Acta Math., 182 (1999), 207–241

On the structure of the Selberg class, I: $0 \leqslant d \leqslant 1$

by

JERZY KACZOROWSKI and ALBERTO PERELLI

A. Mickiewicz University
Poznań, Poland

University of Genova
Genova, Italy

Contents

Notation

We write, as usual, $s = \sigma + it$ and $\bar{f}(s) = \overline{f(\bar{s})}$. Selberg's class $\mathcal{S}$ consists of the functions $F(s)$ satisfying the following conditions.

(i) (*Dirichlet series*) For $\sigma > 1$, $F(s)$ is an absolutely convergent Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

(ii) (*Analytic continuation*) For some integer $m \geqslant 0$, $(s-1)^m F(s)$ is an entire function of finite order.

(iii) (*Functional equation*) $F(s)$ satisfies a functional equation of the form

$$\Phi(s) = \omega \bar{\Phi}(1-s)$$

The first author was partially supported by the CNR Visiting Professors Program and the KBN Grant 2 P03A 02809.

where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

with $Q>0$, $\lambda_j>0$, $\operatorname{Re}\mu_j \geqslant 0$ and $|\omega|=1$.

(iv) (*Ramanujan hypothesis*) For every $\varepsilon>0$, $a(n) \ll n^{\varepsilon}$.

(v) (*Euler product*) For σ sufficiently large,

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

where $b(n)=0$ unless n is a positive power of a prime, and $b(n) \ll n^{\theta}$ for some $\theta<\frac{1}{2}$.

We call Q, λ_j, μ_j and ω in (iii) the *data* of $F(s)$. Observe that (v) implies that the coefficients $a(n)$ are multiplicative. We denote by $\mathcal{S}^\sharp$ the larger class of *not identically vanishing* functions $F(s)$ satisfying (i)–(iii) above.

A function $F \in \mathcal{S}$ is *primitive* if $F(s)=F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}$ implies that $F_1=1$ or $F_2=1$.

The *weight* Λ of $F \in \mathcal{S}^\sharp$ is defined as $\Lambda = \sum_{j=1}^{r} \lambda_j$, and the *degree* $\deg F = d$ of $F \in \mathcal{S}^\sharp$ is defined as $d=2\Lambda$. We denote by $\mathcal{S}_d$ and $\mathcal{S}_d^\sharp$ the subclasses of $F \in \mathcal{S}$ and $F \in \mathcal{S}^\sharp$, respectively, of given degree d.

If $d=0$, i.e., there are no Γ-factors in the functional equation, we write $q=Q^2$ and denote by $\mathcal{S}_0^\sharp(q,\omega)$ the subclass of $F \in \mathcal{S}_0^\sharp$ with given ω and q. We also write $V_0^\sharp(q,\omega) = \mathcal{S}_0^\sharp(q,\omega) \cup \{0\}$.

If $d=1$, we use the notation

$$\beta = \prod_{j=1}^{r} \lambda_j^{-2\lambda_j}, \quad \xi = 2\sum_{j=1}^{r} (\mu_j - \tfrac{1}{2}) = \eta + i\theta, \quad q = \frac{2\pi Q^2}{\beta},$$

$$\omega^* = \omega e^{-i\pi(\eta+1)/2} \left(\frac{Q^2}{\beta} \right)^{i\theta} \prod_{j=1}^{r} \lambda_j^{-2i \operatorname{Im} \mu_j}, \quad \varkappa = \operatorname*{res}_{s=1} F(s).$$

Moreover, we denote by $\mathcal{S}_1^\sharp(q,\xi,\omega^*)$ the subclass of $F \in \mathcal{S}_1^\sharp$ with given parameters q, ξ and ω^*, and write $V_1^\sharp(q,\xi,\omega^*) = \mathcal{S}_1^\sharp(q,\xi,\omega^*) \cup \{0\}$.

If χ is a Dirichlet character $(\bmod\, q)$, we denote by f_χ its conductor, and by χ^* the primitive character inducing χ. We denote by ω_{χ^*} and Q_{χ^*} the ω-factor and the Q-factor in the standard functional equation for $L(s,\chi^*)$, i.e., $\omega_{\chi^*} = \tau(\chi^*)/i^{\mathfrak{a}}\sqrt{f_\chi}$, where $\tau(\chi^*)$ is the Gauss sum, $\mathfrak{a}=0$ if $\chi(-1)=1$ and $\mathfrak{a}=1$ if $\chi(-1)=-1$, and $Q_{\chi^*} = \sqrt{f_\chi/\pi}$. Moreover, we write

$$\mathfrak{X}(q,\xi) = \begin{cases} \{\chi \ (\bmod\, q) \text{ with } \chi(-1)=1\} & \text{if } \eta=-1, \\ \{\chi \ (\bmod\, q) \text{ with } \chi(-1)=-1\} & \text{if } \eta=0. \end{cases}$$

As usual, χ_0 denotes the principal character $(\operatorname{mod} q)$.

The value of an empty sum will be 0, and the value of an empty product will be 1. We shall also use the notation

$$\int_{(c)} = \int_{c-i\infty}^{c+i\infty},$$

$$(a)_k = \begin{cases} 1 & \text{if } k=0, \\ a(a+1)\ldots(a+k-1) & \text{if } k\geqslant 1, \end{cases}$$

$$\|\Omega\| = \sup_{s\in\Omega} |\operatorname{Re} s|$$

for a bounded domain $\Omega\subset\mathbf{C}$, $d(n)$= number of divisors of n, $[x]$= integral part of x, $\{x\}$= fractional part of x, $e(x)=e^{2\pi i x}$, $|\mathcal{A}|$= cardinality of the set $\mathcal{A}$, and $\varphi(n)$ denotes Euler's function.

1. Introduction

In [10], Selberg introduced the class $\mathcal{S}$ and made several important conjectures and remarks regarding it. On the one hand, he initiated the study of primitive functions in $\mathcal{S}$ and made the fundamental orthonormality conjecture, see Conjecture 1.1 and 1.2 of [10]. On the other hand, Selberg raised, amongst others, the problem of characterizing the shape of admissible functional equations in $\mathcal{S}$. These two aspects of the structure of Selberg's class are probably deeply related.

It is well known that the orthonormality conjecture has several interesting implications, such as the unique factorization in $\mathcal{S}$ and Artin's conjecture, see, respectively, Conrey–Ghosh [3] and Murty [8]. However, very little is known unconditionally about primitive functions, as is indeed the case about the structure of admissible functional equations, although it is conjectured that $\deg F$ is always an integer.

From an unconditional viewpoint, the structure of $\mathcal{S}_d$ has been completely determined only for $0\leqslant d<1$. In fact, it follows from work of Conrey–Ghosh [3], see also Bochner [1], that $\mathcal{S}_0=\{1\}$ and $\mathcal{S}_d=\varnothing$ for $0<d<1$. In the case $d=1$, it is conjectured that the functions $F\in\mathcal{S}_1$ are of the form $F(s)=\zeta(s)$ or $F(s)=L(s+i\theta,\chi)$ with some primitive Dirichlet character χ and $\theta\in\mathbf{R}$. This has been proved by Conrey–Ghosh [3] in the case $r=1$, although related results can be found in papers by Bochner [1], Vignéras [14] and Gérardin–Li [5].

In this paper we settle the case $d=1$ in full generality. In fact, it turns out that this case is better understood in the framework of the larger class $\mathcal{S}^\sharp$, where Ramanujan's hypothesis and Euler's product are dropped. The analytic properties (i)–(iii) alone allow a complete characterization of the functions $F\in\mathcal{S}_1^\sharp$, see Theorem 2 below. The introduction of the Euler product axiom (v) imposes an *arithmetic condition* in $\mathcal{S}$, namely the multiplicativity of the coefficients $a(n)$, which, in view of the structure of $\mathcal{S}_1^\sharp$, restricts the functions $F\in\mathcal{S}_1$ to either the Riemann zeta function or shifted Dirichlet L-functions.

We also show that Ramanujan's hypothesis plays no role in the case $0\leqslant d\leqslant 1$, in the sense that every function $F\in\mathcal{S}_d^\sharp$ with $0\leqslant d\leqslant 1$ already satisfies Ramanujan's hypothesis (iv).

In order to state our results, we introduce the notion of *invariant* of $F\in\mathcal{S}^\sharp$. Since the form of the functional equation satisfied by a function $F\in\mathcal{S}^\sharp$ is not unique, the parameters depending on the data of $F(s)$ are not necessarily uniquely determined by $F(s)$ itself, see, e.g., §2 of [3]. Therefore, we say that a parameter, or a set of parameters, is an invariant of $F(s)$ if it is uniquely determined by $F(s)$, independently of the particular form of the functional equation.

We start with the description of the structure of $\mathcal{S}_d^\sharp$ in the simpler case $0\leqslant d<1$.

THEOREM 1. (i) *If $0<d<1$, then $\mathcal{S}_d^\sharp=\varnothing$. If $F\in\mathcal{S}_0^\sharp$, then $q\in\mathbf{N}$, the pair (q,ω) is an invariant of $F(s)$ and $\mathcal{S}_0^\sharp$ is the disjoint union of the subclasses $\mathcal{S}_0^\sharp(q,\omega)$, with $q\in\mathbf{N}$ and $|\omega|=1$.*

(ii) *Every $F\in\mathcal{S}_0^\sharp(q,\omega)$, with q and ω as above, is a Dirichlet polynomial of the form*

$$F(s)=\sum_{n|q}\frac{a(n)}{n^s}.$$

(iii) *For q and ω as above, $V_0^\sharp(q,\omega)$ is a vector space over $\mathbf{R}$ and*

$$\dim_{\mathbf{R}} V_0^\sharp(q,\omega)=d(q).$$

Theorem 1 will be proved, essentially, by the argument in Theorem 3.1 of [3].

The main part of the paper is devoted to the proof of the following result, which completely characterizes the functions $F\in\mathcal{S}_1^\sharp$. We refer to the Notation section for the definition of q, $\xi=\eta+i\theta$, ω^* and $\mathfrak{X}(q,\xi)$.

THEOREM 2. (i) *If $F\in\mathcal{S}_1^\sharp$, then $q\in\mathbf{N}$ and $\eta\in\{-1,0\}$. The triple (q,ξ,ω^*) is an invariant of $F(s)$, and $\mathcal{S}_1^\sharp$ is the disjoint union of the subclasses $\mathcal{S}_1^\sharp(q,\xi,\omega^*)$ with $q\in\mathbf{N}$, $\eta\in\{-1,0\}$, $\theta\in\mathbf{R}$ and $|\omega^*|=1$. Moreover, $a(n)n^{i\theta}$ is periodic with period q.*

(ii) *Every $F\in\mathcal{S}_1^\sharp(q,\xi,\omega^*)$, with q, ξ and ω^* as above, can be uniquely written as*

$$F(s)=\sum_{\chi\in\mathfrak{X}(q,\xi)}P_\chi(s+i\theta)L(s+i\theta,\chi^*)$$

where $P_\chi\in\mathcal{S}_0^\sharp(q/f_\chi,\omega^*\bar{\omega}_{\chi^*})$. Moreover, $P_{\chi_0}(1)=0$ if $\theta\neq 0$.

(iii) *For q, ξ and ω^* as above, $V_1^\sharp(q,\xi,\omega^*)$ is a vector space over* $\mathbf{R}$ *and*

$$\dim_{\mathbf{R}} V_1^\sharp(q,\xi,\omega^*)=\begin{cases}\left[\frac{1}{2}q\right]+1 & \text{if } \xi=-1,\\ \left[\frac{1}{2}(q-1-\eta)\right] & \text{otherwise.}\end{cases}$$

A well-known theorem of Hamburger, see §2.13 of Titchmarsh [13] with $g(s)=\bar{f}(s)$, gives a characterization of the Riemann zeta function in terms of its functional equation. In our notation, Hamburger's theorem may be phrased as $\dim_{\mathbf{R}} V_1^\sharp(1,-1,1)=1$, and hence it is an immediate consequence of (iii) of Theorem 2. Moreover, (iii) of Theorem 2 should be compared with the results of Bochner [1].

We also remark that the argument in §8, see Lemma 8.3 and Proposition 8.2, can be easily adapted to show that the coefficients λ_j in any functional equation of $F\in\mathcal{S}_1^\sharp$ must be *rational numbers* of the form $1/2m$, with $m\in\mathbf{N}$. Moreover, in view of Theorem 2 and of the multiplication formula for the Γ-function

$$\Gamma(s)=m^{s-1/2}(2\pi)^{(1-m)/2}\prod_{k=0}^{m-1}\Gamma\left(\frac{s+k}{m}\right),\quad m=2,3,\dots,$$

the coefficients λ_j can in fact attain any such value.

Our method for the proof of Theorem 2 is inspired by Linnik's [7] approach to the analytic continuation and functional equation of the Dirichlet L-functions by means of the analytic properties of the Riemann zeta function. We refer to §2 for a brief discussion of Linnik's and our arguments, and of the role played by the condition $d=1$.

By further imposing the Euler product axiom, from Theorem 2 we obtain the conjectured structure of $\mathcal{S}_1$.

THEOREM 3. *Let $F\in\mathcal{S}_1$. If $q=1$, then $F(s)=\zeta(s)$. If $q\geqslant 2$, then there exists a primitive Dirichlet character $\chi\ (\mathrm{mod}\, q)$ with $\mathfrak{a}=\eta+1$ such that $F(s)=L(s+i\theta,\chi)$.*

We explicitly state the following simple consequence of Theorems 1, 2, 3 and of Theorem 3.1 of [3].

