Quasiconformal maps in metric spaces with controlled geometry

by

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1. Introduction

This paper develops the foundations of the theory of quasiconformal maps in metric spaces that satisfy certain bounds on their mass and geometry. The principal message is that such a theory is both relevant and viable.

The first main issue is the problem of definition, which we next describe. Quasiconformal maps are commonly understood as homeomorphisms that distort the shape of infinitesimal balls by a uniformly bounded amount. This requirement makes sense in every metric space. Given a homeomorphism f from a metric space X to a metric space Y, then for $x \in X$ and $r > 0$ set

$$H_f(x,r) = \frac{\sup\{|f(x)-f(y)| : |x-y| \leqslant r\}}{\inf\{|f(x)-f(y)| : |x-y| \geqslant r\}}. \tag{1.1}$$

Here and hereafter we use the distance notation $|x-y|$ in any metric space.

Definition 1.2. A homeomorphism $f: X \to Y$ is called *quasiconformal* if there is a constant $H < \infty$ so that

$$\limsup_{r \to 0} H_f(x, r) \leqslant H \tag{1.3}$$

for all $x \in X$.

This definition is easy to state, but not easy to use. It does follow easily from the definition and classical theorems in real analysis that quasiconformal homeomorphisms in Euclidean spaces are almost everywhere differentiable. But it is not clear whether, for instance, the inverse of a given quasiconformal map is quasiconformal; nor is it easy to ascertain desired stronger properties such as Hölder continuity or the compactness of a suitably normalized family of quasiconformal homeomorphisms. The difficulties stem from the fact that (1.3) is a local, infinitesimal condition.

Let us look at a stronger, global requirement.

Definition 1.4. A homeomorphism $f: X \to Y$ is called *quasisymmetric* if there is a constant $H < \infty$ so that

$$H_f(x, r) \leqslant H \tag{1.5}$$

for all $x \in X$ and all $r > 0$.

It is not difficult to demonstrate starting from the definition that quasisymmetric homeomorphisms between reasonable spaces enjoy many strong properties: they are Hölder continuous, inverse maps are quasisymmetric as well, normal families are common and quasisymmetry carries over to limit homeomorphisms. In fact, much of the classical quasiconformal theory can be done by directly exploiting condition (1.5), or its local versions. Quasisymmetric maps made their first official appearance in the 1956 paper [BA] by Beurling and Ahlfors, who were concerned about maps of the real line, and quasiconformal extensions thereof. The concept was later promoted by Tukia and Väisälä, who introduced and studied quasisymmetric maps between arbitrary metric spaces in [TV]. Recently Väisälä [V5], [V6] has developed a "dimension-free" theory of quasiconformal maps in infinite-dimensional Banach spaces based on the idea of quasisymmetry. See also [V2].

It is a fundamental fact that quasiconformal homeomorphisms between Euclidean spaces of dimension at least two are quasisymmetric; that is, (1.3) implies (1.5) for a homeomorphism $f: \mathbf{R}^n \to \mathbf{R}^n$ if $n \geqslant 2$. (The value of the constant H in (1.5) may differ, but it only depends on the constant appearing in (1.3) and possibly—this is an open problem—on the dimension n.) If we assume that f is a diffeomorphism, then this fact is not overly difficult to establish, albeit still nontrivial. That global bounds can be obtained without any a priori regularity assumptions is a deep result that reflects certain

special properties of Euclidean space, worth seeking elsewhere. This result was first proved by Gehring in [G1] for $\mathbf{R}^2$, with a method that extends to higher dimensions; see [V1] for a full account. For $n=1$, the statement is false; consider, for example, $f(x)=x+e^x$.

The problem whether (1.3) and (1.5) are equivalent for a given self-homeomorphism of a space can be phrased in more intrinsic terms as follows. Let X be a space with metric d, which we regard as the fixed "conformal" structure on X. Suppose then that we are given an infinitesimal quasiconformal structure on X. By this we mean a new metric on X whose balls, at the limit when the radius goes to zero, are not too different in shape from the balls in the metric d. Is it then true that these two structures are globally quasisymmetrically equivalent? In other words, can we recapture the global quasisymmetric structure of a space from a local or infinitesimal quasiconformal structure? This question comes up naturally in the quasiisometry classification of negatively curved spaces. It is known that the quasiisometry type of a negatively curved space (in the sense of Gromov [GH]) is in many cases determined by the quasisymmetric type of its boundary; thus we would like to know whether it is already determined by the infinitesimal quasiconformal type. See the survey article by Gromov and Pansu [GP] for an excellent discussion. (See also [P3] and [Pau].)

For a long time it was not clear whether this infinitesimal-to-global principle was valid in spaces that are sufficiently distinct from $\mathbf{R}^n$. In fact, examples of relatively nice spaces were found where it fails; for instance, one can take X to be $\mathbf{R}^2$ and Y to be a certain smooth hypersurface in $\mathbf{R}^3$, cf. [HK1, Example 4.7] or [V2, §5]. As a consequence of the recent work of Mostow [Mo2], Pansu [P1], [P2], and others, it followed that (1.3) implies (1.5) for homeomorphisms between the spaces that occur as conformal boundaries of rank-one symmetric spaces. In particular, Korányi and Reimann [KR1], [KR2] have conducted a thorough study of the quasiconformal maps on the Heisenberg group, and a careful treatment of various definitions for quasiconformal maps on the Heisenberg group is given in [KR2]. The authors showed in [HK1] that (1.3) implies (1.5) in an arbitrary Carnot group (see §6.2 below), or more generally in the case where X is a Carnot group and Y is a metric space with similar homogeneity and local connectivity properties to X. Recently, Margulis and Mostow [MM] established important absolute continuity properties of quasiconformal maps on general Carnot–Carathéodory spaces. Their results can be used to derive (semi-)global distortion properties of quasiconformal maps on those spaces. See also [VG].

One of the main goals of the present paper is to show that the two concepts, quasiconformality and quasisymmetry, are quantitatively equivalent in a large class of metric spaces, which includes all the previously known examples and more. Such spaces are

discussed next.

The most important tool in the quasiconformal theory is the conformal modulus, or capacity. This is a global conformal invariant that attaches a real number to each pair of disjoint continua in a given space. (By a *continuum* we mean a compact, connected set.) The crucial property of this invariant in $\mathbf{R}^n$ is that it has a uniform lower bound, which depends only on the dimension n and on the relative position of the two continua. In the case $n=2$, this fact was known already to Grötzsch and Teichmüller. For $n\geqslant 3$ it was first observed by Loewner [L] in 1959; he used this property of modulus to show that one cannot map $\mathbf{R}^n$ quasiconformally onto a proper subset. In §3, we shall define a *Loewner space* to be a space where a similar lower bound for the modulus holds. Then, in §4, we shall show that a quasiconformal map from a Loewner space X into a space Y is quasisymmetric, if the Hausdorff measures of X and Y are both Ahlfors–David regular of the same dimension larger than one, and if Y satisfies a (necessary) linear local connectivity condition.

Let us recall the definition of an Ahlfors–David regular space.

Definition 1.6. A metric space X is said to be *Ahlfors–David regular* of dimension $Q>0$ if there is a constant $C\geqslant 1$ so that

$$C^{-1}R^Q \leqslant \mathcal{H}_Q(B_R) \leqslant CR^Q \tag{1.7}$$

for all balls B_R in X of radius $R<\operatorname{diam} X$. Here, and hereafter, $\mathcal{H}_Q$ denotes the Q-Hausdorff measure in the metric space X. We often call X simply Q-*regular*, or just *regular*, if the dimension is not important to the discussion.

It is easy to see that X satisfies (1.7) if it satisfies a similar condition for *some* Borel-regular measure μ; only the constant C may change slightly. David and Semmes [DS2], [DS3] have conducted an extensive study of regular spaces, usually furnished with some additional properties. Regular spaces are particular examples of spaces of homogeneous type in the sense of Coifman and Weiss [CW]. Although a lot of harmonic analysis can be done in homogeneous spaces, they are usually too general for the kind of questions we want to address in this paper; for us, it is important that the spaces have good connectivity properties. (This is not to say that many of our results, especially in §4, would not hold in more generality, or when differently interpreted; there are interesting questions left open in this respect. Our techniques fail if the spaces admit few or no rectifiable curves.)

The main new point introduced in [HK1] was that one can use modulus estimates to study maps even without a differentiable structure of any kind in the underlying space. The usual analytic change of variables procedure was replaced there by a discrete,

combinatorial argument. In the end, all one needs is a lower bound for the modulus. This idea is pursued further here, and the main question is: what spaces admit such a lower bound for the modulus? Equivalently, in our present terminology, what spaces are Loewner spaces? The answer will be given in terms of a Poincaré inequality.

Recall that the usual Poincaré inequality in $\mathbf{R}^n$ implies, by way of Hölder's inequality, that

$$\inf_{a \in \mathbf{R}} \int_B |u-a| \, dx \leqslant C(n)(\operatorname{diam} B)^n \left(\int_B |\nabla u|^n \, dx \right)^{1/n} \tag{1.8}$$

for any bounded smooth or Lipschitz function u in a ball B (see [GT, p. 164]). We shall formulate a version of (1.8) in a general, rectifiably connected metric space. We then show (see §5) that if X is a proper and regular space that in addition satisfies a local quasiconvexity condition, then X is a Loewner space if and only if X admits a Poincaré inequality. (A *proper* space is one whose closed balls are compact; for the quasiconvexity condition, see §2.15.)

The search for Poincaré-type inequalities in various situations has been intensive in recent years. Spaces that admit the kind of Poincaré inequality we are looking for include Riemannian manifolds of nonnegative Ricci curvature and Euclidean volume growth, as well as various Carnot-type geometries. See [Bu], [DS1], [Gr], [J], [MSC], [SC], [VSC]. Semmes [S4] has shown recently that any n-regular, complete metric space, that is also an oriented (homology) n-manifold satisfying a linear local contractibility condition, admits a Poincaré inequality. (Added in December 1997: For an interesting new geometry which admits a Poincaré inequality, see the recent paper by Bourdon and Pajot [BP].) We shall show in this paper that any connected, finite simplicial complex admits a Poincaré inequality if it is of pure dimension $n > 1$ and if it has the (obviously necessary) property that the link of every vertex is connected (i.e. a removal of a point does not locally disconnect the space).

Consequently, in all these spaces we can go from an infinitesimal quasiconformal structure to a global one. Note that in the case of a Riemannian manifold, quasiconformal maps are always quasisymmetric in small coordinate charts by Gehring's theorem (if the dimension is larger than one), but in general there need not be any control over the global distortion. Our result says that we can control the global distortion in the presence of appropriate volume bounds and a Poincaré inequality. Recall that certain Sobolev–Poincaré inequalities carry information about the isoperimetric profile of a space, which is closely connected with the quasiconformal theory [GLP]. The inequalities we need in this paper are weaker than those related to isoperimetric inequalities.

There are many important cases where the existence of a Poincaré inequality is not known. If $\widetilde{M}$ is the universal cover of a negatively curved compact Riemannian

n-manifold M, then the ideal boundary $\partial \widetilde{M}$ of $\widetilde{M}$ is topologically a sphere of dimension $n-1$. One would like to understand the metrics on $\partial \widetilde{M}$ where the fundamental group $\pi_1(M)$ acts uniformly quasiconformally. (Note that such metrics always exist [GH].) A particularly interesting case is related to the problem of recognizing the fundamental groups of compact hyperbolic three-manifolds. A long standing conjecture is that every negatively curved or Gromov hyperbolic group whose boundary is a two-sphere is such a group. Cannon, Floyd, Parry, and Swenson [C], [CFP], [CS] have showed that this conjecture can be solved affirmatively if a certain combinatorial modulus on the boundary two-sphere has roughly Euclidean behavior. This is a Loewner-type requirement. In [HK1], the authors employed a discrete modulus similar to that of Cannon et al., but in the present paper the concepts are defined in continuous terms. Many arguments below, however, can be seen as combinatorial.

It remains an open problem precisely under what circumstances the Loewner condition is a quasisymmetric invariant. We conjecture that the Loewner property is a quasisymmetric invariant of a locally compact Q-regular space for $Q>1$. In §8, we prove this under an additional hypothesis. (After this paper was submitted, Tyson [Ty] verified the conjecture; see Remark 8.7 (a).)

After this study of definitions, we come to the second main issue of the paper, which is the actual theory of quasiconformal maps between spaces with appropriate control on mass and geometry. A regular metric space X that admits a Poincaré-type inequality appears to be an amenable environment where the quasiconformal theory works much in the same way it does in Euclidean space. We shall show that a quasiconformal map between two such spaces is not only absolutely continuous in that it preserves sets of measure zero, but it induces an A_∞-weight in the sense of Muckenhoupt. Moreover, under similar assumptions, quasiconformal maps are absolutely continuous on Q-modulus almost every curve. The assumptions are general enough to encompass the results of Pansu [P2], and Margulis and Mostow [MM] on absolute continuity on lines. We also show that a quasiconformal map between such spaces belongs to a Sobolev space of higher degree than a priori is expected. These results extend the celebrated theorems of Bojarski [Bo] in $\mathbf{R}^2$ and Gehring [G3] in $\mathbf{R}^n$. By a Sobolev space, we mean a space as defined either by Hajłasz [Ha], or by Korevaar and Schoen [KS].

Finally, we should warn the reader that what we call *quasisymmetry* in Definition 1.4 is called *weak quasisymmetry* by Tukia and Väisälä in [TV]; they demand that quasisymmetric maps satisfy a stronger distortion condition, valid for points in all locations. See (4.5) for the precise definition. We chose to ignore this difference in this introduction, for we shall deal only with metric spaces where these two definitions of quasisymmetry are quantitatively equivalent. This point is clarified later in §4. Also, in the case of

compact or bounded spaces, the concept of a *quasi-Möbius map* as defined by Väisälä in [V3] would appear more natural than that of a quasisymmetric map. (Recall that the conformal automorphism group of the unit disk is not uniformly quasisymmetric in the Euclidean metric.) However, for simplicity of exposition we have decided not to deal with quasi-Möbius maps in this paper.

Some of the results of this paper were announced in [HK2].

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2. Modulus and capacity in a metric space

In this section, we first recall the definition for the modulus of a curve family in a metric measure space (X, μ). Then we introduce the concept of capacity between two continua in X. The latter requires an appropriate substitute for the gradient of a smooth or Lipschitz function. This done, we proceed to show that the two notions are equal in certain important situations. Just as in $\mathbf{R}^n$, the modulus is more general and flexible in use, while it is easier to give estimates for the more concrete capacity.

2.1. *Definitions and conventions.* All metric spaces in this paper are assumed to be rectifiably connected and all measures are assumed to be locally finite and Borel regular with dense support. A metric space is called *rectifiably connected* if every pair of two points in it can be joined by a rectifiable curve (see §2.2 below). We shall denote by (X, μ) such a metric measure space. We do not assume in general that X be locally compact or complete.

Open balls are written as $B(x, r)$, and if $B = B(x, r)$, then $CB = B(x, Cr)$ for $C > 0$. The closure of a set A is denoted $\bar{A}$.

2.2. *Curves and line integrals in a metric space.* We recall the basic concepts of rectifiability and line integration in a metric space. Let X be a metric space as in §2.1.

By a *curve* we mean either a continuous map γ of an interval $I \subset \mathbf{R}$ into X, or the image $\gamma(I)$ of such a map. We usually abuse notation by writing $\gamma = \gamma(I)$. If $I = [a, b]$ is

a closed interval, then the *length* of a curve $\gamma\colon I \to X$ is

$$l(\gamma) = \operatorname{length}(\gamma) = \sup \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i+1})|,$$

where the supremum is over all finite sequences $a = t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant t_{n+1} = b$. If I is not closed, then we set

$$l(\gamma) = \sup l(\gamma|J),$$

where the supremum is taken over all closed subintervals J of I. We call a curve γ *rectifiable* if its length is a finite number. Similarly, a curve $\gamma\colon I \to X$ is *locally rectifiable* if its restriction to each closed subinterval of I is rectifiable.

Any rectifiable curve $\gamma\colon I \to X$ has a unique extension $\bar{\gamma}$ to the closure $\bar{I}$ of I; we ignore the fact that the values of $\bar{\gamma}$ at the endpoints of I may not lie in X but rather in the completion of X. If I is unbounded, the extension is understood in a generalized sense. From now on, if γ is rectifiable, we automatically consider its extension $\bar{\gamma}$ and do not distinguish these two curves in notation. For any rectifiable γ there are its associated length function $s_\gamma\colon I \to [0, l(\gamma)]$ and a unique 1-Lipschitz continuous map $\gamma_s\colon [0, l(\gamma)] \to X$ such that $\gamma = \gamma_s \circ s_\gamma$. The curve γ_s is the *arc length parametrization* of γ.

If γ is a rectifiable curve in X, the line integral over γ of each nonnegative Borel function $\varrho\colon X \to [0, \infty]$ is

$$\int_\gamma \varrho\, ds = \int_0^{l(\gamma)} \varrho \circ \gamma_s(t)\, dt.$$

If γ is only locally rectifiable, we set

$$\int_\gamma \varrho\, ds = \sup \int_{\gamma'} \varrho\, ds,$$

where the supremum is taken over all rectifiable subcurves γ' of γ. If γ is not locally rectifiable, no line integrals are defined.

A detailed treatment of line integrals in the case $X = \mathbf{R}^n$ can be found in [V1, Chapter 1]. The general case is only ostensibly different, cf. [Fe, 2.5.16]. The length of a curve γ as defined above agrees with its 1-Hausdorff measure in X provided the map $\gamma\colon I \to X$ is injective [Fe, 2.10.13].

2.3. *Modulus of a curve family.* Suppose that (X, μ) is a metric measure space as in §2.1. Let Γ be a family of curves in X and let $p \geqslant 1$ be a real number. The *p-modulus* of Γ is defined as

$$\operatorname{mod}_p \Gamma = \inf \int_X \varrho^p\, d\mu,$$

where the infimum is taken over all nonnegative Borel functions $\varrho\colon X\to[0,\infty]$ satisfying

$$\int_\gamma \varrho\,ds \geqslant 1 \tag{2.4}$$

for all locally rectifiable curves $\gamma\in\Gamma$. Functions ϱ satisfying (2.4) are called *admissible* (*metrics*) for Γ. Note that by definition the modulus of all curves in X that are not locally rectifiable is zero. We observe that

$$\operatorname{mod}_p(\varnothing) = 0, \tag{2.5}$$

$$\operatorname{mod}_p\Gamma_1 \leqslant \operatorname{mod}_p\Gamma_2, \tag{2.6}$$

if $\Gamma_1\subset\Gamma_2$, and

$$\operatorname{mod}_p\left(\bigcup_{i=1}^{\infty}\Gamma_i\right) \leqslant \sum_{i=1}^{\infty}\operatorname{mod}_p\Gamma_i. \tag{2.7}$$

Moreover, if Γ_0 and Γ are two curve families such that each curve $\gamma\in\Gamma$ has a subcurve $\gamma_0\in\Gamma_0$, then

$$\operatorname{mod}_p\Gamma \leqslant \operatorname{mod}_p\Gamma_0. \tag{2.8}$$

These properties of modulus are easily proven, cf. [Fu], [V1, pp. 16–17]. They will be used repeatedly, and usually without extra fanfare, throughout this paper.

