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Harmonic measures for compact negatively curved manifolds

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1. Introduction

Let M be an n-dimensional compact Riemannian manifold of negative sectional curvature. The *geodesic flow* Φ^t is a smooth dynamical system on the unit tangent bundle T^1M of M, generated by the *geodesic spray* X.

Recall that T^1M admits four natural foliations W^{su}, W^u, W^s, W^{ss} which are invariant under the geodesic flow. The leaf $W^{ss}(v)$ containing $v\in T^1M$ of the *strong stable foliation* W^{ss} consists of all points $w\in T^1M$ with the property that the distance between $\Phi^t w$ and $\Phi^t v$ converges to zero as $t\to\infty$ (where we may use the distance on T^1M induced by the Sasaki metric). The leaf $W^s(v)$ through v of the *stable foliation* W^s is $W^s(v)=\bigcup_{t\in\mathbf{R}}\Phi^t W^{ss}(v)$, and the *strong unstable foliation* W^{su} (or the *unstable foliation* W^u) is the image of W^{ss} (or W^s) under the *flip* $\mathcal{F}: w\to -w$. The leaf $W^i(v)$ of W^i $(i=ss, su, s, u)$ is a smoothly immersed submanifold of T^1M depending continuously on v in the C^∞-topology (see [Sh]). Moreover the tangent bundle TW^i of W^i is a Hölder-continuous subbundle of TT^1M.

The purpose of this paper is to investigate ergodic and analytic properties of second-order differential operators L on T^1M with Hölder-continuous coefficients and without zero-order terms which are subordinate to the stable foliation in the following sense:

Definition. A *differential operator subordinate to* W^s is a differential operator L on T^1M with continuous coefficients and such that for every smooth function α on T^1M the value of $L\alpha$ at $v\in T^1M$ only depends on the restriction of α to $W^s(v)$.

If L is subordinate to W^s, then L restricts to a differential operator L^v on $W^s(v)$ for all $v\in T^1M$. Call L *leafwise elliptic* if L^v is elliptic for every $v\in T^1M$. A standard example of such a leafwise elliptic operator can be obtained as follows: Fix a positive semi-definite bilinear form g of class C^1 on T^1M with the property that the restriction of g to the tangent bundle TW^s of W^s is positive definite, i.e. that g induces a Riemannian metric on TW^s. The restriction to every leaf of W^s of this Riemannian metric is of class C^1 and hence g induces for every $v \in T^1M$ a Laplace operator Δ^v on $W^s(v)$. By our assumption on W^s and g these leafwise Laplacians group together to a differential operator Δ on T^1M with continuous coefficients which is subordinate to W^s.

Moreover every second-order leafwise elliptic operator L subordinate to W^s whose principal coefficients are leafwise continuously differentiable can be obtained in this way up to terms of order $\leqslant 1$: Namely for such an operator we can find a continuous, leafwise C^1 Riemannian metric $\bar{g}$ on TW^s such that L coincides with the leafwise Laplacian of $\bar{g}$ up to lower-order terms. This follows from the basic computations for standard elliptic operators as in [IW]. Formally this representation also holds for second-order elliptic operators whose principal coefficients are just continuous.

Recall that a section Y of TW^s over T^1M is said to be *of class* $C_s^{k,\alpha}$ for some $k \geqslant 0$ and some $\alpha \in [0,1)$ if Y as well as its leafwise jets up to order k along the leaves of W^s are Hölder continuous with exponent α. Let as before g be a positive semi-definite bilinear form on T^1M of class $C^{2,\alpha}$ whose restriction to TW^s is positive definite, and denote by Δ the leafwise Laplacian induced by g. Let Y be a section of TW^s of class $C_s^{1,\alpha}$. Then $L = \Delta + Y$ is a second-order leafwise elliptic operator subordinate to W^s with Hölder-continuous coefficients.

Now the leaves of W^s equipped with the metric g are complete Riemannian manifolds of bounded geometry, and for every $v \in T^1M$ the operator L^v is uniformly elliptic with respect to g with uniformly bounded coefficents. Thus L^v defines a conservative diffusion process on $W^s(v)$, given by a Markovian family $\{P^y\}_{y \in W^s(v)}$ of probability measures with initial distribution δ_y on the space Ω_+ of continuous paths $\xi: [0,\infty) \to T^1M$, equipped with the smallest σ-algebra for which the projections $R_t: \xi \to \xi(t)$ are measurable. The full collection of probability measures $\{P^v\}_{v \in T^1M}$ then defines a stochastic process on T^1M which we call the *L-process.*

A Borel probability measure η on T^1M is called *harmonic* for L if it is an invariant measure for the L-process. Harmonic measures always exist ([Ga]); they are precisely those Borel measures η on T^1M which satisfy $\int (L\alpha)\, d\eta = 0$ for every smooth function α on T^1M. Another characterization can be given as follows: Recall that the semi-group $[0,\infty)$ acts on Ω_+ by the *shift transformations* $(t,\xi) \to T^t\xi$ where $T^t\xi(s) = \xi(s+t)$. Then η is invariant for the L-process if and only if the induced probability measure P on Ω_+ which is defined by $P(B) = \int P^v(B)\, d\eta(v)$ is invariant under the shift transformations (see [Ga]).

Since η is harmonic for L we can reverse the time of the diffusion to obtain a new process on T^1M defined by a $\{T^t\}$-invariant probability measure Q on Ω_+. This process is generated by a leafwise elliptic operator L^* which we call the *η-adjoint* of L. Notice that a priori L^* may depend on the choice of an invariant measure for L; it is characterized by $\int (L^*\alpha)\beta\, d\eta = \int \alpha(L\beta)\, d\eta$ for all smooth functions α, β on T^1M.

Call L *self-adjoint* with respect to η if $\int \alpha(L\beta)\, d\eta = \int \beta(L\alpha)\, d\eta$ for all smooth functions α, β on T^1M. We also say that η is a self-adjoint harmonic measure for L. In general self-adjoint measures do not exist; but if self-adjoint measures exist, they are unique (this is shown in §2).

Now L lifts naturally to a differential operator on the unit tangent bundle $T^1\widetilde{M}$ of the universal covering $\widetilde{M}$ of M which we denote again by L. Let $\langle\cdot,\cdot\rangle$ be the Riemannian metric on M and $\widetilde{M}$; for every $v\in T^1\widetilde{M}$ the restriction of L to $W^s(v)$ then projects to a uniformly elliptic operator L_v on $(\widetilde{M}, \langle\cdot,\cdot\rangle)$ with pointwise uniformly bounded coefficients. Call L *weakly coercive* if the operators L_v are weakly coercive in the sense of Ancona ([An]) for all $v\in T^1M$, i.e. if there is a number $\varepsilon>0$ and a positive $(L_v+\varepsilon)$-superharmonic function on $\widetilde{M}$.

Let $\mathcal{M}$ be the space of Borel probability measures on T^1M which are invariant under the geodesic flow Φ^t. For $\varrho\in\mathcal{M}$ denote by h_ϱ the entropy of ϱ. Recall that the *pressure* $\mathrm{pr}(f)$ of a continuous function f on T^1M is defined by $\mathrm{pr}(f)=\sup\{h_\varrho - \int f\, d\varrho \mid \varrho\in\mathcal{M}\}$.

If η is a harmonic measure for L, then the *Kaimanovich entropy* h_L of the diffusion induced by L on (T^1M, η) is defined. We have $h_L=0$ if and only if for η-almost every $v\in T^1M$ the leaf $W^s(v)$ does not admit any non-constant bounded L^v-harmonic functions ([Ka2]).

Recall that the Riemannian metric g on TW^s defines an isomorphism between TW^s and its dual bundle T^*W^s. If φ is a section of T^*W^s of class $C_s^{1,\alpha}$ for some $\alpha>0$, then for every $v\in T^1M$ the exterior differential $d\varphi(v)$ of the restriction of φ to $W^s(v)$ is defined at v and the assignment $v\to d\varphi(v)$ is a section of $\bigwedge^2 T^*W^s$ of class C^α. We call φ *stably-closed* if $d\varphi=0$. With these notations we show

THEOREM A. *Let $L=\Delta+Y$ be as above and assume that Y is g-dual to a stably-closed section of T^*W^s. Then we have:*

(1) *If* $\mathrm{pr}(g(X,Y))>0$ *then L is weakly coercive, L admits a unique harmonic measure η and the Kaimanovich entropy h_L is positive.*

(2) *If* $\mathrm{pr}(g(X,Y))=0$ *then L is not weakly coercive, L admits a unique self-adjoint harmonic measure η and the Kaimanovich entropy h_L vanishes.*

(3) *If* $\mathrm{pr}(g(X,Y))<0$ *then L is weakly coercive and the Kaimanovich entropy h_L vanishes.*

If $\mathrm{pr}(g(X,Y))<0$ then in general a harmonic measure for L is not unique: In [H3] we give examples of operators as above which admit harmonic measures in uncountably many measure classes.

Denote by $P: T^1M \to M$ (or $P: T^1\widetilde{M} \to \widetilde{M}$) the canonical projection. The kernel of the differential dP of P equals the *vertical bundle* T^v, i.e. the tangent bundle of the *vertical foliation* of T^1M whose leaves are just the fibres of the fibration $T^1M \to M$.

Denote by g_0 the smooth positive semi-definite bilinear form on T^1M which is defined by $g_0(Y,Z)=\langle dP(Y), dP(Z)\rangle$. Since the foliation W^s is transversal to the vertical foliation the bilinear form g_0 restricts to a Hölder-continuous Riemannian metric g^s on the tangent bundle TW^s of W^s in such a way that the restriction of g^s to every leaf of W^s is smooth. These data then define a leafwise Laplacian Δ^s on T^1M subordinate to W^s.

Theorem A implies that a harmonic measure ω for Δ^s is unique. This fact was earlier derived by Ledrappier ([L3]) and Yue ([Y2]). In the case that M is a hyperbolic surface the corresponding result is contained in the paper [Ga] of Garnett; her proof easily generalizes for the stable Laplacian Δ^s of an arbitrary compact manifold M of negative curvature (and in fact, Ledrappier and Yue independently rediscover her argument).

§5 of our paper is devoted to a generalization of a result of Ledrappier ([L4]). For this let $\partial\widetilde{M}$ be the *ideal boundary* of $\widetilde{M}$ and let dist be the distance function on $\widetilde{M}$ induced by the Riemannian metric. Let $\pi: T^1\widetilde{M} \to \partial\widetilde{M}$ be the natural projection which maps $v \in T^1\widetilde{M}$ to the asymptoticy class $\pi(v)$ of the geodesic γ_v with initial velocity $\gamma_v'(0)=v$. For $x\in\widetilde{M}$ and $v\neq w \in T^1_x\widetilde{M}$ define the *Gromov product* $(v|w)$ of v and w by

$$(v|w)=\lim_{\substack{y\to\pi(v)\\ z\to\pi(w)}} \tfrac{1}{2}(\mathrm{dist}(x,y)+\mathrm{dist}(x,z)-\mathrm{dist}(y,z)).$$

For sufficiently small $\tau>0$ the assignment $(v,w)\to e^{-\tau(v|w)}$ defines a distance on the fibres of the fibration $T^1\widetilde{M}\to\widetilde{M}$, the so called *Gromov distances* ([GH]), which are invariant under the action of the fundamental group $\pi_1(M)$ of M on $T^1\widetilde{M}$ and hence project to a family of distances on the fibres of $T^1M\to M$ which we denote by the same symbol. Define a (Hölder) norm $\|\cdot\|_\tau$ on the space of continuous functions $f: T^1M\to\mathbf{R}$ by

$$\|f\|_\tau=\sup_v |f(v)|+\sup_x\{\sup |f(v)-f(w)|e^{\tau(v|w)} \mid v,w\in T^1_xM\}.$$

Then we show in §5:

THEOREM B. *Let $L=\Delta+Y$ be as above such that* $\mathrm{pr}(g(X,Y))>0$. *Denote by Q_t ($t\geqslant 0$) the action of $[0,\infty)$ on functions on T^1M which describes the L-diffusion. Let η be the unique harmonic measure for L. Then for sufficiently small $\tau>0$ there are numbers $C>0$ and $\zeta<1$ such that $\|Q_tf-\int f\,d\eta\|_\tau\leqslant C\zeta^t\|f\|_\tau$ for all continuous functions $f: T^1M\to\mathbf{R}$ with $\|f\|_\tau<\infty$ and all $t>0$.*

Theorem B for $L=\Delta^s$ is due to Ledrappier ([L4]); moreover it implies a central limit theorem for the L-diffusion (see [L4] for details and further applications).

The appendices contain a discussion of solutions of families of elliptic and parabolic equations. These more technical results are used for the proof of the above theorems.

Before we proceed we introduce a few more notations which are used throughout the paper.

For every $x\in\widetilde{M}$ the exponential map at x induces local coordinates on the ball $B(x,1)$ of radius 1 about x. These coordinates then induce for every integer $k\geqslant 0$ and every $\alpha\in[0,1)$ a $C^{k,\alpha}$-norm for functions on $B(x,1)$. For a function f on $\widetilde{M}$ define $\|f\|_{k,\alpha}$ to be the supremum of these $C^{k,\alpha}$-norms of the restrictions of f to balls of radius 1 in $\widetilde{M}$ (whenever this exists).

The bilinear form g_0 restricts to Hölder-continuous Riemannian metrics g^i on the leaves of the foliations W^i $(i=su,u,s,ss)$. For $v\in T^1M$ and $r>0$ denote by $B^i(v,r)$ the open ball of radius r about v in $(W^i(v),g^i)$.

The foliations W^i lift to foliations on $T^1\widetilde{M}$ which we denote by the same symbol. For $v\in T^1\widetilde{M}$ let θ_v be the *Busemann function* at the point $\gamma_v(\infty)$ of the ideal boundary $\partial\widetilde{M}$ which is normalized by $\theta_v(\gamma_v(0))=0$. The canonical projection $P\colon T^1\widetilde{M}\to\widetilde{M}$ then maps $W^{ss}(v)$ diffeomorphically onto the *horosphere* $\theta_v^{-1}(0)$ and $W^s(v)$ diffeomorphically onto $\widetilde{M}$. For $\alpha\in(0,\pi)$ denote moreover by $C(v,\alpha)$ the open cone of angle α and direction v in $\widetilde{M}$, i.e. $C(v,\alpha)=\{P\Phi^t w\mid w\in T^1_{Pv}\widetilde{M},\ \angle(v,w)<\alpha,\ t\in(0,\infty)\}$ where $\angle$ is the angle of $\langle\cdot,\cdot\rangle$.

Define

$$\widetilde{D}=\{(v,w)\in T^1\widetilde{M}\times T^1\widetilde{M}\mid w\in W^s(v)\}.$$

Since any two points in $\widetilde{M}$ can be joined by a unique minimizing geodesic, the set $\widetilde{D}$ can naturally be identified with the bundle TW^s over $T^1\widetilde{M}$. In particular, $\widetilde{D}$ carries a natural Hölder structure and a natural foliation $\mathcal{F}$ with smooth leaves. Here the leaf of $\mathcal{F}$ through $(v,w)\in\widetilde{D}$ is just the tangent bundle of the manifold $W^s(v)$. The leaf of $\mathcal{F}$ through (v,w) depends Hölder continuously in the C^∞ topology on the point (v,w), i.e. the jet bundles of arbitrary degree are Hölder continuous. Let moreover D be the projection of $\widetilde{D}$ under the natural action of $\pi_1(M)$ on $T^1\widetilde{M}\times T^1\widetilde{M}\supset\widetilde{D}$. Clearly D is naturally homeomorphic to the bundle TW^s over T^1M.

Recall that an open subset C of $T^1\widetilde{M}$ admits a *local product structure* if for $v\in C$ there are open, relative compact neighborhoods A of v in $W^s(v)$, B of v in $W^{su}(v)$ and a homeomorphism $\Lambda\colon A\times B\to C$ with the following properties:

(i) $\Lambda(w,v)=w$ for all $w\in A$.

(ii) $\Lambda(v,z)=z$ for all $z\in B$.

(iii) $\Lambda(\{w\}\times B\})$ is contained in a leaf of W^{su} for all $w\in A$.

(iv) For every $z\in B$ the map $\Lambda_z: A\to W^s(z)$ which is defined by $\Lambda_z(w)=\Lambda(w,z)$ is a homeomorphism of A into $W^s(z)$.

The maps Λ_z are called *canonical maps* for the local product structure.

2. Harmonic measures for the stable foliation

As in the introduction, let M be an arbitrary compact Riemannian manifold of negative sectional curvature and let g be a positive semi-definite bilinear form on T^1M of class $C^{2,\alpha}$ for some $\alpha>0$ whose restriction to TW^s is positive definite. Denote by ν^s the Lebesgue measure on the leaves of W^s induced by g. Let Δ be the leafwise Laplacian induced by g and let $L=\Delta+Y$ for a section Y of TW^s of class $C_s^{1,\alpha}$. Lift L to an operator on $T^1\widetilde{M}$ which we denote by the same symbol. For $v\in T^1\widetilde{M}$ the restriction L^v of L to $W^s(v)$ admits a unique fundamental solution $p(v,w,t)$ $(w\in W^s(v), t>0)$ of the heat equation $L^v-\partial/\partial t=0$ relative to the volume element $d\nu^s$. Since the coefficients of L are Hölder continuous, the function $p\colon \widetilde{D}\times(0,\infty)\to(0,\infty)$ is Hölder continuous (see Appendix A) and it projects to a Hölder-continuous function on D which we denote again by p.

Let $\widetilde{\Omega}_+$ be the space of paths $\xi\colon[0,\infty)\to T^1\widetilde{M}$, equipped with the smallest σ-algebra $\tilde{\mathcal{A}}$ for which the projections $R_t\colon\xi\to R_t(\xi)=\xi(t)$ are measurable. For $v\in T^1\widetilde{M}$ the L^v-process on $W^s(v)$ is given by a Markovian family $\{P^w\}_{w\in W^s(v)}$ of probability measures P^w on $\widetilde{\Omega}_+$. Namely for every $t>0$ and every Borel set $A\subset T^1\widetilde{M}$ we have $P^v\{\xi\mid\xi(t)\in A\}=\int_{A\cap W^s(v)}p(v,w,t)\,d\nu^s(w)$; moreover P^v-almost every path in $\widetilde{\Omega}_+$ is continuous.

Let $\Pi\colon T^1\widetilde{M}\to T^1M$ be the canonical projection. Then Π induces a measurable projection of $\widetilde{\Omega}_+$ onto the space Ω_+ of paths ξ in T^1M. For every $w\in T^1\widetilde{M}$ the measure P^w projects to a probability measure on Ω_+ which only depends on $\Pi w=v$ and will be denoted by P^v. These measures describe the L-process on T^1M (see [Ga] and the introduction).

Let η be a harmonic measure for L on T^1M. Then η is absolutely continuous with respect to the stable and the strong unstable foliation (see [Ga]), and the conditionals on the leaves of W^s are contained in the Lebesgue measure class. More precisely, let $\tilde{\eta}$ be the lift of η to a σ-finite Borel measure on $T^1\widetilde{M}$. For $v\in T^1\widetilde{M}$ and $r>0$ let again $B^s(v,r)$ be the open ball of radius r about v in $(W^s(v),g^s)$. For $r\in(0,\infty)$ we then can desintegrate $\tilde{\eta}$ to a measure $\tilde{\eta}^{su}$ on $W^{su}(v)$ by defining $\tilde{\eta}^{su}(B)=\tilde{\eta}\big(\bigcup_{w\in B}B^s(w,r)\big)$. This measure is locally finite and projects via the projection π to a measure on $\partial\widetilde{M}$. The measure class of this projection does not depend on $r>0$ or on the base point v and is invariant under the action of $\Gamma=\pi_1(M)$ (these facts follow from the results in [Ga]). We denote it by $\mathrm{mc}(\eta,\infty)$.

