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Lax equations, weight lattices, and Prym–Tjurin varieties

by

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Introduction

The importance of juxtaposing the two approaches to integrable systems—by Lie algebras and by algebraic curves—was laid out by Adler and van Moerbeke [AM1], [AM2]. This paper illuminates the interplay of these two ingredients. First, the line bundles on the algebraic curves that give the evolution of the system are shown to be pullbacks of the line bundles of the Borel–Weil theory. Secondly, the Weyl group action on the Jacobian of the master spectral curve (see [MS1], [MS2]) picks out a sub-abelian variety. We show that the flow of the system takes place in this sub-abelian variety. In the periodic Toda lattice, for example, this result applies to F_4 and the E-family as well as the better understood A, B, C, D, and G_2. This paper is the conclusion of the series [MS1], [MS2].

Here is the setting. Start with a Lax equation, $dA/dt=[A,B]$. The functions $A(s,t)$ and $B(s,t)$ depend on the time t and on a parameter s whose domain is an algebraic curve P. The values of A and B lie in a finite-dimensional Lie algebra and $[A,B]$ is their Lie algebra bracket. Part of the message of [AM1] and [AM2] is that many integrable systems can be written in the form of a Lie algebra-valued Lax equation with a parameter. As is discussed below, we can construct a flow on the Jacobian of the spectral curve which is the normalization of the curve defined by $\det \varrho(A(s))-z=0$. This matrix spectral curve may have several components. Our main theorem considers the flow on the spectral curve associated with the smallest representation as given by the recipe of van Moerbeke and Mumford [MM].

MAIN THEOREM. *Let $\mathfrak{g}$ be a simple Lie algebra and P a compact Riemann surface. Suppose that the pair $A,B\colon P\times\mathbf{R}\to\mathfrak{g}$ satisfies $dA/dt=[A,B]$ and that*

(1) *A,B have entries of the form $\sum c_i(t)m_i(s)$, a finite sum, where $c_i(t)$ is C^2 and $m_i(s)$ is meromorphic,*

(2) $A(s,0)$ *is regular for some* $s \in P$, *and*

(3) *the flow in the Jacobian of a spectral curve associated with the smallest representation is absolutely continuous.*

If $\mathfrak{g}$ *is of type* A, B, C, D, E_6, *or* E_7, *then the flow is in the Prym–Tjurin variety* $\mathrm{Tur}_\lambda Y$.

If P *is the Riemann sphere, then the flow is in the Prym–Tjurin variety* $\mathrm{Tur}_\lambda Y$.

The condition on the entries is that they keep the same form as they evolve. The Lax equation need not be completely integrable, although completely integrable systems provide the motivation for this work. Also the flows on the Jacobian need not be linear, although the raison d'être for transforming the flow $A(t)$ to a flow on the Jacobian of a spectral curve is that for many integrable systems the flow does linearize there. The Prym–Tjurin varieties of the theorem were introduced by Kanev [K] and a detailed analysis is given in [MS2].

For a Lax equation and for each finite-dimensional representation ϱ of the underlying Lie algebra $\mathfrak{g}$, the characteristic polynomial $\det(\varrho(A)-zI)$ is independent of time. The characteristic polynomials define a collection of spectral curves which is analyzed in [MS1]. Here the term spectral curve is used for a connected component of a spectral curve defined by a characteristic polynomial. There is a master spectral curve Y on which the Weyl group W acts and the other curves are all abstractly Y/S_λ for some weight λ where S_λ is the subgroup of W that stabilizes λ. More detail is found in §1, which provides the notation and background for this paper.

The motion of the system is given by the eigenvectors of $\varrho(A)$ which form a line bundle on the appropriate spectral curve and, by pulling back, a line bundle on the master spectral curve. §2 examines the line bundles that give the flow of the system. A flow in the space of line bundles on the master curve is produced for each weight. These flows are shown to be pullbacks via a time-varying map of the bundles over projective homogeneous spaces that occur in the Borel–Weil theory. This ties the flows on the algebraic curves to the Lie theory.

Adler and van Moerbeke also observed that the symmetries of the algebra are reflected in symmetries of the algebraic curves. They observed that for some systems of type B, C, or D the flow took place in a Prym and that the G_2-periodic Toda flow was in a subtorus of a Prym. In [MS3] this subtorus was identified as a Prym–Tjurin. The symmetries of the Lie algebra show up most plainly in the tangent space at the trivial line bundle in the Jacobi variety of the master spectral curve. This is the home of the derivatives of the flows corresponding to the various representations of the algebra. We consider the flows from the component of the spectral curve of the various finite-dimensional representations that arise from the highest weight. The tangent space is a complex Weyl group representation and a major result in this paper is that the derivatives at time $t=0$ of these flows form a copy of the weight lattice as an integral W-representation. It is by considering all the flows at once that it is seen that any given flow is in a Prym–Tjurin. This result is contained in §5. The Prym–Tjurin varieties under consideration were analyzed in terms of W in [MS2]. §1 contains a summary.

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1. Notation and background

We let G denote a simply-connected complex semi-simple Lie group with Lie algebra $\mathfrak{g}$. H is a Cartan subgroup (a maximal torus as G is semi-simple) with Lie algebra $\mathfrak{h}$. W denotes the Weyl group, $W=N_H/C_H$, the normalizer of H modulo the centralizer of H. Since G is semi-simple, $C_H=H$. A choice of Weyl chamber determines a basis of simple roots Δ and a (pair of) Borel subgroup(s), B^+ (and B^-). The real span of $\Delta=\{\alpha_1,\dots,\alpha_r\}$ is a Euclidean subspace of the dual vector space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$ with inner product $\langle\cdot,\cdot\rangle$ given by the dual of the Killing form. The weight lattice is $\{\lambda\in\mathfrak{h}^*\mid 2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle\in\mathbf{Z}\ \forall\alpha\in\Delta\}$. The Weyl group W acts on the weight lattice by the coadjoint action $(nH\cdot\lambda)(X)=\lambda(\mathrm{Ad}_{n^{-1}}X)$ for $X\in\mathfrak{h}$. The stabilizer in W of a weight λ will be denoted S_λ.

Let $\mathfrak{g}_*$ denote the regular elements in the algebra $\mathfrak{g}$. Let $\pi\colon G/H\times\mathfrak{h}_*\to\mathfrak{g}_*$ be given by $\pi(gH,h)=\mathrm{Ad}_g\,h$. This map is a regular cover with group of covering translations W. The action is given by $(nH)\cdot(gH,h)=(gn^{-1}H,\mathrm{Ad}_n\,h)$ for $n\in N_H$. Let A be a morphism from an irreducible algebraic curve P to a Lie algebra $\mathfrak{g}$ with $\mathrm{im}(A)\cap\mathfrak{g}_*\neq\varnothing$. Then $A^{-1}(\mathfrak{g}_*)$ is a Zariski-open set in P (i.e. a P-finite set) denoted P_*. We call P the parameter space as it is the domain for the spectral parameter occurring in the Lax equation.

If ϱ is a representation then the curve $\{(s,z)\in P\times\mathbf{C}\mid\det(\varrho A(s)-z)=0\}$ is in general reducible and decomposes via the dominant weights. These pieces are independent of the representation in that they depend only on the weights. If m_λ is the multiplicity of the weight λ in ϱ then

$$\det(\varrho A(s)-z)=\prod_\lambda(p_\lambda(s,z))^{m_\lambda}$$

where λ runs through the dominant weights (see [MS1]).

Let Y_λ denote the normalization of the variety defined by $p_\lambda(s,z)=0$. This is the notion of spectral curve used by V. Kanev [K]. These curves themselves may decompose.

In [MS1] the irreducible components were referred to as the spectral curves and a complete classification was given. Let $Y_{\lambda *}$ be the inverse image of P_* in Y_λ.

Let Y_* be the pullback of

$$\begin{tikzcd} & G/H \times \mathfrak{h}_* \arrow[d] \\ P_* \arrow[r, "A"] & \mathfrak{g}_* \end{tikzcd}$$

and so Y_* is a principal W-bundle. Let Y be the completion of Y_*. From Theorem 13 in [MS1] and its proof, we have:

PROPOSITION 1. *If λ is a weight then $Y_{\lambda *} \cong Y_*/S_\lambda$. Moreover, $Y_{\lambda *}$ is the pullback of*

$$\begin{tikzcd} & (G/H \times \mathfrak{h}_*)/S_\lambda \arrow[d] \\ P_* \arrow[r, "A"] & \mathfrak{g}_* . \end{tikzcd}$$

We use Y_λ and Y/S_λ interchangeably. Note that if S_λ and S_γ are conjugate subgroups of W then $Y_\lambda \cong Y_\gamma$ as quotients of W-bundles and as varieties.

Call Y the master curve. We introduce some notation associated with Y and Y_λ. Let the composite $Y_* \to G/H \times \mathfrak{h}_* \to \mathfrak{h}_*$ be h. Given a weight λ, the composite $\lambda \circ h: Y_* \to \mathbf{C}$ will be denoted y_λ. This map is meromorphic on Y and descends to $y_\lambda: Y_\lambda \to \mathbf{C}$. The action of W on the curve Y induces an action on the meromophic functions via $(w \cdot f)(y) = f(w^{-1} \cdot y)$. If $w \in W$ and Ω is a 1-form then let $w \cdot \Omega = w^{-1*}\Omega$. In this way W acts on $H^1(Y; \mathcal{O})$.