COROLLARY. *Every function $F\in\mathcal{S}_d^\sharp$ with $0\leqslant d\leqslant 1$ satisfies Ramanujan's hypothesis. Moreover, Selberg's orthonormality conjecture holds for the functions in $\mathcal{S}_d$ with $0\leqslant d\leqslant 1$.*

The paper is organized as follows. In the next section we outline the proof of Theorem 2. In §3 we prove Theorem 1, and the rest of the paper is devoted to the proof of Theorem 2. Theorem 3 is a consequence of Theorem 2, and its proof is given at the end of §8.

In Part II of the series, see [6], we fully characterize the invariants of functions in $\mathcal{S}$. Moreover, we introduce a basic invariant, the *modulus* q_F defined for every $F\in\mathcal{S}$ by

$$q_F=\frac{(2\pi)^d Q^2}{\beta},$$

and investigate its relevance with respect to the problem of twisting by Dirichlet characters in the Selberg class.

We wish to thank the referees for a very accurate checking of the manuscript, for pointing out several inaccuracies and for suggesting how to improve the presentation at several places, especially in §4.

2. Outline of the method

As we have already remarked, Theorem 1 follows by the argument in the proof of Theorem 3.1 in [3], see §3.

The proof of Theorem 2 is based on the study of the analytic properties of the *additive twists*

$$F^{\psi}(s)=\sum_{n=1}^{\infty}\frac{a(n)\psi(n)}{n^s}$$

of any given $F\in\mathcal{S}_1^{\sharp}$, where $\psi(n)=e(-n\alpha)$ with $\alpha\in\mathbf{R}$.

The analytic properties of $F^{\psi}(s)$ are established in §7. We first deal with the approximations $F_N^{\psi}(s)$ to $F^{\psi}(s)$, by expressing $F_N^{\psi}(s)$ in terms of a series involving the *incomplete Fox hypergeometric function* $H_K(z,s)$ at certain points z_n, see (7.2). We recall here that the function $H_K(z,s)$ is built by means of the data of $F(s)$, and we refer to (4.2) below for its definition. Then we let $N\to\infty$, thus obtaining the meromorphic continuations of $F^{\psi}(s)$ to $\sigma>0$.

In order to justify convergence and limit processes, and to keep track of the analytic properties of $F^{\psi}(s)$, we need a rather detailed study of the incomplete Fox function $H_K(z,s)$. To this end, in §4 we obtain the basic properties of the various hypergeometric functions which will enter the picture later on.

In §5 we obtain an explicit expression for the limit of $H_K(z,s)$ as $z\to -i/\beta$, see Theorem 5.1, which in particular assures that such a limit, which we denote by $H_K(-i/\beta,s)$, is meromorphic for $0<\sigma<2$. In fact, $H_K(-i/\beta,s)$ corresponds to the main part of the limit of $F_N^{\psi}(s)$ as $N\to\infty$, see (7.6). Moreover, in §6 we show that $H_K(-i/\beta,s)$ has a simple pole at $s=1-i\theta$, see Theorem 6.1.

As a consequence of the above properties of the incomplete Fox function $H_K(z,s)$, in §7 we obtain the meromorphic continuation of $F^{\psi}(s)$ to $\sigma>0$. Moreover, we link the polar structure of $F^{\psi}(s)$ at $s=1-i\theta$ with the coefficients $a(n)$ of $F(s)$, see (7.10) and Theorem 7.1.

In §8 we exploit the above-mentioned link and the periodicity, with period 1, of the characters ψ with respect to the parameter α to get the periodicity, with period q, of the sequence $a(n)n^{i\theta}$. As a consequence, we obtain the basic representation of $F(s)$ as a linear combination of Dirichlet L-functions over Dirichlet polynomials, see (8.1).

The finer structure of the basic representation (8.1) is then obtained by comparing the functional equations satisfied by the two sides of (8.1), one coming from $F(s)$ and the other coming from the Dirichlet L-functions. This comparison requires a careful analysis, involving simple linear independence and almost periodicity considerations. Such an analysis eventually leads to the proof of Theorem 2.

As indicated in the introduction, our study of additive twists is inspired by Linnik [7]. In fact, in order to obtain the analytic properties of Dirichlet L-functions, Linnik's starting point is to write $L(s,\chi)$ as a linear combination of additive twists of $\zeta(s)$. The analytic properties of such additive twists are easily obtained, in Linnik's case, thanks to the special form of the functional equation of $\zeta(s)$, and the required properties of $L(s,\chi)$ follow at once.

Apparently, Linnik's idea has subsequently been used only by Sprindžuk, see [11] and [12], to show that the Riemann hypothesis for all Dirichlet L-functions is equivalent to the Riemann hypothesis for $\zeta(s)$ plus a certain diophantine property of the imaginary parts of the zeros of $\zeta(s)$.

We point out here two differences between Linnik's and our approach. First, while Linnik's goal is basically the study of additive twists of $\zeta(s)$, in our case the analytic properties of the additive twists of a function $F\in\mathcal{S}_1^\sharp$ are applied to get information on the function $F(s)$ itself. Moreover, due to the general form of the functional equations in $\mathcal{S}_1^\sharp$, in order to study our additive twists we have to derive the properties of a rather general class of incomplete Fox functions.

Apparently, the literature on Fox functions deals only with the behavior in the complex variable z, see Braaksma [2]. In our arguments the s-aspect is crucial, and it is therefore developed in §§4–6. However, the Fox functions arising from additive twists of functions $F\in\mathcal{S}^\sharp$ with $d=1$ have $\mu=0$, where μ (defined and discussed in §4) is the main parameter in the Fox functions theory. This fact makes the case $d=1$ simpler than the general case $d>1$, where the hypergeometric functions associated with additive twists have $\mu>0$ and, in fact, present a definitely more complicated behavior already in the z-variable. We shall deal with the case $d>1$ in a future paper.

Finally, the proof of Theorem 3 follows from Theorem 2 and Theorem 3.1 of [3]. In fact, by Theorem 2 the sequence $a(n)n^{i\theta}$ is in this case both multiplicative and periodic with period q, and hence $F(s)$ reduces to the product of a Dirichlet L-function times a Dirichlet polynomial, see (8.34). Theorem 3 follows then from Theorem 3.1 of [3], since it implies that such a Dirichlet polynomial is trivial.

3. The case $0 \leqslant d < 1$

By Theorem 3.1 of [3] we know that $\mathcal{S}_0 = \{1\}$ and $\mathcal{S}_d = \varnothing$ if $0<d<1$. The same argument shows that $\mathcal{S}_d^\sharp = \varnothing$ if $0<d<1$. It also shows that q is an integer if $F \in \mathcal{S}_0^\sharp$, and that $F(s)$ is a Dirichlet polynomial of the form

$$F(s) = \sum_{n|q} \frac{a(n)}{n^s}.$$

Moreover, $F(s)$ satisfies a functional equation of the form

$$Q^s F(s) = \omega Q^{1-s} \bar{F}(1-s). \tag{3.1}$$

In this case the pair (q,ω) is in fact an invariant of $F(s)$; this can be easily seen by considering the quotient of two functional equations of the type (3.1). Therefore, $\mathcal{S}_0^\sharp$ is the disjoint union of the subclasses $\mathcal{S}_0^\sharp(q,\omega)$ with $|\omega|=1$ and $q \in \mathbf{N}$.

For $\zeta \in \mathbf{C} \setminus \{0\}$ and $F \in \mathcal{S}_0^\sharp(q,\omega)$, we have $\zeta F \in \mathcal{S}_0^\sharp(q, \omega\zeta/\bar{\zeta})$, and hence $V_0^\sharp(q,\omega)$ is a vector space over $\mathbf{R}$. In order to compute its dimension, we write (3.1) as

$$\sum_{n|q} a(n) \left(\frac{q}{n}\right)^s = \sum_{n|q} \frac{\omega n}{\sqrt{q}} \overline{a\left(\frac{q}{n}\right)} \left(\frac{q}{n}\right)^s,$$

and hence by the identity principle for Dirichlet series we have

$$a(n) = \frac{\omega n}{\sqrt{q}} \overline{a\left(\frac{q}{n}\right)} \quad \text{for } n \,|\, q. \tag{3.2}$$

Moreover, if $\sqrt{q} \in \mathbf{N}$ we have

$$a(\sqrt{q}) = \varepsilon a \quad \text{with } a \in \mathbf{R}, \tag{3.3}$$

where ε denotes a fixed square root of ω.

Conversely, given $q \in \mathbf{N}$ and $\omega \in \mathbf{C}$ with $|\omega|=1$, it is easy to see that if a finite sequence of complex numbers $a(n)$, with $n\,|\,q$, satisfies (3.2) and, if $\sqrt{q} \in \mathbf{N}$, (3.3) as well, then the corresponding Dirichlet polynomial belongs to $V_0^\sharp(q,\omega)$.

Hence we can freely choose complex coefficients $a(n)$ for $\sqrt{q} < n \leqslant q$ with $n\,|\,q$, and then determine the remaining $a(n)$ by (3.2). Moreover, if $\sqrt{q} \in \mathbf{N}$ we define $a(\sqrt{q})$ by (3.3). Therefore, choosing $\delta=1$ if $\sqrt{q} \in \mathbf{N}$ and $\delta=0$ otherwise, we have

$$\dim_{\mathbf{R}} V_0^\sharp(q,\omega) = 2 \cdot \tfrac{1}{2}(d(q) - \delta) + \delta = d(q),$$

and Theorem 1 is proved.

4. Hypergeometric functions: basic theory

In this section we collect the basic definitions and results about hypergeometric functions.

We start with the definition of the *incomplete Fox hypergeometric functions.* Let $z, s, w\in\mathbf{C}$, K be a positive integer, λ_j and μ_j, $j=1, ..., r$, be as in axiom (iii) of the Selberg class and

$$\tilde{h}(w,s)=\prod_{j=1}^{r}\frac{\Gamma(\lambda_j(1-s)+\bar{\mu}_j-\lambda_j w)}{\Gamma(\lambda_j s+\mu_j+\lambda_j w)},\quad h(w,s)=\tilde{h}(w,s)\Gamma(w). \tag{4.1}$$

The incomplete Fox hypergeometric function associated with the above data is formally defined by the Barnes integral

$$H_K(z,s)=\frac{1}{2\pi i}\int_{(-K-1/2)}h(w,s)z^w\,dw, \tag{4.2}$$

where

$$z^w=e^{wl(z)} \tag{4.3}$$

and $l(z)$ denotes the *branch of* $\log z$ *on* $\mathbf{C}\setminus(-\infty,0]$ *satisfying* $|\mathrm{Im}\,l(z)|<\pi$. Moreover, let

$$\mu=2\sum_{j=1}^{r}\lambda_j-1=d-1. \tag{4.4}$$

Our treatment of the incomplete Fox functions $H_K(z,s)$ is inspired by Braaksma [2], where a general theory of *Fox hypergeometric functions* $H(z)$ is developed. The Fox functions $H(z)$ are defined by means of Barnes integrals similar to (4.2), and [2] contains a systematic investigation of the analytic properties of such functions in the z-variable. When s is fixed, our functions $H_K(z,s)$ become essentially special cases of the functions $H(z)$.

We remark that the parameter μ in (4.4), defined in §1.1 of Braaksma [2] for every function $H(z)$, is very important in the theory of Fox functions. In particular, the case $\mu=0$ is somewhat simpler than the general case, since it allows an extensive use of Stirling's formula to obtain the convergence properties of the involved Barnes integrals. Moreover, when $\mu>0$ the behavior of the functions $H(z)$ is definitely more complicated. In view of (4.4), this clarifies the role of condition $d=1$ in our argument.

Let us consider the domains

$$\begin{aligned} A&=\{z\in\mathbf{C}:\mathrm{Re}\,z>0\},\\ B_\beta&=\{z\in\mathbf{C}:|z|<1/\beta\}\setminus(-1/\beta,0],\\ C_\beta&=\{z\in\mathbf{C}:|z|>1/\beta\},\\ D_\beta&=A\cup B_\beta\cup C_\beta.\end{aligned} \tag{4.5}$$

The analytic properties of the incomplete Fox functions $H_K(z,s)$ are given by

THEOREM 4.1. *Let $d=1$, $\Omega\subset\mathbf{C}$ be a bounded domain, and $K>\|\Omega\|$ be a positive integer. Then the Barnes integral (4.2) is absolutely and uniformly convergent on compact subsets of $A\times\Omega$, and $H_K(z,s)$ has holomorphic continuation to $D_\beta\times\Omega$ as a single-valued function. Moreover, for $(z,s)\in C_\beta\times\Omega$ we have*

$$H_K(z,s)=\sum_{k=K+1}^{\infty}\frac{(-1)^k}{k!}\tilde{h}(-k,s)z^{-k}, \tag{4.6}$$

the series being absolutely and uniformly convergent on compact sets.

Proof. The proof is based on repeated applications of Stirling's formula and of Cauchy's integral theorem, hence we give only a sketch of it. We refer to Theorems 1 and 2 of Braaksma [2] for further details, in the case of s fixed. We write $w=u+iv$, $s=\sigma+it$ and use the synthetic expression *total convergence* to denote absolute and uniform convergence on compact sets.

Observe that if $K>\|\Omega\|$ then the set of w-poles of $\tilde{h}(w,s)$, $s\in\Omega$, lies to the right of the line $u=-K-\frac{1}{2}$, and has positive distance from it. Moreover, the function $\tilde{h}(w,s)$ is holomorphic for $s\in\Omega$ for any fixed w on the line $u=-K-\frac{1}{2}$. This observation will be tacitly used at several places in what follows.