Often one would like to restrict the pool of admissible metrics to, say, continuous or bounded functions ϱ. Such a reduction generally leads to a different concept. For instance, the n-modulus of the family of all (nonconstant) curves in $\mathbf{R}^n$ that pass through a given point is zero, but there are no admissible bounded metrics for this family. The only concession that can be made is to consider lower semicontinuous functions, for it follows from the Vitali–Carathéodory theorem in real analysis that every function f in $L^p(X)$ can be approximated in $L^p(X)$ by a lower semicontinuous function g with $g\geqslant f$. This requires X to be locally compact; see [Ru, p. 57].

The triple $(E,F;U)$ will denote the family of all curves in an open subset U of X joining two disjoint closed subsets E and F of U, cf. §2.11. For brevity, $(E,F;X)=(E,F)$.

2.9. *Very weak gradients.* Let U be an open set in X and let u be an arbitrary real-valued function in U. We say that a Borel function $\varrho\colon U\to[0,\infty]$ is a *very weak gradient* of u in U if

$$|u(x)-u(y)| \leqslant \int_{\gamma_{xy}} \varrho\,ds \tag{2.10}$$

whenever γ_{xy} is a rectifiable curve joining two points x and y in U. Clearly, a very weak gradient is not unique, and $\varrho=\infty$ is always a very weak gradient. As an example, if X is

a Riemannian manifold, for instance $\mathbf{R}^n$ with its standard metric, and if u is a smooth function on X, then $\varrho = |\nabla u|$ is a very weak gradient of u. It is also not difficult to see that if ϱ is any very weak gradient of a smooth function u in $\mathbf{R}^n$, then $|\nabla u| \leqslant \varrho$ almost everywhere.

Recall that a mapping u between metric spaces is *Lipschitz* if there is a constant $C \geqslant 1$ so that

$$|u(x) - u(y)| \leqslant C|x - y|$$

for all points x and y in the domain of u; moreover, u is *locally Lipschitz* if every point in the domain has a neighborhood where u is Lipschitz. If u is a Lipschitz function on a Riemannian manifold X, it is differentiable almost everywhere, and the function $|\nabla u|$ can be redefined everywhere on X so that it becomes a very weak gradient of u. And, as in the case of a smooth function, $|\nabla u|$ is almost everywhere less than or equal to any given very weak gradient of u.

If X is a Carnot group, then $|\nabla_0 u|$, the length of the horizontal differential of a smooth function u, serves as a very weak gradient of u (see §6.2 and the references there for the terminology). Conversely, if ϱ is any very weak gradient of such a function u, then $|\nabla_0 u| \leqslant \varrho$ almost everywhere, cf. [HK1, proof of Proposition 2.4].

2.11. *Capacity.* Suppose that E and F are closed subsets of an open set U in X. The triple $(E, F; U)$ is called a *condenser* and its *p-capacity* for $1 \leqslant p < \infty$ is defined as

$$\operatorname{cap}_p(E, F; U) = \inf \int_U \varrho^p \, d\mu, \tag{2.12}$$

where the infimum is taken over all very weak gradients of all functions u in U such that $u|E \geqslant 1$ and $u|F \leqslant 0$. Such a function u is called *admissible* for the condenser $(E, F; U)$. If $U = X$, we write $(E, F; X) = (E, F)$ as in the case of modulus.

Remark 2.13. Observe that no a priori regularity of admissible functions is assumed above. In practice, of course, the existence of a very weak gradient in L^p imposes restrictions. We use the notation $\operatorname{cap}_p^c(E, F; U)$ and $\operatorname{cap}_p^L(E, F; U)$ for the quantity in (2.12) if the infimum is taken over all continuous or locally Lipschitz admissible functions, respectively. We trivially have

$$\operatorname{cap}_p(E, F; U) \leqslant \operatorname{cap}_p^c(E, F; U) \leqslant \operatorname{cap}_p^L(E, F; U). \tag{2.14}$$

In $\mathbf{R}^n$, if E and F are compact subsets of an open set U, then equality holds in (2.14); see [He]. We do not know in what generality there is equality in (2.14).

We shall next prove the equality between modulus and capacity, plus an important inequality (2.19) for condensers of certain type. This result is well known in the Euclidean

case, and the proof below is distilled from various works, most notably from [Z]. The generality of the situation forces us to present a detailed argument. First we require some definitions.

2.15. *Quasiconvex and proper spaces.* We say that X is *quasiconvex* if there is a constant $C>0$ so that every pair of two points x and y in X can be joined by a curve γ whose length satisfies $l(\gamma)\leqslant C|x-y|$. Moreover, X is *locally quasiconvex* if every point in X has a neighborhood that is quasiconvex.

More generally, X is said to be *φ-convex* if there is a cover of X by open sets $\{U_\alpha\}$ together with homeomorphisms $\{\varphi_\alpha\colon[0,\infty)\to[0,\infty)\}$ such that any pair of two points x and y in U_α can be joined by a curve in X whose length does not exceed $\varphi_\alpha(|x-y|)$.

We shall not be using the concept of φ-convexity in any serious way in this paper: the functions $\{\varphi_\alpha\}$ will have no quantitative bearing on our discussion. Most of the spaces considered below will be (globally) quasiconvex, but to prove this, something like φ-convexity needs to be assumed first.

We call X *proper* if its closed balls are compact.

Remark 2.16. There is a neat connection between quasiconvexity and very weak gradients of Lipschitz functions. Namely, it is easy to see that a space X is quasiconvex if and only if every function with bounded very weak gradient on X is Lipschitz.

PROPOSITION 2.17. *We always have*

$$\mathrm{cap}_p(E,F;U)=\mathrm{mod}_p(E,F;U). \tag{2.18}$$

Next suppose that X is φ-convex, that E and F are two disjoint closed sets in X with compact boundaries, and that X is proper. Then

$$\mathrm{cap}_p^c(E\cap B,F\cap B;B)\leqslant\mathrm{mod}_p(E,F) \tag{2.19}$$

for each ball B in X. If, moreover, X is locally quasiconvex, (2.19) holds with cap_p^L on the left-hand side.

Proof. To prove the inequality $\mathrm{mod}_p(E,F;U)\leqslant\mathrm{cap}_p(E,F;U)$, take a function u in U such that $u|E\geqslant1$ and $u|F\leqslant0$, and take any very weak gradient ϱ of u. Then

$$\int_\gamma \varrho\,ds\geqslant1 \tag{2.20}$$

for all rectifiable curves γ joining E and F in U, so that

$$\mathrm{mod}_p(E,F;U)\leqslant\int_U \varrho^p\,d\mu.$$

Because u and ϱ were arbitrary, the inequality follows.

To prove the reverse inequality $\mathrm{cap}_p(E,F;U) \leqslant \mathrm{mod}_p(E,F;U)$, fix a function ϱ: $U \to [0, \infty]$ satisfying (2.20). Define

$$u(x) = \inf \int_{\gamma_x} \varrho \, ds \qquad (2.21)$$

for $x \in U$, where the infimum is taken over all rectifiable paths γ_x in U joining x to F; if no such γ_x exists, set $u(x)=1$. Then $u|F=0$ and $u|E \geqslant 1$. Moreover, we have that

$$|u(x) - u(y)| \leqslant \int_{\gamma_{xy}} \varrho \, ds$$

for any rectifiable curve γ_{xy} joining x and y in U. Thus u is admissible and ϱ is a very weak gradient of u, whence

$$\mathrm{cap}_p(E,F;U) \leqslant \int_U \varrho^p \, d\mu.$$

Because ϱ was arbitrary, we conclude the proof of equality (2.18).

Now we turn to the second assertion of the proposition. Fix a ball B in X and fix an admissible metric ϱ for (E,F). We may clearly assume that B is large enough so that the boundaries ∂E and ∂F are both contained in B. We would like to build an appropriate admissible continuous function u in B using ϱ, under the proviso that X is φ-convex. The definition in (2.21) may not work as ϱ need not be bounded, and to circumvent this possibility an approximation argument is needed.

By the remark made in §2.3, we may assume that ϱ is lower semicontinuous, and clearly we may assume that $\varrho|F=0$. By considering the functions $x \mapsto \max\{\varrho(x), 1/m\}$ if $x \in 2B$, and $x \mapsto \varrho(x)$ if $x \notin 2B$, $m=1,2,...$, we may further assume that $\varrho|2B$ is lower semicontinuous and that $\varrho|2B \setminus F$ is bounded away from zero: $\varrho \geqslant \xi$ in $2B \setminus F$, where $\xi > 0$ is a positive constant (use Lebesgue's monotone convergence theorem). Fix a positive integer k and consider the function $\varrho_k = \min\{\varrho, k\}$. Then ϱ_k is bounded and lower semicontinuous in $2B$, $\varrho_k \geqslant \xi$ in $2B \setminus F$ (for we may clearly assume that $\xi < 1$), and vanishes in F. Define

$$u_k(x) = \inf \int_{\gamma_x} \varrho_k \, ds \qquad (2.22)$$

for $x \in B$, where the infimum is taken over all rectifiable paths γ_x joining x to F in B; if no such path exists, we set $u_k(x)=1$. As above, we find that $u_k|F \cap B=0$ and that ϱ_k is a very weak gradient of u_k. Next, it is not difficult to see that u is continuous in B. Indeed, pick a point in B and let x and y be points in some small open φ_α-convex neighborhood

U_α of that point. That is, we can choose a curve γ_{xy} such that $l(\gamma_{xy}) \leqslant \varphi_\alpha(|x-y|)$. By the definition of u_k, we have

$$|u_k(x) - u_k(y)| \leqslant \int_{\gamma_{xy}} \varrho_k \, ds \leqslant k l(\gamma_{xy}) \leqslant k \varphi_\alpha(|x-y|), \tag{2.23}$$

and we conclude that u_k is continuous in U_α. Note here that if X is locally quasiconvex, then u_k is locally Lipschitz.

We would be finished if only ϱ_k were an admissible metric for (E, F), but there is no guarantee for that assumption. Therefore some extra technicalities are due. Denote

$$m_k = \inf u_k | E \cap B.$$

Then the function $v_k = u_k/m_k$ satisfies $v_k | E \geqslant 1$ and $v_k | F = 0$. (Recall that ϱ is assumed to be bounded away from zero in $2B \setminus F$, which fact together with the compactness and disjointness of ∂F and ∂E guarantees that $m_k > 0$.) Because ϱ_k/m_k is a very weak gradient of the continuous function v_k in B, and because

$$m_k^{-p} \int_X \varrho_k^p \, d\mu \leqslant m_k^{-p} \int_X \varrho^p \, d\mu,$$

we infer that it suffices to show

$$\sup_k m_k \geqslant 1.$$

Note also that v_k is Lipschitz if u_k is.

Suppose on the contrary that for each k there are points $x_k \in E \cap B$ and $y_k \in F \cap B$, and curves γ_k joining x_k to y_k in B such that

$$\int_{\gamma_k} \varrho_k \, ds \leqslant 1 - \delta$$

for some positive number δ independent of k. We may assume that each y_k belongs to the compact set ∂F, and that each x_k belongs to the compact set ∂E, and thus by passing to a subsequence we may assume that $y_k \to y \in \partial F$ and $x_k \to x \in \partial E$ as $k \to \infty$. Recall that ∂E and ∂F lie in B. We may also assume that $\gamma_k \subset B \setminus F$ except one end point. Because ϱ_k is bounded away from zero in $2B \setminus F$ by ξ, the lengths of the curves γ_k remain bounded from above by $M = (1-\delta)/\xi$. We assume that each curve $\gamma_k : [0, l(\gamma_k)] \to B \subset X$ is parametrized by its arc length, and then extend $\gamma_k(t) = \gamma_k(l(\gamma_k))$ for $l(\gamma_k) \leqslant t \leqslant M$. We obtain a family of 1-Lipschitz maps $\gamma_k : [0, M] \to X$ with images lying in a fixed compact set $\overline{B}$, because X is proper. The Arzela–Ascoli theorem implies, by passing to a subsequence if necessary, that γ_k converges uniformly on $[0, M]$ to a 1-Lipschitz map $\gamma : [0, M] \to X$. In particular,

γ is a rectifiable curve in X joining the points $x \in \partial E$ and $y \in \partial F$. We may also assume at this point that $l(\gamma_k) \to M$. Hence, for a fixed positive integer k_0, we have that

$$
\begin{aligned}
\liminf_{k \to \infty} \int_{\gamma_k} \varrho_{k_0} \, ds &= \liminf_{k \to \infty} \int_0^{l(\gamma_k)} \varrho_{k_0} \circ \gamma_k(t) \, dt \\
&\geqslant \liminf_{k \to \infty} \int_0^{M-\varepsilon} \varrho_{k_0} \circ \gamma_k(t) \, dt \geqslant \int_0^{M-\varepsilon} \liminf_{k \to \infty} \varrho_{k_0} \circ \gamma_k(t) \, dt \\
&\geqslant \int_0^{M-\varepsilon} \varrho_{k_0} \circ \gamma(t) \, dt,
\end{aligned}
$$

where $\varepsilon > 0$ is arbitrary. Note that the lower semicontinuity of $\varrho_{k_0} | 2B$ was needed here. Thus

$$
\liminf_{k \to \infty} \int_{\gamma_k} \varrho_{k_0} \, ds \geqslant \int_0^M \varrho_{k_0} \circ \gamma(t) \, dt \geqslant \int_\gamma \varrho_{k_0} \, ds.
$$

To justify the last inequality, we use the definition of line integrals together with the fact that $s'(t) \leqslant 1$ for almost every t, where $s = s_\gamma \colon [0, M] \to [0, l(\gamma)]$ is the length function of γ; this follows easily from the 1-Lipschitz continuity of γ. More precisely, we have that

$$
\int_\gamma \varrho_{k_0} \, ds = \int_0^{l(\gamma)} \varrho_{k_0} \circ \gamma_s(t) \, dt = \int_0^M \varrho_{k_0} \circ \gamma_s \circ s(t) s'(t) \, ds \leqslant \int_0^M \varrho_{k_0} \circ \gamma(t) \, dt.
$$

In conclusion,

$$
1 - \delta \geqslant \liminf_{k \to \infty} \int_{\gamma_k} \varrho_k \, ds \geqslant \int_\gamma \varrho_{k_0} \, ds
$$

for all $k_0 = 1, 2, \ldots$. But this is a contradiction as

$$
\lim_{k_0 \to \infty} \int_\gamma \varrho_{k_0} \, ds = \int_\gamma \varrho \, ds \geqslant 1.
$$

This completes the proof of Proposition 2.17.

3. Loewner spaces

Much of the theory of quasiconformal maps in $\mathbf{R}^n$ rests on the fact, observed by Loewner in 1959 [L], that the n-capacity between two nondegenerate continua in $\mathbf{R}^n$ is positive. This motivates the following definition.

Definition 3.1. Suppose that (X, μ) is a metric measure space as in §2.1 of Hausdorff dimension Q. We call X a *Loewner space* if there is a function $\phi \colon (0, \infty) \to (0, \infty)$ so that

$$
\operatorname{mod}_Q(E, F) \geqslant \phi(t) \tag{3.2}
$$

whenever E and F are two disjoint, nondegenerate continua in X and

$$t \geqslant \Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}}. \tag{3.3}$$

Note that the Loewner condition (3.2) depends both on the underlying metric and on the measure μ, which a priori need not be related to each other.

Euclidean space $\mathbf{R}^n$ with its usual metric is a Loewner space, and further examples will be presented in §6. In the present section, we analyze the Loewner condition in some detail.

Remark and convention 3.4. Recall from the introduction that a space (X, μ) is Q-regular if there is a constant $C \geqslant 1$ so that

$$C^{-1} R^Q \leqslant \mu(B_R) \leqslant C R^Q \tag{3.5}$$

for all balls B_R in X of radius $R < \operatorname{diam} X$. In Definition 1.6, we defined the regularity of a space in terms of its Hausdorff measure $\mathcal{H}_Q$. If (3.5) holds, then X has Hausdorff dimension Q and (3.5) holds for $\mathcal{H}_Q$ as well, possibly with different constant C. Moreover, if X is locally compact and if μ is a Borel measure on X satisfying (3.5), then μ and $\mathcal{H}_Q$ are comparable measures on X. See [S4, Appendix C] for a careful discussion on these matters.

From now on, if a space (X, μ) is called Q-regular, we understand that (3.5) holds; if no measure is being specified, we understand that (1.7) holds.

THEOREM 3.6. *Let (X, μ) be a Loewner space of Hausdorff dimension $Q > 1$. Then there is a constant $C_1 \geqslant 1$ such that*

$$C_1^{-1} R^Q \leqslant \mu(B_R) \tag{3.7}$$

for all balls B_R in X of radius $R < \operatorname{diam} X$. If there is a constant $C_2 \geqslant 1$ so that

$$\mu(B_R) \leqslant C_2 R^Q \tag{3.8}$$

for all balls B_R in X of radius $R < \operatorname{diam} X$, then (X, μ) is Q-regular and there is a decreasing homeomorphism $\psi \colon (0, \infty) \to (0, \infty)$ such that

$$\operatorname{mod}_Q(E, F) \geqslant \psi(\Delta(E, F)). \tag{3.9}$$

Moreover, we can select ψ so as to satisfy

$$\psi(t) \approx \log \frac{1}{t} \tag{3.10}$$

for all sufficiently small t, and

$$\psi(t) \approx (\log t)^{1-Q} \tag{3.11}$$

for all sufficiently large t. The statement is quantitative in the sense that the constant C_1 and the homeomorphism ψ depend only on the data associated with (X,μ).