PROPOSITION 2. *If $w \in W$ then $w \cdot y_\lambda = y_{w \cdot \lambda}$.*

Proof. If $y \in Y$ then

$$(w \cdot y_\lambda)(y) = y_\lambda(w^{-1}(y)) = \lambda \circ h \circ w^{-1}(y) = \lambda \circ w^{-1} \circ h(y) = (w \cdot \lambda)(h(y)) = y_{w \cdot \lambda}(y). \qquad \square$$

Let $\operatorname{Jac} X$ be the Jacobian of a curve X. If X is irreducible, $\operatorname{Jac} X$ denotes $\operatorname{Pic}^0 X$, the space of holomorphic line bundles with Chern class zero. An element of $\operatorname{Pic}^0 X$ is determined by the divisor of a meromorphic section. We also view the Jacobian as $\mathbf{C}^g/L$ via the Abel–Jacobi map from the divisors of degree zero. Let $\omega_1, \ldots, \omega_g$ be a basis for the space of holomorphic differentials and $\vec{\omega} = (\omega_1, \ldots, \omega_g)$. Let $c_1, \ldots, c_{2g}$ be cycles that represent generators for the homology group $H_1(X, \mathbf{Z})$. The integrals $\int_{c_i} \vec{\omega} = (\int_{c_i} \omega_1, \ldots, \int_{c_i} \omega_g)$ generate L. If $\mathcal{D}$ is a divisor, then the Abel–Jacobi map sends it to $\int^{\mathcal{D}} \vec{\omega}$.

If G is a group that acts on X, then G acts on the holomorphic differentials, $\operatorname{Jac} X$, and $T_e \operatorname{Jac} X$. The action is given by

$$w\cdot\left(\int^{\mathcal{D}}\omega_1, \ldots, \int^{\mathcal{D}}\omega_g\right) = \left(\int^{w\cdot\mathcal{D}}\omega_1, \ldots, \int^{w\cdot\mathcal{D}}\omega_g\right) = \left(\int^{\mathcal{D}}w^*\omega_1, \ldots, \int^{\mathcal{D}}w^*\omega_g\right).$$

A Prym–Tjurin variety is an abelian variety that arises from a self-correspondence on a curve that satisfies a quadratic polynomial. A Prym variety is a special case that comes from a double cover and satisfies $x^2=1$. In [K], Prym–Tjurin varieties were identified in the Jacobians of some spectral curves. This was generalized and expressed in terms of group theory in [MS2]. In particular, if λ is a weight then there is a Prym–Tjurin $\operatorname{Tur}_\lambda Y \subset \operatorname{Jac} Y$. These respect the action of the Weyl group, $w\cdot\operatorname{Tur}_\lambda Y = \operatorname{Tur}_{w\cdot\lambda} Y$. There is a correspondence $P_\lambda = \sum_{w\in W}\langle\lambda, w^{-1}\lambda\rangle\, w$ such that $P_\lambda\colon \operatorname{Jac} Y \to \operatorname{Tur}_\lambda Y$. Now, $P_\lambda \in \mathbf{Z}[W]$. If M is a $\mathbf{C}[W]$-module, then $P_\lambda\colon M\to M$. If M is not $\mathfrak{h}^*\otimes\mathbf{C}$, then the map is zero. If M is $\mathfrak{h}^*\otimes\mathbf{C}$, then P_λ is a multiple of the projection onto the 1-dimensional subspace containing the weight λ. The subspace $T_e \operatorname{Tur}_\lambda Y \subset T_e \operatorname{Jac} Y$ is the image of P_λ, and so consists of one dimension for each occurrence of $\mathfrak{h}^*\otimes\mathbf{C}$ in $T_e \operatorname{Jac} Y$. It is

$$T_e \operatorname{Tur}_\lambda Y = \bigoplus \mathbf{C}\lambda \subset \bigoplus \mathfrak{h}^*\otimes\mathbf{C} \subset T_e \operatorname{Jac} Y.$$

2. Weight bundles and highest-weight flows

The motion of the Lax equation $dA/dt=[A,B]$ is given by the changing eigenvectors of matrix representations of A. For each time and each weight λ, associated eigenvectors give a line bundle $\mathcal{L}_\lambda(t)$ on the master spectral curve Y. Just as the spectral curves have a representation-free description via $G/H\times\mathfrak{h}_*$ and a representation-reliant description via determinants, so too do these time-dependent line bundles have two descriptions, one via characters and one via eigenvectors. The main theorem of this section states that the eigenvector bundles of the highest-weight representations are pullbacks of weight bundles, $L_{-\lambda}(G/B_\lambda)$. This is stated in Theorem 12. Several definitions are required for a precise statement.

For a simply-connected reductive Lie group G, an integral weight λ determines a character Λ on H by $\Lambda(h)=e^{\lambda(\log h)}$. The function $\Lambda(h)$ is well-defined although log is multivalued (see [V, pp. 357–358]). Suppose that $K\subset G$ is a closed subgroup that contains H and that Λ extends to a homomorphism $\Lambda\colon K\to\mathbf{C}^*$.

Definition. L_λ or $L_\lambda(K)$ is the complex line bundle on G/K which is the quotient of the following left K-action on the trivial bundle $G\times\mathbf{C}$:

$$k\cdot(g,z) = (gk^{-1}, \Lambda(k)^{-1}z).$$

We call these bundles weight bundles.

The case $K=H$ is important for integrable systems since G/H is the generic adjoint orbit, while $K=P$, a parabolic subgroup of G, has the advantage that G/P is a projective variety. For each weight λ of the simple Lie algebra $\mathfrak{g}$ with chosen Cartan subalgebra $\mathfrak{h}$, let B_λ be a positive Borel subgroup relative to a choice of Weyl chamber in which λ is a dominant weight. There is a parabolic subgroup P_λ of G given by $P_\lambda = B_\lambda S_\lambda B_\lambda$. Even though B_λ depends on a choice if λ is not regular, P_λ is completely well-defined. Although the bundle L_λ depends on the particular extension of Λ to K, the following lemma shows that the extension is unique if K is a parabolic. These are the bundles that the Borel–Weil theorem handles. The following lemma is well known.

LEMMA 3. *For any weight λ, let S_λ be its stabilizer in W. Let P be the parabolic subgroup $P=BS_\lambda B$. Then the character $\Lambda: H \to \mathbf{C}^*$ defined by $\Lambda(h)=e^{\lambda(\log h)}$ extends uniquely to a character (i.e. a homomorphism to $\mathbf{C}^*$) on P.*

Proof. The Levi decomposition $P=U \rtimes L$ expresses P as the semi-direct product of its unipotent radical U and a reductive subgroup L called a Levi factor. There is a unique choice of L containing H. Since U is contained in the commutator subgroup $[P,P]$ of P, any character on P must be trivial on U. On the other hand, since U is the normal subgroup in the semi-direct product $P=U \rtimes L$, any character on L extends to P by letting it be trivial on U. The factor L is a direct sum of simple subgroups and an abelian subgroup. Since each simple subgroup is its own commutator, any character on P must be trivial on the simple summands. The abelian summand is contained in H on which Λ is already given, so we see that if Λ has an extension to P, it is unique, and that Λ will have an extension if it acts trivially on the intersection of H with any of the simple summands of L. Now the group S_λ is a direct sum of reflection groups inside W. Let Φ be the set of simple roots $\{\alpha_i \mid r_i(\lambda)=\lambda\}$ where $r_i \in W$ is the reflection in the hyperplane perpendicular to α_i. The reflections Φ generate S_λ. Each α_i corresponds to a dot in the Dynkin diagram for G. Let I be a subset of Φ corresponding to a connected component of the full subdiagram of the Dynkin diagram whose vertices correspond to the elements of Φ. Then the group W_I generated by the reflections $\{r_i \mid i \in I\}$ is the Weyl group of one of the simple summands of L, namely the summand whose Lie algebra is generated by the root spaces in $\mathfrak{g}$ belonging to the roots $\{\pm\alpha_i \mid i \in I\}$. Since $W_I \subset S_\lambda$, S_λ acts irreducibly on the Cartan subalgebra $\mathfrak{h}_I$ of the simple summand S_I. For $X \in \mathfrak{h}_I$, from the invariance of λ under S_λ we have

$$\lambda(X) = \frac{1}{|W_I|} \sum_{w \in W_I} (w \cdot \lambda)(X) = \frac{1}{|W_I|} \sum_{w \in W_I} \lambda(w \cdot X) = \frac{1}{|W_I|} \lambda(0) = 0.$$

This shows that Λ acts trivially on each $H \cap S_I$, and thus extends to P. Details supporting this description may be found in [H, pp. 183–185]. $\square$

LEMMA 4. *If both $L_\lambda(K)$ and $L_\gamma(K)$ are defined, then $L_{\lambda+\gamma}(K)$ and $L_\lambda(K)\otimes L_\gamma(K)$ are isomorphic as holomorphic vector bundles.*

Proof. First note that $e^{(\lambda+\gamma)(\log h)}=e^{\lambda(\log h)}e^{\gamma(\log h)}$ for $h\in\mathfrak{h}$. So if Λ and Γ are characters on K which extend to those determined by λ and γ on H, then $\Lambda\cdot\Gamma$ extends to K, the character on H determined by $\lambda+\gamma$. The proof of the lemma is a check that the map $((g,z),(g,w))\mapsto(g,zw)$ is a well-defined isomorphism from $L_\lambda\otimes L_\gamma$ to $L_{\lambda+\gamma}$, where the elements in each of the line bundles are given by representatives in $G\times\mathbf{C}$. For an alternative pair of representatives with the same first entry, $(gh,\Lambda(h)z),(gh,\Gamma(h)w)$ maps to $(gh,\Lambda(h)z\Gamma(h)w)=(gh,\Lambda\cdot\Gamma(h)zw)$, which equals (g,zw) in $L_{\lambda+\gamma}$. The map is clearly one-to-one and onto. □

PROPOSITION 5. *The line bundles $L_\lambda(K)$ are all distinct topologically as bundles over the base G/K.*

Proof. If $\eta\colon G/H\to G/B$, then $\eta^*\colon H^2(G/B)\to H^2(G/H)$ is an isomorphism, since B/H, the fiber of η, is contractible. Now, $H^2(G/B)=\Lambda$, the weight lattice, and $c_1(L_\lambda(B))=\lambda$. Therefore, $c_1(L_\lambda(H))=c_1(\eta^*(L_\lambda(B)))=\eta^*(c_1(L_\lambda(B)))$ are distinct for all λ. □

PROPOSITION 6. *The action of the Weyl group W on G/H induces an action on the set of line bundles $L_\lambda(H)$. This action is $w\cdot L_\lambda=L_{w\cdot\lambda}$.*

Proof. For each $n\in N_H$ we define a bundle map $f_n\colon L_\lambda(H)\to L_{nH\cdot\lambda}(H)$ covering the action of nH on G/H. The f_n give the action of W on the $L_\lambda(H)$.

