Acta Math., 178 (1997), 109–141

Witt vectors of non-commutative rings and topological cyclic homology

by

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Introduction

Classically, one has for every commutative ring A the associated ring of p-typical Witt vectors $W(A)$. In this paper we extend the classical construction to a functor which associates to any associative ring A an abelian group $W(A)$. The extended functor comes equipped with additive Frobenius and Verschiebung operators. We also define groups $W_n(A)$ of Witt vectors of length n in A. These are related by restriction maps $R: W_n(A) \to W_{n-1}(A)$ and $W(A)$ is the inverse limit. In particular, $W_1(A)$ is defined to be the quotient of A by the additive subgroup $[A, A]$ generated by elements of the form $xy-yx$, $x, y \in A$. There are natural exact sequences

$$0 \to A/[A,A] \xrightarrow{V^{n-1}} W_n(A) \xrightarrow{R} W_{n-1}(A) \to 0$$

which are useful both for proofs and calculations. We use these in Theorem 1.7.10 below to evaluate $W(A)$ when A is a free associative $\mathbf{F}_p$-algebra without unit. The sequences are usually not split exact, but in contrast to the classical case, this is not even true as functors from rings to sets, i.e. $W_n(A)$ is not naturally bijective to the n-fold product of copies of $A/[A,A]$. Finally, the construction $W(-)$ is Morita invariant:

$$W(M_n(A)) \cong W(A).$$

This more general algebraic structure arises naturally in topology: the topological cyclic homology defined by Bökstedt–Hsiang–Madsen, [BHM], associates to any ring A a (-2)-connected spectrum $\mathrm{TC}(A;p)$. We write $\mathrm{TC}_*(A;p) = \pi_* \mathrm{TC}(A;p)$. In §2 below we prove

Partially supported by an NSF grant

THEOREM A. *For any associative ring A,*

$$\mathrm{TC}_{-1}(A;p) \cong W(A)_F,$$

the coinvariants of the Frobenius endomorphism $F: W(A) \to W(A)$.

While it is unlikely that the higher groups $\mathrm{TC}_*(A;p)$ admits an algebraic description in general, this has been expected when A is an $\mathbf{F}_p$-algebra. Indeed, the original calculation of $\mathrm{TC}(\mathbf{F}_p;p)$ in [M] shows that $\mathrm{TC}(A;p)$ is a generalized Eilenberg–MacLane spectrum, i.e. that the k-invariants are trivial. And in the case of a smooth algebra over a perfect field of characteristic p, it follows from [H] that the groups $\mathrm{TC}_*(A;p)$ may be determined up to an extension from the de Rham–Witt complex of [I]. In §3 below, we evaluate the topological cyclic homology of a free associative $\mathbf{F}_p$-algebra without unit. It turns out to be concentrated in degree -1, and since the topological cyclic homology of a simplicial ring may be computed degreewise, we obtain the following general formula:

THEOREM B. *Let A be an associative $\mathbf{F}_p$-algebra. Then*

$$\mathrm{TC}_*(A;p) \cong L_{*+1} W(A)_F,$$

the left derived functors in the sense of Quillen of the functor $W(-)_F$.

In an earlier paper, [H], we evaluated the complex of p-typical curves in Quillen's K-theory in terms of the fixed sets of Bökstedt's topological Hochschild homology. In Corollary 3.3.6 below, we evaluate this complex for a free associative $\mathbf{F}_p$-algebra without unit. The resulting complex bears a close resemblance with the de Rham–Witt complex of a commutative polynomial algebra, as exhibited by Deligne–Illusie, [I], but in contrast to the latter, it is concentrated in degrees 0 and 1.

Let $K_*(A;\mathbf{Z}_p)$ denote the p-adic K-groups of the ring A, that is, the homotopy groups of the p-completion of the spectrum $K(A)$. Similarly, let $\mathrm{TC}_*(A;\mathbf{Z}_p)$ denote the homotopy groups of the p-completion of the spectrum $\mathrm{TC}(A;p)$. The main result of [HM] states that if k is a perfect field of positive characteristic p and A a finite algebra over the Witt ring $W(k)$, then the cyclotomic trace of [BHM] induces isomorphims

$$K_i(A;\mathbf{Z}_p) \cong \mathrm{TC}_i(A;\mathbf{Z}_p), \quad i \geqslant 0.$$

When A is an $\mathbf{F}_p$-algebra, $\mathrm{TC}(A;p)$ is already p-complete, so we have

THEOREM C. *Let k be a perfect field of positive characteristic p and let A be a finite associative k-algebra. Then the p-adic K-groups of A are given by*

$$K_i(A;\mathbf{Z}_p) \cong L_{i+1} W(A)_F,$$

the left derived functors of $W(-)_F$.

We note that at primes l different from p and rationally, one has $K_*(A;\mathbf{Z}_l)\cong K_*(A/J;\mathbf{Z}_l)$, where $J\subset A$ is the radical. Moreover, A/J being semi-simple, the latter splits as a product of l-adic K-groups of finite division algebras over k. These are known when k is a finite field by Wedderburn's theorem and Quillen's original calculation.

All rings considered in this paper will be associative but not necessarily commutative or unital unless otherwise stated. We shall write G for the circle group.

It is a pleasure to thank Mike Hopkins for the original suggestion to evaluate the topological cyclic homology of a free associative algebra as well as for several valuable conversations.

1. Witt vectors

1.1. For any commutative ring A, one has the associated ring of Witt vectors $W(A)$, [W]. In this paragraph, we extend the definition of $W(A)$ to all associative rings. If the ring A is not commutative, $W(A)$ will of course only be an additive group.

Let A be an associative ring and let $[A,A]$ denote the subgroup of commutators, that is, the additive subgroup generated by elements of the form $xy-yx$, $x,y\in A$. We recall the ghost map

$$w\colon A^{\mathbf{N}_0}\to (A/[A,A])^{\mathbf{N}_0} \tag{1.1.1}$$

given by the Witt polynomials

$$\begin{aligned} w_0 &= a_0,\\ w_1 &= a_0^p+pa_1,\\ w_2 &= a_0^{p^2}+pa_1^p+p^2a_2,\\ &\ \vdots \end{aligned}$$

Here p is a fixed prime. One may of course factor w through the equivalence relation which identifies two vectors a and b when their ghost components $w_n(a)$ and $w_n(b)$ are equivalent modulo commutators, for all $n\geqslant 0$. We shall see that it is possible to divide this requirement by p^n.

1.2. Let $B=\widetilde{\mathbf{Z}}\{x_1,\dots,x_s\}$ be the non-unital ring of polynomials in the non-commuting variables $x_1,\dots,x_s$ and recall that the non-unital free monoid Γ of all finite words in the variables $x_1,\dots,x_s$ forms an additive basis for B. The length of a word defines a grading on Γ and we let Γ_n denote the graded piece of degree n. The infinite cyclic group C acts on Γ_n by cyclically permuting the letters in a word. An orbit is called a *circular word* and two words are called *conjugate* if they are in the same orbit. We denote a typical word by $\widetilde{\omega}$ and a typical circular word by ω. Note that words are conjugate precisely when they viewed as monomials in B differ by a commutator. We refer to the element of a circular word which is smallest in the lexicographical order as the preferred representative. For any subset $S\subset\Gamma$, we let $\bar{S}$ denote the set of preferred representatives of the circular words given by the elements of S. We define a partition

$$\Gamma_n = \coprod_{d|n} \Gamma_{n,d}. \tag{1.2.1}$$

Here $\Gamma_{n,1}=\{x_1^n, ..., x_s^n\}$ is the subset of trivial words and $\Gamma_{n,d}$, $d>1$, is defined recursively as follows:

$$\begin{aligned}
\Gamma_{d,d} &= \Gamma_d - \coprod_{e|d,\, e\neq d} \Gamma_{d,e},\\
\Gamma'_{n,d} &= \bar{\Gamma}_{d,d} * \ldots * \bar{\Gamma}_{d,d} \quad (n/d \text{ factors}),\\
\Gamma_{n,d} &= C\cdot \Gamma'_{n,d}.
\end{aligned}$$

We define non-commutative polynomials

$$\Delta_d(x_1, ..., x_s) = \sum_{\widetilde{\omega}\in\bar{\Gamma}_{d,d}} \widetilde{\omega} \tag{1.2.2}$$

and write $\delta_k(x_1, ..., x_s)=\Delta_{p^k}(x_1, ..., x_s)$.

PROPOSITION 1.2.3. *For all* $n\geqslant 1$,

$$\left(\sum_{i=1}^s x_i\right)^n - \sum_{i=1}^s x_i^n = \sum_{d|n,\, d\neq 1} d\Delta_d(x_1, ..., x_s)^{n/d} + \varepsilon_n(x_1, ..., x_s),$$

with $\varepsilon_n(x_1, ..., x_s)$ *a commutator.*

Proof. Let t denote the generator of C and suppose that $\omega\in\Gamma'_{n,d}$. We claim that $t^m\widetilde{\omega}\in\Gamma'_{n,d}$ if and only if $d|m$. If d divides m then obviously $t^m\widetilde{\omega}\in\Gamma'_{n,d}$; we prove the converse. First note that if we write $\widetilde{\omega}=x_{i_0}x_{i_1}\ldots x_{i_{n-1}}$, then

$$i_{dk} \leqslant i_{dk+l},$$

whenever $0\leqslant k\leqslant n/d-1$ and $0\leqslant l\leqslant d-1$. If $m=de+r$ with $1\leqslant r\leqslant d-1$ and $t^m\widetilde{\omega}\in\Gamma'_{n,d}$ then also

$$i_{dk-r} \leqslant i_{dk-r+l},$$

for all $0\leqslant k\leqslant n/d-1$ and $0\leqslant l\leqslant d-1$. It follows that all the letters are equal, contradicting that $\widetilde{\omega}$ is non-trivial. Hence the claim.

From the definitions, $\Delta_d(x_1,...,x_s)^{n/d}=\sum_{\widetilde{\omega}\in\Gamma'_{n,d}}\widetilde{\omega}$, and so the claim shows that

$$d\Delta_d(x_1,...,x_s)^{n/d}=\sum_{\widetilde{\omega}\in\Gamma_{n,d}}\widetilde{\omega}+\varepsilon_{n,d}(x_1,...,x_s)$$

with $\varepsilon_{n,d}(x_1,...,x_s)\in[B,B]$. Now the proposition follows from (1.2.1). □

1.3. Let A be an associative ring and suppose that A has an additive endomorphism $\phi: A\to A$ which preserves the commutator subgroup and satisfies that for all $x\in A$ and all $n\geqslant 1$,

$$x^{p^n}\equiv\phi(x^{p^{n-1}})\quad(\text{modulo } p^nA+[A,A]). \tag{1.3.1}$$

We have the following non-commutative version of a lemma of Dwork.

LEMMA 1.3.2. *A sequence* $(w_0,w_1,...)$ *is in the image of the ghost map*

$$w: A^{\mathbf{N}_0}\to(A/[A,A])^{\mathbf{N}_0}$$

if and only if $w_n\equiv\phi(w_{n-1})$ *modulo* $p^nA+[A,A]$, *for all* $n\geqslant 1$.

