Hypercontractivity over complex manifolds

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1. Introduction

For any probability measure μ on $\mathbf{R}^n$ with a smooth, strictly positive density, define the Dirichlet form operator for μ to be the nonnegative self-adjoint operator A_μ on $L^2(\mathbf{R}^n, \mu)$, with core $C_c^\infty(\mathbf{R}^n)$, satisfying

$$(A_\mu f, g)_{L^2(\mu)} = \int_{\mathbf{R}^n} (\operatorname{grad} f(x), \operatorname{grad} \overline{g(x)}) \, d\mu(x) \tag{1.1}$$

for f and g in $C_c^\infty(\mathbf{R}^n)$. For example, if $d\nu(x) = (2\pi)^{-n/2} e^{-|x|^2/2} \, dx$, then an integration by parts gives

$$A_\nu f(x) = -(\text{Laplacian}) f(x) + (x, \operatorname{grad} f(x)), \quad x \in \mathbf{R}^n,$$

for $f \in C_c^\infty(\mathbf{R}^n)$. Define

$$t_N = t_N(p, q) = \frac{1}{2} \log \frac{p-1}{q-1}, \quad 1 < q \leqslant p < \infty. \tag{1.2}$$

THEOREM 1.1 (E. Nelson [N2]). *Let* $1 < q \leqslant p < \infty$. *Then*

$$\|e^{-tA_\nu}\|_{L^q \to L^p} \leqslant 1 \quad \textit{if } t \geqslant t_N(p, q). \tag{1.3}$$

If $t < t_N$ *then* e^{-tA_ν} *is unbounded as an operator from* $L^q(\nu)$ *to* $L^p(\nu)$. (e^{-tA_ν} *should be extended from* L^2 *to* L^q *if* $1 < q < 2$ *or restricted to* L^q *if* $q > 2$ *for the proper interpretation of this theorem. This comment applies to all following variants of this theorem.*)

This theorem evolved through several stages [N1], [G1], [S], [N2] before reaching the definitive form above. Subsequently, the present author introduced the notion of logarithmic Sobolev inequality, [G1], and showed that families of inequalities such as

(1.3) are equivalent to a single inequality of the following form. We say that a probability measure μ on $\mathbf{R}^n$ satisfies a logarithmic Sobolev inequality if

$$\int_{\mathbf{R}^n} |f|^2 \log |f| \, d\mu \leqslant \int_{\mathbf{R}^n} |\operatorname{grad} f|^2 \, d\mu + \|f\|^2_{L^2(\mu)} \log \|f\|_{L^2(\mu)} \tag{1.4}$$

whenever the weak gradient of f is in $L^2(\mu)$.

THEOREM 1.2 [G1]. *Suppose that μ is a smooth probability measure on $\mathbf{R}^n$ with strictly positive density, and that A_μ is its Dirichlet form operator. Then*

$$\|e^{-tA_\mu}\|_{L^q \to L^p} \leqslant 1 \quad \text{for all } t \geqslant t_N(p,q) \text{ and } 1 < q \leqslant p < \infty \tag{1.5}$$

if and only if the logarithmic Sobolev inequality (1.4) holds.

There are many variants of this theorem that have been developed and applied in a wide variety of contexts. The review papers [B], [G5] survey the state of the art through September, 1992. At that time there were approximately 150 papers dealing with hypercontractivity of semigroups (typically inequalities like (1.5)) and logarithmic Sobolev inequalities (typically inequalities like (1.4)) either separately or in combination and either over manifolds (finite- or infinite-dimensional) or over discrete spaces. The range of applications has continued to expand rapidly since then. Theorem 1.2 and its proof "explains" the particular form of Nelson's shortest time to contractivity $t_N(p,q)$. By and large, this shortest time lurks in all of the applications, even if only by approximation, or as an intermediate step, such as in supercontractivity and ultracontractivity. However, there are four notable exceptions.

Let

$$t_J(p,q) = \frac{1}{2} \log \frac{p}{q}, \quad 0 < q \leqslant p < \infty. \tag{1.6}$$

S. Janson [J1], in a paper aimed at discussing multiplier operators T_ω for orthogonal functions $\{\varphi_n\}$, given by $T_\omega \sum_{n=1}^{\infty} a_n \varphi_n = \sum_{n=1}^{\infty} a_n \omega^n \varphi_n$, where ω is a complex number with $|\omega| \leqslant 1$, discussed an inequality which is in the spirit of (1.3), but operates in spaces of holomorphic functions. In Janson's inequality $t_N(p,q)$ is replaced by the smaller $t_J(p,q)$, while p and q are allowed to run even below one. E. Carlen [C] and Z. Zhou [Z] found two more distinct proofs of Janson's inequality, and in a recent paper [J2] Janson found a fourth distinct proof. The inequality can be phrased in terms of the Gaussian Dirichlet form operator A_ν defined above. Take $n = 2m$ and identify $\mathbf{R}^n$ with $\mathbf{C}^m$. Denote by $\mathcal{H}^p$ the space of holomorphic functions in $L^p(\mathbf{C}^m, \nu)$. We have

THEOREM 1.3 [J1], [J2], [C], [Z]. *Let $0 < q \leqslant p < \infty$. Then*

$$\|e^{-tA_\nu}\|_{\mathcal{H}^q \to \mathcal{H}^p} \leqslant 1 \quad \text{if } t \geqslant t_J(p,q). \tag{1.7}$$

If $t < t_J(p,q)$ then $\|e^{-tA_\nu}\|_{\mathcal{H}^q \to \mathcal{H}^p} = \infty$.

In other words, the restriction of the semigroup e^{-tA_ν} to the holomorphic subspace of L^p has greatly improved boundedness behavior, especially for $p \leqslant 1$. In fact, e^{-tA_ν} is not even definable on all of $L^p(\mathbf{C}^m, \nu)$ in a reasonable way if $p < 1$. (See Example 5.1.)

The four existing proofs of Theorem 1.3 are quite different from one another. Janson's original proof of Theorem 1.3 [J1] uses ordinary hypercontractivity (1.3), the spherical symmetry of the Mehler kernel of $e^{-tA_\nu}(x, dy)$ around $e^{-t}x$, and the fact that $|f(z)|^k$ is subharmonic in z for all $k > 0$ when f is holomorphic. The proof of Zhou [Z] is based on careful estimates of $L^4(\nu)$-norms of holomorphic polynomials $\sum a_k z^k$ and on deep results of Lieb, [Li]. Zhou considers only $p \geqslant 1$. The proof of Carlen [C] is based partly on use of the logarithmic Sobolev inequality (1.4) (for $\mu = \nu$), and partly on special integral identities for holomorphic functions on $\mathbf{C}^m$. Janson's second proof [J2] is based on use of a Brownian motion in $\mathbf{C}^m$ and is genuinely probabilistic.

One of the reasons that Theorem 1.3 is so startling (at least to this writer) is that the proof of the inequality (1.3), via its link to logarithmic Sobolev inequalities (i.e., via Theorem 1.2), is so simple and seemingly tight. Why should holomorphicity make such a difference? The answer lies in a "small" difference between the C^∞- and holomorphic categories in the application of the chain rule. A summary explanation is given in Remark 4.9. In short, a small modification of this author's proof of Theorem 1.2, [G1], yields yet another proof of (1.7).

The significance of this new proof of (1.7) lies not so much in the fact that there is now a fifth proof, but rather that the mechanism of proof does not depend on the linear structure of $\mathbf{C}^m$. It therefore frees one to explore the relation between hypercontractivity and logarithmic Sobolev inequalities in the holomorphic category over general complex manifolds. The resulting theory has some very interesting features not present in the $\mathbf{C}^m$-case and raises a large number of compelling questions about an apparently unexplored class of Dirichlet form operators. The present paper is devoted to the exploration of some of these questions.

Let M be a complex manifold with Hermitian metric g. Denote by h the dual Hermitian metric on the dual spaces $T^*(M) \otimes_{\mathbf{R}} \mathbf{C}$. Let μ be a probability measure on M with strictly positive smooth density (in each coordinate chart). For $f \in C^\infty(M)$ define $d^*df \in C^\infty(M)$ by the identity

$$(d^*df, \varphi)_{L^2(\mu)} = \int_M h(df, d\bar\varphi) \, d\mu, \quad \text{for all } \varphi \in C_c^\infty(M).$$

An integration by parts in a local coordinate chart, U, shows that if $f \in C^\infty(M)$ and f is holomorphic in U then d^*df can be expressed in terms of the first-order derivatives of f because the second-order derivatives are zero by the Cauchy–Riemann equations. Put

more invariantly, there exists a unique complex vector field Z of type $(1,0)$ such that for any function $f \in C^\infty(M)$ one has

$$d^*df = Zf \tag{1.8}$$

in any open set U in M on which f is holomorphic.

Definition. We will say that d^*d is *holomorphic* if for any function $f \in C^\infty(M)$, d^*df is holomorphic in any open set on which f is holomorphic. This is equivalent to the requirement that Z be a holomorphic vector field.

The main goal of this paper is to prove a version of the sufficiency portion of Theorem 1.2, in the holomorphic category, when d^*d is holomorphic. The function t_N will be replaced by t_J, and p and q will run over $(0, \infty)$ as in (1.7).

Unlike the Gaussian case on $\mathbf{C}^m$ the manifold M will be allowed to be incomplete. A self-adjoint version of d^*d must be chosen before a holomorphic version of Theorem 1.2 can be stated. In the present work I have focused on the case of Dirichlet boundary conditions. Let Q be the closed quadratic form in $L^2(M, \mu)$ with core $C_c^\infty(M)$ which is given by

$$Q(f, f) = \int_M h(df, d\bar{f}) \, d\mu, \quad f \in C_c^\infty(M). \tag{1.9}$$

Let A be the associated nonnegative self-adjoint operator in $L^2(M, \mu)$. Denote by $\mathcal{H}$ the space of holomorphic functions on M. Any complex vector field may be written uniquely as $Z = \frac{1}{2}(X - iY)$ where X and Y are real vector fields. Moreover if Z is of type $(1,0)$ then $Zf = Xf$ for any function f in $\mathcal{H}$. So if Z is holomorphic then $X^k f \in \mathcal{H}$, $k = 0, 1, 2, ...$, whenever $f \in \mathcal{H}$. Therefore if we denote by $\exp(-tX)$ the flow of diffeomorphisms of M induced by X (assuming that it exists for $t > 0$), the equation (1.8) then suggests that

$$e^{-tA} f = f \circ \exp(-tX), \quad f \in \mathcal{H} \cap L^2(M, \mu), \tag{1.10}$$

when d^*d is holomorphic. Equation (1.10) fails for interesting reasons. Denote by $\mathcal{H}^2$ the $L^2(\mu)$-closure of $\mathcal{H} \cap (\text{domain } Q)$. It will be shown (Theorem 2.11) that (1.10) holds for f in $\mathcal{H}^2$ when d^*d is holomorphic. More generally, $e^{-(t+is)A} f = f \circ \exp(-tX - sY)$ for $t \geqslant 0$ and s real, when $f \in \mathcal{H}^2$ and d^*d is holomorphic. Even though domain Q is dense in L^2, $\mathcal{H} \cap (\text{domain } Q)$ need not be dense in $\mathcal{H} \cap L^2$ because intersection does not commute with the operation of taking closure. So in general $\mathcal{H}^2$ may be a proper subspace of $\mathcal{H} \cap L^2$. The distinction between $\mathcal{H}^2$ and $\mathcal{H} \cap L^2$ relates in part to the completeness of M. Thus if M is complete and d^*d is holomorphic then $\mathcal{H}^2 = \mathcal{H} \cap L^2$ (Theorem 2.14). An example will be given in [G7] in which $\mathcal{H}^2 \neq \mathcal{H} \cap L^2$. In this example M will be taken to be the n-sheeted Riemann surface for $z^{1/n}$. $\mathcal{H}^2$ is then of codimension $n-1$ in $\mathcal{H} \cap L^2$. All hypotheses of interest in the present work, including logarithmic Sobolev inequalities,

hold in this example. Therefore, the circumstance $\mathcal{H}^2 \neq \mathcal{H} \cap L^2$ should not be regarded as pathology.

In the Gaussian case over $\mathbf{C}^m$, discussed in Theorem 1.3, the equation (1.10) reduces to $(e^{-tA}f)(z) = f(e^{-t}z)$ for $f \in \mathcal{H}^2$ and $z \in \mathbf{C}^m$. This is the identity which underlies all four of the works [J1], [J2], [C], [Z]. But in the present work it is the Y-flow that plays the key technical role. It happens that the Y-flow always preserves the measure μ. Moreover if d^*d is holomorphic and (M, g) is Kählerian then the Y-flow also preserves the metric. It results that in this case the unitary group, $V_s f = f \circ \exp(sY)$, on $L^2(M, \mu)$ preserves Q, preserves $\mathcal{H}$ and coincides on $\mathcal{H}^2$ (and only on $\mathcal{H}^2$) with the unitary group e^{isA}. The unitary group V_s consequently is able to serve as a tool for the global regularization of holomorphic functions. In nonholomorphic categories, inequalities for such second-order differential operators as A are typically proven first for some nice class of functions in $\mathcal{D}(A)$, for example for functions in $C_c^\infty(M)$, and then extended by closure considerations to $\mathcal{D}(A)$ or to all of L^2. But in the holomorphic category one cannot tamper easily with globally holomorphic functions, especially when there are no polynomials handy to approximate them by. In the present work the unitary group V_s will play a vital role in this regard, producing nice, globally holomorphic approximations to functions in $\mathcal{H}^2$, which will allow computations to be made that are valid only for especially nice holomorphic functions.

Define $\mathcal{H}^p$ to be the L^p-closure of $\mathcal{H}^2$ for $0 < p < 2$, and define $\mathcal{H}^p = \mathcal{H}^2 \cap L^p$ for $2 < p < \infty$. The inequality (1.7) implies that e^{-tA} is a contraction on $\mathcal{H}^p$ for all $p \in (0, \infty)$ in the Gaussian context of Theorem 1.3. As is well known, e^{-tA} is a contraction on the full L^p-space for $p \geqslant 1$ because A is a Dirichlet form operator. But in general e^{-tA} does not act in the full L^p-space for $0 < p < 1$. For example, e^{-tA} is typically given by a positive integral kernel, and in the case of Gauss measure it is easy to produce, for any $p \in (0, 1)$, a positive function f in L^p such that $e^{-tA}f$ is identically $+\infty$ (Example 5.1). But if d^*d is holomorphic then $e^{-tA}|\mathcal{H}^2$ extends to a contraction on $\mathcal{H}^p$ for any $p \in (0, 2)$ (Theorem 2.15). In fact, $e^{-(t+is)A}$ is a contraction in $\mathcal{H}^p$ for all $p \in (0, \infty)$, for all $t \geqslant 0$ and for all real s (Theorem 2.16). Even for $p > 1$ this is false in the full L^p-space for Gauss measure when $s \neq 0$. (See the discussion after Theorem 2.15.)

The preceding discussion of the operators $e^{-(t+is)A}$ and their action on each $\mathcal{H}^p$-space does not depend on the validity of logarithmic Sobolev inequalities but only on the fact that d^*d is holomorphic. Proofs of these structure theorems will be given in §3. When d^*d is holomorphic, and in addition a logarithmic Sobolev inequality holds, then one can say, first of all, that the space $\mathcal{H}^p$ is dense in $\mathcal{H}^q$ for $0 < q \leqslant p < \infty$ (Theorem 2.17). In fact, the union of the spectral subspaces for $A|\mathcal{H}^2$, corresponding to bounded intervals, turns out to be an algebra which is thereby naturally associated to the triple

(M, g, μ). This algebra is dense in $\mathcal{H}^p$ for $0<p<\infty$ (Theorem 2.17). In the absence of
a logarithmic Sobolev inequality all such density theorems fail. In fact, all $\mathcal{H}^p$-spaces
can be finite-dimensional with decreasing dimension as p increases (Example 5.1, finite-
dimensional case). Second, when d^*d is holomorphic and a defective logarithmic Sobolev
inequality holds (cf. (2.29)), the operators $e^{-(t+is)A} \colon \mathcal{H}^q \to \mathcal{H}^p$ satisfy bounds similar to
(1.7) (cf. (2.33)) for $t \geqslant t_J(p,q)$ and arbitrary real s (Theorem 2.19 and Corollary 2.20).
These are the main results of this work.

Virtually all of the theorems in the present work depend on the condition that d^*d
be holomorphic: $e^{-tA}\mathcal{H}^2$ need not even be contained in $\mathcal{H}$ otherwise. In particular, e^{-tA}
need not leave $\mathcal{H}^2$ invariant, which it does when d^*d is holomorphic (Theorem 2.11). It
is a strong restriction on the triple (M, g, μ) for d^*d to be holomorphic. How prevalent
are such "holomorphic" triples? §5 is devoted to examples and counterexamples. The
theory is clearly uninteresting if $\mathcal{H}^2$ is trivial, that is, consists only of constant functions.
In Example 5.7 it will be shown that if M and g are given there does not necessarily exist
a smooth probability measure μ on M such that d^*d is holomorphic—even if (M, g) is
complete and Kählerian. In Example 5.6 it will be shown that if M and μ are given, there
does not necessarily exist a Hermitian metric g on M such that d^*d is holomorphic—even
if $M = \mathbf{C}$ and μ is Gaussian. §5 is otherwise devoted to constructing (non-Gaussian) ex-
amples over $\mathbf{C}^m$ for which both key conditions of this paper hold: d^*d is holomorphic and
satisfies a logarithmic Sobolev inequality. An extension of E. Carlen's theorem on the
density of holomorphic polynomials in $\mathcal{H}^p$ is given. The hypothesis that d^*d be holomor-
phic seems fundamental in all questions concerning the strong form of hypercontractivity
embodied in the inequality (1.7). In [G7] an example will be given that provides further
evidence for the necessity of this condition.

A part of the motivation in [C] and [Z] for investigating the behavior of the semi-
group e^{-tA_ν} in $\mathcal{H} \cap L^2(\mathbf{C}^m, \nu)$ comes from the existence of a natural unitary transform
(the Segal–Bargmann transform) of $L^2(\mathbf{R}^m)$ onto $\mathcal{H} \cap L^2(\mathbf{C}^m, \nu)$, which intertwines the
harmonic oscillator Hamiltonian on $L^2(\mathbf{R}^m)$ with $A_\nu | \mathcal{H} \cap L^2(\mathbf{C}^m, \nu)$. An analog of this
transform, that applies to functions over a compact Lie group, K, instead of functions
over $\mathbf{R}^n$, has been found by B. Hall, [Ha1]. The transform maps functions on K to
holomorphic functions over the complexification of K. Recent work on this transform
may be found in [BSZ], [Dr], [DG], [G2], [G3], [G4], [G6], [GM], [Ha1], [Ha2], [Ha3],
[Hij1], [Hij2], [OØ]. The present work is motivated, in part, by the existence of such
natural unitary transforms from full L^2-function spaces to holomorphic function spaces
over some complex manifolds.

It is a pleasure to acknowledge useful discussions with B. Driver, C. Earle, M. Gor-
dina, M. Gross, B. Hall, W. Lewkeeratiyutkul and J. Mitchell.

2. Statement of main results

Notation 2.1. M will denote a complex manifold of complex dimension m. Let g be a Hermitian metric on M, and let μ denote a probability measure on M. It will be assumed throughout that μ has a strictly positive C^∞-density with respect to the Riemann–Lebesgue measure induced by g.

Denote by h the dual Hermitian metric on the complexified dual spaces $T^*_{\mathbf{C}}(M)$. As usual $h(u, v)$ is complex bilinear and symmetric in u and v for u and v in $T^*_x \otimes \mathbf{C}$, and if $0 \neq u$ is of type $(1, 0)$ then $h(u, \bar{u}) > 0$. $C^\infty(M)$ and $C^\infty_c(M)$ will denote complex-valued functions unless reality is specified. If V is a C^∞-section of $T^*_{\mathbf{C}}(M)$ then the divergence d^*V is the function on M determined by

$$\int_M (d^*V)(z)\overline{\varphi(z)}\, d\mu(z) = \int_M h(V(z), d\overline{\varphi(z)})\, d\mu(z) \quad \text{for all } \varphi \in C^\infty_c(M). \qquad (2.1)$$

If f is in $C^\infty(M)$ then df is a smooth section of $T^*_{\mathbf{C}}(M)$, and $d^*df \colon M \to \mathbf{C}$ is therefore well defined. We have

$$\int_M (d^*df)(z)\overline{\varphi(z)}\, d\mu(z) = \int_M h(df(z), d\bar{\varphi}(z))\, d\mu \quad \text{for all } \varphi \in C^\infty_c(M). \qquad (2.2)$$

The operator $d^*d \colon C^\infty(M) \to C^\infty(M)$ is called the *Dirichlet form operator* associated to the *Dirichlet form*

$$Q_0(f, \varphi) = \int_M h(df(z), d\overline{\varphi(z)})\, d\mu(z), \quad f \in C^\infty(M), \varphi \in C^\infty_c(M). \qquad (2.3)$$

Later we will choose a version of d^*d which is a positive self-adjoint operator in L^2.

Definition 2.2. The Dirichlet form Q_0 is *holomorphic* if for any function f in $C^\infty(M)$, d^*df is holomorphic in any open set U on which f is holomorphic.

When Q_0 is holomorphic we will also say that d^*d is holomorphic and also say that the triple (M, g, μ) is holomorphic.

