbf{M}(T)/a$.

THEOREM 8.5 (closure theorem). *Let $(T_h)\subset\mathbf{N}_k(E)$ be a sequence weakly converging to $T\in\mathbf{N}_k(E)$. Then, the conditions*

$$T_h\in\mathcal{R}_k(E),\quad \sup_{h\in\mathbf{N}}\mathbf{N}(T_h)+\mathbf{S}(T_h)<\infty$$

imply $T\in\mathcal{R}_k(E)$, and the conditions

$$T_h\in\mathcal{I}_k(E),\quad \sup_{h\in\mathbf{N}}\mathbf{N}(T_h)<\infty$$

imply $T\in\mathcal{I}_k(E)$.

If E is a w-separable dual space the same closure properties holds for w*-convergence of currents.*

Proof. We argue by induction with respect to k. If $k=0$, we prove the closure theorem first in the case when E is a w*-separable dual space and the currents T_h are w*-converging.

Possibly extracting a subsequence we can assume the existence of an integer p, points $x_h^1,...,x_h^p$ and real numbers $a_h^1,...,a_h^p$ such that

$$T_h(f)=\sum_{i=1}^p a_h^i f(x_h^i)\quad\text{for all } h\in\mathbf{N}. \tag{8.3}$$

We claim that the cardinality of $\operatorname{spt} T$ is at most p. Indeed, if by contradiction $\operatorname{spt} T$ contains $q=p+1$ distinct points $x_1, ..., x_q$, denoting by X the linear span of x_i we can find a w*-continuous linear map $p: E \to X$ whose restriction to X is the identity, and consider, for $r>0$ sufficiently small, the pairwise disjoint sets $C_i=p^{-1}(B_r(x_i))$. Since $q>p$ we can find an integer i such that $C_i \cap \operatorname{spt} T_h=\varnothing$ for infinitely many h, and since $x_i \in C_i$ the contradiction will be achieved by showing the lower semicontinuity of the mass in C_i, namely

$$\|T\|(C_i) \leqslant \liminf_{h\to\infty} \|T_h\|(C_i) = 0. \tag{8.4}$$

Let $f: E \to [-1,1]$ be any Lipschitz function with support contained in C_i, and let $f_k: E \to [-1,1]$ be w*-continuous Lipschitz functions converging to f in $L^1(\|T\|)$ (see Theorem 6.2). Choosing a sequence $(\phi_n) \subset C_0(X)$ such that $\phi_n \geqslant 0$ and $\phi_n \uparrow \chi_{B_r(x)}$ we get

$$T(f_k \phi_n \circ p) = \lim_{h\to\infty} T_h(f_k \phi_n \circ p) \leqslant \liminf_{h\to\infty} \|T_h\|(C_i).$$

Letting first $k\uparrow\infty$ and then $n\uparrow\infty$, we obtain $|T(f)| \leqslant \liminf_h \|T_h\|(C_i)$, and since f is arbitrary we obtain (8.4). In the case when T_h are integer-rectifiable, since the cardinality of $\operatorname{spt} T_h$ is p, for any $x \in \operatorname{spt} T$ we can easily find a w*-continuous Lipschitz function $f: E \to [0,1]$ such that $f(x)=1$, $f(y)=0$ for any $y \in \operatorname{spt} T \setminus \{x\}$, and $\{0<f<1\}$ does not intersect $\operatorname{spt} T_h$ for infinitely many h (it suffices to consider $p+1$ functions f_j of the form $g_j \circ p$ such that $\{0<f_j<1\}$ are pairwise disjoint). Hence

$$a_x = T(f) = \lim_{h\to\infty} T_h(f) = \lim_{h\to\infty} \sum_{i=1}^{p} a_h^i f(x_h^i)$$

is an integer.

In the metric case the proof could be easily recovered using the isometric embedding of the closure of the union of $\operatorname{spt} T_h$ into l_∞; we prefer, however, to give a simpler independent proof, not relying on Theorem 6.2. If $x^1, ..., x^n$ are distinct points in $\operatorname{spt} T$, we can find $\varepsilon>0$ such that the balls $B_\varepsilon(x^i)$ are pairwise disjoint and obtain from the lower semicontinuity of mass that

$$B_\varepsilon(x^i) \cap \operatorname{spt} T_h \neq \varnothing \quad \text{for all } i=1, ..., n,$$

for h large enough. This implies that T is representable by a sum $\sum a_x \delta_x$ with at most p terms, and hence $T \in \mathcal{R}_0(E)$. In the integer case we argue as in the proof of the closure property for w*-convergence.

Let now $k \geqslant 1$ and let us prove that T fulfils (8.1): let $\pi \in \operatorname{Lip}(E, \mathbf{R}^k)$, let L be a σ-compact set on which T is concentrated, and set $\pi=(\pi_1, \pi')$ with $\pi': E \to \mathbf{R}^{k-1}$, $S=T \llcorner d\pi_1$, $S_h=T_h \llcorner d\pi_1$ and

$$S_t := \langle T, \pi_1, t\rangle, \quad S_{ht} := \langle T_h, \pi_1, t\rangle.$$

By Proposition 8.3 we obtain that, for $\mathcal{L}^1$-a.e. $t\in\mathbf{R}$, the current S_t is the limit of a bounded subsequence of (S_{ht}), with $\mathbf{S}(S_{ht})$ equi-bounded. Hence, the induction assumption and Theorem 5.7 give that $S_t\in\mathcal{R}_{k-1}(E)$ for $\mathcal{L}^1$-a.e. $t\in\mathbf{R}$. For any such t, $\langle S_t,\pi',y\rangle\in\mathcal{R}_0(E)$ for $\mathcal{L}^{k-1}$-a.e. $y\in\mathbf{R}^{k-1}$. By Lemma 5.8 we conclude that

$$\langle T,\pi,x\rangle=\langle S_t,\pi',y\rangle \quad \text{for } \mathcal{L}^k\text{-a.e. } x=(y,t)\in\mathbf{R}^k,$$

and hence that

$$\langle T,\pi,x\rangle\in\mathcal{R}_0(E) \quad \text{for } \mathcal{L}^k\text{-a.e. } x=(y,t)\in\mathbf{R}^k.$$

Since π is arbitrary this proves that T is rectifiable. If T_h are integer-rectifiable the proof follows the same lines, using the second part of the statement of Theorem 8.1.

Finally, if E is a w*-separable dual space, the same induction argument based on Remark 8.4 gives

$$\langle T,\pi,x\rangle\in\mathcal{R}_0(E) \quad \text{for } \mathcal{L}^k\text{-a.e. } x\in\mathbf{R}^k,$$

for any w*-continuous map $\pi\in\mathrm{Lip}(E,\mathbf{R}^k)$. Using Remark 8.2 we conclude. □

THEOREM 8.6 (boundary-rectifiability theorem). *Let $k\geqslant 1$ and let $T\in\mathbf{I}_k(E)$. Then $\partial T\in\mathbf{I}_{k-1}(E)$.*

Proof. We argue by induction on k. If $k=1$, by Theorem 4.3 (i) we have only to show that $\partial T(\chi_A)\in\mathbf{Z}$ for any open set $A\subset E$. Setting $\varphi(x)=\mathrm{dist}(x,E\setminus A)$ and $A_t=\{\varphi>t\}$, we notice that

$$\partial T(\chi_{A_t})=\partial T\llcorner A_t(1)=\partial(T\llcorner A_t)(1)+\langle T,\varphi,t\rangle(1)=\langle T,\varphi,t\rangle(1)\in\mathbf{Z}$$

for $\mathcal{L}^1$-a.e. $t>0$. By the continuity properties of measures, letting $t\downarrow 0$ we obtain that $\partial T(\chi_A)=\partial T(\chi_{\{\varphi>0\}})$ is an integer.

Assume now the statement true for $k\geqslant 1$, and let us prove it for $k+1$. Let $\pi=(\pi_1,\tilde\pi)\in\mathrm{Lip}(E,\mathbf{R}^k)$ with $\pi_1\in\mathrm{Lip}(E)$, $\tilde\pi\in\mathrm{Lip}(E,\mathbf{R}^{k-1})$ and $S_t=\langle T,\pi_1,t\rangle$; the currents S_t are normal and integer-rectifiable for $\mathcal{L}^1$-a.e. $t\in\mathbf{R}$, and hence

$$\langle\partial T,\pi_1,t\rangle=-\partial\langle T,\pi_1,t\rangle=-\partial S_t\in\mathbf{I}_{k-1}(E)$$

for $\mathcal{L}^1$-a.e. $t\in\mathbf{R}$ by the induction assumption. The same argument used in the proof of Theorem 8.5, based on Lemma 5.8, shows that $\langle\partial T,\pi,x\rangle\in\mathbf{I}_0(E)$ for $\mathcal{L}^k$-a.e. $x\in\mathbf{R}^k$. By Theorem 8.1 we conclude that $\partial T\in\mathbf{I}_k(E)$. □

As a corollary of Theorem 8.1, we can prove rectifiability criteria for k-dimensional currents based either on the dimension of the measure-theoretic support or on Lipschitz projections on $\mathbf{R}^k$ or $\mathbf{R}^{k+1}$; we emphasize that the current structure is essential for the validity of these properties, which are false for sets (see the counterexample in [7]).

THEOREM 8.7. *Let $T\in \mathbf{N}_k(E)$. Then $T\in \mathcal{R}_k(E)$ if and only if T is concentrated on a Borel set S σ-finite with respect to $\mathcal{H}^k$.*

Proof. Let $\pi\in \mathrm{Lip}(E,\mathbf{R}^k)$ and $S'\subset S$ with $\mathcal{H}^k(S')<\infty$; by Theorem 2.10.25 of [23] we have

$$\int_{\mathbf{R}^k} \mathcal{H}^0(S'\cap\pi^{-1}(x))\,dx \leqslant c(k)[\mathrm{Lip}(\pi)]^k\mathcal{H}^k(S')<\infty,$$

and hence $S'\cap\pi^{-1}(x)$ is a finite set for $\mathcal{L}^k$-a.e. $x\in\mathbf{R}^k$. Since S is σ-finite with respect to $\mathcal{H}^k$ we obtain that $S\cap\pi^{-1}(x)$ is at most countable for $\mathcal{L}^k$-a.e. $x\in\mathbf{R}^k$. Hence, the currents $\langle T,\pi,x\rangle$, being supported in $S\cap\pi^{-1}(x)$, belong to $\mathcal{R}_0(E)$ for $\mathcal{L}^k$-a.e. $x\in\mathbf{R}^k$, whence $T\in\mathcal{R}_k(E)$. □

THEOREM 8.8 (rectifiability and rectifiability of projections). *Let $T\in\mathbf{N}_k(E)$. Then*

(i) *$T\in\mathcal{I}_k(E)$ if and only if $\phi_\# T\in\mathcal{I}_k(\mathbf{R}^{k+1})$ for any $\phi\in\mathrm{Lip}(E,\mathbf{R}^{k+1})$;*

(ii) *$T\in\mathcal{I}_k(E)$ if and only if $\pi_\#(T\llcorner A)\in\mathcal{I}_k(\mathbf{R}^k)$ for any $\pi\in\mathrm{Lip}(E,\mathbf{R}^k)$ and any $A\in\mathcal{B}(E)$;*

(iii) *if E is a finite-dimensional vector space then $T\in\mathcal{R}_k(E)$ if and only if $\phi_\# T\in \mathcal{R}_k(\mathbf{R}^{k+1})$ for any $\phi\in\mathrm{Lip}(E,\mathbf{R}^{k+1})$.*

Proof. (i) Let $\pi\in\mathrm{Lip}(E,\mathbf{R}^k)$ be fixed. By Theorem 8.1 we need only to prove that $T_x=\langle T,\pi,x\rangle$ are integer-rectifiable for $\mathcal{L}^k$-a.e. $x\in\mathbf{R}^k$. Let S be a σ-compact set on which T is concentrated, let $\mathcal{A}$ be the countable collection of open sets given by Lemma 5.5, and let us denote by φ_A, for $A\in\mathcal{A}$, the distance function from the complement of A.

By applying Lemma 5.9 with $n=k+1$ and $\varphi=\varphi_A$ we obtain an $\mathcal{L}^k$-negligible set $N\subset\mathbf{R}^k$ such that

$$\varphi_{A\#}T_x = q_\#\langle(\varphi_A,\pi)_\# T,p,x\rangle\in\mathcal{I}_0(\mathbf{R})$$

for any $A\in\mathcal{A}$ and any $x\in\mathbf{R}^k\setminus N$. In particular, for any $x\in\mathbf{R}^k\setminus N$ we have

$$T_x(\chi_A)=\varphi_{A\#}T_x(\chi_{(0,\infty)})\in\mathbf{Z}\quad\text{for all } A\in\mathcal{A},$$

and, by our choice of $\mathcal{A}$, the same is true for any $A\in\mathcal{B}(E)$. Then, the integer rectifiability of T_x follows by Theorem 4.3 (i).