Proof. If $(w_0,w_1,...)$ is in the image of w, then

$$\phi(w_{n-1})=\phi(a_0^{p^{n-1}})+p\phi(a_1^{p^{n-2}})+...+p^{n-1}\phi(a_{n-1}),$$

and so $w_n\equiv\phi(w_{n-1})$ modulo $p^nA+[A,A]$.

Conversely, we may assume by induction that there exist elements $a_0,...,a_{n-1}\in A$ such that

$$w_{n-1}\equiv a_0^{p^{n-1}}+pa_1^{p^{n-2}}+...+p^{n-1}a_{n-1}$$

modulo commutators. Since ϕ maps commutators to commutators and satisfies (1.3.1), we get that

$$\phi(w_{n-1})\equiv a_0^{p^n}+pa_1^{p^{n-1}}+...+p^{n-1}a_{n-1}^p$$

modulo $p^nA+[A,A]$. By assumption, $w_n\equiv\phi(w_{n-1})$ modulo $p^nA+[A,A]$, so we see that there exists a_n such that

$$w_n\equiv a_0^{p^n}+pa_1^{p^{n-1}}+...+p^na_n$$

modulo $[A,A]$. We note that the class of p^na_n modulo commutators is uniquely determined. □

Let $A=\widetilde{\mathbf{Z}}\{S\}$ be the free associative ring without unit generated by the set S and define $\phi: A\to A$ by

$$\phi\left(\sum a_{\widetilde{\omega}}\widetilde{\omega}\right)=\sum a_{\widetilde{\omega}}\widetilde{\omega}^p, \tag{1.3.3}$$

where the sum runs over finite words in S. Then ϕ is an additive endomorphism, which preserves commutators and (1.2.3) furnishes an induction argument which shows that (1.3.1) holds. Given a linear ordering of the set of variables S, we define a preferred section

$$\sigma_0: A/[A,A] \to A$$

of the projection as follows: a basis for $A/[A,A]$ is given by the set of circular words ω with letters in S. We define $\sigma_0\omega$ to be the preferred representative in the class ω and extend by linearity. In the proof of (1.3.2) we may choose a_n to be this preferred representative of its class modulo commutators. Therefore, we have

ADDENDUM 1.3.4. *If $A=\widetilde{\mathbf{Z}}\{S\}$ is the free associative ring without unit on a linearly ordered set S, then there is a preferred set section*

$$\sigma: w(A^{\mathbf{N}_0}) \to A^{\mathbf{N}_0}$$

of the ghost map. □

We define a new series of non-commutative polynomials r_s, $s\geqslant 0$. Here r_s is a polynomial in the variables x_{ij}, y_{ij}, where $i=0,1,\dots,s$; $j=0,1,\dots,n_i$. Let

$$\varepsilon_i = \sum_{j=0}^{n_i} x_{ij}y_{ij} - y_{ij}x_{ij}.$$

Then $r_0=\varepsilon_0$ and r_s, for $s\geqslant 1$, is defined by the recursion formula

$$r_n = \varepsilon_n - \sigma_0(p^{-n}(w_n(r_0,\dots,r_{n-1},0) - \phi(w_{n-1}(r_0,\dots,r_{n-1})))). \tag{1.3.5}$$

Now let A be any associative ring. The values of the polynomials r_s, $s\geqslant 0$, as the variables x_{ij} and y_{ij}, $i\geqslant 0$, $j=1,\dots,n_i$, run through all elements of A, define a subset

$$N(A) \subset A^{\mathbf{N}_0}. \tag{1.3.6}$$

We note that $N(A)$ is non-canonically bijective to $[A,A]^{\mathbf{N}_0}$. Indeed, choose a representation $\varepsilon=\sum x_jy_j - y_jx_j$ of each commutator in A. Then for a given $(r_0,r_1,\dots)\in N(A)$, we can define ε_i recursively using (1.3.5). The commutators ε_i, $i\geqslant 0$, obtained in this way are uniquely determined by the vector $(r_0,r_1,\dots)\in N(A)$. In particular, if $A'\to A$ is a surjective ring homomorphism then $N(A')\to N(A)$ is also surjective.

LEMMA 1.3.7. *The preimage of zero under the ghost map*

$$w\colon A^{\mathbf{N}_0} \to (A/[A,A])^{\mathbf{N}_0}$$

contains $N(A)$, and the two subsets are equal if $A/[A,A]$ has no p-torsion.

Proof. We show by induction that $w_n(r_0, ..., r_n) = 0$, the case $n=0$ being trivial. We have

$$w_n(r_0, ..., r_n) = w_n(r_0, ..., r_{n-1}, 0) + p^n r_n$$

and by (1.3.5) and induction $p^n r_n = -w_n(r_0, ..., r_{n-1}, 0)$ modulo commutators. Hence $N(A)$ is mapped to zero by the ghost map.

Conversely, choose a bijection of $N(A)$ and $[A,A]^{\mathbf{N}_0}$, and suppose that $(a_0, a_1, ...)$ is mapped to zero by the ghost map. Let us write $r'_n(\varepsilon_0, ..., \varepsilon_{n-1})$ for the second term on the right in (1.3.5). We must find a sequence $(\varepsilon_0, \varepsilon_1, ...)$ of commutators such that the following equality holds

$$a_n = \varepsilon_n - r'_n(\varepsilon_0, ..., \varepsilon_{n-1}).$$

We are given that for every $n \geqslant 0$,

$$a_0^{p^n} + p a_1^{p^{n-1}} + ... + p^n a_n \equiv 0$$

modulo commutators and may assume by induction that $a_i = r_i(\varepsilon_0, ..., \varepsilon_i)$, for $i \leqslant n-1$. The argument above implies that

$$r_0^{p^n} + p r_1^{p^{n-1}} + ... + p^{n-1} r_{n-1}^p \equiv p^n r'_n(\varepsilon_0, ..., \varepsilon_{n-1})$$

modulo commutators, and hence $p^n(a_n + r'_n(\varepsilon_0, ..., \varepsilon_{n-1}))$ is a commutator. Therefore, if $A/[A,A]$ has no p-torsion, $a_n + r'_n(\varepsilon_0, ..., \varepsilon_{n-1})$ is a commutator ε_n. This proves the induction step. □

1.4. We now consider the free associative ring without unit,

$$U = \widetilde{\mathbf{Z}}\{a_0, b_0, a_1, b_1, ...\},$$

with the generators ordered as indicated, and let $\phi\colon U \to U$ be as above. We define non-commutative sum and difference polynomials s_i and d_i by

$$\begin{aligned}(s_0, s_1, ...) &= \sigma(w(a_0, a_1, ...) + w(b_0, b_1, ...)),\\ (d_0, d_1, ...) &= \sigma(w(a_0, a_1, ...) - w(b_0, b_1, ...)).\end{aligned} \tag{1.4.1}$$

Then s_i is a non-commutative polynomial in the variables $a_0, b_0, ..., a_i, b_i$ and similarly for d_i. We note that if we map U to the commutative polynomial ring on the same set of generators, the polynomials $s_i(a,b)$ and $d_i(a,b)$ are mapped to the classical Witt sum and difference polynomials.

We define an equivalence relation on the set $A^{\mathbf{N}_0}$ of vectors in A by

$$a \sim b \quad \Leftrightarrow \quad d(a,b) \in N(A) \tag{1.4.2}$$

and note that the ghost map factors through it. For if $a \sim b$, then

$$w(a) - w(b) = w(d(a,b)) = 0.$$

LEMMA 1.4.3. *Let $\pi: A' \to A$ be a surjective ring homomorphism, and let a and b be two equivalent vectors in A. Then there exists equivalent vectors a' and b' in A' such that $\pi a' = a$ and $\pi b' = b$.*

Proof. In the equation $r = d(a,b)$ either two of the indeterminates determines the third. Indeed,

$$d_i(a_0, b_0, \dots, a_i, b_i) = a_i - b_i + \text{polynomial in } a_0, b_0, \dots, a_{i-1}, b_{i-1}.$$

Now let a and b be equivalent vectors in A and let $r = d(a,b)$; then $r \in N(A)$. We choose a vector a' in A' and $r' \in N(A')$ such that $\pi a' = a$ and $\pi r' = r$, and let b' be the unique solution of the equation $r' = d(a', b')$. Then $\pi b'$ satisfies the equation $r = d(a, \pi b')$, and hence $\pi b' = b$. $\square$

PROPOSITION 1.4.4. *The set of equivalence classes*

$$W(A) = A^{\mathbf{N}_0}/\sim$$

with the composition $a + b = s(a,b)$ is a functor from associative rings to abelian groups such that the ghost map $w: W(A) \to (A/[A,A])^{\mathbf{N}_0}$ is a natural homomorphism.

Proof. If $A/[A,A]$ has no p-torsion, then Lemma 1.3.7 shows that the ghost map

$$w: W(A) \to (A/[A,A])^{\mathbf{N}_0}$$

is an injection onto a subgroup and $w(s(a,b)) = w(a) + w(b)$. For a general A, we choose $\pi: A' \to A$ surjective with $A'/[A', A']$ p-torsion free. If a_1 and a_2 are equivalent vectors in A then by Lemma 1.4.3 we can find equivalent vectors a_1' and a_2' in A' such that $\pi a_1' = a_1$ and $\pi a_2' = a_2$. This shows that the composition $a + b = s(a,b)$ factors through the equivalence relation of (1.4.2). To prove associativity, given vectors a, b and c in A, we choose lifts a', b' and c' to vectors in A' and calculate

$$s(a, s(b,c)) = \pi s(a', s(b', c')) = \pi s(s(a', b'), c') = s(s(a,b), c).$$

The additive inverse of a is given by $d(0,a)$. One shows that this factors through the equivalence relation and verifies the remaining abelian group axioms in a similar manner. $\square$

Definition 1.4.5. We call $W(A)$ the group of Witt vectors over the ring A.

We note that for any pair of rings $W(A\times B)\cong W(A)\times W(B)$. In particular, if A is an algebra over a commutative ring k, we get a pairing

$$W(k)\times W(A)\to W(A), \tag{1.4.6}$$

and an argument similar to the proof of (1.4.4) shows that this makes $W(A)$ a module over the ring $W(k)$.

1.5. We let $U'=\widetilde{\mathbf{Z}}\{a_0,a_1,\dots\}$ with the variables ordered as indicated and note that U' contains the Witt polynomials $w_i=a_0^{p^i}+\dots+p^ia_i$. Let $f_i=f_i(a_0,a_1,\dots,a_{i+1})$ be the non-commutative polynomials given by

$$(f_0,f_1,\dots)=\sigma(w_1,w_2,\dots),$$

where σ is the section of Addendum 1.3.4. We have

$$f_0=a_0^p+pa_1,\quad f_1=(1-p^{p-1})a_1^p-\delta_1(a_0^p,pa_1)+pa_2,\quad \dots,$$

and in general the Kummer congruences show that $f_n\equiv a_n^p$ modulo p. Now let A be a ring and define the Frobenius operator by

$$F\colon W(A)\to W(A),\quad F(a_0,a_1,\dots)=(f_0,f_1,\dots). \tag{1.5.1}$$

An argument similar to the proof of Proposition 1.4.4 shows that F is well-defined and additive. We also note that if A is an $\mathbf{F}_p$-algebra, then $F(a_0,a_1,\dots)=(a_0^p,a_1^p,\dots)$. We define the accompanying Verschiebung operator by

$$V\colon W(A)\to W(A),\quad V(a_0,a_1,\dots)=(0,a_0,a_1,\dots). \tag{1.5.2}$$

Again this is well-defined and additive. Moreover,

$$FV=p.$$

Indeed, if F' and V' are operators on $(A/[A,A])^{\mathbf{N}_0}$ such that $F'w=wF$ and $V'w=wV$, then one easily calculates $F'V'=p$. Hence we have $FV=p$ on $W(A)$ whenever $A/[A,A]$ has no p-torsion. But taking Witt vectors preserves surjections, so the formula holds in general.