We start with the convergence properties of the Barnes integral defining $H_K(z,s)$. Recalling (4.3), from condition $d=1$,

$$|z^w|=|z|^u e^{-v\,\mathrm{Im}\,l(z)} \tag{4.7}$$

and Stirling's formula in the form

$$|\Gamma(w)|=\sqrt{2\pi}\,|v|^{u-1/2}e^{-\pi|v|/2}(1+O(1/|v|)) \quad \text{as } |v|\to\infty, \text{ uniformly for } u_1\leqslant u\leqslant u_2,$$

for w running on the line $u=-K-\frac{1}{2}$ we have

$$h(w,s)z^w \ll |v|^{-\sigma}|z|^{-K-1/2}e^{-|v|(\pi/2-|\mathrm{Im}\,l(z)|)} \quad \text{as } |v|\to\infty, \tag{4.8}$$

uniformly for (z,s) in any compact set $\mathcal{K}\subset A\times\Omega$, where the implicit constant depends on λ_j, μ_j, K and $\mathcal{K}$. The Barnes integral (4.2) is therefore totally convergent on $A\times\Omega$, and hence $H_K(z,s)$ is holomorphic on $A\times\Omega$.

For $(z,s)\in B_\beta\times\Omega$, we consider the contour $\mathcal{C}$ consisting of the vertical segment $[-K-\frac{1}{2}-iV,-K-\frac{1}{2}+iV]$ and of the two horizontal half-lines $[-K-\frac{1}{2}\pm iV,+\infty\pm iV)$. Here $V>0$ is a sufficiently large constant such that the absolute value of the imaginary part of each w-pole of $\tilde{h}(w,s)$, $s\in\Omega$, is at most $V-1$. In order to deal with the integral

$$H_{\mathcal{C}}(z,s)=\frac{1}{2\pi i}\int_{\mathcal{C}}h(w,s)z^w\,dw$$

we use (4.7), the identity

$$\Gamma(w)\Gamma(1-w)=\frac{\pi}{\sin \pi w} \tag{4.9}$$

and Stirling's formula

$$\log\Gamma(w+a)=\left(w+a-\tfrac{1}{2}\right)\log w-w+\tfrac{1}{2}\log 2\pi+O(1/|w|) \quad \text{as } |w|\to\infty, \tag{4.10}$$

uniformly for $|\arg w|\leqslant\pi-\varepsilon$ and a in a compact subset of $\mathbf{C}$, where $\varepsilon>0$ is fixed. Thanks to our condition $d=1$, for w running on $\mathcal{C}$ we have

$$h(w,s)z^{w}\ll(\beta|z|)^{u}u^{\sigma-1} \quad \text{as } u\to+\infty, \tag{4.11}$$

uniformly for (z,s) in any compact set $\mathcal{K}\subset B_\beta\times\Omega$, where the implicit constant depends on λ_j, μ_j, $\mathcal{C}$ and $\mathcal{K}$. The integral $H_{\mathcal{C}}(z,s)$ is therefore totally convergent on $B_\beta\times\Omega$ and represents a holomorphic function.

Now we observe that $H_K(z,s)=H_{\mathcal{C}}(z,s)$ for $(z,s)\in(A\cap B_\beta)\times\Omega$. This follows from Cauchy's theorem applied to the two closed contours obtained by joining the half-lines $\left[-K-\frac{1}{2}\pm iV,-K-\frac{1}{2}\pm i\infty\right)$ and $\left[-K-\frac{1}{2}\pm iV,+\infty\pm iV\right)$ by two arcs of the circle $|w|=R$, $R\to\infty$. In fact, an argument based on (4.9) and (4.10), similar to those leading to (4.8) and (4.11), shows that the integral over such arcs tends to 0 as $R\to\infty$. Since $H_{\mathcal{C}}(z,s)$ is holomorphic on $B_\beta\times\Omega$, in this way we obtain the holomorphic continuation of $H_K(z,s)$ to $(A\cup B_\beta)\times\Omega$.

Finally, again by a Stirling's formula estimate similar to (4.11), we have that the series on the right-hand side of (4.6) is totally convergent on $C_\beta\times\Omega$, and thus it represents a single-valued holomorphic function. Moreover, once again by an argument similar to those leading to (4.8) and (4.11), for $(z,s)\in(A\cap C_\beta)\times\Omega$ we can shift the line $u=-K-\frac{1}{2}$ to $-\infty$ and apply the residue theorem to show that (4.6) holds on $(A\cap C_\beta)\times\Omega$. This gives the holomorphic continuation of $H_K(z,s)$ to a single-valued function on $(A\cup C_\beta)\times\Omega$, and Theorem 4.1 follows. □

We explicitly remark that, although the Fox functions $H(z)$ are in general multi-valued functions, see Braaksma [2], in our case $H_K(z,s)$ is single-valued on the domain $D_\beta\times\Omega$. This is due to the special form of our function $h(w,s)$ in (4.1), and in particular to its factor $\Gamma(w)$. This fact is also reflected by the form of the series on the right-hand side of (4.6), which is in fact a power series in z^{-1}.

Now we turn to the *Gauss hypergeometric functions*, see §2.1 of Erdélyi–Magnus–Oberhettinger–Tricomi [4]. Given $a,b,c\in\mathbf{C}$ with

$$c\neq 0,-1,..., \tag{4.12}$$

the Gauss hypergeometric function is formally defined by

$$F(a,b,c,z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k. \tag{4.13}$$

The series in (4.13) is absolutely and uniformly convergent on compact sets for $|z|<1$, and hence $F(a,b,c,z)$ is holomorphic for $|z|<1$, see §2.1.1 of [4]. Moreover, $F(a,b,c,z)$ has analytic continuation to $\mathbf{C}\setminus[1,\infty)$ as a single-valued holomorphic function, see §2.1.4 of [4]. On $\mathbf{C}\setminus[0,\infty)$ we consider the determination of z with $|\arg(-z)|<\pi$.

We have the following formulae. If $a-b\notin\mathbf{Z}$, c satisfies (4.12) and $z\in\mathbf{C}\setminus[0,\infty)$ we have

$$\begin{aligned} F(a,b,c,z)=&\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F(a,1-c+a,1-b+a,z^{-1})\\ &+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F(b,1-c+b,1-a+b,z^{-1}),\end{aligned} \tag{4.14}$$

see (17) of §2.1.4 of [4]. Moreover, if c satisfies (4.12) and $z\in\mathbf{C}\setminus[0,\infty)$ we have

$$F(a,b,c,z)=(1-z)^{-a}F\Big(a,c-b,c,\frac{z}{z-1}\Big), \tag{4.15}$$

see (22) of §2.1.4 of [4].

The behavior of $F(a,b,c,z)$ at $z=1$ is given by

LEMMA 4.1. *Let c satisfy* (4.12), $\mathrm{Re}(c-a-b)>0$, $c-a-b\neq 1,2,\dots$, *and* $\varrho>0$. *Then, uniformly for* $\phi\neq0$, *we have*

$$\lim_{\varrho\to0^+}F(a,b,c,1+\varrho e^{i\phi})=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Proof. Writing $z=1+\varrho e^{i\phi}$ we have $z/(z-1)=1+e^{-i\phi}/\varrho$. Hence by (4.15) and (4.14) we have

$$\begin{aligned} F(a,b,c,z)&=(-\varrho e^{i\phi})^{-a}F\Big(a,c-b,c,1+\frac{e^{-i\phi}}{\varrho}\Big)\\ &=(1+\varrho e^{i\phi})^{-a}\frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}F\Big(a,1-c-a,1+a+b-c,\frac{\varrho}{\varrho+e^{-i\phi}}\Big)\\ &\quad+\varrho^{c-a-b}(-e^{i\phi})^{-a}(-\varrho-e^{-i\phi})^{b-c}\\ &\qquad\times\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F\Big(c-b,1-b,1-a-b+c,\frac{\varrho}{\varrho+e^{-i\phi}}\Big).\end{aligned} \tag{4.16}$$

Since $\mathrm{Re}(c-a-b)>0$, we have $|\varrho^{c-a-b}|\to0$ as $\varrho\to0^+$. Moreover, $F(a,b,c,0)=1$ provided c satisfies (4.12). Hence the lemma follows letting $\varrho\to0^+$ in (4.16). □

We remark that if c satisfies (4.12) and $\operatorname{Re}(c-a-b)>0$, the series in (4.13) is in fact absolutely convergent for $|z|\leqslant 1$, see §2.1.1 of [4]. Moreover, in this case Gauss' formula

$$F(a,b,c,1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

holds, see last formula of §14.11 of Whittaker–Watson [15].

For $a,b\neq -\frac{1}{2}v$, $v=0,1,\ldots$, we consider the function

$$G(a,b,z)=\sum_{k=0}^{\infty}\frac{1}{k!}\Gamma\bigl(a+\tfrac{1}{2}k\bigr)\Gamma\bigl(b+\tfrac{1}{2}k\bigr)z^k,$$

the series being absolutely and uniformly convergent on compact sets for $|z|<1$ by Stirling's formula. The relation between the function $G(a,b,z)$ and the Gauss hypergeometric function is given by

LEMMA 4.2. *Let $a,b\neq -\frac{1}{2}v$, $v=0,1,\ldots$, and let $z\in\mathbf{C}\setminus\{(-\infty,-2]\cup[2,+\infty)\}$. Then $G(a,b,z)$ is a single-valued holomorphic function and satisfies*

$$G(a,b,z)=\Gamma(a)\Gamma(b)F\bigl(a,b,\tfrac{1}{2},\tfrac{1}{4}z^2\bigr)+z\Gamma\bigl(a+\tfrac{1}{2}\bigr)\Gamma\bigl(b+\tfrac{1}{2}\bigr)F\bigl(a+\tfrac{1}{2},b+\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{4}z^2\bigr).$$

Proof. Assume first that $|z|<1$. For $l\in\mathbf{N}$ we have

$$(2l+1)!=4^l l!\bigl(\tfrac{3}{2}\bigr)_l,\quad (2l)!=4^l l!\bigl(\tfrac{1}{2}\bigr)_l\quad\text{and}\quad \Gamma(a+l)=\Gamma(a)(a)_l.$$

Hence

$$\frac{z^{2l}}{(2l)!}\Gamma(a+l)\Gamma(b+l)=\Gamma(a)\Gamma(b)\frac{(a)_l(b)_l}{l!\bigl(\frac{1}{2}\bigr)_l}\left(\frac{z^2}{4}\right)^l$$

and

$$\frac{z^{2l+1}}{(2l+1)!}\Gamma\bigl(a+\tfrac{1}{2}+l\bigr)\Gamma\bigl(b+\tfrac{1}{2}+l\bigr)=z\Gamma\bigl(a+\tfrac{1}{2}\bigr)\Gamma\bigl(b+\tfrac{1}{2}\bigr)\frac{\bigl(a+\frac{1}{2}\bigr)_l\bigl(b+\frac{1}{2}\bigr)_l}{l!\bigl(\frac{3}{2}\bigr)_l}\left(\frac{z^2}{4}\right)^l,$$

and hence the result follows easily if $|z|<1$. Lemma 4.2 follows then by analytic continuation. □

5. Hypergeometric functions: behavior at $z=-i/\beta$

From now on we assume $d=1$. Let Ω be a bounded domain contained in the strip $0<\sigma<2$ and let K be a sufficiently large integer. For $s\in\Omega$ write

$$H_K(-i/\beta,s)=\lim_{z\to -i/\beta}H_K(z,s),\tag{5.1}$$

where the limit is taken along a path where $\operatorname{Re}z>0$. Moreover, let

$$a=\tfrac{1}{2}(1-s)+\tfrac{1}{2}\bar{\xi},\quad b=\tfrac{1}{2}s+\tfrac{1}{2}\xi\quad\text{and}\quad A(s)=\sum_{j=1}^{r}(\lambda_j(1-2s)-2i\operatorname{Im}\mu_j)\log 2\lambda_j.$$

In this section we prove

THEOREM 5.1. *Let $d=1$. For $s\in\Omega$ the limit (5.1) exists, does not depend on the path, and is meromorphic. Moreover,*

$$H_K\Big(-\frac{i}{\beta},s\Big)=e^{A(s)}\Big\{-\sum_{k=0}^{K}\frac{1}{k!}\frac{\Gamma\big(a+\frac12+\frac12k\big)}{\Gamma\big(b+\frac12-\frac12k\big)}\Big(\frac{i}{2}\Big)^{-k}+\sqrt{\pi}\,\frac{\Gamma\big(a+\frac12\big)\Gamma\big(b-a-\frac12\big)}{\Gamma(-a)\Gamma(b)\Gamma\big(\frac12+b\big)}-i\sqrt{\pi}\,\frac{\Gamma(a+1)\Gamma\big(b-a-\frac12\big)}{\Gamma\big(\frac12-a\big)\Gamma(b)\Gamma\big(\frac12+b\big)}\Big\}+f_1(s)$$

where $f_1(s)$ is holomorphic for $0<\sigma<2$.

In particular, the expression in Theorem 5.1 provides the meromorphic continuation of $H_K(-i/\beta,s)$ to the strip $0<\sigma<2$.

We start the proof of Theorem 5.1 by the following key lemma.

LEMMA 5.1. *Let $d=1$ and $\mathrm{Re}\,w=-K-\frac12$. Then*

$$\tilde h(w,s)=e^{A(s)}\frac{\Gamma\big(a+\frac12-\frac12w\big)}{\Gamma\big(b+\frac12+\frac12w\big)}\Big(\frac{\beta}{2}\Big)^{w}(1+f(w,s))$$

where $f(w,s)$ is holomorphic for $0<\sigma<2$ and satisfies

$$f(w,s)=O(1/|w|)\quad \text{as } |w|\to\infty,$$

uniformly for $0<\sigma<2$.

