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Fixed points and circle maps

by

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Introduction

One of the main purposes of this article is to make rigorous the old heuristic relation between holomorphic dynamics near an indifferent fixed point and dynamics of analytic circle homeomorphisms. This folkloric relation has long been known to specialists of the subject. For instance, V. I. Arnold in the introduction to his celebrated paper "Small denominators I" [Ar1] writes:

> *"... The problem of the center is a singular case of the problem of a map of the circumference into itself whose radius, in the singular case, is equal to zero ..."*

In other words, an analytic circle diffeomorphism of an imbedded circle in $\mathbf{C}$ looks like an indifferent fixed point from far away (it is easier to "see" this for myopic people).

It is also well known that this heuristic relation is very fruitful. Almost all the techniques that have proved useful in one of the problems have also been successfully applied to the other. This includes KAM techniques (see [Bo]) and more recently the geometric techniques initiated by J.-Ch. Yoccoz ([Yo3], [Yo4]). In both cases the proofs of the results deal with difficulties associated with "small divisors" and are long and technical. The geometric construction we present can be applied to avoid repetition of proofs. The cornerstone of this geometric construction is the following theorem.

THEOREM 1 (semi-local invariant compacta or Siegel compacta). *Let $f(z)=\lambda z+\mathcal{O}(z^2)$, $|\lambda|=1$, be a local holomorphic diffeomorphism. Let U be a Jordan neighborhood of the indifferent fixed point 0. Assume that f as well as f^{-1} are defined and univalent on a neighborhood of the closure of U. Then there exists a set K such that* (*Figure* 1):

(i) *K is compact, connected and full* (*i.e.* *$\mathbf{C}\setminus K$ is connected*),

(ii) $0\in K\subset \overline{U}$,

(iii) $K\cap\partial U\neq\varnothing$,

(iv) $f(K)=K$, $f^{-1}(K)=K$.

Moreover, if f is not of finite order, f is linearizable at 0 if and only if $0\in\mathring{K}$.

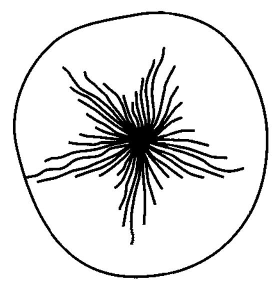

Fig. 1

We will call the compact sets K obtained from this theorem *Siegel compacta*, since they can be viewed as degenerate Siegel domains. Recent work shows how fruitful this point of view is ([Pe6]). When the fixed point is indifferent irrational and K is not contained in the closure of a linearization domain then K is called a *hedgehog*. This theorem can be compared with G. D. Birkhoff's theorem for surface transformations f having a Lyapunov unstable fixed point for f and f^{-1}. Birkhoff shows ([Bir]) the existence of such a compact set K_+ (or K_-) which is positive (or negative) invariant by f, i.e. property (iv) is replaced by $f(K_+)\subset K_+$ (or $f^{-1}(K_-)\subset K_-$). In this general setting there is no totally invariant compact set, as the case of a hyperbolic fixed point shows.

Our theorem asserts that Birkhoff's theorem can be improved in the holomorphic situation to obtain a totally invariant compactum. The proof (§III.2) does not involve any technique unknown to Fatou or Julia. It is quite surprising that it has been ignored for so long. There is a short proof of the theorem using small-divisor results (§§II.3 and III.2), but it is definitely not a small-divisor theorem. We give another elementary proof (§III.3) which consists of the semi-local extension of the classical study of Leau and Fatou of parabolic fixed points. These are the only proofs known to the author of the existence of Siegel compacta. The Siegel compacta thus obtained are related to Siegel disks, similarly as Aubry–Mather sets are to KAM invariant curves for a twist map of the annulus.

This theorem answers a question in the book *Celestial Mechanics* by C.-L. Siegel and J. Moser ([SM, Chapter III, p. 187]). In the terminology of these authors, the existence of the totally invariant compacta of Theorem 1 shows that an indifferent fixed point for a holomorphic map is always mixed, i.e. in any neighborhood of the fixed point there exist points distinct from the fixed point itself, whose full orbit (positive and negative) is contained in the neighborhood.

Using these compacta we can now associate an analytic circle diffeomorphism to an indifferent fixed point. Let K be a compact set given by the theorem associated to a

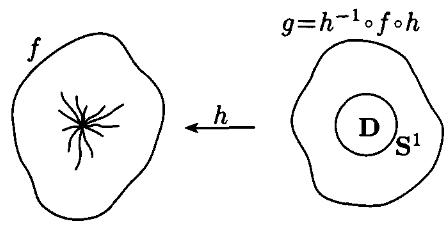

Fig. 2

small disk U centered at 0. Consider a conformal representation (Figure 2)

$$h: \overline{\mathbf{C}}-\overline{\mathbf{D}} \to \overline{\mathbf{C}}-K, \quad h(\infty)=\infty,$$

where $\mathbf{D}$ is the unit disk and $\overline{\mathbf{C}}$ is the Riemann sphere.

The map $g=h^{-1}\circ f\circ h$ is univalent and well defined in an open annulus surrounding $\mathbf{D}$ for which $\mathbf{S}^1=\partial\mathbf{D}$ is a component of its boundary. Using Carathéodory's extension theorem and Schwarz's reflection principle, it is now straightforward (§III.3) to prove that g extends continuously to an analytic circle diffeomorphism of $\mathbf{S}^1$. The main property of g is that its rotation number $\varrho(g)$ is equal to α where $f(z)=e^{2\pi i\alpha}z+\mathcal{O}(z^2)$ (§III.3). A more precise version of this construction is done in §§ III.3 and III.4. Observe the fundamental role in the construction played by the total invariance of the Siegel compactum K. Thus we obtain

THEOREM 2 (fundamental construction). *Assuming the hypothesis in Theorem* 1, *let K be the Siegel compactum given by that theorem. Let $h: \overline{\mathbf{C}}-\overline{\mathbf{D}}\to\overline{\mathbf{C}}-K$, $h(\infty)=\infty$, be a conformal representation of the exterior of K. Then the map $g=h^{-1}\circ f\circ h$ extends to an analytic circle diffeomorphism of $\mathbf{S}^1$ with rotation number*

$$\varrho(g)=\alpha.$$

Using the fundamental construction we obtain a dictionary between the two problems. In §I we review the classical dynamical results in both problems, and we give the theorem correspondence (§I.4) implied by our fundamental construction. To prove the general existence of Siegel compacta (§III.2), we start proving the result for a dense class of holomorphic germs (§§ II.2 and II.3), and then this implies the general case (§§ III.1 and III.2). Two distinct approaches are presented in the proof for a dense class: one via rational rotation numbers, and the other one via "good" irrational rotation numbers. In the first approach we generalize the classical study by Leau and Fatou of the dynamics near a parabolic fixed point.

Several other applications of the fundamental construction are presented in §IV. We give a new natural proof of Naishul's theorem (§IV.1) based on Poincaré's invariance of the rotation number of a circle homeomorphism by conjugation by orientation-preserving homeomorphisms. We prove Dulac–Moussu's conjecture (§IV.2). We give a new proof and a generalization of the snail lemma (§IV.3). We prove the non-existence of periodic points on the boundary of Siegel disks of the first type (§IV.4). We prove that the closure of a Siegel disk of the first type is always full. We apply the fundamental construction to the problem of centralizers (§IV.5). In §V.1 we introduce Herman compacta associated to analytic circle diffeomorphisms and the class of hedgehogs. We give a glimpse of its complex topological structure and its dynamical properties. Their detailed study is in progress and will be presented in future work. We give in §V.2 a converse of the fundamental construction, which generalizes a construction by E. Ghys. Using this generalization we show the intrinsic meaning of the fundamental construction (§V.3).

The fundamental construction of this paper was discovered during a visit to the KTH in Stockholm in the fall of 1991. I thank the KTH Mathematics Department for this stimulating visit. I would like to thank the following people who have contributed with useful discussions: J.-Ch. Yoccoz, M. Benedicks, L. Carleson, M. Herman, R. Moussu. And again, J.-Ch. Yoccoz for showing me the way to attack the general case of Dulac–Moussu's conjecture. And also J. Rogers who has pointed out some mistakes in the original preprint. The English of this article has improved thanks to C. Cleveland, T. Gamelin, J. Kim, and B. Waldron. E. Risler has corrected some misprints from the original preprint. I thank all of them for their help.

I. Theorem correspondence

In §§ I.2 and I.3 we present a list of propositions. Some of them are theorems, others are open questions or conjectures, on the dynamics of analytic circle diffeomorphisms and indifferent fixed points. In §I.4 we give a list of implications concerning these statements which follow from the fundamental construction. We start with some notation and definitions.

I.1. Notation and definitions

For general background on circle diffeomorphisms we refer the reader to [He1] or [Yo5]. For indifferent fixed points we refer to [He2] or [Pe3].

Let us recall that an analytic circle diffeomorphism g is (analytically) *linearizable* if it is conjugate by an analytic circle diffeomorphism to a rotation. In fact, we always have a topological conjugacy on the circle when the diffeomorphism is without periodic orbits (Denjoy's theorem). The angle of the rotation is the *rotation number*, denoted by $\varrho(g)$, and can be defined by the uniform limit

$$\varrho(g)=\lim_{n\to+\infty}\frac{\tilde{g}^n-\mathrm{id}}{n}\quad[\mathrm{mod}\,\mathbf{Z}],$$

where $\tilde{g}$ is a lift of g to $\mathbf{R}$, the universal covering of the circle.

In the same way, a holomorphic germ in $\mathrm{Diff}(\mathbf{C},0)$, $f(z)=e^{2\pi i\alpha}z+\mathcal{O}(z^2)$, $\alpha\in\mathbf{R}$, is said to be *linearizable* if there exists a local holomorphic diffeomorphism $h(z)=z+\mathcal{O}(z^2)$ which conjugates f to the rotation $h^{-1}\circ f\circ h=R_\alpha\colon z\mapsto e^{2\pi i\alpha}z$. By analogy with the preceding situation we define the *rotation number* of f as $\varrho(f)=\alpha$. The *domain of linearization*, or *Siegel domain*, or *Siegel disks*, of f is the maximal open domain where f is conjugate to a rotation by a holomorphic diffeomorphism. We denote it by $\mathcal{S}(f)$. It is easy to show ([Si]) that f is linearizable if and only if the indifferent fixed point is Lyapunov stable, i.e. for any neighborhood U of the fixed point there is a neighborhood V such that all the iterates $(f^n)_{n\geqslant 0}$ are defined on V and $f^n(V)\subset U$. More precisely, if $f|_U$ is a diffeomorphism then V is contained in $\mathcal{S}(f)$.

We denote by $\mathcal{L}$ the set of all linearizable circle diffeomorphisms and holomorphic germs. Let $\mathcal{L}_0$ be the set of linearizable holomorphic germs for which there is no Jordan neighborhood U of $\overline{\mathcal{S}(f)}$ satisfying the hypothesis of Theorem 1.

In this section the rotation number $\alpha\in\mathbf{R}$ will be irrational. For $\Delta>0$, A_Δ is the annulus $\{z\in\mathbf{C}: e^{-2\pi\Delta}<|z|<e^{2\pi\Delta}\}$. We define the sets

$$\begin{aligned}S(\alpha)&=\{f\in\mathrm{Diff}(\mathbf{C},0):f(z)=e^{2\pi i\alpha}z+\mathcal{O}(z^2)\text{ univalent on }\mathbf{D}\},\\ S(\alpha,\Delta)&=\{g\colon\mathbf{S}^1\to\mathbf{S}^1\text{ analytic diffeomorphism}: g\text{ univalent in }A_\Delta,\ \varrho(g)=\alpha\},\\ S(\alpha,0)&=\bigcup_{\Delta>0}S(\alpha,\Delta)\quad(\text{set of analytic diffeomorphisms of }\mathbf{S}^1).\end{aligned}$$

We will be interested in analytic circle diffeomorphisms or holomorphic germs without periodic orbits (different from the fixed point 0 for the germs). Thus we define

$$\begin{aligned}S'(\alpha)&=\{f\in S(\alpha):f\text{ has no periodic orbits distinct from }\{0\}\},\\ S'(\alpha,\Delta)&=\{g\in S(\alpha,\Delta):g\text{ has no periodic orbits}\},\\ S'(\alpha,0)&=\bigcup_{\Delta>0}S'(\alpha,\Delta).\end{aligned}$$

Arithmetic conditions. We briefly recall the main arithmetic conditions appearing in problems of small divisors. For $0<\alpha<1$, $\alpha\in\mathbf{R}-\mathbf{Q}$, the denominators of the convergents $(p_n/q_n)_{n\geqslant 0}$ of α given by the continued fraction algorithm can be obtained by putting $q_0=1$ and

$$q_{n+1}=\min\{q\geqslant 1:\|q\alpha\|<\|q_n\alpha\|\},$$

where $\|x\|=d(x,\mathbf{Z})=\inf_{p\in\mathbf{Z}}|x-p|$ ([La] and [Sch] are classical references for diophantine approximation).

The diophantine conditions can be expressed by the growth properties of $(q_n)_{n\geqslant 0}$:

$$\alpha\in \text{D.C.} \Leftrightarrow \log q_{n+1}=\mathcal{O}(\log q_n),$$

$$\alpha\in\mathcal{B} \Leftrightarrow \sum_{n\geqslant 0}\frac{\log q_{n+1}}{q_n}<+\infty,$$

$$\alpha\in\mathcal{B}' \Leftrightarrow \sum_{n\geqslant 1}\frac{\log\log q_{n+1}}{q_n}<+\infty.$$

When $\alpha\in$D.C. we say that α satisfies a diophantine condition. This is the classical arithmetic condition appearing in the works of C.-L. Siegel ([Si]) and V. I. Arnold ([Ar1]). The Brjuno condition $\mathcal{B}$ arises in the linearization problem ([Br], [Ch], [Yo3]) and $\mathcal{B}'$ in the linearization problem for germs with no (non-trivial) periodic orbits ([Pe1]). There is another important arithmetic condition, denoted by $\mathcal{H}$, discovered by Yoccoz [Yo4] in the study of the global linearization problem for analytic circle diffeomorphisms. We follow the definition of condition $\mathcal{H}$ in [Yo4] and [Yo5].

For $\alpha\in\mathbf{R}-\mathbf{Q}$ we define the sequence $(\alpha_n)_{n\geqslant 0}$ by the continued fraction algorithm. We put $\alpha_0=\{\alpha\}$, where $\{x\}$ denotes the fractional part of $x\in\mathbf{R}$, and for $n\geqslant 0$,

$$\alpha_{n+1}=\{\alpha_n^{-1}\}.$$

For $\alpha\in]0,1[$ we consider the continuous function

$$\sigma_\alpha(x)=\begin{cases}\alpha^{-1}(x-\log(\alpha^{-1})+1) & \text{if } x\geqslant\log(\alpha^{-1}),\\ e^x & \text{if } x\leqslant\log\alpha^{-1}.\end{cases}$$

Then we define the sequence $(\Delta_n(\alpha))_{n\geqslant 0}$ by $\Delta_0(\alpha)=10$ and

$$\Delta_{n+1}(\alpha)=\sigma_{\alpha_n}(\Delta_n(\alpha)).$$

Now we can define $\mathcal{H}_0=\{\alpha\in\mathbf{R}-\mathbf{Q}:\exists n_0,\,\forall n\geqslant n_0,\,\Delta_n(\alpha)\geqslant\log(\alpha_n^{-1})\}$. And finally

$$\mathcal{H}=\{\alpha\in\mathbf{R}-\mathbf{Q}:\forall n\geqslant 0,\,\alpha_n\in\mathcal{H}_0\}.$$

We have that D.C.$\subset\mathcal{H}\subset\mathcal{B}\subset\mathcal{B}'\subset\mathbf{R}-\mathbf{Q}$, all the inclusions being strict. Another important arithmetic condition is $\mathcal{H}'$ (defined in §I.2), which is not explicitly determined in terms of the continued fraction algorithm. From the definition of $\mathcal{H}'$ it can be proved that

$$\mathcal{H}\subset\mathcal{H}'\subset\mathcal{B}',$$

and in fact it should be true that all inclusions are strict.

I.2. Circle diffeomorphisms

We survey and comment the main theorems and conjectures on the dynamics of analytic circle diffeomorphisms. We consider the first pair of propositions:

$$\alpha \in \mathcal{B} \Rightarrow (\exists \Delta(\alpha) > 0, S(\alpha, \Delta(\alpha)) \subset \mathcal{L}), \qquad \text{A.1 (i)}$$

$$\alpha \notin \mathcal{B} \Rightarrow (\forall \Delta > 0, S(\alpha, \Delta) \not\subset \mathcal{L}). \qquad \text{A.1 (ii)}$$

Proposition A.1 (i) is the Arnold–Rüssman–Yoccoz local linearization theorem for analytic circle diffeomorphisms ([Ar1], [Rü], [Yo4]). The arithmetic condition in this theorem has undergone progressive improvement by the various authors. V. I. Arnold's first proof of the theorem was for $\alpha \in$D.C. Then, according to [Do2], H. Rüssman improved the proof for α satisfying

$$\sum_{n \geqslant 1} \frac{\log q_{n+1} \log \log q_{n+1}}{q_n} < +\infty.$$

Finally Yoccoz proved the theorem with the hypothesis $\alpha \in \mathcal{B}$ and proved its optimality, which is the meaning of A.1 (ii). The local hypothesis is usually formulated differently. We refer to the appendix for equivalent statements.

$$\alpha \in \mathcal{B}' \Rightarrow (\exists \Delta(\alpha) > 0, S'(\alpha, \Delta(\alpha)) \subset \mathcal{L}), \qquad \text{A.2 (i)}$$

$$\alpha \notin \mathcal{B}' \Rightarrow (\forall \Delta > 0, S'(\alpha, \Delta) \not\subset \mathcal{L}). \qquad \text{A.2 (ii)}$$

The first is an unwritten proposition but it can be obtained in the same way as the analogous theorem for holomorphic germs given in [Pe1]. The second proposition is a theorem which will be proved in this work as a result of our fundamental construction and the theorem in [Pe1] which will be discussed in §I.3 (Proposition B.2 (ii)). Note that the arithmetic condition in A.2 (ii) improves the condition appearing in one of the main theorems in [Pe1] (which was stated without an arithmetic condition). Here is the precise formulation of this result:

THEOREM I.2.1. *For $\alpha \notin \mathcal{B}'$ and any $\Delta > 0$ there exists an analytic circle diffeomorphism g univalent in A_Δ and for which any positive orbit $(g^n(z))_{n \geqslant 0}$ remaining in A_Δ accumulates $\mathbf{S}^1$. In particular, g is not linearizable and has no periodic orbits.*

The third pair of propositions we consider is

$$\alpha \in \mathcal{H} \Rightarrow S(\alpha, 0) \subset \mathcal{L}, \qquad \text{A.3 (i)}$$

$$\alpha \notin \mathcal{H} \Rightarrow S(\alpha, 0) \not\subset \mathcal{L}. \qquad \text{A.3 (ii)}$$

The first proposition is a theorem by M. Herman [He1] and J.-Ch. Yoccoz [Yo4]. In his thesis ([He1]), M. Herman solved a conjecture of V. I. Arnold proving the linearizability of analytic circle diffeomorphisms for almost every rotation number. J.-Ch. Yoccoz later improved the arithmetic condition ([Yo2]) and recently ([Yo4]) has found the optimal condition $\mathcal{H}$. So the second part is also due to J.-Ch. Yoccoz.

$$\alpha \in \mathcal{H}' \Rightarrow S'(\alpha, 0) \subset \mathcal{L}, \tag{A.4 (i)}$$

$$\alpha \notin \mathcal{H}' \Rightarrow S'(\alpha, 0) \not\subset \mathcal{L}. \tag{A.4 (ii)}$$

These two propositions can be seen as the definition of the arithmetic condition $\mathcal{H}'$. An open question consists of determining $\mathcal{H}'$ explicitly. It is not difficult to show that such an arithmetic condition exists, i.e. that it is invariant by the action of $\mathrm{PSL}(2, \mathbf{Z})$ (see [Pe1]). Moreover, from [Pe1], this condition is not trivial, $\mathcal{H}'$ is not empty and not equal to $\mathbf{R}-\mathbf{Q}$.