The above theorem contains the important fact that if two nondegenerate continua of fixed size in a regular Loewner space are moved towards each other, then the modulus between them tends to infinity. This is of course a much stronger condition than (3.2), and not true, in general, if the space is not regular; see Remark 3.28. The asymptotic behavior we obtain for ψ is the correct one in $\mathbf{R}^n$. Estimates of this kind were first proved in $\mathbf{R}^n$ by Gehring [G2] by a symmetrization argument. For a detailed study of the function ψ in $\mathbf{R}^n$, see [Vu].

Also as a part of Theorem 3.6, we see that the lower bound (3.7) on the mass is a consequence of the Loewner condition. The upper bound (3.8) need not be: $\mathbf{R}^n$ equipped with its Euclidean metric and with the measure $d\mu(x)=(1+|x|)\,dx$ is a Loewner space of Hausdorff dimension n, but it is not n-regular.

We show that regular Loewner spaces enjoy a number of useful geometric properties.

3.12. *Linear local connectivity.* A metric space X is said to be *linearly locally connected* if there is a constant $C \geqslant 1$ so that for each $x \in X$ and $r>0$ the following two conditions hold:

(1) any pair of points in $B(x,r)$ can be joined in $B(x,Cr)$;

(2) any pair of points in $X \setminus \overline{B}(x,r)$ can be joined in $X \setminus \overline{B}(x,r/C)$.

By *joining*, we mean joining by a continuum. For the next proposition, recall the concept of quasiconvexity from §2.15.

THEOREM 3.13. *Let (X,μ) be a Loewner space of Hausdorff dimension $Q>1$ satisfying (3.8). Then X is linearly locally connected and quasiconvex. The statement is quantitative in the sense that the constants associated with the conclusion depend only on the data associated with X.*

In fact, more is true than indicated in Theorem 3.13. There is a large family of curves in $B(x,Cr)$ joining points in different components of $B(x,r)$, and similarly for $X \setminus \overline{B}(x,r)$. See Lemma 3.17 below.

We begin the proofs of Theorems 3.6 and 3.13 by giving three modulus estimates. Many of the ideas used below are rather standard in the quasiconformal theory in $\mathbf{R}^n$, cf. [N], [GM]. In Lemmata 3.14–3.17, we shall assume that (X,μ) is a Loewner space of Hausdorff dimension $Q>1$ satisfying the upper mass bound (3.8). As usual, $C, C', \dots$ will denote positive constants that depend only on the data associated with X.

LEMMA 3.14. *Let $0<2r<R$ and let $y\in X$. Then*

$$\operatorname{mod}_Q(\bar{B}(y,r),X\backslash B(y,R))\leqslant C\left(\log\frac{R}{r}\right)^{1-Q}.$$

Proof. Define $\varrho(x)=(|x-y|\log(R/r))^{-1}$ when $x\in B(y,R)\backslash B(y,r)$ and extend ϱ as zero to the rest of X. Then ϱ is an admissible metric, and hence we have that

$$\operatorname{mod}_Q(\bar{B}(y,r),X\backslash B(y,R))\leqslant\int_X\varrho^Q\,d\mu.$$

Let k be the least integer with $2^kr\geqslant R$. Then, using the assumption (3.8), we compute

$$\int_X\varrho^Q\,d\mu\leqslant C\left(\log\frac{R}{r}\right)^{-Q}\sum_{j=0}^k(2^jr)^{-Q}(2^{j+1}r)^Q\leqslant C\left(\log\frac{R}{r}\right)^{1-Q},$$

and the lemma follows.

LEMMA 3.15. *Let Γ be a family of curves in a ball B_R such that $l(\gamma)\geqslant L>0$ for each $\gamma\in\Gamma$. Then*

$$\operatorname{mod}_Q\Gamma\leqslant\mu(B_R)L^{-Q}\leqslant CR^QL^{-Q}. \tag{3.16}$$

Proof. Use the density $\varrho(x)=L^{-1}$ if $x\in B_R$ and $\varrho(x)=0$ if $x\notin B_R$, and remember that X is assumed to satisfy (3.8). Note that the first inequality in (3.16) holds without assumption (3.8).

LEMMA 3.17. *There exist positive constants $C\geqslant2$ and δ, depending only on the data associated with X, such that*

$$\operatorname{mod}_Q(E,F;B(x,Cr)\backslash\bar{B}(x,r/C))\geqslant\delta \tag{3.18}$$

whenever E and F are disjoint continua in $B(x,r)\backslash\bar{B}(x,\frac{1}{2}r)$, both of diameter no less than $\frac{1}{8}r$.

Proof. Let $x\in X$ and $0<r$. Then fix two continua E and F in $B(x,r)\backslash\bar{B}(x,\frac{1}{2}r)$ as above. For the three path families $\Gamma_1=(E\cup F,\bar{B}(x,r/C))$, $\Gamma_2=(E\cup F,X\backslash B(x,Cr))$ and $\Gamma_3=(E,F;B(x,Cr)\backslash\bar{B}(x,r/C))$, we have by the basic properties (2.6)–(2.8) of the modulus that

$$\operatorname{mod}_Q(E,F)\leqslant\sum_{j=1}^3\operatorname{mod}_Q\Gamma_j.$$

Moreover, by the Loewner condition (3.2) we have that

$$\operatorname{mod}_Q(E,F)\geqslant2\delta\equiv\phi(16)>0.$$

Finally, by Lemma 3.14 we can choose a constant C such that

$$\text{mod}_Q \, \Gamma_1 + \text{mod}_Q \, \Gamma_2 \leqslant \delta,$$

and the claim follows by combining the last three inequalities.

Proof of Theorem 3.13. Fix a pair x_1, x_2 of points in $X \setminus \overline{B}(x, r)$ and pick a rectifiable curve γ joining x_1, x_2 in X. If γ lies in $X \setminus \overline{B}(x, r/C)$, where C is the constant in Lemma 3.17, then condition (2) in §3.12 holds. If γ meets $\overline{B}(x, r/C)$, then by (3.18) we can find two disjoint subcontinua E and F of γ in $B(x, r) \setminus B(x, \frac{1}{2}r)$, both of diameter at least $\frac{1}{8}r$, such that the modulus of the curve family joining E and F in $B(x, Cr) \setminus \overline{B}(x, r/C)$ is positive. In particular, we can join x_1 and x_2 in the complement of $\overline{B}(x, r/C)$, and (2) of §3.12 again holds.

The proof for the first condition (1) of §3.12 is similar. Of course, (1) is implied by the quasiconvexity, which we shall prove next.

Fix two distinct points x_1, y_1 in X. Write $r = |x_1 - y_1|$ and pick a continuum E_1 joining x_1 to $X \setminus B(x_1, \frac{1}{4}r)$ in $\overline{B}(x_1, \frac{1}{4}r)$; then select F_1 corresponding to y_1 analogously. Using the Loewner condition, estimate (3.16), and an argument similar to that in Lemma 3.17, we easily infer that E_1 and F_1 can be joined by a curve γ whose length does not exceed Cr. Next, let $x_2 \in \gamma \cap E_1$, write $r_1 = |x_1 - x_2| \leqslant \frac{1}{4}r$, and pick a continuum E_2 joining x_1 to $X \setminus B(x_1, \frac{1}{4}r_1)$ in $\overline{B}(x_1, \frac{1}{4}r_1)$. Select similarly a subcurve $E_2' \subset \overline{B}(x_2, \frac{1}{4}r_1)$ of γ that joins x_2 to $X \setminus B(x_2, \frac{1}{4}r_1)$. As above, we infer that E_2 and E_2' can be joined by a curve γ_1 whose length does not exceed $Cr_1 \leqslant \frac{1}{4}Cr$. Continuing inductively we obtain a connected set $\gamma \cup \gamma_1 \cup \ldots$ joining x_1 to x_2 whose length does not exceed Cr. We see from the construction that this set contains a curve that joins x_1 and x_2. The claim follows by symmetry, and we conclude the proof for Theorem 3.13.

Remark 3.19. The proof of Theorem 3.13 shows that the following stronger version of linear local connectivity is true as well: any pair x_1, x_2 of points in $B(x, r) \setminus B(x, \frac{1}{2}r)$ can be joined by a curve γ in $B(x, Cr) \setminus B(x, r/C)$ such that the length of γ does not exceed $C|x_1 - x_2|$, where the constant C depends only on the data associated with X. We are assuming here that X is a Loewner space satisfying (3.8) as in Theorem 3.13.

Proof of Theorem 3.6. We prove first that X satisfies the lower mass bound (3.7). Take a ball $B_R = B(x, R)$ with $R < \text{diam} \, X$. Then there is a point $y \in X \setminus B(x, \frac{1}{2}R)$. Join y to x by a curve, and then choose two subcurves γ_1 and γ_2 that lie in $B(x, \frac{1}{2}R) \setminus B(x, \frac{1}{4}R)$ and $B(x, \frac{1}{8}R)$, respectively, with

$$\frac{\text{dist}(\gamma_1, \gamma_2)}{\min\{\text{diam} \, \gamma_1, \text{diam} \, \gamma_2\}} \leqslant 16.$$

By the Loewner property and the basic properties of modulus (see (2.8)), we have that

$$\phi(16) \leqslant \operatorname{mod}_Q(\gamma_1, \gamma_2; X) \leqslant \operatorname{mod}_Q\left(\gamma_2, \partial B\left(x, \tfrac{1}{4}R\right); B\left(x, \tfrac{1}{2}R\right)\right).$$

But because every curve joining γ_2 to $\partial B\left(x, \tfrac{1}{4}R\right)$ has length at least $\tfrac{1}{8}R$, the function $\varrho(x) = 8/R$ for $x \in B\left(x, \tfrac{1}{2}R\right)$, and $\varrho = 0$ elsewhere, is admissible, and whence

$$C^{-1}R^Q \leqslant \mu\left(B\left(x, \tfrac{1}{2}R\right)\right) \leqslant \mu(B_R)$$

as desired.

We conclude that X is Q-regular provided the upper mass bound (3.8) holds.

To prove the existence of a homeomorphism ψ together with the asymptotic estimates (3.10) and (3.11), fix two disjoint, nondegenerate continua E and F in X. We first show that

$$\operatorname{mod}_Q(E, F) \geqslant C' \log(1/\Delta(E, F)), \tag{3.20}$$

provided that $\Delta(E, F)$ is sufficiently small. Let C be the constant in Lemma 3.17; recall that $C \geqslant 2$. We are free to assume that $\operatorname{diam} E \leqslant \operatorname{diam} F$. Now pick a point $x \in E$ with $\operatorname{dist}(x, F) = \operatorname{dist}(E, F) = d$. Choose k to be the largest integer that is both positive and satisfies

$$C^{k+2} \Delta(E, F) \leqslant 1;$$

such an integer k can be found if $\Delta(E, F)$ is sufficiently small. For each positive integer $j \leqslant k$ pick a continuum

$$E_j \subset E \cap B(x, C^{j+1}d) \setminus \bar{B}(x, C^{j-1}d)$$

of diameter at least $C^j d$, and select F_j analogously; such continua exist by our assumptions on $\Delta(E, F)$ and k (see Theorem 2.16 in [HY]). Because no point in X belongs to more than three of the annular sets $B(x, C^{j+1}d) \setminus \bar{B}(x, C^{j-1}d)$, we find that

$$3 \operatorname{mod}_Q(E, F) \geqslant \sum_{j=1}^k \operatorname{mod}_Q(E_j, F_j; B(x, C^{j+1}d) \setminus \bar{B}(x, C^{j-1}d)).$$

Thus Lemma 3.17 shows that

$$3 \operatorname{mod}_Q(E, F) \geqslant k\delta,$$

from which (3.20) follows if $\Delta(E, F)$ is sufficiently small.

It remains to establish (3.11). For this, we can assume that $\Delta(E, F) \geqslant M$ for some large constant M. We can also make the assumption

$$\operatorname{diam} E \leqslant \operatorname{diam} F \leqslant 2 \operatorname{diam} E$$

by replacing F by an appropriate subcontinuum [HY, 2.16]. We claim that

$$\operatorname{mod}_Q(E,F) \geqslant C'(\log \Delta(E,F))^{1-Q}, \tag{3.21}$$

provided M is sufficiently large.

To this end, pick points $x_1 \in E$ and $x_2 \in F$ such that $|x_1 - x_2| = \operatorname{dist}(E,F)$. Let C be the constant in Lemma 3.17; we assume that $C \geqslant 3$. Consider the balls $B_j(i) = B(x_i, C^j \operatorname{diam} E)$ and the annuli

$$A_j(i) = B_{j+1}(i) \setminus \bar{B}_{j-1}(i)$$

for $i = 1, 2$ and $j = 1, ..., k-2$, where k is the least integer so that $B_k(1) \cap B_k(2) \neq \varnothing$. We use the notation $B_0(1) = E$ and $B_0(2) = F$. Note that

$$3 \leqslant k \leqslant C' \log \Delta(E,F), \tag{3.22}$$

provided M is sufficiently large. Next observe that

$$\operatorname{mod}_Q \Gamma_j(i) \geqslant \delta > 0, \tag{3.23}$$

where $\Gamma_j(i)$ is the family of all rectifiable curves joining $B_{j-1}(i)$ to $X \setminus B_j(i)$ inside $A_j(i)$, and where δ depends only on C and on the data for the Loewner space X. (Notice that the modulus of all nonrectifiable curves joining $B_{j-1}(i)$ and $X \setminus B_j(i)$ in $A_j(i)$ is zero, because $A_j(i)$ has finite measure.) Similarly,

$$\operatorname{mod}_Q(B_{k-2}(1), B_{k-2}(2)) \geqslant \delta > 0 \tag{3.24}$$

for some δ as above.

Let then Γ denote the family of all locally rectifiable curves joining E and F in X, and let ϱ be an admissible density for Γ. Set

$$a_j(i) = \inf_\gamma \int_\gamma \varrho \, ds,$$

where the infimum is taken over all curves $\gamma \in \Gamma_j(i)$. We may clearly assume that each $a_j(i)$ is finite. We shall consider two cases depending on whether the sum

$$\sum_{i=1}^{2} \sum_{j=1}^{k-2} a_j(i) \tag{3.25}$$

is less than $\frac{1}{2}$ or not. Assume first that it is no less than $\frac{1}{2}$. If $a_j(i) > 0$, then the function $\varrho/a_j(i)$ restricted to $A_j(i)$ is an admissible density for $\Gamma_j(i)$, and so by (3.23),

$$3 \int_X \varrho^Q \, d\mu \geqslant \sum_{i=1}^{2} \sum_{j=1}^{k-2} \int_{A_j(i)} \varrho^Q \, d\mu \geqslant \delta \sum_{i=1}^{2} \sum_{j=1}^{k-2} a_j(i)^Q.$$

Since

$$\sum_{i=1}^{2} \sum_{j=1}^{k-2} a_j(i)^Q \geqslant C' k^{1-Q}$$

by Hölder's inequality, (3.21) follows from (3.22) in the case when the sum (3.25) is at least $\frac{1}{2}$.

Suppose then that the sum (3.25) is less than $\frac{1}{2}$. We can find rectifiable curves $\gamma_j(i) \in \Gamma_j(i)$ so that

$$\sum_{i=1}^{2} \sum_{j=1}^{k-2} \int_{\gamma_j(i)} \varrho \, ds \leqslant \tfrac{2}{3}. \tag{3.26}$$

Furthermore, we can assume that there is a rectifiable curve γ_{k-1} joining $B_{k-2}(1)$ and $B_{k-2}(2)$ such that

$$\int_{\gamma_{k-1}} \varrho \, ds \leqslant \tfrac{1}{6}, \tag{3.27}$$

for otherwise we easily conclude from (3.24) that (3.21) holds for M sufficiently large. Set

$$b_j(i) = \inf_{\gamma} \int_{\gamma} \varrho \, ds,$$

where the infimum here is over all rectifiable curves γ joining $\gamma_j(i)$ and $\gamma_{j+1}(i)$ in $A_j(i) \cup A_{j+1}(i)$, and where we define $\gamma_{k-1}(i) = \gamma_{k-1}$ and $A_{k-1}(i) = A_{k-2}(i)$. Because ϱ is an admissible metric for the curve family joining E and F, we infer from (3.26) and from (3.27) that

$$\sum_{i=1}^{2} \sum_{j=1}^{k-1} b_j(i) \geqslant \tfrac{1}{6}.$$

Now the argument of the preceding paragraph applies with obvious modifications; simply replace estimate (3.23) with Lemma 3.17. Hence (3.21) follows in this case as well.

Finally, we choose an appropriate homeomorphism ψ of the positive real axis such that $\psi(t) \leqslant \phi(t)$ for $t > 0$, and such that (3.10) and (3.11) hold. This completes the proof of Theorem 3.6.

Remark 3.28. We already pointed out after the statement of Theorem 3.6 that Q-regularity is not a consequence of the Loewner condition although the lower mass bound

(3.7) is. On the other hand, the upper mass bound (3.8) is necessary if we are to obtain the conclusions of Theorems 3.6 and 3.13. This is seen by the following example.

Let X be the plane domain $\{x=(x_1,x_2):|x_1|<|x_2|+1\}$. Let μ be the measure $d\mu(x)=P(|x|)\,dx$ given by some positive increasing weight function P. If $P(t)$ grows sufficiently fast as $t\to\infty$, then (X,μ) satisfies the Loewner condition (3.2); note that the metric in X is the Euclidean metric. On the other hand, it is not difficult to check that one cannot choose ϕ so as to satisfy the estimates in Theorem 3.6. In fact, $\mathrm{mod}_Q(E,F)$ will not necessarily tend to infinity as $\Delta(E,F)$ tends to zero. (Consider $E_t=\{x_1=0, 1\leqslant x_2\leqslant t\}$ and $F_t=\{x_1=0, -t\leqslant x_2\leqslant -1\}$, and let $t\to\infty$.) Moreover, X fails to be linearly locally connected.

4. Quasiconformality vs. quasisymmetry

In this section, we study the fundamental question when quasiconformal maps are quasisymmetric. The main theorems are Theorem 4.7 and Theorem 4.9. They imply, for instance, that quasiconformal maps between Q-regular Loewner spaces are quasisymmetric if Q is bigger than one. This extends the main result of [HK1], where one of the spaces was assumed to be a Carnot group. The crucial idea needed for the proofs of Theorems 4.7 and 4.9 can already be found in [HK1] (see Main Lemma 4.12 below).