$L^2(\mu)$-versions of Q_0 and d^*d will be the principal objects of interest to us. We say that a locally integrable function $f \colon M \to \mathbf{C}$ has locally square-integrable weak first derivatives if its restriction to each coordinate patch U is a weakly differentiable function of the coordinates $x_1, ..., x_{2m}$ whose first derivatives are in $L^2_{\mathrm{loc}}(U)$. That is,

$$\int_K \sum_{j=1}^{2m} \left| \frac{\partial f}{\partial x_j} \right|^2 dx_1 \ldots dx_{2m} < \infty$$

for each compact subset K of U. This class is clearly independent of the choice of local coordinate systems. For such f, $df(z)$ is a well-defined element of $T^*_{z,\mathbf{C}}(M)$ for almost all $z \in M$, and we may define

$$Q(f) = \int_M h(df(z), d\overline{f(z)}) \, d\mu(z). \tag{2.4}$$

We restrict the domain of Q to be the form closure of $C_c^\infty(M)$. Thus

Definition 2.3.

$$\begin{aligned} \mathcal{D}(Q) = \{f \in L^2(M, \mu) \text{ with locally square-integrable first} \\ \text{derivatives such that } Q(f) < \infty \text{ and for which} \\ \exists f_n \in C_c^\infty(M) \text{ with } \|f_n - f\|_{L^2}^2 + Q(f_n - f) \to 0\}. \end{aligned} \tag{2.5}$$

With this domain Q is a closed quadratic form and is densely defined in $L^2(M, \mu)$. By the standard theory of Dirichlet forms [Da], [F], [MR], [RS] there exists a unique nonnegative self-adjoint operator A on $L^2(M, \mu)$ such that $\mathcal{D}(A^{1/2}) = \mathcal{D}(Q)$ and $\|A^{1/2}f\|_{L^2}^2 = Q(f)$. In particular,

$$(Af, g)_{L^2} = \int_M h(df, d\bar{g}) \, d\mu \quad \text{for } f \in \mathcal{D}(A) \text{ and } g \in \mathcal{D}(Q). \tag{2.6}$$

Furthermore the semigroup e^{-tA} preserves nonnegativity of functions and

$$\|e^{-tA}f\|_p \leqslant \|f\|_p, \quad 1 \leqslant p \leqslant \infty, \, f \in L^2(M, \mu). \tag{2.7}$$

Thus the operators e^{-tA} restrict to contractions on L^p for $2 \leqslant p \leqslant \infty$, and extend to contractions on L^p for $1 \leqslant p \leqslant 2$. The resulting semigroups on L^p are strongly continuous for $1 \leqslant p < \infty$. A_p will denote the infinitesimal generator in L^p.

Notation 2.4. Write $\mathcal{H} = \mathcal{H}(M)$ for the space of holomorphic functions on M. For $0 < p < 1$, $\int_M |f(x)|^p \, d\mu(x)$ is a metric on $L^p(M, \mu)$. Convergence in this space will refer to convergence in this metric [Ru, Section 1.47]. Define

$$\mathcal{H}^2 = L^2\text{-closure of } \mathcal{H} \cap \mathcal{D}(Q), \tag{2.8}$$

$$\mathcal{H}^p = \mathcal{H}^2 \cap L^p \quad \text{for } 2 < p < \infty \tag{2.9}$$

and

$$\mathcal{H}^p = \text{closure of } \mathcal{H}^2 \text{ in } L^p \quad \text{for } 0 < p < 2. \tag{2.10}$$

Remark 2.5. Since closure does not commute with intersection it can happen that $\mathcal{H}^2 \neq \mathcal{H} \cap L^2$. In [G7] an example will be given, the n-sheeted Riemann surface for $z^{1/n}$, in

which $\mathcal{H}^2$ detects the highly singular point at the origin. One finds in this case a proper containment: $\mathcal{H}^2 \subsetneq \mathcal{H} \cap L^2$. In this example d^*d is holomorphic, but M is not complete. On the other hand, if M is complete and d^*d is holomorphic one always has the equality $\mathcal{H}^2 = \mathcal{H} \cap L^2$, as will be shown in Theorem 2.14. In general, our theorems will hold in the spaces $\mathcal{H}^p$ rather than in the spaces $\mathcal{H} \cap L^p$. If $\mathcal{H}^2 \neq \mathcal{H} \cap L^2$ then $\mathcal{H} \cap \mathcal{D}(A)$ is not even dense in $\mathcal{H} \cap L^2$!

Let us recall that a complex vector field Z on M of type $(1,0)$ is *holomorphic* if in any local holomorphic coordinate system $z_1, ..., z_m$, it is given by

$$Z = \sum_{r=1}^m \varphi_r(z) \frac{\partial}{\partial z_r} \tag{2.11}$$

with $\varphi_1, ..., \varphi_m$ holomorphic.

THEOREM 2.6. *Let (M, g, μ) be a complex manifold with Hermitian metric and smooth probability measure as above. There exists a unique vector field Z of type $(1,0)$ such that for any function $f \in C^\infty(M)$,*

$$d^*df = Zf \tag{2.12}$$

in any open set U in which f is holomorphic. Denote by ω the $(1,1)$-form associated to the Hermitian metric g and let ν be the density of μ with respect to the Riemannian volume measure. Then Z is given by

$$Zf = h(\partial f, -\bar{\partial} \log \nu + ih \cdot \bar{\partial}\omega), \quad f \in C^\infty(M), \tag{2.13}$$

where $h \cdot \bar{\partial}\omega$ is the form of type $(0,1)$ defined by

$$(h \cdot \bar{\partial}\omega)(\alpha) = \sum_s (\bar{\partial}\omega)(e_s, \bar{e}_s, \alpha), \quad \alpha \in T_z^{0,1}, \tag{2.14}$$

and $e_1, ..., e_m$ is an orthonormal basis of $T_z^{1,0}$. (See Lemma 3.1 for the pairing convention used in (2.14).) If g is Kähler then

$$Zf = h(\partial f, -\bar{\partial} \log \nu), \quad f \in C^\infty(M). \tag{2.15}$$

Furthermore d^*d is holomorphic if and only if Z is holomorphic.

Notation 2.7. Any complex vector field Z can be written uniquely in the form

$$Z = \tfrac{1}{2}(X - iY) \tag{2.16}$$

where X and Y are real vector fields. One has $X=Z+\bar{Z}$ and $Y=i(Z-\bar{Z})$. Throughout this work Z will denote the complex vector field on M determined by g and μ as in (2.12). X and Y will denote the real vector fields defined by (2.16).

Z is of type $(1,0)$ in our case. If J denotes the almost complex structure associated to the given complex structure of M then $JZ=iZ$ while $J\bar{Z}=-i\bar{Z}$. Hence

$$JX = Y. \tag{2.17}$$

In particular, X and Y are mutually orthogonal real vector fields on M.

COROLLARY 2.8. $\bar{Z}$, X and Y are given by

$$\bar{Z}f = h(\bar{\partial}f, -\partial\log\nu + i(h\cdot\partial\omega)), \quad f\in C^\infty(M), \tag{2.18}$$

$$Xf = h(df, -d\log\nu + i(h\cdot d\omega)), \tag{2.19}$$

$$Yf = ih(df, -(\bar{\partial}-\partial)\log\nu + ih\cdot(\bar{\partial}-\partial)\omega), \tag{2.20}$$

where $h\cdot\partial\omega$ is defined as in (2.14) but with $\alpha\in T_z^{1,0}$, and $h\cdot d\omega = h\cdot\bar{\partial}\omega + h\cdot\partial\omega$.

In particular, if (M,g) is Kähler then

$$\bar{Z}f = h(\bar{\partial}f, -\partial\log\nu), \tag{2.21}$$

$$Xf = h(df, -d\log\nu) \tag{2.22}$$

and

$$Yf = ih(df, -(\bar{\partial}-\partial)\log\nu). \tag{2.23}$$

The flow determined by a vector field Y is the one-parameter group $\exp tY\colon M\to M$ of diffeomorphisms of M (if they exist) such that

$$\frac{df_t}{dt} = Yf_t, \quad f\in C^\infty(M), \tag{2.24}$$

where $f_t = f\circ\exp(tY)$. If such a diffeomorphism group exists (i.e., if the flow determined by (2.24) exists globally), one says that the vector field Y is complete [KN1].

Standing assumption. The vector field Y defined by (2.16) will always be assumed to be complete.

THEOREM 2.9. *For any Dirichlet form operator d^*d the flow of Y preserves the measure μ.*

THEOREM 2.10. *Assume that d^*d is holomorphic and that the metric g is Kähler. Then the Y-flow preserves the metric. That is, Y is a Killing vector field.*

Remark. In all of the examples that I have, Kähler or not Kähler, the vector field Y is Killing when d^*d is holomorphic. See Example 5.1. I conjecture that Theorem 2.10 holds without the assumption that g is Kähler.

THEOREM 2.11. *Assume that d^*d is holomorphic and that Y is Killing. Then:*

(a) $\mathcal{H} \cap \mathcal{D}(A) \subset \mathcal{H}^2$ *and is dense in* $\mathcal{H}^2$. *Moreover* $Af \in \mathcal{H}^2$ *if* $f \in \mathcal{H} \cap \mathcal{D}(A)$.

(b) $Af = Zf$ *for* $f \in \mathcal{H} \cap \mathcal{D}(A)$.

(c) $e^{-\zeta A} \mathcal{H}^2 \subset \mathcal{H}^2$ *for* $\operatorname{Re} \zeta \geqslant 0$.

(d) $e^{itA} f = f \circ \exp tY$ *for* $f \in \mathcal{H}^2$ *and for all real* t.

(e) *If* $f \in \mathcal{H}^2$ *then* $f \in \mathcal{D}(A)$ *if and only if* $Zf \in L^2$.

It is interesting that the condition "$f \in \mathcal{H}^2$" in part (e) captures the Dirichlet boundary condition in the holomorphic category. For example, if $f \in \mathcal{H} \cap L^2$ and $Zf \in L^2$ then f need not be in $\mathcal{D}(A)$. An example is given in [G7].

The previous theorem relates the unitary group $e^{isA} | \mathcal{H}^2$ to the Y-flow by part (d). The next corollary relates the semigroup $e^{-tA} | \mathcal{H}^2$ to the (typically one-sided) flow of X. A relevant example of a one-sided flow (in the unit disc) is given in [G7]. There the flow $\exp(-tX)$ exists for $t \geqslant 0$ but not for $t < 0$.

COROLLARY 2.12. *In addition to the hypotheses of Theorem* 2.11 *assume that the flow* $\exp(-tX)$ *exists globally for* $t \geqslant 0$. *Then for all* $z \in M$,

$$
\begin{aligned}
(e^{-(t+is)A}) f(z) &= f(\exp(-tX)\exp(-sY)z) \\
&= f(\exp(-sY)\exp(-tX)z) \quad \text{for } f \in \mathcal{H}^2, t \geqslant 0, s \in \mathbf{R}.
\end{aligned}
\tag{2.25}
$$

If moreover the diffeomorphism semigroup $r \to \exp(-r(aX+bY))$ *exists globally for all* $r \geqslant 0$, *all* $a \geqslant 0$ *and all* $b \in \mathbf{R}$ *then*

$$
e^{-(t+is)A} f = f \circ \exp(-tX - sY) \quad \text{for } f \in \mathcal{H}^2, t \geqslant 0, s \in \mathbf{R}.
\tag{2.26}
$$

Remark 2.13. When the vector field Z is holomorphic the real vector fields X and Y (cf. equation (2.16)) commute. This follows from a straightforward computation in local coordinates. The well-known example of Nelson [RS, p. 273] shows that the flows of commuting vector fields need not commute. However, equation (2.25) shows that if X and Y arise from a holomorphic Dirichlet form then the diffeomorphism semigroup $\exp(-tX)$ and the diffeomorphism group $\exp(-sY)$ do indeed commute, provided that $\mathcal{H}^2$ is ample enough to separate points of M. It is not automatic, however, that $\mathcal{H}^2$ will separate points. Example 5.1 (finite-dimensional case) shows that $\mathcal{H}^2$ could consist of constants even when d^*d is holomorphic.

THEOREM 2.14. *Assume that d^*d is holomorphic and that Y is Killing. If (M,g) is complete then*

$$\mathcal{H}^p = \mathcal{H} \cap L^p, \quad 2 \leqslant p < \infty. \tag{2.27}$$

The proofs of the preceeding theorems will be given in §3. These theorems deal almost entirely with the holomorphic L^2-theory. The main results of this paper are the following contractivity and hypercontractivity theorems for the operators $e^{-\zeta A}$ in the spaces $\mathcal{H}^p$, $0<p<\infty$, for $\mathrm{Re}\,\zeta \geqslant 0$. The operators $e^{-\zeta A}$ will be defined in $\mathcal{H}^p$ by restriction from $\mathcal{H}^2$ when $p \geqslant 2$, and by extension by continuity when $0<p<2$. There is a resulting dichotomy in the techniques for the intervals $p>2$ or $p<2$. Even when $\zeta = t > 0$ is real, e^{-tA} does not in general act in L^p when $0<p<1$. Restriction to the holomorphic category is essential. Example 5.1 (Gaussian case) will illustrate this. The following theorems will be proved in §4. The next theorem contains the basis for the extension of $e^{-\zeta A}$ to $\mathcal{H}^p$ for $0<p<2$.

THEOREM 2.15. *Assume that d^*d is holomorphic and that Y is Killing. Then*

$$\|e^{-(t+is)A}f\|_p \leqslant \|f\|_p \quad \text{for } 0 < p \leqslant 2, \ t \geqslant 0, \ s \text{ real}, \ f \in \mathcal{H}^2. \tag{2.28}$$

We wish now to study the operators $e^{-\zeta A}$ in the spaces $\mathcal{H}^p$ when $\mathrm{Re}\,\zeta \geqslant 0$. Of course, $e^{-\zeta A}$ is a bounded operator on L^2, and by Theorem 2.11 it leaves $\mathcal{H}^2$ invariant. Theorem 2.15 shows that it can be extended to $\mathcal{H}^p$ for $0<p<2$. It is well known, as already mentioned at the beginning of this section, that e^{-tA} is a contraction in the full L^p-space for $1 \leqslant p \leqslant \infty$ and $t \geqslant 0$. The next theorem shows that $e^{-(t+is)A}$ leaves the subspaces $\mathcal{H}^p$ invariant for all $p \in (0, \infty)$ and acts as a contraction in these spaces for $t \geqslant 0$ and arbitrary real s. This behavior contrasts with the action of $e^{-(t+is)A}$ in the full L^p-space. In the classical example ($M = \mathbf{C}$, $\mu = $ Gauss measure, $A = $ Ornstein–Uhlenbeck operator), it is known [E], [J1], [J2], [Li], [We] that if $s \neq 0$ then $e^{-(t+is)A}$ is an unbounded operator on L^p for sufficiently large p.

THEOREM 2.16. *Suppose that d^*d is holomorphic and that Y is Killing. Let $\mathrm{Re}\,\zeta \geqslant 0$. Then $e^{-\zeta A}$ has a unique continuous extension to an operator on $\mathcal{H}^p$ for $0<p<2$. For $2 \leqslant p < \infty$ the restriction of $e^{-\zeta A}$ to $\mathcal{H}^p$ takes $\mathcal{H}^p$ into itself. For all p in $(0, \infty)$, $e^{-\zeta A}$ is a contraction on $\mathcal{H}^p$ and $e^{-\zeta A}\mathcal{H}^p$ is dense in $\mathcal{H}^p$. Furthermore, for $0<p<\infty$, the map $\mathbf{R} \ni s \to e^{isA}$ is a strongly continuous one-parameter group of isometries on $\mathcal{H}^p$. In particular, when $1 \leqslant p < \infty$, $\mathcal{H}^p \cap \mathcal{D}(A_p)$ is dense in $\mathcal{H}^p$, where A_p is the infinitesimal generator of e^{-tA} in L^p. Finally, the equations (2.25) and (2.26) hold for all f in $\mathcal{H}^p$, $0<p<\infty$, for the extended (or restricted) operators $e^{-\zeta A}$, under the same hypotheses as in Corollary 2.12.*

Next, assume that there exist constants $c>0$ and $\beta\geqslant 0$ such that the following logarithmic Sobolev inequality holds:

$$\int_M |f(z)|^2 \log|f(z)|\,d\mu(z) \leqslant cQ(f)+\beta\|f\|_{L^2}^2+\|f\|_{L^2}^2\log\|f\|_{L^2}, \quad f\in\mathcal{D}(Q). \tag{2.29}$$

THEOREM 2.17. *Assume that d^*d is holomorphic, that Y is Killing, and that the logarithmic Sobolev inequality* (2.29) *holds. Let H_a be the spectral subspace for the interval $[0,a]$ for the self-adjoint operator $A|\mathcal{H}^2$. Let $\mathcal{R}=\bigcup_{a<\infty}H_a$. Then, for $0<q<\infty$, $\mathcal{R}\subset\mathcal{H}^q$ and $\mathcal{R}$ is dense in $\mathcal{H}^q$. In particular, $\bigcap_{p<\infty}\mathcal{H}^p$ is dense in $\mathcal{H}^q$, and each $\mathcal{H}^p$ is also dense in $\mathcal{H}^q$ for $0<q\leqslant p<\infty$. Assume further that $\mathcal{H}^2=\mathcal{H}\cap L^2$. Then $\mathcal{R}$ is an algebra under pointwise multiplication and is dense in $\mathcal{H}\cap L^p$ for $2\leqslant p<\infty$. We have also*

$$H_a\cdot H_b\subset H_{a+b}. \tag{2.30}$$

Let

$$\mathcal{H}_\infty = \bigcap_{2\leqslant p<\infty}\bigcap_{n=1}^\infty \mathcal{D}((A_p|\mathcal{H}^p)^n).$$

Then $\mathcal{H}_\infty$ is also an algebra under pointwise multiplication and is dense in $\mathcal{H}\cap L^p$ for all p in $[2,\infty)$.

Remark 2.18. If the hypothesis that the logarithmic Sobolev inequality (2.29) holds is omitted in Theorem 2.17 then all of the conclusions can fail. An example of this will be given in §5 (cf. Example 5.1, finite-dimensional case). In that example the spaces $\mathcal{H}^p$ are finite-dimensional, with occasional jumps in dimension as p decreases. In [G7] an example will be given in which $\mathcal{R}$ is an algebra even though $\mathcal{H}^2\neq\mathcal{H}\cap L^2$. I don't know whether the hypothesis that $\mathcal{H}^2=\mathcal{H}\cap L^2$ is really required to obtain (2.30).

The main theorem of this paper is the following.

THEOREM 2.19. *Assume that d^*d is a holomorphic Dirichlet form operator, that Y is Killing, and that the logarithmic Sobolev inequality* (2.29) *holds. Let $0<q\leqslant p<\infty$. Define*

$$t_J = t_J(p,q) = \frac{1}{2}c\log\frac{p}{q} \tag{2.31}$$

and

$$M(p,q) = \exp[2\beta(q^{-1}-p^{-1})]. \tag{2.32}$$

Then

$$\|e^{-tA}f\|_p \leqslant M(p,q)\|f\|_q \quad \text{if } t\geqslant t_J \text{ and } f\in\mathcal{H}^q. \tag{2.33}$$

COROLLARY 2.20. *Suppose that $0<q\leqslant p<\infty$. Under the hypothesis of Theorem 2.19 one has*

$$\|e^{-\zeta A}\|_{\mathcal{H}^q\to\mathcal{H}^p} \leqslant M(p,q) \quad \text{if } |e^{-\zeta}| \leqslant (q/p)^{c/2}. \tag{2.34}$$

3. Proofs of structure theorems for holomorphic Dirichlet form operators

The results of the preceding section through Theorem 2.15 are concerned not with hyper-contractivity, but rather with the structure of the Hilbert space $\mathcal{H}^2$ of "properly holomorphic" square-integrable functions on M, and with the action of $e^{-\zeta A}$ in this subspace for $\operatorname{Re}\zeta \geqslant 0$. The proofs of these results will be given in this section.