Recall that the semi-group $[0,\infty)$ acts on Ω_+ by the *shift transformations* $\{T^t \mid t>0\}$ via $(T^t\xi)(s)=\xi(s+t)$. The measure $P=\int P^v\,d\eta(v)$ on Ω_+ induced by η is invariant under the shift.

The next lemma describes the ergodic components of a harmonic measure for L, i.e. it translates the results of [Ga] into our geometric context.

LEMMA 2.1. *The measure on Ω_+ induced by η is ergodic under the shift if and only if* $\mathrm{mc}(\eta,\infty)$ *is ergodic under the action of* Γ.

Proof. Let again P be the measure on Ω_+ induced by the L-process and the measure η. Assume first that $\mathrm{mc}(\eta,\infty)$ is ergodic under the action of Γ and let $A\subset\Omega_+$ be a measurable set which is invariant under the transformations T^t $(t\geqslant 0)$. We have to show that $\alpha=P(A)$ equals 0 or 1. Define a function $\psi: T^1M\to[0,1]$ by $\psi(v)=P^v(A)+1$. This function is measurable and lifts to a function $\tilde\psi$ on $T^1\widetilde M$. By the definition of P and the T^t-invariance of A we have for every $u\in T^1\widetilde M$ and every $t\geqslant 0$ that

$$\tilde\psi(u)=P^u\{\xi \mid \Pi T^t\xi\in A\}+1=\int p(u,w,t)\,\tilde\psi(w)\,d\nu^s(w). \qquad (*)$$

For $v\in T^1M$ let ψ^v be the restriction of ψ to the stable manifold $W^s(v)$. By $(*)$ the function ψ^v satisfies $L^v\psi^v=0$. Thus ψ is a bounded positive Borel function on T^1M which is L^v-harmonic for η-almost every $v\in T^1M$.

The Riemannian metric g on TW^s induces a continuous Riemannian metric on the dual bundle T^*W^s of TW^s which we denote again by g. Then

$$(\Delta+Y)(\log\psi)=\psi^{-1}(\Delta+Y)(\psi)-g(d\psi,d\psi)\,\psi^{-2}$$

and hence $\int g(d\psi,d\psi)\,\psi^{-2}\,d\eta=-\int L(\log\psi)\,d\eta=0$, i.e. ψ is constant along η-almost every leaf of T^1M and consequently ψ is constant η-almost everywhere on T^1M by ergodicity. This constant then equals $\alpha+1$ where $\alpha=P(A)$.

Now the finite intersections of sets of the form $R_t^{-1}(B)$ $(B\subset T^1M$ Borel, $t\in(0,\infty))$ form a $\cap$-stable generator for the σ-algebra on Ω_+. Thus under the assumption $\alpha\in(0,1)$ there are for every $\varepsilon>0$ some Borel sets $B_1^i,\dots,B_k^i\subset T^1M$ and numbers $t_1^i,\dots,t_k^i\in(0,\infty)$ $(k>0$ and $i=1,\dots,l)$ with the following properties:

(i) The sets $B_i=\bigcap_{j=1}^k R_{t_j^i}^{-1}(B_j^i)$ are pairwise disjoint.

(ii) $P\big(\bigcup_{i=1}^l B_i\big)>1-\alpha-\varepsilon$.

(iii) $P\big(A\cap\big(\bigcup_{i=1}^l B_i\big)\big)<\varepsilon$.

But since ψ is constant η-almost everywhere on T^1M we have by the Markov property and the definition of P that $P(A\cap B_i)=\alpha P(B_i)$ for all $i\in\{1,\dots,l\}$, i.e. $P\big(A\cap\big(\bigcup_{i=1}^l B_i\big)\big)=$

$\alpha P(\bigcup_{i=1}^{l} B_i)$. If $\alpha \neq 0, 1$ then we can choose $\varepsilon < \alpha(1-\alpha)/(1+\alpha)$ and obtain a contradiction. Hence either $P(A)=1$ or $P(A)=0$, i.e. P is indeed ergodic with respect to the shift.

On the other hand, if $\mathrm{mc}(\eta, \infty)$ is not ergodic under the action of Γ, then we can find a subset A of T^1M consisting of full stable leaves and such that $0<\eta(A)<1$. Then $\{\xi \in \Omega_+ \mid \xi(0) \in A\}$ is a shift-invariant subset of Ω_+ whose measure coincides with $\eta(A)$, i.e. the measure induced on Ω_+ is not ergodic under the shift. □

Next let again η be a harmonic measure for L with lift $\tilde{\eta}$ to $T^1\widetilde{M}$ and let $\tilde{\eta}(\infty)$ be a Borel probability measure on $\partial\widetilde{M}$ which defines the measure class of $\mathrm{mc}(\eta, \infty)$. For $v \in T^1\widetilde{M}$ we then can represent the measure $\tilde{\eta}$ near v in the form $d\tilde{\eta} = \alpha\, d\nu^s \times d\tilde{\eta}(\infty)$ where $\alpha: T^1\widetilde{M} \to (0, \infty)$ is a Borel function and we identify $\tilde{\eta}(\infty)$ with its projections to the leaves of W^{su} under the restrictions of the map π. For $(v, w) \in \widetilde{D}$ define $l_\eta(v, w) = l(v, w) = \alpha(w)/\alpha(v)$; this function is called the *growth of η relative to ν^s* and it is independent of the choice of $\tilde{\eta}(\infty)$.

For a continuous section Z of TW^s over T^1M (or $T^1\widetilde{M}$) which is of class C^1 along the leaves of the stable foliation write $\operatorname{div} Z$ to denote the function on T^1M (or $T^1\widetilde{M}$) whose restriction to a leaf $W^s(v)$ of W^s equals the divergence of $Z|_{W^s(v)}$ with respect to the volume element ν^s. Moreover for a function f of class C^1_s on T^1M denote by ∇f the section of TW^s whose restriction to the leaf $W^s(v)$ equals the g-gradient of $f|_{W^s(v)}$. Then we have

LEMMA 2.2. $\Delta(\alpha) - \operatorname{div}(\alpha Y) = 0$.

Proof. Consider a smooth function f on $T^1\widetilde{M}$ with compact support. Partial integration then shows

$$0 = \int (\Delta + Y)(f)(v)\alpha(v)\, d\nu^s \times d\tilde{\eta}(\infty)(v) = \int f(\Delta(\alpha) - \operatorname{div}(\alpha Y))\, d\nu^s \times d\tilde{\eta}(\infty)$$

and from this the lemma immediately follows. □

By Lemma 2.2 the function α is differentiable along the leaves of the stable foliation. Hence we can define the *g-gradient* of η to be the η-measurable section Z of TW^s whose restriction to the leaf $W^s(v)$ is just the g-gradient of the η-measurable function $w \in W^s(v) \to \log \alpha(w) \in \mathbf{R}$.

Next we describe the self-adjoint harmonic measures in terms of their growth:

LEMMA 2.3. *The measure η is self-adjoint for L if and only if $p(v, w, t)l(w, v) = p(w, v, t)$ for $\tilde{\eta}$-almost every $v \in T^1\widetilde{M}$ and $w \in W^s(v)$, all $t \in (0, \infty)$.*

Proof. Let $(t, u) \to \Lambda_t u$ be the action of $[0, \infty)$ on functions u on $T^1\widetilde{M}$ which describes the L-process on $T^1\widetilde{M}$. Then η is self-adjoint for L if and only if for all continuous functions φ, u on $T^1\widetilde{M}$ with compact support and all $t>0$ we have $\int \varphi(\Lambda_t u)\, d\tilde{\eta} = \int u(\Lambda_t \varphi)\, d\tilde{\eta}$ (this follows as in the case of the trivial foliation, see [IW]). But

$$\begin{aligned}\int \varphi(\Lambda_t u)\, d\tilde{\eta} &= \iint \varphi(v) p(v,w,t) u(w)\, d\nu^s(w)\, \alpha(v)(d\nu^s \times d\tilde{\eta}(\infty))(v) \\ &= \iint u(w) p(v,w,t) \varphi(v) \alpha(w) l(w,v)\, d\nu^s(w)(d\nu^s \times d\tilde{\eta}(\infty))(v) \\ &= \int \Big(\int p(v,w,t) \varphi(v) l(w,v)\, d\nu^s(v) \Big) u(w)\, d\tilde{\eta}(w)\end{aligned}$$

and this is equal to $\int u(\Lambda_t \varphi)\, d\tilde{\eta} = \int (\int p(w,v,t) \varphi(v)\, d\nu^s(v)) u(w)\, d\tilde{\eta}(w)$ for all functions φ, u as above if and only if $p(v,w,t) l(w,v) = p(w,v,t)$ for $\tilde{\eta}$-almost every $v \in T^1\widetilde{M}$, $w \in W^s(v)$ and all $t>0$. □

Recall that the fundamental solution $p(v,w,t)$ of the heat equation for L is a Hölder-continuous function on $D\times(0,\infty)$ (see the appendix). For $t\in(0,\infty)$ and $v\in T^1\widetilde{M}$ define

$$\alpha_t(v) = \frac{d}{ds}(p(v, \Phi^s v, t) p(\Phi^s v, v, t)^{-1})\Big|_{s=0};$$

the function $\alpha_t: T^1\widetilde{M} \to \mathbf{R}$ is Hölder continuous.

COROLLARY 2.4. *There is at most one self-adjoint harmonic measure η for L. Such a measure exists if and only if $\alpha_t = \alpha_s = \alpha$ for all $t, s>0$ and if the pressure of α vanishes.*

Proof. Let η be a self-adjoint harmonic measure for L and write $d\eta = d\nu^s \times d\eta^{su}$ where η^{su} is a quasi-invariant family of locally finite Borel measures on the leaves of W^{su}. Lemma 2.3 shows that

$$\alpha_t(v) = \frac{d}{ds} \frac{d(\eta^{su} \circ \Phi^s)}{d\eta^{su}}(v)\Big|_{s=0} \quad \text{for every } t>0;$$

in particular, $\alpha_t = \alpha_s = \alpha$ for all $s, t>0$. Since the function α is Hölder continuous there is a unique Gibbs equilibrium state defined by α which admits the measures η^{su} as a family of conditionals on strong unstable manifolds. But this just means that the pressure of α vanishes and that a self-adjoint harmonic measure for L is unique.

Vice versa, assume that $\alpha_t = \alpha_s = \alpha$ and that the pressure of α vanishes. Then there is a family of conditionals η^{su} on the leaves of W^{su} of the unique Gibbs equilibrium state defined by α with the property that

$$\frac{d}{dt}\{\eta^{su} \circ \Phi^t\}\Big|_{t=0} = \alpha.$$

Define a finite measure η on T^1M by $d\eta = d\nu^s \times d\eta^{su}$.

By the definition of η, the growth of η relative to ν^s is well defined and can be viewed as a function l on $\widetilde{D}$ which satisfies $l(v, \Phi^s v) = p(v, \Phi^s v, t)p(\Phi^s v, v, t)^{-1}$ for all $s \in \mathbf{R}$ and all $t > 0$. But l is a Hölder-continuous function, and since p is Hölder continuous on $\widetilde{D} \times (0, \infty)$ we necessarily have $l(v, w) = p(v, w, t)p(w, v, t)^{-1}$ for all $(v, w) \in \widetilde{D}$ and all $t > 0$ (compare the considerations in [H2]). By Lemma 2.3 this just means that η is a self-adjoint harmonic measure for L. □

Call a section φ of $\Lambda^p T^* W^s \subset \Lambda^p T^*(T^1 M)$ *of class* C_s^j for some integer $j \in [0, \infty]$ if the restriction of φ to every leaf of W^s is of class C^j and if the jets of order $\leqslant j$ of these restrictions are continuous. If φ is of class C_s^j for some $j \geqslant 1$, then for every $v \in T^1 M$ the exterior differential $d\varphi(v)$ of the restriction of φ to $W^i(v)$ is defined at v, and the assignment $v \to d\varphi(v)$ is a section of $\Lambda^{p+1} T^* W^s$ of class C_s^{j-1}.

Let η be an arbitrary Borel probability measure on $T^1 M$ which is absolutely continuous with respect to the stable and the strong unstable foliation, with conditionals on the leaves of W^s contained in the Lebesgue measure class. More precisely, we assume that there is a Borel probability measure $\tilde{\eta}(\infty)$ on $\partial \widetilde{M}$ and a function $\alpha: T^1 \widetilde{M} \to (0, \infty)$ which is measurable and leafwise differentiable, with measurable leafwise differential such that the lift $\tilde{\eta}$ of η to a σ-finite Borel measure on $T^1 \widetilde{M}$ is locally of the form

$$d\tilde{\eta} = \alpha \, d\nu^s \times d\tilde{\eta}(\infty)$$

where as before we identify $\tilde{\eta}(\infty)$ with its projections to the leaves of W^{su} under the restrictions of the map π. Let Z be the g-gradient of η.

Recall that the Riemannian metric g on TW^s naturally extends to a Riemannian metric on the continuous vector bundles $\Lambda^p T^* W^s$ over $T^1 M$ $(p \geqslant 0)$.

Define an inner product $(\cdot, \cdot)$ on the vector space $C_s^\infty(\bigwedge^p T^* W^s)$ of sections of $\bigwedge^p T^* W^s$ of class C_s^∞ by $(\varphi, \psi) = \int g(\varphi(v), \psi(v)) \, d\eta(v)$, and denote by H_p^0 the completion of $C_s^\infty(\bigwedge^p T^* W^s)$ with respect to this inner product. Then d is a densely defined linear operator of H_p^0 into H_{p+1}^0, and hence its adjoint d^* is well defined. We want to determine d^*; for this let $*$ be the Hodge star operator on the leaves of W^s with respect to the metric g, viewed as a bundle isomorphism of $\bigwedge^p T^* W^s$ onto $\bigwedge^{n-p} T^* W^s$. For a section φ of $\bigwedge^p T^* W^s$ and a section E of TW^s denote by $E \rfloor \varphi$ the inner product of φ and E. Then we have

LEMMA 2.5. *Let Z be the g-gradient of η. Then*

$$d^* \varphi = (-1)^{np+n+1} * d * \varphi - Z \rfloor \varphi \quad \textit{for every } \varphi \in C_s^\infty\Big(\bigwedge^p T^* W^s\Big) \ (p \geqslant 1);$$

in particular, η is a self-adjoint harmonic measure for $\Delta + Z$.

Proof. If η_i $(i = 1, \ldots, k)$ is a finite smooth partition of unity for $T^1 M$, then $d^* \varphi = \sum_i d^*(\eta_i \varphi)$, $*d*\varphi = \sum_i *d*(\eta_i \varphi)$ and $Z \rfloor \varphi = \sum_i Z \rfloor (\eta_i \varphi)$ for all $\varphi \in C_s^\infty(\bigwedge^p T^* W^s)$, and hence it suffices to show the lemma for forms which are supported in an open subset C of T^1M with a local product structure, given by $v\in T^1M$ and open, relative compact neighborhoods A of v in $W^s(v)$, B of v in $W^{su}(v)$ and a homeomorphism $\Lambda: A\times B\to C$ as in the introduction.

Let η^{su} be a conditional of η on B and define a measure $\tilde\eta$ on $A\times B$ by $d\tilde\eta(\tilde v,w)=d\nu^s(\Lambda(\tilde v,w))\times d\eta^{su}(w)$. The map Λ is absolutely continuous with respect to the measure η on C, the measure $\tilde\eta$ on $A\times B$ and its Jacobian with respect to these measures is given by the growth $l=l_\eta: D\cap(C\times C)\to(0,\infty)$ of η with respect to ν^s, where $D\subset T^1M\times T^1M$ is as in the introduction. For $z\in B$ and $w\in W^s(z)$ write $l_z(w)=l(z,w)$.

Let now φ be a section of $\bigwedge^p T^*W^s$ of class C_s^1 with support in C. For a section $\psi\in C_s^1(\bigwedge^{p-1}T^*W^s)$ we then have

$$\begin{aligned}\int g(d\psi,\varphi)\,d\eta &= \int_{z\in B}\int_{w\in W^s(z)} g(d\psi,\varphi)(w)\,l_z(w)\,d\nu^s(w)\,d\eta^{su}(z)\\ &= \int_{z\in B}\left[\int_{W^s(z)} l_z\,d\psi\wedge *\varphi\right] d\eta^{su}(z)\\ &= \int_{z\in B}\left[\int_{W^s(z)} d(l_z\psi\wedge *\varphi)\right] d\eta^{su}(z)\\ &\quad -\int_{z\in B}\left[\int_{W^s(z)} l_z(d\log l_z\wedge\psi\wedge *\varphi+(-1)^{p-1}\psi\wedge d*\varphi)\right] d\eta^{su}(z)\\ &= (-1)^{np+n+1}\int g(\psi,*d*\varphi)\,d\eta-\int g(d\log l_z\wedge\psi,\varphi)\,d\eta\end{aligned}$$

by Stokes' theorem. The lemma now follows from the fact that $g(d\log l_z\wedge\psi,\varphi)=g(\psi,Z\rfloor\varphi)$. □

Now we can characterize self-adjoint harmonic measures as follows:

CORALLARY 2.6. *For a Borel probability measure η on T^1M the following are equivalent*:

(1) *η is a self-adjoint harmonic measure for $L=\Delta+Y$.*

(2) *The g-gradient of η equals Y; in particular, Y is g-dual to a stably-closed section of T^*W^s.*

(3) *$\int(\operatorname{div}(Z)+g(Y,Z))\,d\eta=0$ for all sections Z of TW^s of class C_s^1.*

Proof. The equivalence of (2) and (3) is a consequence of the proof of Lemma 2.5; moreover (3) implies (1). Thus we are left with showing that (3) is a consequence of (1). For this let η be a self-adjoint harmonic measure for $L=\Delta+Y$, let Z be the g-gradient of η and φ, ψ be smooth functions on T^1M. Then

$$\int \varphi(L\psi)\,d\eta = \int (\operatorname{div}(\varphi\nabla\psi) + g(\varphi\nabla\psi, Y) - g(\nabla\varphi, \nabla\psi))\,d\eta$$
$$= \int \psi(L\varphi)\,d\eta = \int (\operatorname{div}(\psi\nabla\varphi) + g(\psi\nabla\varphi, Y) - g(\nabla\psi, \nabla\varphi))\,d\eta$$

and consequently

$$\int (\operatorname{div}(\varphi\nabla\psi - \psi\nabla\varphi) + g(\varphi\nabla\psi - \psi\nabla\varphi, Y))\,d\eta = 0.$$

On the other hand, we have $\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$ and $\int L(\varphi\psi)\,d\eta = 0$, and from this and the above formula we conclude that $\int (\operatorname{div}(\varphi\nabla\psi) + g(\varphi\nabla\psi, Y))\,d\eta = 0$ for all smooth functions φ, ψ on T^1M. Since smooth functions are dense in the space of functions of class C^1_s on T^1M, this identity also holds whenever φ is a function of class C^1_s and ψ is smooth. On the other hand, using a suitable smooth partition of unity for T^1M and local coordinates it is easy to see that every section A of TW^s of class C^1_s can be written as a finite sum of sections of the form $\varphi\nabla\psi$ where φ is of class C^1_s and ψ is smooth. Thus the above equation implies that $\int (\operatorname{div}(A) + g(Y, A))\,d\eta = 0$ for every section A of TW^s of class C^1_s which is (3). □

Let $\mathcal{M}$ be the space of Φ^t-invariant Borel probability measures on T^1M, and for $\varrho \in \mathcal{M}$ denote by h_ϱ the entropy of ϱ. Recall that the *pressure* $\operatorname{pr}(f)$ of a continuous function f on T^1M is defined by $\operatorname{pr}(f) = \sup\{h_\varrho - \int f\,d\varrho \mid \varrho \in \mathcal{M}\}$. If f is Hölder continuous then f admits a unique Gibbs equilibrium state $\varrho_f \in \mathcal{M}$, i.e. ϱ_f is the unique element of $\mathcal{M}$ such that $h_{\varrho_f} - \int f\,d\varrho_f = \operatorname{pr}(f)$. Then ϱ_f admits a family ϱ_f^{su} of conditional measures on strong unstable manifolds which transform under the geodesic flow via

$$\frac{d}{dt}\{\varrho_f^{su} \circ \Phi^t\}\Big|_{t=0} = f + \operatorname{pr}(f).$$

Let X be the geodesic spray on T^1M. As an immediate consequence of Corollary 2.6 we now obtain

COROLLARY 2.7. *$L = \Delta + Y$ admits a self-adjoint harmonic measure if and only if the following is satisfied*:

(1) *Y is g-dual to a stably-closed section of T^*W^s.*

(2) *The pressure of $g(Y, X)$ vanishes.*

Proof. Assume that Y is g-dual to a stably-closed section of T^*W^s and that the pressure of $g(Y, X)$ vanishes. Let η^{su} be a family of conditional measures on strong unstable manifolds of the Gibbs equilibrium state of $g(Y, X)$ with the property that $d\{\eta^{su} \circ \Phi^t\}/dt|_{t=0} = g(Y, X)$. Define a finite Borel measure η on T^1M by $d\eta = d\nu^s \times d\eta^{su}$.