(ii) By Theorem 8.1 we need only to show that, for $\pi\in\mathrm{Lip}(E,\mathbf{R}^k)$ given, $\mathcal{L}^k$-almost all currents $T_x=\langle T,\pi,x\rangle$ are integer-rectifiable. Let $\mathcal{A}$ be given by Lemma 5.5; by (5.15) and (5.18) we can find an $\mathcal{L}^k$-negligible set $N\subset\mathbf{R}^k$ such that

$$\varphi_\#(T_x\llcorner A)=\varphi_\#\langle T\llcorner A,\varphi,x\rangle=\langle\varphi_\#(T\llcorner A),p,x\rangle\in\mathcal{I}_0(\mathbf{R}^k)$$

for any $x\in\mathbf{R}^k\setminus N$ and any $A\in\mathcal{A}$. By Lemma 5.5 we infer that

$$T_x(A)=T_x\llcorner A(1)=\varphi_\#(T_x\llcorner A)(1)\in\mathbf{Z}\quad\text{for all } A\in\mathcal{B}(E),\ x\in\mathbf{R}^k\setminus N.$$

The integer rectifiability of T_x now follows by Theorem 4.3 (i).

(iii) Assuming $E=\mathbf{R}^N$, the proof is analogous to that of statement (i), using the countably many maps $f_{x,\lambda}$ of Lemma 4.4. □

9. Rectifiable currents in Banach spaces

In this section we improve Theorem 4.6, recovering in w*-separable dual spaces Y the classical representation of Euclidean currents by the integration on an oriented rectifiable set, possibly with multiplicities. Moreover, for $T\in\mathcal{R}_k(Y)$, we compare $\|T\|$ with $\mathcal{H}^k \llcorner S_T$ and see to what extent these results still hold in the metric case.

The results of this section depend on some extensions of the Rademacher theorem given in [38] and [7]. Assume that Y is a w*-separable dual space; we proved that any Lipschitz map $f: A\subset\mathbf{R}^k\to Y$ is *metrically* and *w*-differentiable* $\mathcal{L}^k$-a.e., i.e. for $\mathcal{L}^k$-a.e. $x\in A$ there exists a linear map $L:\mathbf{R}^k\to Y$ such that

$$\text{w}^*\text{-}\lim_{y\to x}\frac{f(y)-f(x)-L(y-x)}{|y-x|}=0$$

and, at the same time,

$$\lim_{y\to x}\frac{\|f(y)-f(x)\|-\|L(y-x)\|}{|y-x|}=0.$$

Notice that the second formula is not an obvious consequence of the first, since the difference quotients are only w*-converging to 0. The map L is called w*-differential and denoted by wd_xf, while $\|L\|$ is called metric differential, and denoted by md_xf. The metric differential actually exists $\mathcal{L}^k$-a.e. for any Lipschitz map f from a subset of $\mathbf{R}^k$ into any metric space (E,d), and is in this case defined by

$$md_xf(v):=\lim_{t\to 0}\frac{d(f(x+tv),f(x))}{|t|}\quad\text{for all } v\in\mathbf{R}^k.$$

This result, proved independently in [38] and [40], has been proved in [7] using an isometric embedding into l_∞ and the w*-differentiability theorem.

(1) *Approximate tangent space.* Using the generalized Rademacher theorem one can define an approximate tangent space to a countably $\mathcal{H}^k$-rectifiable set $S\subset Y$ by setting

$$\operatorname{Tan}^{(k)}(S,f(x)):=wd_xf(\mathbf{R}^k)\quad\text{for } \mathcal{L}^k\text{-a.e. } x\in A_i$$

whenever f_i satisfy (4.1). It is proved in [7] that this is a good definition, in the sense that $\mathcal{H}^k$-a.e. the dimension of the space is k and that different choices of f_i produce approximate tangent spaces which coincide $\mathcal{H}^k$-a.e. on S: this is achieved by comparing this definition with more intrinsic ones, related for instance to w*-limits of the secant vectors to the set. Moreover, the approximate tangent space is *local*, in the sense that

$$\mathrm{Tan}^{(k)}(S_1,x)=\mathrm{Tan}^{(k)}(S_2,x) \quad \text{for } \mathcal{H}^k\text{-a.e. } x\in S_1\cap S_2,$$

for any pair of countably $\mathcal{H}^k$-rectifiable sets S_1, S_2.

(2) *Jacobians and area formula.* Let V, W be Banach spaces, with $\dim(V)=k$, and $L: V\to W$ linear. The k-Jacobian of L is defined by

$$\mathbf{J}_k(L):=\frac{\omega_k}{\mathcal{H}^k(\{x:\|L(x)\|\leqslant 1\})}=\frac{\mathcal{H}^k(\{L(x):x\in B_1\})}{\omega_k}.$$

It can be proved that $\mathbf{J}_k$ satisfies the natural product rule for Jacobians, namely

$$\mathbf{J}_k(L\circ M)=\mathbf{J}_k(L)\,\mathbf{J}_k(M) \tag{9.1}$$

for any linear map $M: U\to V$. If s is a seminorm in $\mathbf{R}^k$ we define also

$$\mathbf{J}_k(s):=\frac{\omega_k}{\mathcal{H}^k(\{x:s(x)\leqslant 1\})}.$$

These notions of Jacobian are important in connection with the area formulas

$$\int_{\mathbf{R}^k}\theta(x)\,\mathbf{J}_k(md_xf)\,dx=\int_E\sum_{x\in f^{-1}(y)}\theta(x)\,d\mathcal{H}^k(y) \tag{9.2}$$

for any Borel function $\theta: \mathbf{R}^k\to[0,\infty]$ and

$$\int_A\theta(f(x))\,\mathbf{J}_k(md_xf)\,dx=\int_E\theta(y)\,\mathcal{H}^0(A\cap f^{-1}(y))\,d\mathcal{H}^k(y) \tag{9.3}$$

for $A\in\mathcal{B}(\mathbf{R}^k)$ and any Borel function $\theta: E\to[0,\infty]$.

(3) *k-vectors and orientations.* Let $\tau=\tau_1\wedge...\wedge\tau_k$ be a simple k-vector in Y; we denote by $L_\tau: \mathbf{R}^k\to Y$ the induced linear map, given by

$$L_\tau(x_1,...,x_k):=\sum_{i=1}^k x_i\tau_i \quad \text{for all } x\in\mathbf{R}^k.$$

We say that τ is a *unit* k-vector if L_τ has Jacobian 1; notice that L_τ depends on the single τ_i rather than the k-vector τ, so our compact notation is a little misleading. It is justified, however, by the following property:

$$\tau=\lambda\tau' \;\Rightarrow\; \mathbf{J}_k(L_\tau)=|\lambda|\,\mathbf{J}_k(L_{\tau'}). \tag{9.4}$$

This property follows at once from the chain rule for Jacobians, noticing that we can represent L_τ as $L_{\tau'}\circ M$ for some linear map $M:\mathbf{R}^k\to\mathbf{R}^k$ with $\mathbf{J}_k(M)=|\lambda|$. The same argument proves that any simple k-vector τ with $\mathbf{J}_k(L_\tau)>0$ can be normalized dividing τ_i by constants $\lambda_i>0$ such that $\prod_i\lambda_i=\mathbf{J}_k(L_\tau)$. We also notice that (9.1) gives

$$|\det(L_i(\tau_j))|=\mathbf{J}_k(L\circ L_\tau)=\mathbf{J}_k(L) \tag{9.5}$$

for any unit k-vector τ and any linear function $L:\operatorname{span}\tau\to\mathbf{R}^k$.

An orientation of a countably $\mathcal{H}^k$-rectifiable set $S\subset Y$ is a unit simple k-vector $\tau=\tau_1\wedge...\wedge\tau_k$ such that $\tau_i(x)$ are Borel functions spanning the approximate tangent space to S for $\mathcal{H}^k$-almost every $x\in S$.

(4) *k-covectors and tangential differentiability.* Let Z be another w*-separable dual space, let $S\subset Y$ be a countably $\mathcal{H}^k$-rectifiable set and let $\pi\in\operatorname{Lip}(S,Z)$. Then, for $\mathcal{H}^k$-a.e. $x\in S$ the function π is *tangentially* differentiable on S and we denote by

$$d_x^S\pi:\operatorname{Tan}^{(k)}(S,x)\to Z$$

the tangential differential. This differential can be computed using suitable approximate limits of the difference quotients of π, but for our purposes it is sufficient to recall that it is also characterized by the property

$$wd_y(\pi\circ f)=d_{f(y)}^S\pi\circ wd_yf \quad \text{for } \mathcal{L}^k\text{-a.e. } y\in D, \tag{9.6}$$

whenever $f:D\subset\mathbf{R}^k\to S$ is a Lipschitz map. Clearly in the case $Z=\mathbf{R}^p$ the map $d_x^S\pi$ induces a simple p-covector in $\operatorname{Tan}^{(k)}(S,x)$, whose components are the tangential differentials of the components of π; this p-covector will be denoted by $\bigwedge_p d_x^S\pi$. Notice that, in the particular case $p=k$, (9.6) gives

$$\det(\nabla(\pi\circ f)(y))=\langle\textstyle\bigwedge_k d_{f(y)}^S\pi,\tau_y\rangle \quad \text{for } \mathcal{L}^k\text{-a.e. } y\in D, \tag{9.7}$$

where $\langle\cdot,\cdot\rangle$ is the standard duality between k-covectors and k-vectors, and

$$\tau_y=wd_yf(e_1)\wedge...\wedge wd_yf(e_k).$$

Taking into account the chain rule for Jacobians, from (9.7) we infer that

$$\mathbf{J}_k(d_x^S\pi)=\frac{|\det(\nabla(\pi\circ f))|}{\mathbf{J}_k(L_{\tau_y})}=\left|\left\langle\textstyle\bigwedge_k d_x^S\pi,\frac{\tau_y}{\mathbf{J}_k(L_{\tau_y})}\right\rangle\right| \quad \text{for } \mathcal{L}^k\text{-a.e. } y\in D,$$

with $x=f(y)$. Since $f:D\to S$ is arbitrary we conclude that

$$\mathbf{J}_k(d_x^S\pi)=|\langle\textstyle\bigwedge_k d_x^S\pi,\sigma(x)\rangle| \quad \text{for } \mathcal{H}^k\text{-a.e. } x\in S, \tag{9.8}$$

where σ is any orientation of S.

The following result shows that, as in the Euclidean case, any rectifiable k-current in a w*-separable dual space is uniquely determined by three intrinsic objects: a countably $\mathcal{H}^k$-rectifiable set S, a multiplicity function $\theta>0$ and an orientation τ of the approximate tangent space (notice, however, that in the extreme cases $k=0$ and $k=m$, $E=\mathbf{R}^m$, we allow for a negative multiplicity, because in these cases the orientation is canonically given).

THEOREM 9.1 (intrinsic representation of rectifiable currents). *Let Y be a w^*-separable dual space and let $T\in\mathcal{R}_k(Y)$ (resp. $T\in\mathcal{I}_k(Y)$). Then, there exist a countably $\mathcal{H}^k$-rectifiable set S, a Borel function $\theta: S\to(0,\infty)$ (resp. $\theta: S\to\mathbf{N}_+$) with $\int_S\theta\,d\mathcal{H}^k<\infty$, and an orientation τ of S, such that we have*

$$T(f\,d\pi_1\wedge...\wedge d\pi_k)=\int_S f(x)\theta(x)\langle\textstyle\bigwedge_k d_x^S\pi,\tau\rangle\,d\mathcal{H}^k(x) \tag{9.9}$$

for any $f\,d\pi\in\mathcal{D}^k(Y)$. Conversely, any triplet (S,θ,τ) induces via (9.9) a rectifiable current T.