Proof of Theorem 2.6. Choose a local holomorphic coordinate system $z_1, ..., z_m$ in a coordinate neighborhood V. Let $g_{ij}(z) = g_z(\partial/\partial z_i, \partial/\partial \bar{z}_j)$ and let $h_{ij}(z) = h(dz_i, d\bar{z}_j)$. Then h_{ij} is the transpose matrix of the inverse of $\{g_{ij}(z)\}_{i,j=1}^m$. Write $z_j = x_j + iy_j$, $j = 1, ..., m$, and $dx = dx_1\, dy_1 \ldots dx_m\, dy_m$. Let $\hat{\mu}(z)$ be the density of the measure μ with respect to dx in V. If $f \in C^\infty(M)$ and $\varphi \in C_c^\infty(M)$ with $\operatorname{supp}\varphi \subset V$ then by (2.2),

$$\int_M (d^*df)(z)\overline{\varphi(z)}\, d\mu = \int_V h(df(z), d\bar{\varphi}(z))\, \hat{\mu}(z)\, dx. \tag{3.1}$$

Suppose that f is holomorphic on a set U which intersects V, and that $\operatorname{supp}\varphi \subset U$. Then (3.1) reduces to

$$\int_M (d^*df)(z)\overline{\varphi(z)}\, d\mu = \int_V h(\partial f(z), \bar{\partial}\bar{\varphi}(z))\, \hat{\mu}(z)\, dx$$
$$= \int_V \sum_{r,s=1}^m \frac{\partial f}{\partial z_r} \cdot \frac{\partial \bar{\varphi}}{\partial \bar{z}_s} h(dz_r, d\bar{z}_s)\, \hat{\mu}(z)\, dx.$$

Writing $\partial_r = \partial/\partial z_r$ and $\partial_{\bar{r}} = \partial/\partial\bar{z}_r$ we have, by an integration by parts,

$$\int_M (d^*df)(z)\overline{\varphi(z)}\, d\mu = -\int_V \left[\hat{\mu}^{-1}\sum_{r,s}\partial_{\bar{s}}\{(\partial_r f)h_{rs}\hat{\mu}\}\right]\bar{\varphi}(z)\hat{\mu}\, dx.$$

Since f is holomorphic on $\operatorname{supp}\varphi$ we have $\partial_{\bar{s}}\partial_r f = 0$ on $\operatorname{supp}\varphi$. Hence

$$\int_M (d^*df)(z)\overline{\varphi(z)}\, d\mu = -\int_V \sum_r \left[\hat{\mu}^{-1}\sum_s \partial_{\bar{s}}\{h_{rs}\hat{\mu}\}\right](\partial_r f)\overline{\varphi(z)}\hat{\mu}\, dx.$$

Since $\varphi|U \cap V$ is arbitrary in $C_c^\infty(U \cap V)$ it follows that

$$(d^*df)(z) = -\sum_r \left[\hat{\mu}^{-1}\sum_s \partial_{\bar{s}}\{h_{rs}\hat{\mu}\}\right]\partial_r f(z), \quad z \in U \cap V, \tag{3.2}$$

when $f \in C^\infty(M)$ and is holomorphic on U. We will transform (3.2) to a coordinate-independent form. But first observe that the coefficients of the functions $\partial_r f(z)$ are already uniquely determined by (3.2) because one can take f to be a function in $C^\infty(M)$ which is equal to z_r in a neighborhood of a point $\hat{z} \in V$. The right-hand side of (3.2)

reduces then at $\hat{z}$ to the coefficient of $\partial_r f(\hat{z})$. The coordinate-dependent form of the desired vector field Z is, by (3.2),

$$Z = -\sum_{r=1}^{m} \hat{\mu}(z)^{-1} \sum_{s=1}^{m} \partial_{\bar{s}} \{h_{rs} \hat{\mu}(z)\} \frac{\partial}{\partial z_r}. \tag{3.3}$$

Now let $G(z) = 2^m \det\{g_{ij}(z)\}$. Then the Riemannian volume element is $d\,\mathrm{Vol} = G\,dx$. Since ν is the density of μ with respect to $d\,\mathrm{Vol}$ we have $\hat{\mu}(z)\,dx = \nu\,d\,\mathrm{Vol} = \nu G\,dx$. So $\hat{\mu} = \nu G$. Equation (3.2) gives

$$\begin{aligned}
(d^* df)(z) &= -\sum_r \left[(\nu G)^{-1} \sum_s \partial_{\bar{s}} \{h_{rs} G \nu\} \right] \partial_r f(z) \\
&= -\sum_r \left[\sum_s h_{rs} \nu^{-1} \partial_{\bar{s}} \nu + G^{-1} \sum_s \partial_{\bar{s}} \{h_{rs} G\} \right] \partial_r f(z) \\
&= -h(\partial f, \bar{\partial} \log \nu) - \sum_r \left[G^{-1} \sum_s \partial_{\bar{s}} \{h_{rs} G\} \right] \partial_r f(z),
\end{aligned} \tag{3.4}$$

which is valid for $f \in C^\infty(M)$ when f is holomorphic in a neighborhood of z, and z is in the coordinate patch U.

We need to transform the last term in (3.4) to a coordinate-independent form. The computation is a variant of well-known manipulations. It will be carried out in the following lemma.

LEMMA 3.1.

$$G^{-1} \sum_{s=1}^{m} \partial_{\bar{s}} \{h_{rs} G\} = -i \sum_k h_{rk} (h \cdot \bar{\partial} \omega)(\partial_{\bar{k}}). \tag{3.5}$$

Proof. Write $g(z)$ for the matrix $\{g_{ij}(z)\}_{i,j=1}^m$. The matrix $h(z) = \{h_{ij}(z)\}_{i,j=1}^m$ is given by $h = (g^{-1})^t$ (transpose of g^{-1}). Therefore

$$\begin{aligned}
\sum_s \partial_{\bar{s}} h_{rs} &= \sum_s (\partial_{\bar{s}} g^{-1})_{sr} \\
&= -\sum_s (g^{-1} (\partial_{\bar{s}} g) g^{-1})_{sr} \\
&= -\sum_{s,p,k} (g^{-1})_{sp} (\partial_{\bar{s}} g_{pk})(g^{-1})_{kr} \\
&= -\sum_{s,p,k} (g^{-1})_{sp} \{\partial_{\bar{k}} g_{p,s} + (\partial_{\bar{s}} g_{pk} - \partial_{\bar{k}} g_{ps})\} h_{rk} \\
&= -\sum_k h_{rk} \operatorname{trace}(g^{-1} \partial_{\bar{k}} g) - \sum_k h_{rk} \sum_{p,s} h_{ps} \{\partial_{\bar{s}} g_{pk} - \partial_{\bar{k}} g_{ps}\}.
\end{aligned} \tag{3.6}$$

Applying the product rule to the left-hand side of (3.5) and using the identity $G^{-1}\partial_{\bar{s}}G = \text{trace}(g^{-1}\partial_{\bar{s}}g)$ we see that the terms $\sum_{s}h_{rs}G^{-1}\partial_{\bar{s}}G$ cancel with the first sum in the last line of (3.6) leaving

$$G^{-1}\sum_{s=1}^{m}\partial_{\bar{s}}\{h_{rs}G\} = -\sum_{k}h_{rk}\sum_{p,s}h_{ps}\{\partial_{\bar{s}}g_{pk}-\partial_{\bar{k}}g_{ps}\}. \tag{3.7}$$

Now the $(1,1)$-form ω associated to the metric g is given by $\omega = i\sum_{a,b=1}^{m}g_{ab}\,dz_{a}\wedge d\bar{z}_{b}$ [GH, p. 107] for the choice of g_{ab} made above. We will follow the pairing convention $\langle\alpha_{1}\wedge...\wedge\alpha_{k},\xi_{1}\otimes...\otimes\xi_{k}\rangle = \det(\langle\alpha_{i},\xi_{j}\rangle)$ used in [GH].

If $e_{1},...,e_{m}$ is an orthonormal basis of $T_{z}^{1,0}$ then, expanding each e_{r} in terms of the basis $\partial_{1},...,\partial_{m}$, we see that we can write $\sum_{r}e_{r}\otimes\bar{e}_{r} = \sum_{j,k}a_{jk}\partial_{j}\otimes\partial_{\bar{k}}$. Evaluating the left-hand side on $dz_{p}\otimes d\bar{z}_{s}$ one gets $\sum_{r}dz_{p}(e_{r})\overline{dz_{s}(e_{r})}$ which is $h(dz_{p},dz_{s}) = h_{ps}$. So $\sum_{r}e_{r}\otimes\bar{e}_{r} = \sum_{p,s}h_{ps}\partial_{p}\otimes\partial_{\bar{s}}$. Therefore

$$(h\cdot\bar{\partial}\omega)(\partial_{\bar{k}}) = (\bar{\partial}\omega)\left(\sum_{r}e_{r}\otimes\bar{e}_{r}\otimes\partial_{\bar{k}}\right)$$

$$= \sum_{p,s}(\bar{\partial}\omega)(\partial_{p}\otimes\partial_{\bar{s}}\otimes\partial_{\bar{k}})h_{ps}$$

$$= i\sum_{p,s}\sum_{a,b,c}(\partial_{\bar{c}}g_{ab})(d\bar{z}_{c}\wedge dz_{a}\wedge d\bar{z}_{b})(\partial_{p}\otimes\partial_{\bar{s}}\otimes\partial_{\bar{k}})h_{ps}.$$

Since the only nonzero pairing of ∂_{p} is with dz_{a}, for $a=p$, we have

$$(h\cdot\bar{\partial}\omega)(\partial_{\bar{k}}) = -i\sum_{p,s}\sum_{b,c}(\partial_{\bar{c}}g_{pb})h_{ps}(d\bar{z}_{c}\wedge d\bar{z}_{b})(\partial_{\bar{s}}\otimes\partial_{\bar{k}})$$

$$= -i\sum_{p,s}h_{ps}\sum_{b,c}(\partial_{\bar{c}}g_{pb})(\delta_{cs}\delta_{bk}-\delta_{ck}\delta_{bs})$$

$$= -i\sum_{p,s}h_{ps}\{\partial_{\bar{s}}g_{pk}-\partial_{\bar{k}}g_{ps}\}.$$

Thus, multiplying $-i(h\cdot\bar{\partial}\omega)(\partial_{\bar{k}})$ by h_{rk} and summing over k gives the right-hand side of (3.7) and proves the lemma. $\qquad\square$

We may now complete the proof of Theorem 2.6. The last term in (3.4) may be rewritten, in view of Lemma 3.1, as $i\sum_{r,k}(\partial_{r}f)h_{rk}(h\cdot\bar{\partial}\omega)(\partial_{\bar{k}})$ which is $ih(\partial f,h\cdot\bar{\partial}\omega)$. Defining now a vector field Z of type $(1,0)$ by (2.13), the equation (3.4) shows that $d^{*}df(z) = Zf(z)$ if f is holomorphic in a neighborhood of z. Thus (2.12) holds. It is clear from (2.12) that if Z is holomorphic then $d^{*}d$ is holomorphic. Conversely if $d^{*}d$ is holomorphic and $Z = \sum_{k=1}^{m}\varphi_{k}(z)\partial/\partial z_{k}$ in a local chart (V,z) then we may choose f in

$C^\infty(M)$ such that $f(z)=z_r$ in a neighborhood of a point $\hat{z}\in V$. It follows that $\varphi_r(z)$ is holomorphic in a neighborhood of $\hat{z}$. $\square$

Proof of Corollary 2.8. It suffices to prove (2.18) because the formulas for X and Y then follow from $X=Z+\bar{Z}$ and $Y=i(Z-\bar{Z})$. If $\alpha\in T_z^{1,0}$ then

$$\overline{(h\cdot\bar{\partial}\omega)(\bar{\alpha})}=\overline{\sum_s(\bar{\partial}\omega)(e_s\otimes\bar{e}_s\otimes\bar{\alpha})}=\sum_s(\partial\omega)(\bar{e}_s\otimes e_s\otimes\alpha)$$
$$=-\sum_s(\partial\omega)(e_s\otimes\bar{e}_s\otimes\alpha)=-(h\cdot\partial\omega)(\alpha).$$

Hence $\overline{(h\cdot\bar{\partial}\omega)}=-h\cdot\partial\omega$. Therefore, for $f\in C^\infty(M)$,

$$\bar{Z}f=\overline{Z\bar{f}}=\overline{h(\partial\bar{f},-\bar{\partial}\log\nu+ih\cdot\bar{\partial}\omega)}$$
$$=h(\bar{\partial}f,-\partial\log\nu-i\overline{h\cdot\bar{\partial}\omega})$$
$$=h(\bar{\partial}f,-\partial\log\nu+ih\cdot\partial\omega)$$

since $\log\nu$ is real. This proves Corollary 2.8. $\square$

Proof of Theorem 2.9. Let $\psi_t=\exp tY$. The assertion of the theorem is equivalent to the identity $\int_M f\circ\psi_t\,d\mu=\int_M f\,d\mu$ for all real $f\in C_c^\infty(M)$. This in turn is equivalent to the assertion that the derivative of the left-hand side is zero. But, putting $f_t=f\circ\psi_t$, we have $df_t(z)/dt=(Yf_t)(z)$. Since f_t is just another real-valued function in $C_c^\infty(M)$ we see that we must prove that $\int_M(Yf)(z)\,d\mu(z)=0$ for all real-valued functions $f\in C_c^\infty(M)$. By using a partition of unity we may assume that f is supported in a holomorphic coordinate patch and may then use the expression (3.3) for Z. Thus we need to prove that $\int_V(i(Z-\bar{Z})f)\,d\mu=0$ when f is supported in the coordinate neighborhood V. That is, we need to show that $\int(Zf)(z)\hat{\mu}\,dx=\int(\bar{Z}f)\hat{\mu}\,dx$. These integrals are of course extended over some open set $z(V)$ in $\mathbf{C}^m$. Since f and $\hat{\mu}$ are real the last integral is $\overline{\int(Zf)\hat{\mu}\,dx}$. Thus we need to show that $\int(Zf)\hat{\mu}\,dx$ is real. But by (3.3) and an integration by parts,

$$\int(Zf)(z)\hat{\mu}(z)\,dx=-\int\sum_{r=1}^m\sum_{s=1}^m(\partial_{\bar{s}}\{h_{rs}\hat{\mu}(z)\})\partial_r f(z)\,dx$$
$$=\int\sum_{r,s}(\partial_r\partial_{\bar{s}}\{h_{rs}\hat{\mu}\})f(z)\,dx.\tag{3.8}$$

Since $\hat{\mu}$ is real and h_{rs} is a Hermitian matrix, the conjugate of $\partial_r\partial_{\bar{s}}\{h_{rs}\hat{\mu}\}$ is $\partial_s\partial_{\bar{r}}\{h_{sr}\hat{\mu}\}$. The double sum in the last line of (3.8) is therefore real. $\square$

Proof of Theorem 2.10. Y is a Killing vector field if and only if

$$Yg(A,B)=g([Y,A],B)+g(A,[Y,B])\tag{3.9}$$

for all smooth vector fields A and B. It is sufficient to verify (3.9) in a local coordinate patch and it is convenient to compute in complex form. Moreover it is sufficient to verify (3.9) in case A is one of the local vector fields $\{\partial/\partial z_j\}_{j=1}^m$ or $\{\partial/\partial \bar{z}_k\}_{k=1}^m$, and B also is chosen among these $2m$ vector fields.

The proof breaks into two parts. In the first part, A and B will be taken to be of the same type and only the holomorphy of Z need be used, not the Kähler property of g. In the second part, A and B will be taken to be of opposite type and only the Kähler hypothesis will be used, not the holomorphy of Z.

Suppose that $A=\partial/\partial z_j$ and $B=\partial/\partial z_k$. Then $g(A,B)=0$ so the left-hand side of (3.9) is zero. Furthermore, using the representation (2.11) of Z we find $[Z,\partial/\partial z_j]=-\sum_{r=1}^m(\partial\varphi_r/\partial z_j)\partial/\partial z_r$ while $[\bar{Z},\partial/\partial z_j]=0$ because each function φ_r is holomorphic. Therefore $[Y,\partial/\partial z_j]$ is of type $(1,0)$, and hence $g([Y,\partial/\partial z_j],\partial/\partial z_k)=0$. Similarly $g(\partial/\partial z_j,[Y,\partial/\partial z_k])=0$. So all three terms in (3.9) are zero. Similarly we have $[Y,\partial/\partial \bar{z}_j]=i\sum_r(\overline{\partial\varphi_r/\partial z_j})\partial/\partial \bar{z}_r$ which is type $(0,1)$. So the same argument now shows that (3.9) is satisfied if $A=\partial/\partial \bar{z}_j$ and $B=\partial/\partial \bar{z}_k$.

Next we take A and B to be of opposite type and assume now that g is Kähler. We will verify (3.9) at a point P in the coordinate chart. We may take the holomorphic coordinate system $z_1,...,z_m$ to be such that $g_{ij}(P)=\delta_{ij}$ and such that the first derivatives of g_{ij} are zero at P [GH, p. 107]. Since $h=(g^{-1})^t$ we have $h_{ij}(P)=\delta_{ij}$, and the first derivatives of h_{ij} are also zero at P. Now take $A=\partial/\partial z_j$ and $B=\partial/\partial \bar{z}_k$ in (3.9). At P the left-hand side of (3.9) is a combination of first derivatives of g_{jk} and is therefore zero. To evaluate the commutators on the right observe that $\bar{\partial}\omega=0$ because g is Kähler and (2.13) therefore reduces to $Zf=-\sum_r h_{rs}(\partial\log\nu/\partial\bar{z}_s)\partial f/\partial z_r$. Let $\varphi_r=-\sum_s h_{rs}\partial_{\bar{s}}(\log\nu)$. In the present coordinate system we then have $\varphi_r(p)=-\partial_{\bar{r}}\log\nu$ and $\partial\varphi_r/\partial z_j|_p=-\partial_j\partial_{\bar{r}}\log\nu$ because $(\partial_j h_{rs})(P)=0$ for all j,r,s. Z is given by (2.11). So $[Z,A]=[Z,\partial/\partial z_j]=-\sum_r(\partial_j\varphi_r)\partial_r=\sum_r(\partial_j\partial_{\bar{r}}\log\nu)\partial_r$ at P. Now

$$[\bar{Z},A]=\sum_r[\bar{\varphi}_r\partial_{\bar{r}},\partial_j]=-\sum_r(\partial_j\bar{\varphi}_r)\partial_{\bar{r}}$$

is a vector field of type $(0,1)$. (It is zero if Z is holomorphic, which we are not assuming in this part of the proof.) In computing $g([Y,\partial_j],\partial_{\bar{k}})$ the contribution to $[Y,\partial_j]$ from $[\bar{Z},\partial_j]$ therefore doesn't enter because $\partial_{\bar{k}}$ is also of type $(0,1)$. Hence, at P,

$$g([Y,A],B)=ig([Z,\partial_j],\partial_{\bar{k}})=ig\left(\sum_r(\partial_j\bar{\partial}_r\log\nu)\partial_r,\partial_{\bar{k}}\right)=i\partial_j\partial_{\bar{k}}\log\nu.$$

It remains to show that $g(A,[Y,B])$ is the negative of what has just been computed.

Since $A=\partial_j$ is of type $(1,0)$ we need only compute the part of $[Y,\bar\partial_k]$ of type $(0,1)$. But

$$
\begin{aligned}
[Y,\partial_{\bar{k}}] &= i[Z,\partial_{\bar{k}}] - i[\bar{Z},\partial_{\bar{k}}] \\
&= \text{term of type } (1,0) - i\overline{[Z,\partial_k]} \\
&= \text{term of type } (1,0) + i\sum_r \overline{(\partial_k\varphi_r)}\partial_{\bar{r}}.
\end{aligned}
$$

Hence, at P,

$$
g(\partial_j,[Y,\partial_{\bar{k}}]) = ig\left(\partial_j,\sum_r \overline{(\partial_k\varphi_r)}\partial_{\bar{r}}\right) = i\overline{\partial_k\varphi_j} = -i\overline{\partial_k\partial_{\bar{j}}\log\nu} = -i\partial_{\bar{k}}\partial_j\log\nu
$$

because $\log\nu$ is real. $\qquad\square$

The proof of Theorem 2.11 will be broken into its nonholomorphic and holomorphic parts. Moreover, in addition to the vector field Y defined in (2.16), there are other vector fields in our examples which satisfy the hypotheses of Lemmas 3.2 and 3.3 below. Since these vector fields may be useful, the following two lemmas will be stated in more generality than needed for the proof of Theorem 2.11.

LEMMA 3.2. *Let Y_0 be a C^∞ real vector field on M. Suppose that Y_0 is complete and that its flow, $\exp tY_0$, preserves both the measure μ and the metric g. For any measurable function f on M let*

$$
V_t f = f\circ\exp tY_0, \quad t\in\mathbf{R}. \tag{3.10}
$$

Then:

(a) *V_t is isometric on $L^p(\mu)$ for $0<p<\infty$.*

(b) *$\mathcal{D}(Q)$ and $\mathcal{D}(A)$ are both invariant under V_t.*

(c) *V_t is unitary on $\mathcal{D}(Q)$ in the energy norm, $(Q(f)+\|f\|_{L^2}^2)^{1/2}$.*

(d) *The operators V_t form a strongly continuous one-parameter group of isometries in L^p (in the L^p-metric) for $0<p<\infty$, and in the Hilbert space $\mathcal{D}(Q)$ for the energy norm.*

Proof. (a) follows immediately from the assumption that the diffeomorphism $\exp tY_0$ preserves the measure μ. To prove (b) and (c) suppose that $f\in C_c^\infty(M)$. Then, for fixed real t, $f\circ\exp tY_0$ is also in $C_c^\infty(M)$. Moreover

$$
\begin{aligned}
Q(V_t f, V_t f) &= \int_M h_x(d(f\circ\exp tY_0), d(\bar{f}\circ\exp tY_0))\,d\mu(x) \\
&= \int h_{(\exp tY_0)(x)}(df, d\bar{f})\,d\mu(x) \\
&= \int h_y(df, d\bar{f})\,d\mu(y) \\
&= Q(f,f).
\end{aligned}
\tag{3.11}
$$

The second equality uses the assumption that Y_0 is Killing, and the third equality uses the assumption that Y_0 preserves μ. Now if f is in $\mathcal{D}(Q)$ then there is a sequence f_n in $C_c^\infty(M)$ such that $f_n \to f$ in energy norm. By (3.11), $V_t f_n$ is Cauchy in Q-norm and by (a), $V_t f_n \to V_t f$ in L^2-norm. Since Q is closed, $V_t f \in \mathcal{D}(Q)$. Moreover the equality

$$Q(V_t f) = Q(f), \quad f \in \mathcal{D}(Q), \tag{3.12}$$

holds. Now suppose that $f \in \mathcal{D}(A)$. This is equivalent to the assertion that $f \in \mathcal{D}(Q)$ and that the map $C_c^\infty(M) \ni \varphi \to Q(f, \varphi)$ is continuous in L^2-norm [RS, Theorem VIII.15]. But by (3.12),

$$|Q(V_t f, \varphi)| = |Q(f, V_{-t}\varphi)| \leqslant \mathrm{const} \cdot \|V_{-t}\varphi\|_{L^2} \leqslant \mathrm{const} \cdot \|\varphi\|_{L^2}.$$

Hence $V_t f \in \mathcal{D}(A)$. Since $V_t^{-1} = V_{-t}$ both (b) and (c) follow.