Let $f_n(g,z)=(gn^{-1},z)$ where $(g,z)\in G\times\mathbf{C}$ represents a point in $L_\lambda(H)=(G\times\mathbf{C})/H$ while (gn^{-1},z) represents a point in $L_{nH\cdot\lambda}$. We will show that this action is well-defined. An alternative representative $(gh,e^{\lambda(\log h)}z)$ goes to $(ghn^{-1},e^{\lambda(\log h)}z)=(gn^{-1}(nhn^{-1}),e^{\lambda(\log h)}z)$. Now, $e^{(nH\cdot\gamma)(\log h)}=e^{\gamma(n^{-1}(\log h)n)}=e^{\gamma(\log(n^{-1}hn)}$ for any $\gamma\in P$. Hence the alternative representative maps to $(gn^{-1}(nhn^{-1}),e^{(nH\cdot\lambda)(\log nhn^{-1})}z)$, which is equivalent to (gn^{-1},z) in $L_{nH\cdot\lambda}(H)$. □

The weight bundles over G/H form a $\mathbf{Z}[W]$-module. Addition is by tensor product and the W-action is induced by the group action described above.

PROPOSITION 7. *The assignment $\lambda\mapsto L_\lambda(H)$ is a homomorphism of $\mathbf{Z}[W]$-modules.*

Proof. By Lemma 4, $L_{\lambda+\gamma}(H)=L_\lambda(H)\otimes L_\gamma(H)$. By Proposition 6, $w\cdot L_\lambda(H)=L_{w\cdot\lambda}(H)$. □

In defining $L_\lambda(B)$ the equivalence relation used is $(gb,z)\sim(g,\Lambda^{-1}(b)z)$ instead of $(gb,z)\sim(g,\Lambda(b)z)$. This choice is consistent with the mechanics developed in [AM1],

[AM2] and [G]. However, it is not consistent with most presentations of the Borel–Weil theorem. In our context the theorem states that if ϱ is the representation with highest weight λ then the space of holomorphic sections of $L_\lambda(B)$ is the representation ϱ. A step in the proof of the Borel–Weil theorem is to realize $L_\lambda(B)$ as the pullback of the hyperplane bundle. We require this construction and now describe it.

View $\mathbf{C}P^{N-1}$ as the space of complex lines through the origin in $\mathbf{C}^N$. The hyperplane bundle over $\mathbf{C}P^{N-1}$ has for its fiber over a line in $\mathbf{C}^N$ the linear functionals on that line. In the correspondence between line bundles and equivalence classes of divisors, the hyperplane bundle is the line bundle whose corresponding divisor class is the set of hyperplanes. It is the only bundle over $\mathbf{C}P^{N-1}$ with Chern class 1 and we denote it $\mathcal{O}_{\mathbf{C}P^{N-1}}(1)$. The dual of the hyperplane bundle is the canonical line bundle over $\mathbf{C}P^{N-1}$ whose fiber over a line in $\mathbf{C}^N$ is that line itself. The canonical line bundle is a subbundle of the trivial bundle $\mathbf{C}P^{N-1}\times\mathbf{C}^N$. Denote it $\mathcal{O}_{\mathbf{C}P^{N-1}}(-1)$.

If the irreducible representation of G with highest weight λ has highest-weight vector v_λ then $P_\lambda=B^+S_\lambda B^+$ acts on v_λ by multiplication by the character Λ. Denote by $[\,\cdot\,]\colon\mathbf{C}^N\to\mathbf{C}P^{N-1}$ the map sending a point to the line it spans.

LEMMA 8. *Let ϱ be the irreducible representation of G acting on $\mathbf{C}^N$ with highest weight λ and highest-weight vector v_λ. Then the line bundle $L_\lambda(P_\lambda)$ is isomorphic to the pullback of the hyperplane bundle along the map $G/P_\lambda\to\mathbf{C}P^{N-1}$ that maps gP_λ to the line through $\varrho(g)v_\lambda$. This map gives an imbedding of G/P_λ into $\mathbf{C}P^{N-1}$. Furthermore the pullback of the canonical line bundle is $L_{-\lambda}(P_\lambda)$.*

Proof. Let ϱ^* be the representation dual to ϱ and $v^*_{-\lambda}$ the lowest-weight vector for ϱ^*. The map of L_λ to the hyperplane bundle given by

$$(g,z)\mapsto([\varrho(g)v_\lambda],z\varrho^*(g)v^*_{-\lambda})$$

realizes L_λ as a pullback of the hyperplane bundle. Here $z\varrho^*(g)v^*_{-\lambda}$ means the restriction of this linear functional on $\mathbf{C}^N$ to the line spanned by $\varrho(g)v_\lambda$.

We will check that the map is well-defined. Any other representative for (g,z) has the form $(gp,\Lambda(p)z)$. We already know that $\varrho(gp)v_\lambda$ and $\varrho(g)v_\lambda$ lie on the same line, since for a highest-weight vector $\varrho(p)v_\lambda=\Lambda(p)v_\lambda$. Let $k\varrho(g)v_\lambda$ be an arbitrary point on this line and $\langle\cdot,\cdot\rangle$ be the pairing of dual spaces. We must check that $\langle\Lambda(p)z\varrho^*(gp)v^*_{-\lambda},k\varrho(g)v_\lambda\rangle=\langle z\varrho^*(g)v^*_{-\lambda},k\varrho(g)v_\lambda\rangle$. On the right, by definition, $\langle\varrho^*(g)v^*,v\rangle=\langle v^*,\varrho(g^{-1})v\rangle$, so that $\langle z\varrho^*(g)v^*_{-\lambda},k\varrho(g)v_\lambda\rangle=zk\langle v^*_{-\lambda},v_\lambda\rangle$. On the left-hand side is

$$\langle\Lambda(p)z\varrho^*(gp)v^*_{-\lambda},k\varrho(g)v_\lambda\rangle=\Lambda(p)zk\langle v^*_{-\lambda},\varrho((gp)^{-1})\varrho(g)v_\lambda\rangle=\Lambda(p)zk\langle v^*_{-\lambda},\varrho(p^{-1})v_\lambda\rangle.$$

But the action of p^{-1} on v_λ is multiplication by the number $\Lambda(p)^{-1}$ so the two expressions for the map to the hyperplane bundle agree. The map gives an imbedding of G/P_λ since the stabilizer of $[v_\lambda]$ under the action of G on $\mathbf{C}P^{N-1}$ determined by ϱ is P_λ. The map of $L_{-\lambda}(P)$ to the canonical line bundle given by

$$(g,z)\mapsto([\varrho(g)v_\lambda], z\varrho(g)v_\lambda)$$

realizes $L_{-\lambda}$ as a pullback of the canonical line bundle. Suppose that $(gp, \Lambda(p)^{-1}z)$ is another representation of (g,z). Then its image is $([\varrho(gp)v_\lambda], z\Lambda(p)^{-1}\varrho(gp)v_\lambda)$. Since λ is the highest weight, $\varrho(p)=\Lambda(p)v_\lambda$. Hence $([\varrho(gp)v_\lambda], z\Lambda(p)^{-1}\varrho(gp)v_\lambda)=([\varrho(g)v_\lambda], z\varrho(g)v_\lambda)$. □

The discussion now switches to describing the line bundles by means of representations and eigenvectors of matrices representing the element A in the Lax equation. The following lemma is well known.

LEMMA 9. *Suppose that R is a smooth Riemann surface and that $f\colon R\to\mathbf{C}P^{N-1}$ is a meromorphic map in each coordinate, i.e. $f=(f_0,\dots,f_n)$ in homogeneous coordinates with each f_i meromorphic. Then there exists a unique completion of f to $f\colon R\to\mathbf{C}P^{N-1}$.*

Proof. Suppose that $x\in R$ is an isolated singularity and that f_j has the largest-order pole among the f_i's. Then in affine coordinates, $(f_0/f_j,\dots,f_n/f_j)$, the map clearly extends since the singularities are removable. □

For a meromorphic map $A\colon P\to\mathfrak{g}$ on the parameter space, denote the induced map (see the diagram below) on the spectral curve $Y_{\lambda*}$ as $\tilde{A}_\lambda\colon Y_{\lambda*}\to(G/H\times\mathfrak{h}_*)/S_\lambda$ and its composite with the projection to G/P_λ as $u_\lambda\colon Y_{\lambda*}\to G/P_\lambda$. By Lemma 9, u_λ extends uniquely to a map on Y_λ:

$$\begin{tikzcd}
Y_\lambda \arrow[rrd, "u_\lambda"] & & \\
\cup & & \\
Y_{\lambda*} \arrow[r, "\tilde{A}_\lambda"] \arrow[d] & (G/H\times\mathfrak{h}_*)/S_\lambda \arrow[r] \arrow[d] & G/P_\lambda \\
P_* \arrow[r, "A"] & \mathfrak{g}_*. &
\end{tikzcd}$$

The function $y_{w\cdot\lambda}$ is defined on $Y_{\lambda*}$ as introduced in §1. For each point $x\in Y_{\lambda*}$ with coordinates s and $z=y_{w\cdot\lambda}(x)$ there is a one-dimensional eigenspace of $\varrho(A(s))$ in $\mathbf{C}^N$ associated to the eigenvalue z, where ϱ is the irreducible representation of $\mathfrak{g}$ acting on $\mathbf{C}^N$ whose highest weight is λ and w is an element of the Weyl group.

Definition. Suppose that ϱ is an irreducible representation of $\mathfrak{g}$ on $\mathbf{C}^N$ with highest weight λ. The eigenvector mappings, denoted $f_{w\cdot\lambda}\colon Y_\lambda\to\mathbf{C}P^{N-1}$, are defined by the equation $\varrho(A(s))\bar{f}_{w\cdot\lambda}(x)=y_{w\cdot\lambda}(x)\bar{f}_{w\cdot\lambda}(x)$ for $x\in Y_{\lambda*}$ and $\bar{f}_{w\cdot\lambda}\colon Y_{\lambda*}\to\mathbf{C}^N$ such that $f_{w\cdot\lambda}=[\bar{f}_{w\cdot\lambda}]$.