To set up some notation, we recall that a homeomorphism $f:X\to Y$ between metric spaces X and Y is said to be quasiconformal if there is a constant $H<\infty$ so that

$$\limsup_{r\to 0}\frac{L_f(x,r)}{l_f(x,r)}\leqslant H \tag{4.1}$$

for all $x\in X$, where

$$L_f(x,r)=\sup_{|x-y|\leqslant r}|f(x)-f(y)| \tag{4.2}$$

and

$$l_f(x,r)=\inf_{|x-y|\geqslant r}|f(x)-f(y)|. \tag{4.3}$$

Recall also that a homeomorphism $f:X\to Y$ as above is said to be quasisymmetric if there is a constant $H<\infty$ so that

$$|x-a|\leqslant|x-b|\quad\text{implies}\quad|f(x)-f(a)|\leqslant H|f(x)-f(b)| \tag{4.4}$$

for each triple x,a,b of points in X. This requirement is the same as (1.5) in the introduction. The slightly different formulation used here can easily be turned into the

following stronger quasisymmetry condition. A homeomorphism $f\colon X \to Y$ is called η-*quasisymmetric* if there is a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ so that

$$|x-a| \leqslant t|x-b| \quad \text{implies} \quad |f(x)-f(a)| \leqslant \eta(t)|f(x)-f(b)| \tag{4.5}$$

for each $t>0$ and for each triple x, a, b of points in X. Obviously, (4.5) implies quasisymmetry as defined in (4.4), and in general these two notions are not equivalent. However, we have the following lemma due to Väisälä [V4, 2.9]:

LEMMA 4.6. *Suppose that X and Y are pathwise connected doubling metric spaces (defined in §5.3). Then each homeomorphism f from X onto Y that satisfies (4.4) also satisfies (4.5). The statement is quantitative in that the function η will only depend on H in (4.4) and on the data associated with X and Y.*

Our standing assumption in §2.1 entails that all metric spaces be pathwise connected. Moreover, in connection with quasisymmetric maps, we only consider doubling spaces in this paper. Thus the two notions of quasisymmetry can and will be used interchangeably in what follows.

For the following discussion, recall the standing assumptions from §2.1, and also the definitions in Definition 3.1, (3.5) and §3.12.

THEOREM 4.7. *Suppose that X and Y are Q-regular metric spaces with $Q>1$, that X is a Loewner space, and that Y is linearly locally connected. If f is a quasiconformal map from X onto Y as defined in (4.1), then each point x in X has a neighborhood U where f is η-quasisymmetric as defined in (4.5). We can take $U=B(x,r)$ if $Y \setminus \overline{B}(f(x), 2L_f(x,2r)) \neq \varnothing$. This statement is quantitative in that the function η depends only on H in (4.1) and on the data associated with X and Y.*

The proof of Theorem 4.7 will show that the linear local connectivity of Y could be replaced by a weaker condition that only requires "local linear local connectivity". We leave such generalizations to the reader. Remember also that X as a Loewner space is linearly locally connected (Theorem 3.13).

COROLLARY 4.8. *Suppose that X and Y are unbounded Q-regular metric spaces with $Q>1$, that X is a Loewner space, and that Y is linearly locally connected. If f is a quasiconformal map from X onto Y that maps bounded sets to bounded sets, then f is quasisymmetric. This statement is quantitative in the same sense as in Theorem 4.7.*

Theorem 4.7 does not directly apply for bounded spaces, which need to be handled with a separate argument.

THEOREM 4.9. *Suppose that X and Y are bounded Q-regular metric spaces with $Q>1$, that X is a Loewner space, and that Y is linearly locally connected. If f is a quasiconformal map from X onto Y, then f is quasisymmetric.*

Theorem 4.9 cannot be made quantitative, for there need not be a bound for the quasisymmetry constant in terms of the data of X and Y even if f is conformal. (Think of the group of conformal transformations on the n-sphere.) Similarly, conformal or quasiconformal maps need not map bounded spaces onto bounded spaces, and so there is no counterpart to Theorem 4.9 in the case when only one of the spaces is bounded (the quasisymmetric image of a bounded space is always bounded).

Combining Corollary 4.8, Theorem 4.9 and the simple observation that linear local connectivity is preserved under quasisymmetric maps, we arrive at the following corollary.

COROLLARY 4.10. *Suppose that X and Y are Q-regular metric spaces with $Q>1$ and that X is a Loewner space. Assume that X and Y are simultaneously bounded or unbounded. Then a quasiconformal map f from X onto Y that maps bounded sets to bounded sets is quasisymmetric if and only if Y is linearly locally connected. This statement is quantitative if X and Y are both unbounded, but not so if they are both bounded.*

There is one immediate important application of the above results. Even in $\mathbf{R}^n$, $n \geqslant 2$, it is difficult to verify directly from the definition (4.1) that the inverse of a quasiconformal map is quasiconformal; standard proofs of this fact use rather deep analytic properties of quasiconformal maps. In contrast, the inverse of an η-quasisymmetric map is easily seen to be quasisymmetric, hence quasiconformal, and therefore we obtain the following corollary to Theorem 4.7.

COROLLARY 4.11. *Suppose that X and Y are Q-regular metric spaces with $Q>1$, that X is a Loewner space, and that Y is linearly locally connected. Then the inverse of a quasiconformal map f from X onto Y is quasiconformal. The statement is quantitative in the sense that the constant for f^{-1} depends only on the constant of f and on the data associated with X and Y.*

The proofs rely on the following crucial lemma.

MAIN LEMMA 4.12. *Suppose that X and Y are Q-regular metric spaces with $Q>1$ and that f is a quasiconformal map from X onto Y as defined in (4.1). If E and F are two continua in X such that $y \in f(E) \subset B(y,r)$ and such that $f(F) \subset Y \setminus \bar{B}(y,R)$ for some $y \in Y$ and for some $R>2r$, then*

$$\operatorname{mod}_Q(E, F; X) \leqslant C \left(\log \frac{R}{r} \right)^{1-Q}. \tag{4.13}$$

The constant $C \geqslant 1$ only depends on H from (4.1) *and on the data associated with* X *and* Y.

Proof. The proof of the lemma is essentially contained in the proof of Theorem 1.7 in [HK1]. In that paper, we assumed that X is a Carnot group, but the only property of a Carnot group that was used in the argument there was Q-regularity. We shall not repeat the somewhat lengthy details here. However, because the assertion (4.13) is not directly stated in [HK1], to ease the reader's task, we outline the main steps in the proof.

First, the quasiconformality condition (4.1) guarantees that the images of all sufficiently small balls about each point in X have a uniformly roundish shape. We cover the complement of the sets E and F in X by countably many such small balls B_j, $j=1,2,...$, and obtain in this way a cover for the image of $X \setminus (E \cup F)$ in Y by fairly round objects $f(B_j)$. The selection of the balls B_j is relatively simple if X is $\mathbf{R}^n$, because we can use the Besicovitch covering theorem. In the general case, we have to resort to weaker covering theorems and the selection becomes more delicate. The process is explained in detail in [HK1, pp. 70–71].

Next, one shows that with the given choice of the balls B_j, the function

$$\varrho(x) = C \left(\log \frac{R}{r} \right)^{-1} \sum_{j=1}^{\infty} \frac{\operatorname{diam} f(B_j)}{\operatorname{diam} B_j} \cdot \frac{\chi_{2B_j}(x)}{\operatorname{dist}(f(B_j), y)}$$

is an admissible metric for the condenser $(E, F; X)$; the constant $C \geqslant 1$ depends only on the data. See [HK1, p. 67 and p. 72, in particular formula (2.10) and §3.17 there].

Finally, the indicated bound (4.13) for the modulus follows by estimating the integral of ϱ^Q from above by using the Q-regularity of X and Y, and a maximal function argument. See [HK1, p. 73 and p. 67, and especially formula (2.11)] for this. We thus conclude our discussion of the Main Lemma.

Proof of Theorem 4.7. Fix $x \in X$ and let $r > 0$ be such that $Y \setminus \overline{B}(f(x), 2L_f(x, 2r))$ is not empty. Notice that such an r can be found since f is a homeomorphism. Suppose then that w, a, b are points in $B(x, r)$ such that

$$|w - a| \leqslant |w - b| \tag{4.14}$$

and such that

$$s = |f(w) - f(a)| > M|f(w) - f(b)|. \tag{4.15}$$

We shall show that M cannot be too large in (4.15). This suffices in light of Lemma 4.6.

To this end, notice that $f(b) \in B(f(w), s/M)$, that $f(a) \notin B(f(w), s)$ and that there is a point z in $X \setminus \overline{B}(x, 2r)$ with

$$f(z) \notin \overline{B}\left(f(w), \tfrac{1}{2}s\right). \tag{4.16}$$

To prove the last statement (4.16), note first that

$$s = |f(w) - f(a)| \leqslant |f(w) - f(x)| + |f(x) - f(a)| \leqslant 2L_f(x, 2r)$$

and that there is $z' = f(z)$ with $|f(z) - f(x)| > 2L_f(x, 2r)$, so that $z \in X \setminus \overline{B}(x, 2r)$. We obtain

$$2L_f(x, 2r) < |f(z) - f(w)| + |f(w) - f(x)| \leqslant |f(z) - f(w)| + L_f(x, 2r),$$

and so $|f(z) - f(w)| > L_f(x, 2r) \geqslant \frac{1}{2}s$. This proves (4.16).

Since Y is linearly locally connected, we can join $f(w)$ to $f(b)$ in $B(f(w), Cs/M)$ by a continuum E', and we can join $f(a)$ to $f(z)$ in $Y \setminus \overline{B}(f(w), s/C)$ by a continuum F'. Write $E = f^{-1}(E')$ and $F = f^{-1}(F')$. Then E joins w to b and F joins a to z. We find from (4.14) that

$$\Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}} \leqslant \frac{|w - a|}{\min\{|w - b|, r\}} \leqslant 2.$$

Because X is a Loewner space, we conclude that

$$\operatorname{mod}_Q(E, F; X) \geqslant \phi(2) > 0.$$

On the other hand, for $M > 2C^2$, we have

$$\operatorname{mod}_Q(E, F; X) \leqslant C \left(\log \frac{M}{C^2} \right)^{1-Q}$$

by Main Lemma 4.12. A bound for M follows from these last two estimates as desired, and the theorem is proved.

Proof of Theorem 4.9. The proof in this case is basically the same as above, only slightly more awkward because we have to observe the behavior of f near a fixed "base point". This extra complication reflects the fact that there is no quantitative version of the theorem. Thus, fix a point $x_0 \in X$. By Theorem 4.7 we can pick a ball $B_0 = B(x_0, r_0)$ such that f is quasisymmetric in $4B_0$. Then let x, a, b be a triple of points in X such that

$$|x - a| \leqslant |x - b|. \tag{4.17}$$

We need to show that

$$|f(x) - f(a)| \leqslant H|f(x) - f(b)|$$

for some constant H independent of the points x, a and b.

We shall consider two cases depending on whether x is in B_0 or not. Assume first that $x \in B_0$. If b is in $2B_0$, then a is in $4B_0$, and the desired quasisymmetry estimate follows. If b is not contained in $2B_0$, then the quasisymmetry of f in $4B_0$ shows that

$$\operatorname{diam} f(B_0) \leqslant C|f(x) - f(b)|. \tag{4.18}$$

To see this, assume that

$$M|f(x) - f(b)| \leqslant |f(x) - f(w)| \tag{4.19}$$

for some $w \in B_0$ and for some large M. Because Y is linearly locally connected, we can join $f(x)$ to $f(b)$ in $B(f(x), C|f(x) - f(b)|)$; in particular, there is a point $z \in \partial 2B_0$ such that $|f(x) - f(z)| \leqslant C|f(x) - f(b)|$. Because f is quasisymmetric in $4B_0$, we infer from (4.19) that

$$M|f(x) - f(b)| \leqslant |f(x) - f(w)| \leqslant C|f(x) - f(z)| \leqslant C|f(x) - f(b)|.$$

Thus M in (4.19) cannot be too large, and (4.18) follows. It follows from (4.18) that

$$|f(x) - f(a)| \leqslant \operatorname{diam} Y \leqslant C \operatorname{diam} Y (\operatorname{diam} f(B_0))^{-1}|f(x) - f(b)| \leqslant C|f(x) - f(b)|.$$

We have thus verified the quasisymmetry in the case when x lies in B_0.

Suppose now that x is not contained in B_0. Notice that if

$$|f(x) - f(x_0)| \leqslant M|f(x) - f(b)|, \tag{4.20}$$

then by reasoning as in the above paragraph we conclude that (4.18) holds, and hence that the desired quasisymmetry estimate holds; in this case the constant will depend on M from (4.20). Thus we assume that

$$|f(x) - f(x_0)| \geqslant M|f(x) - f(b)| \tag{4.21}$$

for some large M whose value will be determined momentarily. Suppose that

$$|f(x) - f(a)| \geqslant M|f(x) - f(b)| \tag{4.22}$$

for the same value of M as in (4.21). By the linear local connectivity of Y we may join $f(x)$ to $f(b)$ by a continuum E in $B(f(x), C|f(x) - f(b)|)$ and $f(a)$ to $f(x_0)$ by a continuum F in $Y \setminus B(f(x), M|f(x) - f(b)|/C)$.

We again separate two cases, this time depending on the location of a. Suppose first that $a \notin \delta B_0$, where $0 < \delta < 1$ is a constant such that X is $1/2\delta$-linearly locally connected. Notice that such a constant exists because X is Loewner (Theorem 3.13). Then by (4.17),

$$\begin{aligned}
\min\{\operatorname{diam} f^{-1}(F), \operatorname{diam} f^{-1}(E)\} &\geqslant \min\{\delta r_0, |x - b|\} \geqslant \min\{\delta r_0, |x - a|\} \\
&\geqslant \min\{\delta r_0, \operatorname{dist}(f^{-1}(F), f^{-1}(E))\} \\
&\geqslant \frac{\delta r_0 \operatorname{dist}(f^{-1}(F), f^{-1}(E))}{\operatorname{diam} X}.
\end{aligned} \tag{4.23}$$

Because X is a Loewner space, we obtain from (4.23) and from Main Lemma 4.12 that

$$0 < C \leqslant \operatorname{mod}_Q(f^{-1}(E), f^{-1}(F); X) \leqslant C\left(\log \frac{M}{C^2}\right)^{1-Q},$$

provided $M > 2C^2$. Consequently, a bound for M follows and the proof is complete in the case $a \notin \delta B_0$.

Assume finally that a lies in δB_0, while (4.21) holds. Let F' be a continuum in $\frac{1}{2}B_0$ which joins a and x_0 and has diameter at least δr_0; such a continuum exists by the choice of δ. Let $F = f(F')$. We claim that

$$d = \operatorname{dist}(f(x), F) \geqslant C^{-1} \operatorname{diam} f(B_0), \tag{4.24}$$

where $C \geqslant 1$ depends only on the linear local connectivity constant of Y and on the quasisymmetry constant of f in $4B_0$. To prove (4.24), let $w \in F$ be such that $|f(x) - f(w)| = d$. Then we can join $f(x)$ and $f(w)$ in $B(f(x), Cd)$ by a continuum; in particular, because $x \notin B_0$, we can find points z_1 and z_2 in X such that

$$|x_0 - z_1| = \tfrac{1}{2}r_0, \quad |x_0 - z_2| = r_0 \quad \text{and} \quad f(z_i) \in B(f(x), Cd)$$

for $i = 1, 2$. Because f is η-quasisymmetric in $4B_0$, we have, for any $y \in B_0$, that

$$|f(y) - f(x_0)| \leqslant C|f(z_2) - f(x_0)| \leqslant C|f(z_1) - f(x_0)| \leqslant C|f(z_1) - f(z_2)| \leqslant Cd.$$

This gives (4.24). Next, (4.22) implies that

$$|f(x) - f(b)| \leqslant M^{-1}|f(x) - f(a)| \leqslant M^{-1} \operatorname{diam} X,$$

and hence we can find a continuum E joining $f(x)$ to $f(b)$ in $B(f(x), C \operatorname{diam} X/M)$. Thus an upper bound for M follows as in the above paragraph upon observing that (4.23) holds for the present continua E and F as well. This completes the proof of Theorem 4.9.

Remark 4.25. (a) An inspection of the proof of Theorem 4.9 gives that the quasisymmetry constant of f will depend, in addition to the usual data, on the quantities

$$\frac{r_0}{\operatorname{diam} X}, \quad \frac{\operatorname{diam} f(B_0)}{\operatorname{diam} Y},$$

and nothing else. These are natural parameteres that necessarily show up (and would disappear if quasi-Möbius maps [V3] were used instead of quasisymmetric maps).

(b) In the proof of Main Lemma 4.12 one only needs a *local* lower bound on the mass in Y; that is, the bound (3.7) is only required to hold for R sufficiently small (depending

on the center of the ball). Thus in the case where X and Y are domains (i.e. open connected subsets) in $\mathbf{R}^n$, $n \geqslant 2$, Theorems 4.7 and 4.9 directly generalize some results proved by Gehring and Martio [GM], Väisälä [V4] and the first author [H1]. For those who are familiar with the lingo, we recall that it was proved in [GM] that quasiconformal maps from QED-domains onto linearly locally connected domains are quasisymmetric (see also [V3]); similar results were proved in [V4], [H1] with respect to the internal metrics of the domains in question.

(c) If the space X in Main Lemma 4.12 satisfies a stronger, Besicovitch-type covering theorem, where every cover of a set by balls admits a countable subcover with finite (depending on X) amount of overlap, then one can replace "lim sup" in the definition of quasiconformality (4.1) by "lim inf"; the conclusion remains the same. In particular, under the stronger covering hypothesis in each of the results of this section this weaker notion of quasiconformality is sufficient to imply quasisymmetry. This follows from the proof of Theorem 1.7 in [HK1], where comments in the case $X = \mathbf{R}^n$ have been made. Besides $\mathbf{R}^n$, many other "Riemannian-type" spaces have this covering property, for example compact polyhedra (cf. Theorem 6.13). See [Fe, 2.8] for a thorough discussion of covering theorems.

5. Poincaré inequalities and the Loewner condition

Suppose throughout this section that (X, μ) is a metric measure space as in §2.1. We shall show that the validity of a Poincaré-type inequality in X is tantamount to X being a Loewner space as defined in Definition 3.1, provided X is proper, regular and φ-convex.

5.1. *Poincaré inequalities.* Let $p \geqslant 1$ be a real number. We say that X admits a *weak $(1, p)$-Poincaré inequality* if

$$\fint_B |u - u_B| \, d\mu \leqslant C_p (\operatorname{diam} B) \left(\fint_{C_0 B} \varrho^p \, d\mu \right)^{1/p} \tag{5.2}$$

whenever u is a bounded continuous function in a ball $C_0 B$ and ϱ is its very weak gradient there. The constants $C_p \geqslant 1$ and $C_0 \geqslant 1$ should be independent of B and u. We use the standard notation

$$u_A = \fint_A u \, d\mu = \frac{1}{\mu(A)} \int_A u \, d\mu$$

for the mean value of a function u in a measurable set A of positive measure.