I.3. Indifferent fixed point dynamics

We survey classical and recent results on linearization of indifferent fixed points.

$$\alpha \in \mathcal{B} \Rightarrow S(\alpha) \subset \mathcal{L}, \tag{B.1 (i)}$$

$$\alpha \notin \mathcal{B} \Rightarrow S(\alpha) \not\subset \mathcal{L}. \tag{B.1 (ii)}$$

Proposition B.1 (i) is the celebrated linearization theorem of C.-L. Siegel [Si] and A. Brjuno [Br]. C.-L. Siegel proved this theorem for $\alpha \in$D.C. and later, in 1965, A. Brjuno improved the proof for $\alpha \in \mathcal{B}$. Recently, in 1987, J.-Ch. Yoccoz ([Yo3]) has shown the optimality of Brjuno's condition, i.e. Proposition B.1 (ii) ([Pe3] for a survey on this).

$$\alpha \in \mathcal{B}' \Rightarrow S'(\alpha) \subset \mathcal{L}, \tag{B.2 (i)}$$

$$\alpha \notin \mathcal{B}' \Rightarrow S'(\alpha) \not\subset \mathcal{L}. \tag{B.2 (ii)}$$

These two propositions are theorems due to the author ([Pe1]). They mean that if the holomorphic germs are without periodic orbits in $\mathbf{D}-\{0\}$ then we can improve Brjuno's condition ($\mathcal{B}$ is strictly included in $\mathcal{B}'$). Our geometric construction will show that B.2 (ii) implies A.2 (ii), and we will obtain Theorem I.2.1 by the following more precise version of B.2 (ii) proved in [Pe1]:

THEOREM I.3.1. *For $\alpha \notin \mathcal{B}'$ there exists a holomorphic germ $f(z)=e^{2\pi i\alpha}z+\mathcal{O}(z^2)$ defined and univalent in $\mathbf{D}$ such that any positive orbit $(f^n(z))_{n\geqslant 0}$ remaining in $\mathbf{D}$ accumulates at 0. In particular, f is not linearizable and has no periodic orbits.*

$$\alpha \in \mathcal{H} \Rightarrow S(\alpha) \subset \mathcal{L}_0, \tag{B.3 (i)}$$

$$\alpha \notin \mathcal{H} \Rightarrow S(\alpha) \not\subset \mathcal{L}_0. \tag{B.3 (ii)}$$

The first proposition is a corollary of a theorem by M. Herman [He3] which is an extension of a theorem by E. Ghys ([Gh]). The second proposition will be proved in §V.4 using the generalized Ghys construction. We may conjecture the more general fact:

If $\alpha\in\mathcal{B}-\mathcal{H}$ then the quadratic polynomial $P_\alpha(z)=e^{2\pi i\alpha}z+z^2$ is linearizable ($\alpha\in\mathcal{B}$) and univalent in a neighborhood of the closure of its Siegel domain.

$$\alpha\in\mathcal{H}' \Rightarrow S'(\alpha)\subset\mathcal{L}_0, \tag{B.4 (i)}$$

$$\alpha\notin\mathcal{H}' \Rightarrow S'(\alpha)\not\subset\mathcal{L}_0. \tag{B.4 (ii)}$$

We will show that B.4 (i) follows from A.4 (i). So taking $\mathcal{H}'$ as in the definition given in §I.2 we obtain

THEOREM I.3.2. *If $\alpha\in\mathcal{H}'$ then no holomorphic germ $f\in S(\alpha)$ without periodic orbits distinct from $\{0\}$ can be univalent on a Jordan neighborhood of the closure of its domain of linearization.*

Indeed, as follows from M. Herman's proof of his theorem in [He3], we have the following more general result:

THEOREM I.3.3. *Let $\alpha\in\mathcal{H}'$ and let $f\in S(\alpha)$ have no periodic orbit distinct from $\{0\}$, and be such that $\overline{S(f)}$ is compact and $f|_{\partial S(f)}$ is injective. Then there is a critical point of f in $\partial S(f)$ or a point in $\partial S(f)$ where f is not defined.*

Finally B.4 (ii), as B.3 (ii), will be proved in §V.4 using the generalized Ghys construction.

I.4. Theorem correspondence

The similarity between the linearization theorems for circle diffeomorphism and holomorphic germs is striking. In §§ I.2 and I.3 we have given simplified versions of the theorems. They usually have precise estimates or supplementary dynamical properties which have been omitted here for brevity. We will discuss this in Appendix 2, but let us remark that the full statements of the theorems can also be recovered by the geometric construction.

All the propositions stated before come in pairs and some arithmetic condition is associated with each pair. The first one (i) of each pair is roughly a linearization result. The second one (ii) states the existence of non-linearizable examples when the arithmetic condition fails. The first and second pair of propositions for each problem can be thought of as *local* propositions. The term local coming from the vocabulary employed for circle diffeomorphisms. The third and fourth pairs of propositions are of a *global* character.

A surprising feature which seems to have been missed by the folklore in the subject (an exception is [He3]) is the fact that we can interpret the global linearization theorems in the setting of holomorphic germs. It is commonly accepted that the Siegel problem (the study of the dynamics of an indifferent fixed point) is a local problem. But with a precise formulation we can also obtain global theorems. For a more detailed discussion of local versus global see Appendix 1.

The following theorem (proved in §III.4) gives the full theorem correspondence, which, for instance, avoids many double proofs.

THEOREM I.4.1 (theorem correspondence). *Linearization results for analytic circle diffeomorphisms imply linearization results for holomorphic germs; and conversely, existence of non-linearizable examples for holomorphic germs implies the existence of non-linearizable examples for analytic circle diffeomorphisms. More precisely, for* $k=1,2,3,4$ *we have*

$$\text{A.k (i)} \Rightarrow \text{B.k (i)} \quad \textit{and} \quad \text{B.k (ii)} \Rightarrow \text{A.k (ii)}.$$

This theorem follows from our geometric construction. As a general principle, for any reasonable result on one problem we obtain a corresponding result in the other. Some other applications of this general philosophy are given in §IV, and applications to the topology of Julia sets are discussed in [Pe4]. They illustrate how the construction sheds light on both problems, providing new proofs of classical results and also new theorems.

Some of the implications given in the theorem are straightforward from the construction and its properties. For example, if g is not linearizable in the construction presented in the introduction, then f cannot be linearizable (use the fact that the property of being linearizable is equivalent to topological stability), so B.1 (ii) $\Rightarrow$ A.1 (ii). The main property of the construction is to recover the rotation numbers, i.e. to prove the relation $\varrho(g)=\varrho(f)$. For all the applications except the proof of Naishul's theorem in §IV.1, it is enough to find an invariant continuum K for which this holds (Lemma III.3.3). This is simpler than proving the result for every K given by Theorem 1 (Lemma III.3.4).

II. Semi-local study

II.1. The fundamental construction

We consider K, a compact connected set of the Riemann sphere which does not contain the point at infinity. Denote by Ω_K the component of the complement which contains ∞. The map f is a holomorphic diffeomorphism such that f and its inverse f^{-1} are defined in a neighborhood U of K such that $f(K)=f^{-1}(K)=K$. We also assume that $f(U\cap\Omega_K)$ is a neighborhood of $\partial\Omega_K$ in $\overline{\Omega}_K$.

We can associate to the above an analytic circle diffeomorphism which corresponds to the action induced by f on the prime ends of Ω_K. The construction is as follows:

Consider a conformal representation $h: \overline{\mathbf{C}}-\overline{\mathbf{D}} \to \Omega_K$. We normalize h such that $h(\infty)=\infty$ and h is tangent to the identity at ∞ (this normalization is not important for the construction itself, but it determines h uniquely and this will be used later). The map $g=h^{-1}\circ f\circ h$ is a holomorphic diffeomorphism defined in an annulus having $\mathbf{S}^1$ as one boundary component. The diffeomorphism g maps this annulus into another with the same property. It follows from Carathéodory's extension theorem ([Po, p. 24]) that g extends continuously to the unit circle $\mathbf{S}^1$. Using Schwarz's reflection principle ([Car, p. 75]) g extends analytically to $\mathbf{S}^1$. We still denote by g the analytic extension in a neighborhood of $\mathbf{S}^1$ obtained in this way. It is clear that this extension, which is also denoted by g, is a holomorphic diffeomorphism in a neighborhood of $\mathbf{S}^1$ such that $g(\mathbf{S}^1)=\mathbf{S}^1$. This means that $g|_{\mathbf{S}^1}$ is an analytic circle diffeomorphism.

The reader familiar with Carathéodory's theory of prime ends will recognize that g is the action induced by f on the space of prime ends of Ω_K. To obtain g we only need to use the uniform continuity of g in a compact neighborhood of K. This type of construction was done (for the first time?) for planar homeomorphisms by Cartwright and Littlewood ([CLi]). A purely topological theory of prime ends related to these types of questions has been developed by J. Mather in [Ma]. This action is analytic with respect to the analytic structure on the circle of prime ends induced by the embedding of Ω_K in the Riemann sphere (according to Carathéodory, this analytic structure comes from the identification of the circle of prime ends with $\mathbf{S}^1$ via the conformal representation h).

Continuity of the fundamental construction. We consider the space $\mathcal{K}_c$ of compact connected sets endowed with Carathéodory's topology. This is by definition the minimal topology making continuous the map $K\mapsto\Omega_K$ from $\mathcal{K}_c$ onto the space of simply-connected neighborhoods of ∞ endowed with Carathéodory's kernel topology (see [Du, p. 76]).

LEMMA II.1.1. *Let $(K_i)_{i\geqslant 0}$ be a sequence of elements in $\mathcal{K}_c$ converging to $K\in\mathcal{K}_c$. Let $(f_i)_{i\geqslant 0}$ be a sequence of holomorphic diffeomorphisms, f_i and f_i^{-1} being defined in an open neighborhood U_i of K_i, K_i being totally invariant by f_i, and $f(U_i\cap\Omega_{K_i})$ is a neighborhood of $\partial\Omega_{K_i}$ in $\overline{\Omega}_{K_i}$. We assume that the sequence $(U_i)_{i\geqslant 0}$ contains in its kernel an open neighborhood U of K, and the sequence $(f_i)_{i\geqslant 0}$ (or $(f_i^{-1})_{i\geqslant 0}$) converges uniformly on compact sets in U to a holomorphic diffeomorphism f (or f^{-1}) leaving K invariant and such that $f(U\cap\Omega_K)$ is a neighborhood of $\partial\Omega_K$ in $\overline{\Omega}_K$. Let $(g_i)_{i\geqslant 0}$ (or g) be the sequence of analytic circle diffeomorphisms obtained from the fundamental construction applied to f_i and K_i (or f and K).*

Then the sequence $(g_i)_{i\geqslant 0}$ converges uniformly to g in a neighborhood of $\mathbf{S}^1$.

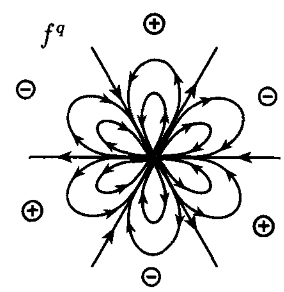

Fig. II.1

Observation. It is easy to prove a similar continuity property for Cartwright and Littlewood's purely topological construction.

Proof. Since the open set U is contained in the kernel of the sequence $(U_i)_{i\geqslant 0}$, the analytic circle diffeomorphisms of the sequence $(g_i)_{i\geqslant 0}$ will be defined for $i\geqslant i_0$ in a fixed annular neighborhood A of $\mathbf{S}^1$. We can assume that A is symmetric with respect to $\mathbf{S}^1$. Consider a circle C in A homotopic to $\mathbf{S}^1$ and exterior to $\mathbf{S}^1$, as well as its reflection C' with respect to $\mathbf{S}^1$. The mappings $(h_i)_{i\geqslant 0}$ are converging uniformly to h on C, as well as $(h_i^{-1})_{i\geqslant i_1}$ to h^{-1} (for some $i_1\geqslant i_0$) on a compact neighborhood of $f(h(C))$. It follows that $(g_i)_{i\geqslant i_1}$, $g_i=h_i^{-1}\circ f\circ h_i$, converges uniformly on C to $g=h^{-1}\circ f\circ h$. By reflection with respect to $\mathbf{S}^1$, the same is true on C'. By the maximum principle, the sequence $(g_i)_{i\geqslant i_1}$ converges uniformly to g in the annulus between C and C'. □

II.2. Semi-local study of the rational case

II.2 (a). Local study

In order to carry out the semi-local study in §II.2, we need some classical facts from the dynamics of holomorphic germs $f(z)=\lambda z+\mathcal{O}(z^2)$ with $\lambda=e^{2\pi i\alpha}$ a root of unity (this is the rational case, $\alpha\in\mathbf{Q}$), which are well known since the work of L. Leau [Le], P. Fatou [Fa] and G. Julia [Ju]. The complete topological classification has been obtained by C. Camacho [Cam]; J. Ecalle [Ec] and S. M. Voronin [Vo] have classified the analytic conjugacy classes.

We fix the notation $f(z)=\lambda z+\mathcal{O}(z^2)$, $\lambda=e^{2\pi i\alpha}$ is not of finite order, and $\alpha=p/q$, $p\in\mathbf{Z}$, $q\in\mathbf{N}^*$, $p\wedge q=1$. The dynamics in a neighborhood of the origin is quite simple and can be described in the following way (Figure II.1):

There exists an integer $n\geqslant 1$ and a *local flower* formed by $2n$ *local cycles* of *local petals.* The petals are Jordan domains invariant by f^q. In Figure II.1, we have drawn the dynamics for f^q. Note the existence of two kinds of local petals corresponding to a

Fig. II.2

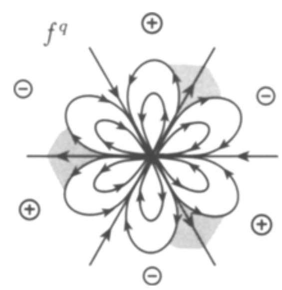

Fig. II.3

positive or negative orientation of the dynamics of f^q. We will refer to positive or negative local petals. The two kinds of petals come in pairs. The petals are not canonically determined, i.e. using the non-wandering dynamics we can modify the domains and still get the same picture. We introduce some concepts:

Definition. Let l be a germ of arc landing at 0. The germ of arc defined by l is f-invariant if $f(l)$ represents the same germ of landing arc.

Definition. An f-invariant bouquet of germs of f-invariant analytic arcs landing at 0 is a family of analytic arcs foliating a domain U containing 0 in its boundary, covering two adjacent petals as well as the local attracting region between them (Figure II.2).

In the semi-local study we will need the following lemma:

LEMMA II.2.1. *Let f be as above. There exists nq f^q-invariant bouquets which form the local flower of f at 0. For each bouquet there exists an analytic arc γ which is transverse to all the analytic arcs forming the bouquet and which crosses each one only once. Thus we have a one-to-one parametrization of the f-invariant arcs forming the bouquet.*

Observe that in general the union of these bouquets is not a foliation of a neighborhood of 0 (in Figure II.3 they do not foliate the shaded zone).

The proof of this lemma is simple using classical results (Fatou coordinates). More precise estimates can be found in [Fa] for example. We leave the proof to the reader.

II.2 (b). Semi-local study

We consider a meromorphic germ $f(z)=\lambda z+\mathcal{O}(z^2)$, $\lambda=e^{2\pi ip/q}$, $p,q\geqslant 1$, $(p,q)=1$, defined on a neighborhood of 0. When the germ defined by f at 0 is not of finite order, there exists a local flower composed by $2n$ local cycles of q local petals $\mathcal{C}=(P_i)_{1\leqslant i\leqslant q}$ such that $0\in\overline{P}_i$, $f^q(P_i)=P_i$ and $f^q|_{P_i}$ is univalent (according to the local study in §II.2). Each petal P_i is a Jordan domain. An extension of the cycle $\mathcal{C}$ will be a cycle $\widehat{\mathcal{C}}=(\widehat{P}_i)_{1\leqslant i\leqslant q}$ such that $P_i\subset\widehat{P}_i$, $\widehat{P}_i$ is a Jordan domain, $0\in\overline{\widehat{P}}_i$, $\widehat{P}_i$ is contained in the domain of definition of f^q, $f^q(\widehat{P}_i)=\widehat{P}_i$ and $f^q|_{\widehat{P}_i}$ is univalent. We prove in this section the following semi-local generalization of the classical local description of Fatou:

THEOREM II.2.1. *Let $f(z)=\lambda z+\mathcal{O}(z^2)$, $\lambda=e^{2\pi ip/q}$, $p\in\mathbf{Z}$, $q\geqslant 1$, $(p,q)=1$, be a meromorphic germ defined on a Jordan neighborhood $U\subset\overline{\mathbf{C}}$ of 0 which is not of finite order. We assume that f and f^{-1} are defined and univalent in a neighborhood of the closure of U. If $\mathcal{C}=(P_i)_{1\leqslant i\leqslant q}$ is a local cycle of petals at 0, then there exists an extension of $\mathcal{C}$ contained in U, $\widehat{\mathcal{C}}=(\widehat{P}_i)_{1\leqslant i\leqslant q}$, which has a petal $P=\widehat{P}_i$ whose closure intersects the boundary of U. Moreover, there exists such a P where ∂P is a Jordan arc containing 0, which is analytic except at 0.*

The set P is called a *principal petal* of the cycle $\mathcal{C}$ of f relative to U, $\widehat{\mathcal{C}}$ is a maximal cycle of $\mathcal{C}$ relative to U.