Now if $f \in C_c^\infty(M)$ then $V_t f$ converges to f pointwise and boundedly as $t \to 0$. Moreover $d(V_t f)$ converges to df pointwise and boundedly also. Since $C_c^\infty(M)$ is dense in all L^p-spaces $(0 < p < \infty)$ as well as in $\mathcal{D}(Q)$ (in energy norm), it follows that the one-parameter group $t \to V_t$ is strongly continuous in all of these spaces. $\square$

LEMMA 3.3. *In addition to the hypotheses of Lemma 3.2 suppose that $Z_0 \equiv \frac{1}{2}(X_0 - iY_0)$ is holomorphic, where $X_0 = -JY_0$ and J is the almost complex structure on M. Then for each real t, V_t leaves invariant $\mathcal{H}(\equiv \mathcal{H}(M))$, $\mathcal{H} \cap \mathcal{D}(Q)$, $\mathcal{H} \cap \mathcal{D}(A)$, $\mathcal{H} \cap L^p$ and $\mathcal{H}^p$ for all $p \in (0, \infty)$. Moreover the one-parameter group $t \to V_t$ is a strongly continuous group of isometries in $\mathcal{H} \cap \mathcal{D}(Q)$ (in energy norm), in $\mathcal{H} \cap L^p$ and in $\mathcal{H}^p$ (in the L^p-metric) for all $p \in (0, \infty)$.*

Proof. In a local holomorphic coordinate system (z, U) we may write $Z_0 = \sum_{k=1}^m \varphi_k(z)\partial/\partial z_k$ because a vector field of the form $\frac{1}{2}(X_0 - iJX_0)$ is of type $(1, 0)$. Moreover, by assumption, the coefficients $\varphi_k(z_1, ..., z_m)$ are holomorphic. The local flow equations for the coodinate functions $z_1, ..., z_m$ are $dz_j(t)/dt = Y_0 z_j = i\varphi_j(z(t))$ because $Y_0 = i(Z_0 - \bar{Z}_0)$. It now follows by a standard argument that the diffeomorphism group $\exp tY_0$ consists of holomorphic maps of M into M. So if f is holomorphic then so is $f \circ \exp tY_0$. That is, $V_t \mathcal{H} \subset \mathcal{H}$.

Thus, in view of Lemma 3.2, we see that V_t leaves invariant $\mathcal{H}$, $\mathcal{D}(Q)$ and L^p for $0 < p < \infty$, and is isometric in L^p and in $\mathcal{D}(Q)$ (energy norm). Therefore V_t leaves invariant $\mathcal{H} \cap L^p$ and $\mathcal{H} \cap \mathcal{D}(Q)$, and is isometric in these spaces. Moreover, since $\mathcal{H}^2$ is the L^2-closure of $\mathcal{H} \cap \mathcal{D}(Q)$ and V_t is unitary on L^2 it follows that $V_t \mathcal{H}^2 \subset \mathcal{H}^2$. Therefore, for $2 \leqslant p < \infty$,

$$V_t \mathcal{H}^p = V_t(\mathcal{H}^2 \cap L^p) \subset \mathcal{H}^2 \cap L^p = \mathcal{H}^p.$$

For $0 < p < 2$ we have

$$V_t \mathcal{H}^p = V_t(L^p\text{-closure of } \mathcal{H}^2) = L^p\text{-closure of } V_t \mathcal{H}^2 \subset L^p\text{-closure of } \mathcal{H}^2 = \mathcal{H}^p.$$

Thus V_t leaves invariant all the asserted spaces. Since $V_{(\cdot)}$ forms a strongly continuous one-parameter isometry group in each of the ambient spaces L^p or $\mathcal{D}(Q)$ (energy norm), the same holds in the invariant subspaces. $\qquad\square$

Most of the proof of Theorem 2.11 follows from the next lemma. The crucial use of the Y-flow for regularizing in all three holomorphic function Hilbert spaces, $\mathcal{H}\cap\mathcal{D}(Q)$, $\mathcal{H}^2$ and $\mathcal{H}\cap L^2$, bears noticing.

LEMMA 3.4. *Assume the hypotheses of Theorem* 2.11.

(a) *If* $f\in\mathcal{H}\cap\mathcal{D}(A)$ *then* $Af=Zf$.

(b) *If* $f\in\mathcal{H}\cap\mathcal{D}(Q)$ *then* $f\in\mathcal{D}(A)$ *if and only if* $Zf\in L^2$.

(c) $\mathcal{H}\cap\mathcal{D}(A)$ *is dense in* $\mathcal{H}^2$.

(d) $e^{itA}f=f\circ\exp tY$ *if* $f\in\mathcal{H}^2$.

(e) $e^{itA}\mathcal{H}^2=\mathcal{H}^2$.

(f) $e^{itA}\mathcal{H}^p=\mathcal{H}^p$ *for* $2\leqslant p<\infty$, *and* e^{itA} *is a strongly continuous one-parameter group of isometries in* $\mathcal{H}^p$ *for* $2\leqslant p<\infty$.

(g) *If* $\operatorname{Re}\zeta\geqslant0$ *then* $e^{-\zeta A}\mathcal{H}^2\subset\mathcal{H}^2$.

(h) *If* f *is in* $\mathcal{H}\cap\mathcal{D}(A)$ *then* Af *is in* $\mathcal{H}^2$.

Proof. Suppose that $f\in\mathcal{H}\cap\mathcal{D}(Q)$. Let $\varphi\in C_c^\infty(M)$. Then

$$Q(f,\varphi)=\int_M h(df,d\bar\varphi)\,d\mu=\int(d^*df)\bar\varphi\,d\mu$$

because $f\in C^\infty(M)$. Since f is also in $\mathcal{H}$, the equation (2.12) shows that $Q(f,\varphi)=\int_M(Zf)\bar\varphi\,d\mu$. Therefore if $Zf\in L^2$ then the map $C_c^\infty(M)\ni\varphi\to Q(f,\varphi)$ is continuous in L^2-norm, and f is consequently in $\mathcal{D}(A)$. Conversely, if $f\in\mathcal{H}\cap\mathcal{D}(A)$ then $f\in\mathcal{H}\cap\mathcal{D}(Q)$, and the equation

$$(Af,\varphi)=Q(f,\varphi)=\int_M(Zf)\bar\varphi\,d\mu,\quad\text{for all }\varphi\in C_c^\infty(M),$$

shows that $Zf=Af$ which is in L^2. This proves (a) and (b).

To prove (c) it suffices to prove that $\mathcal{H}\cap\mathcal{D}(A)$ is dense in $\mathcal{H}\cap\mathcal{D}(Q)$ in L^2-norm because $\mathcal{H}\cap\mathcal{D}(Q)$ is dense in $\mathcal{H}^2$. (Cf. Notation 2.4.) Define

$$V_tF=F\circ\exp tY \tag{3.13}$$

for any measurable function F. The hypotheses on Y_0 in Lemmas 3.2 and 3.3 are satisfied by our present choice $Y_0=Y$. Let $u\in\mathcal{H}\cap\mathcal{D}(Q)$. Choose $\psi\in C_c^\infty(\mathbf{R})$ and define

$$f=\int_{\mathbf{R}}\psi(s)V_su\,ds. \tag{3.14}$$

This exists as a strong integral into $\mathcal{H}\cap\mathcal{D}(Q)$ (in energy norm) because V_s is a strongly continuous unitary group in this Hilbert space by Lemma 3.3. Thus $f\in\mathcal{H}\cap\mathcal{D}(Q)$. Moreover $f\circ\exp tY=V_tf=\int_{\mathbf{R}}\psi(s)V_{t+s}u\,ds=\int_{\mathbf{R}}\psi(s-t)V_su\,ds$. By an elementary and standard argument the derivative of the right-hand side with respect to t exists in energy norm, and therefore in L^2-norm. Thus

$$\frac{d(f\circ\exp tY)}{dt}=\frac{d(V_tf)}{dt}=-\int_{-\infty}^{\infty}\psi'(s-t)V_su\,ds,$$

with the derivatives existing in the sense of the L^2-norm. But the derivative on the left also exists pointwise everywhere on M and is equal to $Y(f\circ\exp tY)$. Thus, taking $t=0$ we see that $Yf\in L^2$. Since $f\in\mathcal{H}$ we have $Yf=i(Z-\bar{Z})f=iZf$. Hence $Zf\in L^2$. By part (b) it now follows that $f\in\mathcal{H}\cap\mathcal{D}(A)$. To conclude the proof of part (c) choose a sequence $\psi_n\in C_c^\infty(\mathbf{R})$ such that $\psi_n\geqslant 0$, $\int\psi_n(s)\,ds=1$ and $\operatorname{supp}\psi_n\subset(-n^{-1},n^{-1})$. Let f_n be the corresponding sequence constructed from u as in (3.14). Then the inequality $\|f_n-u\|=\int\psi_n(s)\|V_su-u\|\,ds$ and the strong continuity of V_s show that $f_n\to u$ in L^2-norm (and also in energy norm, actually). This completes the proof of part (c).

To prove part (d) let us return to the function f defined in (3.14). We saw that f is in both $\mathcal{H}\cap\mathcal{D}(A)$ and in the domain of the infinitesimal generator of the unitary group V_t (as a unitary group in L^2). So V_tf is also in $\mathcal{H}$ and in both domains (e.g., by Lemma 3.3 or the discussion following (3.14)). Moreover we saw that $dV_tf/dt=YV_tf=iZV_tf$, which equals iAV_tf by part (a). Hence

$$\frac{d(e^{-itA}V_tf)}{dt}=\{e^{-itA}(-iA)\}V_tf+e^{-itA}\{iAV_tf\}=0. \qquad (3.15)$$

So $e^{-itA}V_tf=f$. Thus $e^{itA}f=V_tf$ for a dense set in $\mathcal{H}^2$. Hence the two unitary groups agree on $\mathcal{H}^2$. This proves part (d). Since $V_t\mathcal{H}^2=\mathcal{H}^2$ by Lemma 3.3, part (e) now follows. To prove part (f) observe that $\mathcal{H}^p\subset\mathcal{H}^2$ for $p\geqslant 2$. So $e^{itA}f=V_tf$ for $f\in\mathcal{H}^p$. Thus part (f) follows from Lemma 3.3 for $p\geqslant 2$. By part (e) the unitary group e^{itA} leaves $\mathcal{H}^2$ invariant. By the spectral theorem any bounded function of A also leaves $\mathcal{H}^2$ invariant. In particular, part (g) holds. To prove part (h) let $f\in\mathcal{H}\cap\mathcal{D}(A)$. Then $f\in\mathcal{H}^2$ and $e^{itA}f\in\mathcal{H}^2$ by part (e). Hence $(it)^{-1}(e^{itA}f-f)\in\mathcal{H}^2$. Let t go to zero to conclude that $Af\in\mathcal{H}^2$. $\qquad\square$

Proof of Theorem 2.11. All parts of Theorem 2.11 have been proven in Lemma 3.4 except part (e). If $f\in\mathcal{H}^2$ and $f\in\mathcal{D}(A)$ then $f\in\mathcal{H}\cap\mathcal{D}(A)$. Hence by Lemma 3.4, part (a), $Zf=Af$ which is in L^2. So half of Theorem 2.11 (e) is immediate. Now suppose that $u\in\mathcal{H}^2$ and $Zu\in L^2$. We will show that u is in $\mathcal{D}(A)$. Choose $\psi\in C_c^\infty(\mathbf{R})$ and define f by (3.14), interpreting the integrand this time as a continuous function into $\mathcal{H}^2$ in

L^2-norm. This is a valid interpretation because $V_s \colon \mathcal{H}^2 \to \mathcal{H}^2$ is strongly continuous by Lemma 3.3. By Lemma 3.4 (d) we may replace $V_s u$ by $e^{isA} u$ in (3.14) because u is in $\mathcal{H}^2$. Hence $f = \int_{\mathbf{R}} \psi(s) e^{isA} u \, ds$. Since $\psi \in C_c^\infty(\mathbf{R})$ a standard argument shows that f is in $\mathcal{D}(A)$ and that $Af \in \mathcal{H}^2$. Moreover $e^{itA} f = \int \psi(s-t) e^{isA} u \, ds = \int \psi(s-t) V_s u \, ds$. Pick $z \in M$. Since pointwise evaluation is a continuous linear functional on $\mathcal{H}^2$ we may evaluate both sides of the last equality at z and bring the evaluation under the integral sign, obtaining $(e^{itA} f)(z) = \int \psi(s-t) u(\exp(sY)z) \, ds$. Since $d(e^{itA} f)/dt$ exists in the $\mathcal{H}^2$-sense and pointwise evaluation is continuous on $\mathcal{H}^2$, we may differentiate the last equality at $t=0$ and get $i(Af)(z)$ on the left. Thus an integration by parts on the numerical-valued integrand gives

$$i(Af)(z) = -\int_{\mathbf{R}} \psi'(s) u(\exp(sY)z) \, ds = \int_{\mathbf{R}} \psi(s)(Yu)(\exp(sY)z) \, ds$$

$$= i \int \psi(s)(Zu)(\exp(sY)z) \, ds = i \int \psi(s)[V_s(Zu)](z) \, ds.$$

Now Zu is in $\mathcal{H} \cap L^2$ and V_s is a strongly continuous unitary group on $\mathcal{H} \cap L^2$ by Lemma 3.3. (We don't actually know at this stage that $Zu \in \mathcal{H}^2$.) Moreover pointwise evaluation is a continuous linear functional on $\mathcal{H} \cap L^2$ and these linear functionals separate points of $\mathcal{H} \cap L^2$. We may therefore remove the evaluation functional, obtaining

$$Af = \int_{\mathbf{R}} \psi(s) V_s(Zu) \, ds.$$

The integrand on the right is a continuous function into $\mathcal{H} \cap L^2$ and the integral should be interpreted as a Riemann integral into $\mathcal{H} \cap L^2$. Now replace ψ by a sequence $\psi_n \in C_c^\infty$ converging to the δ-function. The resulting functions f_n are in $\mathcal{H} \cap \mathcal{D}(A)$ and converge to u in L^2-norm by (3.14). Moreover the last displayed equation shows that Af_n converges to Zu in L^2-norm because V_s is strongly continuous on $\mathcal{H} \cap L^2$. Since A is a closed operator in L^2 it follows that $u \in \mathcal{D}(A)$ and $Au = Zu$. $\qquad\square$

Proof of Corollary 2.12. Let $t \geqslant 0$ and let $s \in \mathbf{R}$. Let $W = -tX - sY$. Assume that the flow $\exp(rW)$ exists for all $r \geqslant 0$. Let $f \in \mathcal{H} \cap \mathcal{D}(A)$ and define $f_r(z) = (e^{-r(t+is)A} f)(z)$. Then $f_r \in \mathcal{H}^2$ by Theorem 2.11. Since it is also in $\mathcal{D}(A)$ we have $f_r \in \mathcal{H} \cap \mathcal{D}(A)$ for all $r \geqslant 0$. Furthermore $df_r(z)/dr = -(t+is)(Af_r)(z)$. The derivative exists in the strong $L^2(\mu)$-sense. But since pointwise evaluation of holomorphic functions is a continuous linear functional on L^2 the derivative on the left also exists for each z. Now since $f_r \in \mathcal{H} \cap \mathcal{D}(A)$ we have $Af_r = Zf_r = (Z + \bar{Z})f_r = Xf_r$, and similarly $Af_r = (Z - \bar{Z})f_r = -iYf_r$. So $(t+is)Af_r = (tX + sY)f_r = -Wf_r$. Hence

$$\frac{\partial f_r(z)}{\partial r} = (Wf_r)(z) \quad \text{for each } z \in M \text{ and } r \geqslant 0.$$

Let $\varphi_r = \exp(rW)$ for $r \geqslant 0$. Then for $0 < r \leqslant 1$ we have

$$\frac{\partial f_r(\varphi_{1-r}(z))}{\partial r} = (Wf_r)(\varphi_{1-r}(z)) - (Wf_r)(\varphi_{1-r}(z)) = 0.$$

So $f_r(\varphi_{1-r}(z))$ is constant in r in $[0,1]$. That is, $f_1(z) = f(\varphi_1(z))$. So

$$(e^{-(t+is)A}f)(z) = f(\exp(-tX - sY)z), \quad f \in \mathcal{H} \cap \mathcal{D}(A). \tag{3.16}$$

Next, if f is in $\mathcal{H}^2$ then by Theorem 2.11, part (a), there is a sequence f_n in $\mathcal{H} \cap \mathcal{D}(A)$ which converges in L^2-norm to f. Applying (3.16) to f_n and relying once more on the fact that L^2-convergence implies pointwise convergence for holomorphic functions, it follows that both sides of (3.16), for the sequence f_n, converge pointwise to the corresponding expression for f. This proves equation (2.26). Now suppose that we only have separate information about the X- and Y-flows. That is, we assume that the semigroup $\exp(-tX)$ exists globally for $t \geqslant 0$ and that Y is complete. We may apply the previous discussion to the vector field $W = -tX$ and $s = 0$. We may conclude that

$$(e^{-tA}f)(z) = f(\exp(-tX)z), \quad f \in \mathcal{H}^2, t \geqslant 0. \tag{3.17}$$

But by Lemma 3.4, part (d), we may apply e^{-isA} to both sides of this equation to obtain

$$e^{-isA}e^{-tA}f = f \circ \exp(-tX) \circ \exp(-sY). \tag{3.18}$$

On the other hand, we may apply the equation (3.17) to the function $e^{-isA}f$ to obtain (3.18) but with the factors $\exp(-tX)$ and $\exp(-sY)$ reversed. $\qquad\square$

LEMMA 3.5. *Assume that M is complete. If $f \in C^\infty(M) \cap L^2(\mu)$ and $d^*df \in L^2(\mu)$ then $f \in \mathcal{D}(Q)$.*

Proof. This proof, for the weighted Laplacian d^*d, uses a technique that is standard for the ordinary Laplacian. Because M is complete there exists a sequence $\varphi_n \in C_c^\infty(M)$ with range in $[0,1]$ and an increasing sequence U_n of open sets such that $\bigcup U_n = M$, $\varphi_n|U_n = 1$ and $\sup_{n,x}|d\varphi_n(x)| \leqslant C$ for some constant C. Suppose that $f \in C^\infty(M) \cap L^2$ and that d^*df (defined by (2.2)) is in L^2. Let $b = \|d^*df\|_{L^2}\|f\|_{L^2}$. Then

$$\left|\int \{\varphi_n^2 |df|^2 + 2\varphi_n \bar{f} h(df, d\varphi_n)\} \, d\mu\right| = \left|\int h(df, d(\varphi_n^2 \bar{f})) \, d\mu\right| = \left|\int (d^*df)\varphi_n^2 \bar{f} \, d\mu\right| \leqslant b.$$

So

$$\int \varphi_n^2 |df|^2 \, d\mu \leqslant b + 2 \int \varphi_n |f| \cdot |df| \cdot |d\varphi_n| \, d\mu$$

$$\leqslant b + 2\left(\int \varphi_n^2 |df|^2 \, d\mu\right)^{1/2}\left(\int |f|^2 |d\varphi_n|^2 \, d\mu\right)^{1/2}.$$

If $\int |df|^2 \, d\mu \neq 0$ we obtain for large n,

$$\left(\int \varphi_n^2 |df|^2 \, d\mu \right)^{1/2} \leqslant b \left(\int \varphi_n^2 |df|^2 \, d\mu \right)^{-1/2} + 2 \left(\int |f|^2 |d\varphi_n|^2 \, d\mu \right)^{1/2}.$$

Since $|d\varphi_n|$ is uniformly bounded and converges to zero everywhere while $|f|^2$ is in L^1, the last term goes to zero. Applying Fatou's lemma twice gives

$$\int h(df, d\bar{f}) \, d\mu \leqslant \|d^* df\|_{L^2} \|f\|_{L^2} < \infty.$$

This implies that $\varphi_n f$ converges to f in energy norm because

$$\left(\int |d\{\varphi_n f - f\}|^2 \, d\mu \right)^{1/2} = \left(\int |(\varphi_n - 1) \, df + f \, d\varphi_n|^2 \, d\mu \right)^{1/2}$$
$$\leqslant \left(\int (\varphi_n - 1)^2 |df|^2 \, d\mu \right)^{1/2} + \left(\int |f|^2 |d\varphi_n|^2 \, d\mu \right)^{1/2}$$

and each term on the right goes to zero as $n \to \infty$. Of course, $\varphi_n f$ converges to f in L^2-norm also. So $f \in \mathcal{D}(Q)$. $\qquad \square$

Proof of Theorem 2.14. Let $u \in \mathcal{H} \cap L^2$. Define $V_t u = u \circ \exp tY$. By Lemma 3.3, V_t is a strongly continuous unitary group in $\mathcal{H} \cap L^2$. Let $\psi \in C_c^\infty(\mathbf{R})$ and define $f = \int_{-\infty}^\infty \psi(s) V_s u \, ds$. Then $f \in \mathcal{H} \cap L^2$. The same argument used in the proof of Lemma 3.4, part (c), shows that $Yf \in L^2$. Since $f \in \mathcal{H}$ we have $Yf = iZf = id^* df$. So $d^* df \in L^2$. By Lemma 3.5, $f \in \mathcal{H} \cap \mathcal{D}(Q)$. Now choose a sequence $\psi_n \in C_c^\infty(\mathbf{R})$ converging as in the proof of Lemma 3.4, part (c), to $\delta(s)$. The corresponding sequence f_n is in $\mathcal{H} \cap \mathcal{D}(Q)$ and converges in L^2 to u. Hence $u \in \mathcal{H}^2$. This proves (2.27) in case $p=2$. If $\infty > p > 2$ and $u \in \mathcal{H} \cap L^p$ then it is in $\mathcal{H} \cap L^2$, hence in $\mathcal{H}^2$ and therefore in $\mathcal{H}^p$. $\qquad \square$

4. Proofs of hypercontractivity theorems

LEMMA 4.1 (differentiability). *Assume that either $0 < p \leqslant 2$ and f is in the L^2-domain of A, or $2 < p < \infty$ and f is in the L^p-domain of A. Let $\varepsilon > 0$ and write*

$$g_t(z) = (e^{-tA} f)(z) \tag{4.1}$$

and

$$k_t(z) = |g_t(z)|^2 + \varepsilon. \tag{4.2}$$

Then the map $t \to k_t^{p/2}$ is a continuously differentiable function into $L^1(\mu)$ and

$$\frac{d}{dt} \int_M k_t^{p/2} \, d\mu(z) = -p \operatorname{Re} \int_M (Ag_t)(z) \overline{g_t(z)} k_t^{p/2-1} \, d\mu, \quad 0 < p < \infty. \tag{4.3}$$

Proof. Allowing, for the moment, the validity of differentiating under the integral on the left of (4.3), the identity (4.3) follows immediately. Moreover the integrability of the integrand on the right can be seen, in case $0<p\leqslant 2$, by observing that the first two factors are in $L^2(\mu)$ while the third factor $k_t^{p/2-1}$ is bounded by $\varepsilon^{p/2-1}$. In case $p>2$ the first two factors are in L^p. It suffices, by Hölder's inequality, to show that the third factor is in $L^r(\mu)$, where $p^{-1}+p^{-1}+r^{-1}=1$. That is, $r=p/(p-2)$. But $\left(k_t^{p/2-1}\right)^r=k_t^{p/2}\leqslant (|g_t|+\varepsilon^{1/2})^p$ which is indeed integrable. So the integrand on the right-hand side of (4.3) is integrable in all cases. Its continuity as a function of t into L^1 is elementary.