The span of $f_{w\cdot\lambda}(x)$ is the one-dimensional eigenspace of $\varrho(A(s))$ with eigenvalue $y_{w\cdot\lambda}(x)$. Consequently, $f^*_{w\cdot\lambda}(\mathcal{O}_{\mathbf{C}P^{N-1}}(+1))$ is a bundle of eigenvectors. However, we shall follow the custom of other authors and use $f^*_{w\cdot\lambda}(\mathcal{O}_{\mathbf{C}P^{N-1}}(-1))$, pulling back the hyperplane bundle instead of its dual, the canonical line bundle.

LEMMA 10. *The maps $f_{w\cdot\lambda}$ are well-defined.*

Proof. Write z for the eigenvalue $y_{w\cdot\lambda}(x)$ of $\varrho(A(s))$ and f for the eigenvector $f_{w\cdot\lambda}$. The homogeneous coordinates of $f\colon Y_{\lambda*}\to\mathbf{C}P^{N-1}$ are minors of the map $Y_{\lambda*}\to \mathrm{Mat}_{N\times N}(\mathbf{C})$ given by $x\to\varrho(A(s))-z$. The coordinates s and z on $Y_{\lambda*}$ are meromorphic maps to P_* and $\mathbf{C}$ respectively, since $Y_{\lambda*}$ is a Zariski-open subset of the zero set of the polynomial $p_\lambda(s,z)$. By assumption the map $A\colon P\to\varrho(\mathfrak{g}_*)$ is meromorphic. Hence f is meromorphic on $Y_{\lambda*}$, and so extends to a map on Y_λ by Lemma 9. $\square$

LEMMA 11. *If $w\cdot\lambda$ is a maximal weight for the irreducible representation ϱ, then the map $f_{w\cdot\lambda}\colon Y_\lambda\to\mathbf{C}P^{N-1}$ factors through the map $u_{w\cdot\lambda}\colon Y_\lambda\to G/P_{w\cdot\lambda}$.*

Proof. It is sufficient to check this with domain $Y_{\lambda*}$. The weight $w\cdot\lambda$ is in the dominant chamber for $P_{w\cdot\lambda}$. Let $x\in Y_{\lambda*}$, $s=\pi(x)$, and $z=y_{w\cdot\lambda}(x)$. Write $\tilde{A}(x)=(gH,h)$ mod S_λ, which is in $(G/H\times\mathfrak{h}_*)/S_\lambda$. Then $A(s)=\mathrm{Ad}_g\,h$ and $z=w\cdot\lambda(h)$. Let $v_{w\cdot\lambda}$ be a highest-weight vector of ϱ. Then $\varrho(h)v_{w\cdot\lambda}=w\lambda(h)v_{w\cdot\lambda}$. It follows that

$$\varrho(\mathrm{Ad}_g\,h)\varrho(g)v_{w\cdot\lambda}=\varrho(g)\varrho(h)\varrho(g^{-1})\varrho(g)v_{w\cdot\lambda}=\varrho(g)\varrho(h)v_{w\cdot\lambda}=z\varrho(g)v_{w\cdot\lambda}.$$

Since $w\lambda$ is dominant for $P_{w\cdot\lambda}$, we have $\varrho(gp)v_{w\cdot\lambda}=w\Lambda(p)\varrho(g)v_{w\cdot\lambda}$ for $p\in P_{w\cdot\lambda}$. Therefore, $f_{w\lambda}(x)=[u_{w\cdot\lambda}(x)v_{w\cdot\lambda}]$. $\square$

The following theorem is our main theorem on eigenvector bundles.

THEOREM 12. *Suppose that λ is a maximal weight for the irreducible representation ϱ. The eigenvector bundle $f^*_\lambda(\mathcal{O}_{\mathbf{C}P^{N-1}}(-1))$ is isomorphic to $u^*_\lambda(L_{-\lambda}(P_\lambda))$. The pullback of the hyperplane bundle along the eigenvector map, $f^*_\lambda(\mathcal{O}_{\mathbf{C}P^{N-1}}(1))$, is isomorphic to the pullback of the weight bundle along u_λ, $u^*_\lambda(L_\lambda(P_\lambda))$.*

Proof. Take a choice of chamber so that λ is dominant. The factorization of Lemma 11 applies. By Lemma 8, the pullbacks of $\mathcal{O}_{\mathbf{C}P^{N-1}}(-1)$ and $\mathcal{O}_{\mathbf{C}P^{N-1}}(1)$ under $G/P_\lambda\to\mathbf{C}P^{N-1}$ are $L_{-\lambda}$ and L_λ respectively. Precomposing this map with u_λ yields f_λ. The theorem follows. $\square$

Definition. $\mathcal{L}_\lambda$ denotes the bundle $f^*_\lambda(\mathcal{O}_{\mathbf{C}P^{N-1}}(1))$ obtained by pulling back the hyperplane bundle along the eigenvector mapping $f_\lambda\colon Y_\lambda\to\mathbf{C}P^{N-1}$ or any of its pullbacks further up the hierarchy of spectral curves. When it is necessary to be explicit which curve is the base of the bundle, denotations such as $\mathcal{L}_\lambda(Y_\lambda)$ or $\mathcal{L}_\lambda(Y)$ will serve.

PROPOSITION 13. $w\cdot\mathcal{L}_\lambda(Y)=\mathcal{L}_{w\cdot\lambda}(Y)$.

Proof. $\mathcal{L}_\lambda(Y)$ is the pullback of $L_\lambda(B_\lambda)$ along the top row of the diagram

$$\begin{tikzcd} Y_* \arrow[r, "u_\lambda"] \arrow[d, "w"'] & G/B_\lambda \arrow[d, "w"] \\ Y_* \arrow[r, "u_{w\cdot\lambda}"] & G/B_{w\cdot\lambda} \end{tikzcd}$$

while $\mathcal{L}_{w\cdot\lambda}(Y)$ is the pullback of $L_{w\cdot\lambda}(B_{w\cdot\lambda})$ along the bottom row. The map $w\colon G/B_\lambda\to G/B_{w\cdot\lambda}$ is induced from $g\mapsto gn^{-1}$ since $gb\mapsto gbn^{-1}=gn^{-1}(nbn^{-1})$ and $nB_\lambda n^{-1}=B_{w\cdot\lambda}$. We check commutativity on Y_* and Y follows by continuity. If $\tilde{A}(x)=(gH,h)$ for $x\in Y_*$, then $u_\lambda(x)=g \bmod B_\lambda$. By the definition of Y_*, $\tilde{A}(nH\cdot x)=(gn^{-1}H,\operatorname{Ad}_n h)$, so we have $u_{w\lambda}(w\cdot x)=gn^{-1} \bmod B_{w\cdot\lambda}$ for $w=nH$. □

PROPOSITION 14. *If λ and γ are in the same Weyl chamber, then on the master curve Y the bundles $\mathcal{L}_\lambda\otimes\mathcal{L}_\gamma$ and $\mathcal{L}_{\lambda+\gamma}$ are isomorphic.*

Proof. For a Borel B all of λ, γ, and $\lambda+\gamma$ are dominant. Hence,

$$u_\lambda=u_\gamma=u_{\lambda+\gamma}\colon Y\to G/B.$$

By Lemma 4, $L_{\lambda+\gamma}(B)=L_\lambda(B)\otimes L_\gamma(B)$, and so $u^*_{\lambda+\gamma}L_{\lambda+\gamma}(B)=u^*_\lambda L_\lambda(B)\otimes u^*_\gamma L_\gamma(B)$. The proposition follows from Theorem 12 and the definition of $\mathcal{L}_\lambda$. □

The solution to a Lax equation $dA/dt=[A,B]$ with a parameter is a flow in the space of functions A_t from the parameter space to $\mathfrak{g}$. The flow of the equation transforms to a flow on $\operatorname{Pic} Y_\lambda$ by pulling the hyperplane bundle back along the associated eigenvector map $f_\lambda(t)$. This flow may be translated to $\operatorname{Jac} Y_\lambda$ beginning at the trivial bundle. The flow is linear when the system is algebraically completely integrable.

Definition. Let $\xi_\lambda(t)=\mathcal{L}_\lambda(t)\otimes\mathcal{L}_\lambda(0)^{-1}\in\operatorname{Jac} Y$.

$\xi_\lambda(t)$ is defined for any weight λ of $\mathfrak{g}$.

PROPOSITION 15. *For any $w\in W$ and any weight λ,*

$$w\cdot\xi_\lambda(t)=\xi_{w\cdot\lambda}(t).$$

Proof. The action on the left-hand side is the $\mathbf{Z}[W]$-action on $\operatorname{Jac} Y=\operatorname{Pic}^0 Y$. In fact, $\mathbf{Z}[W]$ acts on $\operatorname{Pic} Y$ too. Since multiplication by w distributes over addition,

$$w\cdot\xi_\lambda(t)=w\cdot\mathcal{L}_\lambda(t)\otimes(w\cdot\mathcal{L}_\lambda(0))^{-1}.$$

By Proposition 13 this equals

$$\mathcal{L}_{w\cdot\lambda}(t)\otimes(\mathcal{L}_{w\cdot\lambda}(0))^{-1}=\xi_{w\cdot\lambda}(t). \qquad □$$

PROPOSITION 16. *If λ and γ are in the same Weyl chamber, then $\xi_\lambda(t)\otimes\xi_\gamma(t)=\xi_{\lambda+\gamma}(t)$ in* $\operatorname{Jac} Y$.

Proof. This follows from Proposition 14. □

If the previous proposition were known to hold for all pairs of weights, then combining it with Proposition 15 would say that the flows $\xi_\lambda(t)$ form a $\mathbf{Z}[W]$-module isomorphic to the weight lattice. When the $\mathbf{Z}[W]$-module of flows is isomorphic to the weight lattice, a consequence is that each flow $\xi_\lambda(t)$ takes place in the Prym–Tjurin variety $\operatorname{Tur}_\lambda(Y)$.