Proof. Writing $\alpha_j=\lambda_j(1-s)+\bar\mu_j$ and $\beta_j=\lambda_js+\mu_j$ we have $a=\sum_{j=1}^r\big(\alpha_j-\frac12\big)$, $b=\sum_{j=1}^r\big(\beta_j-\frac12\big)$, $A(s)=\sum_{j=1}^r(\alpha_j-\beta_j)\log 2\lambda_j$ and

$$\tilde h(w,s)=\prod_{j=1}^r\frac{\Gamma(\alpha_j-\lambda_jw)}{\Gamma(\beta_j+\lambda_jw)}.$$

Hence, by Stirling's formula, for $0<\sigma<2$ and $\mathrm{Re}\,w=-K-\frac12$ we have

$$\begin{aligned}\log\tilde h(w,s)&=\sum_{j=1}^r\big\{\big(\alpha_j-\lambda_jw-\tfrac12\big)\log\big(-\tfrac12w\big)+\big(\alpha_j-\lambda_jw-\tfrac12\big)\log 2\lambda_j+\lambda_jw+\tfrac12\log 2\pi\big\}\\&\quad-\sum_{j=1}^r\big\{\big(\beta_j+\lambda_jw-\tfrac12\big)\log\big(\tfrac12w\big)+\big(\beta_j+\lambda_jw-\tfrac12\big)\log 2\lambda_j-\lambda_jw+\tfrac12\log 2\pi\big\}\\&\quad+f_1(w,s)\\&=\big(a-\tfrac12w\big)\log\big(-\tfrac12w\big)+\tfrac12w+\tfrac12\log 2\pi-\big\{\big(b+\tfrac12w\big)\log\big(\tfrac12w\big)-\tfrac12w+\tfrac12\log 2\pi\big\}\\&\quad+w\log\big(\tfrac12\beta\big)+A(s)+f_1(w,s)\\&=\log\Gamma\big(a+\tfrac12-\tfrac12w\big)-\log\Gamma\big(b+\tfrac12+\tfrac12w\big)+w\log\big(\tfrac12\beta\big)+A(s)+f_2(w,s),\end{aligned}$$

where $f_1(w,s)$ and $f_2(w,s)$ are holomorphic for $0<\sigma<2$, and satisfy

$$f_i(w,s)=O(1/|w|) \quad \text{as } |w|\to\infty,\ i=1,2,$$

uniformly for $0<\sigma<2$. Writing $f(w,s)=e^{f_2(w,s)}-1$, the result follows immediately. □

By Lemma 5.1, for $s\in\Omega$ and $z\in A$, where A is defined in (4.5), we have

$$\begin{aligned}H_K(z,s)&=\frac{e^{A(s)}}{2\pi i}\int_{(-K-1/2)}\frac{\Gamma\big(a+\frac{1}{2}-\frac{1}{2}w\big)}{\Gamma\big(b+\frac{1}{2}+\frac{1}{2}w\big)}\Gamma(w)\big(\tfrac{1}{2}\beta z\big)^w\,dw\\&\quad+\frac{e^{A(s)}}{2\pi i}\int_{(-K-1/2)}\frac{\Gamma\big(a+\frac{1}{2}-\frac{1}{2}w\big)}{\Gamma\big(b+\frac{1}{2}+\frac{1}{2}w\big)}\Gamma(w)\big(\tfrac{1}{2}\beta z\big)^w f(w,s)\,dw\\&=H_K^{(1)}(z,s)+H_K^{(2)}(z,s),\end{aligned}\tag{5.2}$$

say. From Lemma 5.1 and Stirling's formula, we see that the integrand in $H_K^{(2)}(z,s)$ is $O(|w|^{-1-\sigma})$ on the line $\operatorname{Re} w=-K-\frac{1}{2}$, uniformly for $0<\sigma<2$ and $z\neq 0$ with $|\arg z|\leqslant\frac{1}{2}\pi$. Hence $H_K^{(2)}(z,s)$ is holomorphic for $0<\sigma<2$ and $z\neq 0$ with $|\arg z|\leqslant\frac{1}{2}\pi$. Observe that, in this case, the values $\arg z=\pm\frac{1}{2}\pi$ are allowed. Therefore, writing

$$f_1(s)=\lim_{z\to -i/\beta}H_K^{(2)}(z,s)\tag{5.3}$$

with the above convention about the path, we see that $f_1(s)$ is holomorphic for $0<\sigma<2$.

In order to deal with $H_K^{(1)}(z,s)$, we observe that

$$H_K^*(z,s)=e^{-A(s)}H_K^{(1)}(2z/\beta,s)\tag{5.4}$$

is also an incomplete Fox function, with

$$\tilde{h}(w,s)=\frac{\Gamma\big(a+\frac{1}{2}-\frac{1}{2}w\big)}{\Gamma\big(b+\frac{1}{2}+\frac{1}{2}w\big)},\quad d=1,\tag{5.5}$$

and having the β-parameter equal to 2. Hence, in view of Theorem 4.1, for $s\in\Omega$ and $|z|>\frac{1}{2}$ we write

$$H_K^*(z,s)=-\sum_{k=0}^{K}\frac{(-1)^k}{k!}\frac{\Gamma\big(a+\frac{1}{2}+\frac{1}{2}k\big)}{\Gamma\big(b+\frac{1}{2}-\frac{1}{2}k\big)}z^{-k}+H^*(z,s)\tag{5.6}$$

with

$$H^*(z,s)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma\big(a+\frac{1}{2}+\frac{1}{2}k\big)}{\Gamma\big(b+\frac{1}{2}-\frac{1}{2}k\big)}z^{-k}.\tag{5.7}$$

Now we express the series in (5.7) in terms of Gauss hypergeometric functions. This step is important in our argument, since it will allow the explicit computation of $H_K(-i/\beta,s)$. Observing that

$$\frac{1}{\Gamma\left(b+\frac{1}{2}-\frac{1}{2}k\right)}=\frac{e^{i\pi b}}{2\pi}\Gamma\left(\tfrac{1}{2}-b+\tfrac{1}{2}k\right)(-i)^k+\frac{e^{-i\pi b}}{2\pi}\Gamma\left(\tfrac{1}{2}-b+\tfrac{1}{2}k\right)i^k,$$

from Lemma 4.2 we have

$$\begin{aligned}H^*(z,s)=&\frac{\cos\pi b}{\pi}\Gamma\left(a+\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}-b\right)F\left(a+\tfrac{1}{2},\tfrac{1}{2}-b,\tfrac{1}{2},-1/4z^2\right)\\&-\frac{\sin\pi b}{\pi z}\Gamma(a+1)\Gamma(1-b)F\left(a+1,1-b,\tfrac{3}{2},-1/4z^2\right)\end{aligned}\tag{5.8}$$

for $s\in\Omega$, $|z|>\frac{1}{2}$ and $a+\frac{1}{2},\frac{1}{2}-b\neq-\frac{1}{2}v$, $v=0,1,\ldots$. Moreover, observing that, in view of the properties of Gauss hypergeometric functions, (5.8) holds for $z\in\mathbf{C}\setminus\left[-\frac{1}{2}i,\frac{1}{2}i\right]$ by analytic continuation, from (5.6) and (5.8) we get

$$\begin{aligned}H_K^*(z,s)=&-\sum_{k=0}^{K}\frac{(-1)^k}{k!}\frac{\Gamma\left(a+\frac{1}{2}+\frac{1}{2}k\right)}{\Gamma\left(b+\frac{1}{2}-\frac{1}{2}k\right)}z^{-k}\\&+\frac{\cos\pi b}{\pi}\Gamma\left(a+\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}-b\right)F\left(a+\tfrac{1}{2},\tfrac{1}{2}-b,\tfrac{1}{2},-1/4z^2\right)\\&-\frac{\sin\pi b}{\pi z}\Gamma(a+1)\Gamma(1-b)F\left(a+1,1-b,\tfrac{3}{2},-1/4z^2\right)\end{aligned}\tag{5.9}$$

for $s\in\Omega$, $z\in\mathbf{C}\setminus\left[-\frac{1}{2}i,\frac{1}{2}i\right]$ and $a+\frac{1}{2},\frac{1}{2}-b\neq-\frac{1}{2}v$, $v=0,1,\ldots$.

Since for $z\to-\frac{1}{2}i$ with $\operatorname{Re}z>0$ we have that $-1/4z^2$ is of the form $1+\varrho e^{i\phi}$ with $\varrho\to0$ and $\phi\neq0$, from (5.9) and Lemma 4.1 we obtain

$$\begin{aligned}\lim_{z\to-i/2}H_K^*(z,s)=&-\sum_{k=0}^{K}\frac{1}{k!}\frac{\Gamma\left(a+\frac{1}{2}+\frac{1}{2}k\right)}{\Gamma\left(b+\frac{1}{2}-\frac{1}{2}k\right)}\left(\frac{i}{2}\right)^{-k}\\&+\frac{\cos\pi b}{\sqrt{\pi}}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-b\right)\Gamma\left(b-a-\frac{1}{2}\right)}{\Gamma(-a)\Gamma(b)}\\&-i\frac{\sin\pi b}{\sqrt{\pi}}\frac{\Gamma(a+1)\Gamma(1-b)\Gamma\left(b-a-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-a\right)\Gamma\left(\frac{1}{2}+b\right)}\end{aligned}\tag{5.10}$$

for $s\in\Omega$ with $\sigma>1$, $a+\frac{1}{2},\frac{1}{2}-b\neq-\frac{1}{2}v$, $v=0,1,\ldots$, and $s-1+2i\operatorname{Im}\xi\neq1,2,\ldots$. Since the right-hand side of (5.10) is a meromorphic function, the left-hand side is also meromorphic, and hence (5.10) holds, in particular, for $s\in\Omega$.

Moreover,

$$\frac{\cos\pi b}{\sqrt{\pi}}\Gamma\left(\tfrac{1}{2}-b\right)=\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+b\right)}\quad\text{and}\quad\frac{\sin\pi b}{\sqrt{\pi}}\Gamma(1-b)=\frac{\sqrt{\pi}}{\Gamma(b)},\tag{5.11}$$

and hence Theorem 5.1 follows from (5.2), (5.3), (5.4), (5.10) and (5.11).

6. Hypergeometric functions: behavior at $s=1-i\theta$

Let $d=1$ and write

$$g(s,\xi)=e^{-A(s)}H_K(-i/\beta,s). \tag{6.1}$$

In this section we detect the behavior of $g(s,\xi)$ at $s=1-i\theta$. From Theorem 5.1, for $0<\sigma<2$ we have

$$\begin{aligned}g(s,\xi)&=-\sum_{k=0}^{K}\frac{1}{k!}\frac{\Gamma\big(1-\frac12 s+\frac12 k+\frac12\bar\xi\big)}{\Gamma\big(\frac12(s+1)-\frac12 k+\frac12\xi\big)}\left(\frac{i}{2}\right)^{-k}\\&\quad+\sqrt{\pi}\,\frac{\Gamma\big(1-\frac12 s+\frac12\bar\xi\big)\Gamma(s-1+i\theta)}{\Gamma\big(\frac12(s-1)-\frac12\bar\xi\big)\Gamma\big(\frac12 s+\frac12\xi\big)\Gamma\big(\frac12(s+1)+\frac12\xi\big)}\\&\quad-i\sqrt{\pi}\,\frac{\Gamma\big(\frac12(3-s)+\frac12\bar\xi\big)\Gamma(s-1+i\theta)}{\Gamma\big(\frac12 s-\frac12\bar\xi\big)\Gamma\big(\frac12 s+\frac12\xi\big)\Gamma\big(\frac12(s+1)+\frac12\xi\big)}+f_2(s)\\&=g_1(s,\xi)+g_2(s,\xi)+g_3(s,\xi)+f_2(s),\end{aligned}\tag{6.2}$$

say, where $f_2(s)$ is holomorphic for $0<\sigma<2$.

We need the following

LEMMA 6.1. *Let $m\in\mathbf{N}$. For $m\geqslant 2$ we have*

$$\Sigma_1(m)=\sum_{l=0}^{m-1}(-1)^l\frac{(2m+2l-3)!}{(2l)!\,(m+l-2)!\,(m-l-1)!}=2(-1)^{m+1}4^{m-2},$$

and for $m\geqslant 1$ we have

$$\Sigma_2(m)=\sum_{l=0}^{m-1}(-1)^l\frac{(2m+2l-1)!}{(2l+1)!\,(m+l-1)!\,(m-l-1)!}=(-1)^{m+1}4^{m-1}.$$

Proof. We use the following identity, see equation 64 on p. 620 of [9]:

$$\sum_{l=0}^{[n/2]}(-1)^l\binom{a}{l}\binom{2a-2l}{2a-n}=2^n\binom{a}{n} \tag{6.3}$$

for non-negative integers a and n.