Proof. Let us first assume that $T=\varphi_\#[\![g]\!]$ for some $g\in L^1(\mathbf{R}^k)$ vanishing outside of a compact set C and some one-to-one function $\varphi\in\mathrm{Lip}(C,Y)$. Let $L=\varphi(\mathbf{R}^k)$ and let τ be a given orientation of L; by (9.7) we get

$$\det(\nabla(\pi\circ\varphi)(y))=\langle\textstyle\bigwedge_k d_{\varphi(y)}^L\pi,\eta_y\rangle\,\mathbf{J}_k(wd_y\varphi)$$

for $\pi=(\pi_1,...,\pi_k)\in\mathrm{Lip}(Y,\mathbf{R}^k)$, where

$$\eta_y=\frac{wd_y\varphi(e_1)\wedge...\wedge wd_y\varphi(e_k)}{\mathbf{J}_k(wd_y\varphi)}\in\{\tau_{\varphi(y)},-\tau_{\varphi(y)}\}$$

and $e_1,...,e_k$ is the canonical basis of $\mathbf{R}^k$. Defining $\sigma(y)=1$ if η_y and $\tau_{\varphi(y)}$ induce the same orientation of $\mathrm{Tan}^{(k)}(L,\varphi(y))$, and $\sigma(y)=-1$ if they induce the opposite orientation, the identity can be rewritten as

$$\det(\nabla(\pi\circ\varphi)(y))=\sigma(y)\langle\textstyle\bigwedge_k d_{\varphi(y)}^L\pi,\tau_{\varphi(y)}\rangle\,\mathbf{J}_k(wd_y\varphi).$$

By applying the area formula and using the identity above we obtain

$$\begin{aligned}T(f\,d\pi_1\wedge...\wedge d\pi_k)&=\int_{\mathbf{R}^k}g(f\circ\varphi)\det(\nabla(\pi\circ\varphi))\,dy\\&=\int_L f(x)\Big(\sum_{y\in\varphi^{-1}(x)}g(y)\sigma(y)\Big)\langle\textstyle\bigwedge_k d_x^L\pi,\tau_x\rangle\,d\mathcal{H}^k(x)\end{aligned}$$

for any $f\,d\pi \in \mathcal{D}^k(Y)$. Setting

$$\theta(x) := \sum_{y \in \varphi^{-1}(x)} g(y)\sigma(y), \tag{9.10}$$

possibly changing the sign of τ (which induces a change of sign of σ) we can assume that $\theta \geqslant 0$. Setting $S = L \cap \{\theta > 0\}$ the representation (9.9) follows. The case of a general current $T \in \mathcal{R}_k(Y)$ easily follows by Theorem 4.5, taking into account the locality properties of the approximate tangent space.

Conversely, if T is defined by (9.9) then T has finite mass and the linearity and the locality axioms are trivially satisfied; the continuity axiom can be checked first in the case $E = \mathbf{R}^k$ (see Example 3.2), then in the case when S is bi-Lipschitz-equivalent to a compact subset of $\mathbf{R}^k$ and then, using Lemma 4.1, in the general case. □

We will denote by $[\![S, \theta, \tau]\!]$ the current defined by (9.9). In order to show that the triplet is uniquely determined, modulo $\mathcal{H}^k$-negligible sets, we want to relate the mass with $\mathcal{H}^k \llcorner S$ and with the multiplicity θ. As a by-product, we will prove that $S = S_T$, modulo $\mathcal{H}^k$-negligible sets. The main difference with the Euclidean case is the appearance in the mass of an additional factor λ_V (V being the approximate tangent space to S), due to the fact that the local norm need not be induced by an inner product.

Let V be a k-dimensional Banach space; we call *ellipsoid* any set $R = L(B)$, where B is any Euclidean ball and $L: \mathbf{R}^k \to V$ is linear. Analogously, we call *parallelepiped* any set $R = L(C)$, where C is any Euclidean cube and $L: \mathbf{R}^k \to V$ is linear. We will call *area factor* of V and denote by λ_V the quantity

$$\lambda_V := \frac{2^k}{\omega_k} \sup\left\{ \frac{\mathcal{H}^k(B_1)}{\mathcal{H}^k(R)} : V \supset R \supset B_1 \text{ parallelepiped} \right\}, \tag{9.11}$$

where B_1 is the unit ball of V. The computation of λ_V is clearly related to the problem of finding optimal rectangles enclosing a given convex body in $\mathbf{R}^k$ (in our case the body is any linear image of B_1 in $\mathbf{R}^k$ through an onto map). The first reference we are aware of on the area factor is [11]. The maximization problem appearing in the definition of the area factor has also recently been considered in [9] in connection with Riemannian geometry and in [55] in connection with geometric number theory. In the following lemma we show a different representation of λ_V, and show that it can be estimated from below and from above with constants depending only on k; the upper bound is optimal, and we refer to [51] for better lower bounds.

LEMMA 9.2. *Let V be as above. Then*

$$\lambda_V = \sup\{\mathbf{J}_k \zeta : \zeta = (\zeta_1, ..., \zeta_k): V \to \mathbf{R}^k \text{ linear, } \mathrm{Lip}(\zeta_i) \leqslant 1\}.$$

Moreover, $\lambda_V=1$ if B_1 is an ellipsoid, $\lambda_V=2^k/\omega_k$ if B_1 is a parallelepiped, and in general $k^{-k/2}\leqslant\lambda_V\leqslant 2^k/\omega_k$.

Proof. We can consider with no loss of generality only onto linear maps ζ; notice that the parallelepiped $\{v:\max_i|\zeta_i(v)|\leqslant 1\}$ contains B_1 if and only if $\max_i \mathrm{Lip}(\zeta_i)\leqslant 1$. Taking into account the area formula we obtain

$$\mathbf{J}_k\zeta=\frac{2^k}{\mathcal{H}^k(\{v:\max_i|\zeta_i(v)|\leqslant 1\})},$$

and this proves the first part of the statement, since $\mathcal{H}^k(B_1)=\omega_k$.

Any parallelepiped $R\subset V$ can be represented by $\zeta^{-1}(W)$ for some parallelepiped $W\subset\mathbf{R}^k$. Since, by translation invariance, $\mathcal{L}^k$ is a constant multiple of $\zeta_{\#}\mathcal{H}^k$, we obtain that λ_V is also given by

$$\frac{2^k}{\omega_k}\sup\left\{\frac{\mathcal{L}^k(C)}{\mathcal{L}^k(W)}:\mathbf{R}^k\supset W\supset C\text{ parallelepiped}\right\}$$

where $C=\zeta(B_1)$. If B_1 is an ellipsoid so is C, and an affine change of variables reducing C to a ball, together with a simple induction in k, shows that the supremum above is equal to 1. If B_1 is a parallelepiped, choosing $W=C$ we see that the supremum is $2^k/\omega_k$.

Due to a result of John (see [52, Chapter 3]) C is contained in an ellipsoid E such that $\mathcal{L}^k(E)\leqslant k^{k/2}\mathcal{L}^k(C)$; this gives the lower bound for λ_V. □

Remark 9.3. The area factor can be equal to 1 even though the norm is not induced by an inner product; as an example one can consider the family of Banach spaces V_y whose unit balls are the hexagons in $\mathbf{R}^2$ obtained by intersecting $[-1,1]^2$ with the strip $-t<y-x<t$, with $t\in[1,2]$. It is not hard to see that $\pi\lambda_{V_t}=4-(2-t)^2$; hence there exists $t_0\in(1,2)$ such that $\lambda_{V_{t_0}}=1$. Moreover, for $t=1$ the area factor equals $3/\pi$, and in [51] it has been proved that $\lambda_V\geqslant 3/\pi$ for any 2-dimensional Banach space V.

COROLLARY 9.4. *Let Y be a w^*-separable dual space and let $\Pi_k(Y)$ be the collection of all w^*-continuous linear maps*

$$\pi=(\pi_1,...,\pi_k):Y\to\mathbf{R}^k$$

with $\pi_i\in\mathrm{Lip}(Y)$ and $\dim(\pi(Y))=k$. There exists a sequence $(\pi^j)\subset\Pi_k(Y)$ such that $\mathrm{Lip}(\pi_i^j)=1$ for any $i\in\{1,...,k\}$, $j\in\mathbf{N}$ and

$$\sup_{j\in N}\mathbf{J}_k(\pi^j|_V)=\sup\{\mathbf{J}_k(\pi|_V):\pi\in\Pi_k(Y),\,\mathrm{Lip}(\pi_i)\leqslant 1\}$$

for any k-dimensional subspace $V\subset Y$.

Proof. In Lemma 6.1 of [7] we proved that $\Pi_k(Y)$, endowed with the pseudometric

$$\gamma(\pi,\pi'):=\sup_{\|x\|\leqslant 1}\big||\pi(x)|-|\pi'(x)|\big|,$$

is separable. Since $\gamma(\pi_h,\pi)\to 0$ implies

$$\mathcal{H}^k(\{v\in V:|\pi(v)|\leqslant 1\})=\lim_{h\to\infty}\mathcal{H}^k(\{v\in V:|\pi_h(v)|\leqslant 1\})$$

we obtain that

$$\pi\mapsto \mathbf{J}_k(\pi|_V)=\frac{\omega_k}{\mathcal{H}^k(\{v\in V:|\pi(v)|\leqslant 1\})}$$

is γ-continuous, and the statement follows choosing a dense subset of

$$\{\pi\in\Pi_k(Y):\mathrm{Lip}(\pi_i)=1\}. \qquad \square$$

Using Corollary 9.4, and still assuming that Y is a w*-separable dual space, we can easily get a representation formula for the mass of a rectifiable current.

THEOREM 9.5 (representation of mass). *Let* $T=[\![S,\theta,\tau]\!]\in\mathcal{R}_k(Y)$. *Then*

$$\|T\|=\theta\lambda\mathcal{H}^k\llcorner S,$$

where $\lambda(x)=\lambda_{\mathrm{Tan}^{(k)}(S,x)}$. *In particular, S is equivalent, modulo $\mathcal{H}^k$-negligible sets, to the set S_T in* (4.2).

Proof. The inequality $\leqslant$ follows by (9.9) and Lemma 9.2, recalling that by (9.8)

$$\left|\langle \textstyle\bigwedge_k d^S\pi,\tau\rangle\right|=\mathbf{J}_k(d^S\pi)\leqslant\lambda(x)\prod_{i=1}^k\mathrm{Lip}(\pi_i).$$

In order to show the opposite inequality we first notice that for any choice of 1-Lipschitz functions $\pi_1,...,\pi_k:Y\to\mathbf{R}$ we have

$$\|T\|\geqslant\pm\theta\langle\textstyle\bigwedge_k d^S\pi,\tau\rangle\,\mathcal{H}^k\llcorner S,$$

whence $\|T\|\geqslant\theta\mathbf{J}_k(d^S\pi)\mathcal{H}^k\llcorner S$. Now we choose π^j according to Corollary 9.4; since any real-valued linear map from a subspace of Y can be extended to Y preserving the Lipschitz constant (i.e. the norm) we have

$$\lambda_V=\sup_{j\in\mathbf{N}}\mathbf{J}_k(\pi^j|_V)$$

for any k-dimensional subspace $V\subset Y$, and hence

$$\|T\|\geqslant\bigvee_j\theta\mathbf{J}_k(d^S\pi^j)\mathcal{H}^k\llcorner S=\theta\sup_j\mathbf{J}_k(d^S\pi^j)\mathcal{H}^k\llcorner S=\theta\lambda_{\mathrm{Tan}^{(k)}(S,x)}\mathcal{H}^k\llcorner S.$$

Now we consider the case of a current $T\in\mathcal{R}_k(E)$ when E is any metric space; let $S=S_T$ as in (4.2) and let us assume, without any loss of generality, that E is separable. In this case, as explained in [7], an approximate tangent space to S can still be defined using an isometric embedding j of E into a w*-separable dual space Y ($Y=l^\infty$, for instance), and setting

$$\mathrm{Tan}^{(k)}(S,x):=\mathrm{Tan}^{(k)}(j(S),j(x)) \quad \text{for } \mathcal{H}^k\text{-a.e. } x\in S.$$

This definition is independent of j and Y, in the sense that $\mathrm{Tan}^{(k)}(S,x)$ is uniquely determined $\mathcal{H}^k$-a.e. up to linear isometries; hence $\mathrm{Tan}^{(k)}(S,x)$ can be thought $\mathcal{H}^k$-a.e. as an equivalence class of k-dimensional Banach spaces. Since the mass is invariant under isometries and the area factor λ_V is invariant under linear isometries, by applying Theorem 9.5 to $j_\# T$ we obtain that

$$\|T\|=\theta\lambda_{\mathrm{Tan}^{(k)}(S,\,\cdot\,)}\mathcal{H}^k \llcorner S$$

and T is integer-rectifiable if and only if $\theta>0$ is an integer $\mathcal{H}^k$-a.e. on S.

In order to formulate the proper extension of Theorem 9.1 to the general metric case we need the following definition: we say that two oriented rectifiable sets with multiplicities (S_1,θ_1,τ_1) and (S_2,θ_2,τ_2) contained in w*-separable dual spaces are *equivalent* if there exist $S_1'\subset S_1$, $S_2'\subset S_2$ with $\mathcal{H}^k(S_1\setminus S_1')=\mathcal{H}^k(S_2\setminus S_2')=0$ and an isometric bijection $f\colon S_1'\to S_2'$ such that $\theta_1=\theta_2\circ f$ and

$$d^{S_1}f_x(\tau_1(x))\wedge\ldots\wedge d^{S_1}f_x(\tau_k(x))=\tau_1'(x)\wedge\ldots\wedge\tau_k'(x) \quad \text{for all } x\in S_1'. \tag{9.12}$$

We can now state a result saying that any $T\in\mathcal{R}_k(E)$ induces an equivalence class of oriented rectifiable sets with multiplicities in w*-separable dual spaces; conversely, any equivalence class can canonically be associated to a rectifiable current T.