Consider the lift $\tilde{\eta}$ of η to $T^1\widetilde{M}$. The growth of $\tilde{\eta}$ with respect to ν^s is a Hölder-continuous function $l: \widetilde{D}\to(0,\infty)$ such that $dl(v,\Phi^t v)/dt|_{t=0}=g(Y,X)(v)$ for all $v\in T^1\widetilde{M}$.

By assumption on Y, for every $v\in T^1\widetilde{M}$ there is a function f_v on $W^s(v)$ of class C^1 such that df_v is g-dual to $Y|_{W^s(v)}$. Then f_v is uniformly Hölder continuous and satisfies $f_v(\Phi^t w)-f_v(w)=\log l(w,\Phi^t w)$ for all $w\in W^s(v)$ and all $t\in\mathbf{R}$. From Hölder continuity we then conclude that $\log l(w,z)=f_v(z)-f_v(w)$ for all $w,z\in W^s(v)$ (compare the arguments in [H2]). But this just means that Y is the g-gradient of η and hence by Corollary 2.6, η is a self-adjoint harmonic measure for $\Delta+Y$. □

Lemma 2.5 shows that the adjoint d^* of d with respect to $(\cdot,\cdot)$ is defined on the dense subspace $C_s^\infty(\bigwedge^p T^*W^s)$ of $(H_p^0,(\cdot,\cdot))$. Define a bilinear form Q on $C_s^\infty(\bigwedge^p T^*W^s)$ by $Q(\varphi,\psi)=(\varphi,\psi)+(d\varphi,d\psi)+(d^*\varphi,d^*\psi)$. Then Q is the form of the self-adjoint extension of $\mathrm{Id}+\mathcal{L}$ where $\mathcal{L}=dd^*+d^*d$ (we denote this extension again by $\mathrm{Id}+\mathcal{L}$). The completion H_p^1 of $C_s^\infty(\bigwedge^p T^*W^s)$ with respect to Q just coincides with the domain of $(\mathrm{Id}+\mathcal{L})^{1/2}$.

Let $i: H_p^1\to H_p^0$ be the natural inclusion.

LEMMA 2.8. *There is a continuous linear map* $G: H_p^0\to(H_p^1,Q)$ *with the following properties*:

(i) $i\circ G$ *is self-adjoint and commutes with the operators* d *and* d^*.

(ii) $(\mathrm{Id}+\mathcal{L})\circ G=\mathrm{Id}$.

Proof. The existence of a continuous linear map G with property (ii) follows as in the case of elliptic differential operators from the Riesz representation theorem. Clearly $i\circ G$ is self-adjoint. To show that G commutes with d^* let $\alpha\in H_p^1$ and let $\psi=G\alpha$. Then

$$(\mathrm{Id}+\mathcal{L})d^*\psi=(\mathrm{Id}+dd^*+d^*d)d^*\psi=d^*(\mathrm{Id}+dd^*)\psi=d^*(\mathrm{Id}+\mathcal{L})\psi=d^*\alpha$$

and hence $d^*\psi=Gd^*\alpha=d^*G\alpha$. In the same way we see that G commutes with d as well. □

Denote by $\mathcal{H}^p$ the vector space of *harmonic* p-forms, i.e. the space of forms φ which satisfy $d\varphi=d^*\varphi=0$. Then $\mathcal{H}^p$ coincides with the orthogonal complement in H_p^0 of the subspace $dH_{p-1}^1+d^*H_{p+1}^1$; in particular, $\mathcal{H}^p$ is closed. Now dH_{p-1}^1 and $d^*H_{p+1}^1$ are clearly orthogonal as well and hence we obtain an orthogonal decomposition $H_p^0=\mathcal{H}^p\oplus\overline{dH_{p-1}^1}\oplus\overline{d^*H_{p+1}^1}$ where $\overline{dH_{p-1}^1}$ denotes the closure of dH_{p-1}^1 in H_p^0. Next we investigate the spaces dH_{p-1}^1 and $\overline{dH_{p-1}^1}$ in more detail.

LEMMA 2.9. (i) $dd^*(\sum_{i=1}^k G^i\alpha)\to\alpha$ $(k\to\infty)$ *for every* $\alpha\in\overline{dH_{p-1}^1}$.

(ii) $d^*d(\sum_{i=1}^k G^i\alpha)\to\alpha$ $(k\to\infty)$ *for every* $\alpha\in\overline{d^*H_{p+1}^1}$.

Proof. We show the lemma for $\overline{dH_{p-1}^1}$, the statement for $\overline{d^*H_{p+1}^1}$ follows in the same way. Denote by $\|\cdot\|$ the norm on H_p^0 induced from the inner product $(\cdot,\cdot)$. Let $\alpha\in\overline{dH_{p-1}^1}$

be an element of unit norm $\|\alpha\|^2=1$, and let $\alpha_i=G^i\alpha\in\overline{dH^1_{p-1}}$. Then $d\alpha_i=0$ for $i\geqslant 1$ and hence $\alpha_i=(\mathrm{Id}+\mathcal{L})\alpha_{i+1}=\alpha_{i+1}+dd^*\alpha_{i+1}$, i.e. inductively $\alpha=\alpha_i+\sum_{j=1}^i dd^*\alpha_j$ for all $i\geqslant 1$. Moreover

$$\|\alpha_i\|^2=\|(\mathrm{Id}+\mathcal{L})\alpha_{i+1}\|^2=\|\alpha_{i+1}\|^2+2(\alpha_{i+1},dd^*\alpha_{i+1})+\|dd^*\alpha_{i+1}\|^2,$$

i.e. again inductively we see that $\|\alpha_i\|^2=1-\sum_{j=1}^i(2\|d^*\alpha_j\|^2+\|dd^*\alpha_j\|^2)$. This shows that the sequence $(\|\alpha_i\|)_{i\geqslant 1}$ is decreasing and the sequence $(d^*\alpha_j)_{j\geqslant 1}$ converges to zero in H^0_0.

We want to show that $\alpha_i\to 0$ $(i\to\infty)$ and for this it suffices to show that $\nu^2=\inf_{i\geqslant 1}\|\alpha_i\|^2=0$. Since $(\alpha_{2i})_{i>0}$ is a bounded sequence in the Hilbert space $\overline{dH^1_{p-1}}$ it admits a subsequence converging weakly to some α_∞. Then $d^*\alpha_i\to 0$ $(i\to\infty)$ implies $\alpha_\infty=0$.

Now a convex combination of a weakly convergent sequence is strongly convergent. This means that for every $\varepsilon>0$ there is a number $k=k(\varepsilon)>0$, integers $1\leqslant i(1)<\ldots<i(k)$ and numbers $\beta_j>0$ $(j=1,\ldots,k)$ such that $\sum_{j=1}^k\beta_j=1$ and $\left\|\sum_j\alpha_{2i(j)}\beta_j\right\|^2<\varepsilon$. But

$$\left\|\sum_j\alpha_{2i(j)}\beta_j\right\|^2=\sum_j\beta_j^2\|\alpha_{2i(j)}\|^2+2\sum_{j<l}\beta_j\beta_l\|\alpha_{i(j)+i(l)}\|^2\geqslant\nu^2$$

and consequently $\nu^2=0$; in particular, the sequence $dd^*\sum_{i=1}^k G^i\alpha$ converges strongly in H^0_1 to α $(k\to\infty)$. □

COROLLARY 2.10. (i) *$\alpha\in\overline{dH^1_{p-1}}$ is contained in dH^1_{p-1} if and only if the sequence $\left(d^*\left(\sum_{i=1}^k G^i\alpha\right)\right)_{k>0}$ is bounded in H^0_{p-1}.*

(ii) *$\alpha\in\overline{d^*H^1_{p+1}}$ is contained in $d^*H^1_{p+1}$ if and only if the sequence $\left(d\left(\sum_{i=1}^k G^i\alpha\right)\right)_{k>0}$ is bounded in H^0_{p-1}.*

Proof. Let $\alpha\in\overline{dH^1_{p-1}}$ and for $k>0$ write $\beta_k=d^*\sum_{i=1}^k G^i\alpha$. Assume that the sequence $(\beta_k)_{k>0}$ is bounded in H^0_{p-1}; by passing to a subsequence we may assume that the sequence $(\beta_k)_{k>0}$ converges weakly in H^0_{p-1} to a form β. We then have $\beta\in\overline{d^*H^1_p}$ and for every $\eta\in H^1_p$ moreover $(\beta_k,d^*\eta)\to(\beta,d^*\eta)$. On the other hand, Lemma 2.9 shows that $(\beta_k,d^*\eta)=(d\beta_k,\eta)\to(\alpha,\eta)$ $(k\to\infty)$ and consequently $\beta\in H^1_{p-1}$ and $d\beta=\alpha$.

Vice versa, let $\alpha=d\beta$ for some $\beta\in H^1_{p-1}$. Since $(\mathcal{H}_{p-1}\oplus\overline{dH^1_{p-2}})\cap H^1_{p-1}$ is contained in the kernel of d we may assume that $\beta\in\overline{d^*H^1_p}$. Then $d^*\left(\sum_{i=1}^k G^i\alpha\right)=d^*d\left(\sum_{i=1}^k G^i\beta\right)\to\beta$ $(k\to\infty)$ by Lemma 2.9; in particular, this sequence is bounded. This shows (i), and (ii) follows in the same way. □

The above considerations show that we may only consider operators of the form $\Delta+Y$ where Y is g-dual to a stably-closed section of T^*W^s. Namely, if Y is an arbitrary section of TW^s and if η is a harmonic measure for $L=\Delta+Y$, then we can decompose $Y=Y_1+Y_2$, where Y_1 is g-dual to an element of $\mathcal{H}^1\oplus\overline{dH_0^1}$, and Y_2 is g-dual to an element of $\overline{d^*H_2^1}$. Then $\int Y_2(f)\,d\eta=0$ for every smooth function f on T^1M and hence η is also a harmonic measure for $L+Y_1$. Notice however that there is a problem of regularity here: In general we can not expect that the sections Y_1, Y_2 are of class $C_s^{1,\alpha}$ for some $\alpha>0$ if this is true for Y.

Denote again by L the lift of L to $T^1\widetilde{M}$. For every $v\in T^1\widetilde{M}$ the restriction of L to $W^s(v)$ projects to a uniformly elliptic operator L_v on $(\widetilde{M}, \langle\cdot,\cdot\rangle)$ with pointwise uniformly bounded coefficients. Recall from the introduction that L is called *weakly coercive* if the operators L_v are weakly coercive in the sense of Ancona for all $v\in T^1\widetilde{M}$. The next lemma shows that weakly coercive operators do not admit self-adjoint harmonic measures.

LEMMA 2.11. *If* $\operatorname{pr}(g(X,Y))=0$ *then* L *is not weakly coercive.*

Proof. Assume that L is weakly coercive. Then there is a number $\delta>0$ such that $L+\delta$ is weakly coercive as well. This implies by the considerations in Appendix B that there is a Hölder-continuous section Z of TW^s over T^1M which satisfies

$$\operatorname{div}(Z)+g(Y,Z)+\|Z\|^2+\delta=0;$$

namely if $\widetilde{Z}$ denotes the lift of Z to $T^1\widetilde{M}$, then for every $v\in T^1\widetilde{M}$ the restriction of $\widetilde{Z}$ to $W^s(v)$ projects to the g-gradient of the logarithm of a minimal positive $(L_v+\delta)$-harmonic function with pole at $\pi(v)$.

Now assume to the contrary that L admits a self-adjoint harmonic measure η. Then $0=\int(\operatorname{div}(Z)+g(Y,Z))\,d\eta=-\int(\|Z\|^2+\delta)\,d\eta$ which is a contradiction and shows the lemma. □

Call $L=\Delta+Y$ *of gradient type* if Y is g-dual to a stably-closed section of T^*W^s. Next we describe the g-gradient of an arbitrary harmonic measure η for such an operator.

Namely, denote by L' the operator which is adjoint to L with respect to η, i.e. L' is defined by requiring that $\int(L'f)\psi\,d\eta=\int f(L\psi)\,d\eta$ for all smooth functions f,ψ on T^1M. Then we have

LEMMA 2.12. *Let* η *be a harmonic measure for* L *with* g*-gradient* $Y+Z$. *Then* Z *is* g*-dual to a harmonic section of* T^*W^s, *i.e. to an element of* $\mathcal{H}^1$, *and* $L'=L+2Z=\Delta+Y+2Z$.

Proof. Let α,β be smooth functions on T^1M. Since the operator $\Delta+Y+Z$ is self-adjoint with respect to η we have

$$\begin{aligned}\int\alpha(L\beta)\,d\eta&=\int\alpha(\Delta+Y+Z)(\beta)\,d\eta-\int\alpha(Z\beta)\,d\eta\\&=\int\beta(\Delta+Y+2Z)(\alpha)\,d\eta-\int Z(\alpha\beta)\,d\eta.\end{aligned}$$

But η is a harmonic measure for $\Delta+Y$ and $\Delta+Y+Z$, and this implies that $\int (Zf)\,d\eta=0$ for every smooth function f on T^1M. In particular, since Z is g-dual to a stably-closed section of T^*W^s this means that $Z\in\mathcal{H}^1$. From this the lemma follows. $\square$

Let now Q be the probability measure on the space Ω_+ of paths on T^1M which is obtained from P by a reversal of time. Let Λ_t (or Λ_t') be the action of $[0,\infty)$ on functions u on T^1M which describes the L-process (or the L'-process) on T^1M. For Borel subsets A, B of T^1M with characteristic functions χ_A, χ_B we then have

$$P\{\omega \mid \omega(0)\in A, \omega(t)\in B\} = \int \chi_A(\Lambda_t\chi_B)\,d\eta$$
$$= \int (\Lambda_t'\chi_A)\chi_B\,d\eta = Q\{\omega \mid \omega(0)\in B, \omega(t)\in A\},$$

and Q is induced by the L'-diffusion. In other words we have

COROLLARY 2.13. *The reversal of time of the L-diffusion on* $(T^1\widetilde{M},\eta)$ *is the* L'-*diffusion with* $L'=L+2Z$.

We conclude this section with the basic examples which were considered earlier in the literature.

Recall that the *Bowen–Margulis measure* μ on T^1M is the Gibbs equilibrium state of a constant function. There are families μ^i of conditional measures on the leaves of W^i $(i=ss, su)$ such that $d\mu=d\mu^{ss}\times d\mu^{su}\times dt$ (with respect to a local product structure) where dt is the one-dimensional Lebesgue measure on the flow lines of the geodesic flow. The measures μ^u on the leaves of W^u which are defined by $d\mu^u=d\mu^{su}\times dt$ are in fact *invariant* under canonical maps.

The above considerations are in particular valid for the Borel probability measure σ on T^1M which is locally the product of the Lebesgue measure λ^s on the leaves of W^s and the (normalized) conditionals of the Bowen–Margulis measure on the leaves of W^{su}, i.e. $d\sigma=d\lambda^s\times d\mu^{su}=d\lambda^{ss}\times d\mu^{su}\times dt$. Let Δ^s be the stable Laplacian, i.e. the leafwise Laplacian induced by the lift g_0 of the Riemannian metric on M.

From Lemma 2.5 we obtain immediately

COROLLARY 2.14. σ *is a self-adjoint harmonic measure for* Δ^s+hX.

Remark. We can also investigate harmonic measures for operators subordinate to the strong stable foliation. Namely, define an inner product $(\cdot,\cdot)_{ss}$ on the vector space $C^\infty_{ss}(\bigwedge^p T^*W^{ss})$ of sections of $\bigwedge^p T^*W^{ss}$ of class C^∞_{ss} by $(\varphi,\psi)_{ss}=\int g^{ss}(\varphi(v),\psi(v))\,d\sigma(v)$ where σ is defined as above and g^{ss} is the restriction of g_0 to TW^{ss}. Let $H^0_{p,ss}$ be the completion of $C^\infty_{ss}(\bigwedge^p T^*W^{ss})$ with respect to this inner product. As before, we can define a natural exterior derivation d_{ss} which is a densely defined linear operator of

$H^0_{p,ss}$ into $H^0_{p+1,ss}$; we denote its adjoint with respect to $(\cdot,\cdot)_{ss}$ by d^*_{ss}. Let $*_{ss}$ be the Hodge star operator on the leaves of W^{ss} with respect to the metric g^{ss}, viewed as a bundle isomorphism of $\bigwedge^p T^*W^{ss}$ onto $\bigwedge^{n-p-1} T^*W^{ss}$. As in the proof of Lemma 2.5 we obtain (see also [Kn], [L3] and [Ka2]):

The restriction of d^*_{ss} to $C^\infty_{ss}(\bigwedge^p T^*W^{ss})$ equals $(-1)^{(n-1)p+n} *_{ss} d_{ss} *_{ss}$, and σ is a self-adjoint harmonic measure for Δ^{ss}.

In fact, the measure σ is the unique harmonic measure for Δ^{ss}. Namely, the strong stable foliation is of subexponential growth and consequently *every* harmonic measure for Δ^{ss} is *fully invariant* ([Ka2]), i.e. it defines a transverse measure for the strong stable foliation which is *invariant* under canonical maps. On the other hand, an invariant transverse measure for W^{ss} is unique (up to a constant) and induces the measures μ^u on the transversals $W^u(v)$ $(v\in T^1M)$ to the strong stable foliation ([BM]).