3. The velocity vector of the flow

The derivative of the λ_1-flow can be found in coordinates using a version of Theorem 5.3 in [M, p. 71]. The idea is from [AM2, pp. 322–323] which in turn utilizes [MM].

We introduce some notation followed by a list of technical hypotheses necessary for the intermediate results. Let $A, B\colon P\to\mathfrak{g}$, P a compact Riemann surface. We will often refer to them as matrices, i.e. consider them in some finite-dimensional representation. Let X be the normalization of an irreducible component of multiplicity 1 of the curve defined by $\det(A-zI)=0$. Write $\pi\colon X\to P$ for the projection. Denote by X_∞ the set of points of X where either $z=\infty$ or one of entries of A or B has a pole (or both). Let $\widehat{X}$ be a finite set that contains $X_\infty\cup\pi^{-1}A^{-1}(\mathfrak{g}-\mathfrak{g}_*)$. Let f be an eigenvector associated with X, i.e. f is a column vector whose entries are functions on X and which satisfies the equation $(A\circ\pi)f=zf$, which below we write $Af=zf$.

HYPOTHESES. (1) *The entries in A and B are meromorphic in $s\in P$ and C^2 in time.*

(2) *The set $P_\infty=\{s\in P\mid$ an entry in $A(s,t)$ or $B(s,t)$ is $\infty\}$ is independent of time.*

(3) *$dA/dt=[A,B]=AB-BA$ for t in a neighborhood of $t=0$ and for all $s\in P-P_\infty$.*

(4) *For some value of the parameter s, the matrix $A(s,0)$ is regular.*

(5) *The eigenvector f has entries that are C^2 in time on $X-\widehat{X}$ and meromorphic on X.*

The first four conditions are independent of the particular representation since a representation $\varrho\colon\mathfrak{g}\to\operatorname{Aut} V$ is a complex linear map. Note that the set $X_\infty\subset X$ is independent of time under the hypotheses above. If $z=\infty$ then $z=\infty$ for all time as z is a constant of the motion for a Lax equation. $A^{-1}(\mathfrak{g}-\mathfrak{g}_*)$ is independent of time since A is isospectral and $\mathfrak{g}_*$ is a union of orbits. If an entry has a pole then it is invariant under time by assumption (2). We consider a finite set that contains $X_\infty\cup\pi^{-1}A^{-1}(\mathfrak{g}-\mathfrak{g}_*)$ since later we will wish it to be invariant under a group action. Also note that $\widehat{X}$ contains the branch points of $X\to P$.

THEOREM 17. *Let $dA/dt=[A,B]$ be a matrix equation satisfying the hypotheses above. Let $\mathcal{D}(t)$ be the divisor $(f(t))$. If the restriction of $\mathcal{D}(t)$ to $\widehat{X}$ is independent of the time t in some neighborhood of $t=0$, then*

(1) *the equation $Bf+\dot{f}=f\Lambda$ defines a function $\Lambda(t)\colon X\to\mathbf{C}$ for t near 0, and*

(2) *for any holomorphic differential ω on X,*

$$\lim_{t\to 0}\frac{1}{t}\int_{\mathcal{D}(0)}^{\mathcal{D}(t)}\omega=-\sum_{P\in\widehat{X}}\operatorname*{Res}_P(\Lambda(0)\omega)$$

where Res_P denotes taking the residue at P of the differential.

Before proceeding with the proof some remarks are in order.

(1) The divisor (f) of a scalar-valued function is its zeros minus its poles. By the divisor $\mathcal{D}=(f)$ of a vector-valued function is meant the maximal divisor $\mathcal{D}$ such that $\mathcal{D}\leqslant(f_i)$ for each of the components f_i of f. This is because f is really a section of the eigenvector bundle and (f) is its divisor as a section.

(2) The present variation of the theorem is limited to eigenspaces of dimension 1, i.e. the component X has multiplicity 1. This restriction is not necessary, merely sufficient for the purposes of this paper. Theorem 5.3 of [M] does not have this restriction.

Proof. Let a dot denote d/dt. Let $X_a=X-\widehat{X}$.

The equations $\dot{U}=-BU$, $U(0)=I$, define an invertible matrix $U(t)$ for small t which is C^2 on X_a since B is C^2 on X_a. Application of $\dot{A}=AB-BA$, $(U^{-1})^{\cdot}=-U^{-1}\dot{U}U^{-1}$, and $\dot{U}=-BU$ gives $d(U^{-1}AU)/dt=0$. Hence $A(t)=U(t)A(0)U^{-1}(t)$.

We verify that $Bf+\dot{f}$ is an eigenvector of $A(t)$ with eigenvalue z. Since $Af=zf$, differentiating and plugging in the Lax equation gives $ABf-BAf+A\dot{f}=z\dot{f}$. Since $BAf=Bzf=zBf$, we have $A(Bf+\dot{f})=z(Bf+\dot{f})$. Thus $(Bf+\dot{f})$ is a multiple of f and so $Bf+\dot{f}=\Lambda f$ defines the function $\Lambda(t)$.

We shall find (define) a function $\eta(t)$ with the property that $(\eta)_a=\mathcal{D}(t)-\mathcal{D}(0)$ where $(\eta)_a$ means the divisor of η restricted to X_a. Since $U^{-1}(t)A(t)U(t)=A(0)$, i.e. $A(t)U(t)=U(t)A(0)$, we have $A(t)(U(t)f(0))=U(t)A(0)f(0)=U(t)zf(0)=z(U(t)f(0))$. Therefore $U(t)f(0)$ is a multiple of $f(t)$. Define $\eta(t)$ by $U(t)f(0)=\eta(t)f(t)$.

This defines $\eta(t)\colon X_a\to\mathbf{C}$, but since U is not defined off X_a, neither is η. Observe that $U(t)f(0)=f(t)\eta(t)$ on X_a, and for small t, $U(t)$ is holomorphic and invertible. Hence the divisor of the section $U(t)f(0)$ is the same as the divisor of the section $f(0)$ when the two are restricted to X_a. Also $(f(t)\eta(t))_a=(f(t))_a+(\eta(t))_a$. Hence $(f(0))_a=(f(t))_a+(\eta(t))_a$ or $(-\eta(t))_a=(f(t))_a-(f(0))_a=(f(t))-(f(0))$ since by hypothesis $(f(t))$ and $(f(0))$ agree on $\widehat{X}$. We show that $\dot{\eta}=\Lambda\eta$ on X_a. Differentiate $U(t)f(0)=\eta(t)f(t)$ and plug in $\dot{U}=-BU$ to get $-B(t)U(t)f(0)=\dot{\eta}(t)f(t)+\eta(t)\dot{f}(t)$. Substituting

$U(t)f(0)=\eta(t)f(t)$ into this equation, it rearranges to $-\eta(t)(B(t)f(t)+\dot{f}(t))=\dot{\eta}(t)f(t)$. Thus $-\eta(t)\Lambda(t)f(t)=\dot{\eta}(t)f(t)$. So on X_a, the holomorphic functions $\eta(t)\Lambda(t)$ and $\dot{\eta}(t)$ agree. From this differential equation, $\eta(0)=1$ since $U(0)f(0)=\eta(0)f(0)$, and $U(0)=I$. Therefore $\dot{\eta}(0)=\Lambda(0)$.

We claim that $\eta(t)=1+\Lambda(0)t+O(t^2)$. This is the Taylor expansion for η. It is sufficient that η have a continuous second derivative. The differentiability of η is inherited from that of f and B, via U.

Now, $(1-\Lambda(0)t)_a=\mathcal{D}'(t)-\mathcal{D}'(0)$ where $\mathcal{D}'(0)$ is the divisor of poles and $\mathcal{D}'(t)$ is the divisor of zeros on X_a of $1-\Lambda(0)t$. We apply the following claim from p. 73 of [M].

Claim. $\mathcal{D}'(t)$ and $\mathcal{D}'(0)$ have the same order and for any holomorphic differential ω on X,

$$\int_{\mathcal{D}(0)}^{\mathcal{D}(t)} \omega = \int_{\mathcal{D}'(0)}^{\mathcal{D}'(t)} \omega + O(t^2).$$

Now compute

$$\lim_{t\to 0} \frac{1}{t} \int_{\mathcal{D}'(0)}^{\mathcal{D}'(t)} \omega.$$

The function $1-\Lambda(0)t$ has poles where $\Lambda(0)$ does. At each point $P\in\widehat{X}$, $\Lambda(0)$ may have a pole of order r and, for small t, r nearby zeros. Call them $Z_1,\ldots,Z_r$. By Abel's theorem,

$$\int_{\mathcal{D}'(0)}^{\mathcal{D}'(t)} \omega = -\sum_{P\in\widehat{X}} \int_{Z_1,\ldots,Z_r}^{rP} \omega.$$

By Lemma 20 of [MM], this limit is $-\sum_{P\in\widehat{X}} \operatorname{Res}_P(\Lambda(0)\omega)$. □

The following formula is needed.

LEMMA 18. *For $\omega\in\Omega^{1,0}(Y)$, f a function on Y, and $w\in W$ a covering translation,*

$$\operatorname*{Res}_{Q}(f\circ w\cdot\omega) = \operatorname*{Res}_{w(Q)}(f\cdot w^{-1*}\omega).$$

PROPOSITION 19. *Assume the hypotheses from the beginning of the section. Let $\widehat{Y}$ be a finite W-equivariant set such that $\pi_X^{-1}(\widehat{X})\subset\widehat{Y}$. Let $z\colon Y\to\mathbf{C}$ be the composition of $\pi_X\colon Y\to X$ with $z\colon X\to\mathbf{C}$. Let $f\colon Y\to\mathbf{C}^N$ satisfy $(A\circ\pi)f=zf$ and let $\mathcal{D}(t)$ be the divisor of the meromorphic section $f(t)$.*

Then

(1) *there is a unique time-dependent function $\lambda\colon Y\to\mathbf{C}$ such that $Bf+\dot{f}=\lambda f$, and*

(2) *for any holomorphic differential ω on Y,*

$$\lim_{t\to 0} \frac{1}{t} \int_{\mathcal{D}(0)}^{\mathcal{D}(t)} \omega = \sum_{Q\in\widehat{Y}} \operatorname*{Res}_{Q} \lambda(0)\omega.$$

Proof. First note that $\widehat{Y}$ is an equivariant version of $\pi_X^{-1}(X_\infty)$. The unique function λ is $\lambda=\Lambda\circ\pi_X$ where Λ is defined in Theorem 17. Write $f_1\colon X\to\mathbf{C}$ for the function on X satisfying $Af_1=zf_1$. Then $f=f_1\circ\pi_X$. Let $\mathcal{D}_1(t)$ be the divisor of f_1. Then $\mathcal{D}(t)=\pi_X^{-1}(\mathcal{D}_1(t))$.