THEOREM 9.6. *Let $T\in\mathcal{R}_k(E)$ and let S,θ be as above. For $i=1,2$, let $j_i\colon E\to Y_i$ be isometric embeddings of E into w*-separable dual spaces Y_i, and let τ_i be unit k-vectors in Y_i such that*

$$j_{i\#}T=[\![j_1(S),\theta\circ j_i^{-1},\tau_i]\!].$$

Then $(j_1(S),\theta\circ j_1^{-1},\tau_1)$ and $(j_2(S),\theta\circ j_2^{-1},\tau_2)$ are equivalent.

Conversely, if (S,θ,τ) and (S',θ',τ') are equivalent, and $f\colon S\to S'$ is an isometry satisfying $\theta=\theta'\circ f$ and (9.12), then

$$f_\#[\![S,\theta,\tau]\!]=[\![S',\theta\circ f^{-1},\tau']\!].$$

Since our proofs use only the metric structure of the space, we prefer to avoid the rather abstract representation of rectifiable currents provided by Theorem 9.6; for this reason we will not give the proof, based on a standard blow-up argument, of Theorem 9.6.

We now consider the properties of the slicing operator, proving that it preserves the multiplicities. We first recall some basic facts about the coarea formula for real-valued Lipschitz functions defined on rectifiable sets.

Let X be a k-dimensional Banach space and let $L: X \to \mathbf{R}$ be linear. The *coarea factor* of L is defined by the property

$$\mathbf{C}_1(L)\,\mathcal{H}^k(A) = \int_{-\infty}^{+\infty} \mathcal{H}^{k-1}(A \cap L^{-1}(x))\,dx \quad \text{for all } A \in \mathcal{B}(X).$$

In [7] we proved that if L is not identically 0 the coarea factor can be represented as a quotient of Jacobians, namely

$$\mathbf{C}_1(L) := \frac{\mathbf{J}_k(q)}{\mathbf{J}_{k-1}(p)}$$

with $q(x) = (p(x), L(x))$ for any one-to-one linear map $p: \mathrm{Ker}(L) \to \mathbf{R}^{k-1}$. Using (9.5) we obtain also an equivalent representation as

$$|\langle \textstyle\bigwedge_{k-1} p, \tau' \rangle|\,\mathbf{C}_1(L) = |\langle \textstyle\bigwedge_k q, \tau \rangle|, \tag{9.13}$$

where τ is any unit k-vector in X, and τ' is any unit $(k-1)$-vector whose span is contained in $\mathrm{Ker}(L)$, with no restrictions on the rank of p and the rank of L; moreover, representing τ as $\tau' \wedge \varepsilon$ for some $\varepsilon \in X$, since we can always choose a one-to-one map p we obtain

$$\mathbf{C}_1(L) = |L(\varepsilon)|. \tag{9.14}$$

Let now Y be a w*-separable dual space, let $S \subset Y$ be a countably $\mathcal{H}^k$-rectifiable set and let $\pi: S \to \mathbf{R}$ be a Lipschitz function. Then, we proved in [7] that the sets $S_y = S \cap \pi^{-1}(y)$ are countably $\mathcal{H}^{k-1}$-rectifiable and

$$\mathrm{Tan}^{(k-1)}(S_y, x) = \mathrm{Ker}(d_x^S \pi) \quad \text{for } \mathcal{H}^{k-1}\text{-a.e. } x \in S_y$$

for $\mathcal{L}^1$-a.e. $y \in \mathbf{R}^n$; moreover

$$\int_S \theta(x)\,\mathbf{C}_1(d_x^S \pi)\,d\mathcal{H}^k(x) = \int_{\mathbf{R}} \left(\int_{S \cap \pi^{-1}(y)} \theta(x)\,d\mathcal{H}^{k-1}(x) \right) dy \tag{9.15}$$

for any Borel function $\theta: S \to [0, \infty]$.

THEOREM 9.7 (slices in w*-separable dual spaces). *Let $T = [\![M, \theta, \tau]\!] \in \mathcal{R}_k(Y)$ and let $\pi \in \mathrm{Lip}(Y, \mathbf{R}^m)$, with $m \leqslant k$. Then, for $\mathcal{L}^m$-a.e. $x \in \mathbf{R}^m$ there exists an orientation τ_x of $M \cap \pi^{-1}(x)$ such that*

$$\langle T, \pi, x \rangle = [\![M \cap \pi^{-1}(x), \theta, \tau_x]\!].$$

Proof. By an induction argument based on Lemma 5.8 we can assume that $m=1$. Let $f\,dp\in\mathcal{D}^{k-1}(Y)$ and set $M_x=M\cap\pi^{-1}(x)$; by the homogeneity of $\tau\mapsto\mathbf{J}_k(L_\tau)$ we can assume that $\tau(y)$ is representable by $\xi(y)\wedge\tau'_x(y)$, with $\tau'_x(y)$ a unit $(k-1)$-vector in $\mathrm{Tan}^{(k-1)}(M_x,y)$ for $\mathcal{H}^{k-1}$-a.e. $y\in M_x$, and for $\mathcal{L}^1$-a.e. x. Taking into account (9.13), and possibly changing the signs of τ'_x and ξ, we obtain

$$\langle \textstyle\bigwedge_{k-1} d_y^{M_x}p,\tau'_x(y)\rangle\,\mathbf{C}_1(d_y^M\pi)=\langle\textstyle\bigwedge_k d_y^M q,\tau(y)\rangle \quad \text{for } \mathcal{H}^{k-1}\text{-a.e. } y\in M_x,$$

for $\mathcal{L}^1$-a.e. x. Using the coarea formula we find that

$$\begin{aligned} T\llcorner(\psi\circ\pi)\,d\pi(f\,dp) &= \int_M \theta\psi\circ\pi f\langle\textstyle\bigwedge_k d^M q,\tau\rangle\,d\mathcal{H}^k \\ &= \int_{\mathbf{R}}\psi(z)\biggl(\int_{M_z}\theta f\langle\textstyle\bigwedge_{k-1} d^{M_z}p,\tau'_z\rangle\,d\mathcal{H}^{k-1}\biggr)\,dz \\ &= \int_{\mathbf{R}}\psi(z)[\![M_z,\theta,\tau'_z]\!](f\,dp)\,dz \end{aligned}$$

for any $\psi\in C_c(\mathbf{R})$. From statement (ii) of Theorem 5.6 we can conclude that $\langle T,\pi,x\rangle$ coincides with $[\![M_x,\theta,\tau'_x]\!]$ for $\mathcal{L}^1$-a.e. $x\in\mathbf{R}$. □

10. Generalized Plateau problem

The compactness and closure theorems of §8 easily lead to an existence result for the generalized Plateau problem

$$\min\{\mathbf{M}(T):T\in\mathbf{I}_{k+1}(E),\,\partial T=S\} \tag{10.1}$$

in any compact metric space E for any $S\in\mathbf{I}_k(E)$ with $\partial S=0$, provided the class of admissible currents is not empty. It may happen, however, that the class of rectifiable currents is very poor, or that there is no $T\in\mathbf{I}_{k+1}(E)$ with $\partial T=S$.

In this section we investigate the Plateau problem in the case when $E=Y$ is a Banach space, not necessarily finite-dimensional. Under this assumption the class of rectifiable currents is far from being poor, and the cone construction, studied in the first part of the section, guarantees that the class of admissible T is not empty, at least if S has bounded support.

For $t\geqslant 0$ and $f:Y\to\mathbf{R}$ we define $f_t(x)=f(tx)$, and notice that $\mathrm{Lip}(f_t)=t\,\mathrm{Lip}(f)$ and $|\partial f_t/\partial t|(x)\leqslant\|x\|\,\mathrm{Lip}(f)$ for $\mathcal{L}^1$-a.e. $t>0$ if $f\in\mathrm{Lip}(Y)$.

Definition 10.1 (cone construction). Let $S\in\mathbf{M}_k(Y)$ with bounded support; the cone C over S is the $(k+1)$-metric functional defined by

$$C(f\,d\pi):=\sum_{i=1}^{k+1}(-1)^{i+1}\int_0^1 S\left(f_t\frac{\partial\pi_{it}}{\partial t}\,d\hat{\pi}_{it}\right)dt$$

where, by definition, $d\hat{q}_i=dq_1\wedge...\wedge dq_{i-1}\wedge dq_{i+1}\wedge...\wedge dq_{k+1}$. We denote the cone C by $S\times[0,1]$.

The definition is well posed because for $\mathcal{L}^1$-a.e. $t\geqslant 0$ the derivatives $\partial\pi_{it}/\partial t(x)$ exist for $\|S\|$-a.e. $x\in Y$. This follows by applying Fubini's theorem with the product measure $\|S\|\times\mathcal{L}^1$, because for x fixed the derivatives $\partial\pi_{it}/\partial t(x)$ exist for $\mathcal{L}^1$-a.e. $t\geqslant 0$. In general we can not say that $S\times[0,1]$ is a current, because the continuity axiom seems hard to prove in this generality. We can, however, prove this for normal currents.

PROPOSITION 10.2. *If $S\in\mathbf{N}_k(Y)$ has bounded support then $S\times[0,1]$ has finite mass and $\mathbf{M}(S\times[0,1])\leqslant R\mathbf{M}(S)$, where R is the radius of the smallest ball $\overline{B}_R(0)$ containing* $\operatorname{spt} S$. *Moreover, $S\times[0,1]\in\mathbf{N}_{k+1}(Y)$ and*

$$\partial(S\times[0,1])=-\partial S\times[0,1]+S.$$

Proof. Let $f\,d\pi\in\mathcal{D}^{k+1}(Y)$ with $\pi_i\in\mathrm{Lip}_1(Y)$; using the definition of mass we find

$$|S\times[0,1](f\,d\pi)|\leqslant R(k+1)\int_0^1 t^k\int_Y|f_t|\,d\|S\|\,dt.$$

This proves that $f\mapsto S\times[0,1](f\,d\pi)$ is representable by integration with respect to a measure. We also get

$$\|S\times[0,1]\|(A)\leqslant R(k+1)\int_0^1 t^k\|S\|(A/t)\,dt\quad\text{for all } A\in\mathcal{B}(Y),$$

and therefore $\mathbf{M}(S\times[0,1])\leqslant R\mathbf{M}(S)$.

In order to prove the continuity axiom we argue by induction on k. In the case $k=0$ we simply notice that

$$S\times[0,1](f\,d\pi)=\int_0^1\left(\int_Y f_t\frac{\partial\pi_t}{\partial t}\,dS\right)dt=\int_Y\left(\int_0^1 f_t\frac{\partial\pi_t}{\partial t}\,dt\right)dS$$

and use the fact that, for bounded sequences $(u_j)\subset W^{1,\infty}(0,1)$, uniform convergence implies w*-convergence in $L^\infty(0,1)$ of the derivatives. Assuming the property true for

$(k-1)$-dimensional currents, we will prove it for k-dimensional ones by showing the identity

$$\partial(S\times[0,1])(f\,d\pi) = -\partial S\times[0,1](f\,d\pi)+S(f\,d\pi) \tag{10.2}$$

for any $f\,d\pi\in\mathcal{D}^k(Y)$.

We first show that $t\mapsto S(f_t\,d\pi_t)$ is a Lipschitz function in $[0,1]$, and that its derivative is given by

$$S\left(\frac{\partial f_t}{\partial t}\,d\pi_t\right)+\sum_{i=1}^{k}(-1)^i\left[S\left(\frac{\partial \pi_{it}}{\partial t}\,df_t\wedge d\hat{\pi}_{it}\right)-\partial S\left(f_t\frac{\partial \pi_{it}}{\partial t}\,d\hat{\pi}_{it}\right)\right] \tag{10.3}$$

for $\mathcal{L}^1$-a.e. $t>0$. Assume first that, for $t>0$, $\partial f_t/\partial t$ and $\partial\pi_{it}/\partial t$ are Lipschitz functions in Y, with Lipschitz constants uniformly bounded for $t\in(\delta,1)$ with $\delta>0$; in this case we can use the definition of boundary to reduce the above expression to

$$S\left(\frac{\partial f_t}{\partial t}\,d\pi_t\right)+\sum_{i=1}^{k}(-1)^{i+1}S\left(f_t\,d\frac{\partial \pi_{it}}{\partial t}\wedge d\hat{\pi}_{it}\right). \tag{10.4}$$

Under this assumption a direct computation and the continuity axiom on currents show that the classical derivative of $t\mapsto S(f_t\,d\pi_t)$ is given by (10.4). In the general case we approximate both f and π_i by

$$f^\varepsilon(x):=\int_0^\infty f(sx)\,\varrho_\varepsilon(s)\,ds,\quad \pi_i^\varepsilon(x):=\int_0^\infty \pi_i(sx)\,\varrho_\varepsilon(s)\,ds,$$

where ϱ_ε are convolution kernels with support in $\left(\frac{1}{2},2\right)$, w*-converging as measures to δ_1. By Fubini's theorem we get

$$\lim_{\varepsilon\to0}\frac{\partial f_t^\varepsilon}{\partial t}(x)=\frac{\partial f_t}{\partial t}(x),\quad \lim_{\varepsilon\to0}\frac{\partial \pi_{it}^\varepsilon}{\partial t}(x)=\frac{\partial \pi_{it}}{\partial t}(x)\quad\text{for }(\|S\|+\|\partial S\|)\text{-a.e. }x,$$

for $\mathcal{L}^1$-a.e. $t\geqslant0$. Hence, we can use the continuity properties of currents to obtain $\mathcal{L}^1$-a.e. convergence of the derivatives of $t\mapsto S(f_t^\varepsilon\,d\pi_t^\varepsilon)$ to (10.3). As

$$\partial(S\times[0,1])(f\,d\pi)+\partial S\times[0,1](f\,d\pi)$$

is equal to the integral of the expression in (10.3) over $[0,1]$, and $S(f_0\,d\pi_0)=0$, the proof of (10.2) is achieved.