In order to prove differentiability of the map $t\to k_t^{p/2}$ into L^1 and thereby justify the preceding computation write $\varphi(u)=(|u|^2+\varepsilon)^{p/2}$ for complex u and put $\psi(s)=\varphi(u_0+s(u-u_0))$. The identity $\psi(1)=\psi(0)+\psi'(0)+\int_0^1(1-s)\psi''(s)\,ds$, combined with the easily established inequality

$$|\psi''(s)|\leqslant C_p|u-u_0|^2|u_0+s(u-u_0)|^{p/2-1},$$

shows, upon inserting $u=g_t(z)$ and $u_0=g_{t_0}(z)$, that

$$\|(\varphi(g_t)-\varphi(g_{t_0}))(t-t_0)^{-1}-p\operatorname{Re}(g_t-g_{t_0})(t-t_0)^{-1}\overline{g_{t_0}}\,[|g_{t_0}(z)|^2+\varepsilon]^{p/2-1}\|_1$$

$$\leqslant C_p\int_0^1(1-s)\,ds\int_M\frac{g_t(z)-g_{t_0}(z)}{t-t_0}\,(\overline{g_t(z)-g_{t_0}(z)}) \tag{4.4}$$

$$\times[|g_{t_0}+s(g_t-g_{t_0})|^2+\varepsilon]^{p/2-1}\,d\mu.$$

Now if $0<p\leqslant 2$ then the last factor in the integrand on the right is bounded by $\varepsilon^{p/2-1}$, the middle factor goes to zero in L^2-norm as $t\to t_0$, and the first factor converges in L^2 as $t\to t_0$. The right-hand side therefore goes to zero. The same argument shows that the second term on the left of (4.4) converges in L^1 to $-p\operatorname{Re}(-Ag_{t_0})(z)\overline{g_{t_0}(z)}k_{t_0}^{p/2-1}$. This establishes the lemma in case $0<p\leqslant 2$. For $p>2$ the argument is similar: the first factor in the integrand on the right-hand side of (4.4) converges in L^p-norm as $t\to t_0$ because f is in the L^p-domain of A. The second factor goes to zero in L^p-norm and the third factor remains bounded in L^r-norm for $r=p/(p-2)$ by the same argument noted in the first paragraph of this proof. $\square$

PROPOSITION 4.2 (integration by parts). *Let f be a holomorphic function on M. Choose $\varepsilon>0$ and write*

$$k(z)=|f(z)|^2+\varepsilon. \tag{4.5}$$

Assume that either $0<p\leqslant 2$ and f is in the L^2-domain of A, or $2<p<\infty$ and f is in the L^p-domain of A. Then

$$\int_M(Af)(z)\overline{f(z)}k^{p/2-1}\,d\mu=\int_M h(\partial f(z),\overline{\partial\{fk^{p/2-1}\}})\,d\mu(z),\quad 0<p<\infty, \tag{4.6}$$

and the integrand on the right-hand side is nonnegative.

Remark 4.3. In view of the definition (2.6) of A the identity (4.6) seems like little more than the definition. Of course, one must verify that $fk^{p/2-1}$ is in $\mathcal{D}(Q)$ to use (2.6). For $p>2$ some approximation scheme must be used. The following method of justification of this integration by parts formula relies heavily on the fact that f is holomorphic. In spite of the fact that f is in the L^p-domain of A I do not know, for example, that ∂f is in L^p, for $p>2$. Such information might be useful for some more general approach to the proof but would probably require detailed knowledge of the "coefficients" of the elliptic operator A and the use of some kind of Sobolev coercivity inequalities for the measure μ. Instead, positivity of the integrands for our approximation will play the key role. Such positivity fails in the nonholomorphic case.

Notation 4.4. Let $\varepsilon>0$, $p>0$ and $a>0$. Define a function $\tau_p\colon\mathbf{C}\to\mathbf{C}$ by

$$\tau_p(\zeta)=\zeta(|\zeta|^2+\varepsilon)^{p/2-1}, \quad \zeta\in\mathbf{C}. \tag{4.7}$$

Then $\tau_p\in C^\infty(\mathbf{C})$. The following easily computed derivatives will be used repeatedly:

$$\frac{\partial\tau_p}{\partial\zeta} = \left(\tfrac{1}{2}p|\zeta|^2+\varepsilon\right)(|\zeta|^2+\varepsilon)^{p/2-2}, \tag{4.8}$$

$$\frac{\partial\tau_p}{\partial\bar\zeta} = \left(\tfrac{1}{2}p-1\right)\zeta^2(|\zeta|^2+\varepsilon)^{p/2-2}. \tag{4.9}$$

Let ψ be a function in $C^\infty([0,\infty))$ such that

$$\psi(r)\geqslant 0 \quad\text{for all } r\in[0,\infty), \tag{4.10}$$

$$\psi(r)=r, \quad 0\leqslant r\leqslant a, \tag{4.11}$$

$$0\leqslant\psi'(r)\leqslant 1 \quad\text{for all } r, \tag{4.12}$$

$$\psi(r)=\text{constant} \quad\text{for } r\geqslant a+1. \tag{4.13}$$

Define

$$\sigma(r)=1 \quad\text{for } 0\leqslant r\leqslant a, \quad \sigma(r)=\frac{\psi(r)}{r} \quad\text{for } r>a. \tag{4.14}$$

Then we clearly have

$$0\leqslant\sigma(r)\leqslant 1 \quad\text{for all } r\geqslant 0 \quad\text{and}\quad \sigma(r)\leqslant\frac{a+1}{r} \quad\text{for } r\geqslant a+1. \tag{4.15}$$

Moreover $\sigma\in C^\infty([0,\infty))$ and $\sigma'(r)=\psi'(r)/r-\psi(r)/r^2$. It will be convenient to write this in the form

$$r\sigma'(r)=\psi'(r)-\sigma(r), \quad r\geqslant 0. \tag{4.16}$$

Define $\varphi_a \colon \mathbf{C} \to \mathbf{C}$ by

$$\varphi_a(w) = w\sigma(|w|), \quad w \in \mathbf{C}. \tag{4.17}$$

Then φ_a is in $C^\infty(\mathbf{C})$ and is bounded by (4.15). Moreover

$$\varphi_a(w) = w \quad \text{if } |w| \leqslant a. \tag{4.18}$$

The following derivatives are easily verified, using (4.16):

$$\begin{aligned}
\frac{\partial \varphi_a}{\partial w} &= \frac{1}{2}\left(\sigma(|w|) + \psi'(|w|)\right) \\
&= 1 \quad \text{if } |w| \leqslant a,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
\frac{\partial \bar{\varphi}_a(w)}{\partial w} &= \frac{1}{2} \cdot \left(\frac{\bar{w}}{|w|}\right)^2 (\psi'(|w|) - \sigma(|w|)) \\
&= 0 \quad \text{if } |w| \leqslant a.
\end{aligned} \tag{4.20}$$

LEMMA 4.5. *Suppose that $f \in \mathcal{D}(Q)$. If $0 < p \leqslant 2$ then $\tau_p \circ f \in \mathcal{D}(Q)$. If $0 < p < \infty$ then $\tau_p \circ \varphi_a \circ f \in \mathcal{D}(Q)$ for all $a > 0$.*

Proof. $\tau_p(0) = 0$. Moreover for $0 < p \leqslant 2$ the first derivatives of τ_p are bounded, as one sees from (4.8) and (4.9). An elementary and standard argument shows that if f_n is a sequence in $C_c^\infty(M)$ which converges in Q-graph norm to f then $\tau_p \circ f_n$ is a sequence in $C_c^\infty(M)$ which converges in Q-graph norm to $\tau_p \circ f$. For general p in $(0, \infty)$ the same argument applies to $\tau_p \circ \varphi_a$ because φ_a has bounded first derivatives by (4.19) and (4.20) ($\partial \varphi_a / \partial \bar{w}$ is the conjugate of (4.20)) and τ_p has bounded first derivatives on the range of φ_a. $\square$

Proof of Proposition 4.2. For $0 < p \leqslant 2$, $fk^{p/2-1} = \tau_p \circ f$ is in $\mathcal{D}(Q)$ by Lemma 4.5. Therefore (4.6) follows from (2.6) in this case. Moreover the positivity of the integrand on the right-hand side of (4.6) follows from the following computation which is valid for all f in $\mathcal{H}(M)$ and all $p \in (0, \infty)$:

$$\begin{aligned}
\partial\{f(z)k(z)^{p/2-1}\} &= (\partial f)k^{p/2-1} + f(z)\left[\tfrac{1}{2}p - 1\right]k^{p/2-2}\partial k \\
&= k^{p/2-2}\left\{k\,\partial f + \left[\tfrac{1}{2}p - 1\right]f(z)\overline{f(z)}\,\partial f\right\} \\
&= k^{p/2-2}\left\{|f|^2 + \varepsilon + \left[\tfrac{1}{2}p - 1\right]|f|^2\right\}\partial f.
\end{aligned} \tag{4.21}$$

Therefore

$$\partial\{f(z)k(z)^{p/2-1}\} = k^{p/2-2}\left\{\tfrac{1}{2}p|f|^2 + \varepsilon\right\}\partial f, \quad 0 < p < \infty,\ f \in \mathcal{H}(M). \tag{4.22}$$

It seems worth emphasizing at this point that this computation contains the distinction between the holomorphic and nonholomorphic theories. (4.22) now gives

$$h(\partial f(z), \overline{\partial\{f(z)k^{p/2-1}\}}) = k^{p/2-2}\{\tfrac{1}{2}p|f|^2+\varepsilon\}\,h(\partial f,\overline{\partial f})$$
$$\geqslant 0, \quad 0<p<\infty,\ f\in\mathcal{H}(M).$$
(4.23)

Next, suppose that $p>2$. Suppose that f is holomorphic and is in the L^p-domain of A. Then f is also in the L^2-domain of A. In order to justify the integration by parts (4.6) we will tamper with the factors $fk^{p/2-1}$ on both sides of (4.6) while leaving the factors Af and ∂f unchanged. Note first that $\tau_p\circ\varphi_a\circ f$ is in $\mathcal{D}(Q)$ by Lemma 4.5. Therefore, since f is in the L^2-domain of A, (2.6) gives

$$\int_M (Af)(z)\overline{(\tau_p\circ\varphi_a\circ f)(z)}\,d\mu(z) = \int h(\partial f(z), \overline{\partial\{(\tau_p\circ\varphi_a\circ f)(z)\}})\,d\mu(z).$$
(4.24)

Observe next that as $a\to\infty$, $(\tau_p\circ\varphi_a\circ f)(z)$ converges to $(\tau_p\circ f)(z)\equiv fk^{p/2-1}$ for all $z\in M$ by (4.18). Moreover since $|\varphi_a(w)|\leqslant|w|$ we have $|(\tau_p\circ\varphi_a\circ f)(z)|\leqslant|f(z)|k^{p/2-1}$ for all a and z. We have already seen, in the proof of Lemma 4.1, that $|Af(z)|\cdot|f(z)|k^{p/2-1}$ is integrable. Hence, by the dominated convergence theorem the left-hand side of (4.24) converges to the left-hand side of (4.6) as $a\to\infty$. It therefore suffices to show convergence of the right-hand side of (4.24) to the right-hand side of (4.6). As already noted in Remark 4.3 the difficulty lies in a lack of knowledge of the L^p-behavior of ∂f. We will show and use positivity of the integrand on the right-hand side of (4.24) to show that the integral is well approximated by the integral over $\{z:|f(z)|<a\}$. On this set the function φ_a can be removed. This will allow us to take the limit as $a\to\infty$.

By the chain rule we have

$$\partial\{\tau_p\circ\varphi_a\circ f\} = \frac{\partial\tau_p}{\partial\zeta}\partial(\varphi_a\circ f) + \frac{\partial\tau_p}{\partial\bar\zeta}\partial\overline{\varphi_a\circ f} = \frac{\partial\tau_p}{\partial\zeta}\cdot\frac{\partial\varphi_a}{\partial w}\partial f + \frac{\partial\tau_p}{\partial\bar\zeta}\cdot\frac{\partial\bar\varphi_a}{\partial w}\partial f$$

since $\partial\bar f=0$. Applying equations (4.8) and (4.9) we find

$$\partial\{\tau_p\circ\varphi_a\circ f\}(z) = (|\zeta|^2+\varepsilon)^{p/2-2}\Big\{\big[\tfrac{1}{2}p|\zeta|^2+\varepsilon\big]\frac{\partial\varphi_a}{\partial w}\partial f + \big[\tfrac{1}{2}p-1\big]\zeta^2\frac{\partial\bar\varphi_a}{\partial w}\partial f\Big\}$$

where $\zeta=\varphi_a(w)=w\sigma(|w|)$ and $w=f(z)$. Applying equations (4.19) and (4.20) it follows that

$$\partial\{\tau_p\circ\varphi_a\circ f\}(z) = (|\zeta|^2+\varepsilon)^{p/2-2}\Big\{\big[\tfrac{1}{2}p|\zeta|^2+\varepsilon\big]\tfrac{1}{2}(\sigma(|w|)+\psi'(|w|))$$
$$+\big[\tfrac{1}{2}p-1\big]\zeta^2\cdot\tfrac{1}{2}(\bar w/|w|)^2(\psi'(|w|)-\sigma(|w|))\Big\}\partial f(z)$$
$$= (|w|^2\sigma(|w|)^2+\varepsilon)^{p/2-2}\Big\{\big[\tfrac{1}{2}p|w|^2\sigma(|w|)^2+\varepsilon\big]\tfrac{1}{2}(\sigma+\psi')$$
$$+\big[\tfrac{1}{2}p-1\big]\sigma(|w|)^2\tfrac{1}{2}|w|^2(\psi'-\sigma)\Big\}\partial f(z)$$

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where $w=f(z)$. Note that in spite of the appearance of $|w|^{-2}$ at an intermediate step in the computation the end result is valid for all w because the term $(\overline{w}/|w|)^2$ is absent for $|w|<a$, as we see from the second line in (4.20). The last expression in braces is $\{|w|^2\sigma^2[\frac{1}{2}\sigma+\frac{1}{2}(p-1)\psi']+\varepsilon\cdot\frac{1}{2}(\sigma+\psi')\}$. Therefore,

$$h(\partial f(z),\overline{\partial\{\tau_p\circ\varphi_a\circ f\}(z)}) = h(\partial f(z),\overline{\partial f(z)})(|w|^2\sigma(|w|)^2+\varepsilon)^{p/2-2}$$
$$\times\{|w|^2\sigma(|w|)^2\cdot\frac{1}{2}[\sigma+(p-1)\psi']+\varepsilon\cdot\frac{1}{2}(\sigma+\psi')\} \tag{4.25}$$

where $w=f(z)$. Since σ and ψ' are both nonnegative and $p\geqslant 1$ all the terms on the right-hand side of (4.25) are nonnegative. Hence the integrand on the right-hand side of (4.24) is nonnegative. Since $(\tau_p\circ\varphi_a\circ f)(z)=(\tau_p\circ f)(z)$ when $|f(z)|<a$ the nonnegativity of the integrand now shows that

$$\int_{|f(z)|<a}h(\partial f,\overline{\partial(\tau_p\circ f)})\,d\mu \leqslant \int_M Af(z)\overline{(\tau_p\circ\varphi_a\circ f)(z)}\,d\mu(z).$$

Letting $a\to\infty$ gives

$$\int_M h(\partial f,\overline{\partial\{\tau_p\circ f\}})\,d\mu \leqslant \int_M Af(z)\overline{(\tau_p\circ f)(z)}\,d\mu. \tag{4.26}$$

We have already shown that the right-hand side is finite. Returning now to (4.25) observe that $\sigma(|f(z)|)\leqslant 1$ and $\psi'(|f(z)|)\leqslant 1$, and as $a\to\infty$, $\sigma(|f(z)|)\to 1$ and $\psi'(|f(z)|)\to 1$ for all z in M. Therefore the right-hand side of (4.25) converges to the right-hand side of (4.23) for all z, and moreover is dominated by the right-hand side of (4.23) for all a. In view of (4.26) we may now apply the dominated convergence theorem to the right-hand side of (4.24) to conclude the proof of Proposition 4.2.   □

LEMMA 4.6. *Let $\varepsilon>0$, let $f\in\mathcal{H}(M)$ and write $k(z)=|f(z)|^2+\varepsilon$. Then*

$$h(dk^{p/4},dk^{p/4})+\tfrac{1}{4}p\varepsilon k^{p/2-2}h(\partial f,\overline{\partial f}) = \tfrac{1}{4}ph(\partial f,\overline{\partial\{fk^{p/2-1}\}}), \tag{4.27}$$

for $0<p<\infty$, $z\in M$, $f\in\mathcal{H}(M)$. In particular,

$$h(dk^{p/4},dk^{p/4}) \leqslant \tfrac{1}{4}ph(\partial f,\overline{\partial\{fk^{p/2-1}\}}), \quad 0<p<\infty. \tag{4.28}$$

Proof.

$$dk^{p/4} = \tfrac{1}{4}pk^{p/4-1}d|f|^2 = \tfrac{1}{4}pk^{p/4-1}(\bar{f}\,\partial f+f\,\overline{\partial f}).$$

Since $h(\partial f(z),\partial f(z))=h(\overline{\partial f(z)},\overline{\partial f(z)})=0$ we find

$$h(dk^{p/4},dk^{p/4}) = \left(\tfrac{1}{4}p\right)^2 k^{p/2-2}\cdot 2|f|^2 h(\partial f,\overline{\partial f}). \tag{4.29}$$

Multiply (4.23) by $\tfrac{1}{4}p$, and use (4.29) to get (4.27).   □

COROLLARY 4.7. *Let* $f \in \mathcal{H}(M)$. *Assume that either* $0 < p \leqslant 2$ *and* f *is in the* L^2-*domain of* A, *or* $p > 2$ *and* f *is in the* L^p-*domain of* A. *Then* $k^{p/4}$ *is in* $\mathcal{D}(Q)$ *and*

$$\int_M h(dk^{p/4}, dk^{p/4}) \, d\mu(z) \leqslant \tfrac{1}{4} p \int_M (Af)(z) \overline{f(z)} k^{p/2-1} \, d\mu(z). \qquad (4.30)$$

Proof. Combine (4.28) with (4.6). □

The following proposition provides the transition from a logarithmic Sobolev inequality of index 2, namely (2.29), to a logarithmic Sobolev inequality of index p, namely (4.32). It is the holomorphic analog of [G1, Lemma 4.6].

PROPOSITION 4.8 (holomorphic transition lemma). *Suppose that the logarithmic Sobolev inequality* (2.29) *holds. Let* $f \in \mathcal{H}(M)$ *and choose* $\varepsilon > 0$. *Assume that* $0 < p \leqslant 2$ *and* f *is in the* L^2-*domain of* A, *or that* $p > 2$ *and* f *is in the* L^p-*domain of* A. *Let*

$$\gamma(z) = (|f(z)|^2 + \varepsilon)^{1/2}. \qquad (4.31)$$

Then

$$\int_M \gamma^p \log \gamma \, d\mu \leqslant \tfrac{1}{2} c(Af, f\gamma^{p-2}) + \frac{2\beta}{p} \|\gamma\|_p^p + \|\gamma\|_p^p \log \|\gamma\|_p, \quad 0 < p < \infty. \qquad (4.32)$$

Proof. Whether $0 < p \leqslant 2$ or $2 < p < \infty$, $\gamma^{p/2} \equiv k^{p/4}$ is in $\mathcal{D}(Q)$ by Corollary 4.7. Replacing f in (2.29) by $\gamma^{p/2}$ and using (4.30) we find, replacing $k^{p/4}$ by $\gamma^{p/2}$ in all previous formulas,

$$\int_M \gamma^p \log \gamma^{p/2} \, d\mu \leqslant c \int h(d\gamma^{p/2}, d\gamma^{p/2}) \, d\mu + \beta \|\gamma^{p/2}\|_2^2 + \|\gamma^{p/2}\|_2^2 \log \|\gamma^{p/2}\|_2$$

$$\leqslant c \cdot \tfrac{1}{4} p(Af, f\gamma^{p-2}) + \beta \|\gamma\|_p^p + \tfrac{1}{2} p \|\gamma\|_p^p \log \|\gamma\|_p.$$

Multiplying by $2/p$ gives (4.32). □

Remark 4.9. To understand the distinction between the holomorphic and nonholomorphic cases one should compare (4.32) to [G1, equation (4.9)]. When $\varepsilon = 0$ in (4.31) the function $f\gamma^{p-2}$ reduces to $f|f|^{p-2}$ which was denoted f_p in [G1]. The coefficient $\tfrac{1}{2} cp/(p-1)$ in [G1, equation (4.9)] has been replaced by $\tfrac{1}{2} c$ in (4.32), allowing the method of [G1] to apply to all $p \in (0, \infty)$ not just $p \in (1, \infty)$. The origin of the difference in these inequalities lies in the difference between the calculus inequality [G1, equation (4.8)] (which has best possible coefficient in the C^∞-category) and the holomorphic calculus inequality (4.28) which (if one puts $\varepsilon = 0$ and stays away from the zeroes of f) is actually an equality, as one sees from (4.27). In one real, respectively complex, variable the

distinction derives from the difference between the identities $(d/dx)f^{p-1}=(p-1)f^{p-2}f'$
for $0<f\in\operatorname{Real} C^\infty(\mathbf{R})$ and $(\partial/\partial z)(f|f|^{p-2})=\frac{1}{2}p|f|^{p-2}f'$ when $f\in\mathcal{H}(C)$.