When f is of finite order we have a similar theorem:

THEOREM II.2.2. *Let $f(z)=\lambda z+\mathcal{O}(z^2)$ be a meromorphic germ defined on a Jordan neighborhood of $U\subset\overline{\mathbf{C}}$ of 0, of finite order, i.e. $f^q=\mathrm{id}$ for a minimal $q\geqslant 1$ as germ at 0. Then the connected component V of 0 of the set $\{z\in U: 0\leqslant i\leqslant q-1,\ f^i(z)\in U\}$ is simply-connected and composed by the fixed point 0 and periodic orbits of period exactly q and if $U\neq\overline{\mathbf{C}}$, for every $z\in\partial V$ there exists $0\leqslant i\leqslant q-1$ such that $f^i(z)\in\partial U$. In particular, $\partial V\cap\partial U\neq\varnothing$.*

Proof of Theorem II.2.2. Using analytic continuation of the equation $f^q=\mathrm{id}$ we see that V is composed of periodic points of period q (not necessarily minimal). Since U is a Jordan domain, the maximum principle relative to U shows the simple connectivity of V. Now we argue by contradiction. Suppose that $z\in\partial V$ and for any $0\leqslant i\leqslant q-1$, $f^i(z)\notin\partial U$. Then in some neighborhood of z the equation $f^q=\mathrm{id}$ holds, contradicting $z\in\partial V$.

It remains to prove that the minimal period of every $z\in V-\{0\}$ is exactly q. There is nothing to prove when $q=1$, so consider the case $q\geqslant 2$. Let $F=\{z\in V:$

$f^k(z)=z$ for some $1\leqslant k<q\}$. The set F is composed of isolated points in V. Otherwise for some $1\leqslant k<q$ the equation $f^k=\mathrm{id}$ will hold on some open subset of V and then $F=V$, contradicting the minimality of $q\geqslant 1$. We prove that $F=\{0\}$. If not, consider $z_0\in F$, $z_0\neq 0$. Consider a path $\gamma\subset V-F$ joining 0 to z_0. The curves $\gamma, f(\gamma), ..., f^{q-1}(\gamma)$ cut the Riemann sphere into q disks $\Omega_1, ..., \Omega_q$. Because $U\neq\overline{\mathbf{C}}$ we have for some index i, $\Omega_i\cap(\overline{\mathbf{C}}-V)\neq\varnothing$. The transitive action of f on the domains shows that this property holds for any index and, since $q\geqslant 2$, this contradicts the simple connectivity of V. □

Proof of Theorem II.2.1. We start with several lemmata:

LEMMA II.2.2. *Let g be a homeomorphism of $[0,1]$ which is real-analytic in a neighborhood of $[0,1]$, $g(0)=0$, $g(1)=1$, and such that the iterates of any point in $]0,1]$ converge to 1 under iteration by g.*

Then there exists a continuous (in the Hausdorff topology) family of Jordan domains $(U_t)_{0<t\leqslant 1}$ in $\mathbf{C}$ such that

(1) *U_t contains $]0,1[$,*

(2) *the boundary of U_t is composed of 0, 1, an analytic arc γ in the upper half-plane, and its reflection with respect to the real line,*

(3) *the analytic extension of g is defined in a neighborhood of the closure of U_t and $g|_{U_t}$ is a homeomorphism of U_t.*

Proof. Let $I=[a,g(a)[$ be a fundamental interval in $]0,1[$ for the dynamics of g. Considering a small transversal l to the real line at a and its image $g(l)$ we can construct a fundamental rectangle for the dynamics of g closing the strip determined by l and $g(l)$ using two symmetric arcs J and J' whose end points correspond by g. If we choose J close enough to the real axes, then the translation arc $\bigcup_{n\in\mathbf{Z}} g^n(J)$ is well defined, lies in the upper half-plane and lands at 0 and 1. This follows from the local structure of the dynamics of g near 0 and 1, which are holomorphic fixed points. With points 0 and 1, this translation arc and its reflection enclose an invariant Jordan domain which has the properties of the lemma. Clearly by the same construction we can obtain a continuous family of domains with the required properties (using a continuous family of arcs (J_t)). □

LEMMA II.2.3. *Let g be a non-linearizable analytic diffeomorphism of the circle $\mathbf{S}^1$ with rational rotation number $\varrho(g)=p/q$, $q\geqslant 1$, $p\in\mathbf{Z}$, $(p,q)=1$. There exists an annular Fatou flower F containing $\mathbf{S}^1$ invariant under g. More precisely, F is a continuum, the interior is composed of Jordan domains bounded by two analytic arcs landing at consecutive fixed points of $g^q|_{\mathbf{S}^1}$, each one containing a component of $\mathbf{S}^1-\mathrm{Fix}(g^q|_{\mathbf{S}^1})$, and each one being invariant under g^q. Moreover, there is a continuous and strictly monotone family of these annular Fatou flowers $(F_t)_{0\leqslant t\leqslant 1}$ such that $F_{1/2}=F$ and $F_t\to F_0=\mathbf{S}^1$ when*

$t \to 0$. By strictly monotone we mean that the closure of F_t minus the end points of the two analytic components of the boundary is contained in the interior of $F_{t'}$ for $t<t'$.

Proof. The diffeomorphism $g|_{\mathbf{S}^1}$ is analytic and non-linearizable, so $\mathrm{Fix}(g^q)$ is a finite set on the circle. The components of $\mathbf{S}^1 - \mathrm{Fix}(g^q|_{\mathbf{S}^1})$ are permuted by g as a (p/q)-rotation. On each cycle we choose a component and we consider g^q. The dynamics of g^q on this component is wandering, and using the previous lemma (II.2.2) we construct an invariant Jordan domain bounded by two analytic arcs with the specified properties. Then transporting by g we obtain the Jordan domains corresponding to the other components of the cycle. Similarly we construct a family of annular Fatou flowers, considering a strictly nested family of Jordan domains containing a component in each cycle (using Lemma II.2.2) and transporting by g. □

LEMMA II.2.4. *Let Ω be a simply-connected domain, relatively compact in $\mathbf{C}$. The map f is a holomorphic diffeomorphism defined in a neighborhod of $\bar{\Omega}$ and leaving Ω invariant, i.e. $f(\Omega)=\Omega$.*

Let $z_1 \in \Omega$ and $z_0 \in \partial\Omega$ be an accumulation point of $(f^n(z_1))_{n\geqslant 0}$. Then z_0 is a fixed point of f and $f^n \to z_0$ uniformly on compact sets of Ω.

Proof. The family of iterates $(f^n|_\Omega)_{n\geqslant 0}$ is uniformly bounded, so it is a normal family. Any limit of a subsequence is constant or a diffeomorphism of Ω (Hurwitz). Since z_0 is an accumulation point of $(f^n(z_1))_{n\geqslant 0}$, the limit must be constant. If $c=\lim_{k\to+\infty} f^{n_k}$, passing to the limit $k\to+\infty$ in the equation $f\circ f^{n_k}=f^{n_k}\circ f$ we get that c is a fixed point of f. So the possible limits of subsequences of $(f^n|_\Omega)_{n\geqslant 0}$ are finite since they are contained in the set of fixed points of the holomorphic map f (since $(f^n(z_1))_{n\geqslant 0}$ accumulates at $z_0 \neq z_1$, f is not the identity). These limits being fixed points of f, it is easy to see that there is at most one limit. □

LEMMA II.2.5. *Let g be an analytic circle diffeomorphism with rational rotation number p/q, $q\geqslant 1$, $(p,q)=1$, obtained by the fundamental construction from a pair (f,K) as in* §I.1. *Let γ be an analytic arc exterior to $\mathbf{S}^1$ bounding laterally a petal of an annular Fatou petal F of g given by Lemma* II.2.3.

The image of γ by the conformal representation h is a crosscut of Ω_K (using the notation of §I.1).

Proof. Let F_1 be the annular Fatou flower strictly containing F (it exists by Lemma II.2.3). Then γ is in the interior of F_1. Let U be the Jordan domain which is the component of the interior of F_1 containing γ and consider $V=h(U)$. The domain V is simply-connected and f^q leaves V invariant. If $z_1\in\gamma$ then $(f^{qn}(z_1))_{n\geqslant 1}$ accumulates at some point $z_0\in\partial V$. Using Lemma II.2.4, it follows that the sequence $(f^{qn})_{n\geqslant 0}$

converges uniformly to z_0 in V. Since $f(h(\gamma))=h(\gamma)$, one end of $h(\gamma)$ lands at z_0. The same argument for f^{-q} shows that the other end of γ also lands, so $h(\gamma)$ is a crosscut of Ω_K. □

Observation. Using this lemma it is easy to classify those continua K invariant by a holomorphic diffeomorphism f defined on a neighborhood of K whose action on prime ends has a rational rotation number. We call them generalized Fatou flowers. It is possible to give a combinatorial topological classification, as well as an analytic one, generalizing the classical Mather–Ecalle–Voronin invariants. For these results we refer to [Pe6] and [Pe7]. There we also determine those full compacta for which the action on prime ends has an irrational rotation number. These are always Siegel compacta.

If K is contained in a Fatou flower associated to a parabolic fixed point z_0, we call K a *Fatou continuum.* In this case, z_0 is the common landing point of all the crosscuts $h(\gamma)$ where γ runs over the external crosscuts bounding the annular Fatou flower F. Conversely if this holds then K is a Fatou continuum. Obviously a Fatou continuum has a neighborhood with no periodic orbit other than the parabolic fixed point, and this property characterizes them in the larger class of generalized Fatou flowers. Moreover, the landing points of γ in the previous lemma are fixed points of f^q. Thus, fixing the holomorphic diffeomorphism f with a parabolic fixed point, the space of generalized Fatou flowers is closed in the Hausdorff topology. This observation is used in the proof of Theorem II.2.1 below.

Proof of Theorem II.2.1. From the classical local study (§II.1 (a)) we obtain the existence of a local, continuous and monotone family of Fatou flowers parametrized by an interval, and generated by the sheaf of germs of landing arcs as described in Lemma II.2.1. Let $t\in]-\infty, t_0[$ be the real parameter of the local family of Fatou flowers. We take $t_0\in\mathbf{R}$ to be maximal. We prove that there is a Fatou flower generated by the parameter value t_0, and that flower cannot be relatively compact in U. Otherwise if K is the filled closure of the union of the F_t for $t<t_0$, K is relatively compact in U and f induces a rational rotation number action on the prime ends of the unbounded component of its complement (this follows from the continuity property of Lemma II.1.1). From Lemma II.2.5 and the previous observation it follows that K is a generalized Fatou flower contained in a Fatou flower generated by some $t_1>t_0$. Also using Lemma II.2.4 we see that all $t<t_1$ generate a Fatou flower. This contradicts the maximality of t_0.

II.3. Semi-local study of the irrational case

We give in this section a simple proof of a weak form of M. Herman's theorem in [He3] which is sufficient for our purposes. The proof is simple but not elementary since we need

Siegel's linearization theorem for holomorphic germs, as well as the global linearization theorem for analytic circle diffeomorphisms.

THEOREM II.3.1. *Let* $f(z)=\lambda z+\mathcal{O}(z^2)$, $\lambda=e^{2\pi i\alpha}$, $\alpha\in\mathbf{R}-\mathbf{Q}$, $\alpha\in$D.C., *be holomorphic and* U *a Jordan neighborhood of* 0 *satisfying the hypothesis of Theorem* 1. *Then the closure of the maximal linearization domain* $\mathcal{S}(f)$ *of* f *touches the boundary of* U, $\overline{\mathcal{S}(f)}\cap\partial U\neq\varnothing$.

As we mentioned, this result follows from:

THEOREM (M. Herman [He3]). *Let* $f(z)=\lambda z+\mathcal{O}(z^2)$, $\varrho(f)=\alpha\in$D.C., *be a holomorphic map in a neighborhood of* $\overline{\mathcal{S}(f)}$, *such that* $f|_{\partial\overline{\mathcal{S}(f)}}$ *is injective. Then* $\partial\overline{\mathcal{S}(f)}$ *contains a critical point of* f.

Remark. These theorems hold for $\alpha\in\mathcal{H}$.

Proof of Theorem II.3.1. By Siegel's theorem, $\mathcal{S}(f)\neq\varnothing$. Arguing by contradiction, we assume that $\mathcal{S}(f)$ is relatively compact in U. Let $K=\widehat{\overline{\mathcal{S}(f)}}$ be the fill of the closure of the linearization domain of f. We apply the fundamental construction to K and f. Let $h\colon\overline{\mathbf{C}}-\overline{\mathbf{D}}\to\Omega_K$ be the normalized conformal representation. We conjugate f in the exterior of $\overline{\mathcal{S}(f)}$ to an analytic circle diffeomorphism g.

Let K_n be an exhausting sequence of invariant closed disks in $S(f)$. We apply the fundamental construction to (K_n, f). Observe that the circle diffeomorphisms (g_n) that we obtain have rotation number $\varrho(g_n)=\alpha$ since they are linearizable.

By continuity of the fundamental construction (Lemma II.1.1) the sequence (g_n) converges to g uniformly on $\mathbf{S}^1$. So we have proved that $\varrho(g)=\lim_{n\to+\infty}\varrho(g_n)=\alpha\in$D.C. Now the global linearization theorem implies that g is linearizable. Transporting an invariant annulus by h, we obtain that the linearization domain for f is larger than K. This contradicts the definition of K and finishes the proof.

III. Proof of the theorem correspondence

III.1. Compact spaces and Hausdorff distance

Let (X,d) be a compact metric space. For $A\subset X$ and $\varepsilon>0$, we denote by $V_\varepsilon(A)=\{x\in X: d(x,A)<\varepsilon\}$ the ε-neighborhood of A. The Hausdorff distance between two compact sets $K_1, K_2\subset X$ is

$$d_H(K_1,K_2)=\inf\{\varepsilon>0: K_1\subset V_\varepsilon(K_2) \text{ and } K_2\subset V_\varepsilon(K_1)\}.$$

It is easy to see that d is a distance on the space $\mathcal{K}(X)$ of compact subsets of X. Moreover, $\mathcal{K}(X)$ endowed with the topology defined by the Hausdorff distance is a compact space ([HY, p. 102] or [MZ, p. 17]). A compact set $K\subset X$ is connected if and only if for every $\varepsilon>0$ there exists an ε-chain in K joining two arbitrary points of K ([Ne, p. 81]). Using this criterium it is easy to prove that the space $\mathcal{K}_c(X)$ of continua (i.e. compact connected non-empty sets) of X is a closed subset of $\mathcal{K}(X)$. Also, fixing a compactum $K\subset X$, the space of compact sets of X containing K is closed in $\mathcal{K}(X)$.

Now, suppose that X and Y are compact metric spaces and $f\in C^0(X,Y)$, i.e. $f\colon X\to Y$ is a continuous map. The map f is uniformly continuous by compactness of X. We denote by ω the modulus of continuity of f, so

$$\forall\varepsilon>0,\ \exists\delta=\omega(\varepsilon)>0,\ \forall x,x'\in X,\quad d(x,x')\leqslant\delta\Rightarrow d(f(x),f(x'))\leqslant\varepsilon.$$

Let $f_*\colon\mathcal{K}(X)\to\mathcal{K}(Y)$ be defined by $f_*(K)=f(K)$. We have

LEMMA III.1.1. *The map f_* is uniformly continuous with the same modulus of continuity as f.*

Proof. If $K_1\subset V_\delta(K_2)$, then for $x_1\in K_1$ there exists $x_2\in K_2$ such that $d(x_1,x_2)\leqslant\delta$. Then $d(f(x_1),f(x_2))\leqslant\varepsilon$ if $\delta=\omega(\varepsilon)$ and $f(K_1)\subset V_\varepsilon(f(K_2))$. Also if $K_2\subset V_\delta(K_1)$ then $f(K_2)\subset V_\varepsilon(f(K_1))$, and finally

$$d_H(K_1,K_2)\leqslant\delta=\omega(\varepsilon)\ \Rightarrow\ d_H(f(K_1),f(K_2))\leqslant\varepsilon.$$

Conversely, a modulus of continuity for f_* is also a modulus of continuity for f. □

We consider the topology of uniform convergence in $C^0(X,Y)$ obtained from the distance δ defined, for $f,g\in C^0(X,Y)$, by

$$\delta(f,g)=\sup_{x\in X} d(f(x),g(x)).$$

Observe that for $x\in X$, $\{f(x)\}\subset V_{\delta(f,g)}(\{g(x)\})$, so for $K\in\mathcal{K}(X)$, $f(K)\subset V_{\delta(f,g)}(g(K))$ and

$$d_H(f(K),g(K))\leqslant\delta(f,g).$$

The following lemma will be useful:

LEMMA III.1.2. *The map*

$$C^0(X,Y)\times\mathcal{K}(X)\to\mathcal{K}(Y),\qquad (f,K)\mapsto f(K)$$

is continuous.

Proof. Consider a sequence $((f_n,K_n))_{n\geqslant 0}$ such that $\lim_{n\to+\infty}(f_n,K_n)=(f,K)$. Let $\varepsilon>0$ and $N\geqslant 1$ be large enough such that for $n\geqslant N$, $d_H(K_n,K)\leqslant\delta=\omega(\varepsilon)$ and $\delta(f_n,f)\leqslant\varepsilon$, where ω is the modulus of continuity for f. Then we have

$$\begin{aligned} d_H(f_n(K_n),f(K)) &\leqslant d_H(f_n(K_n),f(K_n))+d_H(f(K_n),f(K)),\\ &\leqslant \delta(f_n,f)+\varepsilon\leqslant\varepsilon+\varepsilon=2\varepsilon. \end{aligned}$$ □

III.2. The main theorem

We prove Theorem 1 in this section. Observe that the last sentence in this theorem follows from the general theory of indifferent fixed points. Fix a Jordan neighborhood of 0, U, as in Theorem 1. Let $\mathcal{F}_U$ be the set of all continuous functions f in $\overline{U}$ satisfying the hypothesis of Theorem 1. Endow the space $\mathcal{F}_U$ with the topology of uniform convergence on $\overline{U}$. Observe that if $f\in\mathcal{F}_U$ then $f^{-1}\in\mathcal{F}_U$, and $f\mapsto f^{-1}$ is continuous.

PROPOSITION III.2.1. *The set of elements of $\mathcal{F}_U$ satisfying the conclusions of Theorem 1 is closed.*

Proof. Take a sequence $(f_n)_{n\geqslant 0}$ in $\mathcal{F}_U$ satisfying the conclusions of Theorem 1 and converging to $f\in\mathcal{F}_U$. More precisely, for each f_n there is an f_n-invariant compact connected set $0\in K_n\subset\overline{U}$ such that $K_n\cap\partial U\neq\varnothing$. Using the compactness of $\mathcal{K}_c(\overline{U})$ we extract a convergent subsequence $K_{n_k}\to K\in\mathcal{K}_c(\overline{U})$. The set K is compact, connected and nonempty. Property (ii) as well as (iii) clearly hold in the limit. Using Lemma III.1.2 we obtain the total invariance of K by f. The filled compactum $\widehat{K}$ satisfies Theorem 1 for f and U. □

So it is enough to prove Theorem 1 for a dense class. Now Theorem II.2.1 and Theorem II.3.1 prove Theorem 1 for holomorphic germs having a rational rotation number and holomorphic germs having an irrational rotation number satisfying a diophantine condition, respectively. In view of these considerations the next lemma, III.2.1, finishes the proof.