Now we can complete the proof, showing that $S\times[0,1]$ satisfies the continuity axiom. Let f^i,π^i be as in Definition 3.1 (ii) and let us prove that

$$\lim_{i\to\infty}S\times[0,1](f^i\,d\pi_1^i\wedge\ldots\wedge d\pi_{k+1}^i)=S\times[0,1](f\,d\pi_1\wedge\ldots\wedge d\pi_{k+1}).$$

Denoting by p the cardinality of the integers j such that $\pi^i_j=\pi_j$ for every i, we argue by reverse induction on p, noticing that the case $p=k+1$ is obvious, by the definition of mass. To prove the induction step, assume that $\pi^i_j=\pi_j$ for every i and for any $j=2,\dots,p$, and notice that

$$\begin{aligned}S\times[0,1](f^i\,d\pi^i_1\wedge d\hat{\pi}^i_1)&=S\times[0,1]((f^i-f)\,d\pi^i_1\wedge d\hat{\pi}^i_1)\\&\quad+\partial(S\times[0,1])(f\pi^i_1\,d\hat{\pi}^i_1)-S\times[0,1](\pi^i_1\,df\wedge d\hat{\pi}^i_1).\end{aligned}$$

The first term converges to 0 by the definition of mass, the second one converges to $\partial(S\times[0,1])(f\pi_1\,d\hat{\pi}_1)$ by (10.2) and the continuity property of $\partial S\times[0,1]$, and the third one converges to $-S\times[0,1](\pi\,df\wedge d\hat{\pi}_1)$, by the induction assumption. Since the sum of these terms is $S\times[0,1](f\,d\pi)$, the proof is finished. □

In general the stronger Euclidean cone inequality

$$\mathbf{M}(S\times[0,1])\leqslant\frac{R}{k+1}\mathbf{M}(S)\tag{10.5}$$

does not hold, as the following example shows.

Example 10.3. Let X_p be $\mathbf{R}^2$ endowed with the l^p-norm and define λ_p, B_p as the area factor of X_p and the 1-dimensional Hausdorff measure of the unit sphere of X_p, respectively. We claim that $\pi\lambda_p$ is strictly greater than $\frac{1}{2}B_p$ for $p>2$ and $p-2$ sufficiently small. As equality holds for $p=2$, we need only to check that $2\pi\lambda'_p>B'_p$ for $p=2$, where $'$ denotes differentiation with respect to p. Denoting by A_p the Euclidean volume of the unit ball of X_p (which is contained in $[-1,1]^2$), we can estimate

$$\lambda'_2\geqslant\frac{4}{\pi}\lim_{p\to2}\frac{A_p-2}{4(p-2)}=\frac{A'_2}{\pi},$$

and hence it suffices to prove that $2A'_2>B'_2$.

Since $A_p=4\int_0^1(1-x^p)^{1/p}\,dx$, a simple computation shows that

$$\begin{aligned}A'_2&=\int_0^1\sqrt{1-x^2}\left[\frac{2x^2\ln(1/x)}{1-x^2}-\ln(1-x^2)\right]dx\\&=-2\int_0^{\pi/2}(\cos^2\theta\ln\cos\theta+\sin^2\theta\ln\sin\theta)\,d\theta,\end{aligned}\tag{10.6}$$

with the change of variables $x=\cos\theta$.

Now we compute B_p; using the parametrization $\theta\mapsto(\cos^{2/p}\theta,\sin^{2/p}\theta)$ of the unit sphere of X_p we find

$$B_p=\frac{8}{p}\int_0^{\pi/2}(\cos^{2-p}\theta\sin^p\theta+\sin^{2-p}\theta\cos^p\theta)^{1/p}\,d\theta,$$

and differentiation with respect to p gives

$$B_2' = -\pi + 2\int_0^{\pi/2} (\sin^2\theta - \cos^2\theta)(\ln\sin\theta - \ln\cos\theta)\, d\theta. \tag{10.7}$$

Comparing (10.6) and (10.7) we find that $2A_2' > B_2'$ is equivalent to

$$\int_0^{\pi/2} [\ln\sin\theta(6\sin^2\theta - 2\cos^2\theta) + \ln\cos\theta(6\cos^2\theta - 2\sin^2\theta)]\, d\theta < \pi,$$

which reduces to $\int_0^1 \ln x(4x^2-1)/\sqrt{1-x^2}\, dx < \frac{1}{4}\pi$ by simple manipulations. The value of the above integral, estimated with a numerical integration, is less than 0.5, and hence the inequality is true.

The cone inequality (10.5) is in general false even if mass is replaced by size: a simple example is the 2-dimensional Banach space with the norm induced by a regular hexagon $H \subset \mathbf{R}^2$ with side length 1. If we take S equal to the oriented boundary of H, we find that $\mathbf{S}(S \times [0,1]) = \pi$, while $\frac{1}{2}\mathbf{S}(S) = 3 < \pi$ because on the boundary of H the distance induced by the norm is the Euclidean distance.

Now we prove that the cone construction preserves (integer) rectifiability.

THEOREM 10.4. *If $S = [\![M, \theta, \eta]\!] \in \mathcal{R}_k(Y)$ then $S \times [0,1] \in \mathcal{R}_{k+1}(Y)$, and it belongs to $\mathcal{I}_{k+1}(Y)$ if $S \in \mathcal{I}_k(Y)$. In particular, if $M \subset \partial B_1(0)$ and if we extend both θ and η to the cone*

$$C := \{tx : t \in [0,1],\, x \in M\}$$

by 0-homogeneity we get

$$S \times [0,1] = [\![C, \theta, \tau]\!]$$

with $\tau(x) = (x \wedge \eta(x))/\mathbf{J}_{k+1}(L_{x\wedge\eta(x)})$.

Proof. Let $X = \mathbf{R} \times Y$ be equipped with the product metric, let $\bar{e} = (1,0) \in X$ and define $N = [0,1] \times M$. Since the approximate tangent space to N at (t,x) is generated by $\bar{e}$ and by the vectors $(0,v)$ with $v \in \mathrm{Tan}^{(k)}(M,x)$, setting $\sigma = (0,\eta_1) \wedge \ldots \wedge (0,\eta_k)$ the $(k+1)$-vector

$$\tilde{\tau} := \frac{\bar{e} \wedge \sigma(x)}{\mathbf{J}_{k+1}(L_{\bar{e}\wedge\sigma(x)})}$$

defines an orientation of N, and we can set $R = [\![N, \theta, \tilde{\tau}]\!] \in \mathcal{R}_{k+1}(X)$. We will prove that $S \times [0,1] = j_{\#}R$, where $j(t,x) = tx$. In fact, denoting by $\varrho(t,x) = t$ the projection on the first variable, by (9.14) we get

$$\mathbf{C}_1^N \varrho(x,t) = \left| d^N_{(x,t)} \varrho\left(\frac{\bar{e}}{\mathbf{J}_{k+1}(L_{\sigma(x)\wedge\bar{e}})}\right)\right| = \frac{1}{\mathbf{J}_{k+1}(L_{\sigma(x)\wedge\bar{e}})}.$$

Hence, using the coarea formula we find

$$\begin{aligned} j_{\#}R(f\,d\pi) &= \int_N \theta(x) f(tx) \langle \textstyle\bigwedge_{k+1} d^N(\pi\circ j), \tilde{\tau}\rangle \, d\mathcal{H}^{k+1} \\ &= \int_N \theta(x) f(tx) \langle \textstyle\bigwedge_{k+1} d^N(\pi\circ j), \bar{e}\wedge\sigma\rangle \, \mathbf{C}_1^N \varrho \, d\mathcal{H}^{k+1} \\ &= \sum_{i=1}^{k+1} (-1)^{i+1} \int_0^1 \left(\int_M \theta(x) f(tx) \langle \textstyle\bigwedge_k d^M(\hat{\pi}_i\circ j), \sigma\rangle \, d\mathcal{H}^k(x) \right) dt \\ &= \sum_{i=1}^{k+1} (-1)^{i+1} \int_0^1 S\left(f_t \frac{\partial \pi_{it}}{\partial t} \, d\hat{\pi}_{it} \right) dt. \end{aligned}$$

The proof of the second part of the statement is analogous, taking into account that $j: N \to \overline{B}_1(0)$ is one-to-one on $X \setminus (Y \times \{0\})$. □

Coming back to the Plateau problem, the following terminology will be useful.

Definition 10.5 (isoperimetric space). We say that Y is an isoperimetric space if for any integer $k \geqslant 1$ there exists a constant $\gamma(k, Y)$ such that for any $S \in \mathbf{I}_k(Y)$ with $\partial S = 0$ and bounded support there exists $T \in \mathbf{I}_{k+1}(Y)$ with $\partial T = S$ such that

$$\mathbf{M}(T) \leqslant \gamma(k, Y) [\mathbf{M}(S)]^{(k+1)/k}.$$

We will provide in Appendix B several examples of isoperimetric spaces, including Hilbert spaces and all dual spaces with a Schauder basis. Actually, we do not know whether Banach spaces without the isoperimetric property exist or not. For finite-dimensional spaces, following an argument due to M. Gromov, we prove that an isoperimetric constant depending only on k, and not on Y, can be chosen. This is the place where we make a crucial use of the cone construction.

We can now state one of the main results of this paper, concerning existence of solutions of the Plateau problem in dual Banach spaces.

THEOREM 10.6. *Let Y be a w^*-separable dual space, and assume that Y is an isoperimetric space. Then, for any $S \in \mathbf{I}_k(Y)$ with compact support and zero boundary, the generalized Plateau problem*

$$\min\{\mathbf{M}(T) : T \in \mathbf{I}_{k+1}(Y),\, \partial T = S\} \tag{10.8}$$

has at least one solution, and any solution has compact support.

Proof. Let $R > 0$ be such that $\operatorname{spt} S \subset \overline{B}_R(0)$ and consider the cone $C = S \times [0, 1]$. As $\partial C = S$, this implies that the infimum m in (10.8) is finite and can be estimated from above with $R\mathbf{M}(S)$. Let us denote by $\mathcal{M}$ the complete metric space of all $T\in\mathbf{I}_{k+1}(Y)$ such that $\partial T=S$, endowed with the distance $d(T,T')=\mathbf{M}(T-T')$. By the Ekeland–Bishop–Phelps variational principle we can find for any $\varepsilon>0$ a current $T_\varepsilon\in\mathcal{M}$ such that $\mathbf{M}(T_\varepsilon)<m+\varepsilon$ and

$$T\mapsto\mathbf{M}(T)+\varepsilon d(T,T_\varepsilon),\quad T\in\mathcal{M},$$

is minimal at $T=T_\varepsilon$. The plan of the proof is to show that the supports of T_ε are equi-bounded and equi-compact as $\varepsilon\in(0,\frac{1}{2})$; if this is the case we can apply Theorem 6.6 to obtain a sequence (T_{ε_i}) w*-converging to $T\in\mathbf{I}_{k+1}(Y)$, with $\varepsilon_i\downarrow 0$. Since $\partial T_{\varepsilon_i}=S$ w*-converge to ∂T we conclude that $\partial T=S$, and hence $T\in\mathcal{M}$. The lower semicontinuity of mass with respect to w*-convergence gives $\mathbf{M}(T)\leqslant m$, and so T is a solution of (10.8).

