Acta Math., 184 (2000), 1–39

Mirror symmetry and toric degenerations of partial flag manifolds

by

VICTOR V. BATYREV

Eberhard-Karls-Universität Tübingen
Tübingen, Germany

IONUŢ CIOCAN-FONTANINE

Northwestern University
Evanston, IL, U.S.A.

BUMSIG KIM

Pohang University of Science and Technology
Pohang, The Republic of Korea

and

DUCO VAN STRATEN

Johannes Gutenberg-Universität Mainz
Mainz, Germany

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1. Introduction

Using combinatorial dualities for reflexive polyhedra and Gorenstein cones together with the theory of generalized GKZ-hypergeometric functions, one can extend the calculation of the number n_d of rational curves of degree d on the generic quintic threefold in $\mathbf{P}^4$ by Candelas, de la Ossa, Green and Parkes [10] to the case of Calabi–Yau complete intersections in toric varieties [3], [4], [7], [9].

Another class of examples which includes Calabi–Yau quintic 3-folds are Calabi–Yau complete intersections in homogeneous Fano varieties G/P where G is a semisimple Lie group and P is its parabolic subgroup. It is a priori not clear how to find an appropriate mirror family for these varieties, because G/P is not a toric variety in general. In [6], we described a mirror construction (compatible with [12]) for complete intersections in the Grassmannian $G(k,n)$, which turned out to involve a degeneration of $G(k,n)$ to a certain singular toric Fano variety $P(k,n)$ introduced by Sturmfels in [28].

In this paper we consider the extension of our methods to the case of complete intersections in *arbitrary partial flag manifolds* and give complete proofs of statements from [6].

It turns out that the Plücker embedding of any such flag manifold $F := F(n_1, ..., n_l, n)$ admits a flat degeneration to a Gorenstein toric Fano variety $P(n_1, ..., n_l, n)$. This deformation has been studied recently by Gonciulea and Lakshmibai in [24], [18], [19]. The "mirror-dual" toric variety $\mathbf{P}_{\Delta(n_1,...,n_l,n)}$ associated with a reflexive polyhedron $\Delta(n_1, ..., n_l, n)$ has a nice combinatorial description in terms of a certain graph $\Gamma := \Gamma(n_1, ..., n_l, n)$ that was introduced by Givental for the case of the complete flag manifolds [16]. The idea of toric degenerations has been discussed in a more general framework in [4].

Using the residue formula, we compute explicitly a series $\Phi_F := \Phi_F(q_1, ..., q_l)$ associated with the graph Γ and conjecture that Φ_F gives a solution to the quantum $\mathcal{D}$-module associated with Gromov–Witten classes and quantum cohomology of the partial flag manifold F. We note that there is no essential difficulty in checking the conjecture in each particular case at hand, because it involves only calculations in the small quantum cohomology ring of F, for which explicit formulas are known [11], see also Remark 5.1.12 (ii). Applying the "trick with factorials" (see [6], or §4.2 below) to a Calabi–Yau complete intersection in F, we obtain Φ_F as a specialization of the toric GKZ-hypergeometric series, from which the instanton numbers (i.e., the virtual numbers of rational curves on the Calabi–Yau) can be computed via the standard procedure (see e.g. [7]). As the validity of this trick was shown recently for general homogeneous spaces [23], this implies that any instanton numbers computed via the usual "mirror symmetry method" are automatically proven to be correct in all cases for which our conjecture on Φ_F holds. The series Φ_F of complete flag manifolds has also been investigated by Schechtman [27].

The paper is organized as follows. In §2 we introduce main combinatorial notions used in the definition of a Gorenstein toric Fano variety $P(n_1, ..., n_l, n)$ associated with a given partial flag manifold $F(n_1, ..., n_l, n)$. In §3 we investigate singularities of $P(n_1, ..., n_l, n)$ and show that these singularities can be smoothed by a flat deformation to the partial flag manifold $F(n_1, ..., n_l, n)$. As a consequence of our results, we prove a generalized version of a conjecture of Gonciulea and Lakshmibai about the singular locus of $P(n_1, ..., n_l, n)$ [19]. In §4 we discuss quantum differential systems following ideas of Givental [15], [16], [17]. Finally, in §5 we explain the mirror construction for Calabi–Yau complete intersections in partial flag varieties F and the computations of the corresponding hypergeometric series Φ_F.

Acknowledgement. We would like to thank A. Givental, S. Katz, S.-A. Strømme, and E. Rødland for helpful discussions, and the Mittag-Leffler Institute for hospitality. The second and third named authors have been supported by Mittag-Leffler Institute postdoctoral fellowships.

2. Toric varieties associated with partial flag manifolds

In this section we explain how to associate to an arbitrary partial flag manifold $F(n_1, ..., n_l, n)$ certain combinatorial objects: a graph $\Gamma(n_1, ..., n_l, n)$, a reflexive polytope $\Delta(n_1, ..., n_l, n)$ and a Gorenstein toric Fano variety $P(n_1, ..., n_l, n)$.

2.1. The graph $\Gamma(n_1, ..., n_l, n)$

Let $k_1, k_2, ..., k_{l+1}$ be a fixed sequence of positive integers. We set $n_0=0$, $n_i:=k_1+...+k_i$ $(i=1, ..., l+1)$ and $n:=n_{l+1}$. Denote by $F(n_1, ..., n_l, n)$ the partial flag manifold parametrizing sequences of subspaces

$$0 \subset V_1 \subset V_2 \subset ... \subset V_l \subset \mathbf{C}^n,$$

with $\dim V_i = n_i$ $(i=1, ..., l)$. Then

$$\dim F(n_1, ..., n_l, n) = \sum_{i=1}^{l} (n_i - n_{i-1})(n - n_i).$$

To simplify notations, we shall often write F instead of $F(n_1, ..., n_l, n)$, if there is no confusion about the numbers $n_1, ..., n_l, n$. By a classical result of Ehresmann ([13]), a natural basis for the integral cohomology of F is given by the Schubert classes. These are Poincaré dual to the fundamental classes of the closed Schubert cells $C_w \subset F$, parametrized by permutations $w \in S_n$ modulo the subgroup

$$W(k_1, ..., k_{l+1}) := S_{k_1} \times ... \times S_{k_{l+1}} \subset S_n.$$

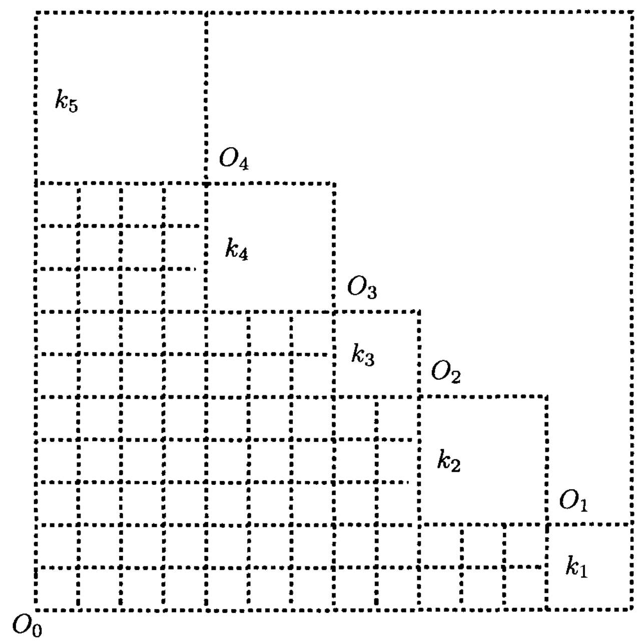


Fig. 1

In particular, the Picard group of F, which is isomorphic to $H^2(F, \mathbf{Z})$, is generated by l divisors $C_1, ..., C_l$, corresponding to the simple transpositions $\tau_i \in S_n$ exchanging n_i and n_i+1.

Definition 2.1.1. Denote by $\Lambda := \Lambda(n_1, ..., n_l, n)$ the standard *ladder diagram* consisting of unit squares (the number of unit squares in Λ is equal to the dimension of F) corresponding to the Schubert cell of maximal dimension in the flag manifold F. We place the ladder diagram Λ in the lower left corner of an $(n \times n)$-square Q. The lower left corner of Λ (or of Q) will be denoted by O_0. We denote by O_i $(i \in \{1, ..., l\})$ the common vertex of the diagonal squares Q_i of size $k_i \times k_i$, and Q_{i+1} of size $k_{i+1} \times k_{i+1}$ (Figure 1 illustrates the case $l=4$).

Definition 2.1.2. Let $\Lambda = \Lambda(n_1, ..., n_l, n)$ be the above ladder diagram. We *associate* with Λ the following:

(i) $D = D(n_1, ..., n_l, n)$, the set of centers of unit squares in Λ: we place a dot at the center of each unit square and call elements of D *dots*.

(ii) $S = S(n_1, ..., n_l, n)$, the set consisting of $l+1$ *stars*: an element of S is obtained by placing a star at the $(\frac{1}{2}, \frac{1}{2})$-shift of the lower left corner of each of the diagonal squares Q_i $(i \in \{1, ..., l+1\})$.

(iii) $E = E(n_1, ..., n_l, n)$, the set of oriented horizontal and vertical segments connect-

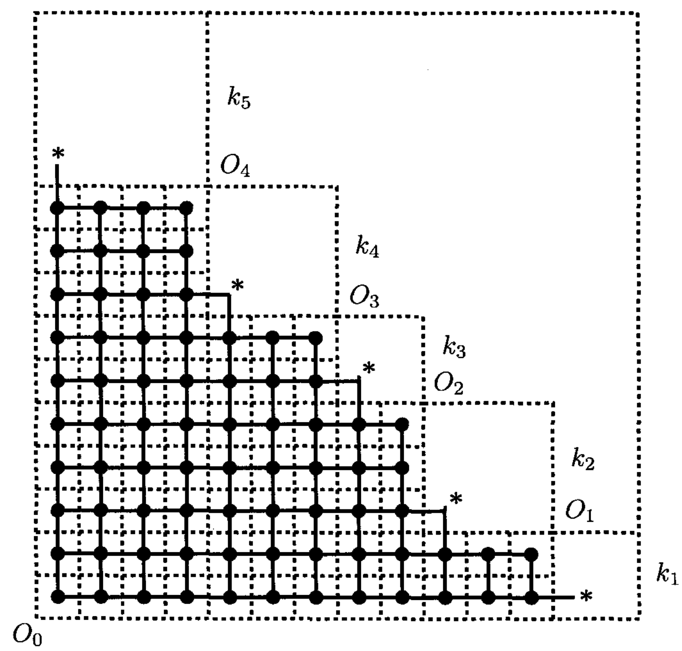


Fig. 2

ing adjacent elements of $D \cup S$: the vertical segments are oriented downwards, and the horizontal segments are oriented to the right.

Definition 2.1.3. $\Gamma := \Gamma(n_1, \ldots, n_l, n)$ is the oriented graph whose set of vertices is $D \cup S$, and whose set of oriented edges is E.

Such a graph Γ (without the orientation!) is shown in Figure 2. The edges of Γ are drawn with solid lines.

Definition 2.1.4. We denote by $L(D) \cong \mathbf{Z}^{|D|}$, $L(S) \cong \mathbf{Z}^{|S|}$ and $L(E) \cong \mathbf{Z}^{|E|}$ the free abelian groups (or lattices) generated by the sets D, S and E.

We remark that the lattices $L(D) \oplus L(S)$ and $L(E)$ can be viewed as the groups of 0-chains and 1-chains of the graph Γ. Then the boundary map in the chain complex is

$$\partial \colon L(E) \to L(D) \oplus L(S), \quad e \mapsto h(e) - t(e),$$

where $h, t \colon E \to D \cup S$ are the maps that associate to an oriented edge $e \in E$ its *head* and its *tail* respectively. See Figure 3.

Definition 2.1.5. A *box* b in Γ is a subset of 4 edges $\{e, f, g, h\} \subset E$ which form together with their endpoints a connected subgraph $\Gamma_b \subset \Gamma$ such that the topological

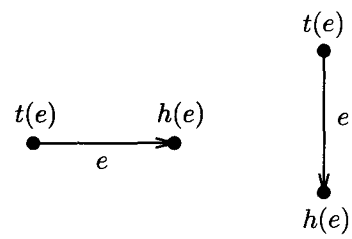


Fig. 3

space associated to Γ_b is homeomorphic to a circle. The set of boxes in Γ will be denoted by B.

It is easy to see that

$$H_0(\Gamma)=\mathrm{Coker}(\partial)\cong \mathbf{Z}, \quad H_1(\Gamma)=\mathrm{Ker}(\partial)\cong \mathbf{Z}^{|B|}.$$

We also consider the projection $\varrho\colon L(D)\oplus L(S)\to L(D)$ and the composed map

$$\delta:=\varrho\circ\partial\colon L(E)\to L(D).$$

Since one can regard the groups $L(E)$ and $L(D)$ together with the homomorphism δ as the relative chain complex of the topological pair (Γ,S), we have

$$H_0(\Gamma,S)=\mathrm{Coker}(\delta)=0, \quad H_1(\Gamma,S)=\mathrm{Ker}(\delta)\cong \mathbf{Z}^{|B|+l}.$$

Definition 2.1.6. A *roof* $\mathcal{R}_i$, $i\in\{1,2,\dots,l\}$, is the set of k_i+k_{i+1} edges of Γ forming the oriented path that runs along the upper right "boundary" of Γ between the ith and the $(i+1)$st stars in S.