Note that we have two different proofs: the first one using approximation by rational rotation numbers and the second by "good" irrational ones. Observe that the first proof based on Theorem II.2.1 is completely elementary. In particular, we do not need any result from the theory of small divisors. On the other hand the second approach uses Theorem II.3.1 whose proof relies on Siegel's linearization theorem and Herman's global linearization theorem for circle diffeomorphisms. The first approach is preferable, although it is more tedious, to establish the theorem correspondence (Theorem II.4.1). This elementary approach is needed to prove Siegel–Brjuno's linearization theorem from Arnold–Rüssmann–Yoccoz's theorem.

LEMMA III.2.1. *Let $E\subset\mathbf{T}$ be a dense set. The set of maps $f\in\mathcal{F}_U$ such that $f(z)=e^{2\pi i\alpha}z+\mathcal{O}(z^2)$ with $\alpha\in E$ is dense in $\mathcal{F}_U$. In particular, the subset of elements of $\mathcal{F}_U$ with rational rotation number (or irrational rotation number satisfying a diophantine condition) is dense in $\mathcal{F}_U$.*

Proof. Let $f\in\mathcal{F}_U$. It is enough to consider $f_n(z)=e^{2\pi i\theta_n}f(z)$ such that $\theta_n+\varrho(f)\in E$ and θ_n small enough. Then $f_n\in\mathcal{F}_U$ and $f_n\to f$. □

III.3. Properties of the fundamental construction

First we do the main construction in a form suitable for the applications. Let $\alpha\in\mathbf{R}$ and $f\in S(\alpha)$. By Koebe's one quarter theorem, for $0<r<\frac{1}{4}$, the map f satisfies the hypothesis of Theorem 1 with $U=\mathbf{D}_r$. In the following, we fix such an r. Let K_r be a Siegel compactum given by Theorem 1 associated to $U=\mathbf{D}_r$, i.e.

(i) K_r is compact connected and full,

(ii) $0\in K_r\subset\overline{\mathbf{D}}_r$,

(iii) $K_r\cap\partial\mathbf{D}_r\neq\varnothing$,

(iv) $f(K_r)=f^{-1}(K_r)=K_r$.

Let $h_{K_r}\colon\overline{\mathbf{C}}-\overline{\mathbf{D}}\to\overline{\mathbf{C}}-K_r$ be a conformal representation satisfying $h_{K_r}(\infty)=\infty$ (Figure 2 in the introduction).

By the total invariance of K_r the conjugated map $g_{K_r}=h_{K_r}^{-1}\circ f\circ h_{K_r}$ is well defined and univalent on an open annulus surrounding $\overline{\mathbf{D}}$ such that $\mathbf{S}^1=\partial\mathbf{D}$ is a component of its boundary.

LEMMA III.3.1. *The map g_{K_r} extends continuously to an analytic circle diffeomorphism of $\mathbf{S}^1$.*

Proof. See §II.1. □

LEMMA III.3.2. *If $\varrho(f)\in\mathbf{Q}$ or $\varrho(f)\in$D.C., then there exists a Siegel compactum K_r such that $\varrho(g_{K_r})=\varrho(f)$.*

Proof. We begin by considering the case $\varrho(f)\in$D.C. Let K_r be the maximal invariant piece of $\mathcal{S}(f)$ inside $\overline{\mathbf{D}}_r$. Theorem II.3.1 guarantees that $\mathcal{S}(f)$ is not contained in $\mathbf{D}_r$. (In this situation it is easy to check the uniqueness of K given by Theorem 1: K is intersected by all the invariant circles forming K_r, thus this shows that necessarily $K=K_r$ since the dynamics of f in these circles is minimal and K is closed.) The invariant closed disk K_r is surrounded by real-analytic invariant Jordan curves where f acts with rotation number $\varrho(f)$. This shows, using reflection with respect to $\mathbf{S}^1$, that $\mathbf{S}^1$ is also surrounded by g_{K_r}-invariant curves. So g_{K_r} is linearizable and $\varrho(g_{K_r})=\varrho(f)$ since g_{K_r} acts with rotation number $\varrho(f)$ on the g_{K_r}-invariant curves surrounding $\mathbf{S}^1$.

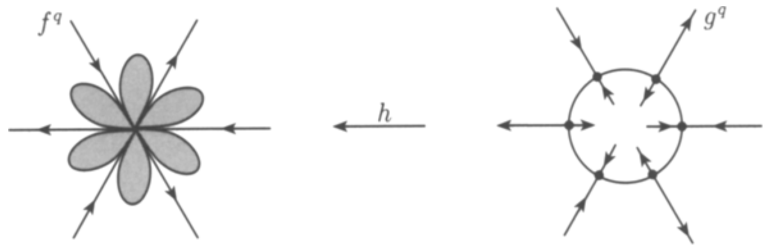

Fig. III.1

Now we consider the case $\varrho(f)\in\mathbf{Q}$. We take K_r to be an invariant subflower of the semi-local Fatou flower given by Theorem 1 associated to $U=\mathbf{D}_r$. We can assume that the boundary of each connected component of its interior is a Jordan curve, analytic except at the point 0 (Figure III.1).

The point 0 is accessible from $\overline{\mathbf{C}}\backslash K_r$ by each one of the nq repelling cusps (repelling for f^q) between the nq bouquets ($\varrho(f)=p/q$, $n>1$). For each landing arc $(\gamma_i)_{i\leqslant i\leqslant nq}$ in these cusps $h_{K_r}^{-1}(\gamma_i)$ is a landing arc in $\mathbf{S}^1$ using Koebe's theorem ([Oh, pp. 270, 285]). Since we can choose γ_i to be f^q-invariant as germs of arcs landing at 0, the arcs $h_{K_r}^{-1}(\gamma_i)$ will be g^q-invariant. Their landing points are fixed points of g^q in $\mathbf{S}^1$. Moreover, the action of f on a q-cycle of arcs γ_i is the same as the action of g_{K_r} on the landing points of $h_{K_r}^{-1}(\gamma_i)$. Since f induces a (p/q)-permutation this shows that g_{K_r} will have a (p/q)-periodic orbit. Thus $\varrho(g_{K_r})=p/q=\varrho(f)$. □

Remark. It is easy to see that the periodic orbits obtained in this way are repulsive (p/q)-periodic orbits of g_{K_r}. Considering attracting cusps we obtain n attracting (p/q)-periodic orbits. If $n\geqslant 2$ we can choose K_r to be a subflower only containing some of the cycles of petals, in order to obtain a circle diffeomorphism with (p/q)-periodic orbits wich are parabolic with convenient multiplicity.

We remove the assumption on the rotation number:

LEMMA III.3.3. *There exists a Siegel compactum K_r such that $\varrho(g_{K_r})=\varrho(f)$.*

Proof. The Siegel compacta K_r given by Theorem 1 are obtained as filled limits of Siegel compacta for germs f with rational rotation number (or irrational satisfying a diophantine condition) according to §III.2. We will prove the lemma for such Siegel compacta: $K_r=\big(\lim_{n\to+\infty}K_r^{(n)}\big)\widehat{\ }$, $f_n\in S(\alpha_n)$, $\alpha_n\in\mathbf{Q}$.

By the continuity property of the fundamental construction (Lemma II.1.1), we have $g_{K_r}=\lim_{n\to+\infty}g_{K_r^{(n)}}$ uniformly on $\mathbf{S}^1$. By Lemma III.3.2 we know that $\varrho(f_n)=\varrho(g_{K_r^{(n)}})$, and by continuity of the rotation number $\varrho(g_{K_r})=\lim_{n\to+\infty}\varrho(g_{K_r^{(n)}})=\varrho(f)$. □

Lemma III.3.3 will be sufficient for all the applications in the sequel except for the proof of Naishul's theorem (§IV.1). We will need there (but only for irrational rotation

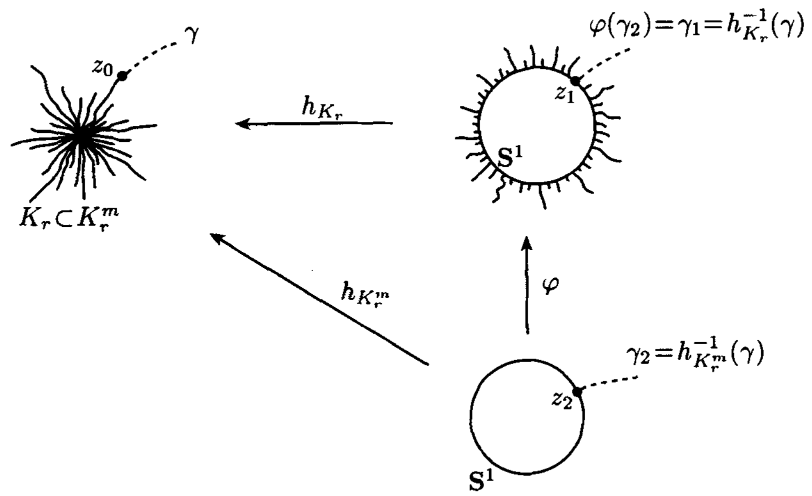

Fig. III.2

numbers) the general rotation number correspondence for arbitrary Siegel compacta. There follows an elementary proof. We now have a new, more natural, proof of this result using the technique of quasi-invariant curves developed in [Pe6].

LEMMA III.3.4. *For any Siegel compactum K_r we have $\varrho(g_{K_r})=\varrho(f)$.*

Proof. Let K_r^m be the maximal Siegel compactum associated to $\overline{\mathbf{D}}_r$. This is the connected component of 0 of the set $\{z\in\overline{\mathbf{D}}_r : \forall n\in\mathbf{Z},\, f^n(z)\in\overline{\mathbf{D}}_r\}$. From Theorem 1 we have $K_r^m\cap\partial\mathbf{D}_r\neq\varnothing$. Any other K_r given by Theorem 1 is a subset of K_r^m. Since $K_r\cap\partial\mathbf{D}_r\neq\varnothing$, let $z_0\in K_r\cap\partial\mathbf{D}_r\subset K_r^m\cap\mathbf{D}_r$ (Figure III.2).

We consider the conformal representation $\varphi\colon \overline{\mathbf{C}}-\overline{\mathbf{D}}\to\overline{\mathbf{C}}-(\overline{\mathbf{D}}\cup h_{K_r}^{-1}(K_r^m-K_r))$, $\varphi= h_{K_r}^{-1}\circ h_{K_r^m}$. The point z_0 is accessible by a germ of arc γ from $\overline{\mathbf{C}}-K_r^m$. The germ of arc $\gamma_1=h_{K_r}^{-1}(\gamma)$ lands at $z_1\in\mathbf{S}^1$ and $\gamma_2=h_{K_r^m}^{-1}(\gamma)$ at $z_2\in\mathbf{S}^1$. We have $\gamma_1=\varphi(\gamma_2)$ and for $n\geqslant 0$, $g_{K_r}^n(\gamma_1)=\varphi(g_{K_r^m}^n(\gamma_2))$. Let $z_1^{(n)}$ and $z_2^{(n)}$ be the landing points in $\mathbf{S}^1$ of $g_{K_r}^n(\gamma_1)$ and $g_{K_r^m}^n(\gamma_2)$. The topological Lemma III.3.5 (below) shows that, when the q points $(z_1^{(n)})_{0\leqslant n\leqslant q}$ are distinct, they have the same ordering in $\mathbf{S}^1$ as the corresponding distinct points $(z_2^{(n)})_{0\leqslant n\leqslant q}$ (to get an annulus in order to apply Lemma III.3.5 blow up the point at infinity and prolongate the germs of arcs landing at the points $z_i^{(n)}$).

Take $1\leqslant q\leqslant+\infty$ maximal such that the points in the sequence $(z_1^{(n)})_{n\geqslant 0}$ are all distinct. If $q=+\infty$ then the ordering of these points determines $\varrho(g_{K_r})$. Also the same holds for the infinite sequence of distinct elements $(z_2^{(n)})_{n\geqslant 0}$ and the rotation number $\varrho(g_{K_r^m})$. Thus these two rotation numbers coincide.

If $q\geqslant 1$ is finite, then $(z_1^{(0)},\ldots,z_1^{(q)})$ is a periodic orbit of g_{K_r}, and g_{K_r} acts on it as a (p/q)-permutation where p/q is the rational rotation number of g_{K_r}. Then $g_{K_r^m}$ acts in the same way on $(z_2^{(0)},\ldots,z_2^{(q)})$, so again $\varrho(g_{K_r^m})=p/q=\varrho(g_{K_r})$.

Now, by Lemma III.3.3, there exists g_{K_r} such that $\varrho(f)=\varrho(g_{K_r})$. We conclude that $\varrho(g_{K_r^m})=\varrho(f)$. Finally for every K_r, $\varrho(g_{K_r})=\varrho(f)$. □

LEMMA III.3.5. *Let A be a topological annulus embedded in the plane, whose boundary has two connected components C_1 and K, with C_1 homeomorphic to a circle and K containing C_2 homeomorphic to a circle. A cut of A will be a path in A landing at two points, one in each component of the boundary of A. Let $\gamma_i,\ldots,\gamma_n$ be n disjoint cuts of A, with γ_i landing at $x_1^i\in C_1$ and at $x_2^i\in C_2$. Let $\mathcal{E}_2$ be the space of prime ends of the boundary of K ($\mathcal{E}_2$ is homeomorphic to a circle). Consider the prime ends $e_2^i\in\mathcal{E}_2$ corresponding to the landing arcs γ_i. We assume that the sequences (x_1^i) and (x_2^i) are composed of distinct points. Then the ordering of the sequence (x_2^i) in C_2 and the ordering of the sequence (e_2^i) in $\mathcal{E}_2$ are the same.*

Proof. Let $\tilde{A}$ be the annulus containing A and bounded by C_1 and C_2. The paths (γ_i) are disjoint cuts of $\tilde{A}$, thus the ordering of (x_1^i) and (x_2^i) in C_1 and C_2 respectively, are the same. We can map the annulus A conformally onto an annulus $\hat{A}$ bordered by two circles. In this way we get an explicit representation of $\mathcal{E}_2$, and by the same argument applied to $\hat{A}$, we obtain that the ordering of (x_1^i) and (e_2^i) in C_1 and $\mathcal{E}_2$ respectively, are the same. □

Finally we prove

PROPOSITION III.3.1. *For $f\in S(\alpha)$ and $0<r<r_0=\frac{1}{8}(3-\sqrt{5})<\frac{1}{4}$ we have*

(1) *$\varrho(f)=\alpha=\varrho(g_{K_r})$,*

(2) *$g_{K_r}\in S(\alpha,(1/2\pi)\log(r_0/r))$,*

(3) *if g_{K_r} is linearizable then f is linearizable,*

(4) *if f is linearizable then there exists $0<r_1(f)<r_0$ such that $\mathbf{D}_{r_1(f)}\subset S(f)$, and for $0<r<r_1(f)$, g_{K_r} is linearizable,*

(5) *if f has no periodic orbits in a pointed neighborhood of 0, $U\setminus\{0\}$, and $\varrho(f)\in\mathbf{R}-\mathbf{Q}$ then there exists $0<r_2(f)<r_0$ such that $\mathbf{D}_{r_2(f)}\subset U$, and for $0<r<r_2(f)$, g_{K_r} has no periodic orbits in a neighborhood of $\mathbf{S}^1$.*

Proof. (1) has already been proved. (3), (4) and (5) are straightforward (same type of argument as in the proof of Lemma III.3.2). We prove the estimate (2). For this we need ([Du])

LEMMA III.3.6. *For $f\in S(\alpha)$, we have for $z\in\mathbf{D}$,*

$$|f(z)|\leqslant\frac{|z|}{(1-|z|)^2}.$$

It follows that $f(\mathbf{D}_{r_0'})\subset\mathbf{D}$ for $r_0'=\frac{1}{2}(3-\sqrt{5})$. If we let $\mathcal{A}_{K_r}=h_{K_r}^{-1}(\mathbf{D}_{r_0'}-K_r)$ for $r<r_0'$ then $\operatorname{mod}\mathcal{A}_{K_r}=\operatorname{mod}(\mathbf{D}_{r_0'}-K_r)\geqslant\operatorname{mod}(\mathbf{D}_{r_0'}-\overline{\mathbf{D}}_r)=(1/2\pi)\log(r_0'/r)$. The maximum modulus in Grötsch's extremal problem ([Ah1]) shows that $\mathcal{A}_{K_r}$ contains the annulus $A_\Delta-\overline{\mathbf{D}}$ with

$$e^{2\pi\Delta}\geqslant\frac{1}{4}e^{2\pi\operatorname{mod}\mathcal{A}_{K_r}}\geqslant\frac{1}{4}\cdot\frac{r_0'}{r}\geqslant\frac{r_0}{r},$$
$$\Delta\geqslant\frac{1}{2\pi}\log\left(\frac{r_0}{r}\right).$$

Now $g_{K_r}=h_{K_r}^{-1}\circ f\circ h_{K_r}$ is well defined and univalent in $A_\Delta-\overline{\mathbf{D}}$, so

$$g_{K_r}\in S\left(\alpha,\frac{1}{2\pi}\log\left(\frac{r_0}{r}\right)\right).$$

III.4. The theorem correspondence

In this section we prove all the implications contained in Theorem II.4.1.

A.1 (i) $\Rightarrow$ B.1 (i). Assuming the Arnold–Rüssman–Yoccoz theorem we prove the Siegel–Brjuno theorem. Let $\alpha\in\mathcal{B}$ and $f\in S(\alpha)$. Let $0<r<r_0$ be small enough such that $(1/2\pi)\log(r_0/r)\geqslant\Delta(\alpha)$. We apply the fundamental construction as described in the previous section. By Proposition III.3.1 (2) we obtain a circle diffeomorphism

$$g_{K_r}\in S\left(\alpha,\frac{1}{2\pi}\log\left(\frac{r_0}{r}\right)\right)\subset S(\alpha,\Delta(\alpha))\subset\mathcal{L}.$$

Thus g_{K_r} is linearizable and by Proposition III.3.1 (3), f will also be linearizable.

B.1 (ii) $\Rightarrow$ A.1 (ii). Let $\alpha\notin\mathcal{B}$ ($\alpha\in\mathbf{R}-\mathbf{Q}$) and f be non-linearizable. For any $\Delta>0$, we can choose $0<r<r_0$ sufficiently small such that $(1/2\pi)\log(r_0/r)\geqslant\Delta$. By the fundamental construction we obtain $g_{K_r}\in S(\alpha,\Delta)$ (Proposition III.3.1 (2)) and g_{K_r} is not linearizable (point (3)).