The minimality of T_ε gives

$$\mathbf{M}(T_\varepsilon)\leqslant\frac{1+\varepsilon}{1-\varepsilon}\mathbf{M}(C)\leqslant 3R\mathbf{M}(S). \tag{10.9}$$

As $K=\operatorname{spt}S$ is compact, the equi-compactness of the supports of T_ε follows by the estimate

$$\|T_\varepsilon\|(B_\varrho(x))\geqslant\frac{(3\gamma)^{-k}}{(k+1)^{k+1}}\varrho^{k+1}\quad\text{for all } x\in\operatorname{spt}T_\varepsilon, \tag{10.10}$$

for any ball $B_\varrho(x)\subset Y\setminus K$, with $\gamma=\gamma(k,Y)$. In fact, let I_ϱ be the open ϱ-neighbourhood of K and let us cover K by finitely many balls $B_\varrho(y_j)$ of radius ϱ; then, we choose inductively points $x_i\in\operatorname{spt}T_\varepsilon\setminus I_\varrho$ in such a way that the balls $B_{\varrho/2}(x_i)$ are pairwise disjoint. By (10.10) and (10.9) we conclude that only finitely many points x_i can be chosen in this way; the balls $B_{2\varrho}(y_j)$ and the balls $B_\varrho(x_i)$ cover the whole of $\operatorname{spt}T_\varepsilon$. We can of course decompose this union of closed balls into connected components. It is easy to see that a component not intersecting K contains a boundary-free part of T_ε, and hence contradicts the minimality assumption for T_ε. On the other hand, all components intersecting K are equi-bounded, and therefore the whole $\operatorname{spt}T_\varepsilon$ is as well.

In order to prove (10.10) we use a standard comparison argument based on the isoperimetric inequalities: let $\varepsilon>0$ and $x\in\operatorname{spt}T_\varepsilon\setminus K$ be fixed, set $\varphi(y)=\|y-x\|$ and

$$\delta:=\operatorname{dist}(x,K),\quad g(\varrho):=\|T_\varepsilon\|(B_\varrho(x))\quad\text{for all }\varrho\in(0,\delta).$$

For $\mathcal{L}^1$-a.e. $\varrho>0$ the slice $\langle T_\varepsilon,\varphi,\varrho\rangle$ belongs to $\mathbf{I}_k(Y)$ and has no boundary; hence, we can find $R\in\mathbf{I}_{k+1}(Y)$ such that $\partial R=\langle T_\varepsilon,\varphi,\varrho\rangle$ and

$$\mathbf{M}(R)\leqslant\gamma[\mathbf{M}(\langle T_\varepsilon,\varphi,\varrho\rangle)]^{(k+1)/k}\leqslant\gamma[g'(\varrho)]^{(k+1)/k}.$$

Comparing T_ε with $T_\varepsilon \llcorner (Y \setminus B_\varrho(x)) + R$ we find

$$\|T_\varepsilon\|(B_\varrho(x)) \leqslant \mathbf{M}(R) + \varepsilon \mathbf{M}(T_\varepsilon \llcorner B_\varrho(x) - R),$$

and hence $g(\varrho) \leqslant 3\gamma [g'(\varrho)]^{(k+1)/k}$. As $g(\varrho)>0$ for any $\varrho>0$, this proves that

$$g(\varrho)^{1/(k+1)} - \frac{(3\gamma)^{-k/(k+1)}\varrho}{k+1}$$

is increasing, and hence positive, in $(0,\delta)$.

Finally, proving for any solution T of (10.8) a density estimate analogous to the one already proved for T_ε, we obtain that $\operatorname{spt} T$ is compact. □

We conclude this section pointing out some extensions of this result, and different proofs. The first remark is that the Gromov–Hausdorff convergence is not actually needed if Y is a Hilbert space: in fact, denoting by E the closed convex hull of $\operatorname{spt} S$, it can be proved that E is compact; hence (10.1) has a solution T_E. If $\pi: Y \to E$ is the metric projection on E, since $\pi_\# S = S$ we get

$$\mathbf{M}(T) \geqslant \mathbf{M}(\pi_\# T) \geqslant \mathbf{M}(T_E) \quad \text{for all } T \in \mathbf{I}_{k+1}(Y),\ \partial T = S;$$

hence T_E, viewed as a current in Y, is a solution of the isoperimetric problem in Y.

A similar argument can be proved to get existence in some nondual spaces such as $L^1(\mathbf{R}^m)$ and $C(K)$:

Example 10.7. (a) $L^1(\mathbf{R}^m)$ can be embedded isometrically in $Y = \mathbf{M}_0(\mathbf{R}^m)$, i.e. the space of measures with finite total variation in $\mathbf{R}^m$; since Y is an isoperimetric space (see Appendix B) and the Radon–Nikodym theorem provides a 1-Lipschitz projection from Y to $L^1(\mathbf{R}^m)$, the Plateau problem has a solution for any $S \in \mathbf{I}_k(L^1(\mathbf{R}^m))$ with compact support.

(b) In the same vein, an existence result for the Plateau problem can be obtained in $E = C(K)$, where (K,δ) is any compact metric space; it suffices to notice that any compact family $\mathcal{F} \subset E$ is equi-bounded and has a common modulus of continuity $\omega(t)$, defined by

$$\omega(t) := \sup\{|f(x) - f(y)| : f \in \mathcal{F},\ \delta(x,y) \leqslant t\} \quad \text{for all } t \geqslant 0.$$

Let $\widetilde{\omega}$ be the smallest concave function greater than ω; since for any $\varepsilon>0$ the function $\varepsilon + Mt$ is greater than ω for M large enough, it follows that $\widetilde{\omega}(0)=0$, and hence $\widetilde{\omega}$ is subadditive. Using the subadditivity of $\widetilde{\omega}$ it can be easily checked that

$$f(x) \mapsto \min_{y \in K} [f(y) + \widetilde{\omega}(\delta(x,y))]$$

provides a 1-Lipschitz projection from E into the compact set

$$\Big\{ f \in E : \|f\|_\infty \leqslant \sup_{g \in \mathcal{F}} \|g\|_\infty,\ |f(x) - f(y)| \leqslant \widetilde{\omega}(d(x,y)) \text{ for all } x, y \in K \Big\}.$$

Since any function in $\mathcal{F}$ has $\omega \leqslant \widetilde{\omega}$ as modulus of continuity, the map is the identity on $\mathcal{F}$.

11. Appendix A: Euclidean currents

The results of §9 indicate that in the Euclidean case $E=\mathbf{R}^m$ our class of (integer-) rectifiable currents coincides with the Federer–Fleming one. In this section we compare our currents to *flat* currents with finite mass of the Federer–Fleming theory. In the following, when talking of Federer–Fleming currents (shortened to FF currents), k-vectors and k-covectors we adopt systematically the notation of [48] (see also [23], [57]) and take the basic facts of that theory for granted. Since flat FF currents are compactly supported by definition, we restrict our analysis to currents $T\in\mathbf{M}_k(\mathbf{R}^m)$ with compact support. We also assume that $k\geqslant 1$, since $\mathbf{M}_0(\mathbf{R}^m)$ is simply the space of measures with finite total variation in $\mathbf{R}^m$.

We recall that the (possibly infinite) flat seminorm of a FF current T is defined by

$$\mathbf{F}(T):=\sup\{T(\omega):\mathbf{F}(\omega)\leqslant 1\} \tag{11.1}$$

where the flat norm of a smooth k-covector field ω with compact support is given by

$$\mathbf{F}(\omega):=\sup_{x\in\mathbf{R}^m}\max\{\|\omega(x)\|^*,\|d\omega(x)\|^*\},$$

and $\|\cdot\|^*$ is the comass norm. It can be proved (see [23, p. 367]) that

$$\mathbf{F}(T)=\inf\{\mathbf{M}(X)+\mathbf{M}(Y):X+\partial Y=T\}. \tag{11.2}$$

We denote by $\mathbf{F}_k(\mathbf{R}^m)$ the vector space of all FF k-dimensional currents with finite mass which can be approximated, in the flat norm, by normal currents. Using (11.2) it can be easily proved (see [23, p. 374]) that $\mathbf{F}_k(\mathbf{R}^m)$ can also be characterized as the closure, with respect to the mass norm, of normal currents.

In the following theorem we prove that any current T in our sense induces a current $\widetilde{T}$ in the FF sense, and that any $T\in\mathbf{F}_k(\mathbf{R}^m)$ induces a current in our sense. Our conjecture is that actually $\widetilde{T}\in\mathbf{F}_k(\mathbf{R}^m)$, and hence that our class of currents with compact support not only includes but coincides with $\mathbf{F}_k(\mathbf{R}^m)$; up to now we have not been able to prove this conjecture because we do not know any criterion for flatness which could apply to this situation. Since the mass of any k-dimensional flat FF current vanishes on $\mathcal{H}^k$-negligible sets (see [23, 4.2.14]), this question is also related to the problem, discussed in §3, of the absolute-continuity property of mass with respect to $\mathcal{H}^k$. On the other hand, for normal currents we can prove that there really is a one-to-one correspondence between the FF ones and our ones.

THEOREM 11.1. *Any $T\in\mathbf{M}_k(\mathbf{R}^m)$ with compact support induces a FF current $\widetilde{T}$ defined by*

$$\widetilde{T}(\omega):=\sum_{\alpha\in\Lambda(m,k)}T(\omega_\alpha\,dx_{\alpha_1}\wedge\ldots\wedge dx_{\alpha_k})$$

for any smooth k-covector field $\omega\colon \mathbf{R}^m \to \bigwedge^k \mathbf{R}^m$ with compact support. Moreover,

$$\mathbf{M}(\widetilde{T}) \leqslant c(m,k)\mathbf{M}(T).$$

Conversely, any $T \in \mathbf{F}_k(\mathbf{R}^m)$ induces a current $\widehat{T} \in \mathbf{M}_k(\mathbf{R}^m)$ with compact support such that $\mathbf{M}(\widehat{T}) \leqslant \mathbf{M}(T)$. Finally, $T \mapsto \widetilde{T}$ and $T \mapsto \widehat{T}$, when restricted to normal currents, are each the inverse of the other.

Proof. By the continuity axiom (ii) on currents, $\widetilde{T}$ is continuous in the sense of distributions, and hence defines a FF current. Since

$$|\widetilde{T}(\omega)| \leqslant \int_{\mathbf{R}^m} \sum_{\alpha \in \Lambda(m,k)} |\omega_\alpha(x)| \, d\|T\|(x) \leqslant c \int_{\mathbf{R}^m} \|\omega(x)\|^* \, d\|T\|(x)$$

we obtain that $\widetilde{T}$ has finite mass (in the FF sense) and $\mathbf{M}(\widetilde{T}) \leqslant c\mathbf{M}(T)$, where c is the cardinality of $\Lambda(m,k)$.

Conversely, let us define $\widehat{T}$ for a normal FF current T first. Let us first notice that any $f\,d\pi \in \mathcal{D}^k(\mathbf{R}^m)$, with $f \in C_c^\infty(\mathbf{R}^m)$ and $\pi_i \in C^\infty(\mathbf{R}^m)$, induces a smooth k-covector field with compact support $\omega\colon \mathbf{R}^m \to \bigwedge^k \mathbf{R}^m$, given by

$$\omega = f\,d\pi_1 \wedge \ldots \wedge d\pi_k = \sum_{\alpha \in \Lambda(m,k)} f \det\left(\frac{\partial(\pi_1, \ldots, \pi_k)}{\partial(x_{\alpha_1}, \ldots, x_{\alpha_k})}\right) dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_k}.$$

Hence, $T(f\,d\pi)$ is well defined in this case. Moreover, since the covectors $\omega(x)$ are simple, the definition of comass easily implies that

$$\|\omega(x)\|^* \leqslant |f(x)| \prod_{i=1}^k \mathrm{Lip}(\pi_i) \quad \text{for all } x \in \mathbf{R}^m. \tag{11.3}$$

Arguing as in Proposition 5.1, and using (11.3) instead of the definition of mass, if $\mathrm{Lip}(\pi_i) \leqslant 1$ and $\mathrm{Lip}(\pi_i') \leqslant 1$ it can be proved that

$$\begin{aligned} |T(f\,d\pi) - T(f'\,d\pi')| \leqslant & \int_{\mathbf{R}^m} |f - f'| \, d\|T\|_{\mathrm{FF}} \\ & + \sum_{i=1}^k \int_{\mathbf{R}^m} |f| \, |\pi_i - \pi_i'| \, d\|\partial T\|_{\mathrm{FF}} + \mathrm{Lip}(f) \int_{\mathbf{R}^m} |\pi_i - \pi_i'| \, d\|T\|_{\mathrm{FF}}, \end{aligned} \tag{11.4}$$

where $\|T\|_{\mathrm{FF}}$ and $\|\partial T\|_{\mathrm{FF}}$ are now understood in the Federer–Fleming sense.