Choosing $a=n=2m-3$ we get

$$\sum_{l=0}^{m-2}(-1)^l\binom{2m-3}{l}\binom{4m-2l-6}{2m-3}=2\cdot 4^{m-2}, \tag{6.4}$$

and choosing $a=2m-3$ and $n=2m-2$ we have

$$\sum_{l=0}^{m-1}(-1)^l\binom{2m-3}{l}\binom{4m-2l-6}{2m-4}=4^{m-1}\binom{2m-3}{2m-2}=0. \tag{6.5}$$

From (6.4) and (6.5) we obtain

$$\begin{aligned}\Sigma_1(m)&=\sum_{l=0}^{m-1}(-1)^l\binom{2m+2l-3}{2m-3}\binom{2m-3}{m-1-l}\\&=(-1)^{m+1}\sum_{l=0}^{m-1}(-1)^l\binom{2m-3}{l}\binom{4m-2l-5}{2m-3}\\&=(-1)^{m+1}\sum_{l=0}^{m-1}(-1)^l\binom{2m-3}{l}\binom{4m-2l-6}{2m-4}\\&\quad+(-1)^{m+1}\sum_{l=0}^{m-2}(-1)^l\binom{2m-3}{l}\binom{4m-2l-6}{2m-3}=2(-1)^{m+1}4^{m-2},\end{aligned}$$

which proves the first assertion of Lemma 6.1. The second assertion follows arguing in a similar way, using (6.3) with the two choices

$$a=n=2m-2 \quad \text{and} \quad a=2m-2,\ n=2m-1. \qquad \square$$

The behavior of $g(s,\xi)$ at $s=1-i\theta$ is given by

PROPOSITION 6.1. *Let $d=1$. The function $g(s,\xi)$ has a simple pole at $s=1-i\theta$, with residue $\varrho(\xi)$ given by*

$$\varrho(\xi)=\begin{cases}-\dfrac{i}{\sqrt{\pi}}e^{-\pi i\eta/2} & \text{if } \eta>-1 \text{ or } \eta\notin\mathbf{Z},\\ \dfrac{i}{\sqrt{\pi}}e^{\pi i\eta/2} & \text{if } \eta=-2m+1 \text{ and } m=1,2,\ldots,\\ -\dfrac{i}{\sqrt{\pi}}e^{\pi i\eta/2} & \text{if } \eta=-2m \text{ and } m=1,2,\ldots.\end{cases}$$

Proof. It is easy to see that $g(s,\xi)$ has at most a simple pole at $s=1-i\theta$. Moreover, the Γ-factors other than $\Gamma(s-1+i\theta)$ in the numerators of (6.2) may become polar at $s=1-i\theta$ only in the cases

$$\eta=-2m+1 \quad \text{and} \quad \eta=-2m, \quad \text{for } m=1,2,\ldots.$$

Since $f_2(s)$ is holomorphic at $s=1-i\theta$, with obvious notation we write

$$\varrho(\xi)=\varrho_1(\xi)+\varrho_2(\xi)+\varrho_3(\xi). \tag{6.6}$$

We also write

$$r_k = \operatorname*{res}_{s=1-i\theta} \Gamma\left(1-\tfrac{1}{2}s+\tfrac{1}{2}k+\tfrac{1}{2}\bar{\xi}\right),$$

and hence

$$r_k\neq 0 \text{ if and only if } \tfrac{1}{2}+\tfrac{1}{2}\eta+\tfrac{1}{2}k=-v \text{ for some } v=0,1,\dots \tag{6.7}$$

and

$$\varrho_1(\xi) = -\sum_{k=0}^{K} \frac{1}{k!}\frac{\left(\frac{1}{2}i\right)^{-k}}{\Gamma\left(1-\frac{1}{2}k+\frac{1}{2}\eta\right)} r_k. \tag{6.8}$$

We prove Proposition 6.1 by actually computing the value of $\varrho(\xi)$, thus showing that always $\varrho(\xi)\neq 0$. If $\eta\neq -2m+1$ and $\eta\neq -2m$, with $m=1,2,\dots$, we have $\varrho_1(\xi)=0$,

$$\varrho_2(\xi) = \frac{\sqrt{\pi}}{\Gamma\left(-\frac{1}{2}\eta\right)\Gamma\left(1+\frac{1}{2}\eta\right)} = -\frac{\sin\left(\frac{1}{2}\pi\eta\right)}{\sqrt{\pi}}$$

and

$$\varrho_3(\xi) = -\frac{i\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\eta\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\eta\right)} = -i\frac{\cos\left(\frac{1}{2}\pi\eta\right)}{\sqrt{\pi}}.$$

Hence Proposition 6.1 follows in this case from (6.6), since $\varrho(\xi)\neq 0$.

Suppose now that $\eta=-2m+1$ for some $m=1,2,\dots$. From (6.7) we have that $r_k\neq 0$ for $k=2l$ with $0\leqslant l\leqslant m-1$, since we may assume K to be sufficiently large. Therefore, from (6.8) we get

$$\varrho_1(\xi) = -\sum_{l=0}^{m-1} \frac{(-1)^l}{(2l)!}\frac{4^l}{\Gamma\left(\frac{3}{2}-m-l\right)} r_{2l}.$$

Since

$$r_{2l} = -2\frac{(-1)^{m-l-1}}{(m-l-1)!}$$

and

$$\Gamma\left(\tfrac{3}{2}-m-l\right) = \frac{\pi}{(-1)^{m+l-1}\Gamma\left(m+l-\frac{1}{2}\right)} = \frac{\pi}{(-1)^{m+l-1}}\frac{4^{m+l-1}}{2\sqrt{\pi}}\frac{\Gamma(m+l-1)}{\Gamma(2m+2l-2)},$$

for $m\geqslant 2$ we have

$$\varrho_1(\xi) = \frac{4^{2-m}}{\sqrt{\pi}}\sum_{l=0}^{m-1}\frac{(-1)^l(2m+2l-3)!}{(2l)!\,(m+l-2)!\,(m-l-1)!},$$

and for $m=1$ we have

$$\varrho_1(\xi) = \frac{2}{\sqrt{\pi}}.$$

Hence, if $\eta=-2m+1$ from Lemma 6.1 we get

$$\varrho_1(\xi)=\frac{2}{\sqrt{\pi}}(-1)^{m+1}. \tag{6.9}$$

Assume now that $\eta=-2m$ for some $m=1,2,\ldots$. From (6.7) we have that $r_k\neq 0$ for $k=2l+1$ with $0\leqslant l\leqslant m-1$. By the same argument used above we get

$$\varrho_1(\xi)=\frac{2i4^{1-m}}{\sqrt{\pi}}\sum_{l=0}^{m-1}\frac{(-1)^l(2m+2l-1)!}{(2l+1)!\,(m+l-1)!\,(m-l-1)!},$$

and hence from Lemma 6.1, in this case we obtain

$$\varrho_1(\xi)=\frac{2i}{\sqrt{\pi}}(-1)^{m+1}. \tag{6.10}$$

A simple computation shows that if $\eta=-2m+1$ for some $m=1,2,\ldots$, then

$$\varrho_2(\xi)=\frac{\sin\left(\frac{1}{2}\pi\eta\right)}{\sqrt{\pi}} \quad\text{and}\quad \varrho_3(\xi)=-i\frac{\cos\left(\frac{1}{2}\pi\eta\right)}{\sqrt{\pi}}, \tag{6.11}$$

and if $\eta=-2m$ we have

$$\varrho_2(\xi)=-\frac{\sin\left(\frac{1}{2}\pi\eta\right)}{\sqrt{\pi}} \quad\text{and}\quad \varrho_3(\xi)=i\frac{\cos\left(\frac{1}{2}\pi\eta\right)}{\sqrt{\pi}}. \tag{6.12}$$

Proposition 6.1 follows by a computation from (6.6) and (6.9)–(6.12), since $\varrho(\xi)\neq 0$ in all cases. □

The analytic properties of $H_K(-i/\beta,s)$ are summarized by

THEOREM 6.1. *Let $d=1$. The function $H_K(-i/\beta,s)$ is meromorphic for $0<\sigma<2$. It has a simple pole at $s=1-i\theta$ and, if $\theta\neq 0$, is holomorphic at $s=1$.*

Proof. This follows from (6.1), (6.2) and Proposition 6.1, observing that if $\theta\neq 0$, then $g(s,\xi)$ is clearly holomorphic at $s=1$. □

7. Additive twists

Given $\alpha\in\mathbf{R}$, we consider the additive character $\psi=\psi_\alpha$ defined by $\psi(n)=e(-n\alpha)$. For $F\in\mathcal{S}_1^\sharp$ and $\sigma>1$ we form the *additive twist*

$$F^\psi(s)=\sum_{n=1}^{\infty}\frac{a(n)\psi(n)}{n^s}.$$

Let $N>2$ and K be a sufficiently large positive integer. For $0<\sigma<2$ and a fixed $\alpha>0$, a standard argument shows that

$$F_N^{\psi}(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}e^{-n(1/N+2\pi i\alpha)}=\frac{1}{2\pi i}\int_{(2)}F(s+w)\Gamma(w)\left(\frac{1}{N}+2\pi i\alpha\right)^{-w}dw$$
$$=R_{N,\alpha}(s)+\sum_{k=0}^{K}\frac{(-1)^k}{k!}F(s-k)\left(\frac{1}{N}+2\pi i\alpha\right)^k$$
$$+\frac{1}{2\pi i}\int_{(-K-1/2)}F(s+w)\Gamma(w)\left(\frac{1}{N}+2\pi i\alpha\right)^{-w}dw,$$

where

$$R_{N,\alpha}(s)=\operatorname*{res}_{w=1-s}\left(F(s+w)\Gamma(w)\left(\frac{1}{N}+2\pi i\alpha\right)^{-w}\right).$$

By the functional equation of $F(s)$ we get, with the notation of §4, that

$$F_N^{\psi}(s)=R_{N,\alpha}(s)+\sum_{k=0}^{K}\frac{(-1)^k}{k!}F(s-k)\left(\frac{1}{N}+2\pi i\alpha\right)^k$$
$$+\frac{\omega Q^{1-2s}}{2\pi i}\int_{(-K-1/2)}\overline{F}(1-s-w)h(w,s)\left(\frac{Q^2}{N}+2\pi i\alpha Q^2\right)^{-w}dw. \tag{7.1}$$

By the absolute convergence of both the Dirichlet series of $\overline{F}(1-s-w)$ and the integral in (7.1), for $-1<\sigma<K+\frac{1}{2}$ we may replace $\overline{F}(1-s-w)$ by its Dirichlet series and interchange integration and summation. Hence by the convergence properties of the Barnes integral $H_K(z,s)$ in Theorem 4.1 we get

$$F_N^{\psi}(s)=R_{N,\alpha}(s)+\sum_{k=0}^{K}\frac{(-1)^k}{k!}F(s-k)\left(\frac{1}{N}+2\pi i\alpha\right)^k$$
$$+\omega Q^{1-2s}\sum_{n=1}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}H_K\left(\frac{n}{Q^2(1/N+2\pi i\alpha)},s\right). \tag{7.2}$$

Write $n_\alpha=q\alpha$ and define

$$a(n_\alpha)=\begin{cases}a(n_\alpha) & \text{if } n_\alpha\in\mathbf{N},\\ 0 & \text{otherwise.}\end{cases}$$

If $n\neq n_\alpha$, by Theorem 4.1 we have that

$$\lim_{N\to\infty}H_K\left(\frac{n}{Q^2(1/N+2\pi i\alpha)},s\right)=H_K\left(-\frac{i}{\beta}\frac{n}{q\alpha},s\right), \tag{7.3}$$

and $H_K(-in/\beta q\alpha, s)$ is holomorphic on a given bounded domain Ω contained in the strip $0<\sigma<2$. Moreover, if $n>n_\alpha$ we use (4.6) in Theorem 4.1 to control the n-dependence of the right-hand side of (7.3). Indeed, since

$$\frac{1}{\Gamma(\lambda_j s+\mu_j-\lambda_j k)} = -\frac{\lambda_j s+\mu_j-\lambda_j k}{\pi}\sin(\pi(\lambda_j s+\mu_j-\lambda_j k))\Gamma(\lambda_j k-\lambda_j s-\mu_j),$$

a Stirling's formula estimate similar to those used in the proof of Theorem 4.1 gives

$$H_K\Big(-\frac{i}{\beta}\frac{n}{q\alpha}, s\Big) = \sum_{k=K+1}^{\infty}\frac{(-1)^k}{k!}\tilde{h}(-k,s)\Big(-\frac{i}{\beta}\frac{n}{q\alpha}\Big)^{-k} \ll n^{-K} \tag{7.4}$$

uniformly for $s\in\Omega$.

By the trivial bound $a(n)\ll n^{3/2}$, say, from (7.3) and (7.4) we have

$$\lim_{N\to\infty}\omega Q^{1-2s}\sum_{\substack{n=1\\ n\neq n_\alpha}}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}H_K\Big(\frac{n}{Q^2(1/N+2\pi i\alpha)}, s\Big) = f_3(s), \tag{7.5}$$

where $f_3(s)$ is holomorphic on Ω. Since K depends only on $\|\Omega\|$, we see that $f_3(s)$ is in fact holomorphic for $0<\sigma<2$.

Now we deal with the crucial case $n=n_\alpha$. In view of (5.1) and Theorem 5.1, if $n=n_\alpha$ and $n_\alpha\in\mathbf{N}$ we have

$$\lim_{N\to\infty}H_K\Big(\frac{n_\alpha}{Q^2(1/N+2\pi i\alpha)}, s\Big) = H_K\Big(-\frac{i}{\beta}, s\Big), \tag{7.6}$$

and $H_K(-i/\beta, s)$ is meromorphic for $0<\sigma<2$.

Further, it is clear that

$$R_\alpha(s) = \lim_{N\to\infty}R_{N,\alpha}(s) = \operatorname*{res}_{w=1-s}(F(s+w)\Gamma(w)(2\pi i\alpha)^{-w}) \tag{7.7}$$

is a meromorphic function for $0<\sigma<2$.