A.2 (i) $\Rightarrow$ B.2 (i). The same proof as for A.1 (i) $\Rightarrow$ B.1 (i) works choosing $0<r<r_2(f)$ at the beginning and using Proposition III.3.1 (5).

B.2 (ii) $\Rightarrow$ A.2 (ii). We follow the same proof as for B.1 (i) $\Rightarrow$ A.1 (i) using Proposition III.3.1 (5).

A.3 (i) $\Rightarrow$ B.3 (i). We consider $f\in S(\alpha)$, $\alpha\in\mathcal{H}$, and we prove that there is no Jordan neighborhood U of $\overline{S(f)}$ such that f and U satisfy the hypothesis of Theorem 1. Otherwise, we can use the fundamental construction (Theorem 2) taking $K=\widehat{\overline{S(f)}}$ to obtain a circle diffeomorphism g, with rotation number α. Then g will be linearizable and this contradicts the maximality of $S(f)$.

B.3 (ii) $\Rightarrow$ A.3 (ii). Let $f \in S(\alpha)$ with $f \notin \mathcal{L}_0$. Applying the fundamental construction to f with $K=\widehat{S(f)}$, we obtain a non-linearizable circle diffeomorphism g with rotation number α.

A.4 (i) $\Rightarrow$ B.4 (i). The same proof as before, observing that the circle diffeomorphism g has no periodic orbits.

B.4 (ii) $\Rightarrow$ A.4 (ii). For $\alpha \notin \mathcal{H}'$, as before, we construct a non-linearizable circle diffeomorphism with no periodic orbits.

IV. Selected applications

In this section we present some selected applications of the geometric construction linking holomorphic fixed points and analytic circle diffeomorphisms. Other applications concerning the dynamics of rational functions and the topology of Julia sets will appear in [Pe4]. In each application a new result is obtained or a new proof of a classical result is presented.

IV.1. Naishul's theorem

V. I. Naishul proves in [Na] by delicate topological considerations the following theorem:

THEOREM IV.1.1 (V. I. Naishul). *Let*

$$f_1(z)=e^{2\pi i\alpha_1}z+\mathcal{O}(z^2) \quad \textit{and} \quad f_2(z)=e^{2\pi i\alpha_2}z+\mathcal{O}(z^2)$$

be two holomorphic germs topologically conjugated by an orientation-preserving homeomorphism of the plane in a neighborhood of 0. *Then* $\alpha_1=\varrho(f_1)=\varrho(f_2)=\alpha_2$.

This theorem strongly recalls the topological (orientation-preserving) invariance of the rotation number for circle homeomorphisms. Using this result for circle homeomorphisms (which is due to H. Poincaré) and our construction, we give a new short and natural proof. For background on prime ends we refer to [CLo], [Po] or [Oh].

Proof. Let φ be the homeomorphism conjugating f_1 and f_2, $f_1=\varphi^{-1}\circ f_2\circ\varphi$. Consider an invariant continuum K for f_1 given by Theorem 1, associated to a small ball U around 0. Construct the associated circle diffeomorphism g_1 as in §III.3. We consider the invariant continuum for f_2, $\varphi(K)$, and the associated circle diffeomorphism g_2. Since φ is

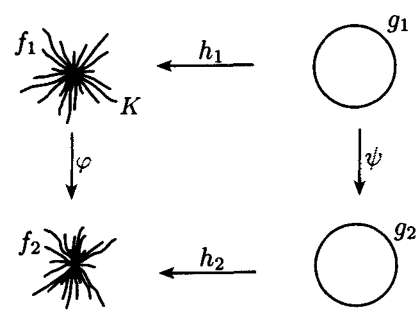

Fig. IV.1

a homeomorphism in a neighborhood of K onto its image, it induces a homeomorphism in the space of prime ends of $\overline{\mathbf{C}}-K$ onto the space of prime ends of $\overline{\mathbf{C}}-\varphi(K)$. This result follows from the uniform continuity of φ on a compact neighborhood of K. Thus the map $\psi=h_2^{-1}\circ\varphi\circ h_1$ extends continuously to an orientation-preserving homeomorphism of $\mathbf{S}^1$ and conjugates g_1 and g_2. It follows that $\varrho(g_1)=\varrho(g_2)$. On the other hand we have proven (Lemma III.3.4) that $\alpha_1=\varrho(f_1)=\varrho(g_1)$ and $\alpha_2=\varrho(f_2)=\varrho(g_2)$ so $\alpha_1=\alpha_2$. □

IV.2. Dulac–Moussu's conjecture

The study of singular points of holomorphic vector fields in the Siegel domain is closely related to the holomorphic dynamics near an indifferent fixed point using the holonomy construction. More specifically, an analytic differential system defined in a neighborhood of $(0,0)\in\mathbf{C}^2$,

$$\begin{cases} \dot{x}=\lambda_1 x+\dots, \\ \dot{y}=\lambda_2 y+\dots \end{cases}$$

(dots meaning higher-order terms), is said to be in the Siegel domain if $\lambda_2\neq 0$ and $\lambda_1/\lambda_2\in\mathbf{R}_-$. These singularities are particularly interesting since they are irreducible, i.e. we cannot decompose them into simpler ones by blowing up the singular point.

When $\lambda_1/\lambda_2<0$ there always exist two invariant solutions (or leaves of the corresponding holomorphic foliation) $\mathcal{F}_x$ and $\mathcal{F}_y$ going through the origin and tangent respectively to $\{x=0\}$ and $\{y=0\}$ (Figure IV.2). For classical references and more detailed explanations we refer to [MM] or [PY].

Considering a small circle γ in $\mathcal{F}_y$ surrounding the singularity at the origin we can lift it (by the "flow-box" theorem) in the neighboring leaves. On a holomorphic transversal disk Σ to $\mathcal{F}_y$ meeting γ, the return map of the lifted paths in the neighboring sheets defines a holomorphic map which has an indifferent fixed point and rotation number $\alpha=-\lambda_2/\lambda_1$ (taking the orientation as in Figure IV.2). The dynamics of f reflects how

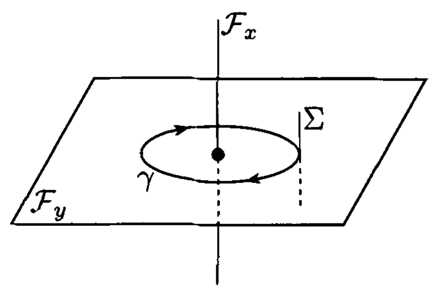

Fig. IV.2

the leaves cut the transversal Σ. The local topology of the holomorphic foliation defined by the differential system is completely determined by the holonomy on Σ by "pushing" Σ along the leaves. J.-F. Mattei and R. Moussu ([MM]) have proven that the conjugacy class of the holonomy determines the conjugacy class (by $\mathrm{Diff}(\mathbf{C}^2, 0)$) of the singularity. Conversely every conjugacy class of holomorphic germs corresponds to a holonomy of some singularity of the preceding type. This was first established by J. Martinet and J.-P. Ramis ([MR]) in the rational case ($\alpha \in \mathbf{Q}$); it results from Siegel–Brjuno linearization theorems in both problems when $\alpha \in \mathbf{R} - \mathbf{Q}$ but $\alpha \in \mathcal{B}$. The general result (without conditions on α) has been obtained by J.-Ch. Yoccoz and the author ([PY]). Thus the study of the two problems are equivalent. The holomorphic vector field is linearizable if it can be holomorphically conjugated to the linear form

$$\begin{cases} \dot{x} = \lambda_1 x, \\ \dot{y} = \lambda_2 y. \end{cases}$$

The linearizability of the singularity of the vector field is equivalent to the linearizability of its holonomy. H. Dulac studies in his remarkable doctoral dissertation [Du], the existence of solutions of the differential system which accumulate at the singular point. This memoir is difficult to find and has been unfairly ignored in the modern literature. There we can find many results which are nowadays wrongly misattributed. An illustrative example is "Seidenberg's desingularisation theorem" (sic), which is well known and used by Dulac, who attributed it to M. Autonne.

It is straightforward to check that the only solutions of the linear system which accumulate at the origin are $\mathcal{F}_x = \{x=0\}$ and $\mathcal{F}_y = \{y=0\}$. This observation suggests the following conjecture, which is implicit in H. Dulac's work as a question(1) and which has been pointed out repeatedly by R. Moussu:

(1) The main question left open by H. Dulac in his doctoral dissertation is the existence of "null solutions". This is similar to the above question. It is equivalent in the setting of holomorphic germs to: *Does there exist an orbit distinct from the indifferent fixed point converging to it by iteration?* This question has now been solved in [Pe6], where the dynamics of hedgehogs is studied in detail.

Dulac–Moussu's conjecture. *Every non-linearizable holomorphic vector field in the Siegel domain has a solution $\mathcal{F}$ distinct from $\mathcal{F}_x$ and $\mathcal{F}_y$ accumulating at the origin, i.e.* $(0,0)\in\overline{\mathcal{F}}$.

By the preceding equivalence of problems we can formulate the equivalent conjecture for holomorphic germs:

> *Every non-linearizable holomorphic germ* $f(z)=\lambda z+\mathcal{O}(z^2)$, $|\lambda|=1$, *has an orbit* $(f^n(z))_{n\in\mathbf{Z}}=\mathcal{O}(z)$, $z\neq 0$, *which accumulates at the origin, i.e.* $0\in\overline{\mathcal{O}(z)}$.

All the attempts to prove the conjecture in the strict setting of holomorphic singularities or holomorphic germs have failed. It is amusing to note that the introduction of an associated analytic circle diffeomorphism to the holomorphic germ, which in turn has been obtained from the holomorphic singularity, will be the cornerstone of the solution.

The rational case ($\lambda=e^{2\pi i\alpha}$, $\alpha\in\mathbf{Q}$) is clear from the local study in §II.2 (see [Cam]). We consider the irrational case. Let f be a holomorphic germ and K a Siegel compactum attached to the fixed point 0 given by Theorem 1 taking U to be a small ball centered at 0. The non-linearizability of f is equivalent to $0\in\partial K$. From the fundamental construction (Theorem 2), points in ∂K correspond heuristically to points of $\mathbf{S}^1$ by the conformal representation h (of course the correspondence only holds for accessible points of ∂K). The dynamics of f outside of K (and perhaps in most of ∂K) corresponds to the dynamics of an analytic circle diffeomorphism g, with $\varrho(g)=\alpha\in\mathbf{R}-\mathbf{Q}$. The diffeomorphism g is minimal in $\mathbf{S}^1$, i.e. every orbit in $\mathbf{S}^1$ is dense in $\mathbf{S}^1$. This is certainly not true for the dynamics of f in ∂K since $0\in\partial K$ is a fixed point. But we can expect this to be true for the orbit of most points in ∂K. Does there exist a point in ∂K having a dense orbit in ∂K? We will give below a positive answer to this question and this will solve Dulac–Moussu's conjecture. First we give a proof in the case where we have some control on the topology of K. I have not succeeded in proving that in general we have this control. It is a difficult problem to understand the topology of the Siegel compacta when they are hedgehogs (see [Pe4] and [Pe5]). In a second step we deal with the general case. I thank J.-Ch. Yoccoz who has indicated to me how to handle the general situation. We will consider the harmonic measure on K, combined with the ergodicity results for circle diffeomorphisms of M. Herman and A. Katok.

IV.2 (a). Topological dynamics

For general background on prime ends and the standard terminology we refer to [Po] and [CLo]. We have

PROPOSITION IV.2.1. *The positive or negative orbit of an impression of a prime end of $\overline{\mathbf{C}}-K$ by f accumulates on a subcontinuum of every other impression of $\overline{\mathbf{C}}-K$.*

Proof. Let e and e' be two prime ends defined respectively by the sequences of crosscuts $(\gamma_i)_{i\geqslant 0}$ and $(\gamma'_i)_{i\geqslant 0}$ of $\overline{\mathbf{C}}-K$. Let $(D_i)_{i\geqslant 0}$ and $(D'_i)_{i\geqslant 0}$ be the decreasing sequences of domains defined by these crosscuts, and $I(e)=\bigcap_{i\geqslant 0}\overline{D}_i$ and $I(e')=\bigcap_{i\geqslant 0}\overline{D}'_i$ be the impressions. For every $i\geqslant 0$ there exist $n_i\geqslant 0$ and $m_i\geqslant 0$ such that $f^{n_i}(I(e))\subset f^{n_i}(\overline{D}_{m_i})\subset D'_i$. This is true since it holds for g because $\varrho(g)\in\mathbf{R}-\mathbf{Q}$ and g is topologically conjugate to an irrational rotation by Denjoy's theorem. Thus $(f^{n_i}(I(e)))_{i\geqslant 0}$ must accumulate at some subcontinuum of $I(e')=\bigcap_{i\geqslant 0}\overline{D}'_i$. □

The principal set $\Pi(e)$ of a prime end e is the subcontinuum in $I(e)$ containing all the accumulation points of sequences of crosscuts (γ_i) defining e. Observing that $f^{n_i}(\gamma_{m_i})\subset D'_i$ in the above proof we prove

PROPOSITION IV.2.2. *The positive or negative orbit of a principal point of a prime end of $\overline{\mathbf{C}}-K$ accumulates on a subcontinuum of every impression of a prime end of $\overline{\mathbf{C}}-K$.*

We also have

PROPOSITION IV.2.3. *The positive or negative orbit of an accessible point of $\overline{\mathbf{C}}-K$ accumulates on a subcontinuum of every impression of a prime end of $\overline{\mathbf{C}}-K$.*

And as a corollary we have

THEOREM IV.2.1. *Suppose that $0\in\partial K$ is exactly the impression of some prime end of $\overline{\mathbf{C}}-K$. Then the positive or negative orbit by f of every principal point accumulates at 0.*

Since accessible points are dense in ∂K and every accessible point is a principal point, it follows that many points have an orbit accumulating at 0 when $0\in\partial K$ (or equivalently when f is non-linearizable). In the non-linearizable situation, we have proven in [Pe4] that the impression of every prime end of $\overline{\mathbf{C}}-K$ contains the fixed point 0. Thus the condition that 0 is the impression of a prime end is equivalent to the fact that there exist prime ends with a degenerate impression (i.e. reduced to a single point). In other words, the assumption of the theorem fails exactly when there are no prime ends of $\overline{\mathbf{C}}-K$ of the first kind. We have proved in [Pe4] that this is also equivalent (assuming f non-linearizable) to the fact that 0 is not accessible from the exterior of K. Thus we have

THEOREM IV.2.2. *When f is non-linearizable the following three conditions are equivalent:*

(i) *$\overline{\mathbf{C}}-K$ has a prime end with an impression reduced to a single point.*

(ii) *The point* 0 *is accessible from* $\overline{\mathbf{C}}-K$.

(iii) *The point* 0 *is the impression of a prime end of* $\overline{\mathbf{C}}-K$.

If one of these conditions holds, the positive or negative orbit of every principal point by f *accumulates at* 0.

I have not been able to rule out the possibility that 0 will not be the impression of some prime end. I conjecture in [Pe4] that this never occurs. For other topological properties we refer to §VI and [Pe4].

Remark. We cannot improve Proposition IV.2.2 by proving that the positive or negative orbit of a principal point accumulates on a subcontinuum of a principal set of any other prime end when 0 is accessible (take $\{0\}$ as principal set!).

IV.2 (b). The general case

We consider the above situation for f non-linearizable, $\varrho(f)\in\mathbf{R}-\mathbf{Q}$, K an invariant continuum attached to the fixed point 0, and we denote by μ_K the harmonic measure at ∞ of K in $\overline{\mathbf{C}}$.

THEOREM IV.2.3. *For* μ_K*-a.e. point* $z\in K$ *we have* $\partial K=\overline{\mathcal{O}(z)}=\overline{(f^n(z))}_{n\in\mathbf{Z}}$. *In particular, when* f *is not linearizable, for* μ_K*-a.e.* $z\in K$, $0\in\overline{\mathcal{O}(z)}$.

We recall the definition and properties of harmonic measures (for details of proofs we refer to [Ts], [Oh], [Ah2], [Ga]). The solution of the Dirichlet problem by the Perron method shows that every continuous function $\varphi\in C^0(K,\mathbf{R})$ can be continuously extended to a harmonic function $\widetilde{\varphi}$ in $\overline{\mathbf{C}}-K$. The linear functional $\varphi\mapsto\widetilde{\varphi}(\infty)$ is positive and by Riesz's representation theorem defines a Borel probability measure μ_K on K, which is called the harmonic measure at ∞, such that

$$\widetilde{\varphi}(\infty)=\int_K \varphi\, d\mu_K.$$

The support of μ_K is ∂K. For example, when $K=\overline{\mathbf{D}}$ the measure $\mu_{\overline{\mathbf{D}}}$ is the linear Lebesgue measure on $\mathbf{S}^1$. The harmonic measure is conformally invariant: if $h:\overline{\mathbf{C}}-\overline{\mathbf{D}}\to\overline{\mathbf{C}}-K$ is a conformal representation, $h(\infty)=\infty$, then up to a set of zero capacity, h and h^{-1} extend radially to the boundaries of their domains of definition. We call this extension h_*, and if $E\subset\partial K$ is a Borel set we have $\lambda(h_*^{-1}(E))=\mu_K(E)$. Now, the proof is based on

LEMMA IV.2.1. *The map* $f|_K$ *is* μ_K*-ergodic, i.e. if* $E\subset K$ *is a Borel set and* $f(E)=E$ *then* $\mu_K(E)=0$ *or* $\mu_K(E)=1$.

In the proof we use the following theorem due to M. Herman ([He1]) and A. Katok ([KH, p. 419, Theorem 12.7.2]):

THEOREM ([He1], [KH]). *Let g be a C^2-circle diffeomorphism with irrational rotation number. Then g is Lebesgue ergodic.*

Proof of Lemma IV.2.1. Let $E\subset K$ be an invariant subset, $f(E)=E$. Then $E'=E\cap\partial K$ is also invariant and $\mu_K(E')=\mu_K(E)$. Now $F=h_*^{-1}(E')\subset\mathbf{S}^1$ is invariant by g and $\lambda(F)=\mu_K(E')=\mu_K(E)$. The lemma follows from the previous theorem. □

Proof of Theorem IV.2.3. Let $(U_n)_{n\geqslant 0}$ be a countable base for the topology of ∂K. For $n\geqslant 0$, let

$$E_n=\{z\in\partial K:\mathcal{O}(z)\cap U_n\neq\varnothing\}.$$

We have $f(E_n)=E_n$ since $\mathcal{O}(f(z))=J\mathcal{O}(z)$. Also, we have $\mu_K(E_n)>0$ since $E_n\supset U_n$ and $\operatorname{supp}\mu_K=\partial K$. We have $\mu_K(E_n)=1$ because f is μ_K-ergodic of f. So $E=\bigcap_{n\geqslant 0}E_n$ is a set of full measure, $\mu_K(E)=1$. This means exactly that for μ_K-a.e. point $z\in K$, $\overline{\mathcal{O}(z)}=\partial K$. Since $0\in\partial K=\operatorname{supp}\mu_K$ for μ_K-a.e. $z\in K$, $0\in\overline{\mathcal{O}(z)}$. □

Note that the result established here is in some sense weaker than the one in the previous section, since we do not characterize topologically the points which have a dense orbit. In general we can ask the question:

Let $z\in K$ be an accessible point from the exterior of K and $z\neq 0$. Is it true that the orbit of z is dense in ∂K?