If $f\,d\pi \in \mathcal{D}^k(\mathbf{R}^m)$ we define

$$\widehat{T}(f\,d\pi) := \lim_{\varepsilon \downarrow 0} T(f * \varrho_\varepsilon \, d(\pi * \varrho_\varepsilon)).$$

By (11.4) the limit exists and defines a metric functional multilinear in $f\,d\pi$: moreover, since for $\varepsilon>0$ fixed the map $f\,d\pi\mapsto T(f*\varrho_\varepsilon\,d(\pi*\varrho_\varepsilon))$ satisfies the continuity axiom (ii) in Definition 3.1, the same estimate (11.4) can be used to show that $\widehat{T}$ retains the same property. Setting $\omega_\varepsilon=f*\varrho_\varepsilon\,d(\pi*\varrho_\varepsilon)$, by (11.3) we obtain

$$\begin{aligned}|\widehat{T}(f\,d\pi)| &= \lim_{\varepsilon\downarrow 0}|T(\omega_\varepsilon)|\leqslant \liminf_{\varepsilon\downarrow 0}\int_{\mathbf{R}^m}\|\omega_\varepsilon(x)\|^*\,d\|T\|_{\mathrm{FF}}\\ &\leqslant \prod_{i=1}^k \mathrm{Lip}(\pi_i)\liminf_{\varepsilon\downarrow 0}\int_{\mathbf{R}^m}|f*\varrho_\varepsilon|\,d\|T\|_{\mathrm{FF}}=\prod_{i=1}^k\mathrm{Lip}(\pi_i)\int_{\mathbf{R}^m}|f|\,d\|T\|_{\mathrm{FF}};\end{aligned}$$

hence $\widehat{T}$ has finite mass and $\|\widehat{T}\|\leqslant\|T\|_{\mathrm{FF}}$. The locality property $\widehat{T}(f\,d\pi)=0$ follows at once from the definition of $\widehat{T}$ if f has compact support and one of the functions π_i is constant in an open set containing spt f; the general case follows now since T is supposed to have compact support. This proves that $\widehat{T}$ is a k-current. The operator $T\mapsto\widehat{T}$ can be extended by continuity to the mass closure of normal currents, i.e. to $\mathbf{F}_k(\mathbf{R}^m)$.

Finally, since $\widehat{T}(f\,d\pi)=T(f\,d\pi)$ if π_i are smooth, for any normal FF current T we get

$$\widetilde{\widehat{T}}(\omega)=\sum_{\alpha\in\Lambda(m,k)}\widehat{T}(\omega_\alpha\,dx_{\alpha_1}\wedge\ldots\wedge dx_{\alpha_k})=\sum_{\alpha\in\Lambda(m,k)}T(\omega_\alpha\,dx_{\alpha_1}\wedge\ldots\wedge dx_{\alpha_k})=T(\omega).\qquad\square$$

12. Appendix B: Isoperimetric inequalities

In this appendix we extend the Euclidean isoperimetric inequality to a more general setting: first, in Theorem 12.2, we consider a finite-dimensional Banach space, proving the existence of an isoperimetric constant depending only on the dimension (neither on the codimension nor on the norm of the space). Then, using projections on finite-dimensional subspaces, we extend in Theorem 12.3 this result to a class of duals of separable Banach spaces. The validity of isoperimetric inequalities in a general Banach space is still an open problem.

We start with the following elementary lemma.

LEMMA 12.1. *Let β: $[0,\infty)\to(0,\infty)$ be an increasing function, let $k\geqslant 2$ be an integer and $c>0$. Then, there exist $\lambda=\lambda(k,\beta(0))<1$ and $T=T(c,k)>0$ such that*

$$(\beta(t)+c[\beta'(t)]^{k/(k-1)})^{(k+1)/k}+(1-\beta(t)+c[\beta'(t)]^{k/(k-1)})^{(k+1)/k}>\lambda \tag{12.1}$$

$\mathcal{L}^1$-a.e. in $(0,T)$ implies $\beta(T)>\frac{1}{2}$.

Proof. Let $\delta=\beta(0)>0$ and define λ as $\sup_{\tau\in[\delta,1/2]}\psi(\tau)$, where

$$\psi(\tau):=\left(\tau+\frac{1}{2k}\tau\right)^{(k+1)/k}+\left(1-\tau+\frac{1}{2k}\tau\right)^{(k+1)/k}.$$

Since ψ is strictly convex and $\psi(0)=1$, $\psi\left(\frac{1}{2}\right)<1$, it follows that $\lambda<1$. Let

$$T > \left[\tfrac{1}{2}(2kc)^{k-1}\right]^{1/k}$$

and assume that (12.1) holds $\mathcal{L}^1$-a.e. in $(0,T)$ and $\beta(T_-)<\frac{1}{2}$; the definition of λ implies that

$$c[\beta']^{k/(k-1)} \geqslant \frac{\beta}{2k}$$

$\mathcal{L}^1$-a.e. in $(0,T)$, and hence

$$\beta(T_-) \geqslant \left(\frac{1}{2kc}\right)^{k-1} T^k > \frac{1}{2}. \qquad \square$$

Now, we recall the isoperimetric inequality in Euclidean spaces: for any current $S\in\mathbf{I}_k(\mathbf{R}^m)$ with compact support and zero boundary there exists $T\in\mathbf{I}_{k+1}(\mathbf{R}^m)$ satisfying $\partial T=S$ and

$$\mathbf{M}(T) \leqslant \gamma(k,m)[\mathbf{M}(S)]^{(k+1)/k}.$$

This result, first proved by H. Federer and W. H. Fleming in [24] by means of the deformation theorem, has been improved by F. J. Almgren in [2], who proved that the optimal value of the isoperimetric constant does not depend on m and corresponds to the isoperimetric ratio of a $(k+1)$-disk.

The proof of the isoperimetric inequality in finite-dimensional Banach spaces follows closely an argument due to M. Gromov (see [32, §3.3]): the strategy is to choose a maximizing sequence for the isoperimetric ratio (which is finite, by the Federer–Fleming result) and to prove, using Lemma 12.1, that almost all the mass concentrates in a bounded region. Using this fact, the cone construction gives an upper bound for the isoperimetric constant which depends only on the dimension of the current.

THEOREM 12.2. *Let $k\geqslant 1$ be integer. There exists a constant γ_k such that for any finite-dimensional Banach space V and any $S\in\mathbf{I}_k(V)$ with $\partial S=0$ there exists $T\in\mathbf{I}_{k+1}(V)$ with $\partial T=S$ and*

$$\mathbf{M}(T) \leqslant \gamma_k[\mathbf{M}(S)]^{(k+1)/k}.$$

Proof. The proof is achieved by induction with respect to k; let $\alpha=(k+1)/k$ and, for $S\in\mathbf{I}_k(V)$ with $\partial S=0$, define

$$\gamma(S) := \inf\left\{\frac{\mathbf{M}(T)}{[\mathbf{M}(S)]^\alpha} : \partial T = S\right\}$$

and $\gamma(0)=0$. Since V is bi-Lipschitz-equivalent to some Euclidean space which is known to be an isoperimetric space we conclude that $L=\sup_S \gamma(S)$ is finite. In the following we consider a maximizing sequence (S_n) and normalize the volumes to obtain $\mathbf{M}(S_n)=1$. A simple compactness argument proves the existence of linear 1-Lipschitz maps $\pi_1, ..., \pi_N$ in V with the property that

$$\operatorname{diam}\Big(\bigcap_{i=1}^{N} \pi_i^{-1}(L_i)\Big) \leqslant 2$$

whenever $\operatorname{diam}(L_i)\leqslant 1$. We define $\beta_i(t)=\|S_n\|(\pi_i^{-1}(-\infty,t))$ for any $i\in\{1,...,N\}$ and n fixed.

Step 1. Let $k=1$; we claim that for any $\varepsilon\in(0,1)$ there exist closed balls B_n with radius less than 4 such that $\|S_n\|(Y\setminus B_n)\leqslant\varepsilon$ for n large enough. In fact, for $\mathcal{L}^1$-a.e. $t\in\mathbf{R}$ such that $\langle S_n,\pi_i,t\rangle\neq 0$ we have

$$\beta_i'(t)\geqslant \mathbf{M}(\langle S_n,\pi_i,t\rangle)\geqslant 1$$

by the boundary-rectifiability theorem. On the other hand, if $\delta\in(0,1)$, $\beta_i(t)\in\big[\frac{1}{2}\delta, 1-\frac{1}{2}\delta\big]$ and $\langle S_n,\pi_i,t\rangle=0$ we can decompose S_n as the sum of two cycles,

$$S_n=S_n^1+S_n^2=S_n\llcorner\{\pi_i<t\}+S_n\llcorner\{\pi_i\geqslant t\},$$

to obtain

$$\gamma(S_n)\leqslant\gamma(S_n^1)(\beta_i(t))^2+\gamma(S_n^2)(1-\beta_i(t))^2\leqslant L\big[1+\delta\big(\tfrac{1}{2}\delta-1\big)\big]<L,$$

and this is impossible for n large enough, depending on δ. Hence, setting $\delta=\varepsilon/N$, $\beta_i'\geqslant 1$ $\mathcal{L}^1$-a.e. in $I_i=\big\{\beta_i\in\big[\frac{1}{2}\delta,1-\frac{1}{2}\delta\big]\big\}$, which implies $\mathcal{L}^1(I_i)\leqslant 1$. Our choice of π_i implies that the intersection of $\pi_i^{-1}(I_i)$ has diameter at most 2.

Step 2. Now we consider the k-dimensional case with $k\geqslant 2$ and set $c=\gamma_{k-1}$. We claim that for any $\varepsilon\in(0,1)$ there exist closed balls B_n with radius less than $r_k=8T(c,k)$ (with T given by Lemma 12.1) such that $\|S_n\|(V\setminus B_n)\leqslant\varepsilon$ for n large enough. For this purpose we set $\delta=\varepsilon/2N$ and observe that

$$(\beta_i(t)+c[\beta_i'(t)]^{k/(k-1)})^\alpha+(1-\beta_i(t)+c[\beta_i'(t)]^{k/(k-1)})^\alpha>\lambda(k,\delta) \tag{12.2}$$

for $\mathcal{L}^1$-a.e. t and n large enough. In fact, for any t such that $L_t=\langle S_n,\pi_i,t\rangle\in\mathbf{I}_{k-1}(V)$ we can find by the induction assumption $R_t\in\mathbf{I}_k(V)$ with $\partial R_t=L_t$ and

$$\mathbf{M}(R_t)\leqslant c[\mathbf{M}(L_t)]^{k/(k-1)}\leqslant c[\beta_i'(t)]^{k/(k-1)}.$$

Writing

$$S_n=S_n^1+S_n^2:=(S_n\llcorner\{\pi<t\}-L_t)+(L_t+S_n\llcorner\{\pi\geqslant t\})$$

if (12.2) does not hold we can estimate $\gamma(S_n)$ by

$$\gamma(S_n^1)(\beta_i(t)+c[\beta_i'(t)]^{k/(k-1)})^\alpha+\gamma(S_n^2)(1-\beta_i(t)+c[\beta_i'(t)]^{k/(k-1)})^\alpha,$$

which is less than $L\lambda$, and this is impossible if $\gamma(S_n)/L>\lambda$. Now we fix n large enough, set

$$t_i:=\inf\{t:\beta_i(t)\geqslant\delta\},\quad s_i:=\sup\{t:\beta_i(t)\leqslant 1-\delta\},$$

and obtain from Lemma 12.1 that $\beta_i(t_i+T)>\frac{1}{2}$ and $\beta_i(s_i-T)<\frac{1}{2}$; hence $s_i-t_i\leqslant 2T$ and

$$\|S_n\|\Bigl(V\setminus\bigcap_{i=1}^N\pi_i^{-1}([t_i,s_i])\Bigr)\leqslant\sum_{i=1}^N\|S_n\|(V\setminus\pi_i^{-1}([t_i,s_i]))\leqslant 2N\delta=\varepsilon.$$

By our choice of N, the intersection of $\pi_i^{-1}([t_i,s_i])$ has diameter less than $4T$, and this concludes the proof of this step.