Inequality (5.2) is termed "weak" because we allow a larger ball on the right-hand side than on the left. In many cases, because of the uniformity of (5.2) (constants are independent of B), this weak estimate can be used to iterate so as to yield an inequality

with the same ball on both sides [J], [HaK]. Moreover, if the measure μ satisfies the doubling condition (5.4) below, then a weak $(1,p)$-Poincaré inequality (5.2) (for all balls) implies the a priori stronger inequality where one replaces (for all balls) the averaged L^1-norm on the left by the averaged L^q-norm for some $q>p$ [HaK]. This explains the terminology; we could speak about a (q,p)-Poincaré inequality. The weak inequality (5.2) is sufficient for our purposes in this paper.

Notice that if X admits a weak $(1,p)$-Poincaré inequality, then it admits a weak $(1,p')$-Poincaré inequality for $p'>p$ by Hölder's inequality. The converse is not true in general; see Remark 6.19.

We emphasize that inequality (5.2) should hold for *all balls* B in X. It may well happen that something like (5.2) is true for $B=X$, but X may still not admit a weak Poincaré inequality in the above sense.

5.3. *Doubling space.* A metric measure space (X,μ) is said to be *doubling* if there is a constant $C \geqslant 1$ so that

$$\mu(B_{2R}) \leqslant C\mu(B_R) \tag{5.4}$$

for all balls B_R in X of radius $0<R<\operatorname{diam} X$. If there is no measure specified on X, we can define X to be doubling if there is a constant $C \geqslant 1$ so that every ball in X can be covered by at most C balls with half the radius. It is easy to see (by Covering Lemma 5.5) that X is doubling in this latter sense if there is a measure μ on X such that (5.4) holds.

Clearly, if X is regular, it is also doubling, but the converse need not be true. For instance, the space $(\mathbf{R}^n, (1+|x|)\,dx)$ is a doubling space with lower mass bound (3.7) (with $Q=n$), but it is not regular. Similarly, any complete Riemannian n-manifold with nonnegative Ricci curvature is doubling with upper mass bound (3.8) (with $Q=n$), but it need not be regular.

Next we record a basic but most useful covering lemma (see [Ma, 2.1] or [St1, p. 9]).

COVERING LEMMA 5.5. *Suppose that X is a metric space and suppose that A is a bounded subset of X. If for each $x \in A$ we are given a radius $r_x>0$ and a ball $B_x = B(x, r_x)$, then we can pick a countable, pairwise disjoint collection $\{B_i = B_{x_i} : i=1,2,...\}$ of balls of this given form such that either*

$$A \subset \bigcup_i 5B_i \tag{5.6}$$

or (r_i) is an infinite sequence that does not converge to zero as $i \to \infty$.

Typically, the spaces X in this paper are such that the second alternative in Covering Lemma 5.5 is ruled out, and hence (5.6) holds. This happens for instance if X is doubling.

We begin with the following theorem, which gives a sufficient condition for a space to be Loewner. Recall that a proper space is one where closed balls are compact; recall also the definition for φ-convexity from §2.15.

THEOREM 5.7. *Suppose that (X, μ) is a proper, doubling and φ-convex space where the lower mass bound (3.7) holds for some $Q \geqslant 1$. If X admits a weak $(1, Q)$-Poincaré inequality, then X is a Loewner space. The statement is quantitative in that the function ϕ in Definition 3.1 only depends on the data associated with X (of which the φ-convexity is not part).*

We do not know to what extent the assumptions "proper, doubling and φ-convex" in Theorem 5.7 are necessary.

Theorem 5.7 follows from the more general Theorem 5.9 below. The latter will be needed later in §6. We record the following corollary to Theorems 5.7 and 3.13. The statement is quantitative, and the φ-convexity plays no role in the conclusion.

COROLLARY 5.8. *Suppose that X is a proper, Q-regular and φ-convex space that admits a weak $(1, Q)$-Poincaré inequality for some $Q > 1$. Then X is linearly locally connected and quasiconvex.*

A smooth submanifold of Euclidean space provides a standard example of a space that is proper and locally quasiconvex. Corollary 5.8 provides a sufficient condition for such a submanifold to be quasiconvex. Related but different sufficient conditions for quasiconvexity of metric manifolds can be found in [S4].

Recall that the *Hausdorff s-content* of a set E in a metric space is the number

$$\mathcal{H}_s^\infty(E) = \inf \sum_i r_i^s,$$

where the infimum is taken over all countable covers of the set E by balls B_i of radius r_i. Thus the s-content of E is less than, or equal to, the Hausdorff s-measure of E, and it is never infinite for E bounded. However, the s-content of a set is zero if and only if its Hausdorff s-measure is zero.

THEOREM 5.9. *Suppose that (X, μ) is a doubling space where the lower mass bound (3.7) holds for some $Q \geqslant 1$. Suppose further that X admits a weak $(1, p)$-Poincaré inequality for some $1 \leqslant p \leqslant Q$. Let E and F be two compact subsets of a ball B_R in X and assume that, for some $Q \geqslant s > Q - p$ and $1 \geqslant \lambda > 0$, we have*

$$\min\{\mathcal{H}_s^\infty(E), \mathcal{H}_s^\infty(F)\} \geqslant \lambda R^{s-Q} \mu(B_R). \tag{5.10}$$

Then there is a constant $C \geqslant 1$, depending only on s and on the data associated with X, so that

$$\int_{B_{CR}} \varrho^p \, d\mu \geqslant C^{-1} \lambda \mu(B_R) R^{-p} \tag{5.11}$$

whenever u is a continuous function in the ball B_{CR} with $u|E \leqslant 0$ and $u|F \geqslant 1$, and ϱ is a very weak gradient of u in B_{CR}.

The reason why (5.11) is not being formulated in terms of capacity, but rather for an individual function, is that we have defined capacity by using arbitrary test functions, whereas a Poincaré inequality is required for continuous functions only, cf. Remark 2.13.

Let us check how Theorem 5.7 follows from Theorem 5.9.

Proof of Theorem 5.7. Let E and F be two disjoint continua in X. Write $d = \operatorname{dist}(E, F)$ and assume without loss of generality that

$$\delta = \operatorname{diam} E = \min\{\operatorname{diam} E, \operatorname{diam} F\}.$$

Fix

$$t \geqslant \Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}} = \frac{d}{\delta}.$$

Choose a point $x \in E$ such that the closed ball $\overline{B}(x, d)$ meets F. Then consider the ball

$$B = B(x, d+2\delta).$$

The compact sets E and $F' = F \cap \overline{B}(x, d+\delta)$ both lie in B and have Hausdorff 1-content at least

$$\delta = \frac{\delta(d+2\delta)^{Q-1}}{\mu(B)}(d+2\delta)^{1-Q}\mu(B).$$

We use this fact and Theorem 5.9 to estimate the modulus between E and F. Because X is assumed to be proper and φ-convex, we can use inequality (2.19) in Proposition 2.17. (This is the only place where the concept of φ-convexity is used in this paper. Moreover, we only use it so that test functions can be taken to be continuous, whence there is no quantitative dependence on φ-convexity.) In conclusion, by (5.10) and (5.11) with $p = Q$ and $s = 1$,

$$\operatorname{mod}_Q(E, F) \geqslant \operatorname{cap}_Q^c(E, F; CB) \geqslant C^{-1}\frac{\delta(d+2\delta)^{Q-1}}{\mu(B)}\mu(B)(d+2\delta)^{-Q} \geqslant C^{-1}\min\{1, 1/t\},$$

and thereby Theorem 5.7 follows.

Proof of Theorem 5.9. Let u be a continuous function in the ball B_{CR}, where $C = 10C_0$, and C_0 is the constant appearing in (5.2). Assume that $u|E \leqslant 0$ and $u|F \geqslant 1$, and let ϱ be a very weak gradient of u in B_{CR}.

The proof splits into two cases depending on whether or not there are points x in E and y in F so that neither

$$|u(x) - u_{B(x,R)}|$$

nor

$$|u(y)-u_{B(y,5R)}|$$

exceeds $\frac{1}{5}$. If such points can be found, then

$$1\leqslant|u(x)-u(y)|\leqslant\tfrac{1}{5}+|u_{B(x,R)}-u_{B(y,5R)}|+\tfrac{1}{5},$$

or

$$1\leqslant C\fint_{B(y,5R)}|u-u_{B(y,5R)}|\,d\mu\leqslant CR\Big(\fint_{B_{CR}}\varrho^p\,d\mu\Big)^{1/p},$$

from which (5.11) follows. Note that $B(x,R)\subset B(y,5R)\subset B_{CR}$ by the choices.

The second alternative, by symmetry, is that for all points x in E we have that

$$\tfrac{1}{5}\leqslant|u(x)-u_{B(x,R)}|.$$

Therefore, because u is continuous,

$$1\leqslant C\sum_{j=0}^{\infty}|u_{B_j(x)}-u_{B_{j+1}(x)}|\leqslant C\sum_{j=0}^{\infty}\fint_{B_j(x)}|u-u_{B_j(x)}|\,d\mu$$
$$\leqslant C\sum_{j=0}^{\infty}(2^{-j}R)\Big(\fint_{C_0B_j(x)}\varrho^p\,d\mu\Big)^{1/p},$$

where $B_j(x)=2^{-j}B(x,R)$. Therefore, if

$$\int_{C_0B_j(x)}\varrho^p\,d\mu\leqslant\varepsilon R^{Q-s-p}(2^{-j}R)^{s-Q}\mu(B_j(x))$$

for a fixed $\varepsilon>0$ and for each $j=0,1,2,\ldots$, we have that

$$1\leqslant C\varepsilon^{1/p}\sum_{j=0}^{\infty}(2^{-j})^{(p+s-Q)/p}\leqslant C\varepsilon^{1/p}$$

because $s>Q-p$. It follows that there is an index j_x such that

$$\int_{C_0B_{j_x}(x)}\varrho^p\,d\mu\geqslant\varepsilon_0 R^{Q-s-p}(2^{-j_x}R)^{s-Q}\mu(B_{j_x}(x))$$

for some $\varepsilon_0>0$ depending only on the data. In particular, using Covering Lemma 5.5 and the fact that X is doubling, we find a pairwise disjoint collection of balls of the form $B_k=B(x_k,r_kR)$ such that

$$E\subset\bigcup_k 5B_k$$

and such that

$$\mu(B_k)(r_kR)^{s-Q} \leqslant CR^{s+p-Q} \int_{B_k} \varrho^p \, d\mu.$$

Hence

$$\lambda R^{s-Q}\mu(B_R) \leqslant \mathcal{H}_s^\infty(E) \leqslant \sum_k (5r_kR)^s \leqslant C\sum_k (r_kR)^{s-Q}(r_kR)^Q$$

$$\leqslant C\sum_k (r_kR)^{s-Q}\mu(B_k) \leqslant CR^{s+p-Q} \int_{B_{CR}} \varrho^p \, d\mu,$$

as desired. This completes the proof of Theorem 5.9.

Next theorem gives a converse to Theorem 5.7.

THEOREM 5.12. *Suppose that X is a locally compact, Q-regular Loewner space. Then X admits a weak $(1,Q)$-Poincaré inequality. The statement is quantitative in that (5.2) will hold with constants $C_Q \geqslant 1$ and $C_0 \geqslant 1$ that depend only on the data associated with X.*

Before we go into the proof of this theorem, it is worthwhile to summarize the equivalence of the Loewner condition and Poincaré inequality in the following corollary.

COROLLARY 5.13. *Suppose that X is proper, Q-regular and φ-convex. Then X is a Loewner space if and only if X admits a weak $(1,Q)$-Poincaré inequality.*

The statement of the above corollary is quantitative, but remember that the φ-convexity assumption has no bearing on the constants. It would be interesting to find analogues of Corollary 5.13 for $1 \leqslant p < Q$.

The proof of Theorem 5.12 consists of two lemmata. The first lemma probably belongs to folklore, but we have no reference to give. In it, an alternative characterization of a Poincaré inequality is given in terms of the maximal function

$$M_R g(x) = \sup_{r<R} \fint_{B(x,r)} g \, d\mu. \tag{5.14}$$

For a future reference, a general p-version is proved.

LEMMA 5.15. *Suppose that (X,μ) is a locally compact doubling space. Then X admits a weak $(1,p)$-Poincaré inequality if and only if there is a constant $C \geqslant 1$ so that*

$$|u(x)-u(y)| \leqslant C|x-y|(M_R\varrho^p(x)+M_R\varrho^p(y))^{1/p} \tag{5.16}$$

whenever u is a continuous function in a ball B_R, $x,y \in C^{-1}B_R$, and ϱ is a very weak gradient of u in B_R. The statement is quantitative in the usual sense.

LEMMA 5.17. *Suppose that X is a Q-regular Loewner space. Then the pointwise estimate* (5.16) *holds for $p=Q$, quantitatively.*

Proof of Lemma 5.15. First we prove the sufficiency. Let u be a continuous function in a ball B_R and let ϱ be a very weak gradient of u there. Denote by B the ball $C^{-1}B_R$, where C is as in (5.16). Pick t such that $\mu(\{x \in B : u \leqslant t\}) \geqslant \frac{1}{2}\mu(B)$ and $\mu(\{x \in B : u \geqslant t\}) \geqslant \frac{1}{2}\mu(B)$. We shall in fact prove a weak (p,p)-Poincaré inequality with u_B replaced by t; that is, we shall prove (5.2) with the averaged L^p-norm of $u-t$ on the left-hand side. By replacing u with $u-t$, we may further assume that $t=0$.

We estimate the integral of $|u|$ over B; by symmetry it suffices to estimate the integral over the set where $u>0$. Thus we can assume that u is nonnegative in B. Let $s>0$. We claim that

$$\mu(\{x \in B : u \geqslant s\}) \leqslant C s^{-p} R^p \int_{B_R} \varrho^p \, d\mu \tag{5.18}$$

for some $C \geqslant 1$ independent of s.

To this end, let x be a point in B such that $u(x) \geqslant s$. Then pick $y \in B$ with $u(y) \leqslant 0$. By assumption (5.16), there is a radius $0 < r < C|x-y|$ so that

$$\mu(B(w,r)) \leqslant C|x-y|^p s^{-p} \int_{B(w,r)} \varrho^p \, d\mu, \tag{5.19}$$

where $w=x$ or $w=y$. Notice that $B(w,r) \subset B_R$. If (5.19) holds with $w=y$ for each choice of y, we apply Covering Lemma 5.5 and conclude that the set in B where u vanishes can be covered by a countable collection of balls of the form $5B(w_i, r_i)$ such that the balls $B(w_i, r_i)$ are pairwise disjoint and satisfy (5.19). Thus

$$\mu(B) \leqslant 2\mu(\{y \in B : u(y) = 0\}) \leqslant 2\sum_i \mu(5B(w_i, r_i))$$

$$\leqslant C\sum_i \mu(B(w_i, r_i)) \leqslant C|x-y|^p s^{-p} \int_{B_R} \varrho^p \, d\mu$$

so that inequality (5.18) follows. Otherwise, given x with $u(x) \geqslant s$, there is always some y such that (5.19) holds for $w=x$. Then we use a covering argument just as above to obtain inequality (5.18) (in this case we really estimate the measure of the set where $u \geqslant s$ and not just $\mu(B)$).

Inequality (5.18) is a weak-type inequality which does not in general lead to the strong-type inequality we are looking for. However, the fact that it holds for a function and its gradient, even if a very weak one, allows one to use a truncation argument which leads to the desired estimate

$$\int_B |u|^p \, d\mu \leqslant C R^p \int_{B_R} \varrho^p \, d\mu. \tag{5.20}$$

This is done, for instance, in [S4, Appendix C]. For convenience, we sketch the argument.

Suppose that j is an integer and consider the set $L_j = \{x \in B_R : 2^j \leqslant u \leqslant 2^{j+1}\}$. Subtract 2^j from u and truncate u to obtain a function v such that $v=0$ when $u \leqslant 2^j$ and $v=2^j$ when $u \geqslant 2^{j+1}$. If U is any open set containing L_j then the function h defined as the restriction of ϱ in U, and zero elsewere, is a very weak gradient of v in B_R (see Lemma C.19 in [S4]). If we now apply the argument above and especially the weak estimate (5.18) to v, we conclude by letting U tend to L_j that

$$\int_{L_{j+1} \cap B} |u|^p \, d\mu \leqslant CR^p \int_{L_j} \varrho^p \, d\mu.$$

(It is in this approximation by open sets that the local compactness of X is needed.) The claim (5.20) follows by summing over j. The sufficiency part of the lemma is thus proved.

To prove the necessity, let u be a continuous function in a ball B_R in X, and let ϱ be its very weak gradient there. Pick $x \in \frac{1}{10} B_R$. Then denote $B_x = B\left(x, \frac{1}{10} R\right)$ and $B_i = 2^{-i} B_x$, and estimate

$$|u(x) - u_{B_x}| \leqslant \sum_{i=0}^{\infty} |u_{B_i} - u_{B_{i+1}}| \leqslant C \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}|$$

$$\leqslant CR \sum_{i=0}^{\infty} 2^{-i} \left(\fint_{B_i} \varrho^p \, d\mu \right)^{1/p} \leqslant CR(M_R \varrho^p(x))^{1/p}.$$

Similarly, for $y \in \frac{1}{10} B_R$ and $B_y = B\left(y, \frac{1}{2} R\right)$, we have

$$|u(y) - u_{B_y}| \leqslant CR(M_R \varrho^p(y))^{1/p}.$$

In conclusion, because $B_x \subset B_y$ and because $\mu(B_y) \leqslant C\mu(B_x)$,

$$|u(x) - u(y)| \leqslant |u(x) - u_{B_x}| + \fint_{B_x} |u - u_{B_y}| \, d\mu + |u(y) - u_{B_y}|$$

$$\leqslant |u(x) - u_{B_x}| + C \fint_{B_y} |u - u_{B_y}| \, d\mu + |u(y) - u_{B_y}|$$

$$\leqslant CR(M_R \varrho^p(x) + M_R \varrho^p(y))^{1/p}$$

as desired. This completes the proof of Lemma 5.15.

Proof of Lemma 5.17. Let u be a continuous function in a ball B in X and let ϱ be a very weak gradient of u in B. We pick two points x and y from $B' = C'^{-1} B$ for some large C', depending only on the data; the value of C' will be determined in the course

of the proof. We need to show that inequality (5.16) holds. As customary, we let below $C, C_1, C_2, ...$ denote various constants depending only on the data.