Remark 4.10. There is a seeming self-improvement in (4.32) when $p=2$ over the
hypothesized inequality (2.29). Take $p=2$ and put $\varepsilon=0$, ignoring zeroes of f. One gets
the same inequality (2.29) that one started with but with c replaced by $\frac{1}{2}c$. The reason
for this is that, wherever $f(z)\neq0$, $h(d|f|,d|f|)=\frac{1}{2}h(\partial f,\overline{\partial f})$ if f is holomorphic.

Proof of Theorem 2.15. Let $f\in\mathcal{H}\cap\mathcal{D}(A)$. By Theorem 2.11, $e^{-tA}f\in\mathcal{D}(A)\cap\mathcal{H}^2=\mathcal{H}\cap\mathcal{D}(A)$ for all $t\geqslant0$. Let $0<p\leqslant2$.

Let $g_t(z)=(e^{-tA}f)(z)$ and

$$\gamma_t(z) = (|g_t(z)|^2+\varepsilon)^{1/2}. \tag{4.33}$$

Then by Lemma 4.1 and Corollary 4.7,

$$\frac{d}{dt}\int_M \gamma_t(z)^p\,d\mu = -p\operatorname{Re}\int_M (Ag_t)(z)\overline{g_t(z)}\gamma_t^{p-2}\,d\mu \leqslant 0.$$

Hence

$$\|\gamma_t\|_p^p \leqslant \|\gamma_0\|_p^p, \quad t\geqslant0.$$

We may now let $\varepsilon\downarrow0$ to obtain

$$\|e^{-tA}f\|_p \leqslant \|f\|_p, \quad 0<p\leqslant2.$$

Since the L^p-norm (or metric) is continuous with respect to the L^2-norm and since, by
Theorem 2.11, $\mathcal{H}^2\cap\mathcal{D}(A)$ is dense in $\mathcal{H}^2$, the inequality (2.28) follows for $s=0$ for all $f\in\mathcal{H}^2$.
Applying that inequality to $e^{isA}f$ with $f\in\mathcal{H}^2$ and using $\|e^{isA}f\|_p=\|f\circ\exp sY\|_p=\|f\|_p$
for $0<p\leqslant2$ when $f\in\mathcal{H}^2$ the inequality (2.28) follows for all real s. $\square$

Proof of Theorem 2.16. Let $0<p<2$ and $\operatorname{Re}\zeta\geqslant0$. By Theorem 2.15, $e^{-\zeta A}$ is a
contraction in L^p-norms on $\mathcal{H}^2$. Since $\mathcal{H}^2$ is by definition dense in $\mathcal{H}^p$, it follows that
$e^{-\zeta A}$ has a unique continuous extension to $\mathcal{H}^p$. We continue to denote the extension by
$e^{-\zeta A}$. On the imaginary ζ-axis we have $e^{itA}f=f\circ\exp tY$ for $f\in\mathcal{H}^2$ by Theorem 2.11.
Since the flow $\exp tY$ preserves the measure μ it follows that $\|e^{itA}f\|_{L^p}=\|f\|_{L^p}$ for $f\in\mathcal{H}^2$
and $0<p<2$. The continuous extension of e^{itA} to $\mathcal{H}^p$ is therefore isometric on $\mathcal{H}^p$ for
$0<p<2$. The group of maps V_t defined in (3.13) is a strongly continuous isometry group
in $\mathcal{H}^p$ by Lemma 3.3. Since V_t and e^{itA} agree on $\mathcal{H}^2$ they also agree on $\mathcal{H}^p$. Hence e^{itA}
is also a strongly continuous isometry group on $\mathcal{H}^p$ for $0<p<2$.

Next, let $2<p<\infty$. For $t\geqslant0$, $e^{-tA}\mathcal{H}^2\subset\mathcal{H}^2$ by Theorem 2.11. As already noted
at the beginning of §2, e^{-tA} is a contraction on all of L^p. In particular, $e^{-tA}\mathcal{H}^p=$

$e^{-tA}(\mathcal{H}^2 \cap L^p) \subset \mathcal{H}^2 \cap L^p = \mathcal{H}^p$. That is, $\mathcal{H}^p$ is invariant under e^{-tA}. Furthermore e^{itA} is a strongly continuous isometry group on $\mathcal{H}^p$ by Lemma 3.4. Thus $e^{-\zeta A}$ is a contraction on $\mathcal{H}^p$ when $\operatorname{Re}\zeta \geqslant 0$.

It remains to prove that for all $p \in (0, \infty)$ and $\operatorname{Re}\zeta \geqslant 0$, $e^{-\zeta A}\mathcal{H}^p$ is dense in $\mathcal{H}^p$. This will clearly imply, by general semigroup theory, that $\mathcal{H} \cap \mathcal{D}(A_p)$ is dense in $\mathcal{H}^p$ in the Banach space cases $1 \leqslant p < \infty$. But e^{itA} is an isometry group in $\mathcal{H}^p$ for all $p \in (0, \infty)$. Therefore it suffices to prove that $e^{-tA}\mathcal{H}^p$ is dense in $\mathcal{H}^p$. For $p \geqslant 2$, e^{-tA} is a strongly continuous semigroup in the ambient spaces L^p and therefore also in the invariant subspaces $\mathcal{H}^p$. Hence $e^{-tA}\mathcal{H}^p$ is dense in $\mathcal{H}^p$. For $0 < p < 2$ we have $e^{-tA}\mathcal{H}^p \supset e^{-tA}\mathcal{H}^2$, which is dense in $\mathcal{H}^2$, which in turn is dense in $\mathcal{H}^p$. Finally, the equations (2.25) and (2.26) hold for $f \in \mathcal{H}^p$ because $\mathcal{H}^p \subset \mathcal{H}^2$ for $p \geqslant 2$, while for $0 < p < 2$ any L^p-convergent sequence of holomorphic functions converges also pointwise. $\qquad\square$

Proof of Theorem 2.17. By the spectral theorem for $A|\mathcal{H}^2$, $\mathcal{R}$ is dense in $\mathcal{H}^2$. Since $\mathcal{H}^2$ is dense in $\mathcal{H}^q$ for $0 < q \leqslant 2$, $\mathcal{R}$ is also dense in $\mathcal{H}^q$ for $0 < q \leqslant 2$. Thus for $0 < q \leqslant 2$ we do not require the logarithmic Sobolev inequality (2.29). For $q > 2$ we are going to make use of the known hypercontractivity of the semigroup e^{-tA} in the full L^p-spaces. Fix $q > 2$. The inequality (2.29) implies that for some $t > 0$, depending on c and q, $e^{-tA}L^2 \subset L^q$ and e^{-tA} is bounded from L^2 into L^q. A proof of this can be given following the method established in [G1]. But the domain considerations that influence the transition lemma arguments in [G1] have been handled better since then. For an efficient approach, based partly on work of D. Stroock, see [G5, Sections 3, 4 and 5, and especially the method of proof of Theorem 5.4] or Theorem 6.1.14 in [DeS]. Now if $f \in \mathcal{R}$ then $f \in H_a$ for some $a > 0$. Thus f is in the domain of the unbounded operator e^{tA}. Let $g = e^{tA}f$. Then $g \in H_a$ also and $f = e^{-tA}g$. So $e^{-tA}\mathcal{R} = \mathcal{R}$. Therefore $\mathcal{R} \subset \mathcal{H}^q$. Suppose that $\mathcal{R}$ is not dense in $\mathcal{H}^q$. Then there exists a continuous linear functional $F: \mathcal{H}^q \to \mathbf{C}$ such that $F(\mathcal{R}) = 0$. Thus $F \circ e^{-tA}\mathcal{R} = 0$. Since $F \circ e^{-tA}$ is continuous on $\mathcal{H}^2$ and $\mathcal{R}$ is dense in $\mathcal{H}^2$, we have $F \circ e^{-tA} = 0$ on $\mathcal{H}^2$ and therefore also on $\mathcal{H}^q$. But by Theorem 2.16, $e^{-tA}\mathcal{H}^q$ is dense in $\mathcal{H}^q$. Hence $F = 0$. Therefore $\mathcal{R}$ is dense in $\mathcal{H}^q$. The density of $\bigcap_{p < \infty} \mathcal{H}^p$ in $\mathcal{H}^q$ now follows.

The fact that $\mathcal{R}$ is an algebra is clear at an informal level because if u and v are eigenfunctions of $A|\mathcal{H}^2$ then u and v are in $\mathcal{H}^4$. So uv is in $\mathcal{H} \cap L^2 = \mathcal{H}^2$. If $Au = \lambda_1 u$ and $Av = \lambda_2 v$ then, by Theorem 2.11 (b), $Z(uv) = (Zu)v + u(Zv) = (\lambda_1 + \lambda_2)(uv)$, which is in L^2. It now follows from Theorem 2.11 (e) that $uv \in \mathcal{D}(A)$, and therefore $A(uv) = (\lambda_1 + \lambda_2)uv$. This would constitute a proof of (2.30) if one knew that $A|\mathcal{H}^2$ had compact resolvent. I am going to give a different proof, partly because I don't have a proof that A has compact resolvent (see [DaS, Appendix A]), and partly because most of the structures and theorems of this paper are likely to go over to infinite-dimensional M, where discrete

spectrum definitely fails in interesting cases, [BSZ].

To prove that $\mathcal{R}$ is an algebra observe that since pointwise evaluation is a continuous linear functional on $\mathcal{H}^2$ there is, for each $z \in M$, a unique vector $w_z \in \mathcal{H}^2$ such that $f(z) = (f, w_z)$ for all $f \in \mathcal{H}^2$. The set $\{w_z\}_{z \in M}$ is fundamental in $\mathcal{H}^2$ because $(f, w_z) = 0$ for all $z \in M$ implies $f(z) = 0$ for all z when $f \in \mathcal{H}^2$. Let $A = \int_0^\infty \lambda\, dE(\lambda)$ be the spectral resolution of $A | \mathcal{H}^2$. If $u \in \mathcal{H}^2$ then $u \in H_a$ if and only if the complex measure $(E(\cdot)u, w)$ is supported in $[0, a]$ for all $w \in \mathcal{H}^2$. If B is a Borel subset of (a, ∞) then $(E(B)u, w) = 0$ for all $w \in \mathcal{H}^2$ if and only if $(E(B)u, w_z) = 0$ for all $z \in M$. Thus $u \in H_a$ if and only if $(E(\cdot)u, w_z)$ is supported in $[0, a]$ for all $z \in M$. Now suppose that $u \in H_a$ and that $v \in H_b$. Then u and v are both in $\mathcal{H} \cap L^4$ by the first part of the theorem. So $uv \in \mathcal{H} \cap L^2$. By assumption, $\mathcal{H} \cap L^2 = \mathcal{H}^2$. So $uv \in \mathcal{H}^2$. Thus we have

$$
\begin{aligned}
\int_0^\infty e^{is\lambda}(E(d\lambda)(uv), w_z) &= (e^{isA}(uv), w_z) \\
&= (e^{isA}(uv))(z) = (uv)(\exp(sY)z) \\
&= u(\exp(sY)z)v(\exp(sY)z) = (e^{isA}u)(z)(e^{isA}v)(z) \\
&= \left[\int_0^\infty e^{is\lambda}(E(d\lambda)u, w_z)\right]\left[\int_0^\infty e^{is\lambda}(E(d\lambda)v, w_z)\right].
\end{aligned}
$$

Thus the Fourier transform of $(E(\cdot)(uv), w_z)$ is the product of the Fourier transforms of two measures with respective supports in $[0, a]$ and $[0, b]$. Therefore $(E(\cdot)(uv), w_z)$ is supported in $[0, a+b]$. This proves (2.30) and shows that $\mathcal{R}$ is an algebra.

A function u in $\mathcal{H}^2$ is in $\mathcal{H}_\infty$ if and only if $e^{isA}u$ is an infinitely differentiable function of s into L^p for all $p \in [2, \infty)$. Since the L^2-norm dominates the L^p-norm on each subspace H_a, it follows that $\mathcal{R} \subset \mathcal{H}_\infty$ and therefore that $\mathcal{H}_\infty$ is dense in $\mathcal{H}^p$ for $2 \leqslant p < \infty$. It remains to show that $\mathcal{H}_\infty$ is an algebra. Now, if u and v are in $\mathcal{H}^{2p}$ then uv is in $\mathcal{H} \cap L^p$ by Schwarz' inequality. Since $\mathcal{H} \cap L^p = \mathcal{H}^p$ for p in $[2, \infty)$, we may conclude that if u and v are in $\mathcal{H}_\infty$ then uv is in $\mathcal{H}^p$ for all $p \in [2, \infty)$. By Theorem 2.11 we therefore have $e^{isA}(uv) = (e^{isA}u)(e^{isA}v)$. Since both factors are infinitely differentiable functions of s into L^{2p} the product is an infinitely differentiable function of s into L^p. $\qquad \square$

It may be useful to note that there are two more natural algebras similar to $\mathcal{R}$ and $\mathcal{H}_\infty$ present in these structures. They may be defined as $\mathcal{H}_\infty$ was, but replacing C^∞-vectors for $A_p | \mathcal{H}^p$ by entire vectors, and analytic vectors respectively, for $A_p | \mathcal{H}^p$. The proof that these spaces are algebras is similar to that for $\mathcal{H}_\infty$. Clearly they lie between $\mathcal{R}$ and $\mathcal{H}_\infty$.

Proof of Theorem 2.19. Let $0 < q < p < \infty$. Define $p(t) = qe^{2t/c}$. Then $p'(t) = (2/c)p(t)$. Choose r so that $\max(2, p) \leqslant r < \infty$. $\mathcal{H} \cap \mathcal{D}(A_r)$ is dense in $\mathcal{H}^q$ because it is dense in $\mathcal{H}^r$ (by Theorem 2.16) which is dense in $\mathcal{H}^q$ by Theorem 2.17. It suffices therefore to prove

(2.33) for $f \in \mathcal{H} \cap \mathcal{D}(A_r)$. Choose $f \in \mathcal{H} \cap \mathcal{D}(A_r)$ and let $g_t(z) = (e^{-tA}f)(z)$. Then $g_t \in \mathcal{H} \cap \mathcal{D}(A_r) \subset \mathcal{H} \cap \mathcal{D}(A_s)$ for $2 \leqslant s \leqslant r$ and all $t \geqslant 0$. Choose $\varepsilon > 0$ and define γ_t by (4.33). Define

$$v(t) = \int \gamma_t(z)^{p(t)} \, d\mu$$

and $\alpha(t) = \|\gamma_t\|_{p(t)}$. Then, suppressing the t-dependence and denoting d/dt by a prime we have, just as in [G1, Section 2],

$$\alpha'(t) = \frac{dv(t)^{1/p(t)}}{dt} = p^{-1}v^{(1/p)-1}v' - \frac{p'}{p^2}v^{1/p}\log v$$

$$= \alpha v^{-1}\left\{p^{-1}v' - \frac{2}{c}v\log\alpha\right\}.$$

But by Lemma 4.1 and (4.32),

$$p^{-1}v' - \frac{2}{c}v\log\alpha = p^{-1}\left[p'\int \gamma_t^{p(t)}\log\gamma_t \, d\mu - p(Ag_t, g_t\gamma_t^{p-2})\right] - \frac{2}{c}v\log\alpha$$

$$= \frac{2}{c}\left\{\int \gamma_t^{p(t)}\log\gamma_t \, d\mu - \|\gamma_t\|_p^p\log\|\gamma_t\|_p\right\} - (Ag_t, g_t\gamma_t^{p-2})$$

$$\leqslant \frac{4\beta}{cp}\|\gamma_t\|_p^p.$$

Hence

$$\alpha'(t) \leqslant \frac{4\beta}{cp(t)}\alpha(t).$$

So $(d/dt)\log\alpha(t) \leqslant 4\beta e^{-2t/c}/(cq)$. This gives upon integration

$$\alpha(t) \leqslant \alpha(0)\exp[(2\beta q^{-1})(1-e^{-2t/c})] = \alpha(0)\exp[2\beta(q^{-1}-p(t)^{-1})].$$

Since $f \in \mathcal{H} \cap \mathcal{D}(A_r)$ this computation is valid at least up to the time t when $p(t) = p$. That is, up to $t = t_J$. (2.33) now follows for $t = t_J$ by letting $\varepsilon \downarrow 0$. For $t > t_J$ we have $\|e^{-tA}f\|_p = \|e^{-(t-t_J)A}e^{-t_JA}f\|_p \leqslant \|e^{-t_JA}f\|_p$ (by Theorem 2.16), which is less than or equal to $M(p,q)\|f\|_q$. $\square$

Proof of Corollary 2.20. If $\zeta = t + is$ then the inequality $|e^{-\zeta}| \leqslant (q/p)^{c/2}$ is equivalent to $t \geqslant t_J$. Since e^{isA} is an isometry on all $\mathcal{H}^p$-spaces the corollary follows from (2.33). $\square$

5. Holomorphic Dirichlet forms on $\mathbf{C}^m$: examples and counterexamples

It is a severe restriction on the triple (M, g, μ) that d^*d be holomorphic. In Example 5.1 we will take $M = \mathbf{C}^m$ and present a class of Hermitian metrics, g, and probability measures, μ, on $\mathbf{C}^m$ for which d^*d is holomorphic. Some of the examples in this class will

be shown to satisfy a logarithmic Sobolev inequality. The main theorem of this paper
is therefore applicable in these cases. The algebra $\mathcal{R}$ of Theorem 2.17 will be identified
(with holomorphic polynomials) and E. Carlen's theorem on the density of holomorphic
polynomials in $\mathcal{H}^p$ will be extended (Theorem 5.5).

In order to appreciate how severely the triple (M, g, μ) is restricted by the require-
ment that d^*d be holomorphic, we will take $M = \mathbf{C}$ in Example 5.6 and give a Gaussian
probability measure μ on $\mathbf{C}$ for which there exists no Hermitian metric g such that d^*d
is holomorphic and such that $\mathcal{H}^2$ is nontrivial. This example is particularly interest-
ing because μ will be taken to be just the heat kernel for a "slightly wrong" Laplacian
on $\mathbf{C}$. This example should be regarded as a guide as to where to seek (more precisely,
where not to seek) measures μ in the form of heat kernels on other complex manifolds
for constructing examples of holomorphic Dirichlet forms. Such heat kernels often satisfy
logarithmic Sobolev inequalities [DH] and seem, therefore, to offer an interesting source
of densities for hypercontractivity over other complex manifolds.

In Example 5.7 a Hermitian metric on $\mathbf{C}$ will be given for which $\mathcal{H}^2$ is necessarily
trivial for any smooth probability measure whose associated Dirichlet form operator d^*d
is holomorphic.

Thus the condition that d^*d be holomorphic imposes constraints on the Hermitian
metric g and on the measure μ separately, as well as a constraint on their relationship.