We can also propose the problem:

Give a topological characterization of the points in K having a dense orbit in the boundary of K.

Observe that there are always many points in a Siegel compactum K having a non-dense orbit. In fact, applying Theorem 1 twice to two neighborhoods $U_1\subset U_2$, with U_1 relatively compact in U_2, we obtain two maximal strictly distinct Siegel compacta with $K_1\subset K_2$. Then, the orbit of any point of K_1 is not dense in K_2 since it is entirely contained inside K_1, which is totally invariant. As a consequence of this, we obtain that $\mu_{K_2}(K_1)=0$, i.e. K_1 is hidden from the exterior of K_2. This is clear when f is linearizable and K_1 and K_2 are compact pieces in the linearization domain. In §V.1 we will see that a Siegel compactum is graduated by smaller ones. This shows the complexity of this graduation in the non-linearizable case.

Using the density of an orbit in K, we can answer another question of R. Moussu: let l be a holomorphic function defined on a punctured neighborhood of 0 which is a "first integral" of $f(z)=e^{2\pi i\alpha}z+\mathcal{O}(z^2)$, $\alpha=\varrho(f)\in\mathbf{R}-\mathbf{Q}$. That means that $l\circ f=l$ in a punctured neighborhood of 0. Does it follow that l is constant?

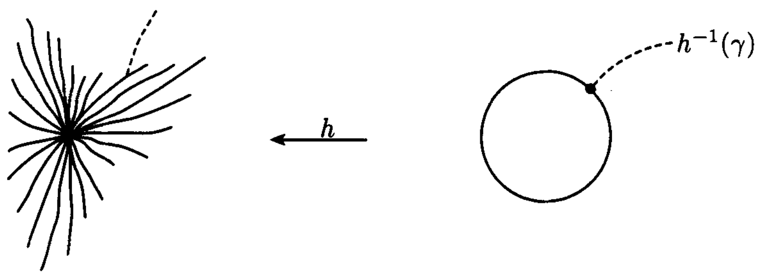

Fig. IV.3

We observe that this does not necessarily hold when $\alpha \in \mathbf{Q}$ because for $f(z)=z/(1+z)$ and $l(z)=e^{-2\pi i z^{-1}}$, we have $l \circ f = l$. The answer is clearly affirmative once we know the existence in this neighborhood of an orbit accumulating at a point distinct from 0.

IV.3. Snail lemma

In the study of rational dynamics D. Sullivan, A. Douady and others need to deal with Cremer points, i.e. indifferent irrational non-linearizable periodic points of a rational map. In particular, in the classification of Fatou components ([Mi], [Su]) or the exhibition of non-locally connected Julia sets ([Do1]) these authors use the following lemma, called the snail lemma:

SNAIL LEMMA. *Let f be a rational function and $z_0 \in \mathbf{C}$ a Cremer point of f lying in the boundary of a Fatou component Ω of f. Suppose that z_0 is accessible from Ω by an f-invariant arc γ. Then $f'(z_0)=1$.*

The name comes from the idea of the proof: If $f'(z_0) \neq 1$ then the invariance of γ forces γ to spiral around z_0, almost enclosing an f-invariant region. We prove here the following stronger result:

THEOREM IV.2.4 (snail lemma). *Let f be a holomorphic germ with an indifferent fixed point at 0. Let K be a totally invariant continuum containing 0 and such that f is univalent on a neighborhood of K. Furthermore, suppose that there exists an arc γ in $\overline{\mathbf{C}}-K$ landing at $z_0 \in K$ which is f-invariant. Then $\varrho(f) \in \mathbf{Z}$, i.e. $f'(0)=1$.*

We obtain the classical snail lemma taking $K=\{0\}$.

Proof. First, suppose $K \neq \{0\}$. By the fundamental construction we obtain an associated circle diffeomorphism g. Now, $h^{-1}(\gamma)$ is a landing arc in $\mathbf{S}^1$ by Koebe's theorem (Figure IV.3).

The arc γ is f-invariant so $h^{-1}(\gamma)$ is g-invariant and the landing point in $\mathbf{S}^1$ is a fixed point for g. From the general properties of circle homeomorphisms, it follows that $\varrho(f)=\varrho(g)\in\mathbf{Z}$ and $f'(0)=e^{2\pi i\varrho(f)}=1$.

Now we deal with the case $K=\{0\}$. Since f has a finite number of fixed points in a neighborhood of 0, $f|_\gamma$ will be locally topologically repulsive or attractive in a small subarc γ' landing at 0. We can assume that $f|_{\gamma'}$ is repulsive considering f^{-1} if necessary. Then applying Theorem 1 to a smaller neighborhood U of 0 we find a compactum $K'\subset\overline{U}$ such that $\bar{\gamma}'\cap K'=\{0\}$. Then we are in the preceding situation. □

IV.4. Dynamics and structure of the boundary of a Siegel disk

The study of the topology of the boundary of Siegel singular domains appears to be a very difficult problem. For example, it is not known if there exists a Siegel domain of a holomorphic map f, relatively compact in $\mathbf{C}$, f being holomorphic in a neighborhood of $\overline{\mathcal{S}(f)}$, which is not a Jordan domain. As our construction suggests, there are two distinct natural classes of Siegel domains: those having a Jordan neighborhood U satisfying the hypothesis of Theorem 1, they are of the first kind, and the rest, which are of the second kind. In other words, Siegel domains of the first kind are those for which $\overline{\mathcal{S}(f)}$ is a Siegel compactum for f.

It is probably much easier to analyze the situation for Siegel compacta of the first kind, and our construction gives some information concerning the dynamics on them. With respect to the topology, it is very likely that Siegel domains of the first kind will always be Jordan domains as Theorem IV.4.3 suggests. For these Siegel domains, f leaves invariant the boundary of the fill of $\overline{\mathcal{S}(f)}$. Lemma IV.2.1 and Theorem IV.2.3 give the following result:

THEOREM IV.4.1. *Let f be a holomorphic map having a Siegel domain $\mathcal{S}(f)\subset\mathbf{C}$ of the first kind. Then $f|_{\partial\widehat{\overline{\mathcal{S}(f)}}}$ is $\mu|_{\widehat{\overline{\mathcal{S}(f)}}}$-ergodic and the full orbit for $\mu|_{\widehat{\overline{\mathcal{S}(f)}}}$-almost every point in $\partial\widehat{\overline{\mathcal{S}(f)}}$ is dense in $\partial\widehat{\overline{\mathcal{S}(f)}}$.*

We recall that $\mu|_{\widehat{\overline{\mathcal{S}(f)}}}$ is the harmonic measure of $\widehat{\overline{\mathcal{S}(f)}}$, the fill of the closure of the Siegel domain. We will see in Theorem IV.4.3 that we have $\widehat{\overline{\mathcal{S}(f)}}=\overline{\mathcal{S}(f)}$ for Siegel domains of the first kind.

We observe that if the Siegel domain is a Jordan domain then the linearization map extends to the boundary, and conjugates topologically the dynamics on the boundary to an irrational rotation. There are no periodic orbits on the boundary. It is natural to conjecture that there cannot exist periodic points on the boundary of a Siegel domain.

This question can be found in [Bie]. We prove this for Siegel domains of the first kind. A similar result has been proved independently by J. Rogers ([Ro]).

THEOREM IV.4.2. *Let $f(z)=e^{2\pi i\alpha}z+\mathcal{O}(z^2)$, $\alpha\in\mathbf{R}-\mathbf{Q}$, be a holomorphic linearizable germ such that $\mathcal{S}(f)$ is of the first kind. Then $\partial\mathcal{S}(f)$ contains no periodic orbits.*

Proof. The proof is similar to the one for Theorem II.3.1. We consider an exhaustion of $\mathcal{S}(f)$ by invariant closed disks $(K_n)_{n\geqslant 0}$, and we apply the fundamental construction to them (Theorem 2). There exist two annuli, $A_1=A_{\Delta_1}$ and $A_2=A_{\Delta_2}$, such that for $n\geqslant 0$, g_n is defined on $\bar{A}_1$ and $g_n(\bar{A}_1)\subset\bar{A}_2$. By contradiction we assume the existence of a periodic orbit $\{z_1, ..., z_q\}$ in $\partial\mathcal{S}(f)$. Let $C_n=h_n^{-1}(\{z_1, ..., z_q\})$. For n large enough, we have $C_n\subset\bar{A}_1$. When $n\to+\infty$, we can extract a convergent subsequence of (h_n) (converging uniformly on compact subsets of $\bar{\mathbf{C}}-\bar{\mathbf{D}}$) and of $(C_n)_{n\geqslant 0}\subset\mathcal{K}(\bar{A}_1)$. Let $C\subset\mathbf{S}^1$ be the limit of this subsequence of compact sets. Then the sequence of circle diffeomorphisms (g_n) converges uniformly on $\mathbf{S}^1$ by the continuity property (Lemma II.1.1). Using Lemma III.1.2 with $X=\bar{A}_1$ and $Y=\bar{A}_2$, we obtain $g(C)=C$, where g is the uniform limit of the sequence (g_n). The map g is also a circle diffeomorphism obtained by the fundamental construction from $K=\widehat{\overline{\mathcal{S}(f)}}$. Being a Hausdorff limit of a sequence of finite sets with cardinal uniformly bounded (by q), C is necessarily a finite set. We conclude that g has a periodic orbit and $\varrho(g)\in\mathbf{Q}$. On the other side we have $\varrho(g)=\lim_{n\to+\infty}\varrho(g_n)=\varrho(f)=\alpha\in\mathbf{R}-\mathbf{Q}$. Contradiction. □

THEOREM IV.4.3. *The closure of a Siegel disk of the first kind is full, i.e.*

$$\widehat{\overline{\mathcal{S}(f)}}=\overline{\mathcal{S}(f)}.$$

The proof of this result uses

LEMMA IV.4.1. *There is no wandering component of the complement of the closure of a Siegel disk of the first kind.*

We do not prove this lemma in full generality here since it relies on some deep tools developed in [Pe6]. The general proof follows from the non-trivial property that the iterates of f accumulate at the identity in the fill of the closure of the Siegel disk in the uniform topology. Here we just show that this result holds when f is a rational map. First, for a Siegel disk of the first kind, the component containing ∞ is never wandering (its intersection with the domain of definition of f is mapped inside this exterior component). If we have a wandering component Ω, then the iterates of f in Ω would be bounded, and Ω would be a component of the Fatou set of f (since $\partial\Omega\subset\partial\mathcal{S}(f)\subset J(f)$). This would contradict Sullivan's non-wandering theorem, and the lemma is proved when f is a rational map.

Proof of Theorem IV.4.3. Let Ω be a bounded component of the complementary of $\mathcal{S}(f)$. By Lemma IV.4.1, there is an integer $m\geqslant 1$ and some iterate $q\geqslant 1$ such that $f^q(f^m(\Omega))=f^m(\Omega)$. From general results in holomorphic dynamics, it follows that $f^m(\Omega)$ contains a periodic point z_0 in its closure. Now, pick a sequence (K_n) as in the proof of Theorem IV.4.2. By the same argument in this proof, with $C_n=h_n^{-1}(z_0)$, we obtain a pre-periodic point of g on the unit circle. Then this point is periodic and $\varrho(g)\in\mathbf{Q}$, contradicting that $\varrho(g)=\lim_{n\to+\infty}\varrho(g_n)=\varrho(f)=\alpha\in\mathbf{R}-\mathbf{Q}$. □

Remark. (1) This result shows that there cannot exist three Siegel domains, one of them of the first kind, the other two bounded, forming Wada lakes. I am grateful to J. Rogers for pointing out this possibility which was overlooked in the preprint version of this article.

(2) In view of the proof, the hypothesis $\mathcal{S}(f)$ of the first kind seems too strong. When f is a polynomial and f is univalent in a neighborhood of $\partial\mathcal{S}(f)$, then $\mathcal{S}(f)$ is a Siegel domain of the first kind. I do not know if this still holds for an arbitrary rational map f.

IV.5. Centralizers

There has been some recent work on the symmetries of non-linearizable holomorphic dynamics. The linearizable situation can be characterized as the one having the maximal group of symmetries. A symmetry of a holomorphic germ is an element of the *centralizer*, $\mathrm{Cent}(f)$ of f in $\mathrm{Diff}(\mathbf{C},0)$, i.e. the subgroup of germs of holomorphic diffeomorphisms commuting with f. For an analytic circle diffeomorphism g, $\mathrm{Cent}(g)$ is the centralizer in the group of analytic circle diffeomorphisms. The study of centralizers generalizes the problem of linearization. The case of rational rotation numbers is completely understood ([Ec], [Vo], [Yo1]). In the case of irrational rotation numbers, $\varrho(f)\in\mathbf{R}-\mathbf{Q}$, the rotation number map $\varrho:\mathrm{Cent}(f)\to\mathbf{T}$ is an injective morphism of groups. In this way, we can identify $\mathrm{Cent}(f)$ with a subgroup $G(f)$ of $\mathbf{T}$. The map f is linearizable if and only if $G(f)=\mathbf{T}$ (this is easy, see [Pe2]). J. Moser has proved ([Mo]) that the rotation numbers of n non-linearizable commuting germs $f_1,...,f_n$, with $\varrho(f_i)=\alpha_i\in\mathbf{R}-\mathbf{Q}$ and $(\alpha_1,...,\alpha_n)$ rationally independent, must have good simultaneous rational approximations: for every $\gamma,\tau>0$, there exists a rational $p/q\in\mathbf{Q}$, such that,

$$\max_{1\leqslant i\leqslant n}\left|\alpha_i-\frac{p}{q}\right|\leqslant\frac{\gamma}{q^\tau}.$$

He proves the same result for analytic (or smooth) circle diffeomorphisms close to a rotation. On the other hand, the author has proven that these commuting germs really do exist ([Pe2]).

We can expect to link the two problems for holomorphic germs and analytic circle diffeomorphisms. Some technical points remain unsolved before we can get the full picture (see below). First we give a taste of how the machinery of Theorem 2 can be used to construct non-linearizable circle diffeomorphisms with an irrational rotation number and with uncountable centralizer, from the same object for holomorphic germs. We have already used this device in [Pe2], where we proved [Pe2]:

THEOREM. *There exists a holomorphic germ* $f\in \mathrm{Diff}(\mathbf{C},0)$, $\varrho(f)\in\mathbf{R}-\mathbf{Q}$, *having an uncountable centralizer. More precisely, there exists a Cantor set* $C\subset\mathbf{T}$, *and a commuting family* $(f_t)_{t\in C}$, $\varrho(f_t)=t$, *with* f_t *defined and univalent in* $\mathbf{D}$, *and a common Siegel compactum* K *for all the holomorphic maps* f_t, $t\in C$.

Using this common Siegel compactum and the fundamental construction, we obtain a commuting family of analytic circle diffeomorphisms parameterized by its rotation numbers.

The main point in the construction is to prove the existence of a common Siegel compactum for the family of commuting germs. We believe that this holds in general:([2])

CONJECTURE. *Let* $(f_t)_{t\in C}$ *be a family of commuting holomorphic germs defined and univalent in a fixed neighborhood of the origin. Then there exists an arbitrarily small common Siegel compactum* K *for all the elements of this family.*

The difficult point consists in proving this conjecture in the irrational non-linearizable situation. We would like to proceed as in the proof of Theorem 1, but we need to find a dense class of commuting families of holomorphic germs, and this is a difficult problem.

Assuming this conjecture we can link both problems using the fundamental construction. For instance, then it is straightforward to prove Moser's result for holomorphic germs from his local result for analytic circle diffeomorphisms.

V. Miscellanea

V.1. Siegel compacta, Herman compacta and hedgehogs

Theorem 1 provides the existence of Siegel compacta associated to an indifferent fixed point. For analytic circle diffeomorphisms we can do a local study in a neighborhood of the circle for rational or irrational rotation numbers satisfying a diophantine condition. With a similar proof we obtain the existence of Herman compacta associated to an

([2]) This conjecture has now been proved ([Pe6]).

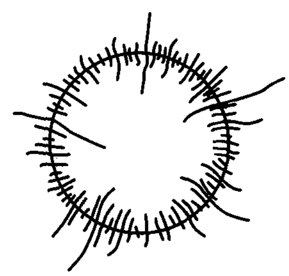

Fig. V.1

analytic circle diffeomorphism. We choose this terminology because we like to think of them as degenerate Herman rings.

THEOREM V.1.1 (Herman compacta). *Let g be an analytic diffeomorphism of the circle $\mathbf{S}^1$ such that g and g^{-1} are univalent in the closure of a Jordan annular neighborhood U of the circle. Then there exists a set K such that (Figure V.1)*

(i) *K is compact, connected and annular ($\mathbf{C}-K$ has two connected components),*

(ii) $\mathbf{S}^1 \subset K \subset \overline{U}$,

(iii) $K \cap \partial U \neq \varnothing$,

(iv) $g(K)=K$, $g^{-1}(K)=K$.

Moreover, if g is not of finite order then g is linearizable if and only if $\mathbf{S}^1 \subset K$.

We can prove the same result when g is holomorphic and leaves invariant a separating continuum in the complex plane with g univalent in a neighborhood of this continuum, and g inducing an action in the boundary of one component of the complement. To see this, we conjugate the map by a conformal representation from one component of the exterior of the continuum into $\mathbf{C}-\overline{\mathbf{D}}$, as in the fundamental construction (§I.1). With this construction we reduce the problem to the case of an analytic invariant circle.

We can formulate Theorem 1 and Theorem V.1.1 in a common statement as follows:

THEOREM V.1.2 (generalized compacta). *Let f and f^{-1} be defined and univalent in a neighborhood of the closure of a Jordan annulus U in an open annulus A with values in A, U being a neighborhood of one of its ends, which is invariant by f ($f(U)$ is a neighborhood of this end). Let $\hat{A}$ be the prime-end compactification of A. The map f has a continuous extension $\hat{f}$ from $\overline{U} \subset \hat{A}$ with values in $\hat{A}$. Then there exists a set $K \subset \hat{A}$ such that*

(i) *K is compact, connected and $\hat{A}-K$ has one connected component,*

(ii) *$C \subset K \subset \overline{U}$, where C is the compact set in $\hat{A}$ corresponding to the end invariant by f,*

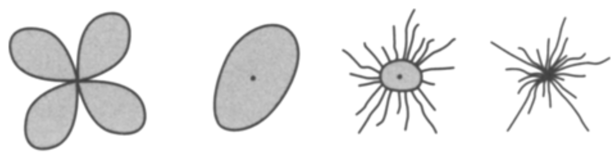

Fig. V.2

(iii) $K\cap(\partial U-C)\neq\varnothing$,
(iv) $f(K)=K$, $f^{-1}(K)=K$.
Moreover, if f is not of finite order, then f is linearizable if and only if $C\subset\mathring{K}$.