The subspaces $d_{ss}H^1_{p,ss}$ are not closed in $H^0_{p+1,ss}$ (or the spaces $d^*_{ss}H^1_{p+1,ss}$ are not closed in $H^0_{p,ss}$). To see this, let $\mathcal{C}$ be the orthogonal complement of the space of constant function with respect to the L^2-inner product defined by σ. Observe that under the assumption that $d_{ss}H^1_{0,ss}$ is closed in $H^0_{1,ss}$, the differential d_{ss} is a continuous one-to-one linear mapping of the Hilbert space $H^1_{0,ss}\cap\mathcal{C}$ onto the Hilbert space $d_{ss}H^1_{0,ss}\subset H^0_{1,ss}$ and hence it admits a continuous linear inverse Ψ. Thus Ψ is in particular bounded, i.e. there is a number $\varrho>0$ such that $(d_{ss}\varphi, d_{ss}\varphi)_{ss}\geqslant\varrho(\varphi,\varphi)_{ss}$ for all $\varphi\in H^1_{0,ss}\cap\mathcal{C}$. On the other hand, if M is a compact locally symmetric space of negative curvature, then σ is just the Lebesgue measure λ, and in particular, σ is invariant under the geodesic flow. Let $f\colon T^1M\to\mathbf{R}$ be any smooth function with $\int f\,d\lambda=0$ and $\int f^2\,d\lambda=1$. For $t\in\mathbf{R}$ define $f_t=f\circ\Phi^t$. Then $(d_{ss}f_t, d_{ss}f_t)\to 0$ $(t\to\infty)$ but $f_t\in\mathcal{C}$ and $(f_t,f_t)_{ss}=1$ for all $t\in\mathbf{R}$ contradicting our assumption that $d_{ss}H^1_{0,ss}$ is closed in $H^0_{1,ss}$.

Recall that for every $y\in\widetilde{M}$ the ideal boundary $\partial\widetilde{M}$ can naturally be identified with the exit boundary for Brownian motion on $\widetilde{M}$ emanating from y. In other words, the Wiener measure on paths starting at y projects to a Borel probability measure ω^y on $\partial\widetilde{M}\sim T^1_y\widetilde{M}$. The measures ω^y transform under $\Gamma=\pi_1(M)$ via $\omega^{\Psi y}=\omega^y\circ(d\Psi)^{-1}$, and hence they project to measures on the fibres T^1_xM of the fibration $T^1M\to M$ $(x\in M)$. Define a Borel probability measure ω on T^1M by $\omega(A)=\int\omega^x(A\cap T^1_xM)\,d\lambda_M(x)$ where λ_M is the normalized Lebesgue measure on M. Then ω is the unique harmonic measure for the stable Laplacian Δ^s ([L3], see also [Y2] and [Ga]).

For $v\in T^1\widetilde{M}$ denote by $Y(v)$ the gradient at Pv of the logarithm of a minimal positive harmonic function with pole at the point $\pi(v)$ of the ideal boundary $\partial\widetilde{M}$. Via the natural identification of $W^s(v)$ with $\widetilde{M}$ the vector $Y(v)$ can be viewed as an element of T_vW^s. The assignment $v\to Y(v)$ is then a section of TW^s of class C^∞_s which is equivariant under the action of the fundamental group Γ of M on $T^1\widetilde{M}$, i.e. Y can be viewed as a vector field on T^1M. Clearly Y is the g_0-gradient of the measure ω. Hence we obtain

LEMMA 2.15. *$d^*\varphi=(-1)^{np+n+1}*d*\varphi-Y\rfloor\varphi$ for every $\varphi\in C_s^\infty(\Lambda^pT^*W^s)$ $(p\geqslant 1)$.*

Let now $\xi\in H_1^0$ be g_0-dual to the vector field Y. The following corollary is an immediate consequence of the above considerations.

COROLLARY 2.16. (i) *$d\xi=d^*\xi=0$, i.e. ξ is harmonic.*

(ii) *$\int\alpha(\Delta^s(\varphi)+Y(\varphi))\,d\omega=\int\varphi(\Delta^s(\alpha)+Y(\alpha))\,d\omega=-\int\langle\nabla^s\alpha,\nabla^s\varphi\rangle\,d\omega$ for all smooth functions α,φ on T^1M; in particular, ω is a self-adjoint harmonic measure for Δ^s+Y.*

(iii) *$\int Y(\alpha)\,d\omega=0$; in particular, $\int\alpha\Delta^s(\varphi)\,d\omega=\int\varphi(\Delta^s(\alpha)+2Y(\alpha))\,d\omega$ for all smooth functions α,φ on T^1M.*

3. Operators of non-zero escape

In this section we consider again an operator L of the form $L=\Delta+Y$ where Δ is the leafwise Laplacian of a positive semi-definite bilinear form g of class $C^{2,\alpha}$ on T^1M whose restriction to TW^s is positive definite and Y is a section of TW^s of class $C_s^{1,\alpha}$ which is g-dual to a stably-closed section of T^*W^s. We assume in addition that $\mathrm{pr}(g(X,Y))\neq 0$. By Corollary 2.7 this is equivalent to the non-existence of a self-adjoint harmonic measure for L. We then call L of *non-zero escape*, a notion which will be justified below.

The purpose of this section is to show that such an operator L is necessarily weakly coercive in the sense of Appendix B. First of all notice the following:

LEMMA 3.1. *For an operator L of non-zero escape there is a number $\varkappa>0$ with the following property: Let η be a harmonic measure for L with g-gradient $Y+Z$. Then $\int\|Z\|^2\,d\eta\geqslant\varkappa$.*

Proof. Assume to the contrary that for every $j>0$ there is a harmonic measure η_j for L with g-gradient $Y+Z_j$ and such that $\int\|Z_j\|^2\,d\eta_j<1/j$. Let η be a weak limit of a subsequence of the sequence $\{\eta_j\}_j$ which we denote again by $\{\eta_j\}$. For every section A of TW^s over T^1M of class C_s^1 we then have

$$\left|\int(\operatorname{div}(A)+g(Y,A))\,d\eta\right|=\lim_{j\to\infty}\left|\int g(Z_j,A)\,d\eta_j\right|$$
$$\leqslant\limsup_{j\to\infty}\left(\int\|A\|^2\,d\eta_j\right)^{1/2}\left(\int\|Z_j\|^2\,d\eta_j\right)^{1/2}=0$$

and hence η is a self-adjoint harmonic measure for L. This contradicts the assumption that $\mathrm{pr}(g(Y,X))\neq 0$. □

Let η be a harmonic measure for $L=\Delta+Y$ with g-gradient $Y+Z$. We use η to define the Hilbert space H_1^1 as in §2. The g-dual φ of Z is pointwise uniformly bounded in norm with pointwise uniformly bounded leafwise differential; in particular, φ is contained in H_1^1. Since $C_s^\infty(T^*W^s)$ is dense in H_1^1 we can approximate φ in H_1^1 by Hölder-continuous leafwise smooth sections of T^*W^s. However, since the harmonic section φ of T^*W^s (in the sense of §2) is in general not continuous it is a priori not clear whether φ can be approximated in H_1^1 by Hölder-continuous leafwise closed sections of T^*W^s. The following lemma answers this question in an affirmative way:

LEMMA 3.2. *Let $Y+Z$ be the g-gradient of η and let φ be g-dual to Z. Then there is a sequence $\{\varphi_i\}\subset C_s^{1,\alpha}(T^*W^s)$ of Hölder-continuous stably-closed forms φ_i with the following properties*:

(1) *$\varphi_i\to\varphi$ in H_1^1 $(i\to\infty)$.*

(2) *The forms φ_i are pointwise uniformly bounded in norm, independent of $i>0$.*

Proof. Write $f=\varphi(X)=g(X,Z)$. Recall that for η-almost every $v\in T^1M$ the restriction of Z to $W^s(v)$ is the g-gradient of the logarithm of a function ψ on $W^s(v)$ which satisfies $\Delta(\psi)+Y(\psi)=0$. In other words, ψ is a solution of an elliptic equation with coefficients of locally uniformly bounded $C^{1,\alpha}$-norm. Schauder theory for elliptic equations then shows that the restriction of the function f to a leaf of W^s is locally uniformly bounded in the $C^{2,\alpha}$-norm.

Choose a smooth partition of unity for T^1M, given by functions $\psi_1,...,\psi_k$ which are supported in open subsets $C_1,...,C_k$ with a local product structure. More precisely, we arrange the set C_i in such a way that the local product structure on C_i is given by a point $p_i\in M$, an open ball A_i about p_i in M, an open subset B_i of $T_{p_i}M$ and a homeomorphism $\Lambda_i: A_i\times B_i\to C_i$ which satisfies $\Lambda_i(y,w)\in W^s(w)$ and $P\circ\Lambda_i(y,w)=y$ for all $(y,w)\in A_i\times B_i$. Then for every $w\in B_i$ the restriction of Λ_i to $A_i\times\{w\}$ is smooth, and its jets of arbitrary degree depend Hölder continuously on w.

Denote by λ_M the Lebesgue measure on M. For every $y\in M$ there is a unique finite Borel measure η^y on T_y^1M such that $\eta(A)=\int\eta^y(A\cap T_y^1M)\,d\lambda_M(y)$ for every Borel set $A\subset T^1M$ (see [H2]). The measures η^y are positive on open sets. For every $i\in\{1,...,k\}$ the map Λ_i is absolutely continuous with respect to the measures $\lambda_M\times\eta^{p_i}$ on $M\times T_{p_i}^1M\supset A_i\times B_i$ and the measure η on $C_i\subset T^1M$, with uniformly bounded Jacobian.

For $w\in T^1M$ and $\varepsilon>0$ write $S(w,\varepsilon)=\{z\in T_{Pw}^1M\mid\angle(z,w)<\varepsilon\}$. Choose $\varepsilon_0>0$ sufficiently small that for every point z in the support of ψ_i the cone $S(z,2\varepsilon_0)$ is contained in C_i. Let $\alpha:\mathbf{R}\to[0,1]$ be a smooth function with $\alpha(t)=1$ for $t\leqslant\frac{1}{2}$, $\alpha(t)=0$ for $t\geqslant1$ and for $\varepsilon\leqslant\varepsilon_0$ and $w\in T^1M$ write

$$\alpha^\varepsilon(w)=\int_{S(w,\varepsilon)}\alpha(\angle(w,z)\varepsilon^{-1})\,d\eta^{Pw}(z)>0.$$

From the explicit description of the measures η^{Pw} $(w\in T^1M)$ ([H2]) it is apparent that the functions α^ε are Hölder continuous. For $i\in\{1,\dots,k\}$ and $\varepsilon<\varepsilon_0$ define a function f_i^ε on T^1M with support in C_i by

$$f_i^\varepsilon(\Lambda_i(y,w))=\alpha^\varepsilon(w)^{-1}\int_{S(w,\varepsilon)}(\psi_i f)(\Lambda_i(y,z))\,\alpha(\angle(w,z)\varepsilon^{-1})\,d\eta^{p_i}(z).$$

Then $f^\varepsilon=\sum_i f_i^\varepsilon$ is Hölder continuous and moreover pointwise uniformly bounded, independent of $\varepsilon>0$. The restriction of f to a leaf of the stable foliation is locally uniformly bounded in the $C^{1,\alpha}$-norm.

Recall from §2 the definition of the Hilbert space H_0^1 of functions on T^1M which are square integrable with respect to η, with square integrable leafwise differential. The functions f^ε converge as $\varepsilon\to0$ in H_0^1 to f. In fact, convergence even holds in the Sobolev-type space of functions which are of class L^{2n} (with respect to η) with leafwise differential again of class L^{2n}. The usual Sobolev embedding theorem then implies that for η-almost every $v\in T^1M$ the restriction of f^ε to $W^s(v)$ converges uniformly on compact subsets of $W^s(v)$ to the restriction of f as $\varepsilon\to0$.

Recall from the introduction the definition of the set $\widetilde{D}\subset T^1\widetilde{M}\times T^1\widetilde{M}$. Let $\tilde{f}^\varepsilon$ be the lift of f_ε to $T^1\widetilde{M}$. Then for every $v\in T^1\widetilde{M}$ the restriction of $\tilde{f}_\varepsilon$ to $W^s(v)$ is locally uniformly Hölder continuous, and hence there is a unique function $\tilde{\beta}^\varepsilon\colon\widetilde{D}\to\mathbf{R}$ such that $\tilde{\beta}^\varepsilon(v,\Phi^t v)=\int_0^t\tilde{f}^\varepsilon(\Phi^s v)\,ds$ for all $v\in T^1\widetilde{M}$ and $t\in\mathbf{R}$. For example, for $w\in W^{ss}(v)$ we have

$$\tilde{\beta}^\varepsilon(v,w)=\lim_{t\to\infty}\int_0^t(\tilde{f}^\varepsilon(\Phi^s w)-\tilde{f}^\varepsilon(\Phi^s v))\,ds$$

(compare [H2]).

The function $\tilde{\beta}^\varepsilon$ is invariant under the diagonal action of $\pi_1(M)=\Gamma$ on $\widetilde{D}\subset T^1\widetilde{M}\times T^1\widetilde{M}$ and satisfies $\tilde{\beta}^\varepsilon(v,z)=\tilde{\beta}^\varepsilon(v,w)+\tilde{\beta}^\varepsilon(w,z)$ for all $v\in T^1\widetilde{M}$ and all $w,z\in W^s(v)$. Moreover $\tilde{\beta}^\varepsilon$ is globally Hölder continuous.

Recall now that $\tilde{f}^\varepsilon$ is differentiable along the leaves of the stable foliation, with uniformly Hölder-continuous leafwise differential. This implies that there is a Hölder-continuous, $\pi_1(M)$-equivariant section $\tilde{\varphi}^\varepsilon$ of T^*W^s over $T^1\widetilde{M}$ such that for every $v\in T^1\widetilde{M}$ the restriction of $\tilde{\varphi}^\varepsilon$ to $W^s(v)$ is the leafwise differential of the function $w\to\tilde{\beta}^\varepsilon(v,w)$. We have $\tilde{\varphi}^\varepsilon(X)=\tilde{f}^\varepsilon$, and if $Y\in T_vW^{ss}$ is tangent to the strong stable foliation at v, then

$$\tilde{\varphi}^\varepsilon(Y)=\lim_{t\to\infty}\int_0^t d\Phi^s(Y)(\tilde{f}^\varepsilon)\,ds$$

(compare [LMM]).

The 1-form $\tilde{\varphi}^\varepsilon$ projects to a section φ^ε of T^*W^s over T^1M. Now φ^ε is in fact a form of class $C_s^{1,\alpha}$, which follows from the fact that f^ε is a function on T^1M of class $C_s^{2,\alpha}$.

For example we obtain the divergence of the g-dual of φ^ε at v simply by computing the derivatives as asymptotic integrals of second derivatives of f^ε as above (compare [LMM]).

Moreover the norm of φ^ε, viewed as an element of H_1^1, is uniformly bounded independent of $\varepsilon>0$.

Let now $\{\varepsilon_i\}_i$ be a sequence such that $\varepsilon_i\to 0$ $(i\to 0)$ and the sections φ^{ε_i} converge weakly in the Hilbert space H_1^1 to a section $\bar\varphi$. Then $\bar\varphi$ is stably-closed and a section of T^*W^s of class L^∞; moreover $\bar\varphi(X)=\varphi(X)$. But this necessarily implies that $\bar\varphi=\varphi$. Then a convex combination of the forms φ^{ε_i} converges strongly to φ in H_1^1 and defines a sequence as stated in the lemma. □

As an immediate corollary we obtain

COROLLARY 3.3. *There is a number $\chi>0$, an integer $k\geqslant 1$ and k sections $A_1, ..., A_k$ of TW^s over T^1M of class C_s^1 with the following properties*:

(1) $\|A_i\|(v)\leqslant 1$ *for all* $v\in T^1M$.

(2) A_i *is g-dual to a stably-closed section of* T^*W^s.

(3) *For every harmonic measure η for L there is $i\in\{1, ..., k\}$ such that*

$$\int(\operatorname{div}(A_i)+g(Y, A_i))\,d\eta\geqslant\chi.$$

Proof. Let η be a harmonic measure for L. By Lemma 3.1 and Lemma 3.2 there is a section A_η of TW^s of class C_s^1 such that $a_\eta=\int(\operatorname{div}(A_\eta)+g(Y, A_\eta))\,d\eta>0$.

Let $\mathcal{E}$ be the space of harmonic measures for L, equipped with the weak*-topology. Then $\mathcal{E}$ is a compact convex subspace of the space of probability measures on T^1M. For every $\eta\in\mathcal{E}$ the set $U_\eta=\{\zeta\in\mathcal{E}\mid\int(\operatorname{div}(A_\eta)+g(Y, A_\eta))\,d\zeta>\frac12 a_\eta\}$ is a weak*-open neighborhood of η in $\mathcal{E}$. Choose finitely many $\eta_1, ..., \eta_k\in\mathcal{E}$ such that $\mathcal{E}\subset\bigcup_{i=1}^k U_{\eta_i}$. Then the corollary is satisfied with $A_i=A_{\eta_i}$ and $\chi=\min\{\frac12 a_{\eta_i}\mid i=1, ..., k\}$. □

As in §2 denote by $\widetilde\Omega_+$ the space of continuous paths $\xi\colon[0,\infty)\to T^1\widetilde M$ and for $v\in T^1\widetilde M$ let $\widetilde P^v$ be the probability measure on $\widetilde\Omega_+$ which describes the diffusion on $W^s(v)$ induced by $L|_{W^s(v)}$ with initial probability δ_v.

Let moreover Ω_+ be the space of continuous paths $\omega\colon[0,\infty)\to T^1M$ and for $v\in T^1M$ denote by P^v the probability measure on Ω_+ which lifts to the measure $\widetilde P^w$ for one and hence every lift w of v to $T^1\widetilde M$.

For $i\in\{1, ..., k\}$ and $t>0$ define now a function $f_t^i\colon\Omega_+\to\mathbf{R}$ as follows: Let $w\in\Omega_+$ and let $\widetilde\omega\in\widetilde\Omega_+$ be a lift of ω. The restriction to $W^s(\widetilde\omega(0))$ of the lift of the section A_i from Corollary 3.3 is the differential of a function α_i. Define $f_t^i(\omega)=\alpha_i(\widetilde\omega(t))-\alpha_i(\widetilde\omega(0))$; this does not depend on the choice of the lift $\widetilde\omega$. If $\{T^t\mid t>0\}$ is the semi-group of shift transformations on Ω_+ then we have $f_{s+t}^i(\omega)=f_s^i(\omega)+f_t^i(T^s\omega)$.

Let again $\chi>0$ be as in Corollary 3.3. The proof of the next lemma is essentially due to Ledrappier ([L4]):

LEMMA 3.4. *For every $\varepsilon>0$ there is a number $T(\varepsilon)>0$ such that*

$$\max_{1\leqslant i\leqslant k}\frac{1}{T}\int f_T^i\,dP^v\geqslant\chi-\varepsilon$$

for all $v\in T^1M$ and all $T\geqslant T(\varepsilon)$.