All of the examples in this section will have the following structure. Let $M = \mathbf{C}^m$.
Denote by $x_1, ..., x_{2m}$ the standard linear coordinates on $\mathbf{C}^m \equiv \mathbf{R}^{2m}$ with $z_k = x_{2k-1} + ix_{2k}$,
$k = 1, 2, ..., m$. Let σ be a strictly positive function in $C^\infty(\mathbf{R}^{2m})$. Define a metric on $\mathbf{R}^{2m}$
by

$$g_x\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = \frac{\delta_{jk}}{\sigma(x)}. \tag{5.1}$$

Then g is a Hermitian metric. That is, it is invariant under the almost complex structure
of $\mathbf{C}^m$. The dual metric is

$$h_x(dx_j, dx_k) = \delta_{jk}\sigma(x), \quad j, k = 1, ..., 2m.$$

g extends complex bilinearly to $T(\mathbf{R}^{2m}) \otimes_{\mathbf{R}} \mathbf{C}$ and its dual metric satisfies

$$h_x(dz_r, d\bar{z}_s) = 2\delta_{rs}. \tag{5.2}$$

Write $dx = dx_1 ... dx_{2m}$. μ will denote a probability measure on $\mathbf{R}^{2m}$ with a smooth
positive density. Thus we put

$$d\mu(x) = \varrho(x)\, dx, \tag{5.3}$$

where ϱ is a strictly positive function in $C^\infty(\mathbf{R}^{2m})$ with integral equal to one. It will be convenient to express the Dirichlet form for $(\mathbf{C}^m, g, \mu)$ in terms of ϱ and w, where

$$w(x) = \sigma(x)\varrho(x), \quad x \in \mathbf{R}^{2m}. \tag{5.4}$$

The Dirichlet form (2.3) may be written

$$Q_0(f, \psi) = \int_{\mathbf{R}^{2m}} h(df, d\bar{\psi})\varrho(x)\,dx, \quad f \in C^\infty(\mathbf{R}^{2m}),\ \psi \in C_c^\infty(\mathbf{R}^{2m}).$$

Writing

$$df \cdot d\bar{\psi} = \sum_{j=1}^{2m} \frac{\partial f}{\partial x_j} \cdot \frac{\partial \bar{\psi}}{\partial x_j}$$

and using (5.1) and (5.4) we may write this as

$$Q_0(f, \psi) = \int_{\mathbf{R}^{2m}} df \cdot d\bar{\psi}\, w(x)\,dx, \quad f \in C^\infty(\mathbf{R}^{2m}),\ \psi \in C_c^\infty(\mathbf{R}^{2m}). \tag{5.5}$$

If f is holomorphic then

$$Q_0(f, \psi) = \int_{\mathbf{R}^{2m}} \partial f \cdot \overline{\partial \psi}\, w(x)\,dx = 2\int \sum_{j=1}^m \frac{\partial f}{\partial z_j} \cdot \overline{\frac{\partial \psi}{\partial z_j}}\, w(x)\,dx$$

$$= -2\int \left[\sum_{j=1}^m \frac{\partial}{\partial \bar{z}_j}\left(w\frac{\partial f}{\partial z_j}\right)\right] \overline{\psi(x)}\,dx.$$

But $\partial^2 f/\partial \bar{z}_j \partial z_j = 0$ for each j. Hence

$$Q_0(f, \psi) = -2\int_{\mathbf{R}^{2m}} \varrho(x)^{-1}\left[\sum_{j=1}^m \frac{\partial w}{\partial \bar{z}_j} \cdot \frac{\partial f}{\partial z_j}\right] \overline{\psi(x)}\,\varrho(x)\,dx.$$

In view of (2.2) one therefore has

$$(d^*df)(z) = -2\sum_{j=1}^m \varrho(x)^{-1}\frac{\partial w}{\partial \bar{z}_j} \cdot \frac{\partial f}{\partial z_j}, \quad f \in \mathcal{H}(\mathbf{C}^m). \tag{5.6}$$

Inserting $f(z) = z_k$ one sees that d^*d is holomorphic if and only if each coefficient

$$\varrho(x)^{-1}\frac{\partial w}{\partial \bar{z}_k} \quad \text{is holomorphic for } k = 1, ..., m. \tag{5.7}$$

Example 5.1. Here is a class of functions w and ϱ satisfying (5.7). Suppose that φ is a strictly positive function in $C^\infty([0, \infty))$. Assume further that its derivative φ' satisfies $\varphi'(s) < 0$ for $0 \leqslant s < \infty$. Define

$$w(z) = \varphi(|z|^2), \tag{5.8}$$

$$\varrho(z) = -b\varphi'(|z|^2), \quad b = \text{constant} > 0, \tag{5.9}$$

$$\sigma(z) = \frac{w(z)}{\varrho(z)}. \tag{5.10}$$

Then $\partial w/\partial \bar{z}_k = z_k \varphi'(|z|^2)$. So

$$\varrho(z)^{-1}\frac{\partial w}{\partial \bar{z}_k} = -b^{-1}z_k. \tag{5.11}$$

Thus (5.7) holds and

$$(d^*df)(z) = 2b^{-1}\sum_{k=1}^{m} z_k \frac{\partial f}{\partial z_k}, \quad f \in \mathcal{H}(\mathbf{C}^m). \tag{5.12}$$

Hence d^*d is holomorphic for $(\mathbf{C}^m, \sigma, \varrho)$.

The vector fields Z, X and Y (cf. (2.12), (2.16)) are given by

$$Z = 2b^{-1}\sum_{k=1}^{m} z_k \frac{\partial}{\partial z_k},$$

$$X = 2b^{-1}\sum_{k=1}^{m}\left(x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k}\right) = 2b^{-1}\sum_{k=1}^{m} r_k \frac{\partial}{\partial r_k}$$

and

$$Y = 2b^{-1}\sum_{k=1}^{m}\left(x_k \frac{\partial}{\partial y_k} - y_k \frac{\partial}{\partial x_k}\right) = 2b^{-1}\sum_{k=1}^{m} \frac{\partial}{\partial \theta_k},$$

where $z_k = r_k e^{i\theta_k}$ defines polar coordinates in the kth complex variable. So $\exp(-tX)z = e^{-2t/b}z$ and $\exp(-sY)z = e^{-2is/b}z$. It follows that

$$\exp(-tX - sY)z = e^{-2(t+is)/b}z, \quad z \in \mathbf{C}^m, \ s, t \in \mathbf{R}. \tag{5.13}$$

Both vector fields X and Y are complete, as is also the sum $tX + sY$. Moreover Y is Killing because $\sigma(z)$ depends only on $|z|^2$, which is invariant under the flow $\exp(-sY)$.

Note that if σ, given by (5.10), is not constant and $m > 1$ then g is not Kählerian. To see this observe that the Kähler form associated to the metric (5.1) is

$$\omega = \sigma(x)^{-1} \cdot \tfrac{1}{2}i \sum_{k=1}^{m} dz_k \wedge d\bar{z}_k.$$

But (5.10) gives $\sigma(z)^{-1} = u(|z|^2)$ for some function u whose derivative, u', is somewhere nonzero. Thus $d\omega = u'(|z|^2)\sum_{j=1}^{m}(\bar{z}_j \, dz_j + z_j \, d\bar{z}_j) \wedge \sum_{k=1}^{m} dz_k \wedge d\bar{z}_k$ and is not identically zero. For example, the coefficient of $dz_1 \wedge dz_2 \wedge d\bar{z}_2$ is $u'(|z|^2)(\bar{z}_1 - \bar{z}_2)$. Nevertheless Y is Killing, as already noted above. So the hypothesis of Theorem 2.10, that g be Kähler, is not a necessary condition.

Notice that if w is given by (5.8) then holomorphicity of d^*d requires ϱ to be given by (5.9) because $\varrho(z)^{-1}\partial w/\partial\bar{z}_k = b^{-1}z_k(\varphi'(|z|^2)/\varrho(z))$ can only be holomorphic if the (clearly real) coefficient of z_k is constant. The constant b may be chosen to normalize ϱ.

In this large class of examples the operator d^*d, as an operator on $\mathcal{H}(\mathbf{C}^m)$, is the same in all cases in spite of the fact that the Hilbert spaces $\mathcal{H}\cap L^2(\mathbf{C}^m,\varrho)$ (respectively $\mathcal{H}^2$) may be quite different. The self-adjoint versions, A, of d^*d as operators in $\mathcal{H}\cap L^2(\mathbf{C}^m,\varrho)$ (respectively $\mathcal{H}^2$) may not be unitarily equivalent for different ϱ. For example, $\mathcal{H}^2$ may be infinite-dimensional as in the Gaussian case below, or finite-dimensional as in the case that follows it.

Gaussian case. Let $c>0$ and take $\varphi(s)=(2\pi c)^{-m}e^{-s/2c}$. Then $\varphi'(s)=-\varphi(s)/2c$. Choose $b=2c$ in (5.9). Then $w(z)=\varrho(z)=(2\pi c)^{-m}e^{-|z|^2/2c}$ and $\sigma(z)=1$. Clearly $(\mathbf{C}^m,g)$ is complete and therefore $\mathcal{H}^p=\mathcal{H}\cap L^p$ for $p\geqslant 2$ by Theorem 2.14. Moreover the exponentials $e^{\gamma\cdot z}$ are in $\mathcal{H}^2$ and are fundamental in $\mathcal{H}^p$ by [Wa, Theorem 3.1] (for $0<p<1$) and by [JPR] (for $p\geqslant 1$). Hence $\mathcal{H}^p=\mathcal{H}\cap L^p$ for all $p\in(0,\infty)$. It is known, [G1], that the logarithmic Sobolev inequality (2.29) holds with this constant c and with $\beta=0$.

We may apply Corollary 2.20 and Corollary 2.12. This yields, for $0<q\leqslant p<\infty$, for all real s and for $f\in\mathcal{H}^q$,

$$\|f(e^{-(t+is)/c}z)\|_{L^p(\mathbf{C}^m,\mu)} \leqslant \|f\|_{L^q(\mathbf{C}^m,\mu)} \quad \text{if } t\geqslant \tfrac{1}{2}c\log(p/q). \tag{5.14}$$

This agrees with [J1, Theorem 11], [C, Theorem 4], [Z, Theorem 1 (for $q\geqslant 1$)] and [J2, Theorem 4]. In (5.14) the operator $e^{-(t+is)A}$ has been written in terms of the flow $\exp(-tX-sY)$, and in this form the constant c can be omitted in both occurrences in (5.14) in this Gaussian case. In all four of the preceding references it is shown that the map $f\to f(e^{-t}\cdot)$ from $\mathcal{H}\cap L^q$ to $\mathcal{H}\cap L^p$ is unbounded if $t<\tfrac{1}{2}\log(p/q)$.

It is illuminating to compare the holomorphic and nonholomorphic actions of the semigroup e^{-tA} in this Gaussian context. For simplicity take $m=1$, $c=\tfrac{1}{2}$ and $b=1$. Then $d\mu(z)=\pi^{-1}e^{-|z|^2}dx\,dy$ and $d^*df(z)=2zf'(z)$ for $f\in\mathcal{H}\cap L^2$. If f is in $L^2(\mu)$ but is not necessarily holomorphic, then $(e^{-tA}f)(z)=\int_\mathbf{C}K_t(z,\zeta)f(\zeta)\,d\zeta$ where K_t is the Mehler kernel, $K_t(z,\zeta)=(\pi(1-e^{-4t}))^{-1}\exp\{-|\zeta-e^{-2t}z|^2/(1-e^{-4t})\}$, [N1]. Now suppose that f is an arbitrary entire function on $\mathbf{C}$. If the last integral is interpreted as an improper Riemann integral, namely as $\lim_{R\to\infty}\int_{|\zeta-e^{-2t}z|<R}K_t(z,\zeta)f(\zeta)\,d\zeta$, then the mean value property of harmonic functions shows that the limit exists and equals $f(e^{-2t}z)$, in agreement with (5.13) and (2.26). This representation of e^{-tA} is the basis for Janson's original proof of (1.7), [J1]. But the integral need not exist as a Lebesgue integral for $f\in\mathcal{H}^p$ if $0<p<1$. For example, for fixed $p\in(0,1)$ choose $p_1\in(p,1)$ and let $f(z)=e^{z^2/p_1}$. Then $f\in\mathcal{H}\cap L^p$. But straightforward estimates show that $\int_\mathbf{C}K_t(z,\zeta)|f(\zeta)|\,d\zeta=\infty$ for all

$z \in \mathbf{C}$ if $1 - e^{-4t} > p_1$. In particular, since p_1 is arbitrary in $(p, 1)$, e^{-tA} seems to have no reasonable interpretation as an operator on all of $L^p(\mu)$ for any $t > 0$, when $p < 1$.

Case of finite-dimensional $\mathcal{H}^2$. Take $m = 1$ for simplicity and let $\varphi(s) = a/(1+s)^\lambda$. Fix $\lambda > 0$. Then $-\varphi'(s) = (a\lambda)/(1+s)^{\lambda+1}$. Define w, ϱ and σ by (5.8), (5.9) and (5.10) with $b = 1$. Then

$$\varrho(z) = \frac{a\lambda}{(1+|z|^2)^{\lambda+1}} \tag{5.15}$$

and

$$\sigma(z) = \lambda^{-1}(1+|z|^2). \tag{5.16}$$

Choose a so that $\int_{\mathbf{C}} \varrho(x)\, dx = 1$. The space $\mathcal{H} \cap L^p(\mathbf{C}, \varrho(x)\, dx)$ is finite-dimensional for all $p \in (0, \infty)$. It is spanned by $1, z, z^2, \ldots, z^n$ where n is the largest integer such that $|z|^{np}(1+|z|^2)^{-\lambda-1}$ is integrable. In particular, $\mathcal{H} \cap L^2(\mathbf{C}, \varrho(x)\, dx)$ is finite-dimensional for all $\lambda > 0$ and consists precisely of constants if and only if $\lambda \leqslant 1$. Now if $f = z^n$ then one can compute that

$$\int_{\mathbf{C}} h(df, d\bar{f})\, d\mu(z) = \int_{\mathbf{C}} |f'(z)|^2 2\sigma(z)\varrho(z)\, dx\, dy = \int_{\mathbf{C}} |f'(z)|^2 2a(1+|z|^2)^{-\lambda}\, dx\, dy,$$

which is finite if and only if $|z|^{2(n-1)}(1+|z|^2)^{-\lambda} \in L^1(\mathbf{C}, dx\, dy)$. An elementary approximation argument now shows that $z^n \in \mathcal{D}(Q)$ if and only if $|z|^{2(n-1)}(1+|z|^2)^{-\lambda} \in L^1(\mathbf{C}, dx\, dy)$. Since this is equivalent to the condition that $|z|^{2n}(1+|z|^2)^{-\lambda-1}$ lie in $L^1(\mathbf{C}, dx\, dy)$ it follows that $\mathcal{H}^2 = \mathcal{H} \cap \mathcal{D}(Q) = \mathcal{H} \cap L^2$. Hence

$$\mathcal{H}^p = \mathcal{H} \cap L^p \quad \text{for } 2 \leqslant p < \infty.$$

For large enough p, $\mathcal{H}^p$ consists only of constants. Therefore if $\mathcal{H}^2 \neq \{\text{constants}\}$ then $\mathcal{H}^p$ is not dense in $\mathcal{H}^2$. It follows from Theorem 2.17 that a logarithmic Sobolev inequality (2.29) cannot hold when $\mathcal{H}^2$ is nontrivial, i.e., when $\lambda > 1$. Actually the method of Herbst inequalities, [GR], shows that a logarithmic Sobolev inequality (2.29) cannot hold for any $\lambda > 0$: one need only let $\psi(z) = \left(\frac{1}{2}\lambda\right)^{1/2} \log(1+|z|^2)$, compute that $h(d\psi(z), d\psi(z)) \leqslant 1$ and that $\int_{\mathbf{C}} e^{\varepsilon\psi(x)^2}\varrho(x)\, dx = \infty$ for all $\varepsilon > 0$, in order to conclude from [GR, Equation 4.3] that (2.29) cannot hold. Although d^*d is given on $\mathcal{H}^2$ by (5.12), which is the same formula as in the Gaussian case, it is clearly not unitarily equivalent to its $\mathcal{H} \cap L^2(\mathbf{C}, \text{Gauss})$-version. However, in both cases the powers z^k form an orthogonal basis (finite in the present case) of eigenvectors for (the self-adjoint version of) d^*d. The spectrum of d^*d is clearly $\{2k\}_{k=0}^n$ in the present case, where $n+1 = \dim \mathcal{H} \cap L^2(\mathbf{C}, \varrho(x)\, dx)$.

Perturbed Gaussian case. There are two kinds of general perturbation theorems for logarithmic Sobolev inequalities [A], [AS], [Hin], [HS], [Le]. They take the following form.

Suppose that μ is a probability measure on a Riemannian manifold (M, g) with dual metric h. Write $Q_\mu(f) = \int_M h(df, d\bar{f}) \, d\mu$. Let $F: M \to \mathbf{R}$ be measurable and suppose that the measure $d\mu_F = e^F \, d\mu$ is normalized. If Q_μ satisfies (2.29) then (2.29) is also satisfied if μ is replaced in all four terms by μ_F, provided the constants c and β are suitably increased and provided F satisfies suitable conditions. These theorems do not discuss explicitly a change of the metric g, but just a change of the measure μ. However, in order to maintain holomorphicity of d^*d we must also change the metric if we change the measure μ. Nevertheless both perturbation theorems are applicable in the present setting under additional restrictions on F. Our aim in this example, rather than to achieve much generality, is to show the existence of a broad class of non-Gaussian measures and corresponding metrics such that d^*d is both holomorphic and satisfies a logarithmic Sobolev inequality. With this in mind we will make use of the easy-to-state perturbation theorem of Holley and Stroock. All that is required for their theorem to apply is that F be bounded. So take $\varphi(s) = (2\pi)^{-m} e^{-(s+v(s))/2}$ in (5.8) and (5.9). Assume that v is bounded on $[0, \infty)$, is in $C^\infty([0, \infty))$ and $\infty > \alpha_2 \geqslant 1 + v'(s) \geqslant \alpha_1 > 0$ for some constants α_1 and α_2. Then $\varphi'(s) = -\frac{1}{2}(1+v'(s))\varphi(s)$. So $w(z) = (2\pi)^{-m} e^{-(|z|^2 + v(|z|^2))/2}$ and $\varrho(z) = \frac{1}{2}b(2\pi)^{-m}(1+v'(|z|^2))e^{-(|z|^2+v(|z|^2))/2}$. Therefore $\sigma(z) = 2b^{-1}(1+v'(|z|^2))^{-1}$. Since v and $1+v'$ are bounded ϱ is integrable. Choose b so as to normalize ϱ. σ is bounded and bounded away from zero. So $\mathbf{C}^m$ is complete in the metric (5.1) and consequently $\mathcal{H}^2 = \mathcal{H} \cap L^2$. $\varrho(z)$ differs from a Gaussian density by a factor $e^F = \frac{1}{2}b(1+v'(|z|^2))e^{-v(|z|^2)/2}$. Thus F is bounded. The L^p-metrics are therefore equivalent to those of the Gaussian case. It follows as in the Gaussian case that

$$\mathcal{H}^p = \mathcal{H} \cap L^p, \quad 0 < p < \infty.$$

Since the metric (5.1) is equivalent to the standard metric the perturbation theorem of Holley and Stroock, [HS], is applicable. One may conclude from their theorem that (2.29) holds for some constant c and for $\beta = 0$. Of course, d^*d is also holomorphic because of our use of (5.8) to (5.10) to define the metric g and measure μ. Thus all the hypotheses of Theorem 2.19 and Corollary 2.20 are satisfied in this class of examples.

Remark 5.2. In any example it is of interest to identify $\mathcal{H}^p$ explicitly because the contractivity theorem, Theorem 2.16, and the strong hypercontractivity theorem, Corollary 2.20, apply only to $\mathcal{H}^p$ and not necessarily to $\mathcal{H} \cap L^p$. In particular, it is desirable to know whether $\mathcal{H}^p = \mathcal{H} \cap L^p$. In Example 5.1 we saw that $\mathcal{H}^p = \mathcal{H} \cap L^p$ for all $p \in (0, \infty)$ in the Gaussian and perturbed Gaussian cases, and for $p \geqslant 2$ in the finite-dimensional case. The equality fails in the finite-dimensional case for small $p > 0$ because $\mathcal{H}^2$ is not dense in $\mathcal{H} \cap L^p$ for small p, these spaces being of different (finite) dimensions. The issue of equality is related to the question of whether the holomorphic polynomials are dense in

$\mathcal{H} \cap L^p$. The following theorem is inspired by E. Carlen's density theorem [C, Theorem 5]. It should be noted, however, that the key tool in the proof below is the holomorphicity of d^*d and the periodicity of the Y-flow, rather than hypercontractivity.

Notation 5.3. For a nonnegative multiindex $\alpha = (\alpha_1, ..., \alpha_m)$ define $z^\alpha = z_1^{\alpha_1} ... z_m^{\alpha_m}$ and $|\alpha| = \alpha_1 + ... + \alpha_m$. A holomorphic polynomial is a function on $\mathbf{C}^m$ of the form $p(z) = \sum_{|\alpha| \leqslant N} a_\alpha z^\alpha$ where each $a_\alpha \in \mathbf{C}$. $\mathcal{P}$ will denote the space of all holomorphic polynomials.

LEMMA 5.4. *Suppose that g and μ are given on $\mathbf{C}^m$ by* (5.1), (5.3), (5.8), (5.9) *and* (5.10). *Then $\mathcal{P} \cap L^p(\mathbf{C}^m, \mu)$ is dense in $\mathcal{H} \cap L^p$ for $1 \leqslant p < \infty$.*

Proof. We will make use of the vector field Y associated to g and μ. Its flow is given by (5.13). Let $(V_\theta f)(z) = f(\exp(\frac{1}{2}b\theta Y) z)$ for measurable f. Then $(V_\theta f)(z) = f(e^{i\theta} z)$ by (5.13). We may apply Lemma 3.3 with $Y_0 = \frac{1}{2}bY$. Thus V_θ, restricted to $\mathcal{H} \cap L^p$, is a strongly continuous one-parameter group of isometries in this space for $1 \leqslant p < \infty$.

Let

$$F_k(\theta) = \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{2\pi} \sum_{n=-j}^{j} e^{in\theta} = \frac{1}{2k\pi} \cdot \frac{\sin^2\left(\frac{1}{2}k\theta\right)}{\sin^2\left(\frac{1}{2}\theta\right)}$$

denote Fejer's kernel, [T, p. 413]. Then F_k is periodic with period 2π, nonnegative, has integral equal to one on $[-\pi, \pi]$, and $\int_{-\pi}^{\pi} F_k(\theta)\varphi(\theta)\, d\theta \to \varphi(0)$ for any continuous function on $[-\pi, \pi]$. Thus for $1 \leqslant p < \infty$ and any function $f \in \mathcal{H} \cap L^p$,

$$\left\| f - \int_{-\pi}^{\pi} F_k(\theta) V_\theta f\, d\theta \right\|_p = \left\| \int_{-\pi}^{\pi} F_k(\theta)(f - V_\theta f)\, d\theta \right\|_p$$

$$\leqslant \int_{-\pi}^{\pi} F_k(\theta) \|f - V_\theta f\|_p\, d\theta \to 0 \quad \text{as } k \to \infty$$

because the isometry group V_θ is strongly continuous in $\mathcal{H} \cap L^p$.

Now since f is in $\mathcal{H}$ we may write $f(z) = \sum_\alpha a_\alpha z^\alpha$ with uniform convergence on bounded sets in $\mathbf{C}^m$. So

$$(V_\theta f)(z) = \sum_\alpha a_\alpha e^{i\theta|\alpha|} z^\alpha.$$

Since pointwise evaluation is a continuous linear functional on $\mathcal{H} \cap L^p$ and the preceding series converges uniformly on compact sets we have

$$f_k(z) \equiv \left(\int_{-\pi}^{\pi} F_k(\theta) V_\theta f\, d\theta \right)(z) = \int_{-\pi}^{\pi} F_k(\theta) f(e^{i\theta} z)\, d\theta$$

$$= \sum_\alpha a_\alpha \left(\int_{-\pi}^{\pi} F_k(\theta) e^{i\theta|\alpha|}\, d\theta \right) z^\alpha = \sum_{|\alpha| \leqslant k} a_\alpha \left(\int_{-\pi}^{\pi} F_k(\theta) e^{i\theta|\alpha|}\, d\theta \right) z^\alpha,$$

which is in $\mathcal{P}$. The integral on the left, being the integral of a continuous function into L^p, is in L^p. So $f_k \in \mathcal{P} \cap L^p$ and converges strongly to f in the L^p-norm. $\qquad\square$

The function φ used in (5.8) to define w is strictly decreasing. If it is bounded away from zero then equation (5.5) shows that the only holomorphic functions in the domain of Q are constants. Since this is an uninteresting case we will assume that $\inf \varphi$ is zero in the following theorem.