Since for $\sigma>1$

$$\lim_{N\to\infty}F_N^\psi(s) = F^\psi(s),$$

from (7.2) and (7.5)–(7.7) we get

$$F^\psi(s) = R_\alpha(s)+\sum_{k=0}^{K}\frac{(-1)^k}{k!}F(s-k)(2\pi i\alpha)^k+\omega Q^{1-2s}\frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s}}H_K\Big(-\frac{i}{\beta}, s\Big)+f_3(s) \tag{7.8}$$

for $1<\sigma<2$. Since the right-hand side of (7.8) is meromorphic for $0<\sigma<2$, it provides the meromorphic continuation of $F^\psi(s)$ to $0<\sigma<2$. We summarize the properties of $F^\psi(s)$ so far obtained by

LEMMA 7.1. *Let $F \in \mathcal{S}_1^\sharp$, $\alpha>0$ and $n_\alpha = q\alpha$. Then the additive twist $F^\psi(s)$ has meromorphic continuation to $\sigma>0$ and satisfies* (7.8) *for $0<\sigma<2$.*

Choose now $\alpha=1$. Hence $F^\psi(s)=F(s)$ and from (7.8) we have

$$R_1(s) = -\sum_{k=1}^{K} \frac{(-1)^k}{k!} F(s-k)(2\pi i)^k - \omega Q^{1-2s} \frac{\overline{a(n_1)}}{n_1^{1-s}} H_K\left(-\frac{i}{\beta}, s\right) - f_3(s). \tag{7.9}$$

From (7.9) and Theorem 6.1 we see that $R_1(s)$ has at most a simple pole at $s=1$. Therefore, from (7.7) with $\alpha=1$ we deduce that $F(s)$ itself has at most a simple pole at $s=1$.

Hence for a given $\alpha>0$ we have

$$R_\alpha = \varkappa \Gamma(1-s)(2\pi i \alpha)^{s-1},$$

and hence

$$\operatorname*{res}_{s=1}\left(R_\alpha(s) + \sum_{k=0}^{K} \frac{(-1)^k}{k!} F(s-k)(2\pi i\alpha)^k\right) = -\varkappa + \varkappa = 0.$$

Moreover, it is clear that

$$R_\alpha(s) + \sum_{k=0}^{K} \frac{(-1)^k}{k!} F(s-k)(2\pi i \alpha)^k$$

is holomorphic for $t \neq 0$. Therefore, writing (7.8) as

$$F^\psi(s) = \omega Q^{1-2s} \frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s}} H_K\left(-\frac{i}{\beta}, s\right) + h_\alpha(s), \tag{7.10}$$

we see that $h_\alpha(s)$ is holomorphic for $0<\sigma<2$.

By (7.10) and Theorem 6.1, we may summarize the analytic properties of $F^\psi(s)$ as follows.

THEOREM 7.1. *Let $F \in \mathcal{S}_1^\sharp$, $\alpha>0$ and $n_\alpha = q\alpha$. Then the additive twist $F^\psi(s)$ has meromorphic continuation to $\sigma>0$ and satisfies* (7.10) *for $0<\sigma<2$. Moreover, $F^\psi(s)$ has a simple pole at $s=1-i\theta$ if and only if $n_\alpha \in \mathbf{N}$ and $a(n_\alpha) \neq 0$.*

We remark that the argument used in the proof of Theorem 7.1 can be suitably modified to provide the meromorphic continuation of $F^\psi(s)$ to the whole complex plane. However, this is not necessary in the proof of Theorem 2, and having Theorem 2 such a meromorphic continuation already follows from the known properties of the additive twists of $\zeta(s)$.

8. The case $d=1$

We begin by establishing the following basic properties of the functions $F\in\mathcal{S}_1^\sharp$.

THEOREM 8.1. *Let $F\in\mathcal{S}_1^\sharp$. Then q is a positive integer, the sequence $a(n)n^{i\theta}$ is periodic with period q and*

$$F(s)=\sum_{\chi\,(\mathrm{mod}\,q)} P_\chi(s+i\theta)L(s+i\theta,\chi^*), \tag{8.1}$$

where the $P_\chi(s)$ are Dirichlet polynomials.

Proof. Let $n\in\mathbf{N}$ be such that $a(n)\neq 0$ and apply Theorem 7.1 with $\alpha=n/q$, thus getting that $F^\psi(s)$ has a simple pole at $s=1-i\theta$. Choosing $\psi'=\psi_{\alpha+1}$ we see that $F^{\psi'}(s)=F^\psi(s)$ has a simple pole at $s=1-i\theta$ too. Therefore, from Theorem 7.1 we obtain that $(\alpha+1)q\in\mathbf{N}$, and hence $q\in\mathbf{N}$, since $\alpha q\in\mathbf{N}$.

Given $n\in\mathbf{N}$, let $\psi=\psi_{n/q}$ and $\psi'=\psi_{(n+q)/q}$. Since $H_K(-i/\beta,s)$ has a simple pole at $s=1-i\theta$ by Theorem 6.1, we have $\mathrm{res}_{s=1-i\theta}\, H_K(-i/\beta,s)\neq 0$. Hence from (7.10) we get

$$\operatorname*{res}_{s=1-i\theta} F^\psi(s)=\omega Q^{2i\theta-1}\frac{\overline{a(n)}}{n^{i\theta}}\operatorname*{res}_{s=1-i\theta} H_K\Big(-\frac{i}{\beta},s\Big)$$

and

$$\operatorname*{res}_{s=1-i\theta} F^{\psi'}(s)=\omega Q^{2i\theta-1}\frac{\overline{a(n+q)}}{(n+q)^{i\theta}}\operatorname*{res}_{s=1-i\theta} H_K\Big(-\frac{i}{\beta},s\Big).$$

Since $F^\psi(s)=F^{\psi'}(s)$, the periodicity of $a(n)n^{i\theta}$ follows at once.

Write $c(n)=a(n)n^{i\theta}$. For $\sigma>1$ we have

$$F(s-i\theta)=\sum_{n=1}^\infty\frac{c(n)}{n^s}=\sum_{d|q}\sum_{\substack{n=1\\(n,q)=d}}^\infty\frac{c(n)}{n^s}=\sum_{d|q}\frac{1}{d^s}\sum_{\substack{n=1\\(n,q/d)=1}}^\infty\frac{c(nd)}{n^s}. \tag{8.2}$$

Since the function

$$c_d(n)=\begin{cases} c(nd) & \text{if } (n,q/d)=1,\\ 0 & \text{otherwise}\end{cases}$$

is periodic with period q/d, we may write

$$c_d(n)=\sum_{\chi\,(\mathrm{mod}\,q/d)} c_{d,\chi}\chi(n) \tag{8.3}$$

with certain $c_{d,\chi}\in\mathbf{C}$. Denoting by $\chi^*\ (\mathrm{mod}\, f_\chi)$ the primitive character inducing χ $(\mathrm{mod}\, q/d)$, from (8.2), (8.3) and the properties of Dirichlet characters we get

$$\begin{aligned} F(s-i\theta) &= \sum_{d|q}\sum_{\chi\,(\mathrm{mod}\, q/d)} \frac{c_{d,\chi}}{d^s} L(s,\chi) \\ &= \sum_{d|q}\sum_{\chi\,(\mathrm{mod}\, q/d)} \frac{c_{d,\chi}}{d^s} \prod_{p|(q/df_\chi)} \left(1-\frac{\chi(p)}{p^s}\right) L(s,\chi^*) \\ &= \sum_{\chi\,(\mathrm{mod}\, q)} P_\chi(s) L(s,\chi^*), \end{aligned}$$

where $\chi^*\ (\mathrm{mod}\, f_\chi)$ induces $\chi\ (\mathrm{mod}\, q)$ and $P_\chi(s)$ is a suitable Dirichlet polynomial. The proof of Theorem 8.1 is complete. □

Now we enter the finer structure of the representation (8.1). To this end, in the following lemma we prove that the Dirichlet L-functions are linearly independent over the Dirichlet polynomials. We say that the Dirichlet characters $\chi_1,\dots,\chi_J$ are *non-equivalent* if $\chi_1^*,\dots,\chi_J^*$ are all distinct, where χ_j^* is the primitive character inducing χ_j.

Lemma 8.1. *For $j=1,\dots,J$, let $P_j(s)$ be Dirichlet polynomials and χ_j be non-equivalent Dirichlet characters such that*

$$\sum_{j=1}^{J} P_j(s) L(s,\chi_j) = 0 \quad \textit{identically}. \tag{8.4}$$

Then $P_j=0$ for $j=1,\dots,J$.

Proof. Write $P_j(s)=\sum_{n\leqslant N} a_j(n) n^{-s}$ and suppose that not all the $P_j(s)$ are identically zero. Let n_0 be the smallest integer n such that $a_j(n)\neq 0$ for some j, and let j_0 be the smallest such j. Let $p>N$ be a prime number, and consider the (n_0p)th coefficient of the left-hand side of (8.4). By the identity principle for Dirichlet series such a coefficient is zero, and hence

$$\sum_{j=1}^{J}\sum_{d|n_0p} a_j(d)\chi_j\left(\frac{n_0p}{d}\right) = 0.$$

But $a_j(d)=0$ for $d<n_0$ and for $d>N$, and therefore

$$\sum_{j=1}^{J} a_j(n_0)\chi_j(p) = 0. \tag{8.5}$$

Let k be the least common multiple of the moduli of the χ_j, and let $\widetilde{\chi}_j\ (\mathrm{mod}\, k)$ be the character induced by χ_j. Hence $\widetilde{\chi}_j(p)=\chi_j(p)$ for $p>k$, and the characters $\widetilde{\chi}_j$ are all distinct. Multiplying by $\overline{\widetilde{\chi}_{j_0}(p)}$ both sides of (8.5) and summing over $\max(N,k)<p\leqslant x$, by a standard argument in prime number theory we get

$$0=\sum_{j=1}^{J} a_j(n_0) \sum_{\max(N,k)<p\leqslant x} \widetilde{\chi}_j(p)\,\overline{\widetilde{\chi}_{j_0}(p)} = a_{j_0}(n_0)\frac{x}{\log x}+o\left(\frac{x}{\log x}\right)$$

as $x\to\infty$. Therefore $a_{j_0}(n_0)=0$, and the lemma follows. □

From Theorem 8.1 and Lemma 8.1 we have that for a given $F\in\mathcal{S}_1^\sharp$,

θ is an invariant of $F(s)$, and the representation (8.1) *is unique.* (8.6)

We shall need the following simple lemma about almost periodic functions.

LEMMA 8.2. *Let $A_0(s)$ and $A_j(s)$, $g_j(s)$, $j=1,\dots,J$, be meromorphic functions on $\mathbf{C}$. Assume that for $j=1,\dots,J$ and for some $\sigma_0\in\mathbf{R}$, the functions $A_0(\sigma_0+it)$ and $A_j(\sigma_0+it)$ are almost periodic in t, and $g_j(s)=o(1)$ as $|s|\to\infty$, uniformly for $\delta\leqslant\arg s\leqslant\pi-\delta$ with some small $\delta>0$. If*

$$\sum_{j=1}^{J} A_j(s)g_j(s)=A_0(s) \quad \textit{identically,}$$

then $A_0=0$.

Proof. Assume that $A_0\neq 0$, and hence $A_0(\sigma_0+it_0)=c\neq 0$ for some t_0. By almost periodicity we find a sequence $t_n\to\infty$ such that for every n,

$$|A_0(\sigma_0+it_n)|\geqslant\left|\tfrac{1}{2}c\right| \quad \text{and} \quad |A_j(\sigma_0+it_n)|=O(1),\ j=1,\dots,J.$$

Therefore

$$\sum_{j=1}^{J} A_j(\sigma_0+it_n)g_j(\sigma_0+it_n)=o(1) \quad \text{as } n\to\infty,$$

a contradiction. □

The Dirichlet polynomials $P_\chi(s)$ in (8.1) can be characterized as follows.

PROPOSITION 8.1. *Let $F(s)\in\mathcal{S}_1^\sharp$. Then the Dirichlet polynomials $P_\chi(s)$ in* (8.1) *belong to $\mathcal{S}_0^\sharp(q/f_\chi,\omega^*\bar{\omega}_{\chi^*})$, for every $\chi\ (\mathrm{mod}\, q)$.*

Proof. Denote by $D_1(q)$ and $D_2(q)$ the set of $\chi\ (\mathrm{mod}\, q)$ with $\chi(-1)=1$ and $\chi(-1)=-1$, respectively. By (8.1) and the functional equation of the Dirichlet L-

functions we have

$$
\begin{aligned}
F(s) &= \frac{\Gamma\big(\frac{1}{2}(1-s-i\theta)\big)}{\Gamma\big(\frac{1}{2}(s+i\theta)\big)} \sum_{\chi\in D_1(q)} \omega_{\chi^*} Q_{\chi^*}^{1-2s-2i\theta} P_\chi(s+i\theta) L(1-s-i\theta,\bar\chi^*) \\
&\quad + \frac{\Gamma\big(\frac{1}{2}(2-s-i\theta)\big)}{\Gamma\big(\frac{1}{2}(s+1+i\theta)\big)} \sum_{\chi\in D_2(q)} \omega_{\chi^*} Q_{\chi^*}^{1-2s-2i\theta} P_\chi(s+i\theta) L(1-s-i\theta,\bar\chi^*) \qquad (8.7) \\
&= \frac{\Gamma\big(\frac{1}{2}(1-s-i\theta)\big)}{\Gamma\big(\frac{1}{2}(s+i\theta)\big)} \Sigma_1(s) + \frac{\Gamma\big(\frac{1}{2}(2-s-i\theta)\big)}{\Gamma\big(\frac{1}{2}(s+1+i\theta)\big)} \Sigma_2(s),
\end{aligned}
$$

say. On the other hand, from (8.1) and the functional equation of $F(s)$ we get

$$
\begin{aligned}
F(s) &= \omega Q^{1-2s} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s)+\bar\mu_j)}{\Gamma(\lambda_j s+\mu_j)} \sum_{\chi \,(\mathrm{mod}\, q)} \bar P_\chi(1-s-i\theta) L(1-s-i\theta,\bar\chi^*) \\
&= \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s)+\bar\mu_j)}{\Gamma(\lambda_j s+\mu_j)} \Sigma_3(s),
\end{aligned} \qquad (8.8)
$$

say.