Definition 2.1.7. The *corner* $\mathcal{C}_b$ of a box $b=\{e,f,g,h\}\in B$ is the pair of edges $\{e,f\}\subset b$ meeting at the lower left vertex of Γ_b. So a corner $\mathcal{C}_b$ contains one vertical edge e and one horizontal edge f such that $h(e)=t(f)$.

The roofs and corners give a decomposition of the set E of edges of the graph Γ into a disjoint union of subsets:

$$E=\mathcal{R}_1\cup\ldots\cup\mathcal{R}_l\cup\bigcup_{b\in B}\mathcal{C}_b.$$

This decomposition is shown in Figure 4.

Definition 2.1.8. The *opposite corner* $\mathcal{C}_b^-$ of a box $b=\{e,f,g,h\}\in B$ is the pair of edges $\{g,h\}\subset b$ meeting at the upper right vertex of Γ_b. An opposite corner $\mathcal{C}_b^-$ contains one vertical edge h and one horizontal edge g such that $h(g)=t(h)$.

By elementary arguments one obtains:

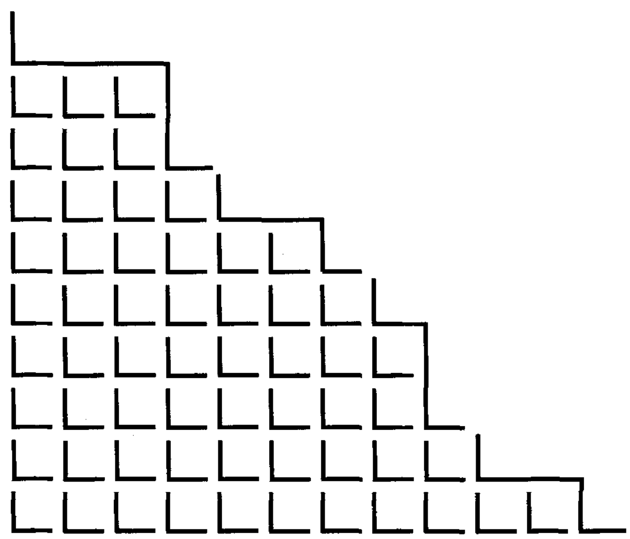


Fig. 4

PROPOSITION 2.1.9. *The elements*

$$\varrho_b = \sum_{e \in \mathcal{C}_b} e - \sum_{e \in \mathcal{C}_b^-} e,$$

where b runs over the set B, form a natural $\mathbf{Z}$*-basis of* $\mathrm{Ker}(\partial) \subset L(E)$. *Moreover, the elements*

$$\varrho_i = \sum_{e \in \mathcal{R}_i} e, \quad i \in \{1, ..., l\},$$

and

$$\varrho_b = \sum_{e \in \mathcal{C}_b} e - \sum_{e \in \mathcal{C}_b^-} e, \quad b \in B,$$

form a natural $\mathbf{Z}$*-basis of* $\mathrm{Ker}(\delta) \subset L(E)$. □

2.2. The toric variety $P(n_1, ..., n_l, n)$

We denote again by δ the $\mathbf{R}$-scalar extension $L(E) \otimes \mathbf{R} \to L(D) \otimes \mathbf{R}$ of the homomorphism $\delta: L(E) \to L(D)$.

Definition 2.2.1. The polyhedron $\Delta := \Delta(n_1, ..., n_l, n)$ associated to F is the convex hull of the set

$$\delta(E) \subset L(D) \otimes \mathbf{R},$$

where the set E is identified with the standard basis of $L(E) \otimes \mathbf{R} \cong \mathbf{R}^{|E|}$.

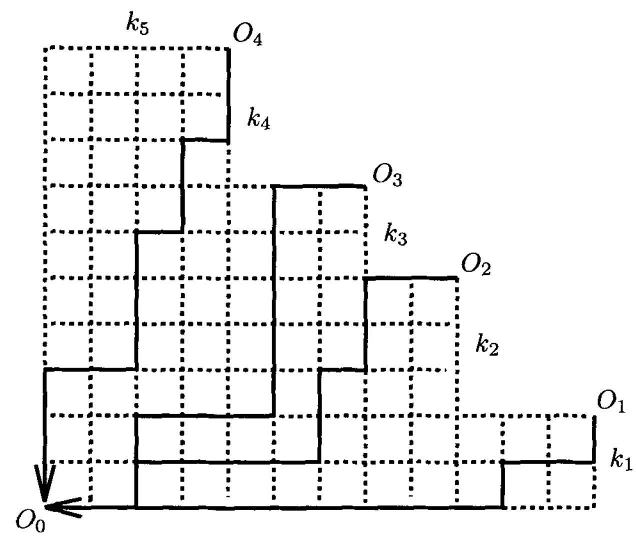


Fig. 5

In order to describe the faces of the polyhedron Δ we introduce some further combinatorial objects associated to the ladder diagram Λ.

Definition 2.2.2. (i) A *positive path* π in the diagram Λ is a path obtained by starting at one of the points O_i $(i=1,...,l)$ and moving either downwards, or to the left along some n edges of Λ, until the lower left corner O_0 is reached (see Figure 5). We denote by Π the set of positive paths, and by Π_i the set of positive paths connecting O_i and O_0, so that

$$\Pi=\Pi_1\cup...\cup\Pi_l.$$

Note that the number of elements in Π_i is

$$N_i=\binom{n}{n_i}.$$

(ii) A *meander* is a collection of positive paths $\{\pi_1,...,\pi_l\}$ $(\pi_i\in\Pi_i)$, with the property that the union

$$\pi_1\cup...\cup\pi_l$$

is a *tree* with endpoints $O_0,O_1,...,O_l$.

The set of all meanders is denoted by $\mathcal{M}$.

THEOREM 2.2.3. *There is a natural bijection between the codimension-1 faces of Δ and the set $\mathcal{M}$ of meanders.*

Proof. Since every face Θ of Δ is given by its supporting hyperplane, it follows from the exact sequence

$$0\to\mathrm{Ker}(\delta)\to L(E)\otimes\mathbf{R}\xrightarrow{\delta}L(D)\otimes\mathbf{R}\to 0$$

that this hyperplane can be described by a linear function

$$\lambda: L(E)\otimes\mathbf{R}\to\mathbf{R}$$

which vanishes on $\mathrm{Ker}(\delta)$ and satisfies the conditions

$$\lambda(v)\leqslant 1, \quad \text{for all } v\in L(E)\otimes\mathbf{R} \text{ with } \delta(v)\in\Delta,$$

and

$$\delta(v)\in\Theta \quad \text{if and only if} \quad \lambda(v)=1 \text{ and } \delta(v)\in\Delta.$$

Let us show that every meander $m=\{\pi_1,...,\pi_l\}\in\mathcal{M}$ defines such a linear function λ_m. We define the value of λ_m on $e\in E$ by the formula

$$\lambda_m(e):=1-\sum_{\{i:\pi_i\cap e\neq\varnothing\}}|\mathcal{R}_i|. \tag{1}$$

It follows that $\lambda_m(e)=1$ if the meander m does not intersect e, and $\lambda_m(e)$ is negative if m intersects e. Now we show that the linear function λ_m satisfies the requirement $\lambda_m|_{\mathrm{Ker}(\delta)}=0$. By Proposition 2.1.9, it suffices to prove that

$$\sum_{e\in\mathcal{R}_i}\lambda_m(e)=0, \tag{2}$$

for all $i\in\{1,2,...,l\}$, and

$$\sum_{e\in\mathcal{C}_b}\lambda_m(e)=\sum_{e\in\mathcal{C}_b^-}\lambda_m(e), \tag{3}$$

for all $b\in B$.

We remark first that every roof $\mathcal{R}_i$, $i\in\{1,2,...,l\}$, contains exactly one edge $e_i\in E$ intersecting the positive path $\pi_i\in m$, for which

$$\lambda_m(e_i):=1-|\mathcal{R}_i|<0.$$

On the other hand, $\lambda(e)=1$ for each $e\in\mathcal{R}_i$, $e\neq e_i$. It follows that

$$\sum_{e\in\mathcal{R}_i}\lambda_m(e)=0 \quad \text{for all } i\in\{1,2,...,l\}.$$

Now let $b\in B$ be an arbitrary box. Since the positive paths of the meander m form a tree, only the following three cases can occur:

Case 1. The meander m does not intersect edges in b. Then $\lambda_m(e)=1$ for all 4 edges of b, and hence (3) holds.

Case 2. The meander m intersects exactly two edges in b. Then m intersects exactly one edge $e' \in b$ belonging to $\mathcal{C}_b$ and exactly one edge $e'' \in b$ belonging to $\mathcal{C}_b^-$. By the formula (1) for λ_m, we have $\lambda_m(e') = \lambda_m(e'')$. So again the relation (3) holds.

Case 3. The meander m intersects exactly three edges in b. Then m intersects both edges $e', e'' \in b$ belonging to $\mathcal{C}_b^-$ and exactly one edge $e''' \in b$ belonging to $\mathcal{C}_b$. By (1),

$$\lambda_m(e''') = \lambda_m(e') + \lambda_m(e'') - 1.$$

Again the relation (3) holds.

Therefore, by Proposition 2.1.9, $\lambda_m|_{\mathrm{Ker}(\delta)} = 0$.

Let Θ_m be the face of Δ defined by the supporting affine hyperplane $\lambda_m(\cdot) = 1$. We claim that Θ_m has codimension 1. Since Θ_m is the convex hull of the lattice points in Δ corresponding to the edges $e \in E$ on which λ_m takes the value 1, it is sufficient to show that any linear function λ' satisfying $\lambda'|_{\mathrm{Ker}(\delta)} = 0$ and $\lambda'(v) = 1$ for all $v \in L(E) \otimes \mathbf{R}$ with $\delta(v) \in \Theta_m$ must coincide with λ_m. Indeed, by Proposition 2.1.9, the value of such a linear function λ' is uniquely determined on each edge e of each roof $\mathcal{R}_i$ $(1 \leqslant i \leqslant l)$:

$$\lambda'(e) = \begin{cases} 1 - |\mathcal{R}_i| & \text{if } \pi_i \cap e \neq \varnothing, \\ 1 & \text{otherwise.} \end{cases}$$

Next we remark that if for some box $b \in B$ we have shown that

$$\lambda'(e) = \lambda_m(e)$$

holds for all $e \in \mathcal{C}_b^-$, then, by Proposition 2.1.9 and (3), we obtain

$$\sum_{e \in \mathcal{C}_b} \lambda'(e) = \sum_{e \in \mathcal{C}_b} \lambda_m(e)$$

and therefore

$$\lambda'(e) = \lambda_m(e) \quad \text{for all } e \in \mathcal{C}_b,$$

since only one edge $e \in \mathcal{C}_b$ can be intersected by m (see Cases 1–3). Since we have established the equality $\lambda'(e) = \lambda_m(e)$ for all $e \in \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_l$, the above arguments imply the equality $\lambda'(e) = \lambda_m(e)$ for all $e \in E$.

Now we prove that any codimension-1 face Θ of Δ can be obtained from some meander $m \in \mathcal{M}$. For this purpose, it suffices to show that if a supporting linear function λ defines a face $\Theta \subset \Delta$, then there exists a meander $m \in \mathcal{M}$ with $\Theta \subset \Theta_m$. The latter is equivalent to the condition $\lambda(e) < 1$ for all edges $e \in E$ such that $e \cap m \neq \varnothing$.

First we remark that the linear function λ cannot attain the value 1 on all edges of the roof $\mathcal{R}_1$, because λ vanishes on the element $\varrho_1 \in \mathrm{Ker}(\delta)$ (see Proposition 2.1.9).

Now start a positive path π_1 at O_1 whose first nonempty intersection with edges of the opposite corner $\mathcal{C}_b^-$ of some box $b \in B$ occurs on an edge $e_1 \in \mathcal{R}_1$ with $\lambda(e_1) < 1$. Since

$$\sum_{e \in \mathcal{C}_b} \lambda(e) = \sum_{e \in \mathcal{C}_b^-} \lambda(e),$$

the value of λ on at least one of the two edges of $\mathcal{C}_b$ has to be strictly less than 1. We prolong our path through that edge and enter a next box, where the same reasoning applies. Continuing this, we complete a positive path π_1 from O_1 to O_0 crossing only edges where λ is strictly less than 1. Now we repeat this construction for each of the O_i in subsequent order, starting at O_2, etc. If in the process of constructing a positive path π_i we collide with some already constructed positive path π_j $(j<i)$, we just follow from this point the path π_j. In the end, we produce a meander with the required property.