As we have already noted, the structure and the topology of Siegel compacta depend crucially on the rotation number. Figure V.2 shows the different kinds of Siegel compacta that we obtain for different rotation numbers.

Hedgehogs. The most difficult situation arises when $\varrho(f)\in\mathbf{R}-\mathbf{Q}$, f is not linearizable or has a "small" linearization domain. We will call them *hedgehogs* because of its complicate topological structure. We formulate the precise definition using Theorem V.1.2. In this respect, note that it is obvious how to define the notions of rotation number, linearizability and linearization domain for holomorphic maps f satisfying the conditions in Theorem V.1.2.

Definition (hedgehog). A hedgehog for a holomorphic map f satisfying the hypothesis of Theorem V.1.2 is an invariant compactum K for $\hat{f}$ obtained by Theorem V.1.2 when $\varrho(f)\in\mathbf{R}-\mathbf{Q}$ and f is non-linearizable or has a linearization domain relatively compact in U.

Hedgehogs having a linearization domain are called *linearizable* hedgehogs. Using the fundamental construction it is straightforward to prove that a hedgehog cannot be locally connected. We have studied the topology of these objects in [Pe4]. We have proven the following pathological properties (the statements are for a non-linearizable hedgehog in the complex plane):

THEOREM ([Pe4, Theorem 5]). *Let K be a hedgehog obtained from a non-linearizable holomorphic germ f with irrational rotation number using Theorem* 1. *We have*:
(i) *K is compact, connected and full,*
(ii) $0\in K$ *and* $\{0\}\neq K$,
(iii) *K is not locally connected at any point distinct from* 0,
(iv) *the impression of every prime end of* $\overline{\mathbf{C}}-K$ *contains* 0,

(v) *for each crosscut* γ *of* $\overline{\mathbf{C}}-K$, *the bounded domain defined by* γ *accumulates at* 0,

(vi) *except for a set of capacity zero, all the prime ends are of the second kind,*

(vii) $K-\{0\}$ *has an uncountable number of constituents,*

(viii) *the only possible biaccessible point in* K *is* 0.

We have a topological model for K which fits nicely with a large class of hedgehogs for a certain type of holomorphic germs (holomorphic germs of quadratic type, [Pe5]), but many questions remain open for the general hedgehog. The most important one is:

CONJECTURE. *For a hedgehog given by Theorem* V.1.2, C *is always accessible from the outside of* K.

Graduation of hedgehogs. In Theorem V.1.2 we can consider a monotone and continuous graduation $(U_r)_{0<r\leqslant 1}$ by open neighborhoods of the end of A fixed by f, which forms a basis of neighborhoods of this end. Applying Theorem V.1.2 to each open set U_r and denoting by K_r the maximal compact set given by this theorem for this neighborhood, we obtain a monotone sequence of continua $(K_r)_{0<r\leqslant 1}$. In this way we obtain a graduation of the compact set K_1 by a monotone sequence of compacta. As we have observed in §IV.2 (b), for $r<1$, K_r is of zero measure in the harmonic measure of K_1. This hints at the complexity of the hedgehog situation.(3)

V.2. Generalized Ghys construction

The fundamental construction (Theorem 2) associates an analytic circle diffeomorphism to a given Siegel compactum of a holomorphic germ. We would like to reverse this construction, and recover a holomorphic germ and a Siegel compactum from a given circle diffeomorphism.

It is easy to achieve this when g is analytically linearizable: let $h: \mathbf{S}^1 \to \mathbf{S}^1$ be the analytic linearization map, $h^{-1}\circ g\circ h = R_\alpha$, and paste $\overline{\mathbf{D}}$ (where R_α acts) to $\overline{\mathbf{C}}-\mathbf{D}$ (where g acts) by h. In this way we obtain a Riemann surface $\mathcal{S}$ which is a topological sphere. Thus by the Poincaré–Koebe uniformization theorem, it is biholomorphic to the Riemann sphere $\overline{\mathbf{C}}$. Choose a uniformization $k: \overline{\mathbf{C}} \to \mathcal{S}$, $k(0)=0$. The initial dynamic induces a dynamic on $\mathcal{S}$, which gives by conjugation by k a holomorphic germ f. The pasted disk $\overline{\mathbf{D}}$ becomes an invariant closed disk in the Siegel domain, i.e. a Siegel compactum of f.

E. Ghys has observed [Gh] that using the Morrey–Ahlfors–Bers rectification theorem it is possible to extend this construction to the case where the circle diffeomorphism is

(3) Recent results have been obtained in [Pe6] on this graduation: It is continuous in the Hausdorff topology and unique up to a parametrization!

only quasi-symmetrically linearizable. We recall that by Denjoy's theorem g is always topologically linearizable on the circle. Let $h: \mathbf{S}^1 \to \mathbf{S}^1$ be the linearization map, which is assumed to be quasi-symmetric. Then, by Beurling–Ahlfors ([Ah1]), h has a quasi-conformal extension $H: \mathbf{D} \to \mathbf{D}$ which is a homeomorphism with $H(0)=0$. We consider the Beltrami form on the Riemann sphere which vanishes on $\overline{\mathbf{C}} - \overline{\mathbf{D}}$ and is equal to the Beltrami form of H in $\mathbf{D}$. We consider the dynamics $F=g$ on $\overline{\mathbf{C}} - \mathbf{D}$ and $F = H \circ R_\alpha \circ H^{-1}$ on $\overline{\mathbf{D}}$. By construction, F preserves this Beltrami form, and using the Morrey–Ahlfors–Bers rectification theorem (the Beltrami form has modulus bounded away from 1 almost everywhere), we find a quasi-conformal homeomorphism $k: \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ which conjugates F to a holomorphic map f. The holomorphic germ f has a Siegel domain which corresponds to the pasted disk.

The first construction and the Ghys construction are conceptually different. We want now to make a construction which works for an arbitrary circle diffeomorphism g. The method is inspired by the first elementary approach and is based on a perturbation method. Fix a $\Delta > 0$ such that g is defined and univalent in the annulus $A_\Delta = \{z \in \mathbf{C} : 1 < |z| < e^{2\pi\Delta}\}$. We pick a sequence of analytic circle diffeomorphisms $g_n: \mathbf{S}^1 \to \mathbf{S}^1$ defined and univalent in A_Δ, converging uniformly to g, and having good rotation numbers, for example $\varrho(g_n) = \alpha_n \in \mathrm{D.C.}$ (we can take $g_n = e^{2\pi i \theta_n} g$ with $\theta_n \to 0$ conveniently chosen). The diffeomorphisms g_n are analytically linearizable by the global linearization theorem. Let $h_n: \mathbf{S}^1 \to \mathbf{S}^1$ be the analytic linearization map. Paste $\overline{\mathbf{D}}$ to $\overline{\mathbf{C}} - \mathbf{D}$ using h_n. We obtain a Riemann surface $\mathcal{S}_n$ with an induced univalent holomorphic dynamic F_n defined on the domain U_n corresponding to $\overline{\mathbf{D}} \bigsqcup_{h_n} A_\Delta$. The map F_n is defined by R_{α_n} on the piece corresponding to $\overline{\mathbf{D}}$, and by g_n on the piece A_n corresponding to A_Δ. The Riemann surface $\mathcal{S}_n$ is biholomorphic to the Riemann sphere. Using the Poincaré–Koebe uniformization theorem, we find $k_n: \overline{\mathbf{C}} \to \mathcal{S}_n$ such that $k_n(0)=0$, $k_n(\infty)=\infty$; and composing on the left by a linear automorphism of $\overline{\mathbf{C}}$, we can normalize k_n such that $k_n^{-1}(U_n) \supset \mathbf{D}$ and $\overline{\mathbf{D}}$ is not contained in $k_n^{-1}(U_n)$.

Conjugating F_n by k_n we obtain a linearizable holomorphic germ

$$f_n(z) = e^{2\pi i \alpha_n} z + \mathcal{O}(z^2)$$

univalent in the unit disk with values in $\mathbf{C}$. The pasted closed disk $\overline{\mathbf{D}}$ gives an invariant closed disk K_n in the linearization domain of f_n. By compactness, we can choose a subsequence such that f_{n_k} converges to a holomorphic map $f(z) = e^{2\pi i \alpha} z + \mathcal{O}(z^2)$ and K_{n_k} to a continuum K' in $\overline{\mathbf{C}}$.

We have $d = \operatorname{diam} K_n \geqslant e^{-2\pi\Delta}$ (diameter for the Euclidean distance in $\mathbf{C}$). Since if $K_n \subset \mathbf{D}_r$ then $\overline{\mathbf{D}} - \mathbf{D}_r \subset k_n^{-1}(A_n)$, so

$$\operatorname{mod}(\overline{\mathbf{D}} - \mathbf{D}_r) = \frac{1}{2\pi} \log r^{-1} \leqslant \operatorname{mod} k_n^{-1}(A_n) = \operatorname{mod} A_n = \Delta,$$

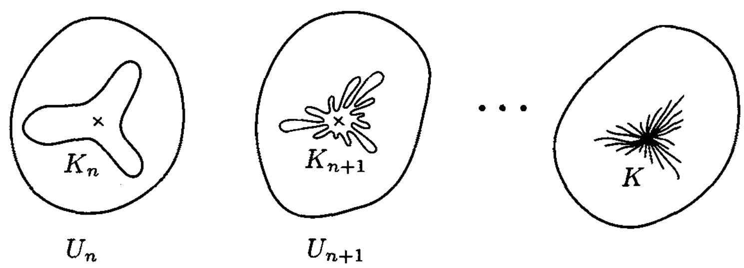

Fig. V.3

and finally $d \geqslant r \geqslant e^{-2\pi\Delta}$. This shows that $K' \neq \{0\}$. The component of the complement of the annulus $k_n^{-1}(A_n)$ in $\overline{\mathbf{C}}$ containing ∞ contains a point on the unit circle by the normalization of k_n that we have chosen. The compact set K_n is the other component and has a spherical diameter bounded below. Since the modulus of the annulus A_n is equal to $\Delta > 0$, we obtain that the spherical distance between the two components is larger than some fixed value $\varepsilon > 0$ depending only on Δ by [LV, Lemma 6.2, p. 34] (we can compute $\varepsilon = \min\left(\delta, 2\arctan\left(e^{-\pi^2\Delta}\tan\left(\frac{1}{2}\delta\right)\right)\right)$ with $\delta = \min\left(\frac{1}{4}\pi, \arctan(e^{2\pi\Delta})\right)\right)$.

Let $K = \widehat{K}'$ be the fill of K' ($\infty \notin K'$). We see that f is univalent in the ε-neighborhood of K for the spherical metric, and K is a Siegel compactum for f. Also the sequence of conformal representations conjugating f_{n_k} to g_{n_k} converges in the Carathéodory sense in an annulus surrounding K to a univalent map conjugating the dynamics of f on the exterior of K to the dynamics of g. Note that by the limiting process K can have empty interior (Figure V.3). Indeed, when f is non-linearizable we have $0 \in \partial K$. We do not have any information on the topology of K in the limit as opposed to Ghys's approach. Certainly this construction poses more questions than it solves. It is not our purpose to discuss these questions here, it is postponed to future work. Finally we can summarize the construction with

THEOREM V.2.1 (generalized Ghys construction). *Let g be an analytic circle diffeomorphism. There exists a holomorphic germ f and a Siegel compactum K associated to f (Theorem 1) such that when we apply the fundamental construction (Theorem 2) to f and K, we obtain an analytic circle diffeomorphism analytically conjugate to g. In terms of* §V.3 (a), *every analytic circle diffeomorphism is in the external class of some holomorphic germ.*

V.3. Intrinsic correspondence

V.3 (a). External equivalence

We consider the space of germs of holomorphic diffeomorphisms of $(\mathbf{C},0)$, denoted by $\mathrm{Diff}(\mathbf{C},0)$. From a dynamical point of view, we focus our interest on the space of analytic conjugacy classes in $\mathrm{Diff}(\mathbf{C},0)$. Two germs f_1 and f_2 are in the same class if there exists $h\in\mathrm{Diff}(\mathbf{C},0)$ such that $h^{-1}\circ f_1\circ h=f_2$. Note that in such case $\varrho(f_1)=\varrho(f_2)$. So the fibers of ϱ, $\varrho^{-1}(\alpha)=\mathrm{Diff}_\alpha(\mathbf{C},0)$, are compatible with this equivalence relation.

For our purposes, we need to consider an apparently different notion of equivalence that we call *external equivalence.*

Definition (external equivalence). Two holomorphic germs f_1 and f_2 are externally equivalent, $f_1\sim_e f_2$, if there exist two Siegel compacta K_1 and K_2, given by Theorem 1 for f_1 and f_2 respectively, such that when we apply the fundamental construction to (f_1,K_1) and (f_2,K_2), we obtain two analytic circle diffeomorphisms, g_1 and g_2 respectively, analytically conjugate on the circle.

We say that the circle diffeomorphism g_1 is in the *external class* of f_1.

This naturally suggests the definition of the external class of an analytic circle diffeomorphism g_1 of $\mathbf{S}^1$, as the set of all analytic circle diffeomorphisms g_2 which are analytically conjugate on the circle to g_2' which can be obtained from the fundamental construction (§V.1) applied to g_1 and a Herman compactum for g_1.

It is clear that the external relation in $\mathrm{Diff}(\mathbf{C},0)$ is reflexive and symmetric. The next lemma proves the transitivity. This shows that it is an equivalence relation.

LEMMA V.3.1. *The relation $\sim_e$ in* $\mathrm{Diff}(\mathbf{C},0)$ *is transitive.*

Proof. Let f_1, f_2 and f_3 in $\mathrm{Diff}(\mathbf{C},0)$ be such that $f_1\sim_e f_2$ and $f_2\sim_e f_3$. We denote by g_1, g_2, g_2' and g_3' the analytically conjugated circle diffeomorphisms obtained from the fundamental construction applied to (f_1,K_1), (f_2,K_2), (f_2,K_2') and (f_3,K_3') respectively. We consider K_2'', the filling of $K_2\cup K_2'$, which is a Siegel compactum for f_2. Since $K_2''\supset K_2$, there corresponds to K_2'', transporting by the conformal representations, a larger Siegel compactum $K_1''\supset K_1$ for f_1, as well as $K_3''\supset K_3$ for f_3. With obvious notation, we have that g_2'' is in the external class of g_2 and g_2', and g_1'' (or g_3'') is in the external class of g_1 (or g_3). The analytic conjugacy between g_2' and g_3' induces an analytic conjugacy between g_2'' and g_3''. The same happens for g_1'' and g_2''. Thus finally, g_1'' and g_3'' are analytically conjugate, showing that $f_1\sim_e f_3$. □

It is obvious from the definition that analytic equivalence implies external equivalence. It is not hard to see that the two notions coincide in $\mathrm{Diff}_\alpha(\mathbf{C},0)$ when $\alpha\in\mathbf{Q}$ and

$\alpha\in\mathcal{B}$ (in this last situation there is only one equivalence class for both relations). This motivates the following

CONJECTURE. *For any* $\alpha\in\mathbf{R}$ *the analytic equivalence coincides with the external equivalence in* $\mathrm{Diff}_\alpha(\mathbf{C},0)$.

Note that this is a strong, dynamical, removability conjecture for dynamical objects. We can define the notion of external equivalence in the space of analytic circle diffeomorphisms, $\mathrm{Diff}^\omega(\mathbf{S}^1)$. Two circle diffeomorphisms $g_1, g_2\in\mathrm{Diff}^\omega(\mathbf{S}^1)$ are externally equivalent, $g_1\sim_e g_2$, if their external classes (as defined previously) have a non-empty intersection. As above, we prove in a similar way that $\sim_e$ is an equivalence relation in $\mathrm{Diff}^\omega(\mathbf{S}^1)$. It is straightforward to show that two analytically equivalent circle diffeomorphisms are indeed externally equivalent. But the converse is false. For rotation numbers $\alpha\in\mathcal{B}-\mathcal{H}$ there exist two externally equivalent circle diffeomorphisms g_1 and g_2, with g_1 linearizable and g_2 non-linearizable. Thus external equivalence for analytic circle diffeomorphisms is a strictly larger equivalence relation.

V.3 (b). The correspondence

The purpose of this section is to distill the exact content of our fundamental construction.

We are interested in the quotient spaces $\mathrm{Diff}(\mathbf{C},0)/\sim_e$ (conjecturally the same as $\mathrm{Diff}(\mathbf{C},0)/\sim$) and $\mathrm{Diff}^\omega(\mathbf{S}^1)/\sim_e$. We endow $\mathrm{Diff}(\mathbf{C},0)$ and $\mathrm{Diff}^\omega(\mathbf{S}^1)$ with the inductive topology associated to an exhaustion by subspaces of elements defined and univalent in neighborhoods forming a basis of neighborhoods of 0 and $\mathbf{S}^1$ respectively. Each subspace is endowed with the topology of uniform convergence on compact sets.

From the results in [Yo3], we know that the quotient spaces are wild topological spaces when $\alpha\notin\mathcal{B}$, $\alpha\in\mathbf{R}-\mathbf{Q}$, because of the high variety of non-linearizable analytic classes accumulated by the linearizable class. The essence of our fundamental construction is to identify these spaces:

THEOREM V.3.1. *The spaces* $\mathrm{Diff}(\mathbf{C},0)/\sim_e$ *and* $\mathrm{Diff}^\omega(\mathbf{S}^1)/\sim_e$ *are isomorphic.*

Note that the quotient made at the level of analytic circle diffeomorphisms is in some sense bigger since the external equivalence classes are larger than the analytic ones. This explains why the Siegel center problem may appear as a singular case of the circle map case.

Proof. Consider a representative f of a class of $\mathrm{Diff}(\mathbf{C},0)$ modulo external equivalence. By the fundamental construction we obtain an element $g\in\mathrm{Diff}^\omega(\mathbf{S}^1)$. If we choose another representative we obtain another circle diffeomorphism which is externally equivalent to g. In this way we obtain a well-defined map Φ from $\mathrm{Diff}(\mathbf{C},0)/\sim_e$

to $\mathrm{Diff}^{\omega}(\mathbf{S}^1)/\sim_e$. By definition of the external equivalence, this map is injective. By the generalized Ghys construction (Theorem V.2.1), this map is surjective. The continuity of Φ is straightforward from the continuity in the fundamental construction, and the continuity of the inverse follows from the continuity of the generalized Ghys construction. □

V.4. Other applications of the generalized Ghys construction

An admissible Jordan domain U for f is a Jordan domain such that f and U satisfy the hypothesis of Theorem 1.