1.6. The relation polynomials r_i and the sum and difference polynomials s_i and d_i all have the property that they only depend on the variables x_{sj}, y_{sj} and a_s, b_s, respectively, with $s \leqslant i$. Therefore, we can repeat the construction of $W(A)$ starting from vectors of length n and get the group of Witt vectors of length n,

$$W_n(A) = A^n/\sim \tag{1.6.1}$$

with addition given by the first n sum polynomials,

$$(a_0, a_1, ..., a_{n-1}) + (b_0, b_1, ..., b_{n-1}) = (s_0, s_1, ..., s_{n-1}).$$

Therefore, $W(A)$ may be identified with the inverse limit of the $W_n(A)$ over the restriction maps

$$R: W_n(A) \to W_{n-1}(A), \quad R(a_0, ..., a_{n-1}) = (a_0, ..., a_{n-2}). \tag{1.6.2}$$

The Frobenius and Verschiebung operators reduce to

$$F: W_n(A) \to W_{n-1}(A), \quad V: W_{n-1}(A) \to W_n(A),$$

which, on the other hand, induce the original operators $F, V: W(A) \to W(A)$ on the limit over R.

PROPOSITION 1.6.3. *The sequences*

$$0 \to W_n(A) \xrightarrow{V^k} W_{n+k}(A) \xrightarrow{R^n} W_k(A) \to 0$$

are exact.

Proof. Let $N_s(A) \subset A^s$ denote the analogue of (1.3.6). We choose representations of all commutators in A such that we get a bijection between $N_s(A)$ and $[A, A]^s$. A representative of a typical element in the kernel of R^n has the form

$$a = (r_0, ..., r_{k-1}, a_k, ..., a_{n+k-1})$$

with $(r_0, ..., r_{k-1}) \in N_k(A)$. Let $\varepsilon_0, ..., \varepsilon_{k-1}$ be the corresponding sequence of commutators; let

$$r = (r_0, ..., r_{k-1}, r_k, ..., r_{n+k-1}) \in N_{n+k-1}(A)$$

be the vector which corresponds to the sequence of commutators $\varepsilon_0, ..., \varepsilon_{k-1}, 0, ..., 0$. We now let b be the unique solution to the equation $r = d(a, b)$. Then b is equivalent to a and by inspection of the difference polynomials, we see that b has the form

$$b = (0, ..., 0, b_k, ..., b_{n+k-1}).$$

This shows that $\ker R^n \subset \operatorname{im} V^k$. The opposite inclusion is trivial. $\square$

Remark 1.6.4. We note that $W_1(A)=A/[A,A]$, so in particular, Proposition 1.6.3 gives exact sequences

$$0 \to W_n(A) \xrightarrow{V} W_{n+1}(A) \xrightarrow{R^n} A/[A,A] \to 0.$$

We recall that when A is commutative, R^n has a natural multiplicative (but of course, non-additive) section given by the Teichmüller character,

$$\tau\colon A \to W_{n+1}(A), \quad \tau(a) = (a, 0, ..., 0). \tag{1.6.5}$$

In the non-commutative setting, we still have the map τ, but it does not in general factor over $A/[A,A]$. In fact, although $W_n(A)$ is bijective to the product $(A/[A,A])^n$, there exists no *natural* bijection. To see this, suppose that there were a natural set section

$$\nu\colon A/[A,A] \to W_2(A)$$

of the restriction, and consider the ring homomorphism $\phi\colon \widetilde{\mathbf{Z}}[c] \to \widetilde{\mathbf{Z}}\{x,y\}$ given by $\phi(c)= xy-yx$. Let us write $A=\widetilde{\mathbf{Z}}\{x,y\}$. By naturality, we would have a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathbf{Z}}[c] \arrow[r, "\nu"] \arrow[d, "\phi"] & W_2(\widetilde{\mathbf{Z}}[c]) \arrow[r, "F"] \arrow[d, "\phi"] & \widetilde{\mathbf{Z}}[c] \arrow[d, "\phi"] \\
A/[A,A] \arrow[r, "\nu"] & W_2(A) \arrow[r, "F"] & A/[A,A].
\end{tikzcd}$$

Since ν is a section of the restriction, $\nu(c)=(c, f(c))$ for some $f(c) \in \widetilde{\mathbf{Z}}[c]$, so $F(\nu(c))= c^p+pf(c)$. By the commutativity of the diagram, we have

$$(xy-yx)^p + pf(xy-yx) \equiv 0 \tag{1.6.6}$$

modulo commutators. We shall see that this is impossible. We have $f(c)=\sum a_n c^n$ and hence

$$f(xy-yx) = \sum_n a_n (xy-yx)^n.$$

Here $(xy-yx)^n$ is a homogeneous polynomial of degree $2n$, which is not a commutator unless $n=1$. Since $(xy-yx)^p$ is homogeneous of degree $2p$ we must have $f(c)=a_p c^p$, and then (1.6.6) becomes

$$(xy-yx)^p + pa_p (xy-yx)^p \equiv 0$$

modulo commutators. Now $(xy-yx)^p$ is divisible by p but not by p^2 modulo commutators. Therefore, this equation is not satisfied for any integer a_p.

1.7. Let S be a linearly ordered set and let $A=\widetilde{\mathbf{F}}_p\{S\}$ be the free associative $\mathbf{F}_p$-algebra without unit generated by S. In this paragraph we evaluate the group of Witt vectors $W(A)$. The calculation is inspired by Illusie's paper [I].

Consider the free associative $\mathbf{Q}_p$-algebra without unit generated by S,

$$L=\widetilde{\mathbf{Q}}_p\{S\};$$

we recall the structure of the Hochschild homology of L. Let Ω_0 be the set of circular words with letters in S, that is, the set of orbits of the action by the infinite cyclic group C on the set Γ of finite non-empty words in S by cyclically permuting the letters in words. The *period* of ω, by which we mean the length $\pi\omega$ of the orbit ω, divides the length $|\omega|$ of the word, and then

$$\mathrm{HH}_*(L/\mathbf{Q}_p)\cong\mathbf{Q}_p\langle\Omega_0\rangle\otimes\Lambda\{\varepsilon\},\quad \deg\varepsilon=1, \tag{1.7.1}$$

with Connes' B-operator given by the formula

$$B(\omega\otimes 1)=(|\omega|/\pi\omega)\,\omega\otimes\varepsilon.$$

The map $\phi\colon L\to L$ of (1.3.3) induces a map of the Hochschild groups which satisfies

$$B\phi=p\phi B. \tag{1.7.2}$$

Let D^* be the complex obtained from $\mathrm{HH}_*(L/\mathbf{Q}_p)$ by inverting ϕ and with the differential given by (1.7.2); we describe D^* in more detail.

Let C_n be the quotient of C of index n and note that

$$\Omega_0=\coprod_{n\geqslant 1}\mathrm{Map}(C_n,S)/C,$$

where the action by C on the set of maps is induced from the action on C_n. We let $\widehat{C}_p$ be the profinite and hence topological group

$$\widehat{C}_p=\varprojlim_n C_{p^n},$$

where the limit is over the natural projections. In other words, $\widehat{C}_p$ is the additive group of p-adic integers written multiplicatively. Then $C\subset\widehat{C}_p$ acts by multiplication and we define

$$\Omega=\coprod_{(d,p)=1}\mathbf{Z}\times\mathrm{Map}(\widehat{C}_p\times C_d,S)/C, \tag{1.7.3}$$

where the action by C on the mapping space is induced from the diagonal action on $\widehat{C}_p \times C_d$. We write elements of Ω as

$$\omega = (d; r, [\alpha]),$$

where d is a natural number prime to p, $r \in \mathbf{Z}$ and $[\alpha]$ is a C-orbit of continuous maps from $\widehat{C}_p \times C_d$ to S. By the *length* of ω we mean the rational number $|\omega| = p^r d$ and by the *period* we mean the length $\pi\omega$ of the orbit $[\alpha]$. The period is finite because α is continuous.

We identify $\Omega_0 \subset \Omega$ with the subset of those ω where $|\omega|$ is an integer and divisible by $\pi\omega$. More generally, we define $\Omega_n \subset \Omega$ to be the subset of those ω such that $p^n|\omega|$ is an integer divisible by $\pi\omega$ and note that

$$\Omega = \bigcup_{n \geqslant 0} \Omega_n. \tag{1.7.4}$$

We have bijections

$$f: \Omega_n \to \Omega_{n-1}, \quad f(d; r, [\alpha]) = (d; r+1, [\alpha]), \quad n \geqslant 1, \tag{1.7.5}$$

which induce a bijection $f: \Omega \to \Omega$, and the inclusion $\Omega_0 \subset \Omega_n$ followed by the bijection $f^n: \Omega_n \to \Omega_0$ is equal to $\phi^n: \Omega_0 \to \Omega_0$. Hence Ω is the set obtained from Ω_0 by inverting ϕ and

$$D^* = \mathbf{Q}_p\langle\Omega\rangle \otimes \Lambda\{\varepsilon\}, \quad \deg \varepsilon = 1,$$

with the differential B given by the formula (1.7.1).

We define a subcomplex $E^* \subset D^*$ as follows: call an element $\tau = \sum_{\omega \in \Omega} x_\omega \omega \otimes \varepsilon^i$ *integral* if $x_\omega \in \mathbf{Z}_p$, for all $\omega \in \Omega$, and let

$$E^* = \{\tau \in D^* \mid \tau \text{ and } B\tau \text{ are integral}\}. \tag{1.7.6}$$

More concretely, E^i, $i = 0, 1$, is a free $\mathbf{Z}_p$-module on the generators $e_i(\omega)$, one for each $\omega \in \Omega$, given by the formulas

$$\begin{aligned} e_0(\omega) &= \begin{cases} \omega \otimes 1, & \text{if } v_p(\pi\omega) \leqslant r, \\ p^{v_p(\pi\omega) - r} \omega \otimes 1, & \text{if } v_p(\pi\omega) > r, \end{cases} \\ e_1(\omega) &= \omega \otimes \varepsilon. \end{aligned} \tag{1.7.7}$$

The linear automorphism $F: D^* \to D^*$ induced from the bijection $f: \Omega \to \Omega$ and the endomorphism $V = pF^{-1}$ both restrict to monomorphisms of the subcomplex E^*, and moreover,

$$FV = VF = p, \quad BF = pFB, \quad VB = pBV. \tag{1.7.8}$$

The complex E^* has a decreasing filtration given by

$$\mathrm{Fil}^n E^i = V^n E^i + BV^n E^{i-1} \tag{1.7.9}$$

and we write $E_n^* = E^*/\mathrm{Fil}^n E^*$. In other words, $\mathrm{Fil}^n E^i$ is the free submodule of E^i on generators $p^m e_i(\omega)$, where $m = \min\{n, n+r-v_p(\pi\omega)\}$. We note that $E_1^* \cong \mathrm{HH}_*(A)$.