Writing

$$
\Delta_1(s) = \frac{\Gamma\big(\frac{1}{2}(1-s-i\theta)\big)}{\Gamma\big(\frac{1}{2}(s+i\theta)\big)} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j s+\mu_j)}{\Gamma(\lambda_j(1-s)+\bar\mu_j)}
$$

and

$$
\Delta_2(s) = \frac{\Gamma\big(\frac{1}{2}(2-s-i\theta)\big)}{\Gamma\big(\frac{1}{2}(s+1+i\theta)\big)} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j s+\mu_j)}{\Gamma(\lambda_j(1-s)+\bar\mu_j)},
$$

from (8.7) and (8.8) we obtain

$$
\Delta_1(s)\Sigma_1(s)+\Delta_2(s)\Sigma_2(s) = \Sigma_3(s). \qquad (8.9)
$$

Assume that $\delta\leqslant\arg s\leqslant\pi-\delta$ for some small $\delta>0$. Hence $-\pi+\delta\leqslant\arg(-s)\leqslant-\delta$ and $\arg(-1)=-\pi$. By a computation, which we omit, based on Stirling's formula and similar to that in the proof of Lemma 5.1 we get

$$
\log\Delta_1(s) = a_1+b_1 s+O(1/|s|) \quad \text{as } |s|\to\infty, \qquad (8.10)
$$

where

$$
a_1 = \tfrac{1}{2} i\pi(\eta+1) + \log\sqrt{\tfrac{1}{2}\beta} + \log 2^{i\theta} + \log\prod_{j=1}^{r} \lambda_j^{2i\,\mathrm{Im}\,\mu_j} \quad \text{and} \quad b_1 = \log\frac{2}{\beta}.
$$

In a similar way we also get

$$\log \Delta_2(s) = a_2 + b_2 s + O(1/|s|) \quad \text{as } |s| \to \infty, \tag{8.11}$$

with certain constants a_2 and b_2.

But

$$\frac{\Delta_1(s)}{\Delta_2(s)} = \frac{\Gamma\big(\frac{1}{2} - \frac{1}{2}(s+i\theta)\big)\Gamma\big(\frac{1}{2} + \frac{1}{2}(s+i\theta)\big)}{\Gamma\big(\frac{1}{2}(s+i\theta)\big)\Gamma\big(1 - \frac{1}{2}(s+i\theta)\big)} = \tan\big(\tfrac{1}{2}\pi(s+i\theta)\big),$$

and hence

$$\log \Delta_1(s) - \log \Delta_2(s) = \log \tan\big(\tfrac{1}{2}\pi(s+i\theta)\big) = O(1/|s|) \quad \text{as } |\mathrm{Im}\, s| \to \infty. \tag{8.12}$$

From (8.10)–(8.12) we see that $a_1 = a_2$ and $b_1 = b_2$. Therefore, writing

$$a = e^{i\pi(\eta+1)/2} \sqrt{\tfrac{1}{2}\beta}\, 2^{i\theta} \prod_{j=1}^{r} \lambda_j^{2i\, \mathrm{Im}\, \mu_j} \quad \text{and} \quad b = \log \frac{2}{\beta}, \tag{8.13}$$

uniformly for $\delta \leqslant \arg s \leqslant \pi - \delta$ we have

$$\Delta_j(s) = a e^{bs}(1 + g_j(s)) \quad \text{and} \quad g_j(s) = O(1/|s|) \text{ as } |s| \to \infty, \quad j = 1, 2,$$

where $g_j(s)$ are meromorphic functions. Hence (8.9) takes the form

$$a e^{bs} \Sigma_1(s)(1 + g_1(s)) + a e^{bs} \Sigma_2(s)(1 + g_2(s)) = \Sigma_3(s). \tag{8.14}$$

Since $ae^{bs}\Sigma_1(s)$, $ae^{bs}\Sigma_2(s)$ and $\Sigma_3(s)$ are almost periodic in t for $\sigma < 0$, from (8.14) and Lemma 8.2 with $J = 2$ we obtain

$$a e^{bs} \Sigma_1(s) + a e^{bs} \Sigma_2(s) = \Sigma_3(s), \tag{8.15}$$

where a and b are given by (8.13).

Since $Q = (\beta q/2\pi)^{1/2}$, $Q_{\chi^*} = (f_\chi/\pi)^{1/2}$ and $b = \log(2/\beta)$, substituting $s + i\theta$ by s we may rewrite (8.15) in the form

$$\sum_{\chi\, (\mathrm{mod}\, q)} a\big(\tfrac{1}{2}\beta\big)^{i\theta} \omega_{\chi^*} Q_{\chi^*} f_\chi^{-s} P_\chi(s) L(1-s, \bar{\chi}^*) = \sum_{\chi\, (\mathrm{mod}\, q)} \omega Q^{1+2i\theta} q^{-s} \overline{P}_\chi(1-s) L(1-s, \bar{\chi}^*).$$

Hence by Lemma 8.1 we get

$$a\big(\tfrac{1}{2}\beta\big)^{i\theta} \omega_{\chi^*} Q_{\chi^*} f_\chi^{-s} P_\chi(s) = \omega Q^{1+2i\theta} q^{-s} \overline{P}_\chi(1-s) \tag{8.16}$$

for every $\chi\ (\mathrm{mod}\, q)$.

From (8.16) and (8.13), for every $\chi \pmod q$ we have

$$P_\chi(s)=\frac{\omega\bar{\omega}_{\chi^*}}{a}\,\frac{Q^{1+2i\theta}}{Q_{\chi^*}}\left(\frac{2}{\beta}\right)^{i\theta}\left(\frac{f_\chi}{q}\right)^{1/2}\left(\sqrt{\frac{q}{f_\chi}}\right)^{1-2s}\overline{P}_\chi(1-s)$$
$$=\omega^*\bar{\omega}_{\chi^*}\left(\sqrt{\frac{q}{f_\chi}}\right)^{1-2s}\overline{P}_\chi(1-s),$$

and Proposition 8.1 is proved. □

Since by (8.6) the Dirichlet polynomials $P_\chi(s)$ are uniquely determined and by Theorem 1 the pair $(q/f_\chi,\omega^*\bar{\omega}_{\chi^*})$ is an invariant of $P_\chi(s)$, we have that

$$q \text{ and } \omega^* \text{ are invariants of } F\in\mathcal{S}_1^\sharp. \tag{8.17}$$

We shall need the following lemma about Dirichlet series.

LEMMA 8.3. *Let $D_j(s)$, $j=1,2$, be Dirichlet series with finite abscissa of convergence, and let s_k be a sequence of complex numbers with $\operatorname{Re} s_k\to\infty$, $\lambda>0$ and $c\in\mathbf{C}$. Assume that*

$$D_1(s_k)+\cot\Big(-\pi\frac{k}{2\lambda}+c\Big)D_2(s_k)=0 \tag{8.18}$$

for k sufficiently large. We have

(i) *if $\lambda=1/2m$ with some $m\in\mathbf{N}$, then*

$$D_1(s)+\cot(c)D_2(s)=0 \quad \textit{identically};$$

(ii) *if $\lambda\neq 1/2m$ for every $m\in\mathbf{N}$, then $D_1=D_2=0$.*

Proof. We shall repeatedly use the following well-known property of Dirichlet series: every non-zero Dirichlet series with finite abscissa of convergence has a zero-free right half-plane.

We first observe that we may assume that $s=-\pi k/2\lambda+c$ is not a pole of $\cot(s)$ for k sufficiently large. Otherwise, $D_2=0$ by the above-mentioned property and hence $D_1=0$ too by the same property, and therefore both (i) and (ii) would follow.

Assume that $\lambda=1/2m$ for some $m\in\mathbf{N}$. By the periodicity of $\cot(s)$, (8.18) becomes

$$D_1(s_k)+\cot(c)D_2(s_k)=0,$$

and (i) follows by the above property of Dirichlet series.

Assume now that $\lambda\neq 1/2m$ for every $m\in\mathbf{N}$, and write $k/2\lambda=[k/2\lambda]+\{k/2\lambda\}$. Hence $\cot(-\pi k/2\lambda+c)=\cot(-\pi\{k/2\lambda\}+c)$ and, by our assumption, $\{k/2\lambda\}$ is not constant.

Moreover, by a well-known elementary result in diophantine approximation, $\{k/2\lambda\}$ does not tend to a limit as $k\to\infty$. Therefore

$$\liminf_{k\to\infty}\cot\Big(-\pi\frac{k}{2\lambda}+c\Big)\neq\limsup_{k\to\infty}\cot\Big(-\pi\frac{k}{2\lambda}+c\Big). \tag{8.19}$$

For k sufficiently large we may expand the Dirichlet series in (8.18), thus getting, with obvious notation, that

$$\sum_{n=1}^{\infty}\frac{a_1(n)}{n^{s_k}}+\cot\Big(-\pi\frac{k}{2\lambda}+c\Big)\sum_{n=1}^{\infty}\frac{a_2(n)}{n^{s_k}}=0. \tag{8.20}$$

Let n_j, $j=1,2$, be the least integer n such that $a_j(n)\neq0$. If $n_1=n_2=n_0$, say, from (8.20) we obtain

$$a_1(n_0)+o(1)+\cot\Big(-\pi\frac{k}{2\lambda}+c\Big)(a_2(n_0)+o(1))=0$$

as $k\to\infty$, and hence from (8.19) we get $a_1(n_0)=a_2(n_0)$. This contradiction shows that $D_1=D_2=0$ in this case.

A similar argument shows that $D_1=D_2=0$ in the case $n_1\neq n_2$ too, and hence Lemma 8.3 is proved. □

The nature of (8.1) is further clarified by

Proposition 8.2. *Let $F\in\mathcal{S}_1^\sharp$. Then $\eta=\mathfrak{a}-1$ and* (8.1) *takes the form*

$$F(s)=\sum_{\chi\in\mathfrak{X}(q,\xi)}P_\chi(s+i\theta)L(s+i\theta,\chi^*).$$

Proof. With the notation in (8.7) write

$$B_j(s)=\sum_{\chi\in D_j(q)}P_\chi(s+i\theta)L(s+i\theta,\chi^*),\quad j=1,2.$$

From (8.1), the functional equation of the Dirichlet L-functions and Proposition 8.1 we get

$$F(s)=\omega^*\Big(\frac{q}{\pi}\Big)^{1/2-s-i\theta}\bigg\{\frac{\Gamma\big(\frac12(1-s-i\theta)\big)}{\Gamma\big(\frac12(s+i\theta)\big)}\bar{B}_1(1-s)+\frac{\Gamma\big(\frac12(2-s-i\theta)\big)}{\Gamma\big(\frac12(s+1+i\theta)\big)}\bar{B}_2(1-s)\bigg\}. \tag{8.21}$$

On the other hand, from (8.1) and the functional equation of $F(s)$ we have

$$F(s)=\omega Q^{1-2s}\prod_{j=1}^{r}\frac{\Gamma(\lambda_j(1-s)+\bar{\mu}_j)}{\Gamma(\lambda_js+\mu_j)}\{\bar{B}_1(1-s)+\bar{B}_2(1-s)\}. \tag{8.22}$$

Comparing (8.21) and (8.22) we get

$$\begin{aligned}\frac{\Gamma\big(\frac{1}{2}(1-s-i\theta)\big)}{\Gamma\big(\frac{1}{2}(s+i\theta)\big)}\,\overline{B}_1(1-s)+\frac{\Gamma\big(\frac{1}{2}(2-s-i\theta)\big)}{\Gamma\big(\frac{1}{2}(s+1+i\theta)\big)}\,\overline{B}_2(1-s)\\ =\omega\bar{\omega}^*\Big(\frac{q}{\pi}\Big)^{i\theta}\big(\tfrac{1}{2}\beta\big)^{1/2-s}\prod_{j=1}^{r}\frac{\Gamma(\lambda_j(1-s)+\bar{\mu}_j)}{\Gamma(\lambda_j s+\mu_j)}\,\{\overline{B}_1(1-s)+\overline{B}_2(1-s)\}.\end{aligned} \tag{8.23}$$

The right-hand side of (8.23) vanishes at $s_k=-(k+\mu_j)/\lambda_j$, $k=1,2,\ldots$ and $j=1,\ldots,r$. Since the zeros of $\Gamma\big(\frac{1}{2}(s+i\theta)\big)^{-1}$ and those of $\Gamma\big(\frac{1}{2}(s+1+i\theta)\big)^{-1}$ are disjoint, fixing a $j=j_0$ we have either

Case I. There exists an infinite subsequence of k's such that s_k is a zero of $\Gamma\big(\frac{1}{2}(s+i\theta)\big)^{-1}$ but not of $\Gamma\big(\frac{1}{2}(s+1+i\theta)\big)^{-1}$, or vice versa,

or

Case II. For k sufficiently large, s_k is not a zero of both $\Gamma\big(\frac{1}{2}(s+i\theta)\big)^{-1}$ and $\Gamma\big(\frac{1}{2}(s+1+i\theta)\big)^{-1}$.