The curve X is a quotient of Y by a subgroup of W; write $X=Y/S$. The space of holomorphic differentials $\Omega^{1,0}(Y)$ decomposes as $\Omega^{1,0}(Y)=\ker\mathrm{Ave}\oplus\pi_X^*(\Omega^{1,0}(X))$ where $\mathrm{Ave}\colon\Omega^{1,0}(Y)\to\Omega^{1,0}(Y)$ is averaging over the orbit of the action of S.

If $\omega=\pi_X^*\eta$ then

$$\lim_{t\to 0}\frac{1}{t}\int_{\mathcal{D}(0)}^{\mathcal{D}(t)}\omega=|S|\lim_{t\to 0}\frac{1}{t}\int_{\mathcal{D}_1(0)}^{\mathcal{D}_1(t)}\eta=|S|\sum_{P\in\widehat{X}}\operatorname*{Res}_P\Lambda(0)\eta=\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q\lambda(0)\pi_X^*\eta,$$

and the formula follows for $\omega\in\pi_X^*\Omega^{1,0}(X)$.

If $\omega\in\ker\mathrm{Ave}$ then

$$\lim_{t\to 0}\frac{1}{t}\int_{\mathcal{D}(0)}^{\mathcal{D}(t)}\omega=\lim_{t\to 0}\frac{1}{t}\cdot\frac{1}{|S|}\sum_{g\in S}\int_{g\mathcal{D}(0)}^{g\mathcal{D}(t)}\omega=\lim_{t\to 0}\frac{1}{t}\cdot\frac{1}{|S|}\int_{\mathcal{D}(0)}^{\mathcal{D}(t)}\sum_{g\in S}g^*\omega=\lim_{t\to 0}\frac{1}{t}\int_{\mathcal{D}(0)}^{\mathcal{D}(t)}\mathrm{Ave}\,\omega=0$$

On the other hand,

$$\begin{aligned}\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q\lambda(0)\omega&=\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q(\lambda(0)\circ g\circ\omega)\qquad(\text{for } g\in S)\\&=\sum_{Q\in\widehat{Y}}\operatorname*{Res}_{g(Q)}(\lambda(0)(g^{-1})^*\omega)\qquad(\text{by Lemma 18})\\&=\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q(\lambda(0)(g^{-1})^*\omega)\qquad(\text{since } g(\widehat{Y})=\widehat{Y}),\end{aligned}$$

and therefore

$$\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q\lambda(0)\omega=\sum_{Q\in\widehat{Y}}\frac{1}{|S|}\sum_{g\in S}\operatorname*{Res}_Q(\lambda(0)(g^{-1})^*\omega)=\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q(\lambda(0)(\mathrm{Ave}\,\omega))=0.$$

So if $\omega\in\ker\mathrm{Ave}$ then the equation holds. Since $\lim_{t\to 0}(1/t)\int_{\mathcal{D}(0)}^{\mathcal{D}(t)}\omega$ and $\sum_{Q\in\widehat{Y}}\mathrm{Res}_Q\,\lambda(0)\omega$ are both linear in ω, the result follows. □

4. Wedge representations

In this section, wedge representations are examined in order to obtain relations between the highest-weight flow vectors. The main result is the following proposition.

PROPOSITION 25. *Suppose that the Lax equation $dA/dt=[A, B]$ satisfies the hypotheses from the beginning of §3 for each fundamental representation, and that for each fundamental wieght λ_i the restriction of the divisor of f_{λ_i} to $\hat{Y}$ is independent of time for t in a neighborhood of 0. Then the velocity vectors v_{λ_i} satisfy the system of equations*

$$v_{\lambda_i}+\sigma_i(v_{\lambda_i})=-\sum A_{ij}v_{\lambda_j}, \quad \text{where } A_{ij} \text{ is the Cartan matrix,}$$
$$w\cdot v_{\lambda_i}=v_{\lambda_i} \quad \text{for all } w\in S_{\lambda_i}.$$

LEMMA 20. *Let λ be a dominant weight and γ the highest weight of the representation $\varrho_\lambda\wedge\varrho_\lambda$. Suppose that $\gamma=\lambda+w\cdot\lambda$. Let $\hat{Y}$ be a finite W-equivariant set containing $\pi^{-1}(\hat{X})$. Additionally suppose that the hypotheses hold for the representation ϱ_λ, and that f_λ is an eigenvector satisfying the conditions*:

(1) $\varrho_\lambda(A)f_\lambda=y_\lambda f_\lambda$ *for all* $y\in Y_*$,

(2) *the entries of* f_λ *are* C^2 *in the time* t *and meromorphic on the curve* Y,

(3) *the restriction of the divisor of* f_λ *to* $\hat{Y}$ *is independent of* t.

Define f_γ to be the projection of $f_\lambda\wedge(f_\lambda\circ w^{-1})$ onto the summand of the representation $\varrho_\lambda\wedge\varrho_\lambda$, which is an irreducible module with highest weight γ.

Then f_γ satisfies the three conditions:

(1) $\varrho_\gamma(A)f_\gamma=y_\gamma f_\gamma$ *for all* $y\in Y_*$,

(2) *the entries of* f_γ *are* C^2 *in* t *and meromorphic on the curve* Y,

(3) *the restriction of the divisor of* f_γ *to* $\hat{Y}$ *is independent of* t.

Let Λ and Γ be defined by the equations

$$\varrho_\lambda(B)f_\lambda+\dot{f}_\lambda=\Lambda f_\lambda \quad \text{for all } y\in Y_*,$$
$$\varrho_\gamma(B)f_\gamma+\dot{f}_\gamma=\Gamma f_\gamma \quad \text{for all } y\in Y_*.$$

Then $\Gamma=\Lambda+\Lambda\circ w^{-1}$.

Proof. Evaluate the equation $\varrho_\lambda(A)f_\lambda=y_\lambda f_\lambda$ at $w^{-1}(y)$. Observe that $\varrho_\lambda(A)=\varrho_\lambda(A(\pi(y))$ is unchanged since $\pi(w^{-1}(y))=\pi(y)$, while $y_\lambda(w^{-1}(y))=(w\cdot y_\lambda)(y)=y_{w\cdot\lambda}(y)$, using Proposition 2. Hence $\varrho_\lambda(A)(f_\lambda\circ w^{-1})=(y_{w\cdot\lambda})(f\circ w^{-1})$.

By assumption the highest-weight vector of $\varrho_\lambda\wedge\varrho_\lambda$ is $\lambda+w\cdot\lambda$. The wedge product $f\wedge(f\circ w^{-1})$ is an eigenvector of $(\varrho_\lambda\wedge\varrho_\lambda)(A)$ since

$$\begin{aligned}((\varrho_\lambda\wedge\varrho_\lambda)(A))(f\wedge(f\circ w^{-1}))&=\varrho_\lambda(A)f\wedge(f\circ w^{-1})+f\wedge\varrho_\lambda(A)(f\circ w^{-1})\\&=(y_\lambda f)\wedge(f\circ w^{-1})+f\wedge y_{w\cdot\lambda}(f\circ w^{-1})\\&=(y_\lambda+y_{w\cdot\lambda})(f\wedge(f\circ w^{-1}))\\&=y_{\lambda+w\cdot\lambda}(f\wedge(f\circ w^{-1}))\\&=y_\gamma(f\wedge(f\circ w^{-1})).\end{aligned}$$

Decompose $\varrho_\lambda \wedge \varrho_\lambda = \varrho_\gamma \oplus$ other. The restriction of the above formula to the ϱ_γ-summand gives $\varrho_\gamma(A) f_\gamma = y_\gamma f_\gamma$ for all $y \in Y_*$. The vector $f \wedge (f \circ w^{-1})$ and hence its projection f_γ inherit the properties of being C^2 in t and meromorphic on Y from the same properties for f_λ. The third assertion follows from the fact that $\widehat{Y}$ is W-invariant combined with the third assumption about f_λ.

We now show the last statement. Since $\varrho_\lambda \wedge \varrho_\lambda$ contains one copy of ϱ_λ as a summand, we may write $f_\lambda \wedge (f_\lambda \circ w^{-1})$ as $\binom{f_\gamma}{0}$. Then

$$\varrho_\lambda \wedge \varrho_\lambda(B)(\pi(y)) \begin{pmatrix} f_\gamma(y) \\ 0 \end{pmatrix} + \begin{pmatrix} \dot{f}_\gamma(y) \\ 0 \end{pmatrix} = \Gamma \begin{pmatrix} f_\gamma(y) \\ 0 \end{pmatrix},$$

by Theorem 17. Switching notation, we have $(\varrho_\lambda \wedge \varrho_\lambda)(B)(f_\lambda \wedge f_\lambda \circ w^{-1}) + (f_\lambda \wedge f_\lambda \circ w^{-1})^{\cdot} = \Gamma f_\lambda \wedge f_\lambda \circ w^{-1}$.