We conclude that Θ_m $(m \in \mathcal{M})$ are all the codimension-1 faces of Δ. □

COROLLARY 2.2.4. *$\Delta(n_1, ..., n_l, n)$ is a reflexive polyhedron.*

Proof. The statement follows immediately from Theorem 2.2.3 and from the integrality of the supporting linear function λ_m (see Definition 4.1.5 in [3]). □

Definition 2.2.5. The complete rational polyhedral fan $\Sigma = \Sigma(n_1, ..., n_l, n)$ is the fan defined as the collection of cones over all faces of Δ. The toric variety $\mathbf{P}_\Sigma$ associated to the fan Σ will be denoted by $P = P(n_1, ..., n_l, n)$.

Using one of the equivalent characterizations of reflexive polyhedra (see Theorem 4.1.9 in [3]), we obtain from Corollary 2.2.4:

PROPOSITION 2.2.6. *$P(n_1, ..., n_l, n)$ is a Gorenstein toric Fano variety.* □

3. Further properties of $P(n_1, ..., n_l, n)$

3.1. Singular locus

Definition 3.1.1. Define $\widehat{P} = \widehat{P}(n_1, ..., n_l, n)$ to be the toric variety $\mathbf{P}_{\widehat{\Sigma}}$ associated to the fan $\widehat{\Sigma}$, obtained by refining the fan Σ to a simplicial one, whose one-dimensional cones are the same as the ones of Σ (i.e., they are generated by the lattice vectors $\{\delta(e), e \in E\} \subset L(D)$) and whose combinatorial structure is given by the following $|B| + l$ primitive collections:

$$\mathcal{R}_1,\ \mathcal{R}_2,\ ...,\ \mathcal{R}_l \text{ and } \mathcal{C}_b, \quad b \in B.$$

In other words, the cones of maximal dimension of the fan $\widehat{\Sigma}$ are defined by taking all edges $e \in E$ except one from each roof and from each corner.

PROPOSITION 3.1.2. *The variety $\widehat{P}$ is a small toric desingularization of P.*

Proof. We have to show that each cone of $\widehat{\Sigma}$ is contained in a cone of Σ, and each cone of $\widehat{\Sigma}$ is generated by a part of a basis. It suffices to prove the above properties for cones of $\widehat{\Sigma}$ of maximal dimension.

Choose an edge e_i in each roof $\mathcal{R}_i$ $(i=1,...,l)$ and an edge f_b in each corner $\mathcal{C}_b$, $b\in B$. This choice determines a $|D|$-dimensional cone σ in $\widehat{\Sigma}$. For each $i=1,...,l$ there exists a unique positive path from O_i to O_0 with the following two properties:

(i) π_i crosses the edge e_i;

(ii) if π_i enters a box b, then it crosses the edge f_b.

It is easy to see that the union $\pi_1\cup...\cup\pi_l$ of these paths is a meander. Indeed, if a union of positive paths as above is not a tree, then there must exist a box $b\in B$ with both edges of the corner $\mathcal{C}_b$ intersecting the union of positive paths. This contradicts the second of the above conditions. Therefore the set of edges $\{e_i\}\cup\{f_b\}$ defines uniquely a meander $m\in\mathcal{M}$, and the cone σ is contained in the cone over the face $\Theta_m\subset\Delta$. On the other hand, the elements $\{\varrho_i\}_{i=1,...,l}$ and $\{\varrho_b\}_{b\in B}$ together with the set

$$G_\sigma := E\setminus(\{e_i\}_{i=1,...,l}\cup\{f_b\}_{b\in B})$$

form a $\mathbf{Z}$-basis of $L(E)$. By Proposition 2.1.9, the set of generators of σ (i.e., the δ-image of G_σ) is a $\mathbf{Z}$-basis of $L(D)$.

The desingularization morphism $\widehat{P}\to P$ induced by the refinement $\widehat{\Sigma}$ of Σ is small (i.e., contracts no divisor), because the sets of 1-dimensional cones in $\widehat{\Sigma}$ and Σ are the same. □

There is another way to describe $\widehat{P}$, namely as an iterated toric fibration over $\mathbf{P}^1$: One starts with the product of projective spaces

$$\mathbf{P}^{|\mathcal{R}_1|-1}\times...\times\mathbf{P}^{|\mathcal{R}_l|-1}$$

corresponding to the roofs. Then one chooses a corner $\mathcal{C}_b$ of a box $b\in B$ whose opposite corner $\mathcal{C}_b^-$ belongs to a roof. This choice allows us to define a toric bundle over $\mathbf{P}^1$ with the fibre $\mathbf{P}^{|\mathcal{R}_1|-1}\times...\times\mathbf{P}^{|\mathcal{R}_l|-1}$. Then one adds a new corner $\mathcal{C}_{b'}$ of a box $b'\in B$ whose opposite corner $\mathcal{C}_{b'}^-$ is contained in the union of roofs and $\mathcal{C}_b$, etc. At each stage of this process one gets a toric fibre bundle over $\mathbf{P}^1$, with fibre the space constructed in the previous step. Using this description of $\widehat{P}$, one obtains an alternative proof of the fact that the anticanonical divisor on P is Cartier and ample, i.e., that the polyhedron Δ is reflexive.

Definition 3.1.3. Let $b\in B$ be an arbitrary box. Define $W_b\subset P$ to be the closure of the torus orbit in P corresponding to the 3-dimensional cone σ_b generated by the δ-image of the 4-element set b.

THEOREM 3.1.4. *The singular locus of P consists of codimension-3 strata W_b, $b \in B$. These are conifold strata, i.e., transverse to a generic point of W_b the variety P has an ordinary double point.*

Proof. Since the desingularization morphism $\varphi: \widehat{P} \to P$ is small, P is smooth in codimension 2. Moreover, the singular locus of P is precisely the union of toric strata in P over which the morphism φ is not bijective. According to the main result of [26], the exceptional locus $Ex(\varphi) \subset \widehat{P}$ (i.e., $\varphi^{-1}(\mathrm{Sing}(P))$) is the union of toric strata covered by rational curves contracted by φ. On the other hand, since $\widehat{P}$ is an iterated toric bundle, the Mori cone $\overline{NE}(\widehat{P})$ is a simplicial cone generated by the classes of the primitive relations

$$\sum_{e \in \mathcal{R}_i} \delta(e) = 0, \quad i = 1, \dots, l,$$

and

$$\sum_{e \in \mathcal{C}_b} \delta(e) = \sum_{e \in \mathcal{C}_b^-} \delta(e), \quad b \in B,$$

(see §§2 and 4 in [2]). Since the morphism φ is defined by the semiample anticanonical class of $\widehat{P}$, it contracts exactly the extremal rays in $\overline{NE}(\widehat{P})$ defined by the primitive relations corresponding to the boxes $b \in B$. The rational curves representing each such class cover the codimension-2 strata $\widehat{W_b}$, $b \in B$, corresponding to the 2-dimensional cones in $\widehat{\Sigma}$ spanned by the δ-images of the edges forming the opposite corner $\mathcal{C}_b^-$. These strata are contracted, with $\mathbf{P}^1$-fibres, to the codimension-3 strata W_b in P corresponding to the 3-dimensional cones $\sigma_b \in \Sigma$ over the quadrilateral faces Θ_b of Δ whose vertices are δ-images of the edges in b ($b \in B$). It follows that $\bigcup_{b \in B} W_b$ is exactly the singular locus of P. □

3.2. Canonical flat smoothing

Let $F = F(n_1, \dots, n_l, n)$ be a partial flag manifold. The semiample line bundles

$$\mathcal{O}(C_1), \ \dots, \ \mathcal{O}(C_l)$$

associated to the Schubert divisors $C_1, \dots, C_l$ define the Plücker embedding of F into a product of projective spaces:

$$\phi: F \hookrightarrow \mathbf{P}^{N_1 - 1} \times \dots \times \mathbf{P}^{N_l - 1}, \quad \text{where } N_i = \binom{n}{n_i}.$$

We will always consider F as a smooth projective variety together with this embedding.

We describe now an embedding of P in the same product of projective spaces.

Definition 3.2.1. For each $e \in E$, let H_e be the toric Weil divisor on P determined by the 1-dimensional cone of Σ spanned by the vector $\delta(e)$.

For every edge $e \in \bigcup_{i=1}^{l} \mathcal{R}_i$ which is part of a roof, denote by $U(e)$ the subset of E consisting of the edge e, together with all edges $f \in E$ which are either directly below e in the graph Γ, if e is horizontal, or directly to the left of e, if e is vertical.

Fix $1 \leqslant i \leqslant l$. For $e \in \mathcal{R}_i$ consider the Weil divisor $\sum_{f \in U(e)} H_f$.

LEMMA 3.2.2. *For each* $e \in \mathcal{R}_i$, *the Weil divisor* $\sum_{f \in U(e)} H_f$ *is Cartier. Moreover, if* $e' \in \mathcal{R}_i$ *is another edge in the same roof, then the associated divisor* $\sum_{f' \in U(e')} H_{f'}$ *is linearly equivalent to* $\sum_{f \in U(e)} H_f$.

Proof. To each edge $e \in \mathcal{R}_i$, and each positive path $\pi \in \Pi_i$ joining O_i with O_0, we associate a linear function $\pi[e]: L(E) \to \mathbf{Z}$ defined by

$$\pi[e](g) = \begin{cases} 0 & \text{if } \pi \cap g = \varnothing \text{ and } g \notin U(e), \\ 0 & \text{if } \pi \cap g \neq \varnothing \text{ and } g \in U(e), \\ -1 & \text{if } \pi \cap g = \varnothing \text{ and } g \in U(e), \\ 1 & \text{if } \pi \cap g \neq \varnothing \text{ and } g \notin U(e). \end{cases}$$

It is an elementary exercise to check that $\pi[e]$ vanishes on the elements

$$\varrho_j = \sum_{g \in \mathcal{R}_j} g, \quad j \in \{1, ..., l\},$$

and

$$\varrho_b = \sum_{g \in \mathcal{C}_b} g - \sum_{g \in \mathcal{C}_b^-} g, \quad b \in B.$$

It follows from Proposition 2.1.9 that $\pi[e]$ descends to a linear function on $L(D)$.

To show that $\sum_{f \in U(e)} H_f$ is Cartier, it suffices to construct for each maximal-dimensional cone σ in Σ an integral linear function

$$\lambda_\sigma: L(E) \to \mathbf{Z}$$

which vanishes on $\ker(\delta)$ and satisfies

(i) $\lambda_\sigma(g) = 0$, for all $g \in E$ such that $\delta(g) \in \sigma$ and $g \notin U(e)$;

(ii) $\lambda_\sigma(g) = -1$, for all $g \in E$ such that $\delta(g) \in \sigma$ and $g \in U(e)$.

By Theorem 2.2.3, every maximal cone σ is determined by a meander $m = (\pi_1, \pi_2, ..., \pi_l)$, and $\delta(g) \in \sigma$ if and only if the meander does not intersect the edge g (cf. the proof of Theorem 2.2.3). It follows that

$$\lambda_\sigma := \pi_i[e]$$

satisfies the above conditions, where π_i is the positive path in m which joins O_i with O_0. Hence $\sum_{f\in U(e)} H_f$ is Cartier. Note that the functional λ_σ defined above does not depend on the positive paths π_j $(j\neq i)$ in m that do not intersect the roof $\mathcal{R}_i$.

To prove the second part of the lemma, define an integral linear function $\mu: L(E)\to\mathbf{Z}$ by

$$\mu(g)=\begin{cases}-1 & \text{if } g\in U(e),\\ 1 & \text{if } g\in U(e'),\\ 0 & \text{otherwise.}\end{cases}$$

As above, one can easily check that μ vanishes on $\ker(\delta)$, and hence it descends to a linear function on $L(D)$. The descended linear function defines a rational function on P, whose divisor is $\sum_{f\in U(e)} H_f-\sum_{f'\in U(e')} H_{f'}$. This finishes the proof of the lemma. □

Definition 3.2.3. For each $i=1,2,\dots,l$, the line bundle associated to the roof $\mathcal{R}_i$ is

$$\mathcal{L}_i:=\mathcal{O}\Bigl(\sum_{f\in U(e)} H_f\Bigr),$$

for some edge $e\in\mathcal{R}_i$.

It follows from Lemma 3.2.2 that $\mathcal{L}_i$ does not depend on the choice of the edge $e\in\mathcal{R}_i$.

We note that for each maximal-dimensional cone σ the linear function λ_σ defined in the proof of Lemma 3.2.2 satisfies $\lambda_\sigma(g)\geqslant 0$ for all $g\in E$ such that $g\notin\sigma$. This implies that the line bundle $\mathcal{O}\bigl(\sum_{f\in U(e)} H_f\bigr)$ is generated by global sections (cf. [14, p. 68]). We will now identify the space of global sections.