Let K be the free $\mathbf{Z}_p$-module generated by Ω_0 and note that $K \subset E^0$. We compose the restriction of the section $\sigma_0: A/[A,A] \to A$ to $\Omega_0 \subset A/[A,A]$ with the map $\tau: A \to W(A)$ to obtain a map $\iota_0: \Omega_0 \to W(A)$. We then extend this by linearity to a map

$$\iota_0: K \to W(A)$$

using that $W(A)$ is a $\mathbf{Z}_p$-module.

THEOREM 1.7.10. *The map ι_0 extends uniquely to a linear embedding $\iota: E^0 \to W(A)$ such that $V\iota = \iota V$. Moreover, this extended map induces isomorphisms*

$$\iota: E_n^0 \to W_n(A),$$

for all $n \geqslant 1$.

Proof. If $\omega \notin \Omega_0$ then we can write

$$e_0(\omega) = p^{v_p(\pi\omega)-r}\omega = V^{v_p(\pi\omega)-r}\omega_0,$$

where $\omega_0 = (d; v_p(\pi\omega), [\alpha]) \in \Omega_0$. It follows that

$$E^0 = \sum_{n \geqslant 0} V^n K, \tag{1.7.11}$$

from which the uniqueness of the extension immediately follows. On the other hand, if an extension exists, it must be given by $\iota = V^n \iota_0$ on the submodule $V^n K \subset E^0$. To see that this gives a well-defined map, we must show that

$$V^m \iota_0 = V^n \iota_0: V^m K \cap V^n K \to W(A).$$

Suppose that $m \leqslant n$. Then $V^m K \cap V^n K = V^m(K \cap V^{n-m} K)$, so we may assume that $m=0$. Suppose that $\omega_0 \in \Omega_0$ and let $x = V^n \omega_0$. We assume that $x \in K$. This means that if we write $x = p^n \omega$ then $\omega \in \Omega_0$. Now recall that Ω_0 is canonically bijective to the set of circular words in S and note that if $\tilde{\omega}$ is the preferred representative of ω, then the preferred representative of ω_0 is $\tilde{\omega}^{p^n}$. We find

$$\iota_0(x) = p^n(\tilde{\omega}, 0, \ldots) = F^n V^n(\tilde{\omega}, 0, \ldots) = (0, \ldots, 0, \tilde{\omega}^{p^n}, 0, \ldots) = V^n(\tilde{\omega}^{p^n}, 0, \ldots) = V^n(\iota_0(\omega_0))$$

and hence we get a well-defined map $\iota: E^0 \to W(A)$ which commutes with V.

To prove the second part of the theorem, we recall that $V: E^0 \to E^0$ is a monomorphism so that

$$V^n: E^0/VE^0 \to V^nE^0/V^{n+1}E^0$$

is an isomorphism. Moreover, we have $K \cap VE^0 \subset pK$ and also $pK = VFK \subset VK \subset K \cap VE^0$. This gives a map $K/pK \to E^0/VE^0$ which is an isomorphism by (1.7.11). Finally, $K/pK = A/[A,A]$ and the map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A/[A,A] & \xrightarrow{V^n} & E^0_{n+1} & \longrightarrow & E^0_n & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \\ 0 & \longrightarrow & A/[A,A] & \xrightarrow{V^n} & W_{n+1}(A) & \xrightarrow{R} & W_n(A) & \longrightarrow & 0 \end{array}$$

furnishes an induction argument which finishes the proof. □

COROLLARY 1.7.12. *The group of Witt vectors $W(A)$ is canonically isomorphic as a $\mathbf{Z}_p$-module to the set of infinite formal sums $\sum_{\omega\in\Omega} x_\omega e_0(\omega)$ with $x_\omega \in \mathbf{Z}_p$ such that $v_p(x_\omega) \geqslant v_p(\pi\omega) - r$ and for every $N \geqslant 0$ the set*

$$\{\omega \in \Omega \mid v_p(x_\omega) \leqslant N\}$$

is finite. Moreover, the Frobenius $F: W(A) \to W(A)$ is induced from the map $f: \Omega \to \Omega$ of (1.7.5).

2. Topological cyclic homology

2.1. In this paragraph we prove Theorem A of the introduction. The result was established for commutative rings in [HM]. The topological cyclic homology functor associates to any unital ring A a (-2)-connected spectrum $\mathrm{TC}(A;p)$. We first recall how this is defined and how one may extend the definition to all associative rings. For a thorough treatment see [HM]. In this paragraph, G will denote the circle group S^1. The finite subgroup of G of order r will be denoted C_r.

For any unital ring A, one has the topological Hochschild spectrum $T(A)$ defined by Bökstedt, [B]. This is a G-equivariant spectrum indexed on a complete G-universe $\mathcal{U}$ in the sense of [LMS]. Therefore, the obvious inclusion map

$$F_r: T(A)^{C_{rs}} \to T(A)^{C_s} \tag{2.1.1}$$

from the C_{rs}-fixed set to the C_s-fixed set is accompanied by a transfer map going in the opposite direction

$$V_r: T(A)^{C_s} \to T(A)^{C_{rs}}.$$

We call the maps F_r and V_r the rth Frobenius and Verschiebung, respectively. However, $T(A)$ has an additional structure: it is a *cyclotomic* spectrum, see [HM, §2]. In particular, there is an extra map, called restriction,

$$R_r: T(A)^{C_{rs}} \to T(A)^{C_s}. \tag{2.1.2}$$

The restriction and Frobenius maps satisfy

$$R_r R_s = R_{rs}, \quad F_r F_s = F_{rs}, \quad R_r F_s = F_s R_r, \tag{2.1.3}$$

and on the level of homotopy groups, one has in addition

$$R_r V_s = V_s R_r, \quad F_r V_r = r.$$

We shall often restrict attention to the p-subgroups C_{p^n} for some prime p. We then simply write R, F and V instead of R_p, F_p and V_p.

In general it is very difficult to analyze the fixed sets of an equivariant spectrum. However, for a cyclotomic spectrum, and in particular for $T(A)$, one has the following fundamental cofibration sequence of spectra

$$T(A)_{hC_{p^n}} \xrightarrow{N} T(A)^{C_{p^n}} \xrightarrow{R} T(A)^{C_{p^{n-1}}}. \tag{2.1.4}$$

The left-hand term is the homotopy orbit spectrum whose homotopy groups are approximated by a strongly convergent first-quadrant homology type spectral sequence

$$E^2_{s,t} = H_s(C_{p^n}; \pi_t T(A)) \Rightarrow \pi_{s+t} T(A)_{hC_{p^n}},$$

where $\pi_t T(A)$ is a trivial C_{p^n}-module.

Consider the functor

$$\mathrm{TR}(A;p) = \operatorname*{\underleftarrow{holim}}_{R} T(A)^{C_{p^n}}. \tag{2.1.5}$$

The Frobenius maps $F: T(A)^{C_{p^n}} \to T(A)^{C_{p^{n-1}}}$ induce a self-map of $\mathrm{TR}(A;p)$, which we also denote F, and now topological cyclic homology is defined as the homotopy-fixed set

$$\mathrm{TC}(A;p) = \mathrm{TR}(A;p)^{h\langle F\rangle}. \tag{2.1.6}$$

It is canonically equivalent to the homotopy fiber of $F - \mathrm{id}: \mathrm{TR}(A;p) \to \mathrm{TR}(A;p)$.

More generally, if $A \to B$ is a map of unital rings, the relative topological Hochschild spectrum

$$T(A \to B) = \mathrm{hofiber}(T(A) \to T(B))$$

is again a cyclotomic spectrum, so the above discussion applies to $T(A \to B)$ as well. Now if A is a possibly non-unital ring, we can form the associated unital ring $\mathbf{Z} \ltimes A$, which is $\mathbf{Z} \oplus A$ with multiplication given by the formula

$$(x,a)(x',a') = (xx', xa' + ax' + aa').$$

This is an augmented ring with augmentation ideal A and we define

$$T'(A) = T(\mathbf{Z} \ltimes A \xrightarrow{\varepsilon} \mathbf{Z}). \tag{2.1.7}$$

If A is unital, we have the ring homomorphism $\phi: \mathbf{Z} \ltimes A \to A$ given by $\phi(x,a) = x \cdot 1 + a$ and hence a ring isomorphism $\varepsilon \times \phi: \mathbf{Z} \ltimes A \to \mathbf{Z} \times A$. Moreover, the topological Hochschild spectrum preserves products such that we get a G-equivariant equivalence $T(\mathbf{Z} \ltimes A) \to T(\mathbf{Z}) \times T(A)$, i.e. for all closed subgroups $C \subset G$, the induced map of C-fixed point spectra is an equivalence. It follows that in this case, we have a canonical G-equivalence

$$T'(A) \to T(A).$$

We shall therefore simply write $T(A)$ for the spectrum in (2.1.7). As already mentioned $T(A)$ is a cyclotomic spectrum, so we can define $\mathrm{TR}(A;p)$ and $\mathrm{TC}(A;p)$ by the formulas (2.1.5) and (2.1.6), respectively. Finally, we note that since homotopy limits commute, we get a cofibration sequence of spectra

$$\mathrm{TC}(A;p) \to \mathrm{TC}(\mathbf{Z} \ltimes A;p) \xrightarrow{\varepsilon} \mathrm{TC}(\mathbf{Z};p). \tag{2.1.8}$$

If k is a unital ring and A is a k-algebra, we could also form the associated unital k-algebra $k \ltimes A$ and define a functor $T''(A) = T(k \ltimes A \xrightarrow{\varepsilon} k)$. However, since the topological Hochschild spectrum only depends on the underlying ring of a k-algebra, $T''(A)$ is canonically G-equivalent to $T'(A)$. So for k-algebras, we can replace $\mathbf{Z}$ by k in (2.1.8).

2.2. We note that for any ring A,

$$\pi_i T(A) = \mathrm{HH}_i(A), \quad i = 0, 1,$$

where we remember that $\mathrm{HH}_0(A) = A/[A,A]$. For commutative rings, $\mathrm{HH}_1(-)$ preserves surjections of rings, but for unital rings in general, this is not true. Instead one has an exact sequence

$$\mathrm{HH}_1(A) \to \mathrm{HH}_1(A/I) \to I/[I,A] \to A/[A,A]$$

for any two-sided ideal $I \subset A$.

LEMMA 2.2.1. *For any unital ring A, there is a unital ring B and a ring homomorphism $B \to A$ such that*

$$\mathrm{HH}_i(B) \to \mathrm{HH}_i(A), \quad i=0,1,$$

are surjections and such that $B/[B,B]$ is a free abelian group.