Now, substitution of $w^{-1}(y)$ for y in

$$\varrho_\lambda(B)(\pi(y)) f_\lambda(y) + \dot{f}_\lambda(y) = \Lambda(y) f_\lambda(y)$$

combined with the observation that $\pi(wy) = \pi(y)$ gives

$$\varrho_\lambda(B)(\pi(wy)) f_\lambda \circ w^{-1}(wy) + \dot{f}_\lambda \circ w^{-1}(wy) = \Lambda \circ w^{-1}(wy) f_\lambda \circ w^{-1}(wy).$$

Since w is independent of time, $(f_\lambda \circ w^{-1})^{\cdot} = \dot{f}_\lambda \circ w^{-1}$ and

$$\varrho_\lambda(B) f_\lambda \circ w^{-1} + (f_\lambda \circ w^{-1})^{\cdot} = \Lambda \circ w^{-1} f_\lambda \circ w^{-1}.$$

Also,

$$\begin{aligned}
&(\varrho_\lambda \wedge \varrho_\lambda)(B)(f_\lambda \wedge f_\lambda \circ w^{-1}) + (f_\lambda \wedge f_\lambda \circ w^{-1})^{\cdot} \\
&\quad = (\varrho_\lambda(B) f_\lambda) \wedge f_\lambda \circ w^{-1} + f_\lambda \wedge (\varrho_\lambda(B) f_\lambda \circ w^{-1}) + \dot{f}_\lambda \wedge f_\lambda \circ w^{-1} + f_\lambda \wedge (f_\lambda \circ w^{-1})^{\cdot} \\
&\quad = (\varrho_\lambda(B) f_\lambda + \dot{f}_\lambda) \wedge (f_\lambda \circ w^{-1}) + f_\lambda \wedge (\varrho_\lambda(B) f_\lambda \circ w^{-1} + (f_\lambda \circ w^{-1})^{\cdot}) \\
&\quad = (\Lambda + \Lambda \circ w^{-1})(f_\lambda \wedge f_\lambda \circ w^{-1}) \quad \text{(by the paragraph above).}
\end{aligned}$$

Therefore, $\Gamma = \Lambda + \Lambda \circ w^{-1}$. □

LEMMA 21. *Let λ be a dominant weight and γ the highest weight of the representation $\varrho_\lambda \wedge \varrho_\lambda$. Suppose that $\gamma = \lambda + w \cdot \lambda$, that the ϱ_λ-matrix representation of the Lax equation satisfies the hypotheses from the beginning of §3, and that the restriction of the divisor of f_λ to $\widehat{Y}$ is independent of time for t in a neighborhood of 0. Let X_γ and X_λ be the highest-weight components of the spectral curves defined by ϱ_λ and ϱ_γ. Let*

$\widehat{Y}=\pi_{X_\lambda}^{-1}(\widehat{X}_\lambda)\cup\pi_{X_\gamma}^{-1}(\widehat{X}_\gamma)$. *If* v_λ *and* v_γ *are the tangent vectors to the* λ- *and* γ-*highest-weight flows on* Y, *then*

$$v_\gamma=(e+w)v_\lambda.$$

Proof. The set $\widehat{Y}$ is W-equivariant and allows application of Proposition 19 to both v_λ and v_γ. By Proposition 19,

$$v_\gamma=\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q \Gamma\circ\pi_{X_\gamma}\cdot\vec{\omega}$$

where $\Gamma\colon X_\gamma\to\mathbf{C}$ and $\Lambda\colon X_\lambda\to\mathbf{C}$ are as in Lemma 20, and $\vec{\omega}$ is the vector of the ordered basis of holomorphic differentials on Y used for coordinates on the Jacobian. This expression is

$$\begin{aligned}
&=\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q \Lambda\circ\pi_{X_\lambda}\cdot\vec{\omega}+\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q \Lambda\circ\pi_{X_\lambda}\circ w^{-1}\cdot\vec{\omega}\quad\text{(by Lemma 20)}\\
&=\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q \Lambda\circ\pi_{X_\lambda}\cdot\vec{\omega}+\sum_{Q\in\widehat{Y}}\operatorname*{Res}_{w(Q)} \Lambda\circ\pi_{X_\lambda}\cdot w^*\vec{\omega}\quad\text{(by Lemma 18)}\\
&=\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q \Lambda\circ\pi_{X_\lambda}\cdot\vec{\omega}+\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q \Lambda\circ\pi_{X_\lambda}\cdot w^*\vec{\omega},
\end{aligned}$$

since w permutes the points of $\widehat{Y}$, and

$$\begin{aligned}
&=\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q \Lambda\circ\pi_{X_\lambda}\cdot\vec{\omega}+\sum_{Q\in\widehat{Y}}\operatorname*{Res}_Q \Lambda\circ\pi_{X_\lambda}\cdot w^*\vec{\omega}\\
&=\lim_{t\to0}\int_{\mathcal{D}(0)}^{\mathcal{D}(t)}\vec{\omega}+\lim_{t\to0}\int_{\mathcal{D}(0)}^{\mathcal{D}(t)}w^*\vec{\omega}\quad\text{(by Proposition 19)}\\
&=\lim_{t\to0}\int_{\mathcal{D}(0)}^{\mathcal{D}(t)}\vec{\omega}+w\cdot\lim_{t\to0}\int_{\mathcal{D}(0)}^{\mathcal{D}(t)}\vec{\omega},
\end{aligned}$$

using the group action discussed in §1,

$$=v_\lambda+wv_\lambda$$ □

LEMMA 22. *If* λ *is the highest weight of the irreducible representation* ϱ_λ *and* μ *is any weight of* ϱ_λ, *then* $\|\lambda\|\geqslant\|\mu\|$ *with equality only if* μ *is in the* W-*orbit of* λ.

Proof. The result follows from Humphreys, [H], by combining Lemmas A and B of §13.2 with Theorem 20.2 (b) and Theorem 21.2. □

PROPOSITION 23. *If λ_i is a fundamental weight, then*

(a) *every weight of $\varrho_{\lambda_i} \wedge \varrho_{\lambda_i}$ has height $\leqslant 2\lambda_i - \alpha_i$,*

(b) *$2\lambda_i - \alpha_i$ is a weight of $\varrho_{\lambda_i} \wedge \varrho_{\lambda_i}$ with multiplicity 1.*

Proof. Since ϱ_{λ_i} is irreducible, every weight of ϱ_{λ_i} has the form $\lambda_i - \sum k_j \alpha_j$ for $k_j \geqslant 0$. Moreover, if the k_j are not all zero, then $k_i > 0$ since

$$\left\|\lambda_i - \sum_{j\neq i} k_j \alpha_j\right\| = \left(\|\lambda_i\|^2 + \left\|\sum_{j\neq i} k_j \alpha_j\right\|^2\right)^{1/2} > \|\lambda_i\|,$$

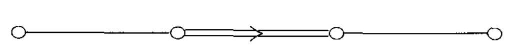
and hence, by Lemma 22, is not a weight of ϱ_{λ_i}. Also, $\sigma_i(\lambda_i) = \lambda_i - \alpha_i$ is in the W-orbit of λ_i, and hence is a weight of ϱ_{λ_i}. This shows that ϱ_{λ_i} not only has a highest weight λ_i but also a second highest weight $\lambda_i - \alpha_i$, although it may not be dominant.

The weights of $\varrho_{\lambda_i} \wedge \varrho_{\lambda_i}$ are the sums of distinct weights of ϱ_{λ_i} (where a weight with multiplicity >1 counts as two distinct weights). Since ϱ_{λ_i} has a highest weight and a second highest weight, their sum $2\lambda_i - \alpha_i$ is the highest weight of $\varrho_{\lambda_i} \wedge \varrho_{\lambda_i}$. The sum can only be obtained in one way, so $2\lambda_i - \alpha_i$ has multiplicity 1. $\square$

The following recipe appears in [BE, p. 36]. In the recipe, (A_{ij}) is the Cartan matrix. These values can be read off of the Dynkin diagram. If the nodes in the Dynkin diagram labelled i and j are not adjacent, then A_{ij} is 0. If they are adjacent, then $-A_{ij}$ is 1 unless there is a multiple edge with an arrow pointing towards node j in which case $-A_{ij}$ equals the multiplicity of the edge.

LEMMA 24. *Let λ_i be a fundamental weight and $\sigma_i \in W$ the corresponding reflection. Then $\lambda_i + \sigma_i(\lambda_i) = -\sum A_{ij}\lambda_j$.*

Example.



Above is the Dynkin diagram for F_4. The recipe gives the following set of equations:

$$\begin{aligned}
\lambda_1 + \sigma_1(\lambda_1) &= \lambda_2,\\
\lambda_2 + \sigma_2(\lambda_2) &= \lambda_1 + 2\lambda_3,\\
\lambda_3 + \sigma_3(\lambda_3) &= \lambda_2 + \lambda_4,\\
\lambda_4 + \sigma_4(\lambda_4) &= \lambda_3.
\end{aligned}$$

We now prove the proposition stated at the beginning of this section.

Proof of Proposition 25. From Proposition 23, the highest weight of $\varrho_{\lambda_i} \wedge \varrho_{\lambda_i}$ is $2\lambda_i - \alpha_i = \lambda_i + \sigma_i(\lambda_i)$. Take $\gamma = \lambda_i + \sigma_i(\lambda_i)$ in Lemma 21. This yields $(e + \sigma_i)v_{\lambda_i} = v_\gamma$. By Lemma 24, $\gamma = -\sum A_{ij}\lambda_j$ and, by Proposition 16, $v_{\lambda_i} + \sigma_i(v_{\lambda_i}) = -\sum A_{ij} v_{\lambda_j}$. The second equation is Proposition 15. $\square$

5. Lattice of flow vectors

THEOREM 26. *Suppose that the Lax equation $dA/dt=[A,B]$ satisfies the hypotheses from the beginning of §3 for each fundamental representation and that the restriction of the divisor of f_{λ_i} to $\widehat{Y}$ is independent of time for t in a neighborhood of 0. Then either the velocities v_λ are all zero or the map from the weight lattice to the set of highest-weight flows given by $\lambda \mapsto v_\lambda$ is an isomorphism of $\mathbf{Z}[W]$-modules.*

Proof. As a $\mathbf{Z}[W]$-module, the weight lattice Λ is $\mathbf{F}/\mathbf{R}$ where $\mathbf{F}=\mathbf{Z}[W][\lambda_1,\dots,\lambda_n]$ is the free module generated by the fundamental weights, and $\mathbf{R}=\langle \sigma_i(\lambda_i)=-\sum A_{ij}\lambda_j-\lambda_i, \sigma_j(\lambda_i)=\lambda_i\rangle$. This presentation follows from the fact that Λ is a free abelian group on the λ_i and that the σ_i generate W. Let $M\subset T_e \operatorname{Jac} Y$ be the $\mathbf{Z}[W]$-submodule generated by $\{v_{\lambda_1},\dots,v_{\lambda_n}\}$, the highest-weight flows of the fundamental weights. Now, $\mathbf{F}\to M$ defined by $\lambda_i \mapsto v_{\lambda_i}$ is an epimorphism. It factors through $\Lambda\to M$ as the relations $\mathbf{R}$ are preserved by Proposition 25. If $N\subset\Lambda$ is any $\mathbf{Z}[W]$-submodule, then Λ/N is torsion as $\Lambda\otimes\mathbf{C}$ and $N\otimes\mathbf{C}$ are both $\mathfrak{h}^*\otimes\mathbf{C}$. Since $M\subset T_e \operatorname{Jac} Y$ is not torsion, $\Lambda\to M$ is an isomorphism.