The Cartier divisor $\sum_{f\in U(e)} H_f$ determines a rational convex polyhedron $\Delta[e]$ in the dual vector space $L(D)^*\otimes\mathbf{R}$, given by

$$\Delta[e]=\{\lambda\in L(D)^*\otimes\mathbf{R}:\lambda(\delta(g))\geqslant -1\ \forall g\in U(e),\ \lambda(\delta(g))\geqslant 0\ \forall g\in E\setminus U(e)\}.$$

The space of global sections of the line bundle $\mathcal{O}\bigl(\sum_{f\in U(e)} H_f\bigr)$ has a natural basis, indexed by the lattice points in $\Delta[e]$. By its very definition, for each positive path $\pi\in\Pi_i$, the linear function $\pi[e]$ introduced in the proof of Lemma 3.2.2 gives such a lattice point.

PROPOSITION 3.2.4. *For each $i=1,2,\dots,l$, the space of global sections of $\mathcal{L}_i$ has a natural basis parametrized by the set Π_i of positive paths connecting O_i and O_0.*

Proof. Choose an edge $e\in\mathcal{R}_i$. We have to show that the only lattice points in $\Delta[e]$ are the ones given by $\pi[e]$, $\pi\in\Pi_i$. Let $\lambda: L(E)\to\mathbf{Z}$ be any linear function vanishing on $\ker(\delta)$, and such that the descended linear function is in $\Delta[e]$.

Since on the one hand λ vanishes on every

$$\varrho_j=\sum_{g\in\mathcal{R}_j} g,\quad j\in\{1,\dots,l\},$$

and on the other hand λ can be negative only on edges in $U(e)$, there are exactly two possibilities:

(I) $\lambda(g)=0$ for all $g\in\bigcup_{j=1}^{l}\mathcal{R}_j$;

(II) $\lambda(e)=-1$, there exists an edge h in the ith roof $\mathcal{R}_i$ with $\lambda(h)=1$, and $\lambda(g)=0$ for all $g\in\bigcup_{j\neq i}\mathcal{R}_j\setminus\{e,f\}$.

If (I) holds, then we start a positive path π at O_i that intersects the roof $\mathcal{R}_i$ at the edge e. Let b be the box containing e in its opposite corner and let f be the other edge in $U(e)$ contained in this box. If $\lambda(f)=0$, we prolong the path through the edge f, and enter a next box b', where we have the same situation as before (i.e., there is another edge $f'\in U(e)$, and if $\lambda(f')=0$, then we prolong the path through f', etc.). So we may assume that $\lambda(f)=-1$. The edge f is part of the corner $\mathcal{C}_b$ of b. Let f'' be the other edge in $\mathcal{C}_b$. Since λ vanishes on all elements

$$\varrho_b=\sum_{g\in\mathcal{C}_b} g-\sum_{g\in\mathcal{C}_b^-} g, \quad b\in B,$$

$\lambda(f'')$ must be strictly positive (hence at least 1). We prolong the path π through the edge f'', and enter a next box b'', for which f'' is part of the opposite corner. Now λ is nonnegative on all four edges of b'', and $\lambda(f'')\geqslant 1$. It follows that there must be an edge f''' in the corner $\mathcal{C}_{b''}$, with $\lambda(f''')\geqslant 1$. We prolong the path through this edge, and enter a next box, where the same reasoning applies. Continuing this, we complete eventually a positive path π. Consider the linear function $\nu:=\lambda-\pi[e]$ on $L(E)$. By construction, and the definition of $\pi[e]$, the functional ν is nonnegative on all edges $g\in E$. On the other hand, ν vanishes on the generators of $\ker(\delta)$ described in Proposition 2.1.9, since both λ and $\pi[e]$ do. We claim that ν is identically zero on $L(E)\otimes\mathbf{R}$. Indeed, since ν is nonnegative and $\nu\left(\sum_{g\in\mathcal{R}_j} g\right)=0$ for $j=1,2,\dots,l$, it follows that ν takes the value zero on each edge in the union of all roofs. Similarly, if ν vanishes on each of the edges of the opposite corner $\mathcal{C}_b^-$ of some box, then it must vanish on each of the edges of the corner $\mathcal{C}_b$ as well. From these two facts, one obtains inductively that ν takes the value zero on every $g\in E$. Hence $\lambda=\pi[e]$.

Assume now that (II) holds. In this case, we start a positive path π at O_i that intersects the roof $\mathcal{R}_i$ at the edge h. A reasoning entirely similar to that in case (I) shows that the path π can be completed such that the functional $\lambda-\pi[e]$ is nonnegative on every edge. Hence we obtain again $\lambda=\pi[e]$. □

Definition 3.2.5. The $(|D|+l)$-dimensional cone $C=C(n_1,\dots,n_l,n)$ associated to the flag manifold F is the convex polyhedral cone in the space $\mathrm{Im}(\partial)\otimes\mathbf{R}$ spanned by the vectors

$$\partial(e)\in\mathrm{Im}(\partial)\otimes\mathbf{R}\cong\mathbf{R}^{|D|+|S|-1}$$

with $e \in E$. We denote by C^* the dual cone in the dual space $\mathrm{Im}(\partial)^* \otimes \mathbf{R}$.

Definition 3.2.6. Let $\pi \in \Pi$ be any positive path $\pi \in \Pi$. We associate to π a linear function

$$\lambda_\pi : L(E) \to \mathbf{Z}$$

by setting $\lambda_\pi(e)=1$ if the path π crosses the edge e, and $\lambda_\pi(e)=0$ if it does not.

Remark 3.2.7. If the path π enters a box $b \in B$, then it does so by crossing an edge which is part of the opposite corner $\mathcal{C}_b^-$, and it has to leave b by crossing an edge which is part of the corner $\mathcal{C}_b^-$. It follows that the corresponding functional λ_π is zero on $\mathrm{Ker}(\partial)=H_1(\Gamma)$, and hence it descends to a functional on $L(E)/\mathrm{Ker}(\partial)=\mathrm{Im}(\partial)$, still denoted by λ_π. By definition, λ_π is a lattice point in the dual cone $C^* \subset \mathrm{Im}(\partial)^* \otimes \mathbf{R}$.

THEOREM 3.2.8. *The semigroup of lattice points in C^* is minimally generated by the set of all λ_π, where π runs over the set Π of positive paths.*

Proof. Let $\lambda: L(E) \to \mathbf{Z}$ with $\lambda|_{\mathrm{Ker}(\partial)}=0$ and $\lambda(e) \geqslant 0$ for all $e \in E$. We define the *weight* of λ to be

$$w(\lambda) = \sum_{e \in E} \lambda(e).$$

It is clear that $w(\lambda) \geqslant 0$, and that $w(\lambda)=0$ if and only if $\lambda=0$. Note also that $w(\lambda_\pi)=n$ for all $\pi \in \Pi$.

The statement of the theorem will be proved if we show that $w(\lambda) \geqslant n$ for all nonzero integral linear functions $\lambda: L(E) \to \mathbf{Z}$ with $\lambda|_{\mathrm{Ker}(\partial)}=0$ and $\lambda(e) \geqslant 0$ for all $e \in E$, and, moreover, any such λ is a nonnegative integral linear combination of λ_π ($\pi \in \Pi$).

By Proposition 2.1.9, the requirement $\lambda|_{\mathrm{Ker}(\partial)}=0$ is equivalent to $\lambda(\varrho_b)=0$ for all $b \in B$, or

$$\sum_{e \in \mathcal{C}_b} \lambda(e) = \sum_{e \in \mathcal{C}_b^-} \lambda(e) \quad \text{for all } b \in B.$$

As in the proof of Proposition 3.2.4, the above condition implies that if $\lambda \neq 0$, then there exists a roof $\mathcal{R}_i$ containing an edge e on which λ is nonzero (hence $\lambda(e) \geqslant 1$). We start to construct a positive path π_i from O_i by choosing its edges in such a way that e is the first edge of the graph Γ intersected by π_i. Let $b \in B$ be a box containing e in its opposite corner (i.e., $e \in \mathcal{C}_b^-$). Since $\lambda(\varrho_b)=0$, there must be an edge $f \in \mathcal{C}_b$ such that $\lambda(f)>0$ (hence $\lambda(f) \geqslant 1$). We prolong the path π_i through f and enter a next box b', for which $f \in \mathcal{C}_{b'}^-$. Again there must exist an edge $g \in \mathcal{C}_{b'}$ such that $\lambda(g) \geqslant 1$, etc. Continuing this process, we eventually obtain a positive path π_i, which only crosses edges e of Γ having the property $\lambda(e) \geqslant 1$. This shows that $\lambda' := \lambda - \lambda_{\pi_i}$ is again an integral nonnegative linear functional on C. On the other hand,

$$w(\lambda') = w(\lambda) - w(\lambda_{\pi_i}) = w(\lambda) - n.$$

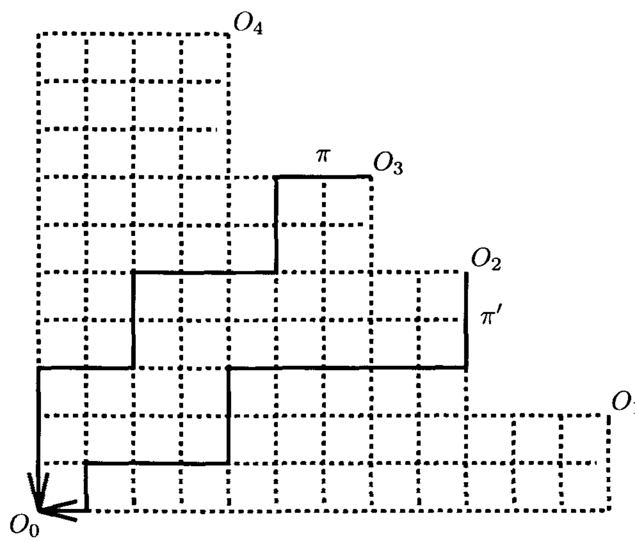


Fig. 6

Since $w(\lambda')\geqslant 0$, this shows that $w(\lambda)\geqslant n$. By induction on $w(\lambda)$, we can assume that λ' is already a nonnegative integral linear combination of λ_π, and hence so is $\lambda=\lambda'+\lambda_{\pi_i}$. □

Definition 3.2.9. We define a partial ordering on the set Π of positive paths by declaring that $\pi\geqslant\pi'$ if the path π runs above the path π'. See Figure 6.

Remark 3.2.10. It is easy to see that the set Π of positive paths together with the above partial ordering is a distributive lattice. The maximum $\max(\pi,\pi')$ for any two paths π and π' is the path bounding the union of the regions under π and π'; similarly, $\min(\pi,\pi')$ bounds the intersection of these regions.

Definition 3.2.11. Consider the partition of the set of independent variables $\{z_\pi\}_{\pi\in\Pi}$ into l disjoint subsets

$$\{z_\pi\}_{\pi\in\Pi_i},\quad i=1,\dots,l,$$

and define $X=X(n_1,\dots,n_l,n)$ to be the subvariety of

$$\mathbf{P}^{N_1-1}\times\mathbf{P}^{N_2-1}\times\dots\times\mathbf{P}^{N_l-1}$$

given by the l-homogeneous quadratic equations

$$z_\pi z_{\pi'}-z_{\min(\pi,\pi')}z_{\max(\pi,\pi')}=0, \tag{4}$$

for all pairs of noncomparable elements $\pi,\pi'\in\Pi$.

The variety X has been investigated by N. Gonciulea and V. Lakshmibai in the papers [18], [19], where the following result has been proved:

THEOREM 3.2.12. (i) $X(n_1, ..., n_l, n)$ *is a* $|D|$*-dimensional, irreducible, normal, toric variety.*

(ii) *There exists a flat deformation*

$$\varrho: \mathcal{X} \to \operatorname{Spec}(\mathbf{C}[t])$$

such that $\varrho^{-1}(0) = X(n_1, ..., n_l, n)$ *and* $\varrho^{-1}(t) = F(n_1, ..., n_l, n)$ *for all* $t \neq 0$.

The next theorem describes an isomorphism

$$X(n_1, ..., n_l, n) \cong P(n_1, ..., n_l, n).$$

THEOREM 3.2.13. *Let* $P = P(n_1, ..., n_l, n)$ *be the toric variety associated with a partial flag manifold* $F = F(n_1, ..., n_l, n)$. *The line bundles* $\mathcal{L}_i$ $(i = 1, ..., l)$ *define an embedding*

$$\psi: P \hookrightarrow \mathbf{P}^{N_1-1} \times \mathbf{P}^{N_2-1} \times ... \times \mathbf{P}^{N_l-1},$$

whose image coincides with the toric variety $X(n_1, ..., n_l, n)$.