Suppose that Case I holds and that s_k is a zero of $\Gamma\big(\frac{1}{2}(s+i\theta)\big)^{-1}$ but not of $\Gamma\big(\frac{1}{2}(s+1+i\theta)\big)^{-1}$. Since $\Gamma\big(\frac{1}{2}(2-s-i\theta)\big)$ is regular at s_k, from (8.23) we find that $\overline{B}_2(1-s_k)=0$ along a subsequence s_k with $\operatorname{Re} s_k\to\infty$. Therefore $B_2=0$ by the property of Dirichlet series stated at the beginning of Lemma 8.3. Clearly, $B_1=0$ in the opposite situation.

Suppose now that Case II holds. Hence from (8.23), for k sufficiently large we have

$$\begin{aligned}0&=\overline{B}_1(1-s_k)+\frac{\Gamma\big(1-\frac{1}{2}(s_k+i\theta)\big)\Gamma\big(\frac{1}{2}(s_k+i\theta)\big)}{\Gamma\big(\frac{1}{2}+\frac{1}{2}(s_k+i\theta)\big)\Gamma\big(\frac{1}{2}-\frac{1}{2}(s_k+i\theta)\big)}\,\overline{B}_2(1-s_k)\\ &=\overline{B}_1(1-s_k)+\cot\Big(-\pi\frac{k}{2\lambda}+c\Big)\,\overline{B}_2(1-s_k),\end{aligned} \tag{8.24}$$

where $\lambda=\lambda_{j_0}$ and $c=-\pi\mu_{j_0}/2\lambda_{j_0}+i\theta$.

If $\lambda=1/2m$ for some $m\in\mathbf{N}$, by (8.24) and Lemma 8.3 we get

$$\overline{B}_1(s)+\cot(c)\overline{B}_2(s)=0 \quad \text{identically},$$

and hence $B_1=B_2=0$ by Lemma 8.1, a contradiction. If $\lambda\neq 1/2m$ for every $m\in\mathbf{N}$, by (8.24) and Lemma 8.3 we have $B_1=B_2=0$, again a contradiction.

Therefore, either $B_1=0$ or $B_2=0$. Moreover, from (8.23) and Stirling's formula we get

$$\sum_{j=1}^{r}\big(\mu_j-\tfrac{1}{2}\big)=\tfrac{1}{2}(\mathfrak{a}-1+i\theta).$$

Hence $\eta=\mathfrak{a}-1$ and Proposition 8.2 follows. □

From Proposition 8.2 we have, in particular, that η is an invariant of $F(s)$, and $P_{\chi_0}(1)=0$ if $\theta\neq 0$. Hence from (8.6) and (8.7) we have that

$$\textit{the triple } (q,\xi,\omega^*) \textit{ is an invariant of } F\in\mathcal{S}_1^\sharp, \textit{ and } P_{\chi_0}(1)=0 \textit{ if } \theta\neq 0. \tag{8.25}$$

Parts (i) and (ii) of Theorem 2 follow now from Theorem 8.1, (8.6), Proposition 8.1, Proposition 8.2 and (8.25). We state part (iii) as

PROPOSITION 8.3. *For $q\in\mathbf{N}$, $\eta\in\{-1,0\}$, $\theta\in\mathbf{R}$ and $|\omega^*|=1$, $V_1^\sharp(q,\xi,\omega^*)$ is a vector space over $\mathbf{R}$. Its dimension is given by*

$$\dim_{\mathbf{R}} V_1^\sharp(q,\xi,\omega^*) = \begin{cases} \left[\frac{1}{2}q\right]+1 & \textit{if } \xi=-1, \\ \left[\frac{1}{2}(q-1-\eta)\right] & \textit{otherwise.} \end{cases}$$

Proof. The same argument as in the proof of Theorem 1 shows that $V_1^\sharp(q,\xi,\omega^*)$ is a vector space over $\mathbf{R}$.

Assume first that $\xi\neq -1+i\theta$ with $\theta\neq 0$. Writing

$$r(f,\xi)=|\{\chi\in\mathfrak{X}(q,\xi): f_\chi=f\}|,$$

by (iii) of Theorem 1 and Proposition 8.1 we have

$$\dim_{\mathbf{R}} V_1^\sharp(q,\xi,\omega^*) = \sum_{\chi\in\mathfrak{X}(q,\xi)} d\left(\frac{q}{f_\chi}\right) = \sum_{f|q} r(f,\xi)\, d\left(\frac{q}{f}\right) = \sum_{d|q}\sum_{f|d} r(f,\xi). \tag{8.26}$$

If $\xi=-1+i\theta$ with $\theta\neq 0$, we have to take into account the condition $P_{\chi_0}(1)=0$. This condition imposes a linear dependence on the coefficients of $P_{\chi_0}(s)$, and hence the dimension of the resulting vector space decreases by 1. Therefore, writing

$$\delta_\xi = \begin{cases} 1 & \text{if } \xi=-1+i\theta \text{ with } \theta\neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

from (8.26) we get

$$\dim_{\mathbf{R}} V_1^\sharp(q,\xi,\omega^*) = \sum_{d|q}\sum_{f|d} r(f,\xi) - \delta_\xi \tag{8.27}$$

in the general case.

Now we compute the right-hand side of (8.27). It is easy to see that

$$\sum_{f|d} r(f,\xi) = \begin{cases} \frac{1}{2}\varphi(d) & \text{if } d>2, \\ 1 & \text{if } d\leqslant 2,\ \eta=-1, \\ 0 & \text{if } d\leqslant 2,\ \eta=0, \end{cases}$$

and hence Proposition 8.3 follows by a simple analysis from the well-known relation $\sum_{d|q}\varphi(d)=q$. □

The proof of Theorem 2 is now complete.

Theorem 3 is a simple consequence of Theorem 2 and Theorem 5.1 of Conrey–Ghosh [3]. However, since we cannot follow the argument in the last paragraph of the proof of Theorem 5.1, see p. 688 of [3], we give here an essentially self-contained proof of Theorem 3, using instead Conrey–Ghosh's [3] result that $\mathcal{S}_0=\{1\}$. We remark that the argument below also provides the missing details in the proof of Theorem 5.1 of [3].

If $F\in\mathcal{S}_1$, then the coefficients $a(n)$ are multiplicative. Hence, by (i) of Theorem 2, $c(n)=a(n)n^{i\theta}$ is both multiplicative and periodic with period q. Therefore, there exists a Dirichlet character $\chi \pmod q$ such that

$$c(n)=\chi(n) \quad \text{for } (n,q)=1. \tag{8.28}$$

In fact, let $(mn,q)=1$ and, by the Chinese remainder theorem, let a be such that $(m+aq,n)=1$. Then

$$c(mn)=c((m+aq)n)=c(m+aq)c(n)=c(m)c(n),$$

and hence $c(n)$ is completely multiplicative on the n's with $(n,q)=1$, and (8.28) follows.

From (8.28), for $\sigma>1$ we have

$$F(s-i\theta)=\prod_{p|q}\left(\sum_{m=0}^{\infty}c(p^m)p^{-ms}\right)L(s,\chi). \tag{8.29}$$

For a fixed $p|q$, let $d=0,\dots,p-1$ and write $A_d=\{m\in\mathbf{N}: p^m\equiv d \pmod q\}$. Hence by the periodicity of $c(n)$ we get

$$\sum_{m=0}^{\infty}c(p^m)p^{-ms}=\sum_{d=0}^{p-1}c(d)\sum_{m\in A_d}p^{-ms}. \tag{8.30}$$

Writing $q=p^\alpha q'$ with $(p,q')=1$, we see that A_d is the set of the solutions m of the system

$$\begin{cases} p^m\equiv d \pmod{q'},\\ p^m\equiv d \pmod{p^\alpha}.\end{cases}$$

For a fixed d, the solutions of the first congruence, when they exist, are $m=a+k\nu_p$, where ν_p is the order of p modulo q', a is a certain integer and $k\in\mathbf{N}$. In order to treat the second congruence we write $d=p^\beta d'$ with $(p,d')=1$, and consider the two cases $\beta\geqslant\alpha$

and $\beta<\alpha$. In the first case the solutions are $m\geqslant\alpha$, while in the second case either there are no solutions or there is only the solution $m=\beta$.

Therefore, either A_d is a finite set or $A_d=\{m=a+k\nu_p$ with $k\in\mathbf{N}$ and $m\geqslant\alpha\}$; denote by k_0 the least such k. Accordingly, the Dirichlet series $\sum_{m\in A_d} p^{-ms}$ is either a Dirichlet polynomial or

$$\sum_{m\in A_d} p^{-ms}=\frac{p^{-(a+k_0\nu_p)s}}{1-p^{-\nu_p s}}.$$

We may therefore write

$$\sum_{d=0}^{p-1} c(d)\sum_{m\in A_d} p^{-ms}=\frac{Q_p(s)}{1-p^{-\nu_p s}}, \tag{8.31}$$

where $Q_p(s)$ is a Dirichlet polynomial. Hence from (8.29)–(8.31) we have

$$F(s)=P(s+i\theta)\prod_{p|q}(1-p^{-\nu_p(s+i\theta)})^{-1}L(s+i\theta,\chi^*), \tag{8.32}$$

where $P(s)$ is a Dirichlet polynomial. Comparing (8.32) with the expression for $F(s)$ in (ii) of Theorem 2 and multiplying both sides by $\prod_{p|q}(1-p^{-\nu_p(s+i\theta)})$ we obtain

$$\sum_{\chi\in\mathfrak{X}(q,\xi)}\prod_{p|q}(1-p^{-\nu_p(s+i\theta)})P_\chi(s+i\theta)L(s+i\theta,\chi^*)=P(s+i\theta)L(s+i\theta,\chi^*), \tag{8.33}$$

where the Dirichlet polynomials $P_\chi(s)$, $\chi\in\mathfrak{X}(q,\xi)$, belong to $\mathcal{S}_0^\sharp$.

From (8.33) and Lemma 8.1 we see that $\chi\in\mathfrak{X}(q,\xi)$,

$$P(s+i\theta)\prod_{p|q}(1-p^{-\nu_p(s+i\theta)})^{-1}=P_\chi(s+i\theta)$$

and $P_\chi\in\mathcal{S}_0^\sharp$. Hence (8.32) becomes

$$F(s)=P_\chi(s+i\theta)L(s+i\theta,\chi^*). \tag{8.34}$$

By (8.34), the coefficients of $P_\chi(s)$ are multiplicative, therefore $P_\chi\in\mathcal{S}_0$, and hence by Theorem 3.1 of [3] we see that $P_\chi=1$. Consequently, (8.34) becomes

$$F(s)=L(s+i\theta,\chi^*),$$

and since q is an invariant of $F(s)$ we have $q=f_\chi$. Hence $\chi=\chi^*$ and Theorem 3 is proved.

References

[1] Bochner, S., On Riemann's functional equation with multiple gamma factors. *Ann. of Math.*, 67 (1958), 29–41.

[2] Braaksma, B. L. J., Asymptotic expansions and analytic continuations for a class of Barnes-integrals. *Compositio Math.*, 15 (1964), 239-341.

[3] Conrey, J. B. & Ghosh, A., On the Selberg class of Dirichlet series: small degrees. *Duke Math. J.*, 72 (1993), 673–693.

[4] Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G., *Higher Transcendental Functions*, Vol. I. McGraw-Hill, New York–Toronto–London, 1953.

[5] Gérardin, P. & Li, W.-C. W., Functional equations and periodic sequences, in *Théorie des nombres* (Québec, PQ, 1987), pp. 267–279. de Gruyter, Berlin–New York, 1989.

[6] Kaczorowski, J. & Perelli, A., On the structure of the Selberg class, II: invariants and conjectures. To appear.

[7] Linnik, Yu. V., On the expression of L-series by the ζ-function. *Dokl. Akad. Nauk SSSR*, 57 (1947), 435–437 (Russian).

[8] Murty, M. R., Selberg's conjectures and Artin L-functions. *Bull. Amer. Math. Soc.*, 31 (1994), 1–14.

[9] Prudnikov, A. P., Bryčkov, Yu. A. & Maričev, O. I., *Integrals and Series: Elementary Functions.* Nauka, Moscow, 1981 (Russian); English translation, Gordon & Breach, New York, 1986.

[10] Selberg, A., Old and new conjectures and results about a class of Dirichlet series, in *Proceedings of the Amalfi Conference on Analytic Number Theory* (Maiori, 1989), pp. 367–385. Univ. of Salerno, Salerno, 1992; *Collected Papers*, Vol. II, pp. 47–63. Springer-Verlag, Berlin, 1991.

[11] Sprindžuk, V. G., On extended Riemann hypothesis, in *Topics in Number Theory* (Debrecen, 1974), pp. 369–372. Colloq. Math. Soc. Janos Bolyai, 13. North-Holland, Amsterdam, 1976.

[12] — The vertical distribution of zeros of the zeta-function and the extended Riemann hypothesis. *Acta Arith.*, 27 (1975), 317–332 (Russian).

[13] Titchmarsh, E. C., *The Theory of the Riemann Zeta-function*, 2nd edition. Edited and with a preface by D. R. Heath-Brown. Oxford Univ. Press, New York, 1986.

[14] Vignéras, M.-F., Facteurs gamma et équations fonctionnelles, in *Modular Functions of One Variable*, VI (Bonn, 1976), pp. 79–103. Lecture Notes in Math., 627. Springer-Verlag, Berlin, 1977.

[15] Whittaker, E. T. & Watson, G. N., *A Course of Modern Analysis*, 4th edition. Cambridge Univ. Press, London, 1927.

Jerzy Kaczorowski
Faculty of Mathematics
and Computer Science
A. Mickiewicz University
PL-60-769 Poznań
Poland
kjerzy@math.amu.edu.pl

Alberto Perelli
Department of Mathematics
University of Genova
Via Dodecaneso 35
IT-16146 Genova
Italy
perelli@dima.unige.it

Received December 3, 1997