THEOREM 5.5. *Suppose that g and μ are given on $\mathbf{C}^m$ by (5.1), (5.3), (5.8), (5.9) and (5.10), and that $\lim_{s\to\infty}\varphi(s)=0$. If $\mathcal{P}\subset L^2(\mu)$ then*

(a) $\mathcal{P}\subset\mathcal{H}^p$, $0<p<\infty$,

(b) $\mathcal{P}$ *is dense in* $\mathcal{H}\cap L^p$ *for* $1\leqslant p<\infty$,

(c) $\mathcal{H}^p=\mathcal{H}\cap L^p$, $1\leqslant p<\infty$,

(d) $\mathcal{P}=\mathcal{R}$, *the algebra defined in Theorem 2.17,*

(e) *the spectrum of $A|\mathcal{H}^2$ is $(2/b)\{0,1,2,...\}$.*

Proof. Since $\mathcal{P}\subset L^2(\varrho(z)\,dz)$ we have $\int_{\mathbf{C}^m}|z|^{2k}\varrho(z)\,dz<\infty$ for $k=0,1,2,....$. Switching to polar coordinates r,ω in $\mathbf{R}^{2m}$ and writing $s=r^2$, we see that $\int_0^\infty s^j(-\varphi'(s))\,ds<\infty$ for $j=m-1,m,m+1,....$. Hence, if $0<t<u<\infty$ then, for some constant c_j independent of t and u, $t^j(\varphi(t)-\varphi(u))=\int_t^u t^j(-\varphi'(s))\,ds\leqslant\int_t^u s^j(-\varphi'(s))\,ds\leqslant c_j$. Letting $u\to\infty$ we find $\varphi(t)\leqslant c_j t^{-j}$. Switching back from polar coordinates it follows that $\int_{\mathbf{C}^m}|f(z)|^2w(z)\,dz<\infty$ for all $f\in\mathcal{P}$. A standard approximation argument now shows that $\mathcal{P}\subset\mathcal{D}(Q)$, and therefore $\mathcal{P}\subset\mathcal{H}^2$. Hence $\mathcal{P}\subset\mathcal{H}^p$ for $0<p<2$. But if $f\in\mathcal{P}$ then $f^k\in\mathcal{P}$ for all positive integers k. So $\mathcal{P}\subset L^p$ for all $p<\infty$. This proves (a). (b) now follows from Lemma 5.4. Thus $\mathcal{H}^p$ is dense in $\mathcal{H}\cap L^p$ for $1\leqslant p<\infty$. Since it is closed in L^p item (c) follows. Now (5.12) shows that $d^*dz^\alpha=(2/b)|\alpha|z^\alpha$. Since also $z^\alpha\in\mathcal{D}(Q)$, Theorem 2.11 (e) shows that $z^\alpha\in\mathcal{D}(A)$. But the functions z^α form an orthogonal and fundamental system in $\mathcal{H}^2$. Hence these are all the eigenfunctions of $A|\mathcal{H}^2$. Parts (d) and (e) now follow. $\qquad\square$

In the following example we will choose the density ϱ on $\mathbf{R}^2$ to be a heat kernel for the operator $a^{-1}\partial^2/\partial x^2+b^{-1}\partial^2/\partial y^2$. That is, $\exp\left[\frac{1}{2}t(a^{-1}\partial^2/\partial x^2+b^{-1}\partial^2/\partial y^2)\right]$ is given by convolution by ϱ when $t=1$. It will be shown that if $a\neq b$ then there exists no Hermitian metric g on $\mathbf{C}$ such that $(\mathbf{C},g,\varrho)$ is holomorphic and nontrivial.

Example 5.6. Let $a>0$ and $b>0$. Assume that $a\neq b$. Let

$$\varrho(x,y)=Ce^{-(ax^2+by^2)/2} \qquad (5.17)$$

where C is chosen so that $\int_{\mathbf{R}^2}\varrho(x,y)\,dx\,dy=1$. If g is a C^2 Hermitian metric on $\mathbf{C}$ such that the Dirichlet form associated to $(\mathbf{C},g,\varrho)$ is holomorphic then $\mathcal{H}^2=\{\text{constants}\}$.

Proof. First observe that the most general Hermitian metric (its dual, actually) on **C** is given by (5.2). What we will actually show is that if (5.7) holds (with $k=m=1$) for $w=\sigma\varrho$ and $w\in C^2(\mathbf{R}^2)$ then w is constant. Assume then that there exists an entire function $G(z)=u(z)+iv(z)$ on **C**, with u and v real, such that $\varrho^{-1}\partial w/\partial\bar{z}=G$. Since $\partial/\partial\bar{z}=\frac{1}{2}(\partial/\partial x+i\partial/\partial y)$ this equation may be written in real form as

$$\frac{\partial w}{\partial x}=2\varrho u, \quad \frac{\partial w}{\partial y}=2\varrho v. \tag{5.18}$$

Since w is twice continuously differentiable the compatibility condition $\partial(2\varrho u)/\partial y=\partial(2\varrho v)/\partial x$ must hold. That is, $\varrho_y u+\varrho u_y=\varrho_x v+\varrho v_x$. But $\varrho_y=-by\varrho$ and $\varrho_x=-ax\varrho$. Hence compatibility requires

$$-byu+u_y = -axv+v_x. \tag{5.19}$$

Since u, v, u_y and v_x are harmonic we may apply $\partial^2/\partial x^2+\partial^2/\partial y^2$ to (5.19) and obtain $-2bu_y=-2av_x$. Since $v_x=-u_y$ this yields $2(a+b)u_y=0$. So $u_y=0$ and $v_x=0$. u is therefore a function of x alone while v depends only on y. Since $u_x=v_y$ we have $u_x=v_y=$ real constant, α, say. So $u=\alpha x+\beta$ and $v=\alpha y+\gamma$ with α, β and γ real. Insert these functions into (5.19). We find $-by(\alpha x+\beta)+0=-ax(\alpha y+\gamma)+0$. Comparing the coefficients of x, y and xy we obtain $\alpha=0$ because $a\neq b$, and also $\beta=\gamma=0$. Thus $G=0$. Equation (5.18) now shows that w is constant. Thus $\mathcal{H}\cap\mathcal{D}(Q)$ consists of those functions $f\in\mathcal{H}(\mathbf{C})$ such that $\int_{\mathbf{C}}|f'(z)|^2\,dx\,dy<\infty$. Only constant functions satisfy this. $\square$

Example 5.7. Let a and b be distinct strictly positive real numbers. Let U be an open disc in **C**. Suppose that $\sigma(z)$ is a strictly positive function in $C^2(\mathbf{R}^2)$ and that

$$\sigma(x,y)=1+ax^2+by^2, \quad (x,y)\in U. \tag{5.20}$$

Define a Hermitian metric on **C** by (5.2). Suppose that $\varrho\in C^2(\mathbf{R}^2)$ and $\varrho>0$ everywhere. If the Dirichlet form d^*d determined by σ and ϱ is holomorphic then $w\equiv\sigma\varrho$ is constant and $\mathcal{H}^2\subset\{\text{constants}\}$.

Proof. Assume that d^*d is holomorphic for the Hermitian metric (5.1) and density ϱ with σ given in U by (5.20). By (5.7) we must have

$$\varrho^{-1}\frac{\partial w}{\partial\bar{z}}=G(z)=u+iv \quad \text{for some entire function } G. \tag{5.21}$$

We may write this as $\partial\log w/\partial\bar{z}=\sigma^{-1}G$. As in Example 5.6 this may be written in real form as

$$\frac{\partial\log w}{\partial x}=2\sigma^{-1}u, \quad \frac{\partial\log w}{\partial y}=2\sigma^{-1}v. \tag{5.22}$$

Compatibility of these two equations requires

$$(\sigma^{-1}u)_y = (\sigma^{-1}v)_x. \tag{5.23}$$

The remainder of these computations are valid in U. Taking into account (5.20) one can compute that (5.23) is equivalent to $(u_y - v_x)(1 + ax^2 + by^2) = 2byu - 2axu$. Using $v_x = -u_y$ we then find

$$u_y(1 + ax^2 + by^2) = byu - axv \tag{5.24}$$

to be equivalent to (5.23). Apply the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$ to (5.24) to find

$$u_y(2a + 2b) + u_{yx}2ax + u_{yy}2by = 2bu_y - 2av_x.$$

Using $v_x = -u_y$ again we find

$$axu_{yx} + byu_{yy} = 0. \tag{5.25}$$

Apply the Laplacian to this equation to find $2au_{yxx} + 2bu_{yyy} = 0$. However, u_y is harmonic so that $u_{yxx} = -u_{yyy}$. Therefore $2(b-a)u_{yyy} = 0$. Since $b - a \neq 0$ we have $u_{yyy} = 0$. As in Example 5.6 it now follows that the holomorphic function $-G''(z) = u_{yy} + iv_{yy}$ is a first-degree polynomial with real leading coefficient. That is, $G''(z) = \alpha z + \beta$ with α real. Hence G is cubic: $G(z) = \frac{1}{6}\alpha z^3 + \frac{1}{2}\beta z^2 + \gamma z + \delta$ with α real. The cubic terms in u are therefore $\alpha(x^3 - 3xy^2)$. Thus $u_{xy} = -6\alpha y$ and $u_{yy} = -6\alpha x$ up to additive constants. Inserting this in (5.25) one obtains $-6\alpha axy - 6\alpha bxy = $ linear terms. Therefore $\alpha = 0$. Next, we may rewrite (5.25) in the form $axu_{xy} - byv_{xy} = 0$ since $u_{yy} = -v_{xy}$. That is, $\mathrm{Re}\{(ax + iby)(u_{xy} + iv_{xy})\} = 0$. But $u_{xy} + iv_{xy} = iG''(z) = i\beta$. Hence $\mathrm{Re}\{(ax + iby)i\beta\} = 0$ in U. Therefore $\beta = 0$. We now know that $G(z) = \gamma z + \delta$ in U with γ and δ complex constants. Hence u_y is constant. Observe that in the polynomial identity (5.24) the only constant term is u_y. Thus $u_y = 0$. Hence γ is real. Therefore $u = \gamma x + \delta_1$ and $v = \gamma y + \delta_2$ with γ, δ_1 and δ_2 real. Inserting this into (5.24), which reads $byu - axv = 0$ because $u_y = 0$, we see that $\gamma = 0$ since $a \neq b$, and also $\delta_1 = \delta_2 = 0$. Hence $G = 0$ in U and therefore $G = 0$ everywhere. Equation (5.22) now shows that $\log w$ is constant on $\mathbf{R}^2$. As in Example 5.6 it follows that the only holomorphic functions that can be in $\mathcal{D}(Q)$ are constant functions. Moreover if $\varrho \equiv \mathrm{const} \cdot \sigma^{-1}$ is not integrable then the only constant functions in $L^2(\mathbf{R}^2, \varrho(x)\,dx)$ are zero. Hence $\mathcal{H} \cap \mathcal{D}(Q)$ is zero- or one-dimensional. So $\mathcal{H}^2$ is also zero- or one-dimensional. $\qquad\square$

Remark 5.8. If (M, g) is a complex manifold with Hermitian metric then the heat kernel, $\varrho_t(x, y)$, associated to the Laplace–Beltrami operator on M, provides a natural source of probability densities ϱ on M to explore for their possible role in a holomorphic triple. One takes $\varrho(y) = \varrho_t(x, y)$ with (t, x) fixed in $(0, \infty) \times M$. But if one takes the function σ in Example 5.7 to be bounded and bounded away from zero then $\mathbf{C}$ is complete in

the Hermitian metric (5.1) and $(\mathbf{C}, g)$ is Kählerian, being of two real dimensions. However, the function ϱ constructed from the heat kernel as above is a probability density on $\mathbf{C}$ for the Riemann–Lebesgue measure on $\mathbf{C}$, and is therefore integrable with respect to Lebesgue measure dy also, which differs from the Riemann–Lebesgue measure only by a factor σ. So $\sigma(z)\varrho(z)$ is not constant. By Example 5.7, $(\mathbf{C}, \sigma, \varrho(\cdot))$ is not holomorphic. Therefore even if (M, g) is a complete Kähler manifold it does not follow that $(M, g, \varrho_t(x, y)\, dy)$ is holomorphic, when $\varrho_t(x, y)\, dy$ is the heat kernel measure. Small-time asymptotics suggest that it is necessary in addition that (M, g) be Ricci flat. But a global topological constraint is also required, as is shown by the example of a cylinder with the flat metric. In this case (M, g) is Ricci flat, but the heat kernel does not give a holomorphic triple.

References

[A]   AIDA, S., Uniform positivity improving property, Sobolev inequalities, and spectral gaps. *J. Funct. Anal.*, 158 (1998), 152–185.

[AMS] AIDA, S., MASUDA, T. & SHIGEKAWA, I., Logarithmic Sobolev inequalities and exponential integrability. *J. Funct. Anal.*, 126 (1994), 83–101.

[AS]   AIDA, S. & SHIGEKAWA, I., Logarithmic Sobolev inequalities and spectral gaps: perturbation theory. *J. Funct. Anal.*, 126 (1994), 448–475.

[B]   BAKRY, D., L'hypercontractivité et son utilisation en théorie des semigroupes, in *Lectures on Probability Theory* (Saint-Flour, 1992), pp. 1–114. Lecture Notes in Math., 1581. Springer-Verlag, Berlin, 1994.

[BSZ] BAEZ, J. C., SEGAL, I. E. & ZHOU, Z., *Introduction to Algebraic and Constructive Quantum Field Theory.* Princeton Univ. Press, Princeton, NJ, 1992.

[C]   CARLEN, E. A., Some integral identities and inequalities for entire functions and their applications to the coherent state transform. *J. Funct. Anal.*, 97 (1991), 231–249.

[Da]   DAVIES, E. B., *Heat Kernels and Spectral Theory.* Cambridge Tracts in Math., 92. Cambridge Univ. Press, Cambridge–NewYork, 1989.

[DaS] DAVIES, E. B. & SIMON, B., Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.*, 59 (1984), 335–395.

[DeS] DEUSCHEL, J.-D. & STROOCK, D. W., *Large Deviations.* Pure Appl. Math., 137. Academic Press, Boston, MA, 1989.

[DG]   DRIVER, B. K. & GROSS, L., Hilbert spaces of holomorphic functions on complex Lie groups, in *New Trends in Stochastic Analysis* (Charingworth, 1994), pp. 76–106. World Sci. Publishing, River Edge, NJ, 1997.

[DH]   DRIVER, B. K. & HU, Y., On heat kernel logarithmic Sobolev inequalities, in *Stochastic Analysis and Applications* (Powys, 1995), pp. 189–200. World Sci. Publishing, River Edge, NJ, 1996.

[Dr]   DRIVER, B. K., On the Kakutani-Itô-Segal-Gross and the Segal-Bargmann-Hall isomorphisms. *J. Funct. Anal.*, 133 (1995), 69–128.

[E]   EPPERSON, J., The hypercontractive approach to exactly bounding an operator with complex Gaussian kernel. *J. Funct. Anal.*, 87 (1989), 1–30.

[F]   FUKUSHIMA, M., *Dirichlet Forms and Markov Processes.* North-Holland Math. Library, 23. North-Holland, NewYork, 1980.

[G1]   GROSS, L., Logarithmic Sobolev inequalities. *Amer. J. Math.*, 97 (1975), 1061–1083.

[G2]   — Uniqueness of ground states for Schrödinger operators over loop groups. *J. Funct. Anal.*, 112 (1993), 373–441.

[G3]   — The homogeneous chaos over compact Lie groups, in *Stochastic Processes, Festschrift in Honor of Gopinath Kallianpur* (S. Cambanis et al., eds.), pp. 117–123. Springer-Verlag, New York, 1993.

[G4]   — Harmonic analysis for the heat kernel measure on compact homogeneous spaces, in *Stochastic Analysis on Infinite-Dimensional Spaces* (Baton Rouge, LA, 1994), pp. 99–110. Pitman Res. Notes Math. Ser., 310. Longman Sci. Tech., Harlow, 1994.

[G5]   — Logarithmic Sobolev inequalities and contractivity properties of semigroups, in *Dirichlet Forms* (Varenna 1992), pp. 54–88. Lecture Notes in Math., 1563. Springer-Verlag, Berlin, 1993.

[G6]   — The mathematical ramifications of Wiener's program in statistical physics, in *Proceedings of the Norbert Wiener Centenary Congress* (East Lancing, MI, 1994), pp. 53–61. Proc. Sympos. Appl. Math., 52. Amer. Math. Soc., Providence, RI, 1997.

[G7]   — Strong hypercontractivity and relative subharmonicity. To appear.

[GH]   GRIFFITHS, P. & HARRIS, J., *Principles of Algebraic Geometry.* Wiley-Interscience, New York, 1978.

[Gl]   GLIMM, J., Boson fields with nonlinear self-interaction in two dimensions. *Comm. Math. Phys.*, 8 (1968), 12–25.

[GM]   GROSS, L. & MALLIAVIN, P., Hall's transform and the Segal–Bargmann map, in *Itô's Stochastic Calculus and Probability Theory*, pp. 73–116. Springer-Verlag, Tokyo, 1996.

[GR]   GROSS, L. & ROTHAUS, O., Herbst inequalities for supercontractive semigroups. *J. Math. Kyoto Univ.*, 38 (1998), 295–318.

[Ha1]   HALL, B., The Segal–Bargmann "coherent state" transform for compact Lie groups. *J. Funct. Anal.*, 122 (1994), 103–151.

[Ha2]   — The inverse Segal–Bargmann transform for compact Lie groups. *J. Funct. Anal.*, 143 (1997), 98–116.

[Ha3]   — Quantum mechanics in phase space, in *Perspectives on Quantization* (South Hadley, MA, 1996), pp. 47–62. Contemp. Math., 214. Amer. Math. Soc., Providence, RI, 1998.

[Hij1] HIJAB, O., Hermite functions on compact Lie groups, I. *J. Funct. Anal.*, 125 (1994), 480–492.

[Hij2] — Hermite functions on compact Lie groups, II. *J. Funct. Anal.*, 133 (1995), 41–49.

[Hin]   HINO, M., Spectral properties of Laplacians on an abstract Wiener space with a weighted Wiener measure. *J. Funct. Anal.*, 147 (1997), 485–520.

[HS]   HOLLEY, R. & STROOCK, D. W., Logarithmic Sobolev inequalities and stochastic Ising models. *J. Statist. Phys.*, 46 (1987), 1159–1194.

[J1]   JANSON, S., On hypercontractivity for multipliers on orthogonal polynomials. *Ark. Mat.*, 21 (1983), 97–110.

[J2]   — On complex hypercontractivity. *J. Funct. Anal.*, 151 (1997), 270–280.

[JPR]   JANSON, S., PEETRE, J. & ROCHBERG, R., Hankel forms and the Fock space. *Rev. Mat. Iberoamericana*, 3 (1987), 61–138.

[KN1]   KOBAYASHI, S. & NOMIZU, K., *Foundations of Differential Geometry*, Vol. 1. Wiley, New York–London, 1963.

[KN2]   — *Foundations of Differential Geometry*, Vol. 2. Wiley, New York–London, 1969.

[Le]   LEWKEERATIYUTKUL, W., Perturbation theorems for supercontractive semigroups. Ph.D. thesis, Cornell University, 1998.

[Li]   LIEB, E., Gaussian kernels have only Gaussian maximizers. *Invent. Math.*, 102 (1990),
          179–208.

[MR]   MA, Z.-M. & RÖCKNER, M., *Introduction to the Theory of (Non-Symmetric) Dirichlet
          Forms*. Springer-Verlag, Berlin, 1992.

[N1]   NELSON, E., A quartic interaction in two dimensions, in *Mathematical Theory of Ele-
          mentary Particles* (Dedham, MA, 1965), pp. 69–73. M.I.T. Press, Cambridge, MA,
          1966.

[N2]   — The free Markov field. *J. Funct. Anal.*, 12 (1973), 211–227.

[OØ]   ÓLAFSSON, G. & ØRSTED, B., Generalizations of the Bargmann transform, in *Lie The-
          ory and Its Applications in Physics* (Clausthal, 1995), pp. 3–14. World Sci. Publish-
          ing, River Edge, NJ, 1996.

[RS]   REED, M. & SIMON, B., *Methods of Modern Mathematical Physics*, Vol. 1. Academic
          Press, New York–London, 1972.

[Ru]   RUDIN, W., *Functional Analysis*. McGraw-Hill, New York–Düsseldorf–Johannesburg,
          1973.

[S]   SEGAL, I. E., Construction of non-linear local quantum processes, I. *Ann. of Math.*, 92
          (1970), 462–481.

[T]   TITCHMARSH, E. C., *The Theory of Functions*. Oxford Univ. Press, New York, 1950.

[Wa]   WALLSTÉN, R., The S^p-criterion for Hankel forms on the Fock space, $0<p<1$. *Math.
          Scand.*, 64 (1989), 123–132.

[We]   WEISSLER, F., Two-point inequalities, the Hermite semigroup, and the Gauss–Weier-
          strass semigroup. *J. Funct. Anal.*, 32 (1979), 102–121.

[Z]   ZHOU, Z., The contractivity of the free Hamiltonian semigroup in the L_p space of entire
          functions. *J. Funct. Anal.*, 96 (1991), 407–425.

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