Proof. Let $V=\mathbf{Z}\{A\}$ be the free associative unital ring on the underlying set of A. Then $V/[V,V]$ is a free abelian group and $\mathrm{HH}_0(V) \to \mathrm{HH}_0(A)$ is surjective. However, $\mathrm{HH}_1(V) \to \mathrm{HH}_1(A)$ need not be a surjection. To obtain this, we first construct, for each cycle $z \in A \otimes A$, a ring homomorphism $U_z \to A$ such that the class of z is in the image of the induced map $\mathrm{HH}_1(U_z) \to \mathrm{HH}_1(A)$. Suppose that

$$z=\sum_{i=1}^{n} a_i \otimes b_i$$

and let U_z be the associative ring on generators $x_{i,z}$, $y_{i,z}$ $(i=1, \dots n_z)$ subject to the relation

$$\sum_{i=1}^{n} (x_{i,z} y_{i,z} - y_{i,z} x_{i,z}) = 0. \tag{2.2.2}$$

Then there is a unique ring homomorphism $U_z \to A$ which sends $x_{i,z}$ and $y_{i,z}$ to a_i and b_i, respectively. Under this map, the cycle $z'=\sum_{i=1}^{n} x_{i,z} \otimes y_{i,z}$ is mapped to z. Hence the class of z' is mapped to the class of z under the induced map

$$\mathrm{HH}_1(U_z) \to \mathrm{HH}_1(A).$$

Now let B be the generalized free product of V and the U_z as z runs through all cycles in $A \otimes A$, that is, the coproduct in the category of unital associative rings. Concretely, B is the associative ring on generators $a \in A$ and $x_{i,z}$, $y_{i,z}$, where z runs through all cycles in $A \otimes A$ and $i=1, \dots, n_z$, subject to the relations (2.2.2). We have ring homomorphisms $B \to A$ and, for each cycle z, $U_z \to B$ such that the composite $U_z \to B \to A$ is the original map $U_z \to A$. Hence the induced map

$$\mathrm{HH}_1(B) \to \mathrm{HH}_1(A)$$

is surjective. Finally, we choose a linear ordering of the set of generators. Then a basis for $B/[B,B]$ is given by the circular words in the variables a, $x_{i,z}$ and $y_{i,z}$ which does not contain a factor $x_{1,z} y_{1,z}$. □

PROPOSITION 2.2.3. *For any associative ring A, the sequence*

$$0 \to \pi_0 T(A) \xrightarrow{V^n} \pi_0 T(A)^{C_{p^n}} \xrightarrow{R} \pi_0 T(A)^{C_{p^{n-1}}} \to 0$$

is exact.

Proof. The fundamental cofibration sequence (2.1.4) induces a long-exact sequence of homotopy groups

$$\ldots \to \pi_1 T(A)^{C_{p^{n-1}}} \xrightarrow{\partial} \pi_0 T(A)_{hC_{p^n}} \xrightarrow{N} \pi_0 T(A)^{C_{p^n}} \xrightarrow{R} \pi_0 T(A)^{C_{p^{n-1}}} \to 0.$$

Moreover, the edge homomorphism of the spectral sequence of (2.1.4) gives an isomorphism

$$\iota_n \colon \pi_0 T(A) \to \pi_0 T(A)_{hC_{p^n}}$$

and [HM, Lemma 3.2] shows that $N \circ \iota_n = V^n$. We must show that the boundary map ∂ is trivial.

Suppose first that $A/[A,A]$ is p-torsion free. By (2.1.3) the composition

$$\pi_0 T(A) \xrightarrow{V^n} \pi_0 T(A)^{C_{p^n}} \xrightarrow{F^n} \pi_0 T(A)$$

is multiplication by p^n, and therefore in this case, V^n is injective. Suppose next that A is a unital ring and let $B \to A$ be as in Lemma 2.2.1. We consider the diagram

$$\begin{array}{ccccccc}
\pi_1 T(B)_{hC_{p^n}} & \xrightarrow{N} & \pi_1 T(B)^{C_{p^n}} & \xrightarrow{R} & \pi_1 T(B)^{C_{p^{n-1}}} & \xrightarrow{\partial} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_1 T(A)_{hC_{p^n}} & \xrightarrow{N} & \pi_1 T(A)^{C_{p^n}} & \xrightarrow{R} & \pi_1 T(A)^{C_{p^{n-1}}}. & &
\end{array}$$

The spectral sequence (2.1.4) gives an exact sequence

$$\mathrm{HH}_1(B) \xrightarrow{\iota_n} \pi_1 T(B)_{C_{p^n}} \to \mathrm{HH}_0(B)/p^n\,\mathrm{HH}_0(B) \to 0$$

and similar for A. It follows that $\pi_1 T(B)_{hC_{p^n}} \to \pi_1 T(A)_{hC_{p^n}}$ is surjective. An induction argument based on the diagram above now shows that $\pi_1 T(B)^{C_{p^m}} \to \pi_1 T(A)^{C_{p^m}}$ is surjective, for all $m \geqslant 0$, and hence

$$\partial \colon \pi_1 T(A)^{C_{p^{n-1}}} \to \pi_1 T(A)_{hC_{p^n}}$$

is trivial. This proves the proposition for any unital ring, and finally, the general case follows from (2.1.7) and the (3×3)-lemma. $\square$

For any ring A, there is a natural map of sets

$$\Delta_r\colon A \to \pi_0 T(A)^{C_r}. \tag{2.2.4}$$

This was defined in [HM, 3.3] for unital rings and extends by (2.1.7) and naturality to all rings. Let $\pi\colon A \to A/[A,A]$ be the projection. Then one has the following formulas, proved in [HM, Lemma 3.3.2]:

$$R_r \circ \Delta_r a = \pi a, \quad F_r \circ \Delta_r a = \pi a^r. \tag{2.2.5}$$

We consider the following map of sets

$$\tilde{I}\colon A^{[n-1]} \to \pi_0 T(A)^{C_{p^{n-1}}}, \quad \tilde{I}(a_0, \ldots, a_{n-1}) = \sum_{i=0}^{n-1} V^i(\Delta_{p^{n-1-i}}(a_i)), \tag{2.2.6}$$

where $[n-1] = \{0, 1, \ldots, n-1\}$. We also consider the map of spectra

$$\tilde{w}\colon T(A)^{C_{p^{n-1}}} \to T(A)^{[n-1]}, \tag{2.2.7}$$

which on the ith factor is given by $R^{n-1-i}F^i$. It induces an additive map on homotopy groups, which we also denote $\tilde{w}$, and the relations (2.2.5) show that

$$\tilde{w} \circ \tilde{I} = w\colon A^{[n-1]} \to (A/[A,A])^{[n-1]}, \tag{2.2.8}$$

where w is the ghost map of (1.1.1).

THEOREM 2.2.9. *For any associative ring A, the map $\tilde{I}$ factors to an isomorphism of abelian groups*

$$I\colon W_n(A) \to \pi_0 T(A)^{C_{p^{n-1}}}$$

which commutes with the operators R, F and V.

Proof. We show by induction that the map $\tilde{I}$ of (2.2.6) is surjective, the case $n=1$ being trivial. The fundamental cofibration sequence (2.1.4) gives an exact sequence

$$A \xrightarrow{V^{n-1}} \pi_0 T(A)^{C_{p^{n-1}}} \xrightarrow{R} \pi_0 T(A)^{C_{p^{n-2}}} \to 0$$

and it follows from [HM, Lemmas 3.3.1 and 3.3.2] that the image of $\tilde{I}(a_0, \ldots, a_{n-1})$ under R is equal to $\tilde{I}(a_0, \ldots, a_{n-2})$. Therefore as $a_0, \ldots, a_{n-2}$ vary, the elements $\tilde{I}(a_0, \ldots, a_{n-1})$ of $\pi_0 T(A)^{C_{p^{n-1}}}$ form a set of coset representatives of the image of A under V^{n-1}. Hence $\tilde{I}$ is surjective. In particular, it follows from (2.2.4) that the image of the homomorphism

$$\tilde{w}\colon \pi_0 T(A)^{C_{p^{n-1}}} \to (A/[A,A])^{[n-1]}$$

is equal to the image of the ghost map.

Suppose that $A/[A,A]$ has no p-torsion. Then by (1.3.6) the ghost map $w\colon W_n(A)\to (A/[A,A])^{[n-1]}$ is a monomorphism. Therefore, to prove the theorem in this case it suffices to prove that the same holds for the map $\tilde{w}$ above. Again we proceed by induction from the trivial case $n=1$. In the induction step, we use the following map of exact sequences

$$\begin{tikzcd} & \pi_0 T(A)_{hC_{p^{n-1}}} \arrow[r, "N"] \arrow[d, "\mathrm{trf}"] & \pi_0 T(A)^{C_{p^{n-1}}} \arrow[r, "R"] \arrow[d, "\tilde{w}"] & \pi_0 T(A)^{C_{p^{n-2}}} \arrow[r] \arrow[d, "\tilde{w}"] & 0 \\ 0 \arrow[r] & \pi_0 T(A) \arrow[r, "\iota"] & \pi_0 T(A)^{[n-1]} \arrow[r, "\pi"] & \pi_0 T(A)^{[n-2]} \arrow[r] & 0. \end{tikzcd}$$

The left-hand vertical map is the transfer associated with the projection

$$\mathrm{pr}\colon T(A)\wedge EC_{p^{n-1}} \to T(A)\wedge_{C_{p^{n-1}}} EC_{p^{n-1}},$$

and ι and π are the inclusion as the last coordinate and the projection away from the last coordinate, respectively. The left-hand square commutes by [HM, Lemma 3.2] and the right-hand square by [HM, Lemmas 3.3.1 and 3.3.2]. Moreover, the composition

$$\pi_0 T(A) \xrightarrow{\mathrm{pr}} \pi_0 T(A)_{hC_{p^{n-1}}} \xrightarrow{\mathrm{trf}} \pi_0 T(A)$$

is multiplication by p^{n-1} and hence trf is injective. Therefore, $\tilde{w}$ is injective by induction and the five lemma.

In the general case, we choose a ring epimorphism $A'\to A$ such that $A'/[A',A']$ has no p-torsion. Then $\pi_0 T(A')\to \pi_0 T(A)$ is onto and an induction argument based on the diagram

$$\begin{tikzcd} \pi_0 T(A') \arrow[r, "V^{n-1}"] \arrow[d] & \pi_0 T(A')^{C_{p^{n-1}}} \arrow[r, "R"] \arrow[d] & \pi_0 T(A')^{C_{p^{n-2}}} \arrow[r] \arrow[d] & 0 \\ \pi_0 T(A) \arrow[r, "V^{n-1}"] & \pi_0 T(A)^{C_{p^{n-1}}} \arrow[r, "R"] & \pi_0 T(A)^{C_{p^{n-2}}} \arrow[r] & 0 \end{tikzcd}$$

shows that so is $\pi_0 T(A')^{C_{p^{n-1}}}\to \pi_0 T(A)^{C_{p^{n-1}}}$. Hence $\tilde{I}$ factors to a surjection of abelian groups

$$I\colon W_n(A) \to \pi_0 T(A)^{C_{p^{n-1}}}.$$

Moreover, we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & W_1(A) \arrow[r, "V^{n-1}"] \arrow[d, "I"] & W_n(A) \arrow[r, "R"] \arrow[d, "I"] & W_{n-1}(A) \arrow[r] \arrow[d, "I"] & 0 \\ 0 \arrow[r] & \pi_0 T(A) \arrow[r, "V^{n-1}"] & \pi_0 T(A)^{C_{p^{n-1}}} \arrow[r, "R"] & \pi_0 T(A)^{C_{p^{n-2}}} \arrow[r] & 0 \end{tikzcd}$$

with the rows exact by (1.6.3) and (2.2.3), respectively. The claim now follows by induction and the five lemma. □

We recall that topological Hochschild homology and its fixed points are Morita invariant. This is proved in [B] and [BHM] for a unital ring, and the non-unital case follows easily from (2.1.7) and the fact that $M_n(\mathbf{Z}\ltimes A)\cong M_n(\mathbf{Z})\ltimes M_n(A)$. We may therefore conclude from (2.2.9) that there is a natural isomorphism

$$W_n(M_m(A))\cong W_n(A). \tag{2.2.10}$$

One would like also to have an algebraic proof of this fact.