Proof. (Compare the proof of Proposition 2 in [L4].) We argue by contradiction and we assume that the lemma is false. Then there are numbers $T_n>0$ such that $T_n\to\infty$ $(n\to\infty)$ and points $v_n\in T^1M$ such that $(1/T_n)\int f_{T_n}^i\,dP^{v_n}<\chi-\varepsilon$ for every $i\in\{1,...,k\}$. By our assumption we can find a number $t_0>0$ small enough that

$$\sup_{0\leqslant t\leqslant t_0}\sup_{w\in T^1M}\left|\int f_t^i\,dP^w\right|\leqslant\tfrac{1}{4}\varepsilon$$

for every $i\in\{1,...,k\}$. By the Markov property for the L-diffusion and the fact that $f_{s+t}^i(\omega)=f_s^i(\omega)+f_t^i(T^s\omega)$ there are then integers $N_j>0$ such that $N_j\to\infty$ $(j\to\infty)$ and

$$\frac{1}{N_jt_0}\int f_{N_jt_0}^i\,dP^{v_j}<\chi-\tfrac{1}{2}\varepsilon.$$

Denote by Q_t the action of $[0,\infty)$ on functions on T^1M which describes the L-diffusion. Take a weak limit μ of a subsequence of the sequence μ_j of probability measures on T^1M defined by $\mu_j=(1/N_j)\sum_{k=0}^{N_j-1}Q_{kt_0}\delta_{v_j}$ where δ_{v_j} is the Dirac mass at v_j. Then μ is Q_{t_0}-invariant and

$$\frac{1}{t_0}\int f_{t_0}^i\,d\mu\leqslant\chi-\tfrac{1}{2}\varepsilon$$

for every $i\in\{1,...,k\}$. Let $\mu'=(1/t_0)\int_0^{t_0}(Q_s\mu)\,ds$. Then μ' is Q_t-invariant and hence a harmonic measure for L. On the other hand we have $(1/t_0)\int f_{t_0}^i\,d\mu'\leqslant\chi-\frac{1}{4}\varepsilon$ for $i=1,...,k$, which is a contradiction to the fact that $\max_{1\leqslant i\leqslant k}\lim_{t\to\infty}(1/t)\int f_t^i\,d\mu'\geqslant\chi$ by Corollary 3.3. This shows the lemma. □

Let again $\omega\in\Omega_+$ and let $\tilde\omega\in\tilde\Omega_+$ be a lift of ω. For $t>0$ define

$$\varphi_t(\omega)=\operatorname{dist}(P\tilde\omega(0),P\tilde\omega(t));$$

this clearly does not depend on the choice of $\tilde\omega$. Since for every $i\in\{1,...,k\}$ the g-norm of A_i is pointwise bounded by 1 there is a constant $\beta>0$ such that

$$\varphi_t(\omega)\geqslant\beta\max_{1\leqslant i\leqslant k}|f_t^i(\omega)|$$

for all $t>0$ and all $\omega\in\Omega_+$. This together with Lemma 3.4 then shows

COROLLARY 3.5. *There are numbers $T_0>0$, $b>0$ such that $(1/T)\int \varphi_T\, dP^v \geqslant b$ for all $v\in T^1M$ and all $T\geqslant T_0$.*

Now by the subadditive ergodic theorem, for every harmonic measure η for L, for η-almost every $v\in T^1M$ and P^v-almost every ω the limit $\varphi_\infty(\omega)=\lim_{t\to\infty}(1/t)\varphi_t(\omega)$ exists. The assignment $\omega\to\varphi_\infty(\omega)$ is measurable and invariant under the shift. We call $\int \varphi_\infty\, dP^v\, d\eta(v)$ the *non-signed escape rate* of the diffusion induced by L and η. By Corollary 3.5 this non-signed escape rate is not smaller than $b>0$ for all η. The arguments of Prat ([Pr]) then imply that for every $v\in T^1\widetilde{M}$ and P^v-almost every path $\omega\in\widetilde{\Omega}_+$ the limit $\lim_{t\to\infty}\omega(t)=\omega(\infty)$ exists and is contained in $\partial\widetilde{M}$ and consequently the measure P^v projects to a probability measure ζ_v on $\partial\widetilde{M}$. The measures ζ_v $(v\in T^1\widetilde{M})$ are then equivariant under the action of $\pi_1(M)$ on $T^1\widetilde{M}$ and $\partial\widetilde{M}$. The following lemma gives some properties of the measures ζ_v.

LEMMA 3.6. *For $L=\Delta+Y$ with $\mathrm{pr}(g(Y,X))\neq 0$ the following are equivalent:*

(1) *There is $v\in T^1\widetilde{M}$ such that the support of ζ_v is not $\pi(v)$.*

(2) *For every $v\in T^1\widetilde{M}$, ζ_v does not have an atom at $\pi(v)$.*

Proof. Clearly (1) above is a consequence of (2). Thus we assume that (1) above is satisfied.

Denote by S the set of all vectors $v\in T^1M$ with the property that for one (and hence every) lift $\tilde{v}$ of v to $T^1\widetilde{M}$ the support of $\zeta_{\tilde{v}}$ is not equal to $\pi(\tilde{v})$. By our assumption S is not empty; moreover S consists of full stable manifolds.

We show first that $S=T^1M$, and for this it is enough to show that for $p\in M$ the intersection of S with T^1_pM is open in T^1_pM.

As in the introduction, denote for $w\in T^1\widetilde{M}$ and $\alpha>0$ by $C(w,\alpha)$ the open cone of angle α about w in $\widetilde{M}$, i.e. $C(w,\alpha)=\{P\Phi^t z\,|\, z\in T^1_{Pw}\widetilde{M},\ \angle(w,z)<\alpha,\ t\in(0,\infty)\}$. Let $\partial C(w,\alpha)$ be the boundary of $C(w,\alpha)$ as a subset of $\widetilde{M}\cup\partial\widetilde{M}$.

Let $v\in T^1\widetilde{M}$ be a lift of a point of S and $\alpha_0\in(0,\pi)$ be such that $\varrho=\zeta_v(\partial C(-v,\alpha_0))>0$. Choose numbers $\alpha_1\in(\alpha_0,\pi)$, $\alpha_2\in(\alpha_1,\pi)$. By Corollary 3.5 and the arguments of Prat ([Pr]) there is a number $\tau>0$ such that for every $w\in T^1\widetilde{M}$ and every $z\in T^1_{P_w}\widetilde{M}$ we have

$$\zeta_w(\partial C(z,\alpha_2))+\tfrac{1}{6}\varrho \geqslant \zeta_w\{\omega\,|\,P\omega(\tau)\in C(z,\alpha_1)\}\geqslant \zeta_w(\partial C(z,\alpha_0))-\tfrac{1}{6}\varrho.$$

By Ito's formula (compare [Pr]) there is a number $R>0$ such that

$$P^w\{\omega\,|\,\mathrm{dist}(\omega(0),\omega(\tau))\geqslant R\}<\tfrac{1}{6}\varrho$$

for every $w \in T^1\widetilde{M}$, where $\tau > 0$ is as before. Let $B \subset \widetilde{M}$ be the open ball of radius R about Pv in $\widetilde{M}$. Then

$$\int_{Pz \in C(-v,\alpha_1)\cap B} p(v,z,\tau)\, d\nu^s(z) \geqslant \tfrac{2}{3}\varrho$$

by the above consideration.

By Corollary A.5 from Appendix A the kernel p is Hölder continuous and hence there is an open neighborhood U of v in $T^1_{Pv}\widetilde{M}$ such that

$$\int_{Pz \in C(-v,\alpha_1)\cap B} p(w,z,\tau)\, d\nu^s(z) \geqslant \tfrac{1}{2}\varrho$$

for every $w \in U$. But this just means by the above that $\zeta_w(\partial C(-v,\alpha_2)) \geqslant \frac{1}{3}\varrho$ for every $w \in U$. In other words, the projection of U to T^1M is contained in S. This then shows that for every $w \in T^1\widetilde{M}$ the support of ζ_w is not $\pi(w)$.

For $v \in T^1M$ write now $A_v = \{\omega \in \Omega_+ \mid \omega(0) = v,\ \lim_{t\to\infty} \widetilde{\omega}(t) = \pi(\widetilde{v})$ for a lift $\widetilde{\omega}$ of ω with $\widetilde{\omega}(0) = \widetilde{v}\}$, and let $A = \bigcup_{v\in T^1M} A_v$. Then A is a subset of Ω_+ which is invariant under the shift, and $P^v(A) < 1$ for every $v \in T^1M$ by the above. But this implies that for every ergodic harmonic measure η for L we have $P(A) = 0$ where $P = \int P^v\, d\eta(v)$. Since ergodic harmonic measures for L are just extremal points in the space of all harmonic measures, this implies that $P(A) = 0$ for every measure P of the form $\int P^v\, d\eta(v)$ where η is an arbitrary harmonic measure for L.

On the other hand, every shift invariant measure for the diffusion induced by L is of this form and thus we conclude that $P^v(A) = 0$ for every $v \in T^1M$. This is equivalent to saying that for every $\widetilde{v} \in T^1\widetilde{M}$ the measure $\zeta_{\widetilde{v}}$ does not have an atom at $\pi(\widetilde{v})$. In other words, (2) above follows from (1), and hence (1) and (2) are equivalent. □

Let now $\overline{X}$ be the section of TW^s over T^1M whose restriction to $W^s(v)$ equals the g-gradient of the negative of a Busemann function at $\pi(v)$. If g is the lift g_0 of the Riemannian metric on M, then $\overline{X}$ just coincides with the geodesic spray X. Let η be a harmonic measure for L and define the *signed escape rate* of the L-diffusion to be

$$l_\eta(L) = -\int (\operatorname{div}(\overline{X}) + g(Y, \overline{X}))\, d\eta.$$

Notice that a priori $l_\eta(L)$ depends on the choice of the harmonic measure η. However we obtain the following.

COROLLARY 3.7. *Assume that L satisfies the assumption in Lemma* 3.6 *and let $b > 0$ be as in Corollary* 3.5. *Then $l_\eta(L) \geqslant b$ for every harmonic measure η for L.*

Proof. It suffices to show the corollary for ergodic harmonic measures η for L. Let η be such a measure, let P be the measure on Ω_+ derived from η and let $\widetilde{\omega} \in \widetilde{\Omega}_+$ be the lift to $T^1\widetilde{M}$ of a typical path for P. Let θ be the lift to $W^s(\widetilde{\omega}(0))$ of the Busemann function at $\pi(\widetilde{\omega}(0))$ which is normalized at $P\widetilde{\omega}(0)$. By Ito's formula and the Birkhoff ergodic theorem we then have

$$\lim_{t\to\infty}\frac{1}{t}(\theta(\widetilde{\omega}(t))-\theta(\widetilde{\omega}(0)))=-\int(\operatorname{div}(\overline{X})+g(Y,\overline{X}))\,d\eta.$$

On the other hand, since $\widetilde{\omega}(\infty)\neq\pi\widetilde{\omega}(0)$ by Lemma 3.6 there are numbers $t_0>0$, $R>0$ such that $\theta(\widetilde{\omega}(t))\geqslant\operatorname{dist}(P\widetilde{\omega}(0),P\widetilde{\omega}(t))-R$ for all $t\geqslant t_0$. This then implies that $l_\eta(L)=-\int(\operatorname{div}(\overline{X})+g(Y,\overline{X}))\,d\eta\geqslant b$ by Corollary 3.5. □

In the sequel we call an operator L which satisfies the assumption of Lemma 3.6 *of positive escape.*

For a number $t>0$ define a function $\sigma_t:\Omega_+\to\mathbf{R}$ as follows: Let $\omega\in\Omega_+$ and let $\widetilde{\omega}$ be a lift of ω to $T^1\widetilde{M}$. Denote again by $\theta^{\widetilde{\omega}(0)}$ the function on $W^s(\widetilde{\omega}(0))$ which satisfies $\theta^{\widetilde{\omega}(0)}(\widetilde{\omega}(0))=0$ and which projects to the negative of a Busemann function on $\widetilde{M}$ at $\pi(v)$. Define $\sigma_t(\omega)=\theta^{\widetilde{\omega}(0)}(\widetilde{\omega}(t))$; this does not depend on the choice of the lift $\widetilde{\omega}$ of ω.

For an operator of positive escape the arguments in the proof of Lemma 3.4 imply (compare also [L4]):

LEMMA 3.8. *If L is of positive escape, then for every $\varepsilon>0$ there is a number $T(\varepsilon)>0$ such that $(1/T)\int\sigma_T\,dP^v\geqslant b-\varepsilon$ for all $v\in T^1M$ and all $T\geqslant T(\varepsilon)$, where $b>0$ is as in Corollary* 3.5.

From Lemma 3.8 we conclude with the arguments of Ledrappier (see Proposition 3 in [L4]):

LEMMA 3.9. *If L is of positive escape, then there is a number $\tau_0>0$ and for every $\tau\in(0,\tau_0]$ a number $\zeta=\zeta(\tau)<1$ such that $\int e^{-\tau\sigma_t}\,dP^v<\zeta^t$ for all sufficiently large $t>0$ and all $v\in T^1M$.*

Proof. Again we follow Ledrappier. By the Markov property and the properties of the functions σ_t it suffices to show the lemma for a fixed time T.

For $t>0$ define a function ψ_t on Ω_+ as follows: Let $\omega\in\Omega_+$ and let $\widetilde{\omega}$ be any lift of ω to $T^1\widetilde{M}$. Then $\psi_t(\omega)=(\operatorname{dist}(P\widetilde{\omega}(0),P\widetilde{\omega}(t)))^2e^{\operatorname{dist}(P\widetilde{\omega}(0),P\widetilde{\omega}(t))}$.

Choose $T>T\left(\frac{1}{2}b\right)$ as in Lemma 3.8. We then have $e^{-\tau\sigma_t}\leqslant1-\tau\sigma_t+2\tau^2\psi_t$ for $t\leqslant T$ and $\tau>0$.

Since the coefficients of the differential operators L_v on $\widetilde{M}$ are uniformly bounded with respect to the Riemannian metric $\langle\cdot,\cdot\rangle$, independent of $v\in T^1\widetilde{M}$, a comparison argument shows that there is a constant $C>0$ such that $\int\psi_t\,dP^v\leqslant C$ for all $v\in T^1M$ and all $t\leqslant T$. By Lemma 3.8 we then have

$$\int e^{-\tau\sigma_T}\,dP^v\leqslant1-\tfrac{1}{2}\tau b+2\tau^2C$$

and moreover

$$\int e^{-\tau\sigma_t}\, dP^v \leqslant 1+\tau C+2\tau^2 C$$

for all $t \leqslant T$.

Choose now $\tau>0$ sufficiently small that $a=1-\frac{1}{2}\tau b+2\tau^2 C<1$. If $k\geqslant 1$ is sufficiently large that $\bar{\zeta}=a^k(1+\tau C+2\tau^2 C)<1$ then we obtain the lemma for this number τ with $\zeta=\bar{\zeta}^{1/T^k}$. □

COROLLARY 3.10. *Let $L=\Delta+Y$ be as before. If L is of positive escape then L is weakly coercive.*

Proof. Assume again that L is of positive escape. Recall the definition of the subset $\widetilde{D}$ of $T^1\widetilde{M}\times T^1\widetilde{M}$ from the introduction and let $p: \widetilde{D}\times(0,\infty)\to(0,\infty)$ be the fundamental solution of the Cauchy problem $L-\partial/\partial t=0$ on $T^1\widetilde{M}$. Let $v\in T^1\widetilde{M}$ and for $r>0$ let B_r be a ball of radius r about v in $W^s(v)$. Let $\tau>0$, $\zeta=\zeta(\tau)<1$ be as in Lemma 3.9. Then $e^{-\tau\theta_v(w)}\geqslant c_r>0$ for all $w\in B_r$.

Choose $t_0>0$ such that for all $t>t_0$ the conclusions of Lemma 3.9 are satisfied, and let $\varepsilon=-\frac{1}{2}\log\zeta>0$. Then

$$\begin{aligned}\int_{B_r} e^{\varepsilon t} p(v,w,t)\, d\nu^s(w) &\leqslant \frac{1}{c_r}\int_{B_r} e^{\varepsilon t} p(v,w,t)\, e^{-\tau\theta_v(w)}\, d\nu^s(w)\\ &\leqslant \frac{1}{c_r} e^{\varepsilon t}\int e^{-\tau\sigma_t}\, dP^v < \frac{1}{c_r} e^{-\varepsilon t}\end{aligned}$$

by Lemma 3.9, and consequently the Harnack inequality for parabolic equations implies that for $v\neq w$ the integral $\int_0^\infty e^{\varepsilon t} p(v,w,t)\, dt$ is finite. But this just means that there is a positive $(L_v+\varepsilon)$-superharmonic function on $\widetilde{M}$; in other words, L is weakly coercive. □

We are left with the investigation of operators $L=\Delta+Y$ as above with $\mathrm{pr}(g(X,Y))\neq 0$ which do not have the properties described in Lemma 3.6. We call such an operator *of negative escape*. In other words, if L is of negative escape, then for every $v\in T^1\widetilde{M}$ the measure P^v projects to the Dirac mass at $\pi(v)$.

For a harmonic measure η for L denote again by $l_\eta(L)=-\int(\mathrm{div}(\overline{X})+g(Y,\overline{X}))\, d\eta$ the signed escape rate of the L-diffusion with respect to η. We want to show that $l_\eta(L)\leqslant -b$ for every harmonic measure η, where $b>0$ is as in Corollary 3.5.

For this denote by DTM the smooth fibre bundle over M whose fibre at $x\in M$ consists of pairs (v,w) of elements of T^1_xM and denote by $DT\widetilde{M}$ the corresponding fibre bundle over $\widetilde{M}$. We then obtain a Hölder-continuous foliation DW^s on $DT\widetilde{M}$ by requiring that the leaf of DW^s through $(v,z)\in DT\widetilde{M}$ consists of all points $(w,u)\in DT\widetilde{M}$ with $\pi(u)=\pi(z)$ and $\pi(v)=\pi(w)$. The first factor projection $R_1: DT\widetilde{M}\to T^1\widetilde{M}$ and the second factor projection $R_2: DT\widetilde{M}\to T^1\widetilde{M}$ map the foliation DW^s to the stable foliation of $T^1\widetilde{M}$; moreover we have a natural embedding $(T^1\widetilde{M}, W^s) \to (DT\widetilde{M}, DW^s)$ of foliated spaces by mapping $v \in T^1\widetilde{M}$ to the element (v, v) of the *diagonal* in $DT\widetilde{M}$. In the sequel we identify $T^1\widetilde{M}$ with this diagonal.

The fundamental group $\pi_1(M)$ of M acts naturally on $DT\widetilde{M}$ and this action preserves the foliation DW^s. Thus we obtain a corresponding foliation DW^s on DTM and an embedding $(T^1M, W^s) \to (DTM, DW^s)$ of foliated spaces as before. The structure of this foliation can be described as follows:

LEMMA 3.11. *Every leaf of $DW^s \subset DTM$ contains the diagonal in its closure.*

Proof. Recall that the closure of every leaf of DW^s in DTM is a union of leaves and that moreover every leaf of the stable foliation of T^1M is dense in T^1M. Thus it suffices to show that the closure of every leaf of DW^s contains a point of the diagonal. For this let $(v, w) \in DT\widetilde{M}$ and let $\zeta \in \partial\widetilde{M} - \{\pi(v), \pi(w)\}$. If $\{x_j\} \subset \widetilde{M}$ is any sequence of points which converges as $j \to \infty$ in $\widetilde{M} \cup \partial\widetilde{M}$ to ζ, then the angle under which the points $\pi(v), \pi(w)$ are seen at x_j tends to zero as $j \to \infty$. From this the lemma follows. □

Recall from the introduction the definition of the Gromov product on $\partial\widetilde{M}$ (see [GH]). Namely for $x \in \widetilde{M}$ and $\zeta, \eta \in \partial\widetilde{M}$ define

$$(\zeta|\eta)_x = \lim_{\substack{y \to \zeta \\ z \to \eta}} \tfrac{1}{2}(\mathrm{dist}(x, y) + \mathrm{dist}(x, z) - \mathrm{dist}(y, z))$$

and for $x \in \widetilde{M}$ and $v \neq w \in T_x^1\widetilde{M}$ write also $(v|w) = (\pi(v)|\pi(w))_x$. There is then a number $c > 0$ only depending on the curvature bounds such that $(\angle(v, w))^c \leqslant e^{-(v|w)} \leqslant (\angle(v, w))^{1/c}$ for all $v, w \in T_x^1\widetilde{M}$ and all $x \in \widetilde{M}$; in particular, for a sufficiently small number $\tau > 0$ the assignment $(v, w) \to e^{-\tau(v|w)}$ defines a distance on the fibres of the fibration $T^1\widetilde{M} \to \widetilde{M}$.