Because X is assumed to be Q-regular and Loewner, it is quasiconvex by Theorem 3.13. Pick a curve γ joining x and y with length not exceeding $C_1|x-y|$; we may assume that γ lies in B by making C' sufficiently large. We may also assume that $|u(x)-u(y)|=1$, because estimate (5.16) has the correct homogeneity properties. Consider the annuli

$$A_j = B(x, C_2^{-3j}d) \setminus B(x, C_2^{-3j-2}d)$$

for $j=0, 1, 2, ...$, where $C_2 \geqslant 2$ is a constant that will be determined momentarily and where d is chosen so that the boundary of the ball $B(x,d)$ meets the midpoint of γ. Note that $d \leqslant C_1|x-y|$. We choose C' so large that the ball $B(x, C_2d)$ is still in B; in particular, the annuli A_j all lie in B. Next, let γ_j in $\bar{A}_j$ be a part of γ joining the boundaries of the two balls that define A_j. Then we infer from Lemma 3.15 and from (the proof of) Lemma 3.17 that, by choosing C_2 sufficiently large, the estimate

$$\mathrm{mod}_Q \Gamma_j \geqslant C^{-1} > 0 \tag{5.21}$$

holds, where Γ_j consists of all those curves in the family $(\gamma_j, \gamma_{j+1}; B_j)$ whose length does not exceed $C_3 C_2^{-3j}d$, where $B_j = B(x, C_2^{-3j+1}d)$.

Next, write

$$a_j = \inf_\sigma \int_\sigma \varrho \, ds, \tag{5.22}$$

where the infimum is taken over all curves σ in Γ_j. By (5.21), we have that

$$\int_{B_j} \varrho^Q \, d\mu \geqslant C^{-1} a_j^Q,$$

and hence that

$$\fint_{B_j} \varrho^Q \, d\mu \geqslant C^{-1} a_j^Q C_2^{3jQ} d^{-Q}.$$

Therefore the desired estimate

$$|u(x)-u(y)| = 1 \leqslant d \left(\fint_{B_j} \varrho^Q \, d\mu \right)^{1/Q} \leqslant C|x-y|(M_{C|x-y|}\varrho^Q(x))^{1/Q}$$

holds provided that there is an index j with

$$a_j^Q C_2^{3jQ} \geqslant C^{-1}$$

for some $C \geqslant 1$ depending only on the data; recall that $d \leqslant C_1|x-y|$. Thus we may assume that

$$a_j \leqslant \varepsilon C_2^{-3j} \tag{5.23}$$

for some small $\varepsilon > 0$ and for all j.

Assuming (5.23), we connect each pair of curves γ_j and γ_{j+1} by a curve $\gamma_j' \in \Gamma_j$ inside the ball B_j such that

$$\int_{\gamma_j'} \varrho \, ds \leqslant 2\varepsilon C_2^{-3j}. \tag{5.24}$$

By the definition of Γ_j, the length of γ_j' does not exceed $C_3 C_2^{-3j} d$. These continua γ_j' are so located that we can repeat the above argument by using γ_j' instead of γ_j. Notice however that in this case the continua γ_j' and γ_{j+1}' need not be disjoint. If they are disjoint, we obtain that the modulus of the family of all those curves that join γ_j' and γ_{j+1}' in some appropriate ball roughly of size $C_2^{-3j} d$, and whose length does not exceed $C_4 C_2^{-3j} d$, has a positive lower bound depending only on the data.

Now we define numbers b_j analogously to (5.22); if the continua γ_j' and γ_{j+1}' meet, we set $b_j = 0$. It then follows as above that the required pointwise estimate holds unless

$$b_j \leqslant \varepsilon C_2^{-3j}$$

for some small $\varepsilon > 0$ and for all j. This means that there are curves γ_j'' joining γ_j' and γ_{j+1}' with length not exceeding $C_4 C_2^{-3j} d$ such that

$$\int_{\gamma_j''} \varrho \, ds \leqslant 2\varepsilon C_2^{-3j}. \tag{5.25}$$

Note that γ_j'' may be a constant curve. By using the curves γ_j' and γ_j'', we obtain a curve γ_x joining x to some point in γ_0. The curve γ_x has finite length, as follows by summing up the lengths of γ_j' and γ_j''; moreover, by (5.24) and (5.25) we have that

$$\int_{\gamma_x} \varrho \, ds \leqslant \frac{2\varepsilon}{1 - C_2^{-3}} < \frac{1}{3}, \tag{5.26}$$

provided ε is small enough, depending only on the data.

Next we repeat the argument for the point y. We find a curve γ_y starting from y such that

$$\int_{\gamma_y} \varrho \, ds < \frac{1}{3} \tag{5.27}$$

and

$$\Delta(\gamma_x, \gamma_y) = \frac{\operatorname{dist}\{\gamma_x, \gamma_y\}}{\min\{\operatorname{diam} \gamma_x, \operatorname{diam} \gamma_y\}} \leqslant C. \tag{5.28}$$

By using (5.28) and the Loewner property as above one more time, we find that either the desired estimate (5.16) holds or there is a curve $\widehat{\gamma}$ joining γ_x and γ_y in B such that

$$\int_{\widehat{\gamma}} \varrho \, ds < \frac{1}{3}. \tag{5.29}$$

By combining (5.26)–(5.29), we arrive at the contradictory inequality

$$1 = |u(x) - u(y)| \leqslant \int_{\gamma_x \cup \gamma_y \cup \hat{\gamma}} \varrho \, ds < 1.$$

The proof of Lemma 5.17 is complete.

6. Examples of Loewner spaces

In this section, we collect examples, old and new, of spaces that satisfy the Loewner condition as defined in Definition 3.1. In particular, we give examples of situations where the local-to-global phenomenon in the quasiconformal mapping theory occurs, cf. §4.

6.1. *Euclidean space and compact manifolds.* Euclidean n-space $\mathbf{R}^n$ is a Loewner space for each $n \geqslant 1$. This is trivial for $n=1$, easy for $n=2$, and due to Loewner [L] for $n \geqslant 3$. The quickest proof of this fact uses the scale invariance of the Loewner condition and the Sobolev embedding theorem on codimension-one spheres in $\mathbf{R}^n$. The argument using the Poincaré inequality in §5 of course applies here as well, but insofar as one does not care about sharp estimates for the function ϕ, there are other simple proofs available, cf. [V1, 10.12].

Modelled by $\mathbf{R}^n$, a compact Riemannian manifold is a Loewner space because a weak Poincaré inequality trivially holds. In fact, one only needs Lipschitz charts so that by Sullivan's theorem [Su] every compact topological manifold outside dimension four admits a metric that makes it a Loewner space. See also §6.10 below.

6.2. *Carnot groups.* *Carnot–Carathéodory spaces* provide examples of regular Loewner spaces whose Hausdorff dimension exceeds the topological dimension. Roughly, these are spaces that are locally modelled by a *Carnot group*. It was in the setting of Carnot groups where quasiconformal maps beyond Riemannian spaces first appeared. Namely, the boundaries of rank-one symmetric spaces can be identified as certain Carnot groups of step two, and Mostow [Mo1] had to develop the basic quasiconformal theory in these groups in order to reach his celebrated rigidity results.

A Carnot group admits a $(1, p)$-Poincaré inequality for all $p \geqslant 1$ by a result of Jerison [J], and hence is a Loewner space. The Loewner property of a Carnot group was known before; see [Re] and [H2]. More information about Carnot spaces, Poincaré inequalities and quasiconformal maps can be found in [Gr], [HK1], [KR1], [KR2], [MM], [P1], [P2], [VG], [VSC].

6.3. *Riemannian manifolds of nonnegative Ricci curvature.* Let M be a complete, noncompact Riemannian n-manifold, $n \geqslant 2$, whose Ricci curvature is nonnegative. Then

M is a Loewner space if and only if M is n-regular; the latter is equivalent to the existence of a constant $C \geqslant 1$ such that

$$C^{-1}R^n \leqslant \mu(B_R), \tag{6.4}$$

where B_R is any metric ball of radius R and μ is the Riemannian volume. Recall that the classical comparison theorem of Bishop implies that we always have the inequality

$$\mu(B_R) \leqslant \Omega_n R^n \tag{6.5}$$

for balls B_R as above, where Ω_n is the volume of the unit ball in $\mathbf{R}^n$; see e.g. [Ch, p. 123]. Thus M is n-regular if it has *Euclidean volume growth*.

The claim follows from Theorem 5.7 and from the fact that M admits a Poincaré inequality; the necessity of (6.4) follows from Theorem 3.6. The validity of a $(1,2)$-Poincaré inequality under the above assumptions was proved by Buser [Bu]. There are several works by several people in this area; see [Ch], [MSC], [SC], and the references there. (See also [CSC].)

Recently, significant advances have been made in understanding the structure of a tangent cone M_∞ of a manifold M as above (in particular, assume that M has Euclidean volume growth). See [CC]. Because quasiconformal maps are insensitive to scaling, and because they easily form compact families, one is led to wonder whether quasiconformal maps can be used to resolve some issues in the study of the tangent cones. One can show without much difficulty that if M and N are two manifolds of nonnegative Ricci curvature and Euclidean volume growth, and if $f: M \to N$ is quasiconformal, then the tangent cones at infinity of M and N are pairwise (quasisymmetrically) homeomorphic in the following sense: for each tangent cone M_∞, there is a tangent cone N_∞ and a quasisymmetric homeomorphism $M_\infty \to N_\infty$. This discussion is very general and well known, and it has nothing to do with Ricci curvature, or even manifolds per se, cf. [GLP]. The perhaps new observation made here is that if f is a quasiconformal map (in the infinitesimal sense) between two manifolds M and N as above, then f is globally quasisymmetric, and hence can be scaled and pushed to a quasisymmetric homeomorphism between appropriate tangent cones.

6.6. *Strong A_∞-geometry.* The concept of a strong A_∞-geometry is due to David and Semmes [DS1], [S1], [S2]. Intuitively, such a geometry results when one changes the standard metric in $\mathbf{R}^n$ by a conformal factor. The main difference is that we do not assume this change to be smooth, or even continuous, but rather to hold on to the minimal requirements that prevent the associated distance function from degenerating. Assume that μ is a doubling measure in $\mathbf{R}^n$ given by a locally integrable density w. That is, $d\mu(x) = w(x)\,dx$, where dx is Lebesgue measure in $\mathbf{R}^n$. (Recall the definition for

doubling measure from §5.3.) We assume here $n \geqslant 2$. To each such density, or *weight*, we associate the distance function

$$d_w(x,y) = \mu(B_{xy})^{1/n} = \left(\int_{B_{xy}} w(x)\, dx \right)^{1/n}, \tag{6.7}$$

where B_{xy} is the smallest closed Euclidean ball containing x and y. The function d_w: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ does not generally define a metric in $\mathbf{R}^n$, nor is it even comparable to one. If there is a metric D_w in $\mathbf{R}^n$ and a constant $C \geqslant 1$ so that

$$C^{-1} d_w(x,y) \leqslant D_w(x,y) \leqslant C d_w(x,y) \tag{6.8}$$

for all x and y in $\mathbf{R}^n$, then w is called a *strong A_∞-weight*. The name derives from the fact that every strong A_∞-weight is an A_∞-weight in the sense of Muckenhoupt, while not every A_∞-weight is a strong one. See [DS1], [S1], [S2] for a thorough discussion of strong A_∞-weights.

Let w be a strong A_∞-weight and let D_w be a metric satisfying (6.8). Then the metric space $(\mathbf{R}^n, D_w)$ with measure $d\mu = w\, dx$ is n-regular and every two points in it can be joined by a rectifiable curve [S1].

THEOREM 6.9. *The n-regular metric space $X = (\mathbf{R}^n, D_w)$ associated to a strong A_∞-weight w as above is a Loewner space. The statement is quantitative in that the function ϕ only depends on n and on the constants associated with w.*

Because X admits the appropriate Poincaré inequalities and is quasiconvex (see [DS1], [S1], [S4], [S5]), Theorem 6.9 follows from Theorem 5.7.

The space $(\mathbf{R}^n, D_w)$ can be quite different from the standard $\mathbf{R}^n$. Semmes [S2] has given examples of strong A_∞-weights w in $\mathbf{R}^3$ such that the associated space $(\mathbf{R}^3, D_w)$ is not bi-Lipschitz equivalent to the standard $\mathbf{R}^3$. Moreover, there is, in some $\mathbf{R}^n$, a strong A_∞-weight w such that the associated space does not admit a bi-Lipschitz embedding into any Euclidean space. On the other hand, the identity map from $(\mathbf{R}^n, D_w)$ to the standard $\mathbf{R}^n$ is always quasisymmetric, and this implies, among other things, quantitative bounds for the contractibility function of the space (cf. [S3], [S4]).

Semmes [S2] has further shown that every doubling metric space X admits a bi-Lipschitz embedding into some $(\mathbf{R}^n, D_w)$, where w is a strong A_∞-weight. Semmes uses partly a theorem of Assouad [A]. From the point of view of quasiconformal mapping theory, a bi-Lipschitz change of coordinates is a harmless procedure. Therefore, in light of the Assouad–Semmes theorem, we could think of all the metric spaces in this paper as subspaces of $(\mathbf{R}^n, D_w)$ for some n and for some strong A_∞-weight w in $\mathbf{R}^n$.

6.10. *Uniformly contractible, regular topological manifolds.* In an interesting and massive recent paper [S4], Semmes has verified Poincaré-type inequalities for a large class of metric spaces.

Let X be a connected and complete n-regular metric space that is also an orientable topological n-manifold, $n \geqslant 2$. Assume that X satisfies the following *local linear contractibility condition*: there is $C \geqslant 1$ so that, for each $x \in X$ and $R < C^{-1} \operatorname{diam} X$, the ball $B(x, R)$ can be contracted to a point inside $B(x, CR)$.

THEOREM 6.11. *Under the above assumptions, X admits a $(1, p)$-Poincaré inequality for all $p \geqslant 1$. In particular, X is a Loewner space. These statements are quantitative in the usual sense.*

The first part of Theorem 6.11 is contained in [S4, Theorem B.10]. The second part follows from Theorem 5.7. (Note that under these assumptions, X is quasiconvex by [S4, Theorem B.6].)

Theorem 6.11 covers a wide variety of spaces, including the strong A_∞-geometries. Moreover, the assumption that X be a topological manifold can be relaxed: it suffices to assume that X is an orientable homology n-manifold, cf. [S4, A.35]. Every finite polyhedron (i.e. a finite simplicial complex) that is an orientable homology n-manifold satisfies the above assumptions (see [S3, Proposition 1.11]) and hence is a Loewner space. These and other examples are discussed in the many papers by Semmes [S2], [S3], [S4], [S5]. We reiterate that there are spaces, even finite polyhedra, that satisfy the assumptions of Theorem 6.11 and are homeomorphic to some Euclidean space, but do not admit even local bi-Lipschitz or quasisymmetric coordinates.

6.12. *Simplicial complexes.* It follows from Theorem 6.11 that every finite simplicial complex that is also a manifold is a Loewner space. In fact, a more general result is true.

THEOREM 6.13. *Suppose that X is a connected, finite simplicial complex of pure dimension $Q > 1$ such that the link of each vertex is connected. Then X admits a $(1, Q)$-Poincaré inequality. In particular, X is a Q-regular Loewner space.*

In the above theorem, we can use in X either the barycentric metric or the metric X inherits by sitting inside some $\mathbf{R}^N$. Simple examples show that as stated this theorem cannot be made quantitative. Similarly, the assumption on links is necessary.

Theorem 6.13 follows from a yet more general result which will be described in the next subsection.

6.14. *Glueing spaces together.* The Loewner condition, or more generally a weak $(1, p)$-Poincaré inequality, rather easily survives under unions.

Suppose that X and Y are two locally compact Q-regular metric measure spaces. Suppose also that A is a closed subset of X that has an isometric copy inside Y, i.e. there is an isometric embedding $i\colon A \to Y$. In the following, we understand this embedding as fixed and think of A as a closed subset of both X and Y. Then form a space

$$X \cup_A Y$$

which is the disjoint union of X and Y with points in the two copies of A identified. This space has a natural metric which extends the metrics from X and Y: the distance between $x \in X \cup_A Y$ and $y \in X \cup_A Y$ is

$$\inf_{a \in A} |x-a| + |a-y|.$$

Furthermore, the measures on X and Y add to a measure μ on $X \cup_A Y$ that is evidently Q-regular. (Note that if X and Y were only doubling spaces, the new space $X \cup_A Y$ is not necessarily doubling.)

THEOREM 6.15. *Let X, Y and A be as above. Suppose that there are numbers $Q \geqslant s > Q-p$ and $C \geqslant 1$ so that*

$$\mathcal{H}_s^\infty(A \cap B_R) \geqslant C^{-1} R^s \tag{6.16}$$

for all balls B_R either in X or in Y that are centered at A with radius $0 < R < \min\{\operatorname{diam} X, \operatorname{diam} Y\}$. If both X and Y admit a weak $(1,p)$-Poincaré inequality, then $X \cup_A Y$ admits a weak $(1,p)$-Poincaré inequality as well. The statement is quantitative in the usual sense.

Proof. The proof is based on Theorem 5.9 and Lemma 5.15. Fix a ball $B = B_{R_0}$ in $X \cup_A Y$, a continuous function u in B, and a very weak gradient ϱ of u in B. We may assume that B is centered at a point in A, as is easily seen. By Lemma 5.15 it suffices to show that

$$|u(x) - u(y)| \leqslant C |x-y| (M_{R_0} \varrho^p(x) + M_{R_0} \varrho^p(y))^{1/p} \tag{6.17}$$

for all $x, y \in C^{-1} B$ for some $C \geqslant 1$ depending only on the data.

To this end, observe that (6.17) has the correct homogeneity properties so that there is no loss of generality in assuming that

$$\max_{C^{-1}B} u = 1 \quad \text{and} \quad \min_{C^{-1}B} u = 0.$$

We assume here for simplicity that the above extreme values are taken at points x and y, so that $u(x) = 1$ and $u(y) = 0$. We may clearly assume that $x \in X$ and $y \in Y$, for if both

points belong to either space, then there is nothing to prove by assumption and by the necessity part of Lemma 5.15. Next, take a ball B' in $X \cup_A Y$ that is centered at A, has radius R comparable to $|x-y|$, and such that the points x and y are contained in $\frac{1}{4}B'$. Such a ball B' clearly exists and can be assumed to lie in B by simply choosing the constant C above sufficiently large.