Proof. We have $X = \operatorname{Proj}(\mathbf{C}[z_\pi; \pi \in \Pi]/\mathcal{I})$, with $\mathcal{I}$ the ideal generated by the quadratic polynomials in (4), and Proj is taken with respect to the $\mathbf{Z}^l$-grading given by

$$\deg(z_\pi) = (0, ..., 0, \overset{i}{1}, 0, ..., 0), \quad \text{if } \pi \in \Pi_i.$$

If we identify Π with the set $\{\lambda_\pi, \pi \in \Pi\} \subset \operatorname{Im}(\partial)^*$, then $\mathcal{I}$ is the *toric ideal* (see the definition in [28, p. 31]) associated to this set (this is a standard fact about the ideals associated to distributive lattices; see for example Theorem 4.3 in [18] for a proof). Let Y be the affine toric variety $\operatorname{Spec}(\mathbf{C}[z_\pi; \pi \in \Pi]/\mathcal{I})$. By Theorem 3.2.8 and Proposition 13.5 in [28], Y coincides with the affine toric variety defined by the cone $C \subset \operatorname{Im}(\partial) \otimes \mathbf{R}$, i.e., $\mathbf{C}[z_\pi; \pi \in \Pi]/\mathcal{I}$ can be identified with the ring $\mathbf{C}[S_C]$ determined by the semigroup S_C of lattice points in the dual cone C^*.

Pick an edge $e_i \in \mathcal{R}_i$ for each $1 \leqslant i \leqslant l$, and identify the line bundle $\mathcal{L}_i$ with $\mathcal{O}(\sum_{f \in U(e_i)} H_f)$. For each $1 \leqslant i \leqslant l$, let $\Delta[e_i] \subset L(D)^* \otimes \mathbf{R}$ be the supporting polyhedron for the global sections of the line bundle $\mathcal{L}_i$ (cf. Proposition 3.2.4); recall that the lattice points in $\Delta[e_i]$ are given by the linear functions $\pi[e_i]$ $(\pi \in \Pi_i)$ defined in the proof of Lemma 3.2.2. Define now for each i a linear function $v[e_i]: L(E) \to \mathbf{Z}$ by

$$v[e_i](f) = \begin{cases} 1 & \text{if } f \in U(e), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $v[e_i]$ descends to a functional on $\operatorname{Im}(\partial)$, and that for every path $\pi \in \Pi_i$ the functional λ_π in Definition 3.2.6 coincides with $\pi[e_i] + v[e_i]$. For each $1 \leqslant i \leqslant l$, let

$\sigma_i \subset \mathrm{Im}(\partial)^* \otimes \mathbf{R}$ be the cone over the translated polyhedron $v[e_i]+\Delta[e_i]$. Then the Minkowski sum $\sigma := \sigma_1 + \ldots + \sigma_l$ of these cones coincides with the cone C^*, since both σ and C^* are generated by the vectors $\{\lambda_\pi, \pi \in \Pi\}$. It follows that $P \cong \mathrm{Proj}(\mathbf{C}[S_C])$, where Proj is taken with respect to the natural $\mathbf{Z}^l$-grading induced by the decomposition of C^* into the Minkowski sum of the σ_i.

For each $1 \leqslant i \leqslant l$, choose an ordering $\{\pi_{i,1}, \pi_{i,2}, \ldots, \pi_{i,N_i}\}$ of the set Π_i. Let $s_{\pi_{i,j}} \in H^0(P, \mathcal{L}_i)$ denote the section determined by $\pi_{i,j}$ $(i=1, \ldots, l,\ j=1, \ldots, N_i)$. The line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_l$ define a morphism

$$\psi: P \to \mathbf{P}^{N_1-1} \times \mathbf{P}^{N_2-1} \times \ldots \times \mathbf{P}^{N_l-1},$$
$$x \mapsto ([s_{\pi_{1,1}}(x) : \ldots : s_{\pi_{1,N_1}}(x)], \ldots, [s_{\pi_{l,1}}(x) : \ldots : s_{\pi_{l,N_l}}(x)]).$$

By the above arguments, ψ is the isomorphism

$$\mathrm{Proj}(\mathbf{C}[S_C]) \to \mathrm{Proj}(\mathbf{C}[z_\pi; \pi \in \Pi]/\mathcal{I}),$$

and the theorem is proved. □

From Theorems 3.2.12 and 3.2.13 we obtain

COROLLARY 3.2.14. *There exists a flat deformation*

$$\varrho: \mathcal{X} \to \mathrm{Spec}(\mathbf{C}[t])$$

such that $\varrho^{-1}(0)=P(n_1, \ldots, n_l, n)$ *and* $\varrho^{-1}(t)=F(n_1, \ldots, n_l, n)$ *for all* $t \neq 0$.

Remark 3.2.15. A description of the singular locus of P was conjectured by N. Gonciulea and V. Lakshmibai in the case when F is a Grassmannian (see [19]). Our Theorem 3.1.4 proves this conjecture and its generalization for arbitrary partial flag manifolds F.

4. Quantum differential systems

4.1. Quantum $\mathcal{D}$-module

In order to explain our mirror construction, we give a short overview of the quantum cohomology $\mathcal{D}$-module. The reader is referred to [15], [23] for details.

Let V be a smooth projective variety. Denote by $\{T_a\}_a$ and $\{T^a\}_a$ two homogeneous bases of $H^*(V, \mathbf{Q})$, dual with respect to the Poincaré pairing, i.e., such that

$$\langle T_a, T^b \rangle = \delta_{a,b}.$$

We will consider only the even-degree part of $H^*(V,\mathbf{Q})$ and will assume that $H^2(V,\mathbf{Z})$ and $H_2(V,\mathbf{Z})$ are torsion-free. We denote by 1 the fundamental class of V.

To simplify the exposition, suppose that there is a basis $\{p_i, i=1,2,...,l\}$ of $H^2(V,\mathbf{Z})$ consisting of nef-divisors. Let $NE(V)$ be the Mori cone of V.

Introduce formal parameters q_i, $i=1,...,l$, and let $\mathbf{Q}[[q_1,...,q_l]]$ be the ring of formal power series. The small quantum cohomology ring of V will be denoted by $QH^*(V)$. This is the free $\mathbf{Q}[[q_1,...,q_l]]$-module $H^*(V,\mathbf{Q})\otimes_{\mathbf{Q}}\mathbf{Q}[[q_1,...,q_l]]$, together with a new multiplication given by

$$T_a\circ T_b=\sum_{\beta\in NE(V)}\prod_{i=1}^{l} q_i^{\langle p_i,\beta\rangle}\Big(\sum_c I^V_{3,\beta}(T_aT_bT_c)T^c\Big),$$

with $I^V_{3,\beta}(T_aT_bT_c)$ the *3-point, genus-0, Gromov–Witten invariants* of V.

Remark 4.1.1. For the case of a partial flag manifold, the small quantum cohomology ring is well understood. A presentation of this ring is known ([1], [20], [21]), as well as explicit formulas for quantum multiplication ([11]).

The operators of quantum multiplication with the generators p_i give the *quantum differential system*, a consistent first-order partial differential system (see e.g. [15]):

$$\hbar\frac{\partial}{\partial t_i}\vec{S}=p_i\circ\vec{S},\quad i=1,...,l,$$
$$\hbar\frac{\partial}{\partial t_0}\vec{S}=1\circ\vec{S},$$

where $\vec{S}$ is an $H^*(V,\mathbf{Q})$-valued function in formal variables t_0 and $t_i=\log q_i$, $i=1,...,l$. Here $\hbar$ is an additional parameter.

Remarkably, a complete set of solutions to this system can be written down explicitly in terms of the so-called *gravitational descendants* [15]:

$$\vec{S}^V_a:=e^{t_0/\hbar}\bigg(e^{pt/\hbar}T_a+\sum_{\beta\in NE(V)-0} q^{\langle p,\beta\rangle}\sum_b T^b\int_{[\overline{M}_{0,2}(V,\beta)]}\frac{e_1^*(e^{pt/\hbar}T_a)}{\hbar-c}\cup e_2^*(T_b)\bigg).$$

Here $\overline{M}_{0,2}(V,\beta)$ is Kontsevich's space of stable maps, with evaluation morphisms e_1,e_2: $\overline{M}_{0,2}(V,\beta)\to V$ at the two marked points, $[\overline{M}_{0,2}(V,\beta)]$ is the *virtual fundamental class* ([8], [25]), and c is the first Chern class of the line bundle over $\overline{M}_{0,2}(V,\beta)$ given by the cotangent line at the first marked point. Finally, pt and $q^{\langle p,\beta\rangle}$ are shorthand notations for $\sum_i p_it_i$ and $\prod_i q_i^{\langle p_i,\beta\rangle}$ respectively.

The *quantum $\mathcal{D}$-module* of V is the $\mathcal{D}$-module generated by the functions $\langle\vec{S},1\rangle$ for all solutions $\vec{S}$ to the above differential system.

A general conjecture about the structure of quantum $\mathcal{D}$-modules is Givental's version of the mirror conjecture [16]:

CONJECTURE 4.1.2. *There exists a family $(M_q, \mathcal{F}_q, \omega_q)$ of (possibly noncompact) complex manifolds M_q, having the same dimension as V, together with holomorphic functions $\mathcal{F}_q$ and holomorphic volume forms ω_q such that the $\mathcal{D}$-module generated by integrals*

$$\int_{\gamma\subset M_q} e^{(\mathcal{F}_q+t_0)/\hbar}\omega_q,$$

where γ are suitable Morse-theoretic middle-dimension cycles of the function $\mathrm{Re}(\mathcal{F}_q)$*, is equivalent to the quantum $\mathcal{D}$-module of V.*

4.2. Complete intersections

Now assume that V is Fano. Let X be the zero-locus of a generic section of a decomposable rank-r vector bundle

$$\mathcal{E} = \bigoplus_{j=1}^{r} L_j,$$

such that each L_j is generated by global sections. In such a situation one can also define a quantum ring $QH^*(\mathcal{E})$ over the coefficient ring $\mathbf{Q}[[q_1, \dots, q_l]]$ which encodes some of the enumerative geometry of rational curves on the complete intersection X. This leads to a quantum differential system for $(V, \mathcal{E})$ (see [17], [23]). We define degrees of q_i's by requiring that

$$c_1(TV) - c_1(\mathcal{E}) = \sum (\deg q_i) p_i.$$

Furthermore, we suppose that all degrees of q_i are nonnegative (this is equivalent to the condition that $-K_X$ is nef). One can write down a similar complete set of solutions to the quantum differential system for $(V, \mathcal{E})$ [17]:

$$\vec{S}_a^{\mathcal{E}} := e^{t_0/\hbar}\Big(e^{pt/\hbar}T_a + \sum_{\beta\in NE(V)-0} q^{\langle p,\beta\rangle} \sum_b T^b \int_{[\overline{M}_{0,2}(V,\beta)]} \frac{e_1^*(e^{pt/\hbar}T_a)}{\hbar - c} \cup e_2^*(T_b) \cup E_\beta\Big),$$

where E_β is the Euler class of the vector bundle on $\overline{M}_{0,2}(V,\beta)$ whose fibre over a point $(C, \mu; x_1, x_2)$ is the subspace of $H^0(\mu^*\mathcal{E})$ consisting of sections vanishing at x_2, and the rest of the notations are as above.

Consider the cohomology-valued functions

$$S_V := \sum_a \langle \vec{S}_a^V, 1\rangle T^a$$

and

$$S_{\mathcal{E}} := \sum_a \langle \vec{S}_a^{\mathcal{E}}, c_r(\mathcal{E})\rangle T^a.$$

These functions are given explicitly by the expressions

$$S_V = e^{(t_0+pt)/\hbar}\left(1+\sum_{\beta\in NE(V)-0} q^{\langle p,\beta\rangle}(e_1)_*\left(\frac{1}{\hbar-c}\right)\right)$$

and

$$S_{\mathcal{E}} = e^{(t_0+pt)/\hbar}\left(c_r(\mathcal{E})+\sum_{\beta\in NE(V)-0} q^{\langle p,\beta\rangle}(e_1)_*\left(\frac{e_1^*(c_r(\mathcal{E}))\cdot E'_\beta}{\hbar-c}\right)\right),$$

where now E'_β is the Euler class of the vector bundle on $\overline{M}_{0,2}(V,\beta)$ whose fibre over a point $(C,\mu;x_1,x_2)$ is the subspace of $H^0(\mu^*\mathcal{E})$ consisting of sections vanishing at x_1.

Remark 4.2.1. If we view X as an abstract variety, the general theory in §4.1 gives an $H^*(X,\mathbf{Q})$-valued function S_X. The functions S_X and $S_{\mathcal{E}}$ are closely related. For example, if $i^*\colon H^2(V,\mathbf{Z})\tilde{\to}H^2(X,\mathbf{Z})$, where $i\colon X\hookrightarrow V$ is the inclusion, then $i_*(S_X)=S_{\mathcal{E}}$.