For $v \in T^1\widetilde{M}$ let again θ_v be the Busemann function at $\pi(v)$ normalized by $\theta_v(Pv) = 0$. Recall the following observation (see [GH]) which we state as a lemma for further reference:

LEMMA 3.12. $(\pi(v)|\pi(w))_y - (\pi(v)|\pi(w))_x = \frac{1}{2}(\theta_v(y) + \theta_w(y))$ *for all $x, y \in \widetilde{M}$ and all $v \neq w \in T_x^1\widetilde{M}$.*

Now the assignment $(v, w) \to (v|w)$ can be viewed as a function on the complement of the diagonal in $DT\widetilde{M}$ which is clearly invariant under the action of the fundamental group of M on $DT\widetilde{M}$ and hence it descends to a function on the complement of the diagonal in DTM which we denote by ϱ.

Notice that ϱ is well defined and continous on $DTM - T^1M$ and $\varrho(v, w) \to \infty$ if and only if (v, w) converges to the diagonal.

Recall that the first factor projection $DTM \to T^1M$ maps DW^s to the stable foliation, and hence the operator L lifts to a leafwise elliptic differential operator DL on (DTM, DW^s), with Hölder-continuous coefficients and without zero-order terms.

In other words, DL induces a diffusion process on DTM which restricts to the L-diffusion on the diagonal.

After this preparation we are ready to show

LEMMA 3.13. *If L is of negative escape, then $l_\eta(L) \leqslant -b$ for every harmonic measure η for L, where $b>0$ is as in Corollary* 3.5.

Proof. We argue by contradiction and we assume that the lemma does not hold. Denote by Q_t the action of $[0,\infty)$ on functions on T^1M which describes the L-diffusion. Then there is $v \in T^1M$ and a sequence $\{t_j\}_j \subset [0,\infty)$ with $t_j \to \infty$ $(j \to \infty)$ and such that the following is satisfied:

(1) The measures $\mu_j = (1/t_j)\int_0^{t_j}(Q_t\delta_v)\,dt$ converge weakly as $j \to \infty$ to a harmonic measure η.

(2) For P^v-almost every path ω the limit $\lim_{t\to\infty}(1/t)\varphi_t(\omega)$ exists and equals $\bar{b} \geqslant b > 0$ where φ_t is defined as in Corollary 3.5.

(3) For P^v-almost every path ω the limit $\lim_{t\to\infty}\sigma_t(\omega)$ exists and equals $c > -b$ where σ_t is as in Lemma 3.8.

Let now $w \neq v$ and consider the restriction of the diffusion induced by DL on the leaf $DW^s(v,w)$ of DW^s. Denote by $P^{(v,w)}$ the corresponding probability measure on the space of paths in DTM with initial condition (v,w). We claim that for $P^{(v,w)}$-almost every path ω the limit

$$\lim_{t\to\infty} \frac{1}{t}\varrho(\omega(t))$$

exists and equals $\frac{1}{2}(\bar{b}+c) > 0$. To see this consider a lift $(\tilde{v},\widetilde{w})$ of (v,w) to $DT\widetilde{M}$. The restriction to $DW^s(\tilde{v},\widetilde{w})$ of the DL-diffusion can be identified with the diffusion induced by L on $W^s(\tilde{v})$. Let $\theta_{\widetilde{w}}$ be the Busemann function at $\pi(\widetilde{w})$ which is normalized by $\theta_{\widetilde{w}}(P\widetilde{w}) = 0$. Since L is of negative escape, $P^{\tilde{v}}$-almost every path converges to $\pi(\tilde{v}) \neq \pi(\widetilde{w})$. But this just means that for $P^{\tilde{v}}$-almost every path ω the limit $\lim_{t\to\infty}\theta_{\widetilde{w}}(\omega(t))/t$ exists and equals $\bar{b}$, where $\bar{b}>0$ is as above. On the other hand, by our assumption (3) above the limit $\lim_{t\to\infty}\theta_{\tilde{v}}(\omega(t))/t$ exists P^v-almost everywhere as well and equals c. It is then immediate from Lemma 3.12 that $\lim_{t\to\infty}\varrho(\omega(t))/t = \frac{1}{2}(\bar{b}+c)/t > 0$ for $P^{(v,w)}$-almost every ω. In other words, $P^{(v,w)}$-almost every path ω of the DL-diffusion approaches the diagonal in DTM as $t \to \infty$. But this contradicts the fact that the projection of $P^{\tilde{v}}$ to $\partial\widetilde{M}$ equals the Dirac mass at $\pi(\tilde{v})$ and $\pi(\widetilde{w}) \neq \pi(\tilde{v})$. This contradiction then finishes the proof of the lemma. □

Now Lemma 3.13 together with the arguments in the proof of Lemma 3.9 and Lemma 3.10 show that an operator L of negative escape is weakly coercive as well. In other words we have shown

PROPOSITION 3.14. *If* $\mathrm{pr}(g(X,Y))\neq 0$ *then* $L=\Delta+Y$ *is weakly coercive.*

4. Weakly coercive operators

In this section we investigate an operator L of gradient type of the form $L=\Delta+Y$ with $\mathrm{pr}(g(X,Y))\neq 0$. Proposition 3.14 shows that L is weakly coercive. We continue to use the assumptions and notations from §2. Our goal is the proof of Theorem A from the introduction. The next lemma is partially a consequence of the considerations in §3.

LEMMA 4.1. *For a weakly coercive operator* $L=\Delta+Y$ *the following are equivalent*:

(1) *There is a harmonic measure* η *for* L *with* $l_\eta(L)<0$.

(2) *For every ergodic harmonic measure* η *for* L, $l_\eta(L)$ *equals the negative of the non-signed escape rate for the diffusion induced by* (L,η).

(3) *There is* $v\in T^1\widetilde{M}$ *such that the minimal positive* L_v*-harmonic function on* $\widetilde{M}$ *with pole at* $\pi(v)$ *is constant.*

(4) *For every* $v\in T^1\widetilde{M}$ *the minimal positive* L_v*-harmonic function with pole at* $\pi(v)$ *is constant.*

Proof. Let $A\subset T^1\widetilde{M}$ be the set of all vectors $v\in T^1\widetilde{M}$ with the property that the minimal positive L_v-harmonic function with pole at $\pi(v)$ is constant. Then A consists of full stable manifolds and is invariant under the action of $\pi_1(M)$ on $T^1\widetilde{M}$.

Assume now that (3) is satisfied, i.e. that $A\neq\varnothing$. Then for every $p\in\widetilde{M}$ the set $A\cap T^1_p\widetilde{M}$ is dense in $T^1_p\widetilde{M}$. Thus for an arbitrary $v\in T^1\widetilde{M}$ and every $\varepsilon>0$ there is a point $w\in T^1_{Pv}\widetilde{M}\cap A$ with $\angle(v,w)<\varepsilon$. Let f be a minimal L_v-harmonic function on $\widetilde{M}$ with pole at $\pi(v)$. Since the constant function is minimal L_w-harmonic with pole at $\pi(w)$ the Harnack inequality at infinity (Corollary B.5 of Appendix B) shows that the restriction of f to the cone $C(-v,\pi-2\varepsilon)$ is bounded from below by a positive constant. Martin's theory then implies that the support of the L_v-harmonic measure at Pv is contained in the intersection with $\partial\widetilde{M}$ of the closure of $C(v,2\varepsilon)$ in $\widetilde{M}\cup\partial\widetilde{M}$. Since $\varepsilon>0$ was arbitrary we conclude that the harmonic measure for L_v is an atom at $\pi(v)$, in other words we have $v\in A$. This shows that (3) and (4) above are equivalent.

Assume now that (4) above is satisfied and let η be an ergodic harmonic measure for L. Since L is weakly coercive, the non-signed escape rate for L is positive; moreover for η-almost every $v\in T^1\widetilde{M}$ the exit boundary of the L_v-diffusion consists of the single point $\pi(v)$ by our assumption (4). With the notations from §3 this just means that L is of negative escape, which implies (2) by the arguments in §3.

On the other hand, (2) clearly implies (1). But if (1) is satisfied, then L does not satisfy the assumption in Lemma 3.6 and hence for every $v\in T^1\widetilde{M}$ the exit boundary of the diffusion induced by L_v is the single point $\pi(v)$ which implies (4). □

As before, we call an operator L as in Lemma 4.1 *of negative escape.*

LEMMA 4.2. *If L is of negative escape then* $\mathrm{pr}(g(X,Y))<0$.

Proof. Since $\mathrm{pr}(g(X,Y))\neq 0$ by Lemma 2.11 we may assume to the contrary that $\alpha=\mathrm{pr}(g(X,Y))>0$. Let ϱ^{ss} be a family of conditional measures on strong stable manifolds for the Gibbs equilibrium state of $g(X,Y)$ such that $d(\varrho^{ss}\circ\Phi^t)/dt|_{t=0}=-g(X,Y)-\alpha$. Choose moreover a harmonic measure η for L and let η^{su} be a family of conditional measures on strong unstable manifolds for η such that $d\eta=d\nu^s\times d\eta^{su}$ with respect to a local product structure. Denote by $Y+Z$ the g-gradient of η. Since L is of negative escape, for every $v\in T^1\widetilde{M}$ the constant function is a minimal L_v-harmonic function with pole at $\pi(v)$ and consequently by the Harnack inequality at infinity and Martin's theory we conclude that there is a number $c>0$ such that $\int_0^t g(X,Z)(\Phi^{-s}v)\,ds\geqslant -c$ for all $v\in T^1\widetilde{M}$ and all $t\geqslant 0$.

Let σ be the Borel measure on T^1M which is defined by $d\sigma=d\varrho^{ss}\times d\eta^{su}\times dt$ with respect to a local product structure; we may assume that $\sigma(T^1M)=1$. Then we have $d(\sigma\circ\Phi^{-t})/dt|_{t=0}=\alpha-g(X,Z)$ and hence for $t>\log(c+2)/\alpha$ the Radon–Nikodym derivative of $\sigma\circ\Phi^{-t}$ with respect to σ is at least 2 at every point $v\in T^1M$. Since σ is finite, this is impossible and shows that $\mathrm{pr}(g(X,Y))<0$. □

Next we consider weakly coercive operators which admit a harmonic measure η such that $l_\eta(L)>0$. As in §3 we call such an operator *of positive escape.* By Lemma 4.2 these operators include all weakly coercive operators with $\mathrm{pr}(g(X,Y))>0$. For $v\in T^1\widetilde{M}$ let ω_v be the hitting probability of the L_v-diffusion (recall that this is well defined) on $\partial\widetilde{M}$. Then $\omega_v(\partial\widetilde{M}-\pi(v))=1$ by Lemma 3.6 and Lemma 4.1, and moreover the measure class of ω_v is independent of $v\in T^1\widetilde{M}$. The next lemma contains a more precise statement of this fact:

LEMMA 4.3. *There is a number $c_1>0$ with the following property: Let $\nu>0$ be as in Corollary* B.3 *of Appendix* B, *let $v\in T^1\widetilde{M}$ and let $w\in T^1_{Pv}\widetilde{M}$ with $\angle(v,w)<\nu$. Then the restrictions to $\partial C(\Phi^1(-v),\frac14\pi)\cap\partial\widetilde{M}$ of the measures ω_v,ω_w are absolutely continuous and their Radon–Nikodym derivatives are contained in the interval $[c_1^{-1},c_1]$.*

Proof. Recall that the sets $B_\infty(v,\frac14\pi)=\partial C(v,\frac14\pi)\cap\partial\widetilde{M}$ $(v\in T^1\widetilde{M})$ form a basis for the topology of $\partial\widetilde{M}$. Since the measures ω_v are Borel it thus suffices by Corollary B.5

to show that there is a constant $\varkappa>0$ such that for all $v\in T^1\widetilde{M}$, all $w\in T^1_{Pv}\widetilde{M}$ with $\angle(-v,w)<\frac{1}{4}\pi$ and all $t>0$ we have

$$\omega_v\big(B_\infty\big(\Phi^t w, \tfrac{1}{4}\pi\big)\big)K_v^*(Pv, P\Phi^t w, \pi(v))^{-1}\in[\varkappa^{-1},\varkappa]$$

where as in the appendix we denote for $v\in T^1\widetilde{M}$ by $K_v\colon \widetilde{M}\times\widetilde{M}\times\partial\widetilde{M}\to(0,\infty)$ the Martin kernel of L_v and by K_v^* the Martin kernel of its formal adjoint L_v^*.

For this let $w\in T^1_{Pv}\widetilde{M}$ with $\angle(-v,w)<\frac{1}{4}\pi$, let $t>0$, $\xi\in B_\infty\big(\Phi^t w,\frac{1}{4}\pi\big)\subset B_\infty\big(-v,\frac{1}{2}\pi\big)$ and write also $x=\Phi^t w$. The Harnack inequality of Ancona, applied to the positive L_v-harmonic functions $y\to K_v(x,y,\pi(w))$ and $y\to K_v(x,y,\xi)$ which are defined on $C\big(-\Phi^t w,\frac{1}{2}\pi\big)$ and vanish on $\partial C\big(-\Phi^t w,\frac{1}{2}\pi\big)\cap\partial\widetilde{M}$, shows that there is a number $c>0$ not depending on v,w,t,ξ such that

$$K_v(x,Pv,\pi(w))\,K_v(x,Pv,\xi)^{-1}\in[c^{-1},c].$$

Let now $\chi>0$ be such that $\omega_z\big(B_\infty\big(\bar z,\frac{1}{4}\pi\big)\big)\geqslant\chi$ for all $z\in T^1\widetilde{M}$ and all $\bar z\in T^1_{Pz}\widetilde{M}$. The existence of such a constant again follows from the uniform estimates of Ancona ([An]). Let $z\in W^s(v)$ be such that $Pz=x$. Then

$$\omega_v\big(B_\infty\big(\Phi^t w,\tfrac{1}{4}\pi\big)\big)=\int_{B_\infty(\Phi^t w,\pi/4)}\frac{d\omega_v}{d\omega_z}(\xi)\,d\omega_z(\xi)=\int K_v(x,Pv,\xi)\,d\omega_z(\xi)$$

by the definition of the Martin kernel K_v, and hence

$$c^{-1}\chi K_v(x,Pv,\pi(w))\leqslant\omega_v\big(B_\infty\big(\Phi^t w,\tfrac{1}{4}\pi\big)\big)\leqslant cK_v(x,Pv,\pi(w))$$

by the above estimates. On the other hand, Lemma B.9 shows that there is a number $c_0>0$ such that

$$c_0^{-1}\leqslant K_v^*(x,Pv,\pi(-w))\,K_v(x,Pv,\pi(w))\leqslant c_0.$$

But for every $w\in T^1_{Pv}\widetilde{M}$ with $\angle(-v,w)<\frac{1}{4}\pi$ the function $y\to K_v^*(Pv,y,\pi(-w))$ is positive and L_v^*-harmonic on $C\big(-v,\frac{1}{2}\pi\big)$ and vanishes on $\partial C\big(-v,\frac{1}{2}\pi\big)\cap\partial\widetilde{M}$. Thus another application of the Harnack inequality at infinity for the weakly coercive operator L_v^* yields

$$K_v^*(Pv,x,\pi(-w))(K_v^*(Pv,x,\pi(v))^{-1}\in[c^{-1},c].$$

This shows that

$$K_v(x,Pv,\pi(w))\leqslant c_0K_v^*(x,Pv,\pi(-w))^{-1}\leqslant c_0cK_v^*(Pv,x,\pi(v))$$

and similarly

$$K_v(x,Pv,\pi(w))\geqslant c_0^{-1}K_v^*(x,Pv,\pi(-w))^{-1}\geqslant c_0^{-1}c^{-1}K_v^*(Pv,x,\pi(v)).$$

From this we obtain that

$$c^{-2}\chi c_0^{-1}K_v^*(Pv,x,\pi(v))\leqslant\omega_v\big(B_\infty\big(\Phi^t w,\tfrac{1}{4}\pi\big)\big)\leqslant c^2 c_0 K_v^*(Pv,x,\pi(v))$$

and this is just the desired inequality. □

Remark. The estimates in the proof of the above lemma imply in particular that the measures ω_v $(v\in T^1\widetilde{M})$ do not have atoms.

Garnett showed in [Ga] that a harmonic measure for the stable Laplacian Δ^s on a compact surface of constant negative curvature defined by the lift g_0 of the Riemannian metric is unique, a fact which was generalized to arbitrary compact negatively curved manifolds M by Ledrappier ([L3]) and Yue ([Y2]) with essentially the same proof. We want to generalize their result to operators $L=\Delta+Y$ of positive escape. For this recall the definition of the set $\widetilde{D}\subset T^1\widetilde{M}\times T^1\widetilde{M}$ from the introduction. Let $K:\widetilde{D}\times\partial\widetilde{M}\to(0,\infty)$ be the function whose restriction to $W^s(v)\times W^s(v)\times\partial\widetilde{M}$ equals the Martin kernel of the operator $L^v=L|_{W^s(v)}$; the function K is invariant under the action of $\Gamma=\pi_1(M)$ on $\widetilde{D}\times\partial\widetilde{M}$. For $v\in T^1\widetilde{M}$ define $\chi(v)=dK(v,\Phi^t v,\pi(v))/dt|_{t=0}$. The function χ is clearly invariant under the action of Γ; moreover by Corollary B.7 (see Appendix B) it is Hölder continuous and hence χ projects to a Hölder-continuous function on T^1M which we denote by the same symbol. Then $\beta=\chi+g(X,Y)$ is Hölder continuous as well.

LEMMA 4.4. *The pressure of β vanishes.*

Proof. For $v\in T^1\widetilde{M}$ denote by ω_v the hitting probability on $\partial\widetilde{M}$ of the diffusion on $\widetilde{M}$ which is induced by the operator L_v and which emanates from Pv. Since ω_v has no atoms we may project ω_v along the geodesics which are asymptotic to $\pi(v)$ to a Borel probability measure $\widetilde{\omega}_v$ on $W^{ss}(v)$. For $w\in W^{ss}(v)$ the measure ω_w is absolutely continuous with respect to ω_v. This means that we can define a family η^{ss} of locally finite Borel measures on the leaves of W^{ss} such that for $v\in T^1\widetilde{M}$ the restriction of η^{ss} to $W^{ss}(v)$ is absolutely continuous with respect to $\widetilde{\omega}_v$ and its Radon–Nikodym derivative with respect to $\widetilde{\omega}_v$ at $w\in W^{ss}(v)$ equals $(d\widetilde{\omega}_w/d\widetilde{\omega}_v)(w)$. By Lemma 4.3 the measures are quasi-invariant under canonical maps; moreover by the estimates in the appendix there is a number $c_1>0$ such that $c_1^{-1}\leqslant\eta^{ss}B^{ss}(v,1)\leqslant c_1$ for all $v\in T^1\widetilde{M}$.

Let now η^{su} be a family of conditionals on strong unstable manifolds of the Gibbs equilibrium state induced by β. The measures η^{su} are well defined on every leaf of $W^{su}\subset T^1M$, they are locally finite, positive on open sets and quasi-invariant under canonical maps. As before there is a number $c_2>0$ such that $c_2^{-1}\leqslant\eta^{su}B^{su}(v,1)\leqslant c_2$ for all $v\in T^1M$.

Now the measures η^{ss} are invariant under the action of $\Gamma=\pi_1(M)$ on $T^1\widetilde{M}$ and hence they project to locally finite Borel measures on the leaves of $W^{ss}\subset T^1M$ which we denote by the same symbol. We then obtain a locally finite Borel measure η on T^1M by defining $d\eta = d\eta^{ss}\times d\eta^{su}\times dt$, where dt is the one-dimensional Lebesgue measure on the flow lines of the geodesic flow. By the above estimates the measure η is in fact finite and positive on open sets.