Step 3. Assuming with no loss of generality that the balls B_n of Step 2 (or Step 1 if $k=1$) are centered at the origin, we can apply the localization lemma with $\varphi(x)=\|x\|$ to choose $t_n\in(r_k,r_k+1)$ such that the currents

$$L_n:=\langle S_n,\varphi,t_n\rangle=\partial(S_n\llcorner B_{t_n})=-\partial(S_n\llcorner(V\setminus B_{t_n}))$$

have mass less than ε for n large ($L_n=0$ if $k=1$); by the induction assumption we can find currents $R_n\in\mathbf{I}_k(V)$ with $\partial R_n=L_n$ and $\mathbf{M}(R_n)\leqslant c\varepsilon^{k/(k-1)}$; we project R_n on the ball $\overline{B}_{t_n}(0)$ with the 2-Lipschitz map

$$\pi(x):=\begin{cases}x & \text{if } \|x\|\leqslant t_n,\\ t_nx/\|x\| & \text{if } \|x\|\geqslant t_n,\end{cases}$$

to obtain $R_n'\in\mathbf{I}_k(V)$ with $\partial R_n'=L_n$, $\operatorname{spt}R_n'\subset\overline{B}_{t_n}$ and $\mathbf{M}(R_n)\leqslant 2^kc\varepsilon^{k/(k-1)}$. Writing

$$S_n=(S_n\llcorner B_{t_n}-R_n')+(R_n'+S_n\llcorner(V\setminus B_{t_n}))$$

and applying the cone construction to $S_n\llcorner B_{t_n}-R_n'$, for n large enough we obtain

$$\gamma(S_n)\leqslant(r_k+1)(1+2^kc\varepsilon^{k/(k-1)})+L(2^kc\varepsilon^{k/(k-1)}+\varepsilon)^\alpha.$$

Letting first $n\to\infty$ and then $\varepsilon\to 0$ we conclude that $L\leqslant r_k+1$, and r_k depends only on k. □

THEOREM 12.3. *Let Y be a w^*-separable dual space, and assume the existence of finite-dimensional subspaces $Y_n \subset Y$ and continuous linear maps $P_n: Y \to Y_n$, such that $P_n(x)$ w^*-converge to x as $n \to \infty$ for any $x \in Y$. Then*

$$\inf\{\mathbf{M}(T) : T \in \mathbf{I}_{k+1}(Y), \partial T = S\} \leqslant \gamma_k C^{k+1} [\mathbf{M}(S)]^{(k+1)/k}$$

for any $S \in \mathbf{I}_k(Y)$ with bounded support, where $C = \sup_n \|P_n\|$ and γ_k is the constant of Theorem 12.2. If S has compact support, the infimum is achieved by some current T with compact support.

Proof. The constant C is finite by the Banach–Steinhaus theorem. Let $S \in \mathbf{I}_k(Y)$ with bounded support, let $S_n = P_{n\#} S$ and notice that by Theorem 12.2 we can find solutions T_n of the Plateau problem

$$\min\{\mathbf{M}(T) : T \in \mathbf{I}_{k+1}(Y_n), \partial T = S_n\}$$

and these solutions satisfy

$$\mathbf{M}(T_n) \leqslant \gamma_k [\mathbf{M}(S_n)]^{(k+1)/k} \leqslant \gamma_k C^{k+1} [\mathbf{M}(S)]^{(k+1)/k}.$$

Since Y_n embeds isometrically in Y we can view T_n as currents in Y and prove, by the same argument as in Theorem 10.6 (but using Theorem 12.2 in place of the assumption that Y is an isoperimetric space), that $\operatorname{spt} T_n$ are equi-bounded and equi-compact. By Theorem 6.6 we can find a subsequence $T_{n(h)}$ w*-converging to some limit T. Since $\partial T_{n(h)}$ w*-converge to ∂T and $S_{n(h)}$ w*-converge to S, we conclude that $\partial T = S$, and the lower semicontinuity of mass gives

$$\mathbf{M}(T) \leqslant \liminf_{h \to \infty} \mathbf{M}(T_h) \leqslant \gamma_k C^{k+1} [\mathbf{M}(S)]^{(k+1)/k}.$$

Finally, since we have just proved that Y is an isoperimetric space, if S has compact support the infimum is a minimum by Theorem 10.6. □

Any dual Banach space Y satisfying the assumptions of Theorem 12.3 is an isoperimetric space. These assumptions are satisfied by Hilbert spaces (in this case the optimal isoperimetric constant is the same as for Euclidean spaces), dual separable spaces with a Schauder basis, and also by some nonseparable spaces such as l^∞.

Also the space $Y = \mathbf{M}_0(\mathbf{R}^m)$ of measures with finite total variation in $\mathbf{R}^m$ has the isoperimetric property: indeed, let us consider regular grids $\mathcal{T}_n$ in $\mathbf{R}^m$ with mesh size $1/n$ and let us define

$$P_n(\mu) := \sum_{Q \in \mathcal{T}_n} n^m \mu(Q) \mathcal{H}^m \llcorner Q \quad \text{for all } \mu \in Y.$$

It is easy to check that $\|P_n\|=1$ and that $P_n(\mu)$ w*-converge to μ as $n\to\infty$ for any $\mu\in Y$. More generally, any dual space $Y=X^*$ is an isoperimetric space if X has a Schauder basis: in fact, denoting by X_n the n-dimensional subspaces generated by the first n vectors of the basis, and denoting by $\pi_n: X\to X_n$ the corresponding projections such that $\|x-\pi_n(x)\|\to 0$ for any $x\in X$, we can define

$$P_n: Y\to Y_n := \{y\in Y: y\circ\pi_n = y\}$$

setting $P_n(y)=y\circ\pi_n$.

13. Appendix C: Mass, Hausdorff measure, lower semicontinuity

In this section we assume that Y is a w*-separable dual space and $k\geqslant 1$ is an integer. We discuss here the possibility to define lower semicontinuous functionals, with respect to the weak convergence of currents, in $\mathbf{I}_k(Y)$. Denoting by $\bigwedge_k Y$ the exterior k-product of Y, and by $\bigwedge_k^s Y$ the subset of simple k-vectors, any function $\lambda\colon \bigwedge_k^s Y\to[0,\infty)$ induces a functional $\mathcal{F}_\lambda$ on $\mathcal{I}_k(Y)\supset\mathbf{I}_k(Y)$: indeed, recall that any $T\in\mathcal{I}_k(Y)$ is representable, essentially in a unique way, as $[\![S,\theta,\tau]\!]$ through (9.9), with $S=S_T$ given by (4.2), θ integer-valued and $\|\tau\|_m=1$ on S, i.e.

$$\mathcal{H}^k\biggl(\biggl\{\sum_{i=1}^k z_i\tau_i(x) : \sum_{i=1}^k z_i^2\leqslant 1\biggr\}\biggr)=\omega_k \quad \text{for all } x\in S.$$

If $T=[\![S,\theta,\tau]\!]$ we define

$$\mathcal{F}_\lambda(T) := \int_S \theta\lambda(\tau)\, d\mathcal{H}^k.$$

Notice that, in order to define $\mathcal{F}_\lambda$, λ needs to be defined only on unit simple vectors; for this reason all the functions λ that we consider later on are positively 1-homogeneous.

In the following, for $\tau\in\bigwedge_k^s Y\neq 0$, $V_\tau\subset Y$ is the k-dimensional Banach space spanned by τ with the induced metric, and B_τ is its unit ball. Several choices of λ are possible, and have been considered in the literature. In particular, we mention the following three (normalized so that they agree if Y is a Hilbert space):

(a) $\lambda_1(\tau)=\|\tau\|_m=\mathcal{H}^k(\{\sum_{i=1}^k z_i\tau_i : \sum_{i=1}^k z_i^2\leqslant 1\})/\omega_k$;

(b) $\lambda_2(\tau)=\lambda_{V_\tau}\|\tau\|_m$, where λ_V is defined in (9.11) (see also Lemma 9.2 for a definition in terms of Jacobians);

(c) $\lambda_3(\tau)=\mathrm{VP}(\tau)\|\tau\|_m/\omega_k^2$, where $\mathrm{VP}(\tau)$ is the so-called volume product of V_τ (see [59, 2.3.2]).

The functional $\mathcal{F}_1$ induced by λ_1 is $\int_S |\theta|\, d\mathcal{H}^k$, i.e. the Hausdorff measure with multiplicities, while, according to Theorem 9.5, the functional $\mathcal{F}_2$ induced by λ_2 is the mass. The functional $\mathcal{F}_3$ induced by λ_3 arises in the theory of finite-dimensional Banach spaces (also called Minkowski spaces) and is the so-called Holmes–Thompson area; we refer to the book by A. C. Thompson [59] and to the book by R. Schneider [56] for a presentation of the whole subject; in this context, the function λ_1 has been studied by H. Busemann, and λ_2 has been studied by R. V. Benson [11].

Coming to the problem of lower semicontinuity, the following definition (adapted from [23, 5.1.2]) will be useful. We recall that the vector space of polyhedral chains is the subspace of $\mathbf{I}_k(Y)$ generated by the normal currents $[\![F, 1, \tau]\!]$ associated to subsets F of k-dimensional planes with multiplicity 1.

Definition 13.1 (semiellipticity). We say that $\lambda\colon \bigwedge_k^s Y \to [0,\infty)$ is semielliptic if

$$\sum_{i=1}^{q} \theta_i \lambda(\tau_i)\mathcal{H}^k(F_i) \geqslant \theta_0 \lambda(\tau_0)\mathcal{H}^k(F_0) \tag{13.1}$$

whenever $T = \sum_{i=1}^q [\![F_i, \theta_i, \tau_i]\!] - [\![F_0, \theta_0, \tau_0]\!]$ is a k-dimensional polyhedral chain satisfying $\partial T = 0$.

Since (13.1) is equivalent to

$$\sum_{i=1}^{q} \mathcal{F}_\lambda([\![F_i, \theta_i, \tau_i]\!]) \geqslant \mathcal{F}_\lambda([\![F_0, \theta_0, \tau_0]\!]),$$

the geometric significance of the semiellipticity condition is that "flat" currents $T_0 = [\![F_0, \theta_0, \tau_0]\!]$ minimize $\mathcal{F}_\lambda$ among all polyhedral chains T with $\partial T = \partial T_0$.

By a simple rescaling argument, it is not difficult to prove that the semiellipticity of λ is a necessary condition for lower semicontinuity of $\mathcal{F}_\lambda$. At least in finite-dimensional spaces Y, using polyhedral approximation results it could be proved, following 5.1.5 of [23], that the condition is also sufficient; we believe that, following the arguments of Appendix B, this fact could be proved in greater generality, but we will not tackle this problem here.

Since we know that the mass is lower semicontinuous, these remarks imply that the Benson function λ_2 is elliptic. We will, however, give a more direct proof of this fact in Theorem 13.2 below (this result has been independently proved by A. C. Thompson in [60]). Concerning the Busemann and Holmes–Thompson definitions, their semiellipticity is a long-standing open problem in the theory of Minkowski spaces (see [59, Problems 6.1.1, 7.1.1]), and it has been established only in the extreme cases $k=1$, $k=\dim(Y)-1$; in these cases, as in the theory of quasiconvex functionals, semiellipticity can be reduced to convexity. We also mention, in this connection, the work [10] by G. Bellettini, M. Paolini and S. Venturini, where the relevance of these results for anisotropic problems in calculus of variations is emphasized.

We define

$$\lambda(\tau) := \frac{1}{\omega_k} \sup\{\mathcal{L}^k(\eta(B_\tau))\|\tau\|_m : \eta\in\Lambda\} \quad \text{for all } \tau\in \textstyle\bigwedge_k^s Y\setminus\{0\}, \tag{13.2}$$

where Λ is the collection of all linear maps $\eta: Y\to\mathbf{R}^k$ with $\mathrm{Lip}(\eta_i)\leqslant 1$, $i=1, ..., k$. By the area formula, the function λ can also written as

$$\lambda(\tau) = \sup\{\mathbf{J}_k(\eta)\|\tau\|_m : \eta\in\Lambda\}, \tag{13.3}$$

and hence Lemma 9.2 gives that $\lambda=\lambda_2$.

THEOREM 13.2. *The function* $\lambda: \bigwedge_k^s Y\to[0,\infty)$ *defined in* (13.2) *is semielliptic.*

Proof. Let T be as in Definition 13.1 and let $\eta\in\Lambda$ be fixed; since

$$T(1\, d\eta) = \partial T(\eta_1\, d\eta_2\wedge...\wedge d\eta_{k-1}) = 0,$$

taking into account (9.9) we obtain

$$\theta_0\left|\int_{F_0} \langle \textstyle\bigwedge_k d^{F_0}\eta, \tau_0\rangle\, d\mathcal{H}^k\right| \leqslant \sum_{i=1}^q \theta_i \left|\int_{F_i} \langle \textstyle\bigwedge_k d^{F_i}\eta, \tau_i\rangle\, d\mathcal{H}^k\right|.$$

Since the definition of the Jacobian together with (9.8) imply that

$$|\langle \textstyle\bigwedge_k d^{F_i}\eta, \tau_i\rangle| = \mathbf{J}_k(L_\eta|_{\mathrm{span}(F_i)}) = \frac{\mathcal{L}^k(\eta(B_{\tau_i}))}{\omega_k},$$

we obtain

$$\frac{\theta_0}{\omega_k}\mathcal{L}^k(\eta(B_{\tau_0}))\mathcal{H}^k(F_0) \leqslant \sum_{i=1}^q \frac{\theta_i}{\omega_k}\mathcal{L}^k(\eta(B_{\tau_i}))\mathcal{H}^k(F_i).$$

This proves that $\theta_0\mathcal{L}^k(\eta(B_{\tau_0}))\mathcal{H}^k(F_0)/\omega_k \leqslant \sum_1^q \theta_i\lambda(\tau_i)\mathcal{H}^k(F_i)$. Since η is arbitrary, the semiellipticity of λ follows. ∎

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