Now consider a new cohomology-valued function

$$I_{\mathcal{E}} = e^{(t_0+pt)/\hbar}\left(c_r(\mathcal{E})+\sum_{\beta\in NE(V)-0} q^{\langle p,\beta\rangle}\prod_j\prod_{m=0}^{\langle c_1(L_j),\beta\rangle}(c_1(L_j)+m\hbar)(e_1)_*\left(\frac{1}{\hbar-c}\right)\right).$$

In general it is very hard to compute S_X or $S_{\mathcal{E}}$ explicitly. However, note that $I_{\mathcal{E}}$ can be computed directly from the function S_V associated to the ambient manifold, which in many cases turns out to be more tractable. It is therefore extremely useful to have a result relating $S_{\mathcal{E}}$ and $I_{\mathcal{E}}$. Extending ideas of Givental, B. Kim [23] has recently proved the following theorem, which applies to the cases considered in this paper:

THEOREM 4.2.2. *If V is a homogeneous space and $X\subset V$ is the zero-locus of a generic section of a nonnegative decomposable vector bundle $\mathcal{E}$, then $S_{\mathcal{E}}$ and $I_{\mathcal{E}}$ coincide up to a weighted homogeneous triangular change of variables*:

$$t_0\to t_0+f_0(q)\hbar+f_{-1}(q),\quad \log q_i\to\log q_i+f_i(q),\quad i=1,\dots,l,$$

where $f_{-1},f_0,f_1,\dots,f_l$ are weighted homogeneous formal power series supported in $NE(V)-0$, with $\deg f_{-1}=1$ and $\deg f_i=0$, $i=0,1,\dots,l$.

In particular, this implies that the coefficient Φ_V of the cohomology class $1\in H^*(V,\mathbf{Q})$ in S_V, and the coefficient Φ_X of $c_r(\mathcal{E})$ in $I_{\mathcal{E}}$ (specialized to $\hbar=1$, $t_0=0$) are related in a very simple way. Namely, if

$$\Phi_V=\sum_{\beta\in NE(V)-0} a_\beta q^{\langle p,\beta\rangle},\quad \Phi_X=\sum_{\beta\in NE(V)-0} b_\beta q^{\langle p,\beta\rangle},$$

then

$$b_\beta = a_\beta \prod_{i=1}^{r} (\langle c_1(L_i), \beta\rangle !). \tag{5}$$

We will refer to Theorem 4.2.2 as the *quantum hyperplane section theorem.* The relation (5) above was called the "trick with factorials" in [6].

5. The mirror construction

In this section we give a partially conjectural mirror construction for partial flag manifolds, and use it to obtain an explicit hypergeometric series as the power-series expansion of the integral representation. The case of Calabi–Yau complete intersections is then discussed in some detail.

5.1. Hypergeometric solutions for partial flag manifolds

Let $F=F(n_1, ..., n_l, n)$ be a partial flag manifold. In the notations of §2, we introduce l independent variables q_i, $i=1,2,...,l$ (each q_i corresponds to the roof $\mathcal{R}_i$), $|B|$ independent variables $\tilde{q}_b$, $b\in B$, and $|E|$ independent variables y_e, $e\in E$. Consider the following set of algebraically independent polynomial equations:

(i) Roof equations: for $i=1,2,...,l$,

$$\mathcal{F}_i := \prod_{e\in\mathcal{R}_i} y_e - q_i = 0. \tag{6}$$

(ii) Box equations: for $b=\{e,f,g,h\}\in B$,

$$\mathcal{G}_b := y_e y_f - \tilde{q}_b y_g y_h = 0, \tag{7}$$

where $\{e,f\}=\mathcal{C}_b$.

This set of equations was discussed by Givental [16], and was used to give an integral representation for the solutions to the quantum cohomology differential equations for the special case of complete flag manifolds. The results in that paper were the starting point for our investigations. We describe below Givental's result and our (conjectural) generalization to a general partial flag manifold.

Let $\mathbf{A}^{|E|}$ be the complex affine space with the coordinates y_e $(e\in E)$. For fixed parameter values of

$$(q,\tilde{q}) := (q_1, ..., q_l, ..., \tilde{q}_b, ...)$$

we obtain an affine variety

$$M_{q,\tilde{q}} := \{u\in\mathbf{A}^{|E|} : \mathcal{F}_i = 0,\ i=1,...,l, \text{ and } \mathcal{G}_b = 0,\ b\in B\}.$$

If all components of $(q,\tilde{q})$ are nonzero, $M_{q,\tilde{q}}$ is isomorphic to the torus $(\mathbf{C}^*)^{|D|}$.

One can define on $M_{q,\tilde{q}}$ a holomorphic volume form

$$\omega_{q,\tilde{q}} := \mathrm{Res}_{M_{q,\tilde{q}}}\left(\frac{\Omega}{\prod_{i=1}^{l}\mathcal{F}_i\prod_{b\in B}\mathcal{G}_b}\right),$$

where

$$\Omega := \bigwedge_{e\in E} dy_e.$$

Let $\mathcal{F}=\sum_{e\in E} y_e$. Consider the integral

$$I_\gamma(q,\tilde{q}) := \int_\gamma e^{\mathcal{F}}\omega_{q,\tilde{q}},$$

where $\gamma\in H_{|D|}(M_{q,\tilde{q}},\mathrm{Re}(\mathcal{F})=-\infty)$. We put

$$\Phi_\gamma(q_1,...,q_l) := I_\gamma(q_1,...,q_l,1,1,...,1).$$

We can now formulate a precise version of Conjecture 4.1.2:

CONJECTURE 5.1.1. *Let $\vec{S}$ be any solution to the quantum differential system for F. Then the component $\langle\vec{S},1\rangle$ can be expressed as $\Phi_\gamma(q)$ for some $\gamma\subset M_{q,1}$.*

Remark 5.1.2. This conjecture generalizes Givental's mirror theorem for complete flag manifolds [16].

Definition 5.1.3. Let W denote the set of edges in the diagram Λ that intersect Γ. We orient the vertical edges in W upwards and the horizontal edges to the right. Let $V:=B\cup\{0,1,2,...,l\}$. For $w\in W$, the *tail* $t(w)$ of w is defined to be the box $b_1\in B$ where w starts. Similarly, the *head* $h(w)$ of w is the box $b_2\in B$ where w ends. If w crosses the roof $\mathcal{R}_i$, so that its "head" is outside the graph Γ, we put $h(w):=i$, and if the "tail" of w is outside Γ, we put $t(w)=0$. In the sense of duality of planar graphs, the graph with vertices V, edges W and incidence given by $h,t\colon W\to V$ is *dual* to the graph Γ with all stars collapsed to one point.

Definition 5.1.4. For each cone $\sigma\in\widehat{\Sigma}$ of maximal dimension we define a cycle $\gamma=\gamma_{q,\tilde{q}}(\sigma)$ in $M_{q,\tilde{q}}$ by

$$\gamma := \{y\in M_{q,\tilde{q}} : |y_e|=1 \text{ for all } e\in E \text{ with } \delta(e)\in\sigma\}.$$

Note that the y_f with $\delta(f)\notin\sigma$ are determined uniquely by the y_e with $\delta(e)\in\sigma$ and the roof and box equations (6), (7).

The cycle γ is a real torus, of dimension equal to $\dim_{\mathbf{C}}(M_{q,\tilde{q}})=\dim_{\mathbf{C}}(F)$. Since it is defined over the entire family of the $M_{q,\tilde{q}}$, it is invariant under monodromy. The integral over this special cycle will be denoted by $I(q,\tilde{q})$.

Definition 5.1.5. The specialization $\Phi_F(q){:=}I(q_1,...,q_l,1,...,1)$ is called the *hypergeometric series of the partial flag manifold F.*

It turns out that $I(q,\tilde{q})$ has a nice power-series expansion.

THEOREM 5.1.6.

$$I(q,\tilde{q})=\sum_{m_1,...,m_l,...,m_b,...} A_{m_1,...,m_l,...,m_b,...}\, q_1^{m_1}...\,q_l^{m_l}\prod_{b\in B}\tilde{q}_b^{m_b},$$

with

$$A_{m_1,...,m_l,...,m_b,...}:=\frac{1}{(m_1!)^{k_1+k_2}}\cdot\frac{1}{(m_2!)^{k_2+k_3}}\cdot...\cdot\frac{1}{(m_l!)^{k_l+k_{l+1}}}B_{m_1,...,m_l,...,m_b,...},$$

$$B_{m_1,...,m_l,...,m_b,...}:=\prod_{w\in W}\binom{m_{h(w)}}{m_{t(w)}}.$$

Proof. By Leray's theorem, the integral is equal to

$$\int_{T(\gamma_{q,\tilde{q}}(\sigma))} e^{\mathcal{F}}\frac{\Omega}{\prod_{i=1}^{l}\mathcal{F}_i\prod_{b\in B}\mathcal{G}_b},$$

where T is the tube map. For $|q|<1$, $|\tilde{q}|<1$, the cycle $T(\gamma_{q,\tilde{q}}(\sigma))$ is homologous to the cycle

$$T:=\{y\in\mathbf{A}^{|E|}:|y_e|=1 \text{ for all } e\in E\}$$

in the complement of the hypersurfaces $y_e{=}0$. We now expand all the terms in the integrand:

$$e^{\mathcal{F}}=\sum_{d=0}^{\infty}\frac{1}{d!}\mathcal{F}^d=\sum_{d_e\geqslant 0}\frac{\prod_{e\in E}y_e^{d_e}}{\prod_{e\in E}d_e!},$$

$$\frac{1}{\mathcal{F}_i}=\frac{1}{\prod_{e\in\mathcal{R}_i}y_e}\sum_{m_i\geqslant 0}\left(\frac{q_i}{\prod_{e\in\mathcal{R}_i}y_e}\right)^{m_i},$$

$$\frac{1}{\mathcal{G}_b}=\frac{1}{y_ey_f}\sum_{m_b\geqslant 0}\left(\frac{\tilde{q}_by_gy_h}{y_ey_f}\right)^{m_b},$$

where $\{e,f\}$ makes up the corner and $\{g,h\}$ the opposite corner of the box $b{=}\{e,f,g,h\}$. The integral picks up precisely the constant coefficient of the following power series in the y_e's, with parameters the q's and $\tilde{q}$'s:

$$\sum_{d_e,m_i,m_b\geqslant 0}\frac{\prod_{e\in E}y_e^{d_e}}{\prod_{e\in E}d_e!}\prod_{i=1}^{l}\left(\frac{q_i}{\prod_{e\in\mathcal{R}_i}y_e}\right)^{m_i}\prod_{b\in B}\left(\frac{\tilde{q}_by_gy_h}{y_ey_f}\right)^{m_b}.$$

Now there are three types of edges.

Type I. $e \in \mathcal{R}_i$ for some $i=1,...,l$. Then e is also edge of the opposite corner of a unique box b. Only the terms with

$$d_e = m_i - m_b$$

will give a contribution.

Type II. $e \in b \cap b'$ for two boxes b and b'. We can then assume that e is part of the corner of b, and the opposite corner of b'. Only the terms with

$$d_e = m_b - m_{b'}$$

will give a contribution.

Type III. e is contained in a unique $b \in B$. In this case e is part of the corner of b. Only the terms with

$$d_e = m_b$$

will give a contribution.

Hence we see that the integral is given by the series

$$\sum_{m_i, m_b \geqslant 0} \frac{1}{\prod_{e \in E} d_e!} \prod_{i=1}^{l} q_i^{m_i} \prod_{b \in B} \tilde{q}_b^{m_b},$$

where for each edge the number d_e is determined by the m_i and m_b by the above equations. We can rewrite this coefficient nicely in terms of binomial coefficients as follows. Each edge $w \in W$ of the diagram Λ intersects precisely one edge $e \in E$ of Type I or Type II. The corresponding coefficient d_e is then given by

$$d_e = m_{h(w)} - m_{t(w)}.$$

Trivially,

$$\frac{1}{\prod_{e \in E} d_e!} = \frac{\prod_{w \in W} m_{h(w)}!}{\prod_{w \in W} m_{h(w)}! \prod_{e \in E} d_e!}.$$

The heads of arrows $w \in W$ which are *not* tails are the heads of arrows intersecting the edges of Type I. The tails of arrows $w \in W$ which are not heads are in bijection to the edges of Type III. Hence, when we pull out a factor $\prod_{i=1}^{l} \prod_{e \in \mathcal{R}_i} m_e!$ from the denominator of the left-hand side of the above equality, the other terms in the numerator and the denominator can precisely be combined into the product

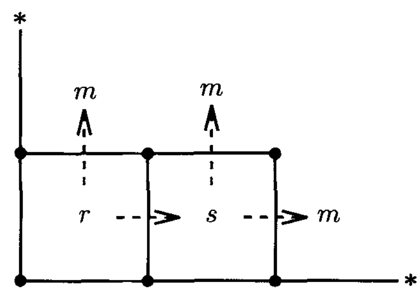
$$\prod_{w \in W} \binom{m_{h(w)}}{m_{t(w)}}.$$

This proves the result. □

Remark 5.1.7. Note that $I(q,\tilde{q})$ is the generalized hypergeometric series for the smooth toric variety $\widehat{P}$ defined in §3. The parameters $q_1,...,q_l$ correspond to the generators of $\mathrm{Pic}(\widehat{P})$ coming from the singular variety P (the pullbacks of the line bundles $\mathcal{L}_1,...,\mathcal{L}_l$), while $\tilde{q}_b$ correspond to the additional generators of $\mathrm{Pic}(\widehat{P})$.