Let $q\in\mathbf{R}$ be the pressure of β. The measures η^{su} are quasi-invariant under the action of Φ^t and they satisfy $d(\eta^{su}\circ\Phi^t)/dt|_{t=0}=\beta+q$. Also, the measures η^{ss} on the leaves of W^{ss} are quasi-invariant under Φ^t and we have

$$\frac{d}{dt}\{\eta^{ss}\circ\Phi^t(v)\}\Big|_{t=0} = \frac{d}{dt}K(v,\Phi^t v,\pi(-v)).$$

In other words, for $t\in\mathbf{R}$ and $v\in T^1M$ the Radon–Nikodym derivative of $\eta\circ\Phi^t$ with respect to η at v equals

$$f_v(\Phi^t v)K(v,\Phi^t v,\pi(v))K(v,\Phi^t v,\pi(-v))e^{qt}$$

where f_v is the unique function on $W^s(v)$ which satisfies $f_v(v)=1$ and such that the g-gradient of its logarithm equals $Y|_{W^s(v)}$.

Recall from Lemma B.8 and Lemma B.9 in the appendix that there is a number $c>0$ such that

$$f_v(\Phi^t v)K(v,\Phi^t v,\pi(v))K(v,\Phi^t v,\pi(-v))\in[c^{-1},c]$$

for all $t\in\mathbf{R}$. Assume that $q\neq 0$ and choose $\tau\in\mathbf{R}$ in such a way that $e^{q\tau}\geqslant 2c$. By the above, the Radon–Nikodym derivative of $\eta\circ\Phi^\tau$ with respect to η is $\geqslant 2$ everywhere on T^1M. But this is a contradiction to the fact that the measure η is finite. From this we conclude that necessarily $q=0$. □

COROLLARY 4.5. *Let ν^s be the family of Lebesgue measures on the leaves of W^s induced by g and let η^{su} be a family of conditional measures on the leaves of W^{su} of the Gibbs measure induced by β. Then the measure η on T^1M defined by $d\eta=d\nu^s\times d\eta^{su}$ is the unique harmonic measure for L (up to a constant).*

Proof. By Lemma 4.4 and its proof, the family η^{su} of conditionals on the leaves of W^{su} of the Gibbs equilibrium state η_0 defined by β transforms under Φ^t via

$$\frac{d}{dt}\{\eta^{su}\circ\Phi^t\}\Big|_{t=0}=\beta.$$

Let η be defined by $d\eta=d\nu^s\times d\eta^{su}$ and let l be the growth of η with respect to ν^s. Then for every $v\in T^1M$ the function $l_v\colon W^s(v)\to\mathbf{R}$ defined by $l_v(w)=l(v,w)$ is L_v^*-harmonic, which means that η is a harmonic measure for L. Notice that $\mathrm{mc}(\eta,\infty)$ is ergodic with respect to Γ since a Gibbs equilibrium state is ergodic with respect to Φ^t.

Now let ϱ be any ergodic harmonic measure for L and denote by $\bar{l}(v,w)$ the growth of ϱ with respect to ν^s. Then for ϱ-almost every $v\in T^1\widetilde{M}$ the function $\alpha_v\colon W^s(v)\to(0,\infty)$,

$w \to \alpha_v(w) = \bar{l}(v, w)$ is $L^*|_{W^s(v)}$-harmonic. Since L^* is weakly coercive this means that for every $v \in T^1\widetilde{M}$ there is a unique Borel probability measure ζ_v on $\partial\widetilde{M}$ such that the function α_v satisfies

$$\alpha_v(w) = \int K^*(v, w, \xi)\, d\zeta_v(\xi).$$

Let η^{ss} be a family of locally finite Borel measures on strong stable manifolds such that the measure η_0 on T^1M defined by $d\eta_0 = d\eta^{ss} \times d\eta^{su} \times dt$ is the Gibbs equilibrium state η_0 of the function β. The measures η^{ss} are well defined on *every* leaf of the strong stable foliation and hence we obtain a finite Borel measure ψ on T^1M by defining

$$d\psi = d\eta^{ss} \times d\varrho^{su} \times dt.$$

Via normalization of the measures ϱ^{su} by a universal constant we may assume that $\psi(T^1M) = 1$. Let $\tilde{\psi}$ be the lift of ψ to $T^1\widetilde{M}$.

For $v \in T^1\widetilde{M}$ and $w \in W^s(v)$ we have $\alpha_w = \alpha_w(v)\alpha_v$; in particular, the measures ζ_v, ζ_w define the same measure class and hence they have the same support. By ergodicity we can assume that for $\tilde{\psi}$-almost every $v \in T^1\widetilde{M}$ the measure ζ_v does not have an atom at $\pi(v)$.

Let $v \in T^1\widetilde{M}$ be such that the function α_v is defined and L^*-harmonic on $W^s(v)$. The Harnack inequality at infinity of Ancona together with the maximum principle shows that there is a number $c > 0$ not depending on v such that $\alpha_v(\Phi^{-t}v) \geqslant cK^*(v, \Phi^{-t}v, \pi(v))$ for all $t \geqslant 0$. But $\alpha_v(\Phi^{-t}v)\, K^*(v, \Phi^{-t}v, \pi(v))^{-1}$ equals the Radon–Nikodym derivative at v of the measure $\psi \circ \Phi^{-t}$ with respect to ψ which implies that $\psi \circ \Phi^{-t} \geqslant c\psi$ on T^1M (compare Lemma B.8 from Appendix B).

Let now $\bar{\omega}$ be an accumulation point of the sequence $\{(1/k)\sum_{i=1}^{k} \psi \circ \Phi^{-i}\}_{k>0}$. Then $\bar{\omega} \geqslant c\psi$, and moreover $\bar{\omega}$ is Φ^t-invariant. Since $\mathrm{mc}(\eta, \infty)$ and $\mathrm{mc}(\varrho, \infty)$ are ergodic with respect to the action of Γ we obtain from this the existence of a Φ^t-invariant ergodic measure ω on T^1M which is contained in the measure class of ψ. If $\tilde{\omega}$ is the lift of ω to $T^1\widetilde{M}$, then for $\tilde{\omega}$-almost every $v \in T^1\widetilde{M}$ we have

$$\liminf_{t \to \infty} K^*(v, \Phi^t v, \pi(v))^{-1} \alpha_v(\Phi^t v) > 0$$

which implies by Martin's theory that the measure ζ_v has an atom at $\pi(v)$. This is a contradiction to our assumption and shows that a harmonic measure for L is unique. □

Remark. Corollary 4.5 shows in particular that we can define the *escape rate* $l(L) > 0$ of the L-diffusion to be the escape rate of L with respect to its unique harmonic measure.

COROLLARY 4.6. *If L is of positive escape, then the pressure of $g(X,Y)$ is positive.*

Proof. For $v\in T^1\widetilde{M}$ let again $\chi(v)=dK(v,\Phi^t v,\pi(v))/dt$ and denote again by χ the projection of χ to T^1M. Since the operator L does not have a zero-order term we obtain from Martin's theory that $\liminf_{t\to\infty}(1/t)\int_0^t\chi(\Phi^s v)\,ds\geqslant 0$ for all $v\in T^1M$. Thus if ϱ is any Φ^t-invariant Borel probability measure on T^1M then

$$h_\varrho-\int g(X,Y)\,d\varrho\geqslant h_\varrho-\int(\chi+g(X,Y))\,d\varrho$$

and hence the pressure of $g(X,Y)$ is non-negative by Lemma 4.4. However the case $\mathrm{pr}(g(X,Y))=0$ is excluded by Lemma 2.11. □

Recall the definition of the functions β and χ on T^1M. We have

LEMMA 4.7. *If L is of positive escape, then there is a number $\varepsilon>0$ such that*

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\chi(\Phi^{-s}v)\,ds\leqslant-\varepsilon\quad\text{and}\quad\limsup_{t\to\infty}\frac{1}{t}\int_0^t\beta(\Phi^{-s}v)\,ds\leqslant-\varepsilon$$

for all $v\in T^1M$.

Proof. We consider first the function χ. Assume to the contrary that there is a sequence $\{v_i\}\subset T^1M$ and a sequence $\{t_i\}\subset\mathbf{R}$ such that $t_i\to\infty$ $(i\to\infty)$ and

$$\frac{1}{t_i}\int_0^{t_i}\chi(\Phi^s v_i)\,ds\leqslant\frac{1}{i}.$$

For a Borel set A of T^1M denote by c_A its characteristic function and define a Borel probability measure ν_i on T^1M by $\nu_i(A)=(1/t_i)\int_0^{t_i}c_A(\Phi^s v_i)\,ds$. Let ν be a weak limit of the measures ν_i. Then ν is invariant under Φ^t, and moreover $\int\chi\,d\nu\leqslant 0$ since χ is continuous.

For $v\in T^1\widetilde{M}$ define a function f_v on $W^s(v)$ by $f_v(w)=K(v,w,\pi(v))$. Let Z be the (Hölder-continuous) section of TW^s over T^1M whose lift $\widetilde{Z}$ to $T^1\widetilde{M}$ restricts on $W^s(v)$ to $\nabla\log f_v$ for every $v\in T^1\widetilde{M}$. Recall that L_v does not have a zero-order term and hence by the maximum principle the Green function G_v of L_v is uniformly bounded on $\{(x,y)\in\widetilde{M}\times\widetilde{M}\,|\,\mathrm{dist}(x,y)\geqslant 1\}$. Since f_v projects to a minimal positive L_v-harmonic function on $\widetilde{M}$ with pole at $\pi(v)$ the Harnack inequality at infinity of Ancona ([An]) implies that there is a number $c>0$ such that $f_v(\Phi^{-t}v)\leqslant e^c$ for all $v\in T^1\widetilde{M}$ and all $t\geqslant 0$. This means that $\int_0^t\chi(\Phi^s v)\,ds\geqslant -c$ for all $v\in T^1\widetilde{M}$ and all $t\geqslant 0$.

By Lemma 4.1, for every $v\in T^1\widetilde{M}$ the harmonic measure for L_v does not have an atom at $\pi(v)$. Martin's theory then shows that $\lim_{t\to\infty}\inf\int_0^t\chi(\Phi^s v)\,ds=\infty$ for all $v\in T^1M$.

For $T\geqslant 0$ define a set $C_T\subset T^1M$ by $C_T=\{v\in T^1M\,|\,\int_0^t\chi(\Phi^s v)\,ds\geqslant 4c$ for all $t\geqslant T\}$. Then $C_T\subset C_\tau$ for $T\leqslant\tau$, and moreover $\bigcup_{T>0}C_T=T^1M$ by the above considerations. Thus there is a number $T>0$ such that $\nu(C_T)\geqslant\frac{1}{2}$. Then

$$\begin{aligned}\int\chi\,d\nu&=\frac{1}{T}\int\left(\int_0^T\chi(\Phi^s v)\,ds\right)d\nu(v)\\&=\frac{1}{T}\left[\int_{C_T}\left(\int_0^T\chi(\Phi^s v)\,ds\right)d\nu(v)+\int_{T^1M-C_T}\left(\int_0^T\chi(\Phi^s v)\,ds\right)d\nu(v)\right]\\&\geqslant\frac{1}{T}\left(2c-\frac{c}{2}\right)=\frac{3c}{2T}>0,\end{aligned}$$

a contradiction. This means that the lemma holds indeed for χ.

Consider now the function β. Observe that for $v\in T^1\widetilde{M}$ and $t>0$ we have

$$\int_0^t\beta(\Phi^s v)\,ds=\log K^*(v,\Phi^t v,\pi(v))$$

where as before K^* is the Martin kernel of the formal adjoint of L. Since the Green function G_v of L_v is uniformly bounded on $\{(x,y)\in\widetilde{M}\times\widetilde{M}\,|\,\mathrm{dist}(x,y)\geqslant 1\}$, the same is true for the Green function $G_v^*\colon(x,y)\to G_v^*(x,y)=G_v(y,x)$ of L_v^*. As before, this means that there is a number $c>0$ such that $\int_0^t\beta(\Phi^s v)\,ds\geqslant -c$ for all $v\in T^1M$ and all $t\geqslant 0$.

We argue by contradiction and assume that the statement for β is false. Then there is a Φ^t-invariant Borel probability measure ϱ on T^1M such that $\int\beta\,d\varrho\leqslant 0$. Since by Lemma 4.4 the pressure of β vanishes, the measure ϱ has vanishing entropy and coincides with the unique Gibbs equilibrium state for β. In particular, we can decompose $d\varrho=d\varrho^{su}\times d\varrho^{ss}\times dt$ where ϱ^i is a family of locally finite Borel measures on the leaves of W^i ($i=ss,su$) and we have $d(\varrho^{su}\circ\Phi^t)/dt|_{t=0}=\beta$. Since the function β is Hölder continuous we obtain moreover from the Birkhoff ergodic theorem that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\beta(\Phi^{-s}w)\,ds=0$$

for every $v\in T^1M$ and ϱ^{ss}-almost every $w\in W^{ss}(v)$.

Consider the lifts of the measures ϱ^i to the leaves of $W^i\subset T^1\widetilde{M}$ which we denote by the same symbols. Then the projections of the measures ϱ^{su} to $\partial\widetilde{M}$ define the measure class $\mathrm{mc}(\eta,\infty)$ where η is the unique harmonic measure for L. The considerations in the proof of Lemma 4.3 show moreover that for every $v\in T^1\widetilde{M}$ the projection of $\varrho^{ss}|_{W^{ss}(v)}$ to $\partial\widetilde{M}$ determines the measure class of the exit measure of the L_v-diffusion on $\widetilde{M}$.

Together with Lemma B.9 from Appendix B this means the following: Let $v\in T^1\widetilde{M}$ and let ζ_v be the exit measure of the L_v-diffusion emanating from Pv. Then for ζ_v-almost every $\xi\in\partial\widetilde{M}$ the minimal positive L_v-harmonic function with pole at ξ grows subexponentially along a geodesic ray with endpoint ξ.

Let now $\widetilde{\omega}\in\widetilde{\Omega}_+$ be a typical path of the L-diffusion on $T^1\widetilde{M}$ for which the limit $\lim_{t\to\infty}P\widetilde{\omega}(t)=\widetilde{\omega}(\infty)$ exists and is contained in $\partial\widetilde{M}-\pi(\widetilde{\omega}(0))$. Let Ψ be a minimal positive $L_{\widetilde{\omega}(0)}$-harmonic function on $\widetilde{M}$ with pole at $\widetilde{\omega}(\infty)$. Then

$$\lim_{t\to\infty}\frac{\log\Psi(\widetilde{\omega}(t))-\log\Psi(\widetilde{\omega}(0))}{t}$$

equals the Kaimanovich entropy h_L of the L-diffusion (see [Ka1], [Ka2]). On the other hand, since a typical path follows a geodesic ([Pr]) this limit has to vanish by the above considerations. But the support of the exit measure for $L_{\widetilde{\omega}(0)}$ is all of $\partial\widetilde{M}$ and hence this entropy is strictly positive ([Ka1], [Ka2]). This gives the required contradiction and finishes the proof of the lemma. □

For $v\in T^1\widetilde{M}$ denote now by G_v the Green function of the operator L_v. Then we have

COROLLARY 4.8. *There are numbers $c>0$, $\alpha>0$ such that $G_v(x,y)\leqslant ce^{-\alpha\,\mathrm{dist}(x,y)}$ for all $v\in T^1\widetilde{M}$ and all $x,y\in\widetilde{M}$ with* $\mathrm{dist}(x,y)\geqslant 1$.

Proof. By Lemma 4.7, Lemma B.9 from Appendix B and the Harnack inequality at infinity of Ancona, for all $v,w\in T^1\widetilde{M}$ with $Pv=Pw$ there is a number $\varepsilon>0$ such that $\lim_{t\to\infty}(1/t)\log G_v(Pv,P\Phi^t w)\leqslant-\varepsilon$. We just have to derive from this a uniform estimate.

For this recall from the results of Ancona ([An]) that there is a number $\alpha>0$ not depending on v and w such that $G_v(Pv,P\Phi^{t+s}w)\leqslant e^{\alpha}G_v(Pv,P\Phi^t w)G_v(P\Phi^t w,P\Phi^{t+s}w)$ for all $v,w\in T^1\widetilde{M}$ with $Pv=Pw$, and all $s,t\geqslant 1$.

Let DTM be the compact subset of $T^1M\times T^1M$ consisting of vectors which project to the same point in M. For $(v,w)\in DTM$ there is then by the above a number $T(v,w)\geqslant 1$ such that $G_u(Pu,P\Phi^{T(v,w)}z)<e^{-2\alpha}$ for every lift (u,z) of (v,w) to $T^1\widetilde{M}\times T^1\widetilde{M}$. By continuity the same is true for every point of an open neighborhood $U(v,w)$ of (v,w) in DTM.

Choose finitely many points $(v_i,w_i)\in DTM$ $(i=1,\dots,k)$ such that the sets $U_i=U(v_i,w_i)$ cover DTM. Write $T_i=T(v_i,w_i)$ and let $T_0=\max\{T_i\,|\,i=1,\dots,k\}$. By the Harnack inequality there is then a number $a>1$ such that $G_u(x,y)\leqslant aG_u(x,z)$ for all $u\in T^1\widetilde{M}$ and all points $x,y,z\in\widetilde{M}$ with $\mathrm{dist}(x,y)\geqslant 1$, $\mathrm{dist}(x,z)\geqslant 1$ and $\mathrm{dist}(y,z)\leqslant T_0$. Let $u\in T^1\widetilde{M}$, $w\in T^1\widetilde{M}$ with $Pu=Pw$ and choose $i_0\in\{1,\dots,k\}$ such that (u,w) projects to a point in U_{i_0}. Define inductively a sequence $\{i_j\}_{j\geqslant 0}\subset\{1,\dots,k\}$ as follows: If i_j is already determined for all $j\leqslant j_0$ and $j_0\geqslant 0$ then let $T=\sum_{j=0}^{j_0}T_{i_j}$, let $\bar{u}\in W^s(u)$ be such that $P\bar{u}=P\Phi^T w$ and choose i_{j_0+1} in such a way that the projection to DTM of the point $(\bar{u},\Phi^T w)\in T^1\widetilde{M}\times T^1\widetilde{M}$ is contained in $U_{i_{j_0+1}}$. The required property now follows from the estimates of Ancona:

Namely, for $t \geqslant 1$ there is a unique integer $l \geqslant 0$ such that $t \in [\sum_{j=0}^{l} T_{i_j}, \sum_{j=0}^{l+1} T_{i_j})$; clearly $t \leqslant (l+1)T_0$. Ancona's inequality then implies inductively that $G_u(Pu, P\Phi^t w) \leqslant ae^{-(l+1)\alpha}$ and hence $G_u(Pu, P\Phi^t w) \leqslant ae^{-\varepsilon t}$ where $\varepsilon = \alpha/T_0$. This shows the corollary. □

As another application of the above results we obtain a better estimate for the fundamental solution p of the Cauchy problem $L - \partial/\partial t = 0$. For this recall again the definition of the Gromov distances on $\partial\widetilde{M}$ (see [GH]). Namely for $x \in \widetilde{M}$ and $\zeta, \eta \in \partial\widetilde{M}$ define

$$(\zeta|\eta)_x = \lim_{\substack{y \to \zeta \\ z \to \eta}} \tfrac{1}{2}(\mathrm{dist}(x,y) + \mathrm{dist}(x,z) - \mathrm{dist}(y,z)).$$

For $x \in \widetilde{M}$ and $v \neq w \in T_x^1\widetilde{M}$ write also $(v|w) = (\pi(v)|\pi(w))_x$. Then we have

COROLLARY 4.9. *Assume that $L = \Delta + Y$ is of positive escape. For $v \in T^1\widetilde{M}$ let $p_v : \widetilde{M} \times \widetilde{M} \times (0,\infty) \to (0,\infty)$ be the fundamental solution of the L_v-Cauchy problem. Then there are numbers $a, b > 0$ and $\delta > 0$ such that for all $t \geqslant 2$ we have*

$$|p_v(x,y,t) - p_w(x,y,t)| \leqslant ae^{-\delta t}[e^{-b(\pi(v)|\pi(w))_x} + e^{-b(\pi(v)|\pi(w))_y}].$$

Proof. By Corollary 4.8 and uniform boundedness of coefficients there is a number $\delta > 0$ such that $L + 2\delta$ is weakly coercive and such that moreover for every $v \in T^1\widetilde{M}$ the Green function $G_v^{2\delta}$ of $L_v + 2\delta$ is bounded on $\widetilde{M} \times \widetilde{M} - \{(x,y) \mid \mathrm{dist}(x,y) \leqslant 1\}$ by a universal constant independent of v. Since $G_v^{2\delta}(x,y) = \int_0^\infty e^{2\delta t} p_v(x,y,t)\,dt$ this implies by the Harnack inequality for parabolic equations that there is a number $c > 0$ such that for every $v \in T^1\widetilde{M}$ and every $x \in \widetilde{M}$, $t \geqslant 1$ the C^0-norm of the function $y \to p_v(x,y,t)$ is bounded from above by $ce^{-2\delta t}$.