If γ is any weight, then we claim that $\gamma\mapsto v_\gamma$. There is a $w\in W$ such that $w\cdot\gamma$ is dominant, $w\cdot\gamma=\sum a_i\lambda_i$ with $a_i\geqslant 0$. Then, $w\cdot\gamma=\sum a_i\lambda_i\mapsto\sum a_i v_{\lambda_i}=v_{\sum a_i\lambda_i}=v_{w\cdot\gamma}$ by Proposition 16. So, $\gamma\mapsto w^{-1}v_{w\cdot\gamma}=v_\gamma$ by Proposition 15. □

We now move from local to global results. We first discuss the question of when the eigenvectors satisfy hypothesis (5) from §3. The issue of the poles (or zeros) of these functions is more subtle than one might initially expect. If $f=(f_1,\dots,f_N)$ is a tuple of meromorphic functions, then using Lemma 9 we get a holomorphic function into $\mathbf{C}P^{N-1}$. If the functions are C^2 in time then one can do this for each given time. However, there is no guarantee that the extension is smooth or even continuous in t as the following example on $\mathbf{C}$ shows. Let $f_t(x)=[(t/x+x,1)]$. $f_0(x)=[(x,1)]$ so that $f_0(0)=[(0,1)]$, but $f_t(0)=[(t,0)]$ and $\lim_{t\to 0} f_t(0)=[(1,0)]$. However, in the case of highest-weight eigenvectors of isospectral flows, we can only lose differentiability at points in $\widehat{Y}$. Let $\overline{Y}_\lambda=Y_\lambda-\widehat{Y}_\lambda$ and $\overline{P}=\pi(\overline{Y}_\lambda)$. As $\widehat{Y}_\lambda$ is a finite set that is fixed over time, there is a map $g(y,t)$ as in the diagram

$$\begin{tikzcd} \overline{Y}_\lambda \arrow[r, "g"] \arrow[d] & (G/H\times\mathfrak{h}_*)/S_\lambda \arrow[r] \arrow[d] & G/P_\lambda\subset\mathbf{CP}^{N-1} \\ \overline{P} \arrow[r, "A"] & \mathfrak{g}_*. & \end{tikzcd}$$

Here Y_λ is the cover of the highest-weight component of multiplicity 1. Since $\overline{Y}_\lambda\to\overline{P}$ and $(G/H\times\mathfrak{h}_*)/S_\lambda\to\mathfrak{g}_*$ are covering spaces, g inherits its smoothness from A. The composition of the top row picks out the class of the eigenvector in projective space; this is $[f]$. In homogeneous coordinates, the entries of f can be taken as ratios of minors of

$\varrho(A(s))-z$, viz.,

$$\left(\frac{\Delta_{i,1}}{\Delta_{i,1}},\dots,\frac{\Delta_{i,N}}{\Delta_{i,1}}\right)$$

is the eigenvector. This is independent of the row as the first entry is normalized to 1. Since the multiplicity of λ is one, expansion of some row yields an eigenvector. Note that U_t in Theorem 17 and g_t above are related by $g_t g_0^{-1}=U_t$.

If $\mathfrak{g}$ has a representation in which all of the entries of A and B are of the form $\sum c_i(t)m_i(x)$, then the same holds for any other representation since the entries of the new representation are linear combinations of the old. We have shown

LEMMA 27. *Suppose that A,B is a Lax pair whose entries (in some representation) are of the form $\sum c_i(t)m_i(s)$, a finite sum, where the $c_i(t)$ are C^2 in time and the $m_i(s)$ are meromorphic on P. Then each highest-weight vector can be represented as a column vector whose entries are functions which are meromorphic on Y_λ and C^2 in time off of a finite set $\widehat{Y}_\lambda$.*

THEOREM 28. *Let $\mathfrak{g}$ be a simple Lie algebra and P a compact Riemann surface. Suppose that the pair $A,B\colon P\times\mathbf{R}\to\mathfrak{g}$ satisfies $dA/dt=[A,B]$ and that*

(1) A,B have entries of the form $\sum c_i(t)m_i(s)$, a finite sum, where $c_i(t)$ is C^2 and $m_i(s)$ is meromorphic,

(2) $A(s,0)$ is regular for some $s\in P$, and

(3) the flow in the Jacobian of a spectral curve is absolutely continuous.

Then the highest-weight flow f_λ is in the Prym–Tjurin variety $\operatorname{Tur}_\lambda Y$.

Proof. Since the entries of the matrix have the form $\sum c_i(t)m_i(s)$, the set P_∞ is a subset of the poles of the m_i, a finite set. Therefore, the first four of the hypotheses from §3 are satisfied. These hold in every finite-dimensional representation of $\mathfrak{g}$. The fifth hypothesis holds by Lemma 27 for each highest-weight vector.

Let $\psi(t)$ be the absolutely continuous λ-flow in $\operatorname{Jac}Y$, for λ a fundamental weight and Y the master curve. Write $\psi'=g\psi$, where $g\colon\mathbf{R}\to T_e\operatorname{Jac}Y$. Then $\psi(t)=\exp(\int_0^t g\,dt)$. Let $T_e\operatorname{Jac}Y=T_e\operatorname{Tur}_\lambda Y\oplus V$, $p_1\colon T_e\operatorname{Jac}Y\to V$, and $p_2\colon T_e\operatorname{Jac}Y\to T_e\operatorname{Tur}_\lambda Y$ be the projections. Then, $\psi(t)=\exp(\int_0^t g\circ p_1\,dt)\exp(\int_0^t g\circ p_2\,dt)$ since $\operatorname{Jac}Y$ is an abelian group. If we write $f_\lambda(t)$ for the highest-weight eigenvector of this flow, then by Theorem 26, $g\circ p(t)=0$ at those times t when the restriction of the divisor of f_λ to $\widehat{Y}$ is independent of time for t in an open set. We claim that $\{t\mid g\circ p(t)\neq 0\}$ is a set of measure zero and so the highest-weight flow evolves in $\operatorname{Tur}_\lambda Y$.

Consider a point of $\widehat{Y}$. Suppose that it projects to (x_0,z_0) in the spectral curve of ϱ_λ. The divisor of f may change at (x_0,z_0,t_0) if the poles or zeros of a minor of $(A(x,t)-z)$ changes at t_0. Take y as a local coordinate about $y_0=(x_0,z_0)$. Expand an entry $f_i(y,t)=(A(x,t)-z)_{i,j}/(A(x,t)-z)_{1,j}$ as a Laurent series about y_0 near time t_0, $\sum_{n\geqslant M} K_n(t)(y-y_0)^n$. The coefficient of $(y-y_0)^n$ is given by the integral

$$K_n(t)=\int_\alpha \frac{f_i(y,t)}{(y-y_0)^{n-1}}\,dy$$

over a path α around y_0 on Y. Hence $K_n(t)$ is continuous. Such a coefficient vanishes on a closed set, and so it is on the boundary of a closed set that a pole or zero can appear or vanish. The boundary of a closed set has measure zero. Since $\widehat{Y}$ consists of a finite number of points, the measure of the set of t such that the divisor of f is not constant on $\widehat{Y}$ in a neighborhood of t is a set of measure zero in time. Therefore, $\psi(t)\subset \mathrm{Tur}_\lambda Y$ for all time. □

In the interest of solving the differential equation, we would like to consider the entire curve defined by the characteristic polynomial of A. The flow will then reflect the motion of all the eigenvectors of A. In many cases the flow from the smallest representation is a highest-weight flow. We now prove the main theorem which is a corollary to Theorem 28.

THEOREM 29. *Let $\mathfrak{g}$ be a simple Lie algebra and P a compact Riemann surface. Suppose that the pair $A,B\colon P\times\mathbf{R}\to\mathfrak{g}$ satisfies $dA/dt=[A,B]$ and that*

(1) *A,B have entries of the form $\sum c_i(t)m_i(s)$, a finite sum, where $c_i(t)$ is C^2 and $m_i(s)$ is meromorphic,*

(2) *$A(s,0)$ is regular for some $s\in P$, and*

(3) *the flow in the Jacobian of a spectral curve associated with the smallest representation is absolutely continuous.*

If $\mathfrak{g}$ is of type A, B, C, D, E_6, or E_7, then the flow is in the Prym–Tjurin variety $\mathrm{Tur}_\lambda Y$.

If P is the Riemann sphere, then the flow is in the Prym–Tjurin variety $\mathrm{Tur}_\lambda Y$.

Proof. In order to show that the eigenvector flow of the whole matrix is in the Prym–Tjurin, we would, ostensively, need to consider all eigenvectors of the representation and not just the highest weight. We consider the matrix in the smallest representation of $\mathfrak{g}$. The only weights are (possibly) the trivial weight and those in the orbit of the highest weight. The result holds for those in the orbit of the highest weight by Theorem 28. If the algebra is of type A, B, C, D, E_6, or E_7, then the smallest representation has no trivial weight and consists only of the orbit of the highest weight. This shows the first statement. The spectral curve for the zero weight is the curve P itself. So this flow is in the moduli space of bundles over P. If P is the Riemann sphere then the moduli space is trivial, i.e. a point. This shows the second statement. □

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