We can now prove Theorem A of the introduction. The homotopy groups of the spectrum $\mathrm{TR}(A;p)$ defined in (2.1.5) are given by Milnor's exact sequence

$$0\to \varprojlim_n{}^{(1)}\pi_{i+1}T(A)^{C_{p^{n-1}}}\to \pi_i\mathrm{TR}(A;p)\to \varprojlim_n \pi_i T(A)^{C_{p^{n-1}}}\to 0.$$

For $i=0$ the maps in the limit system on the left are all surjective and hence the derived limit vanishes. Therefore, we obtain that for any associative ring

$$\mathrm{TR}_0(A;p)\cong W(A). \tag{2.2.11}$$

Finally, $\mathrm{TC}(A;p)$ is the homotopy fiber of $F-\mathrm{id}\colon \mathrm{TR}(A;p)\to\mathrm{TR}(A;p)$ and since $\mathrm{TR}(A;p)$ is a connective spectrum, Theorem A follows.

3. Free algebras

3.1. In this paragraph, we evaluate the topological cyclic homology of a free associative $\mathbf{F}_p$-algebra without unit and prove Theorem B of the introduction.

Let k be a unital ring and let S be a set. The free associative k-algebra with unit generated by S, which we denote $k\{S\}$, is the monoid algebra of the monoid $\langle S\rangle$ of all finite words in S under concatenation. We recall that for any monoid Γ, one has the cyclic bar-construction $N^{\mathrm{cy}}_{\bullet}(\Gamma)$ introduced by Waldhausen. It is a cyclic set in the sense of Connes with n-simplices

$$N^{\mathrm{cy}}_n(\Gamma)=\Gamma^{n+1} \tag{3.1.1}$$

and cyclic structure maps

$$\begin{aligned} d_i(\gamma_0,\dots,\gamma_n)&=\begin{cases}(\gamma_0,\dots,\gamma_i\gamma_{i+1},\dots,\gamma_n), & 0\leqslant i<n,\\ (\gamma_n\gamma_0,\gamma_1,\dots,\gamma_{n-1}), & i=n,\end{cases}\\ s_i(\gamma_0,\dots,\gamma_n)&=(\gamma_0,\dots,\gamma_i,1,\gamma_{i+1},\dots,\gamma_n),\quad 0\leqslant i\leqslant n,\\ t_n(\gamma_0,\dots,\gamma_n)&=(\gamma_n,\gamma_0,\dots,\gamma_{n-1}).\end{aligned}$$

In particular, $N^{\rm cy}_\bullet(\Gamma)$ is a simplicial set so we can take its geometric realization. The amazing fact about cyclic sets is that the realization carries a continuous action by the circle G, see e.g. [J].

We shall use Theorem 7.1 of [HM] to study the topological Hochschild homology of $k\{S\}$. It states that there is a natural equivalence of G-spectra indexed on $\mathcal{U}$,

$$T(k\{S\})\simeq_G T(k)\wedge|N^{\rm cy}_\bullet(\langle S\rangle)|_+, \tag{3.1.2}$$

where the smash product on the right is formed in the category of G-spectra indexed on $\mathcal{U}$. The infinite cyclic group C acts on $\langle S\rangle$ by cyclically permuting the letters in a word, and for each orbit ω, the subset of $N^{\rm cy}_\bullet(\langle S\rangle)$ consisting of those simplices $(\tilde\omega_0,\dots,\tilde\omega_n)$ for which the product $\tilde\omega_0*\dots*\tilde\omega_n\in\omega$ is preserved under the cyclic structure maps. We denote this cyclic subset by $N^{\rm cy}_\bullet(\langle S\rangle;\omega)$ and note the splitting

$$N^{\rm cy}_\bullet(\langle S\rangle)=\coprod_{\omega\in\Omega_0}N^{\rm cy}_\bullet(\langle S\rangle;\omega). \tag{3.1.3}$$

The realization decomposes correspondingly. Recall the notion of length and period of circular words from §1.7.

LEMMA 3.1.4. *Let $C(\omega)$ denote the cyclic group of order $|\omega|/\pi\omega$. There is a G-equivariant equivalence*

$$|N^{\rm cy}_\bullet(\langle S\rangle;\omega)|\simeq_G S^1/C(\omega)_+,$$

which depends on a choice of representative for the circular word ω.

Proof. We choose a representative $\tilde\omega$ for the orbit ω and write $|\omega|=n+1$. If $\sigma=(\tilde\omega_0,\dots,\tilde\omega_k)$ is a simplex in $N^{\rm cy}_\bullet(\langle S\rangle;\omega)$, there exists by definition $u_\sigma\in C$ such that $u_\sigma\cdot\tilde\omega_0*\dots*\tilde\omega_k=\tilde\omega$. Hence the simplex σ is determined by the following data: the ordered partition $(|\omega_0|,\dots,|\omega_k|)$ of $|\omega|$ and the element $u_\sigma\in C$, or equivalently, a weakly increasing function $\theta_\sigma\colon[n]\to[k]$ and the element $u_\sigma\in C$. Moreover, two simplices σ and σ' are equal if and only if θ_σ and $\theta_{\sigma'}$ are equal and the product $u_\sigma^{-1}u_{\sigma'}$ acts trivially on ω. We also note that ω is a transitive C-set with isotropy group the subgroup of C of index $\pi\omega$.

We recall that the cyclic category $\mathbf{\Lambda}$ has the same objects as the simplicial category $\mathbf{\Delta}$ but more maps: the automorphism group of $[n]$ is cyclic of order $n+1$ with a preferred generator τ_n and any morphism $f\in\mathbf{\Lambda}([k],[n])$ decomposes uniquely as $f=u\theta$, with $u\in\mathrm{Aut}_{\mathbf{\Lambda}}([k])$ and $\theta\in\mathbf{\Delta}([k],[n])$. Also recall the standard cyclic set

$$\Lambda[n]=\mathbf{\Lambda}(-,[n]),$$

which is the free cyclic set generated by the identity $\iota_n\colon [n]\to[n]$. Suppose that $\widetilde{\omega}=x_0\dots x_n$ with $x_i\in S$. Then there is a unique cyclic map

$$\alpha\colon \Lambda[n]\to N^{\mathrm{cy}}_\bullet(\langle S\rangle;\omega)$$

which maps ι_n to the n-simplex $(x_0,\dots,x_n)$. The automorphism group of $[n]$ acts on $\Lambda[n]$ through cyclic maps; let $C(\omega)$ denote the subgroup of index $\pi\omega$. It follows readily from the characterization of the simplices in $N^{\mathrm{cy}}_\bullet(\langle S\rangle;\omega)$ that α factors to an isomorphism of cyclic sets

$$\bar{\alpha}\colon \Lambda[n]/C(\omega)\to N^{\mathrm{cy}}_\bullet(\langle S\rangle;\omega).$$

The realization of $\Lambda[n]$ is homeomorphic to $S^1\times\Delta^n$, where G acts by multiplication in the first variable. Moreover, the homeomorphism may be chosen such that $\tau_n\in\mathrm{Aut}_\Lambda([n])$ acts by the formula

$$\tau_n(x;u_0,\dots,u_n)=\left(x-\frac{1}{n+1};u_1,\dots,u_n,u_0\right),$$

when we identify S^1 with $\mathbf{R}/\mathbf{Z}$, see [HM, 7.2]. It follows that we have a G-equivariant homeomorphism

$$|N^{\mathrm{cy}}_\bullet(\langle S\rangle;\omega)|\cong S^1\times_{C(\omega)}\Delta^n.$$

Finally, $S^1/C(\omega)$ is a strong G-equivariant deformation retract of $S^1\times_{C(\omega)}\Delta^n$. □

If the set S is linearly ordered, then we have a preferred representative of ω. For later reference we note that if a word $\widetilde{\omega}$ is a preferred representative then so is any iterated concatenation $\widetilde{\omega}*\dots*\widetilde{\omega}$ of it.

PROPOSITION 3.1.5. *Let k be a ring and let $\tilde{k}\{S\}$ be the free associative k-algebra without unit generated by a linearly ordered set S. Then there is a preferred equivalence of G-equivariant spectra indexed on $\mathcal{U}$,*

$$T(\tilde{k}\{S\})\simeq_G\bigvee_{\omega\in\Omega_0}T(k)\wedge S^1/C(\omega)_+,$$

where the wedge runs over the set Ω_0 of non-empty circular words in S.

Proof. The associative k-algebra with unit $k\{S\}$ is augmented over k and the augmentation ideal is the free associative algebra without unit generated by S. Moreover, the map $T(k\{S\})\to T(k)$ induced by the augmentation corresponds under (3.1.2) to the map which collapses all the non-trivial summands ω in (3.1.3) to the base point. Now the claim follows from (2.1.7) □

3.2. Let T be a G-spectrum indexed on a complete G-universe $\mathcal{U}$ and let $j: \mathcal{U}^G \to \mathcal{U}$ be the inclusion of the trivial universe. We write j^*T for the G-spectrum indexed on $\mathcal{U}^G$ obtained by forgetting the value of T on non-trivial representations. We also call j^*T a spectrum with a G-action or a naive G-spectrum. In this paragraph, we determine the structure of the C_r-fixed point spectrum

$$j^*(T \wedge S^1/C_{s+})^{C_r},$$

which is a spectrum with a G/C_r-action.

The rth root defines a group isomorphism $\varrho: G \to G/C_r$, which allows us to view a spectrum D with a (G/C_r)-action as a spectrum $\varrho^*_{C_r} D$ with a G-action. If T is a spectrum with a G-action, we write $T(i)$ for the spectrum T with G acting through the ith power map.

Proposition 3.2.1. *Let T be a G-spectrum indexed on $\mathcal{U}$. For positive integers r and s, let $d=(r,s)$ be the greatest common divisor and write $r'=r/d$ and $s'=s/d$. Then for every pair of integers m and n such that $mr+ns=d$ there is a natural non-equivariant equivalence*

$$\varrho^*_{C_r} j^*(T \wedge S^1/C_{s+})^{C_r} \simeq \varrho^*_{C_{r'}}(G(r's')_+ \wedge \varrho^*_{C_d} j^* T^{C_d}(mr'))$$

given by a chain of equivariant maps of spectra with a G-action.