Theorem 5.1.6 shows that it is very easy to write down the power-series expansion for $I(q,\tilde{q})$ directly from the diagram.

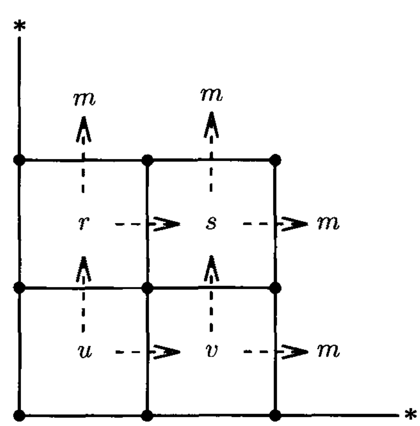
Example 5.1.8. $F(2,5)$ (the Grassmannian of 2-planes in $\mathbf{C}^5$):



Hence we read off:

$$I(q,\tilde{q}) = \sum_{m,r,s\geqslant 0} \frac{1}{(m!)^5}\binom{s}{r}\binom{m}{r}\binom{m}{s}^2 q^m \tilde{q}_1^r \tilde{q}_2^s.$$

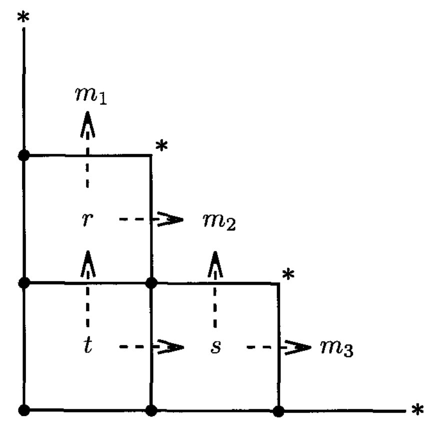
Example 5.1.9. $F(3,6)$ (the Grassmannian of 3-planes in $\mathbf{C}^6$):



Hence we read off:

$$I(q,\tilde{q}) = \sum_{m,r,s,u,v} \frac{1}{(m!)^6}\binom{r}{u}\binom{v}{u}\binom{s}{r}\binom{s}{v}\binom{m}{r}\binom{m}{s}^2\binom{m}{v} q^m \tilde{q}_1^r \tilde{q}_2^s \tilde{q}_3^u \tilde{q}_4^v.$$

Example 5.1.10. $F(1,2,3,4)$ (the variety of complete flags in $\mathbf{C}^4$):



Hence we read off:

$$I(q,\tilde{q})=\sum_{m_1,m_2,m_3,r,s,t} A_{m_1,m_2,m_3,r,s,t}\, q_1^{m_1}q_2^{m_2}q_3^{m_3}\tilde{q}_1^r\tilde{q}_2^s\tilde{q}_3^t,$$

with

$$A_{m_1,m_2,m_3,r,s,t}=\frac{1}{(m_1!)^2(m_2!)^2(m_3!)^2}\binom{r}{t}\binom{s}{t}\binom{m_1}{r}\binom{m_2}{r}\binom{m_2}{s}\binom{m_3}{s}.$$

A weaker version of Conjecture 5.1.1 is

CONJECTURE 5.1.11. *The series $\Phi_F:=I(q,1)$ is the coefficient of the cohomology class 1 in the $H^*(F,\mathbf{Q})$-valued function S_F describing the quantum $\mathcal{D}$-module of F, i.e.,*

$$\Phi_F=1+\sum_{\bar{m}:=(m_1,\dots,m_l)\neq 0}\left(\int_{\overline{M}_{0,2}(F,\bar{m})}\frac{e_1^*(e^{Ct}\Omega_F)}{1-c}\cup e_2^*(1)\right)q_1^{m_1}\dots q_l^{m_l},$$

where Ct stands for $C_1t_1+\dots+C_lt_l$, with $\{C_1,\dots,C_l\}$ the Schubert basis of $H^2(F,\mathbf{Q})$, and Ω_F is the cohomology class of a point.

Remark 5.1.12. (i) Besides the case of complete flag manifolds (cf. Remark 5.1.2), there is another case for which the above conjecture agrees with previously known results. Consider the partial flag manifold $F:=F(1,n-1,n)$ of flags $V^1\subset V^{n-1}\subset\mathbf{C}^n$. The Plücker embedding identifies F with a $(1,1)$-hypersurface in $\mathbf{P}^{n-1}\times\mathbf{P}^{n-1}$. The hypergeometric series for $\mathbf{P}^{n-1}\times\mathbf{P}^{n-1}$ is

$$\sum_{m_1,m_2\geqslant 0}\frac{1}{(m_1!)^n(m_2!)^n}q_1^{m_1}q_2^{m_2}$$

(cf. [17]), and by the quantum hyperplane section theorem ([17], [23]) we obtain that the hypergeometric series for F is

$$\sum_{m_1,m_2\geqslant 0} \frac{(m_1+m_2)!}{(m_1!)^n(m_2!)^n} q_1^{m_1} q_2^{m_2}. \tag{8}$$

On the other hand, the recipe of Theorem 5.1.6 gives the formula

$$\sum_{m_1,m_2\geqslant 0} \frac{\sum_s \binom{m_1}{s}\binom{m_2}{s}}{(m_1!)^{n-1}(m_2!)^{n-1}} q_1^{m_1} q_2^{m_2} \tag{9}$$

for the hypergeometric series of F. The identity

$$\sum_s \binom{m_1}{s}\binom{m_2}{s} = \binom{m_1+m_2}{m_1}$$

implies that the series (8) and (9) coincide.

(ii) The quantum Pieri formula [11] gives explicitly the quantum product of a special Schubert class with a general one, and in particular the quantum product of a Schubert divisor with any other Schubert class. Using this, one can write down in reasonably low-dimensional cases the quantum differential system for F, and reduce this first-order system to higher-order differential equations satisfied by the components. In particular, one can write down the differential operators annihilating the component $\langle \vec{S}, 1\rangle$ of any solution $\vec{S}$, and check by direct computation that the hypergeometric series $\Phi_F(q)$ of Theorem 5.1.6 is annihilated by these operators. In [6] this is done for the Grassmannians containing complete-intersection Calabi–Yau 3-folds. For the complete flag manifolds, the operators are known to be the operators for the quantum Toda lattice (see [22]).

5.2. Calabi–Yau complete intersections in $F(n_1, ..., n_l, n)$

Recall that $\mathrm{Pic}(F)$ is generated by the line bundles $\mathcal{O}(C_i)$, $i=1, ..., l$, which also generate the (closed) Kähler cone. Hence any line bundle $\mathcal{H}$ on F which is globally generated is of the form $\mathcal{O}(\bar{d}) := \mathcal{O}\big(\sum_{i=1}^{l} d^{(i)} C_i\big)$, with $d^{(i)}$ nonnegative. The common zero-locus of r general sections of the line bundles $\mathcal{O}(\bar{d}_1), ..., \mathcal{O}(\bar{d}_r)$ will be denoted by $X := X_{\bar{d}_1,...,\bar{d}_r}$.

Assuming Conjecture 5.1.11, it follows from the quantum hyperplane section theorem that the hypergeometric series Φ_X has the expression

$$\Phi_X = \sum_m \prod_{j=1}^{r} \Big(\sum_{i=1}^{l} d_j^{(i)} m_i\Big)!\, A_{m_1,...,m_l}\, q_1^{m_1} ... q_l^{m_l}, \tag{10}$$

where $A_{m_1,\ldots,m_l}$ are the coefficients of Φ_F in Theorem 5.1.6.

From now on the complete intersection $X_{\bar{d}_1,\ldots,\bar{d}_r}$ is assumed to be a Calabi–Yau manifold. The construction of mirrors described in [6] for the case when F is a Grassmannian can be extended to the case of a general F as follows:

X can be regarded as the intersection of $F\subset\mathbf{P}^{N_1-1}\times\ldots\times\mathbf{P}^{N_l-1}$ with r general hypersurfaces Z_j $(j=1,\ldots,r)$ in $\mathbf{P}^{N_1-1}\times\ldots\times\mathbf{P}^{N_l-1}$, with Z_j of multidegree $(d_j^{(1)},\ldots,d_j^{(l)})$. Let Y be the Calabi–Yau complete intersection of the same hypersurfaces with the toric degeneration P of F.

For each edge $e\in\bigcup_{i=1}^{l}\mathcal{R}_i$ which is part of a roof, define polynomials

$$\varphi_e(y):=\sum_{f\in U(e)} c_f y_f,$$

where c_f are generically chosen complex numbers. (Recall that we have defined $U(e)$ as the set consisting of e, together with all edges in the graph Γ which are either directly below e, if e is horizontal, or directly to the left of e, if e is vertical.)

Partition each of the roofs $\mathcal{R}_i$, $i=1,\ldots,l$, into r disjoint subsets

$$\mathcal{R}_i=\mathcal{R}_{i,1}\cup\ldots\cup\mathcal{R}_{i,r}$$

such that $|\mathcal{R}_{i,j}|=d_i^{(j)}$. It follows from Definition 3.2.3 and Theorem 3.2.13 that the toric Weil divisor

$$\sum_{i=1}^{l}\sum_{e\in\mathcal{R}_{i,j}} H_e$$

is Cartier, and

$$\mathcal{O}\Big(\sum_{i=1}^{l}\sum_{e\in\mathcal{R}_{i,j}} H_e\Big)\cong\mathcal{L}_1^{\otimes d_j^{(1)}}\otimes\ldots\otimes\mathcal{L}_l^{\otimes d_j^{(l)}}.$$

Consider the torus T in the affine space $\cong\mathbf{A}^{|E|}$ given by the following set of equations:

(i) Roof equations: for $i=1,2,\ldots,l$,

$$\prod_{e\in\mathcal{R}_i} y_e=1.$$

(ii) Box equations: for $b=\{e,f,g,h\}\in B$,

$$y_e y_f-y_g y_h=0,$$

where $\{e,f\}$ form the corner C_b of b.

Introduce additional independent variables x_d, $d \in D$, one for each generator of the lattice $L(D)$. For every edge $e \in E$, set

$$x^{\delta(e)} := x_{h(e)}(x_{t(e)})^{-1},$$

where, as before, $h(e)$ (resp. $t(e)$) is the head (resp. tail) of e. The torus T can be identified with $\mathrm{Spec}(\mathbf{C}[x_d, x_d^{-1}; d \in D])$, with the embedding $T \hookrightarrow \mathbf{A}^{|E|}$ induced by the ring homomorphism

$$\mathbf{C}[y_e; e \in E] \to \mathbf{C}[x_d, x_d^{-1}; d \in D], \quad y_e \mapsto x^{\delta(e)}.$$

With this identification, we obtain Laurent polynomials

$$\varphi_e(x) := \sum_{f \in U(e)} c_f x^{\delta(f)}.$$

For $j = 1, ..., r$, let ∇_j be the Newton polyhedron of the Laurent polynomial

$$\mathcal{P}_j := 1 - \sum_{i=1}^{l} \sum_{e \in \mathcal{R}_{i,j}} \varphi_e(x).$$

The polyhedra ∇_j, $j = 1, ..., r$, define a nef-partition of the anticanonical class of P (see definitions in [9], [4]), and according to [7], [9], the mirror family Y^* of the Calabi–Yau complete intersection $Y \subset P$ consists of Calabi–Yau compactifications of the general complete intersections in T defined by the equations

$$1 - \sum_{i=1}^{l} \sum_{e \in \mathcal{R}_{i,j}} \varphi_e(x) = 0, \quad j = 1, ..., r. \tag{11}$$

CONJECTURE 5.2.1. *Let Y_0^* be a Calabi–Yau compactification of a general complete intersection in T defined by the equations* (11)*, with the additional requirement that the coefficients satisfy the relation*

$$c_{f_1} c_{f_2} = c_{f_3} c_{f_4}$$

whenever $\{f_1, f_2, f_3, f_4\}$ make up a box $b \in B$, with $\{f_1, f_2\}$ forming the corner C_b of b. Then a minimal desingularization of Y_0^* is a mirror of a generic complete-intersection Calabi–Yau $X \subset F$.

The main period of the mirror Y^* of Y is given by

$$\Phi_Y = \int_\gamma \mathrm{Res}_{M_q, \bar{q}} \left(\frac{\Omega}{\prod_{j=1}^{r} \mathcal{E}_j \prod_{i=1}^{l} \mathcal{F}_i \prod_{b \in B} \mathcal{G}_b} \right),$$

where the extra factors $\mathcal{E}_j$ come from the nef-partition of the anticanonical class of P described above. Specifically,

$$\mathcal{E}_j := 1 - \sum_{i=1}^{l} \sum_{e \in \mathcal{R}_{i,j}} \sum_{f \in U(e)} y_f, \quad j = 1, ..., r.$$

By direct expansion of the integral defining Φ_Y (as in Theorem 5.1.6), followed by the specialization $\tilde{q}_b = 1$, $b \in B$, one gets exactly the hypergeometric series Φ_X.