THEOREM V.4.1. *If $\alpha\notin\mathcal{H}$ then there exists a holomorphic germ $f(z)=\lambda z+\mathcal{O}(z^2)$, $\varrho(f)=\alpha$, such that $f\notin\mathcal{L}_0$, i.e. such that f is non-linearizable or f has a Siegel disk $\mathcal{S}(f)$ whose closure $\overline{\mathcal{S}(f)}$ has an admissible Jordan neighborhood U.*

Proof. Take $g\in S(\alpha,0)$ to be non-linearizable, and apply the generalized Ghys construction to g. Clearly, from the existence of an admissible Jordan annulus V for g (in the sense of Theorem V.1.1), we derive the existence of an admissible Jordan domain U for the limit map f. □

THEOREM V.4.2. *If $\alpha\notin\mathcal{H}'$ then there exists a holomorphic germ $f(z)=\lambda z+\mathcal{O}(z^2)$, $\varrho(f)=\alpha$, such that f has no periodic orbit than 0, and $f\notin\mathcal{L}_0$, i.e. such that f is non-linearizable or f has a Siegel disk $\mathcal{S}(f)$ whose closure $\overline{\mathcal{S}(f)}$ has an admissible Jordan neighborhood U.*

Proof. We do the same proof as for the preceding theorem. We start from $g\in S(\alpha,0)$ without periodic orbits. We need only to check that f has no periodic orbits except 0. The map f is the limit of univalent maps f_n without periodic orbits. Considering $\varphi_n(z)=(f_n(z)-z)/z$, we see that φ_n has no zeros. Moreover, $\varphi_n\to\varphi\neq\mathrm{id}$, $\varphi(z)=(f(z)-z)/z$, it follows that φ has no zeros in the domain of definition of f. □

Appendix 1. Local hypothesis

In classical linearization theorems for analytic circle diffeomorphisms, the local hypothesis is formulated as follows: Let $B_\delta=\{z\in\mathbf{C}:-\delta<\operatorname{Im} z<\delta\}$, $\mathcal{C}$ be some arithmetic condition, R_α be the translation by α, and let G denote a lift to $\mathbf{R}$ of an analytic circle diffeomorphism. Denote by $\varrho(G)\in\mathbf{R}$ the mean translation of G.

(LH1) Let $\alpha\in\mathcal{C}$ and $\delta>0$ be given. There exists $\varepsilon=\varepsilon(\alpha,\delta)>0$ such that if $\varrho(G)=\alpha$, G is holomorphic in B_δ and $\|G-R_\alpha\|_{C^0(B_\delta)}<\varepsilon$ then ... .

Here the ellipsis means the result implied by the local hypothesis.

The way we state the local hypothesis in this paper is different and was first introduced by J.-Ch. Yoccoz in [Yo4]. It is more geometric and better adapted to geometric techniques.

(LH2) Let $\alpha \in \mathcal{C}$. There exists $\Delta = \Delta(\alpha) > 0$ such that if $\varrho(G) = \alpha$ and G is defined and univalent in B_Δ then ... .

These two statements are apparently different, but are in fact equivalent as Yoccoz indicates ([Yo4]). For completeness, we provide the proof of this equivalence below.

Proof of (LH1) $\Rightarrow$ (LH2). Fix $\delta > 0$. Let $(\Delta_n)_{n \geqslant 0}$ be an increasing sequence of strictly positive numbers tending to $+\infty$, and G_n a sequence of lifts of circle diffeomorphisms with $\varrho(G_n) = \alpha$ which are univalent in B_{Δ_n}. By classical arguments the sequence of univalent maps $(G_n)_{n \geqslant n_0}$ forms a normal family on each annulus B_Δ where n_0 is large enough so that $\Delta_{n_0} > \Delta$. The only possible limit point of this sequence is a Möbius transformation on the Riemann sphere fixing ∞ and leaving invariant $\mathbf{R}$ with mean translation α. This shows that the sequence (G_n) will converge uniformly on each B_δ to the translation $R_\alpha : z \mapsto z + \alpha$. This shows that (LH2) implies (LH1).

Proof of (LH2) $\Rightarrow$ (LH1). We need to use a return-map construction. We can do the construction at the level of the real line $\mathbf{R}$. Pick a point $x_0 \in \mathbf{R}$ and consider the closest returns of its iterates (on the circle) $(G^{q_n}(x_0) - p_n)_n$. Fix some $n \geqslant 1$ and consider the fundamental segment I_n between x_0 and $G^{q_n}(x_0)$. We look at the return map on the corresponding fundamental region in the circle for the dynamics of g. More precisely, considering a vertical transversal l to $\mathbf{S}^1$ crossing at x_0 and its image $G^{q_n}(l) - p_n$, we can define a return map in a rectangle containing the arc I_n. The rotation number α being fixed (so also the sequence (q_n)), if G is close enough to the translation of angle R_α in B_δ then the height of the fundamental rectangle can be chosen to be of the order of δ. Pasting the boundaries l and $G^{q_n}(l) - p_n$ of this fundamental rectangle with height of the order of δ and base of length $G^{q_n}(x_0) - p_n$, we obtain an annulus of large modulus (as large as we want, choosing n large enough at the beginning and ε small enough depending on n). The segment I_n becomes an analytic circle on the annulus, the return map an analytic diffeomorphism on this circle and it is defined and univalent in an annulus of large modulus. When we lift the whole to $\mathbf{R}$, we obtain a lifting of a circle diffeomorphism satisfying (LH2).

Moreover, we should note that in all the applications the dynamical properties of the return map induce the same dynamical properties for the initial map, and this finishes the proof.

Appendix 2. Quantitative aspects of the theorem correspondence

Linearization theorems such as the ones overviewed in §§ I.2 and I.3 usually come with precise geometric estimates, usually on the size of the linearization domain. For these estimates, we need to introduce some arithmetic functions.

Let $\alpha\in\mathbf{R}-\mathbf{Q}$ and let $(\alpha_i)_{i\geqslant 0}$ be the sequence obtained by the continued-fraction algorithm (§I.1). Let $(\beta_i)_{i\geqslant -1}$ be the sequence defined by $\beta_{-1}=1$ and $\beta_i=\alpha_0 \ldots \alpha_i$. We define, over the irrationals, the following two functions with values in $\overline{\mathbf{R}}$:

$$\Phi(\alpha)=\sum_{j=0}^{+\infty}\beta_{j-1}\log\alpha_j^{-1},$$

$$\Psi(\alpha)=\sum_{j=0}^{+\infty}\beta_{j-1}\log^+\log\alpha_j^{-1}.$$

Now, for $f\in S(\alpha)$, we define $R(f)\geqslant 0$ as the radius of convergence of the linearizing map (which is equal to 0 when f is not linearizable), and

$$R(\alpha)=\inf_{f\in S(\alpha)}R(f),$$

$$R'(\alpha)=\inf_{f\in S'(\alpha)}R(f).$$

Then according to [Yo3], a precise version of the two propositions B.1 (i) and B.1 (ii) (Siegel–Brjuno–Yoccoz) is that there exists a universal constant $C>0$ such that

$$|\Phi(\alpha)-\log R(\alpha)^{-1}|\leqslant C.$$

In the same way, for B.2 (i) and B.2 (ii), we have, according to [Pe1, p. 604], that there exists a universal constant $C>0$ such that

$$|\Psi(\alpha)-\log R'(\alpha)^{-1}|\leqslant C.$$

Also, the value of $\Delta(\alpha)$ appearing in A.1 (i) is determined in [Yo4]. We can take $\Delta(\alpha)=(1/2\pi)\Phi(\alpha)+C$ (C is a universal constant which can be taken to be 3.15). Similarly for A.2 (i), we can take the value $\Delta(\alpha)=(1/2\pi)\Psi(\alpha)+C$.

Knowing this value in the circle-diffeomorphism situation, and making a quantitative version of the fundamental construction, as is done in §III.3, we can clearly obtain the optimal minoration for the radius $R(\alpha)$ and $R'(\alpha)$ shown above. This kind of optimal quantitative estimates hold also in the other situations where the fundamental construction applies.

References

[Ah1] AHLFORS, L. V., *Lectures on Quasiconformal Mappings.* Wadsworth & Brooks/Cole, Monterey, CA, 1987.

[Ah2] — *Conformal Invariants: Topics in Geometric Function Theory.* McGraw-Hill, New York–Düsseldorf–Johannesburg, 1973.

[Ar1] ARNOLD, V. I., Small denominators I: Mapping the circle onto itself. *Izv. Akad. Nauk. SSSR Ser. Mat.*, 25 (1961), 21–86; English translation in *Amer. Math. Soc. Transl. Ser. 2*, 46 (1965), 213–284.

[Ar2] — *Chapitres supplémentaires de la théorie des équations différentielles ordinaires.* "Mir", Moscow, 1980.

[Bie] BIELEFELD, B., Conformal dynamics problem list. Preprint, Stony Brook.

[Bir] BIRKHOFF, G. D., Surface transformations and their dynamical applications. *Acta Math.*, 43 (1922), 1–119; Also in: *Collected Mathematical Papers*, Vol. II, pp. 111–229.

[Bo] BOST, J.-B., Tores invariants des systèmes dynamiques hamiltoniens, in *Séminaire Bourbaki*, vol. 1984/85, exp. n° 639. *Astérisque*, 133–134 (1986), 113–157.

[Br] BRJUNO, A. D., Analytical form of differential equations. *Trans. Moscow Math. Soc.*, 25 (1971), 131–288; 26 (1972), 199–239.

[Cam] CAMACHO, C., On the local structure of conformal mapping and holomorphic vector fields in $\mathbf{C}^2$. *Astérisque*, 59–60 (1978), 83–94.

[Car] CARTAN, H., *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes.* Hermann, Paris, 1961.

[Ch] CHERRY, T. M., A singular case of iteration of analytic functions: a contribution to the small divisors problem, in *Non-linear Problems of Engineering* (W. F. Ames, ed.), pp. 29–50. Academic Press, New York, 1964.

[CLi] CARTHWRIGHT, M. L. & LITTLEWOOD, J. E., Some fixed point theorems. *Ann. of Math.*, 54 (1951), 1–37; Also in: *Collected Papers of J. E. Littlewood*, Vol. 1, pp. 38–74.

[CLo] COLLINGWOOD, E. F. & LOHWATER, A. J., *The Theory of Cluster Sets.* Cambridge Univ. Press, Cambridge, 1966.

[Do1] DOUADY, A., Systèmes dynamiques holomorphes, in *Séminaire Bourbaki*, vol. 1982/83, exp. n° 599. *Astérisque*, 105–106 (1983), 39–63.

[Do2] — Disques de Siegel et anneaux de Herman, in *Séminaire Bourbaki*, vol. 1986/87, exp. n° 677. *Astérisque*, 152–153 (1987), 151–172.

[Du] DULAC, H., Recherches sur les points singuliers des équations différentielles. *Journal de l'Ecole Polytechnique Sér. 2*, 9 (1904), 1–125.

[Ec] ECALLE, J., Les fonctions résurgentes et leurs applications. Publ. Math. Orsay, t. I, n° 81-05; t. II, n° 81-06; t. III, n° 85-05.

[Fa] FATOU, P., Sur les équations fonctionnelles. *Bull. Soc. Math. France*, 47 (1919), 161–271; 48 (1920), 33–94; 48 (1920), 208–314.

[Ga] GARNETT, J. B., *Applications of Harmonic Measure.* University of Arkansas Lecture Notes in the Mathematical Sciences, 8. John Wiley & Sons, New York, 1986.

[Gh] GHYS, E., Transformations holomorphes au voisinage d'une courbe de Jordan. *C. R. Acad. Sci. Paris Sér. I Math.*, 298 (1984), 385–388.

[He1] HERMAN, M. R., Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Inst. Hautes Études Sci. Publ. Math.*, 49 (1979), 5–234.

[He2] — Recent results and some open questions on Siegel's linearisation theorem of germs of complex analytic diffeomorphisms of $\mathbf{C}^n$ near a fixed point, in *VIIIth International Congress on Mathematical Physics* (Marseille, 1986), pp. 138–184. World Sci. Publishing, Singapore, 1987.

[He3] — Are there critical points on the boundaries of singular domains? *Comm. Math. Phys.*, 99 (1985), 593–612.

[HY] HOCKING, J. G. & YOUNG, G. S., *Topology*, 2nd edition. Dover, New York, 1988.

[Ju] JULIA, G., Mémoire sur la permutabilité des substitutions rationnelles. *Ann. Sci. École Norm. Sup.*, 39 (1922), 131–215; Also in: *Oeuvres*, vol. I, pp. 335–419. Gauthier-Villars, Paris, 1968.

[KH] KATOK, A. & HASSELBLATT, B., *Introduction to the Modern Theory of Dynamical Systems*. Encyclopedia Math. Appl., 54. Cambridge Univ. Press, Cambridge, 1995.

[La] LANG, S., *Introduction to Diophantine Approximations*. Addison-Wesley, Reading, MA–London–Don Mills, ON, 1966.

[Le] LEAU, L., Étude sur les equations fonctionnelles à une ou à plusieurs variables. *Ann. Fac. Sci. Toulouse Math. Sér. I*, 11 (1897), E.1–E.110.

[LV] LEHTO, O. & VIRTANEN, K. I., *Quasiconformal Mappings in the Plane*, 2nd edition. Grundlehren Math. Wiss., 126. Springer-Verlag, New York–Heidelberg, 1973.

[Ma] MATHER, J., Topological proofs of some purely topological consequences of Carathéodory's theory of prime ends, in *Selected Studies* (Th. M. Rassias, ed.), pp. 225–255. North-Holland, Amsterdam–New York, 1982.

[Mi] MILNOR, J., Dynamics in one complex variable: Introductory lectures. Preprint, Stony Brook, 1990/5.

[MM] MATTEI, J.-F. & MOUSSU, R., Holonomie et intégrales premières. *Ann. Sci. École Norm Sup.* (*4*), 13 (1980), 469–523.

[Mo] MOSER, J., On commuting circle mappings and simultaneous diophantine approximations. *Math. Z.*, 205 (1990), 105–121.

[MR] MARTINET, J. & RAMIS, J.-P., Classification analytique des équations différentielles non linéaires résonnantes du premier ordre. *Ann. Sci. École Norm. Sup.* (*4*), 4 (1984), 571–621.

[MZ] MONTGOMERY, D. & ZIPPIN, L., *Topological Transformation Groups*. Krieger, Huntington, NY, 1974.

[Na] NAISHUL, V. I., Topological invariants of analytic and area preserving mappings and their application to analytic differential equations in $\mathbf{C}^2$ and $\mathbf{CP}^2$. *Trans. Moscow Math. Soc.*, 42 (1983), 239–250.

[Ne] NEWMAN, M. H. A., *Elements of the Topology of Plane Sets of Points*. Dover, New York, 1992.

[Oh] OHTSUKA, M., *Dirichlet Problem, Extremal Length and Prime-ends*. Van Nostrand Reinhold, 1970.

[Pe1] PÉREZ-MARCO, R., Sur les dynamiques holomorphes non linéarisables et une conjecture de V. I. Arnold. *Ann. Sci. École Norm. Sup.* (*4*), 26 (1993), 565–644; See also: Sur la structure des germes holomorphes non linéarisables. *C. R. Acad. Sci. Paris Sér. I Math.*, 312 (1991), 533–536.

[Pe2] — Non-linearizable holomorphic dynamics having an uncountable number of symmetries. *Invent. Math.*, 119 (1995), 67–127; See also: Centralisateurs non dénombrables de germes de difféomorphismes holomorphes non linearisables de $(\mathbf{C}, 0)$. *C. R. Acad. Sci. Paris Sér. I Math.*, 313 (1991), 461–464.

[Pe3] — Solution complète au problème de Siegel de linéarisation d'une application holomorphe au voisinage d'un point fixe (d'après J.-C. Yoccoz), in *Séminaire Bourbaki*, vol. 1991/92, exp. n° 753. *Astérisque*, 206 (1992), 273–310.

[Pe4] — Topology of Julia sets and hedgehogs. Preprint, Université de Paris-Sud, 94-48 (1994).

[Pe5] — Holomorphic germs of quadratic type. To appear.

[Pe6] — Hedgehogs dynamics. To appear; A partial account in: Sur une question de Dulac et Fatou. *C. R. Acad. Sci. Paris Sér. I Math.*, 321 (1995), 1045–1048.

[Pe7] — Classification dynamique des compacts pleins invariants par un difféomorphisme holomorphe. Manuscript, 1996.

[Po] POMMERENKE, CH., *Boundary Behaviour of Conformal Maps.* Grundlehren Math. Wiss., 299. Springer-Verlag, Berlin, 1992.

[PY] PÉREZ-MARCO, R. & YOCCOZ, J.-CH., Germes de feuilletages holomorphes à holonomie prescrite, in *Complex Analytic Methods in Dynamical Systems* (Rio de Janeiro, 1992). *Astérisque*, 222 (1994), 345–371.

[Ro] ROGERS, J., Diophantine conditions imply critical points on the boundaries of Siegel disks of polynomials. Preprint, Tulane University.

[Rü] RÜSSMANN, H., Kleine Nenner II: Bemerkungen zur Newtonschen Methode. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 1972, 1–10.

[Sch] SCHMIDT, W. M., *Diophantine Approximation.* Lecture Notes in Math., 785. Springer-Verlag, Berlin–New York, 1980.

[Si] SIEGEL, C. L., Iterations of analytic functions. *Ann. of Math.*, 43 (1942), 807–812.

[SM] SIEGEL, C. L. & MOSER, J., *Lectures on Celestial Mechanics.* Grundlehren Math. Wiss., 187. Springer-Verlag, New York–Heidelberg, 1971.

[Su] SULLIVAN, D., Conformal dynamical systems, in *Geometric Dynamics* (Rio de Janeiro, 1981), pp. 725–752. Lecture Notes in Math., 1007. Springer-Verlag, Berlin–New York, 1983.

[Ts] TSUJI, M., *Potential Theory in Modern Function Theory.* Maruzen, Tokyo, 1959.

[Vo] VORONIN, S. M., Analytic classification of germs of conformal mappings $(\mathbf{C},0)\to(\mathbf{C},0)$ with identity linear part. *Functional Anal. Appl.*, 15:1 (1981), 1–17.

[Yo1] YOCCOZ, J.-CH., Centralisateurs et conjugaison différentiable des difféomorphismes du cercle. Thèse d'Etat, Université de Paris-Sud, 1985.

[Yo2] — Conjugation différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne. *Ann. Sci. École Norm. Sup.* (*4*), 17 (1984), 333–359.

[Yo3] — Théorème de Siegel, polynômes quadratiques et nombres de Brjuno. *C. R. Acad. Sci. Paris Sér. I Math.*, 306 (1988), 55–58.

[Yo4] — Conjugaison des difféomorphismes analytiques du cercle. Manuscript.

[Yo5] — An introduction to small divisors problems, in *From Number Theory to Physics* (Les Houches, 1989), pp. 659–679. Springer-Verlag, Berlin, 1992.

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