Proof. We recall from [LMS, p. 89] the duality equivalence of G-spectra indexed on $\mathcal{U}$,

$$T \wedge S^1/C_{s+} \simeq_G \Sigma F(S^1/C_{s+}, T).$$

It induces, in particular, an equivalence of spectra with a G-action

$$j^*(T \wedge S^1/C_{s+}) \simeq_G j^* \Sigma F(S^1/C_{s+}, T) = \Sigma F(S^1/C_{s+}, j^*T).$$

We evaluate the C_r-fixed points of the spectrum on the right. To this end, we recall that if G is any group, $H \subset G$ a closed subgroup and X a left G-space, then the function space $F(G/H_+, X)$ carries both a left G-action and a left action by the Weyl group WH. The G-action is by conjugation and the left WH-action is induced from the right action of WH on the canonical orbit G/H. Moreover, evaluation in H defines a WH-equivariant homeomorphism

$$X^H \cong F(G/H_+, X)^G.$$

If we apply this space-wise in the spectrum at hand, we get a (G/C_r)-equivariant isomorphism

$$F(G/C_{s+}, j^*T)^{C_r} \cong F(G/C_{r+}, F(G/C_{s+}, j^*T))^G \cong F((G/C_r \times G/C_s)_+, j^*T)^G.$$

For a spectrum D with a $(G\times G/C_r)$-action and integers i and j, we write $D(i,j)$ for the spectrum D with a new $(G\times G/C_r)$-action given by $(g_1,g_2)\cdot x=g_1^i g_2^j x$. With this notion in hand, the spectrum on the right-hand side of the equation above may be written more precisely as

$$F((G/C_r(1,1)\times G/C_s(1,0))_+, j^*T(1,0))^{G\times C_r}.$$

We note the $(G\times G/C_r)$-equivariant homeomorphism

$$G/C_r(1,1)\times G/C_s(1,0)\cong G(r,r)\times G(s,0),$$

which raises the first and second coordinate to the rth and sth powers, respectively. The choice of m and n with $mr+ns=d$ specifies a linear isomorphism of the torus on the right-hand side above,

$$G(r,r)\times G(s,0)\cong G(d,mr)\times G(0,-rs/d),\quad (z,w)\mapsto (z^m w^n, z^{-s/d}w^{r/d}), \tag{3.2.2}$$

and the map

$$F((G(d,mr)\times G(0,-rs/d))_+, j^*T(1,0))^{G\times C_r}\to F(G(rs/d)_+, j^*T^{C_d}(mr/d)),$$

which takes a function Φ to the function ϕ given by $\phi(w)=\Phi(1,w)$, is a (G/C_r)-equivariant isomorphism when G/C_r acts on the function spectrum on the right by conjugation. We can view this as a spectrum with a $(G/C_{r'})$-action via the dth root map $\varrho_{C_d}\colon G/C_{r'}\to G/C_r$,

$$\varrho_{C_d}^*F(G(rs/d)_+, j^*T^{C_d}(mr/d)) = F(\varrho_{C_d}^*G(rs/d)_+, (\varrho_{C_d}^*j^*T^{C_d})(mr/d)),$$

so in all we obtain an isomorphism of spectra with a G-action

$$\varrho_{C_r}^*F(G/C_{s+}, j^*T)^{C_r}\cong \varrho_{C_{r'}}^*F(G(r's')_+, (\varrho_{C_d}^*j^*T^{C_d})(mr')).$$

Finally, the equivalence of [LMS, p. 89] gives us a chain of G-maps which induces a non-equivariant equivalence

$$\Sigma\varrho_{C_{r'}}^*F(G(r's')_+, (\varrho_{C_d}^*j^*T^{C_d})(mr'))\simeq \varrho_{C_{r'}}^*(G(r's')_+\wedge\varrho_{C_d}^*j^*T^{C_d})(mr')).$$

In effect, this is just Spanier–Whitehead duality, but given by a chain of equivariant maps. □

Remark 3.2.3. Suppose that m',n' is another pair of integers such that $m'r+n's=d$, say, $m'=m+ks$ and $n'=n-kr$. Then there is an isomorphism of spectra with a G-action

$$k_*\colon \varrho_{C_{r'}}^*(G(r's')_+\wedge\varrho_{C_d}^*j^*T^{C_d})(mr'))\to \varrho_{C_{r'}}^*(G(r's')_+\wedge\varrho_{C_d}^*j^*T^{C_d})(m'r')),$$

given by $k_*(g,t)=(g,g^k t)$, and as one readily verifies, the equivalences of (3.2.1) are compatible, for varying choices of m and n, with these isomorphisms.

Let $\sigma\in\pi_1^S(G_+)$ be the element which reduces to zero in $\pi_1^S(S^0)$ and to the identity in $\pi_1^S(S^1)$. We get a degree-one map

$$\delta\colon \pi_*T \xrightarrow{\sigma} \pi_{*+1}(G_+\wedge T) \xrightarrow{\mu} \pi_{*+1}T \tag{3.2.4}$$

as the composition of exterior multiplication by σ and the map induced from the action map. More generally, for $C\subset G$ a finite subgroup, we may apply the construction above to the naive G-spectrum $\varrho_C^* T^C$ and get a map $\delta\colon\pi_*T^C\to\pi_{*+1}T^C$. We recall from [H] that δ is a differential provided that multiplication by $\eta\in\pi_1^S$ on π_*T is trivial. In general, one has $\delta\delta=\eta\delta$.

COROLLARY 3.2.5. *A pair of integers m and n with $mr+ns=d$ determines an isomorphism*

$$\alpha_{mn}\colon \pi_*(T\wedge S^1/C_{s+})^{C_r}\to\pi_*T^{C_d}\oplus\pi_{*-1}T^{C_d},$$

and if also $m'r+n's=d$, then $(\alpha_{m'n'}\circ\alpha_{mn}^{-1})(a,b)=(a+k\delta b,b)$, where $ks=m'-m$.

Proof. The underlying non-equivariant spectrum of the naive G-spectrum on the right-hand side of (3.2.1) is equal to $S^1_+\wedge T^{C_d}$ independently of the choice of m and n. Hence

$$\pi_*(T\wedge S^1/C_{s+})^{C_r}\cong\pi_*T^{C_d}\oplus\pi_{*-1}T^{C_d},$$

where we use σ to identify the right-hand side as a direct sum. The isomorphism, however, depends on m and n, and different choices differ by the isomorphism of (3.2.3). The claim now follows from the definition of δ. $\square$

Suppose that r' is a divisor in r. We next evaluate the map on homotopy groups induced from the obvious inclusion of non-equivariant spectra

$$F_{r/r'}\colon (T\wedge S^1/C_{s+})^{C_r}\to(T\wedge S^1/C_{s+})^{C_{r'}}.$$

Let d and d' be the greatest common divisors of r and s and r' and s, respectively, and let $q=rd'/r'd$.

ADDENDUM 3.2.6. *If m and n are integers with $mr+ns=d$, then $m'=mq$ and $n'=nd'/d$ is a pair of integers with $m'r'+n's=d'$ and*

$$(\alpha_{m'n'}\circ F_{r/r'}\circ\alpha_{mn}^{-1})(a,b)=(qF_{d/d'}a+(q-1)\eta F_{d/d'}b, F_{d/d'}b),$$

where $\eta\in\pi_1^S$.

Proof. The proof of (3.2.1) gives, in particular, an equivalence of non-equivariant spectra

$$e_{mn}\colon (T\wedge S^1/C_{s+})^{C_r}\to\Sigma F(S^1_+,T^{C_d}).$$

Chasing through the argument, one sees that there is, with the particular choices of m' and n', a strictly commutative diagram of non-equivariant spectra

$$\begin{tikzcd}
(T\wedge S^1/C_{s+})^{C_r} \arrow[r, "e_{mn}"] \arrow[d, "F_{r/r'}"] & \Sigma F(S^1_+, T^{C_d}) \arrow[d, "\Sigma F(q, F_{d/d'})"] \\
(T\wedge S^1/C_{s+})^{C_{r'}} \arrow[r, "e_{m'n'}"] & \Sigma F(S^1_+, T^{C_{d'}}),
\end{tikzcd}$$

where $q: S^1 \to S^1$ is the q-fold covering. Now let $q^!: \Sigma^\infty_+ S^1 \to \Sigma^\infty_+ S^1$ be the Becker–Gottlieb transfer and recall that under Spanier–Whitehead duality,

$$\Sigma F(S^1_+, T^{C_d}) \simeq T^{C_d} \wedge \Sigma^\infty_+ S^1,$$

the map $F(q, \mathrm{id})$ on the left corresponds to the map $\mathrm{id}\wedge q^!$ on the right. Finally, we recall that under the isomorphism $\pi^S_*(S^1_+) \cong \pi^S_* \oplus \pi^S_{*-1}$,

$$q^!(a,b) = (qa + (q-1)\eta b, b).$$

Hence the given formula for $F_{r/r'}$. $\square$

Finally, it follows immediately from the definition of δ and (3.2.1) that, for any choice of m and n with $mr+ns=d$, the map

$$\delta: \pi_*(T\wedge S^1/C_{s+})^{C_r} \to \pi_{*+1}(T\wedge S^1/C_{s+})^{C_r} \tag{3.2.7}$$

is given by

$$(\alpha_{mn}\circ\delta\circ\alpha^{-1}_{mn})(a,b) = (0, (rs/d^2)a).$$

3.3. In this paragraph we evaluate the topological cyclic homology of the free associative $\mathbf{F}_p$-algebra without unit generated by a linearly ordered set S. In passing, we also give a calculation of the p-typical curves on $K(A)$ using Theorem A of [H].

Suppose that k is a perfect field of characteristic $p>0$ and recall from [HM, Theorem 5.5] that

$$\pi_* T(k)^{C_{p^{n-1}}} \cong S_{W_n(k)}\{\sigma_n\}, \tag{3.3.1}$$

where σ_n is a polynomial generator of degree 2. The Frobenius, Verschiebung and restriction maps extend the corresponding maps on the coefficient ring $W_n(k)$ and

$$F(\sigma^i_n) = \sigma^i_{n-1}, \quad V(\sigma^i_{n-1}) = p\sigma^i_n, \quad R(\sigma^i_n) = p^i\lambda^i_n\sigma^i_{n-1},$$

where $\lambda_n \in W_{n-1}(\mathbf{F}_p)$ is a unit. For degree reasons, the differential δ of (3.2.4) is trivial. Therefore in this case, the identification of the homotopy groups in (3.2.5) is canonical, i.e. independent of the choice of m and n. Now recall the complex E^* defined in (1.7.6).

THEOREM 3.3.2. *Let A be the free associative $\mathbf{F}_p$-algebra without unit generated by a linearly ordered set S. Then there is a canonical isomorphism*

$$\pi_* T(A)^{C_{p^{n-1}}} \cong E_n^* \otimes S\{\sigma_n\}, \quad \deg \sigma_n = 2,$$

which is compatible with the restriction, Frobenius, Verschiebung and differential when these operators act on the extra generator σ_n as in (3.3.1).