Finally, we discuss some applications to the case when $X \subset F$ is a Calabi–Yau 3-fold.

First, as discussed in [6], our construction can be interpreted via *conifold transitions.* Indeed, by Theorem 3.1.4, if X is generic, then its degeneration $Y \subset P$ is a singular Calabi–Yau 3-fold, whose singular locus consists of finitely many nodes. The resolution of singularities $\widehat{P} \to P$ induces a small resolution $\widehat{Y} \to Y$. In other words the (nonsingular) Calabi–Yau's X and $\widehat{Y}$ are related by a conifold transition, and Conjecture 5.2.1 essentially states that their mirrors are related in a similar fashion.

Second, it is well understood (see e.g. [7]) that the knowledge of the hypergeometric series Φ_X for a Calabi–Yau 3-fold gives the virtual numbers of rational curves on X via a formal calculation. In [6] we have used the hypergeometric series (10) to compute these numbers for complete intersections in Grassmannians.

5.3. List of Calabi–Yau complete-intersection 3-folds

Recall that if $F := F(n_1, ..., n_l, n)$ is a partial flag manifold, then

$$\dim(F) = \sum_{i=1}^{l} (n_i - n_{i-1})(n - n_i). \tag{12}$$

In the Schubert basis of the Picard group, the anticanonical bundle of F is given by

$$\omega_F^{-1} = \mathcal{O}\left(\sum_{i=1}^{l} (n_{i+1} - n_{i-1}) C_i\right). \tag{13}$$

A (general) complete-intersection Calabi–Yau 3-fold in F is the common zero-locus of $r := \dim(F) - 3$ general sections $s_j \in H^0(F, \mathcal{O}(\bar{d}_j))$, where $\mathcal{O}(\bar{d}_j)$, $j = 1, 2, ..., r$, are line bundles with $\bigotimes_{j=1}^{r} \mathcal{O}(\bar{d}_j) = \omega_F^{-1}$. Hence, if F contains a complete-intersection Calabi–Yau 3-fold, then necessarily

$$\dim(F) \leqslant 3 + \sum_{i=1}^{l} (n_{i+1} - n_{i-1}) = n + n_l - n_1 + 3. \tag{14}$$

PROPOSITION 5.3.1. *If $F:=F(n_1,\dots,n_l,n)$ is a partial flag manifold containing a complete-intersection Calabi–Yau 3-fold, and F is not a projective space or one of the manifolds $F(1,n-1,n)$, then $n\leqslant 7$.*

Proof. Using (12), after some manipulation, one can rewrite the inequality (14) as

$$(n_1-1)(n-n_1-1)+(n_2-n_1)(n-n_2-1)+\dots+(n_l-n_{l-1})(n-n_l-1)\leqslant 4. \qquad (15)$$

There are two cases.

(1) $n_1>1$. Then it is easy to see that $(n_1-1)(n-n_1-1)>4$ for $n\geqslant 8$, unless $n_1=n-1$, in which case F is a projective space.

(2) $n_1=1$. If $l=1$, then F is a projective space, so we may assume $l\geqslant 2$. As above, $(n_2-1)(n-n_2-1)>4$ for $n\geqslant 8$, unless $n_2=n-1$, in which case $F=F(1,n-1,n)$. □

Remark 5.3.2. The flag manifold $F(1,n-1,n)$ sits as a $(1,1)$-hypersurface in $\mathbf{P}^{n-1}\times\mathbf{P}^{n-1}$. Hence these cases (as well as the case when F is projective space) can be viewed as particular instances of complete-intersection Calabi–Yau's in toric varieties.

We list below all the partial flag manifolds (not excluded by Proposition 5.3.1) for which the inequality (14) is satisfied. The anticanonical class of F, denoted by $-K_F$, is expressed in terms of the natural Schubert basis of the Picard group. The last column of the table below contains the possible splittings of the anticanonical class into $\dim(F)-3$ nonnegative divisors.

In general, there is a natural duality isomorphism

$$F(n_1,\dots,n_l,n)\cong F(n-n_l,\dots,n-n_1,n). \qquad (16)$$

This is taken into account by listing only one of the two isomorphic flag manifolds. It may also be that the flag manifold is self-dual, i.e., (16) is an automorphism, and two families of complete-intersection Calabi–Yau 3-folds corresponding to different splittings of the anticanonical class are interchanged by the duality automorphism. Whenever this happens (e.g., when F parametrizes complete flags), only one of the two splittings of $-K_F$ is listed.

n	F	$\dim(F)$	$-K_F$	splitting of $-K_F$
7	$F(2,7)$	10	7	$7(1)$
7	$F(1,2,7)$	11	$(2,6)$	$2(1,0)+6(0,1)$
7	$F(1,5,7)$	14	$(5,6)$	$5(1,0)+6(0,1)$
7	$F(1,2,6,7)$	15	$(2,5,5)$	$2(1,0,0)+5(0,1,0)+5(0,0,1)$
6	$F(2,6)$	8	6	$(2)+4(1)$
6	$F(3,6)$	9	6	$6(1)$
6	$F(1,2,6)$	9	$(2,5)$	$(2,0)+5(0,1)$ $(1,0)+(1,1)+4(0,1)$ $2(1,0)+(0,2)+3(0,1)$
6	$F(1,3,6)$	11	$(3,5)$	$3(1,0)+5(0,1)$
6	$F(1,4,6)$	11	$(4,5)$	$(2,0)+2(1,0)+5(0,1)$ $3(1,0)+(1,1)+4(0,1)$ $4(1,0)+(0,2)+3(0,1)$
6	$F(1,2,5,6)$	12	$(2,4,4)$	$(2,0,0)+4(0,1,0)+4(0,0,1)$ $(1,0,0)+(1,1,0)+3(0,1,0)+4(0,0,1)$ $(1,0,0)+(1,0,1)+4(0,1,0)+3(0,0,1)$ $2(1,0,0)+(0,2,0)+2(0,1,0)+4(0,0,1)$ $2(1,0,0)+(0,1,1)+3(0,1,0)+3(0,0,1)$ $2(1,0,0)+4(0,1,0)+(0,0,2)+2(0,0,1)$
6	$F(1,3,5,6)$	13	$(3,4,3)$	$3(1,0,0)+4(0,1,0)+3(0,0,1)$
5	$F(2,5)$	6	5	$(3)+2(1)$ $2(2)+(1)$
5	$F(1,2,5)$	7	$(2,4)$	$(2,0)+(0,2)+2(0,1)$ $(1,0)+(1,1)+(0,2)+(0,1)$ $2(1,1)+2(0,1)$ $2(1,0)+2(0,2)$ $(1,0)+(1,2)+2(0,1)$ $(2,1)+3(0,1)$
5	$F(2,3,5)$	8	$(3,3)$	$(1,0)+(2,0)+3(0,1)$ $2(1,0)+(1,1)+2(0,1)$
5	$F(1,3,5)$	8	$(3,4)$	$(3,0)+4(0,1)$ $(1,0)+(2,1)+3(0,1)$ $(1,1)+(2,0)+3(0,1)$ $(1,0)+2(1,1)+2(0,1)$ $(1,0)+(2,0)+(0,2)+2(0,1)$ $3(1,0)+2(0,2)$ $3(1,0)+(0,1)+(0,3)$ $2(1,0)+(1,2)+2(0,1)$

n	F	$\dim(F)$	$-K_F$	splitting of $-K_F$
5	$F(1,2,4,5)$	9	$(2,3,3)$	$2(1,0,0)+(0,3,0)+3(0,0,1)$
				$2(1,0,0)+3(0,1,0)+(0,0,3)$
				$(2,1,0)+2(0,1,0)+3(0,0,1)$
				$(2,0,1)+3(0,1,0)+2(0,0,1)$
				$(1,2,0)+(1,0,0)+(0,1,0)+3(0,0,1)$
				$2(1,0,0)+(0,2,1)+(0,1,0)+2(0,0,1)$
				$2(1,0,0)+(0,1,2)+2(0,1,0)+(0,0,1)$
				$(1,0,0)+(1,0,2)+3(0,1,0)+(0,0,1)$
				$(1,1,1)+(1,0,0)+2(0,1,0)+2(0,0,1)$
				$(2,0,0)+(0,2,0)+(0,1,0)+3(0,0,1)$
				$(2,0,0)+(0,0,2)+3(0,1,0)+(0,0,1)$
				$2(1,0,0)+(0,2,0)+(0,1,0)+(0,0,2)+(0,0,1)$
				$2(1,1,0)+(0,1,0)+3(0,0,1)$
				$(1,1,0)+(1,0,1)+2(0,1,0)+2(0,0,1)$
				$(1,1,0)+(1,0,0)+(0,1,1)+(0,1,0)+2(0,0,1)$
				$2(1,0,0)+2(0,1,1)+(0,1,0)+(0,0,1)$
				$(1,0,0)+(1,0,1)+(0,1,1)+2(0,1,0)+(0,0,1)$
				$2(1,0,1)+3(0,1,0)+(0,0,1)$
				$(2,0,0)+(0,1,1)+2(0,1,0)+2(0,0,1)$
				$(1,1,0)+(0,2,0)+(1,0,0)+3(0,0,1)$
				$(1,0,1)+(0,2,0)+(1,0,0)+(0,1,0)+2(0,0,1)$
				$2(1,0,0)+(0,2,0)+(0,1,1)+2(0,0,1)$
				$(1,1,0)+(1,0,0)+2(0,1,0)+(0,0,2)+(0,0,1)$
				$(1,0,1)+(1,0,0)+3(0,1,0)+(0,0,2)$
				$2(1,0,0)+(0,1,1)+2(0,1,0)+(0,0,2)$
5	$F(1,2,3,5)$	9	$(2,2,3)$	$(2,0,0)+2(0,1,0)+3(0,0,1)$
				$(0,2,0)+2(1,0,0)+3(0,0,1)$
				$(0,0,2)+2(1,0,0)+2(0,1,0)+(0,0,1)$
				$(1,1,0)+(1,0,0)+(0,1,0)+3(0,0,1)$
				$(1,0,1)+(1,0,0)+2(0,1,0)+2(0,0,1)$
				$(0,1,1)+2(1,0,0)+(0,1,0)+2(0,0,1)$
5	$F(1,2,3,4,5)$	10	$(2,2,2,2)$	$(2,0,0,0)+2(0,1,0,0)+2(0,0,1,0)+2(0,0,0,1)$
				$2(1,0,0,0)+(0,2,0,0)+2(0,0,1,0)+2(0,0,0,1)$
				$(1,1,0,0)+(1,0,0,0)+(0,1,0,0)+2(0,0,1,0)+2(0,0,0,1)$
				$(1,0,0,1)+(1,0,0,0)+2(0,1,0,0)+2(0,0,1,0)+(0,0,0,1)$
				$(1,0,1,0)+(1,0,0,0)+2(0,1,0,0)+(0,0,1,0)+2(0,0,0,1)$
				$(0,1,1,0)+2(1,0,0,0)+(0,1,0,0)+(0,0,1,0)+2(0,0,0,1)$

n	F	$\dim(F)$	$-K_F$	splitting of $-K_F$
4	$F(2,4)$	4	4	(4)
4	$F(1,2,4)$	5	$(2,3)$	$(1,0)+(1,3)$
				$(1,1)+(1,2)$
				$(2,1)+(0,2)$
				$(2,2)+(0,1)$
4	$F(1,2,3,4)$	6	$(2,2,2)$	$(2,0,0)+(0,2,0)+(0,0,2)$
				$(1,1,0)+(1,0,1)+(0,1,1)$
				$(1,2,0)+(1,0,0)+(0,0,2)$
				$(1,2,0)+(1,0,1)+(0,0,1)$
				$(2,1,0)+(0,1,0)+(0,0,2)$
				$(2,1,0)+(0,1,1)+(0,0,1)$
				$(2,0,1)+(0,2,0)+(0,0,1)$
				$(2,0,1)+(0,1,1)+(0,1,0)$
				$2(1,1,0)+(0,0,2)$
				$2(1,0,1)+(0,2,0)$
				$(2,2,0)+2(0,0,1)$
				$(2,0,2)+2(0,1,0)$

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VICTOR V. BATYREV
Mathematisches Institut
Eberhard-Karls-Universität Tübingen
DE-72076 Tübingen
Germany
batyrev@bastau.mathematik.uni-tuebingen.de

IONUŢ CIOCAN-FONTANINE
Department of Mathematics
Northwestern University
Evanston, IL 60208
U.S.A.
ciocan@math.nwu.edu

BUMSIG KIM
Department of Mathematics
Pohang University of Science and Technology (Postech)
Pohang, 790-784
The Republic of Korea
bumsig@euclid.postech.ac.kr

DUCO VAN STRATEN
FB 17, Mathematik
Johannes Gutenberg-Universität Mainz
DE-55099 Mainz
Germany
straten@mathematik.uni-mainz.de

Received May 18, 1998