Now if either $|u(x)-u_{B_X}|$ or $|u(y)-u_{B_Y}|$ exceeds $\frac{1}{5}$, where $B_X = B(x, |x-y|)$ is a ball in X and $B_Y = B(y, |x-y|)$ is a ball in Y, then clearly (6.17) holds by (the proof of) Lemma 5.15. Thus we assume that

$$|u_{B_Y}| \leqslant \tfrac{1}{5} \quad \text{and} \quad |1-u_{B_X}| \leqslant \tfrac{1}{5}.$$

It then follows that the sets

$$A_X = \left\{ w \in B_X : u(w) > \tfrac{3}{4} \right\} \quad \text{and} \quad A_Y = \left\{ w \in B_Y : u(w) < \tfrac{1}{4} \right\}$$

both have measure at least $\frac{1}{5}$ of the measure of B_X and B_Y, respectively. But the measures of these two balls are comparable to R^Q by the regularity assumption. Therefore we find that

$$\min\{\mathcal{H}_Q^\infty(A_X \cap B'), \mathcal{H}_Q^\infty(A_Y \cap B')\} \geqslant C^{-1}R^Q,$$

and hence that

$$\min\{\mathcal{H}_s^\infty(A_X \cap B'), \mathcal{H}_s^\infty(A_Y \cap B')\} \geqslant C^{-1}R^s. \tag{6.18}$$

(We leave this latter deduction to the reader to verify.) Note that $A_X \cup A_Y$ belongs to B' because x and y belong to $\frac{1}{4}B'$.

Next, consider the sets

$$A_1 = \left\{ w \in A \cap B' : u(w) < \tfrac{1}{2} \right\}, \quad A_2 = (A \cap B') \backslash A_1.$$

By assumption, one of these two sets has to have s-content at least $C^{-1}R^s$ both as a subset of X and as a subset of Y. Say A_1 does. (The opposite case is similar.) The function $v = 4\left(u - \tfrac{1}{2}\right)$ satisfies $v \geqslant 1$ on A_X and $v \leqslant 0$ on A_1, and it has 4ϱ as its very weak gradient in B. We combine this observation with (6.18) and deduce from Theorem 5.9 that

$$1 \leqslant CR^{p-Q} \int_B \varrho^p \, d\mu \leqslant CR^p M_{R_0} \varrho^p(x) \leqslant C|x-y|^p M_{R_0} \varrho^p(x),$$

as desired. Note that we may assume $R \leqslant R_0$. This completes the proof of Theorem 6.15.

Remark 6.19. (a) Theorem 6.15 can be used to produce a variety of examples of spaces that admit a Poincaré inequality. Some of these are quite amusing. Take, for instance, X to be $\mathbf{R}^4$ with its standard metric and Y to be the first Heisenberg group

$\mathbf{H}_1$ with its Carnot metric. Then both spaces are 4-regular. Glue them along a geodesic, i.e. along an isometric copy of $\mathbf{R}$. The resulting space $X \cup_{\mathbf{R}} Y$ admits a $(1,p)$-Poincaré inequality for all $p>3$. (It does not admit a weak $(1,3)$-Poincaré inequality, cf. (b) below.) It is a Loewner space, too. Notice that locally the topological dimension of $X \cup_{\mathbf{R}} Y$ is different at different places.

(b) One can use Theorem 6.15 to show that the spaces that admit a weak $(1,p)$-Poincaré inequality are strictly ordered by p. In fact, given $1 \leqslant q < p \leqslant n$, there is an n-regular Riemannian n-manifold M that admits a weak $(1,p)$-Poincaré inequality but does not admit a weak $(1,q)$-Poincaré inequality.

We sketch the proof for this claim. Let $1 \leqslant q < p \leqslant n$. Take a suitable closed Cantor-type set A in $\mathbf{R}^n$ such that A has zero Hausdorff $(n-q)$-measure but the estimate

$$\mathcal{H}_s^\infty (A \cap B_R) \geqslant C^{-1} R^s$$

holds for all balls B_R centered at A, for some $n-q>s>n-p$. Then glue two copies of $\mathbf{R}^n$ along A. The resulting space admits a $(1,p)$-Poincaré inequality by Theorem 6.15. However, it is well known that $\mathcal{H}_{n-q}^\infty (A)=0$ implies that A has zero q-capacity in $\mathbf{R}^n$ (see e.g. [HKM, Chapter 3]). In particular, there are continuous functions ϕ in the Sobolev class $W^{1,q}(\mathbf{R}^n)$ with arbitrarily small $W^{1,q}$-norm such that $\phi|A=1$. By extending such a function ϕ to be identically 1 in the second copy of $\mathbf{R}^n$, we see that no $(1,q)$-Poincaré inequality is possible in $\mathbf{R}^n \cup_A \mathbf{R}^n$.

We leave it to the reader to modify this example so that X can be taken to be a smooth manifold.

One can furthermore show that there are spaces that satisfy, for a given $p>1$, a $(1,p)$-Poincaré inequality, but do not satisfy a $(1,q)$-Poincaré inequality for any $q<p$ [K].

(c) In §8, we show that under certain additional assumptions on a Q-regular space, the validity of a $(1,Q)$-Poincaré inequality is a quasisymmetric invariant. The example in (b) shows that the same statement is not true for $p<n$. Simply take a quasiconformal map of $\mathbf{R}^n$ that transforms one self-similar Cantor set of dimension s to another self-similar Cantor set of dimension q, where $1 \leqslant q < s < n$.

7. Absolute continuity of quasisymmetric maps

In this section, we show that a quasisymmetric map $f: X \to Y$ between two locally compact Q-regular metric spaces induces a measure in X that is A_∞-related to the Hausdorff Q-measure in X, provided that $Q>1$ and that X admits a $(1,p)$-Poincaré inequality for some $p<Q$. This generalizes Gehring's well-known result [G3]. For earlier extensions of Gehring's theorem, see [DS1], [S1] and [KR2].

Notice that by the results of the previous sections, with some (mild) additional assumptions added on X and Y, we could equivalently assume that f is only quasiconformal in the sense of Definition 1.2. We prefer to assume quasisymmetry at the outset here.

7.1. A_∞-*weights.* Let (X, μ) be a metric measure space as in §2.1 and assume that μ is a doubling measure (Definition 5.3). Assume also that X is *locally compact.* Let σ be another doubling Borel measure in X. Then σ is said to be A_∞-*related* to μ if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\mu(E) < \delta \mu(B) \quad \text{implies} \quad \sigma(E) < \varepsilon \sigma(B)$$

whenever E is a measurable subset of a ball B. Clearly σ is absolutely continuous with respect to μ if σ is A_∞-related to μ, so that $d\sigma = w \, d\mu$ for some nonnegative locally μ-integrable weight function w. It turns out that if σ is A_∞-related to μ, then μ is A_∞-related to σ, and that this symmetric relationship between two doubling measures can be expressed in various equivalent ways [ST], [St2]. Consider the following *reverse Hölder inequality*:

There is a locally μ-integrable function w in X together with constants $C \geqslant 1$ and $p > 1$ so that $d\sigma = w \, d\mu$ and

$$\left(\fint_B w^p \, d\mu \right)^{1/p} \leqslant C \fint_B w \, d\mu \tag{7.2}$$

whenever B is a ball in X.

It is well known that a doubling measure σ is A_∞-related to μ if and only if (7.2) is satisfied. This is a quantitative statement. See e.g. [CF] or [ST, Chapter I].

Condition (7.2) has the following important self-improving character.

GEHRING'S LEMMA 7.3. *If a weight w satisfies (7.2), then there is $\varepsilon > 0$ such that*

$$\left(\fint_B w^{p+\varepsilon} \, d\mu \right)^{1/(p+\varepsilon)} \leqslant C \fint_B w \, d\mu \tag{7.4}$$

whenever B is a ball in X. The constants ε and C depend only on the constants appearing in (7.2) and on the doubling constant of μ.

The original proof due to Gehring [G3] in $\mathbf{R}^n$ can be extended to this more general setting; see e.g. [ST, p. 6].

7.5. *Volume derivative of a quasisymmetric map.* Let X and Y be two Q-regular metric measure spaces as in §2.1. Denote the Hausdorff Q-measure in both spaces by $\mathcal{H}_Q$, and to keep the notation simple, we write

$$|A| = \mathcal{H}_Q(A), \quad dx = d\mathcal{H}_Q(x).$$

Suppose now that $f: X \to Y$ is an η-quasisymmetric homeomorphism between X and Y as defined in (4.5). By the Lebesgue–Radon–Nikodym theorem, the *volume derivative*

$$\mu_f(x) = \lim_{r \to 0} \frac{|f(B(x,r))|}{|B(x,r)|} \tag{7.6}$$

exists and is finite for almost every x in X; it is locally integrable and satisfies

$$\int_E \mu_f(x)\,dx \leqslant |f(E)| \tag{7.7}$$

for every measurable subset E of X, with equality if and only if f is absolutely continuous.

The paramount reference to the basic measure theory is [Fe].

7.8. *Maximum derivative of a quasisymmetric map.* Next we introduce another function, which describes the local stretching of f. The situation is assumed to be as in §7.5. For $x \in X$ write

$$L_f(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{r}, \tag{7.9}$$

where $L_f(x,r)$ is defined in (4.2). It is not difficult to see that L_f is a Borel-measurable function in X. Moreover, by the Q-regularity of X and Y, and by the quasisymmetry of f, we find that

$$\left(\frac{L_f(x,r)}{r}\right)^Q \leqslant C \frac{|f(B(x,r))|}{|B(x,r)|}$$

for $0 < r < \operatorname{diam} X$. Therefore

$$L_f(x)^Q \leqslant C \mu_f(x) \tag{7.10}$$

for almost every x in X, where C depends only on the quasisymmetry function η of f and on the constants associated with the Q-regularity of X and Y. With an obvious abuse of language, we call the function L_f the *maximum derivative* of a quasisymmetric map f.

THEOREM 7.11. *Suppose that X and Y are locally compact Q-regular spaces for some $Q > 1$ and that X admits a weak $(1,p)$-Poincaré inequality for some $p < Q$. Let f be a quasisymmetric map from X onto Y. Then the pull-back measure σ_f,*

$$\sigma_f(E) = |f(E)|, \quad E \subset X,$$

is A_∞-related to the Hausdorff Q-measure $\mathcal{H}_Q$ in X. Moreover,

$$d\sigma_f = \mu_f\,dx$$

with $\mu_f(x) > 0$ for $\mathcal{H}_Q$-almost every x in X, and there is $\varepsilon > 0$ such that

$$\left(\fint_B \mu_f^{1+\varepsilon} \, dx \right)^{1/(1+\varepsilon)} \leqslant C \fint_B \mu_f \, dx \qquad (7.12)$$

for all balls B in X. The statement is quantitative in that all the constants involved in the conclusion depend only on the quasisymmetry constant of f, on the constants associated with the Q-regularity of X and Y, and on the constant appearing in the Poincaré inequality.

COROLLARY 7.13. *Under the assumptions of Theorem 7.11,*

$$|E| = 0 \quad \text{if and only if} \quad |f(E)| = 0$$

for $E \subset X$. In other words, both f and its inverse are absolutely continuous.

Because, in the situation of Theorem 7.11, L_f^Q and the volume derivative are comparable for quasisymmetric maps, we also have the following reverse Hölder inequality for the maximum derivative as well.

COROLLARY 7.14. *Under the assumptions of Theorem 7.11, there is $q > Q$ such that*

$$\left(\fint_B L_f^q \, dx \right)^{1/q} \leqslant C \left(\fint_B L_f^Q \, dx \right)^{1/Q} < \infty$$

for all balls B in X. The statement is quantitative similarly to Theorem 7.11.

We do not know whether Theorem 7.11 remains true if X satisfies a weak $(1, Q)$-Poincaré inequality only, cf. Remark 8.7.

To prove Theorem 7.11, we shall show that the reverse Hölder inequality (7.12) holds for the volume derivative. This suffices in view of the discussion in §7.1. In the following, let $f: X \to Y$ be an η-quasisymmetric map between two Q-regular metric spaces with $Q > 1$. For $\varepsilon > 0$ define

$$L_f^\varepsilon(x) = \sup_{0 < r \leqslant \varepsilon} \frac{L_f(x, r)}{r}. \qquad (7.15)$$

Clearly $L_f^\varepsilon(x)$ decreases as ε decreases, and

$$\lim_{\varepsilon \to 0} L_f^\varepsilon(x) = L_f(x), \quad \varepsilon \to 0,$$

for each $x \in X$ by the definition of L_f. Fix a ball $B = B(x_0, R)$ in X with $R < \operatorname{diam} X$.

LEMMA 7.16. *There is a constant $C=C(\eta)\geqslant1$ such that, for each $\varepsilon>0$, the function CL_f^ε is a very weak gradient of the function $u(x)=|f(x)-f(x_0)|$ in B.*

Proof. Fix $\varepsilon>0$ and let γ be a rectifiable curve joining two points x and y in B. Suppose first that $d=\operatorname{diam}\gamma\leqslant\varepsilon$. Then for each $z\in\gamma$ we have

$$L_f^\varepsilon(z)\geqslant\frac{L_f(z,d)}{d}\geqslant C^{-1}\frac{L_f(x,d)}{d}$$

by quasisymmetry. Thus

$$\int_\gamma L_f^\varepsilon\,ds\geqslant C^{-1}\frac{L_f(x,d)}{d}l(\gamma)\geqslant C^{-1}L_f(x,d)\geqslant C^{-1}|f(x)-f(y)|$$
$$\geqslant C^{-1}|\,|f(x)-f(x_0)|-|f(y)-f(x_0)|\,|=C^{-1}|u(x)-u(y)|.$$

If $d=\operatorname{diam}\gamma>\varepsilon$, then pick successive points $x_0,...,x_N$ from γ such that $x_0=x$, $x_N=y$, and such that the diameter of γ_i, the portion of γ between x_{i-1} and x_i, is less than ε for $i=1,...,N$. As above,

$$\int_\gamma L_f^\varepsilon\,ds=\sum_{i=1}^N\int_{\gamma_i}L_f^\varepsilon\,ds\geqslant C^{-1}\sum_{i=1}^N|f(x_i)-f(x_{i-1})|$$
$$\geqslant C^{-1}|f(x)-f(y)|\geqslant C^{-1}|u(x)-u(y)|.$$

The lemma follows.

Remark 7.17. Note that the above proof gives

$$|f(x)-f(y)|\leqslant C(\eta)\int_\gamma L_f^\varepsilon\,ds \tag{7.18}$$

for all points x and y in X and all rectifiable curves γ joining these points. It is important to notice that (7.18) is not necessarily true when L_f^ε is replaced with L_f, for f need not be absolutely continuous on rectifiable curves. In other words, it may well happen that L_f^ε is not integrable on such a curve. The next lemma tells us however that L_f^ε is integrable against the volume measure.

LEMMA 7.19. *The function L_f^ε belongs to the space weak-$L^Q(B)$ with norm independent of ε, provided ε is small enough. More precisely, for $\varepsilon<\frac{1}{10}R$ and $t>0$ we have that*

$$|\{x\in B:L_f^\varepsilon(x)>t\}|\leqslant Ct^{-Q}|f(B)|, \tag{7.20}$$

where $C \geqslant 1$ depends only on η and the data of X and Y. A fortiori, the function L_f belongs to weak-$L^Q(B)$ with norm depending only on the data.

Proof. Denote by E_t the set of points x in B where $L_f^\varepsilon(x) > t$. Then by Covering Lemma 5.5, we can find a countable collection of disjoint balls $B_i = B(x_i, r_i)$ such that $0 < r_i \leqslant \varepsilon$,

$$\frac{L_f(x_i, r_i)}{r_i} > t$$

and

$$E_t \subset \bigcup 5B_i \subset 2B.$$

Thus, by quasisymmetry and Q-regularity,

$$
\begin{aligned}
|E_t| &\leqslant C \sum r_i^Q \leqslant C t^{-Q} \sum L_f(x_i, r_i)^Q \\
&\leqslant C t^{-Q} \sum |f(B_i)| \leqslant C t^{-Q} |f(2B)| \leqslant C t^{-Q} |f(B)|,
\end{aligned}
$$

as desired. Finally, because $L_f \leqslant L_f^\varepsilon$, the lemma follows.

COROLLARY 7.21. *For $1 \leqslant s < Q$ and $0 < \varepsilon < \frac{1}{10} R$, the function L_f^ε is in $L^s(B)$ with*

$$\|L_f^\varepsilon\|_s \leqslant C |B|^{(Q-s)/Qs} |f(B)|^{1/Q}, \tag{7.22}$$

where $C \geqslant 1$ depends only on s, η and the data of X and Y. A similar statement is true for L_f.

Proof of Theorem 7.11. We shall show that L_f satisfies the reverse Hölder inequality

$$\left(\fint_B L_f^Q \, dx \right)^{1/Q} \leqslant C \left(\fint_B L_f^p \, dx \right)^{1/p}, \tag{7.23}$$

where $p < Q$ is as in the assumptions. The claim follows from this and Corollary 7.21 via Gehring's Lemma 7.3, because the Qth power of the maximum derivative is comparable to the volume derivative by quasisymmetry (see (7.10)). Remember that we have fixed a ball $B = B(x_0, R) \subset X$ with $R < \operatorname{diam} X$. The constant $C \geqslant 1$ in (7.23) does not depend on B.