Proof. The decomposition in (3.1.5) is equivariant, so we get an induced decomposition of the C_{p^n}-fixed set. The homotopy groups of each summand are given by (3.2.5),

$$\pi_i T(A)^{C_{p^{n-1}}} \cong \bigoplus_{\omega\in\Omega_0} \mathbf{Z}/p^m, \quad i \geqslant 0, \tag{3.3.3}$$

where $m=m(\omega)=\min\{n, r-v_p(\pi\omega)+1\}$. Moreover, it follows from [HM, Theorem 7.1] that under this decomposition the restriction map

$$R: \pi_* T(A)^{C_{p^{n-1}}} \to \pi_* T(A)^{C_{p^{n-2}}}$$

takes the summand indexed by $\phi(\omega)$ to the summand indexed by ω by the restriction map

$$R: \pi_* T(\mathbf{F}_p)^{C_{p^{m-1}}} \to \pi_* T(\mathbf{F}_p)^{C_{p^{m-2}}} \tag{3.3.4}$$

and annihilates summands which are not indexed by elements in the image of $\phi: \Omega_0 \to \Omega_0$. We can use the bijections of (1.7.5) to index the sum above by Ω_{n-1} rather than Ω_0. One gets

$$\pi_i T(A)^{C_{p^{n-1}}} \cong \bigoplus_{\omega\in\Omega_{n-1}} \mathbf{Z}/p^m, \quad i \geqslant 0, \tag{3.3.5}$$

with $m=\min\{n, n+r-v_p(\pi\omega)\}$. Letting elements in $\Omega-\Omega_{n-1}$ correspond to the trivial group, we may view the sum (3.3.5) as indexed by Ω. In this setup, the restriction map preserves the index ω: it annihilates summands with $\omega\in\Omega-\Omega_{n-2}$ and is given by (3.3.4) on the remaining summands. Comparing this to (1.7.9), we see that the homotopy groups are as stated and that the isomorphism commutes with the restriction map. It remains to be shown that the Frobenius, Verschiebung and differential are as stated.

The Frobenius is induced from the inclusion $F: T(A)^{C_{p^{n-1}}} \to T(A)^{C_{p^{n-2}}}$, and hence it preserves the index in (3.3.3) and covers the bijection $f: \Omega_{n-1} \to \Omega_{n-2}$ in (3.3.5). We consider the summand indexed by $\omega\in\Omega_{n-1}$ and let $s=p^{n-1}|\omega|/\pi\omega$. On this summand, F is the map

$$F: \pi_i(T(\mathbf{F}_p) \wedge S^1/C_{s+})^{C_{p^{n-1}}} \to \pi_i(T(\mathbf{F}_p) \wedge S^1/C_{s+})^{C_{p^{n-2}}}$$

induced from the inclusion, and was evaluated in (3.2.6). Let pr: $\mathbf{Z}/p^m \to \mathbf{Z}/p^{m-1}$ be the projection. Since the fixed point spectra $T(\mathbf{F}_p)^{C_{p^r}}$ are all Eilenberg–MacLane, multiplication by η is trivial, so we get

$$F = \begin{cases} \mathrm{pr}, & \text{if } r \geqslant v_p(\pi\omega), \\ p, & \text{if } r < v_p(\pi\omega) \text{ and } i \text{ is even}, \\ \mathrm{id}, & \text{if } r < v_p(\pi\omega) \text{ and } i \text{ is odd}. \end{cases}$$

The claim for F and V readily follows. Finally, the claim for the differential follows from (3.2.7). □

Let E^* be the complex from §1.7, let $\widehat{E}^*$ be the completed complex

$$\widehat{E}^* = \varprojlim_n E_n^*$$

and let $F, V, \delta\colon \widehat{E}^* \to \widehat{E}^*$ be the operators induced from the Frobenius, Verschiebung and differential operators on E^*.

COROLLARY 3.3.6. *Let A be as above, then*

$$\mathrm{TR}_*(A;p) \cong \widehat{E}^*$$

compatible with the Frobenius, Verschiebung and differential.

Proof. The groups $\mathrm{TR}_i(A;p) = \pi_i \,\mathrm{TR}(A;p)$ are given by Milnor's exact sequence

$$0 \to \varprojlim_n{}^{(1)} \pi_{i+1} T(A)^{C_{p^{n-1}}} \to \mathrm{TR}_i(A;p) \to \varprojlim_n \pi_i T(A)^{C_{p^{n-1}}} \to 0.$$

The extra generator σ_n vanishes in the limit, since $R(\sigma_n^s) = p^s \lambda_n^s \sigma_{n-1}^s$. Finally, for $i = -1, 0$ the maps in the limit system on the left are surjective, so the derived limit vanishes. □

Let

$$\Sigma = \coprod_{(d,k)=1} \mathrm{Map}(\widehat{C}_p \times C_d, S)/C$$

where the dth summand is the set of continuous maps from $\widehat{C}_p \times C_d$ to the discrete set S.

COROLLARY 3.3.7. *The topological cyclic homology of A is concentrated in degree -1 and*

$$\mathrm{TC}_{-1}(A;p) \cong \Big(\bigoplus_{\sigma \in \Sigma} \mathbf{Z}_p\Big)_p^{\wedge},$$

the group of infinite sums $\sum_{\sigma\in\Sigma} a_\sigma \sigma$, $a_\sigma \in \mathbf{Z}_p$, *where, for every* $n \geqslant 0$, *all but finitely many* $a_\sigma \in p^n \mathbf{Z}_p$.

Proof. The previous result gives an exact sequence

$$0 \to \mathrm{TC}_1(A;p) \to \widehat{E}^1 \xrightarrow{F-1} \widehat{E}^1 \to \mathrm{TC}_0(A;p) \to \widehat{E}^0 \xrightarrow{F-1} \widehat{E}^0 \to \mathrm{TC}_{-1}(A;p) \to 0$$

and shows that the remaining groups vanish. We first prove that $F-1: \widehat{E}^1 \to \widehat{E}^1$ is an isomorphism. Recall that $\widehat{E}^1$ consists of infinite sums $\sum_{\omega\in\Omega} a_\omega e_1(\omega)$ such that, for every $n \geqslant 0$, only a finite number of the coefficients a_ω has $v_p(a_\omega) \leqslant \min\{n, n+r-v_p(\pi\omega)\}$. The Frobenius, given by $Fe_1(\omega) = e_1(f\omega)$, acts invertibly making $\widehat{E}^1$ a $\mathbf{Z}[F, F^{-1}]$-module, and the topology on $\widehat{E}^1$ is such that this extends to a $\mathbf{Z}[F][\![F^{-1}]\!]$-module structure. Hence $F-1$ is an isomorphism with inverse

$$(F-1)^{-1} = \sum_{i=1}^{\infty} F^{-i}.$$

Recall that the Frobenius operator on $\widehat{E}^0$ is given by

$$Fe_0(\omega) = \begin{cases} e_0(\omega), & \text{if } r \geqslant v_p(\pi\omega), \\ pe_0(\omega), & \text{if } r < v_p(\pi\omega), \end{cases}$$

and let $\widehat{E}^0_+$ and $\widehat{E}^0_-$ be the submodules of sums $\sum a_\omega e_0(\omega)$ supported on the $\omega \in \Omega$ with $r \geqslant v_p(\pi\omega)$ and $r < v_p(\pi\omega)$, respectively. We define $\varphi: \widehat{E}^0 \to \widehat{E}^0$ to be the automorphism given by $\varphi e_0(\omega) = e_0(f\omega)$ and note the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \widehat{E}^0_+ \arrow[r] \arrow[d, "\varphi-1"] & \widehat{E}^0 \arrow[r] \arrow[d, "F-1"] & \widehat{E}^0_- \arrow[r] \arrow[d, "p\varphi-1"] & 0 \\ 0 \arrow[r] & \widehat{E}^0_+ \arrow[r] & \widehat{E}^0 \arrow[r] & \widehat{E}^0_- \arrow[r] & 0. \end{tikzcd}$$

The right-hand vertical map is an isomorphism with inverse

$$(p\varphi - 1)^{-1} = -\sum_{i\geqslant 0} p^i \varphi^i,$$

and finally, we have a split exact sequence

$$0 \to \widehat{E}^0_+ \xrightarrow{\varphi-1} \widehat{E}^0_+ \xrightarrow{\varepsilon} \Big(\bigoplus_{\sigma\in\Sigma} \mathbf{Z}_p\Big)^\wedge_p \to 0,$$

where ε is induced from the map $\varepsilon: \Omega \to \Sigma$ given by $\varepsilon(d; r, [\alpha]) = (d; [\alpha])$. □

Proof of Theorem B. We choose an equivalence $P_\bullet \to A$ of simplicial rings such that each P_s is a free associative $\mathbf{F}_p$-algebra without unit. Then $W(P_\bullet)_F$ is a simplicial abelian group and by definition $L_* W(A)_F$ is the homology of the associated chain complex, [Q]. We recall that there are equivalences of spectra

$$\mathrm{TC}(A;p) \simeq \mathrm{TC}(P_\bullet;p) \simeq \underset{\Delta^{\mathrm{op}}}{\underrightarrow{\mathrm{holim}}}\, \mathrm{TC}(P_s;p).$$

The skeletal filtration of the homotopy colimit of spectra on the right yields a strongly convergent right half-plane homology type spectral sequence

$$E^1_{s,t} = \pi_s \mathrm{TC}(P_t;p) \Rightarrow \pi_{s+t} \underset{\Delta^{\mathrm{op}}}{\underrightarrow{\mathrm{holim}}}\, \mathrm{TC}(P_s;p).$$

Finally, the E^1-term is concentrated on the line $t=-1$ by (3.3.7) and $E^1_{s,-1} = W(P_s)_F$ by Theorem A. $\square$

Remark 3.3.8. It is in order to note that in contrast to the case of a free associative algebra over $\mathbf{F}_p$ the topological cyclic homology of a free commutative algebra over $\mathbf{F}_p$ is not concentrated in a single degree. We let I'_m denote the set of ordered tuples $\underline{i}=(i_1, ..., i_m)$ with $2 \leqslant i_1 < i_2 < ... < i_m \leqslant n$, for $m \geqslant 1$, and let $I'_0 = \{0\}$. Given $\underline{i} \in I'_m$, we denote by $J(\underline{i})'$ the set of n-tuples $k=(k_1, ..., k_n)$ with $k_s \in \mathbf{N}[1/p]$ such that $k_{i_j} \neq 0$ for all $i_j \in \underline{i}$. The infinite cyclic group C acts on $J(\underline{i})'$ by $t \cdot k = pk = (pk_1, ..., pk_n)$; let $J(\underline{i})$ denote the orbit space. Finally, we let

$$G_m = \{(\underline{i}, [k]) \mid \underline{i} \in I'_m,\ [k] \in J(\underline{i})\}.$$

Then one has

$$\widetilde{\mathrm{TC}}_{m-1}(\mathbf{F}_p[X_1, ..., X_n]) \cong \Big(\bigoplus_{g \in G_m} \mathbf{Z}_p\Big)^\wedge_p. \tag{3.3.9}$$

It is non-zero if and only if $0 \leqslant m \leqslant n-1$.

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Received December 10, 1996