Because X is assumed to admit a weak $(1, p)$-Poincaré inequality and because $C L_f^\varepsilon$ is a very weak gradient of the function $u(x) = |f(x) - f(x_0)|$ in B by Lemma 7.16, we have that

$$\fint_{B'} |u(x) - u_{B'}| \, dx \leqslant C \operatorname{diam} B \left(\fint_B L_f^{\varepsilon p} \, dx \right)^{1/p},$$

where $B' = C_0^{-1} B$, cf. (5.2). Therefore, in fact,

$$\fint_{B'} |u(x) - u_{B'}| \, dx \leqslant C \operatorname{diam} B \left(\fint_B L_f^p \, dx \right)^{1/p}, \tag{7.24}$$

by (7.22) and the Lebesgue Convergence Theorem. On the other hand,

$$u_{B'} = \fint_{B'} |f(x) - f(x_0)|\, dx \geqslant \frac{1}{|B|} \int_{B' \setminus \frac{1}{2}B'} |f(x) - f(x_0)|\, dx$$

$$\geqslant C^{-1} L_f(x_0, R) \frac{|B' \setminus \frac{1}{2}B'|}{|B|} \geqslant C_1^{-1} L_f(x_0, R)$$

because

$$L_f(x_0, R) \leqslant C|f(x) - f(x_0)|$$

for $x \in B' \setminus \frac{1}{2}B'$ by quasisymmetry. For sufficiently small $\delta > 0$, we similarly have that

$$u(x) = |f(x) - f(x_0)| \leqslant \eta(\delta) L_f(x_0, R) \leqslant (2C_1)^{-1} L_f(x_0, R)$$

for $x \in \delta B'$, where $C_1 > 0$ is as above, and so

$$|u(x) - u_{B'}| \geqslant (2C_1)^{-1} L_f(x_0, R)$$

for x in $\delta B'$. Consequently,

$$\int_{B'} |u(x) - u_{B'}|\, dx \geqslant \int_{\delta B'} |u(x) - u_{B'}|\, dx \geqslant C^{-1} L_f(x_0, R)|B|,$$

where $C \geqslant 1$ depends only on η and the data associated with X. Combining this with (7.24) gives

$$\frac{L_f(x_0, R)}{R} \leqslant C\left(\fint_B L_f^p\, dx\right)^{1/p}. \tag{7.25}$$

Finally, we invoke the Lebesgue inequality (7.7) which together with (7.10) and (7.25) implies

$$\left(\fint_B L_f^Q\, dx\right)^{1/Q} \leqslant C\left(\fint_B \mu_f\, dx\right)^{1/Q} \leqslant C\left(\frac{|f(B)|}{|B|}\right)^{1/Q}$$

$$\leqslant C\frac{L_f(x_0, R)}{R} \leqslant C\left(\fint_B L_f^p\, dx\right)^{1/p}. \tag{7.26}$$

This is (7.23), and the proof of Theorem 7.11 is complete as soon as we can show that there is an equality in (7.7); that is, we need to show that f is absolutely continuous. Notice that the A_∞-theory does guarantee that μ_f is positive almost everywhere, cf. [CF].

The absolute continuity of f follows basically from the second line in formula (7.26); it implies by quasisymmetry and by Hölder's inequality that

$$(\operatorname{diam} f(B))^Q \leqslant C\int_B L_f^Q\, dx \tag{7.27}$$

for all balls B in X. Because L_f is locally in L^Q, (7.27) implies the claim by standard arguments. Indeed, for bounded open sets U, a covering argument and quasisymmetry together with (7.27) give

$$|f(U)| \leqslant C \int_U L_f^Q \, dx,$$

from which the claim follows as every set of measure zero in X is contained in an open set of arbitrarily small measure [F, 2.2.2].

This completes the proof of Theorem 7.11.

Remark 7.28. Stephen Semmes pointed out to us that Theorem 7.11 admits a generalization as follows. Suppose that (X, μ) is a locally compact Q-regular space for some $Q > 1$ and suppose that X admits a weak $(1, p)$-Poincaré inequality for some $p < Q$. Suppose next that σ is a metric doubling measure in X; this means that σ is doubling and that there is a metric D_σ in X such that $D_\sigma(x, y)$ is comparable to $\sigma(B(x, |x-y|))^{1/Q}$ (equivalently, to $\sigma(B(y, |x-y|))^{1/Q}$) with constants independent of x and y, cf. the discussion in §6.6.

The conclusion then is that there is a locally μ-integrable density w in X such that $d\sigma = w \, d\mu$ and that w satisfies the reverse Hölder inequality (7.2). Moreover, the space (X, σ, D_σ) is a Q-regular space admitting a weak $(1, q)$-Poincaré inequality for some $p \leqslant q < Q$.

This assertion for $X = \mathbf{R}^n$ was proved in [DS1]; see also [S1], [S5]. Note that it extends Gehring's theorem to the case where no maps are present; Theorem 7.11 follows by substituting $\sigma_f = \sigma$ above. The main point is to change the function $u(x) = |f(x) - f(x_0)|$ to $D_\sigma(x, x_0)$ in the proof of Theorem 7.11. There is also the issue of "smoothing", cf. (7.15); the arguments in [S1] are helpful here.

It is interesting to note that when $X = \mathbf{R}^n$, then one can have $p = q = 1$ in the above conclusion, so there is no loss in the exponent after the deformation of metric [DS1]. In general, one has to allow for values $q > p$, as follows from Remark 6.19 (c). It would be interesting to find general conditions under which $q = p$ is an admissible choice.

8. Quasisymmetric invariance of Loewner spaces

We do not know whether the Loewner condition is a quasisymmetric invariant of an Ahlfors–David regular metric space. Basically, the lack of Fuglede's theorem prevents us from drawing this conclusion. Recall that in $\mathbf{R}^n$, Fuglede's theorem says that $W^{1,p}$-Sobolev maps are absolutely continuous on p-modulus a.e. curve [Fu]. Pansu [P1], [P2] proved that quasisymmetric maps enjoy the same property in spaces that can be foliated by curves in a suitable way. See also [MM].

We shall show next that Fuglede's theorem holds for quasisymmetric maps between Q-regular spaces where an appropriate Poincaré inequality holds. The quasisymmetric invariance of the Loewner condition then follows under these assumptions. The situation here is similar to that in the previous section.

As in §7, we use the notation $|E|=\mathcal{H}_Q(E)$ and $dx=d\mathcal{H}_Q$.

THEOREM 8.1. *Suppose that f is a quasisymmetric map between two Q-regular, locally compact metric spaces X and Y, where $Q>1$. Suppose furthermore that X admits a $(1,p)$-Poincaré inequality for some $p<Q$. Then f is absolutely continuous on Q-almost every curve in X.*

The conclusion of Theorem 8.1 means that the curve family

$$\Gamma_0 = \{\gamma\colon I \to X : f \circ \gamma\colon I \to Y \text{ is not absolutely continuous}\} \tag{8.2}$$

has zero Q-modulus in X. Recall that a rectifiable curve $\gamma\colon I \to X$ is absolutely continuous if $\mathcal{H}_1(E)=0$ implies $\mathcal{H}_1(\gamma(E))=0$ for $E \subset I$; this is equivalent to saying that $s'_\gamma>0$ a.e. on I, where $s_\gamma\colon I \to [0, l(\gamma)]$ is the reparametrization of γ as explained in §2.2.

Proof of Theorem 8.1. The issue is obviously local, so fix a small ball B in X. It follows from Theorem 7.11 that the maximum derivative L_f of f is in $L^q(B)$ for some $q>Q$. We claim that for $\varepsilon<\frac{1}{20}\operatorname{diam} B$ the function L_f^ε is in $L^s(B)$ for all $s<q$, where L_f^ε is defined in (7.15). In fact, the covering argument in Lemma 7.19 together with Corollary 7.14 gives that

$$|\{x \in B : L_f^\varepsilon(x) > t\}| \leqslant C \sum r_i^Q$$
$$\leqslant Ct^{-q} \sum r_i^Q \frac{L(x_i, r_i)^q}{r_i^q} \leqslant Ct^{-q} \sum r_i^Q \left(\frac{|f(B_i)|}{|B_i|}\right)^{q/Q}$$
$$= Ct^{-q} \sum r_i^Q \left(\fint_{B_i} \mu_f\, dx\right)^{q/Q} \leqslant Ct^{-q} \sum r_i^Q \fint_{B_i} L_f^q\, dx$$
$$\leqslant Ct^{-q} \int_B L_f^q\, dx.$$

It follows that L_f^ε is in weak-$L^q(B)$, and hence in $L^s(B)$ for all $s<q$.

Fix ε as above, and pick a curve γ from Γ_0, where Γ_0 is defined in (8.2). We can assume that all curves from Γ_0 lie in B and are locally rectifiable. Then

$$\int_\gamma L_f^\varepsilon\, ds = \infty, \tag{8.3}$$

for otherwise

$$|f(x)-f(y)| \leqslant C \int_{\gamma_{xy}} L_f^\varepsilon\, ds < \infty \tag{8.4}$$

for any subcurve γ_{xy} of γ joining two points x and y (cf. Remark 7.17). In particular, (8.4) implies that f is absolutely continuous on γ, which is a contradiction. Thus (8.3) holds. It follows that λL_f^ε is an admissible function for Γ_0 for any $\lambda > 0$. Because L_f^ε is in $L^Q(B)$, we conclude that $\mathrm{mod}_Q \Gamma_0 = 0$, and the proof of Theorem 8.1 is complete.

THEOREM 8.5. *Suppose that f is a quasisymmetric map between two Q-regular, locally compact metric spaces X and Y, where $Q > 1$. Suppose furthermore that X is a Loewner space that admits a $(1,p)$-Poincaré inequality for some $p < Q$. Then Y is a Loewner space. In particular, Y satisfies a weak $(1,Q)$-Poincaré inequality.*

Proof. By quasisymmetry, it suffices to show that

$$\mathrm{mod}_Q(E,F;X) \leqslant C \, \mathrm{mod}_Q(f(E),f(F);Y) \tag{8.6}$$

whenever E and F are two disjoint continua in X. Indeed, it then follows from (3.2) that Y is a Loewner space, and the second assertion comes from Theorem 5.12.

To this end, let ϱ' be an admissible function for $(f(E),f(F);Y)$. Let γ be a rectifiable path joining E and F in X, parametrized by the arc length. We may assume that f is absolutely continuous on γ by Theorem 8.1. Write

$$\varrho(x) = \varrho' \circ f(x) L_f(x).$$

Standard arguments using the absolute continuity then give that

$$\int_\gamma \varrho \, ds = \int_0^{l(\gamma)} \varrho' \circ f(\gamma(s)) L_f(\gamma(s)) \, ds \geqslant \int_{f \circ \gamma} \varrho' \, ds \geqslant 1.$$

Thus ϱ is admissible for $(E,F;X)$, and hence

$$\mathrm{mod}_Q(E,F;X) \leqslant \int_X \varrho(x)^Q \, dx \leqslant C \int_X \varrho' \circ f(x)^Q \mu_f(x) \, dx = C \int_Y \varrho'(y)^Q \, dy.$$

Here the middle inequality follows because $L_f(x)^Q \leqslant C\mu_f(x)$ almost everywhere, and the last, by standard arguments, because f is absolutely continuous (cf. (7.7)). Because ϱ' was arbitrary, and because $C \geqslant 1$ only depends on the data, the theorem follows.

Remark 8.7. (a) We conjecture that the Loewner condition is preserved under quasisymmetric maps between locally compact Q-regular spaces, $Q > 1$. It is possible that the validity of a weak $(1,Q)$-Poincaré inequality is similarly preserved, cf. Remark 6.19 (c). (Added in July 1997: Tyson [Ty] has recently verified this conjecture. It also follows from his work and from Theorems 5.7 and 5.12 that a weak $(1,Q)$-Poincaré inequality is preserved if the spaces in question are proper and φ-convex.)

(b) Theorem 8.5 can be strengthened. MacManus and the second author [KM] have recently established that, under the assumptions of Theorem 8.5, the space Y admits a $(1, q)$-Poincaré inequality for some $p \leqslant q < Q$. This also follows from the remark made by Semmes in Remark 7.28. Moreover, Koskela and MacManus have shown that $(1, p)$-Poincaré inequalities for $p > Q$ are not quasisymmetric invariants in the following sense: given $p > 2$ there is a compact geodesic 2-regular space X that admits a $(1, p)$-Poincaré inequality and there is a quasisymmetric map f from X onto another 2-regular space Y that does not admit a $(1, q)$-Poincaré inequality for any $q > 2$; moreover, the pullback measure under f is A_∞-related to the Hausdorff 2-measure on X.

9. Quasiconformal maps and Sobolev spaces

The Sobolev space theory occupies a central role in the quasiconformal analysis in $\mathbf{R}^n$, and it is natural to ask how the maps considered in this paper fit in to more general Sobolev space theories. In this last section, we consider two generalizations of classical Sobolev spaces. One is due to Hajłasz [Ha], and the other to Korevaar and Schoen [KS]. The results here are basically corollaries of the results in the previous sections.

9.1. *Sobolev spaces of Hajłasz.* In [Ha] Hajłasz defined, for any metric measure space (X, μ) of finite diameter, the Sobolev space $W^{1,p}(X)$ for $1 < p < \infty$ to be the set of all real-valued measurable functions u in $L^p(X)$ such that there is an $L^p(X)$-function g with the property that

$$|u(x) - u(y)| \leqslant |x - y| (g(x) + g(y)) \tag{9.2}$$

whenever x and y lie outside some set of measure zero. If X is a smoothly bounded domain in some Euclidean space, then $W^{1,p}(X)$ as defined above identifies with the standard Sobolev space; a natural choice for g in this case is an appropriate maximal function of $|\nabla u|$. See [Ha] for a more complete discussion.

THEOREM 9.3. *Suppose that X and Y are locally compact, Q-regular metric spaces for some $Q > 1$, and suppose that X admits a $(1, p)$-Poincaré inequality for some $p < Q$. If f is a quasisymmetric map from X onto Y, then there is $q > Q$ such that the function $u(x) = |f(x) - f(x_0)|$ belongs to the Sobolev space $W^{1,q}(B)$ of Hajłasz whenever $B = B(x_0, R)$ is a ball in X.*

The statement is quantitative in that q depends only on the quasisymmetry constant of f and the data associated with X.

Remark 9.4. Under mild additional assumptions on X and Y, we could have stated Theorem 9.3 for quasiconformal maps in the sense of (1.3), cf. §4.

Proof. By the proof of Theorem 7.11, we have the following weak Poincaré inequality

$$\fint_B |u(x)-u_B|\,d\mu \leqslant C\operatorname{diam}B\left(\fint_{CB}L_f^p\,d\mu\right)^{1/p},\tag{9.5}$$

where L_f is the maximum derivative of f as defined in (7.9), and where $p<Q$ is as in the assumptions. Although L_f need not be a very weak gradient of u, we still obtain from (9.5), by using a chaining argument as in the proof of Lemma 5.15, that the estimate

$$|u(x)-u(y)| \leqslant C|x-y|(M_{CR}L_f^p(x)+M_{CR}L_f^p(y))^{1/p}$$

holds for all pair of points x,y in $C^{-1}B$. Here M_{CR} is the restricted maximal function defined in (5.14). Because M_{CR} maps $L^s(B)$ to $L^s(B)$ for $s>1$ (see [St, Chapter 1]), and because L_f belongs to $L^q(B)$ for some $q>Q$ by Theorem 7.11, we conclude that u lies in the Sobolev space $W^{1,q}(C^{-1}B)$. Because B was an arbitrary ball in X, the factor C^{-1} can be ignored. Theorem 9.3 follows.

9.6. Sobolev spaces of Korevaar and Schoen. In [KS] Korevaar and Schoen considered Sobolev spaces $W^{1,p}(\Omega;X)$ of maps u from a Riemannian domain Ω into a complete metric space X.

THEOREM 9.7. *Suppose that Ω is a connected, open subset of a Riemannian n-manifold M such that its metric completion $\bar{\Omega}$ is a compact subset of M. Suppose that $n>1$ and that f is a quasiconformal map of Ω onto a linearly locally connected (Definition 3.12) n-regular metric space X. Then there is $q>n$ such that $f\in W^{1,q}(\Omega;X)$, where $W^{1,q}(\Omega;X)$ is the Sobolev space of Korevaar and Schoen.*

The statement is quantitative in that q depends only on H in Definition 1.2 and on the data associated with X.

Let us quickly recall the definition for $W^{1,p}(\Omega;X)$ as given in [KS]. For a map u from Ω into X, for points x and y in Ω, and for $\varepsilon>0$, write

$$e_\varepsilon(x,y;u)=\frac{|u(x)-u(y)|}{\varepsilon}.$$

Then for $1\leqslant p<\infty$ the (nonnormalized) averaged ε-approximate density function is

$$e_\varepsilon^p(x;u)=\fint_{B(x,\varepsilon)}e_\varepsilon^p(x,y)\,dy,$$

where dy denotes the Riemannian measure on Ω. We also require that $\operatorname{dist}(x,\partial\Omega)>\varepsilon$. Next, for each compactly supported function φ on Ω, write

$$E_\varepsilon^p(\varphi;u)=\int_\Omega \varphi(x)\,e_\varepsilon^p(x;u)\,dx$$

whenever ε is positive and small enough. The map u is said to be in the Sobolev space $W^{1,p}(\Omega; X)$ if there is a constant $E(u) < \infty$ so that

$$\limsup_{\varepsilon \to 0} E^p_\varepsilon(\varphi; u) \leqslant \max |\varphi| E(u) \tag{9.8}$$

for all compactly supported continuous real-valued functions φ in Ω.

Proof of Theorem 9.7. Because $\bar{\Omega}$ is compact in M, we may assume that Ω is small enough so that a $(1,1)$-Poincaré inequality holds there. It then follows, because X is assumed to be linearly locally connected, that f is quasisymmetric in Ω (§§ 4 and 5). Let $q > n$ be an exponent such that the volume derivative μ_f, defined in (7.6), belongs to $L^{q/n}(\Omega)$ (see Theorem 7.11). We shall show that this choice of q will do here.

To this end, we estimate the average $e_\varepsilon(x; f)^q$ by using the quasisymmetry of f as follows:

$$e^q_\varepsilon(x) = \fint_{B(x,\varepsilon)} \frac{|f(x) - f(y)|^q}{\varepsilon^q} \, dy \leqslant \varepsilon^{-q} L_f(x, \varepsilon)^q$$

$$\leqslant C\varepsilon^{-q} \left(\int_{B(x,\varepsilon)} \mu_f \, dy \right)^{q/n} \leqslant C \left(\fint_{B(x,\varepsilon)} \mu_f \, dy \right)^{q/n} \leqslant CM\mu_f(x)^{q/n},$$

where $M\mu_f$ is the Hardy–Littlewood maximal function of the volume derivative μ_f. (Recall the definition for $L_f(x, \varepsilon)$ from (4.2).) Because the volume derivative belongs to $L^{q/n}(\Omega)$, where $q/n > 1$, the maximal function belongs to $L^{q/n}(\Omega)$ as well (note that $\bar{\Omega}$ is a compact subset of a Riemannian manifold so that the maximal function operator maps L^p to L^p for $p > 1$). In conclusion,

$$\int_\Omega \varphi(x) e^q_\varepsilon(x; f) \, dx \leqslant C \int_\Omega \varphi(x) M\mu_f(x)^{q/n} \, dx$$

for any compactly supported continuous function φ in Ω. Because the right-hand side is dominated by

$$C \max |\varphi| \int_\Omega M\mu_f(x)^{q/n} \, dx,$$

we arrive at the desired conclusion (9.8). The theorem follows.

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