Let now $t \geqslant 1$, $z \in \widetilde{M}$ and define $f_t^z(y) = p_v(y,z,t)$. Schauder theory for parabolic equations then shows that there is a constant $\bar{c} > 0$ not depending on $z \in \widetilde{M}$ and $t \geqslant 1$ such that $\|f_t^z\|_{2,\alpha} \leqslant \bar{c}e^{-2\delta t}$ where the $C^{2,\alpha}$-norm $\|\cdot\|_{2,\alpha}$ is defined as in the introduction.

For $x \in \widetilde{M}$ and $s \geqslant 0$ define now

$$u_v(x,s) = \int p_v(x,y,s) f_t^z(y)\,dy = p_v(x,z,s+t)$$

and $u_w(x,s) = \int p_w(x,y,s) f_t^z(y)\,dy$. Lemma A.4 then implies that

$$|(u_w - u_v)(x,t)| \leqslant \bar{a}e^{-\beta(\pi(v)|\pi(w))_x} e^{-\delta t}$$

where $\bar{a} > 0$ and $\beta > 0$ are constants depending on δ.

Let now L_v^* be the operator on $\widetilde{M}$ which is formally adjoint to L_v. By our assumption on L there is then a positive function f on $\widetilde{M}$ such that $L_v^*(\varphi) = f^{-1}L_v(f\varphi)$ for every smooth function φ on $\widetilde{M}$. Thus if B is a ball in $\widetilde{M}$, if $t>0$ and if ν is a function on $B\times[0,t]$ which satisfies $\nu\leqslant 0$ on $B\times\{0\}\cup\partial B\times[0,t]$ and $(L_v^*-\partial/\partial t)\nu\geqslant 0$ then $f\nu$ is a function on $B\times[0,t]$ with $f\nu\leqslant 0$ on $B\times\{0\}\cup\partial B\times[0,t]$ and $(L_v-\partial/\partial t)(f\nu)\geqslant 0$. The maximum principle for the parabolic operator $L_v-\partial/\partial t$ without zero-order terms then shows that $f\nu\leqslant 0$ on $B\times[0,t]$, and hence $\nu\leqslant 0$ on $B\times[0,t]$. In other words, the argument given in the proof of Lemma A.4 in Appendix A can be applied to L_v^*. Now for $x\in\widetilde{M}$ define $g_t^x(y)=p_w(x,y,t)$; with the same argument as above we have $\|g_t^x\|_{2,\alpha}\leqslant\bar{c}e^{-2\delta t}$.

Let $\tilde{u}_v(z,s)=\int p_v(y,z,s)g_t^x(y)\,dy$ and $\tilde{u}_w(z,s)=\int p_w(y,z,s)g_t^x(y)\,dy=p_w(x,z,s+t)$. The above argument can now be applied to the functions $\tilde{u}_v$ and $\tilde{u}_w$ using the parabolic equation $L_v^*-\partial/\partial t=0$ (which is possible by the above remark) and shows that

$$|(\tilde{u}_w-\tilde{u}_v)(z,t)|\leqslant\bar{a}e^{-\beta(\pi(v)|\pi(w))_z}e^{-\delta t}.$$

Combining the two estimates we then obtain that

$$|p_v(x,z,2t)-p_w(x,z,2t)|\leqslant\bar{a}e^{-\delta t}[e^{-\beta(\pi(v)|\pi(w))_x}+e^{-\beta(\pi(v)|\pi(w))_z}]$$

for all $t\geqslant 1$. □

In a similar way we obtain a better estimate for *all* solutions of the Cauchy problem $L-\partial/\partial t=0$.

COROLLARY 4.10. *There is a number $\chi>0$ with the following properties: Let $v,w\in T^1\widetilde{M}$ with $\pi(v)\neq\pi(w)$ and let $f:\widetilde{M}\to\mathbf{R}$ be a function with $\|f\|_{2,\alpha}<\infty$. Denote by f_v (or f_w) the solution of the parabolic equation $(L_v-\partial/\partial t)f_v=0$ (or $(L_w-\partial/\partial t)f_w=0$) with $f_v(x,0)=f(x)$ (or $f_w(x,0)=f(x)$) for all $x\in\widetilde{M}$. Then*

$$|(f_v-f_w)(x,t)|\leqslant\chi^{-1}\|f\|_{2,\alpha}e^{-\chi(\pi(v)|\pi(w))_x}\quad \textit{for all } (x,t)\in\widetilde{M}\times[0,\infty).$$

Proof. Let $\varepsilon>0$ be sufficiently small that the operator $L+\varepsilon$ is weakly coercive and that moreover there is a number $\alpha>0$ such that for every $v\in T^1\widetilde{M}$ the Green function G_v of $L_v+\varepsilon$ satisfies $G_v(x,y)\leqslant\alpha^{-1}e^{-\alpha\,\mathrm{dist}(x,y)}$ for all $x,y\in\widetilde{M}$ with dist $(x,y)\geqslant 1$; such a number exists by Corollary 4.8.

Let K_v be the Martin kernel of the operator $L_v+\varepsilon$ and define a function φ_v on $\widetilde{M}$ by

$$\varphi_v(y)=K_v(Pv,y,\pi(v))+K_v(Pv,y,\pi(-v)).$$

Since

$$\liminf_{t\to\infty}\frac{1}{t}\log K_v(Pv,P\Phi^t v,\pi(v))\geqslant\alpha,\quad \liminf_{t\to\infty}\frac{1}{t}\log K_v(Pv,P\Phi^{-t}v,\pi(-v))\geqslant\alpha$$

the restriction of φ_v to the geodesic γ with initial velocity $\gamma'(0)=v$ is bounded from below by a number $c_0>0$ not depending on v.

On the other hand, φ_v is a positive $(L_v+\varepsilon)$-harmonic function and hence the gradient of the logarithm of φ_v is pointwise bounded in norm, independent of $v\in T^1\widetilde{M}$. Thus there is a constant $\varrho>0$ such that $\varphi_v(\psi(t))\geqslant c_0e^{-\varrho|t|}$ for every geodesic ψ in $\widetilde{M}$ which meets γ orthogonally in $\psi(0)$ and every $t\in\mathbf{R}$. Since on the other hand we have $e^{-(\pi(v)|\pi(-v))_{\psi(t)}}\leqslant c_1e^{-|t|/2}$ for some $c_1>0$ and every such geodesic ψ, this implies that there are constants $c_2>0$, $\delta>0$ such that $c_2(\varphi_v(y))^\delta\geqslant e^{-(\pi(v)|\pi(-v))_y}$ for all $y\in\widetilde{M}$.

Now by our assumption on L there is a number $\bar{b}>0$ such that $|(L_v-L_{-v})u(x)|\leqslant \bar{b}^{-1}\|u\|_{2,\alpha}e^{-\bar{b}(\pi(v)|\pi(-v))_x}$ for all functions u on $\widetilde{M}$ with $\|u\|_{2,\alpha}<\infty$ and all $v\in T^1\widetilde{M}$. If we choose $b>0$ smaller than $\delta\bar{b}$ and $c_2^{-1}\bar{b}$, then φ_v^b is a L_v-superharmonic function (since L_v does not have zero-order terms) and $|(L_v-L_{-v})u(x)|\leqslant b^{-1}\|u\|_{2,\alpha}(\varphi_v(x))^b$ for all functions u with $\|u\|_{2,\alpha}<\infty$. On the other hand we have $L_v(\varphi_v^b)\leqslant-\bar{\varepsilon}\varphi_v^b$ for some $\bar{\varepsilon}>0$.

We use now the argument in the proof of Lemma A.4 to derive the desired conclusion. Let $f:\widetilde{M}\to\mathbf{R}$ be a function with $\|f\|_{2,\alpha}<\infty$ and let f_v (or f_{-v}) be the solution of the L_v-Cauchy problem (or the L_{-v}-Cauchy problem) with $f_v(x,0)=f(x)$ (or $f_{-v}(x,0)=f(x)$). Following the argument in the proof of Lemma A.4, the $C^{2,\alpha}$-norm of the functions $f_v^t: x\to f_v(x,t)$ and $f_{-v}^t: x\to f_{-v}(x,t)$ is bounded from above by $a\|f\|_{2,\alpha}$, where $a>0$ is a universal constant not depending on v.

As in the proof of Lemma A.4 choose again a non-decreasing function ψ of class C^∞ on $(0,\infty)$ such that $\psi(s)=0$ for $s\in(0,\frac{1}{2}]$ and $\psi(s)=s$ for $s\geqslant1$. Define $\varrho(x)=\psi(\operatorname{dist}(Pv,x))$. Then there is a number $k>0$ not depending on v such that $|L_v\varrho|\leqslant k$. Let $N=2\|f\|_0$ and for $R\geqslant1$, $x\in\widetilde{M}$ and $s\geqslant0$ define

$$\nu(x,s)=(f_v-f_{-v})(x,s)-\frac{N}{R}(\varrho+ks)(x)-a\bar{\varepsilon}^{-1}b^{-1}\|f\|_{2,\alpha}\varphi_v^b(x).$$

Since

$$\left|\left(L_v-\frac{\partial}{\partial t}\right)(f_v-f_{-v})(x,t)\right|=|(L_v-L_{-v})f_{-v}^t(x)|\leqslant b^{-1}a\|f\|_{2,\alpha}\varphi_v^b(x)$$

for all $x\in\widetilde{M}$ we have $(L_v-\partial/\partial t)\nu\geqslant0$, and moreover

$$\nu\leqslant0 \quad \text{on } B(Pv,R)\times\{0\}\cup\partial B(Pv,R)\times[0,t].$$

As in the proof of Lemma A.4 we conclude from this that

$$(f_v-f_{-v})(x,s)\leqslant a\bar{\varepsilon}^{-1}b^{-1}\|f\|_{2,\alpha}\varphi_v^b(x)$$

for all $(x,s)\in\widetilde{M}\times[0,\infty)$.

Let now exp be the exponential map of $\widetilde{M}$, and let

$$A_v = \{\exp sY \mid Y \in T_{P\Phi^t v}\widetilde{M} \cap (\Phi^t v)^\perp \text{ for some } t \in [-1,1],\, s \in \mathbf{R}\}.$$

By the Harnack inequality at infinity of Ancona, applied to the function φ_v on A_v, and the estimates for the Green function G_v, there is then a number $\chi>0$ such that

$$a\bar{\varepsilon}^{-1} b^{-1} \varphi_v^b(y) \leqslant \chi^{-1} e^{-\chi(\pi(v)|\pi(-v))_y}$$

for all $y \in A_v$. On the other hand, for every $t \in \mathbf{R}$ we have $f_{\Phi^t v} = f_v$ and $f_{-\Phi^t v} = f_{-v}$ and consequently the above arguments applied to $\Phi^t v$ then show that $(f_v - f_{-v})(x,s) \leqslant \chi^{-1} \|f\|_{2,\alpha} e^{-\chi(\pi(v)|\pi(-v))_x}$ for all $x \in \widetilde{M}$. Exchange of the role of v and $-v$ then yields $|f_v - f_{-v}|(x,s) \leqslant \chi^{-1} \|f\|_{2,\alpha} e^{-\chi(\pi(v)|\pi(-v))_x}$ for all $v \in T^1\widetilde{M}$, $x \in \widetilde{M}$ and $s \in [0,\infty)$.

Now if $v, w \in T^1\widetilde{M}$ are arbitrary with $\pi(v) \neq \pi(w)$ then there is $z \in T^1\widetilde{M}$ such that $\pi(z) = \pi(v)$ and $\pi(-z) = \pi(w)$. Then $L_v = L_z$, $L_{-z} = L_w$ and hence the corollary follows from the above considerations. □

5. A central limit theorem for operators of positive escape

In his paper [L4] Ledrappier proves a central limit theorem for the leafwise diffusion induced on T^1M by the stable Laplacian Δ^s. In this section we generalize his results to operators $L = \Delta + Y$ of gradient type as in §§ 2–4 with $\operatorname{pr}(g(X,Y)) > 0$.

Recall from §3 the definition of the bundle DTM over T^1M and the definition of the foliation DW^s of DTM.

Recall that the first factor projection $DTM \to T^1M$ maps DW^s to the stable foliation and hence the operator L lifts to a leafwise elliptic differential operator DL on (DTM, DW^s) with Hölder-continuous coefficients without zero-order term. In other words, DL induces a diffusion process on DTM which restricts to the L-diffusion on the diagonal. In the next lemma we describe the harmonic measures for DL; this lemma basically coincides with Proposition 1 of [L4]:

LEMMA 5.1. *Every harmonic measure for DL is supported in the diagonal of DTM.*

Proof (compare the proof of Proposition 1 of [L4]). For $(v,w) \in DT\widetilde{M}$ let $\widetilde{P}^{(v,w)}$ be the probability measure on the space of paths on $DT\widetilde{M}$ which is induced by the lift of DL to $DT\widetilde{M}$, with initial probability the Dirac mass at (v,w). Via the first factor projection the measure $\widetilde{P}^{(v,w)}$ projects to the measure $\widetilde{P}^v$ on the space of paths in $T^1\widetilde{M}$ induced by L and the initial probability the Dirac mass at v.

Now the hitting probability on $\partial\widetilde{M}$ of the L-diffusion on $W^s(v)$ is well defined and does not have an atom (this follows from the explicit description of this hitting probability in §4). In other words, for $\widetilde{P}^v$-almost every path $\widetilde{\omega}$ the limit $\lim_{t\to\infty}\widetilde{\omega}(t)$ exists in $W^s(v)\cup\partial\widetilde{M}$ and is contained in $\partial\widetilde{M}-\{\pi(v),\pi(w)\}$. By the argument in the proof of Lemma 3.11 this just means that for $\widetilde{P}^{(v,w)}$-almost every path $\widetilde{\omega}$ the distance between $\widetilde{\omega}(t)$ and the diagonal goes to zero as $t\to\infty$. From this the lemma immediately follows (compare Proposition 1 of [L4]). □

The unique harmonic measure η for L on T^1M now induces a harmonic measure $D\eta$ for DL on DTM which is supported on the diagonal. Lemma 5.1 together with Corollary 4.5 then imply

COROLLARY 5.2. *$D\eta$ is the unique harmonic measure for DL on DTM.*

Recall that the DL-diffusion on DTM leaves the complement of the diagonal invariant. Thus if Q_t denotes the action of $[0,\infty)$ on functions on DTM which describes the DL-diffusion then we can evaluate $Q_t\varrho$ outside the diagonal. The following evaluation is due to Ledrappier (Proposition 2 of [L4], compare also Lemma 3.3):

LEMMA 5.3. *For every $\varepsilon>0$ there is a number $T(\varepsilon)>0$ such that*

$$\frac{1}{T}(Q_T\varrho-\varrho)(v,w)\geqslant l-\varepsilon$$

for all $(v,w)\in DTM-T^1M$ and all $T\geqslant T(\varepsilon)$, where $l=l(L)$ is the escape rate of the L-diffusion.

Proof. Our lemma is a slightly improved version of Proposition 2 of [L4], so we repeat the proof for the convenience of the reader.

Assume that the lemma is false. Then there are numbers $T_n>0$ such that $T_n\to\infty$ $(n\to\infty)$ and points $(v_n,w_n)\in DTM-T^1M$ such that $(1/T_n)(Q_{T_n}\varrho-\varrho)(v_n,w_n)<l-\varepsilon$.

By Lemma 3.12 and the assumptions on L we can find a number $t_0>0$ small enough that

$$\sup_{0\leqslant t\leqslant t_0}\ \sup_{(v,w)\in DTM-T^1\widetilde{M}} Q_t|\varrho-\varrho(v,w)|(v,w)\leqslant \tfrac{1}{4}\varepsilon.$$

Thus by our assumptions we can find integers $N_j>0$ such that $N_j\to\infty$ $(j\to\infty)$ and

$$\frac{1}{N_jt_0}(Q_{N_jt_0}\varrho-\varrho)(v_j,w_j)<l-\tfrac{1}{2}\varepsilon.$$

Define a function φ on $DTM-T^1M$ by $\varphi(v,w)=(1/t_0)(Q_{t_0}\varrho-\varrho)(v,w)$. Then φ has a continuous extension to the diagonal by defining $\varphi(v,v)=(1/t_0)Q_{t_0}(\psi_v)$ where ψ_v is the function on $W^s(v)\subset T^1M$ which is given by $\psi_v(\Phi^tW^{ss}(v))=-t$.

By the above, there is a sequence of integers N_j such that $N_j \to \infty$ $(j \to \infty)$ and points $(v_j, w_j) \in DTM$ such that

$$\frac{1}{N_j} \sum_{k=0}^{N_j-1} Q_{kt_0} \varphi(v_j, w_j) < l - \tfrac{1}{2}\varepsilon.$$

Take a weak limit μ of a subsequence of the sequence of probability measures μ_j on the compact space DTM defined by $\mu_j = (1/N_j) \sum_{k=0}^{N_j-1} Q_{kt_0} \delta(v_j, w_j)$ where $\delta(v_j, w_j)$ is the Dirac mass at (v_j, w_j). Then μ is Q_{t_0}-invariant and satisfies $\int \varphi \, d\mu \leqslant l - \frac{1}{2}\varepsilon$.

Now $\mu' = (1/t_0) \int_0^{t_0} (Q_s \mu) \, ds$ is Q_t-invariant and we have $\int \varphi \, d\mu \leqslant l - \frac{1}{4}\varepsilon$, a contradiction to Corollary 5.2 and the definition of l. □

Ledrappier uses Proposition 2 in his paper [L4] to deduce a uniform estimate for the speed of contraction of the L-diffusion. The following corollary is the equivalent to Proposition 3 in [L4] and can be proved with exactly the same arguments (compare also the proof of Lemma 3.4):

COROLLARY 5.4. *There is a number $\tau_0 > 0$ and for every $\tau \in (0, \tau_0]$ there is a number $\zeta = \zeta(\tau) < 1$ such that $(Q_t e^{-\tau \varrho})(v, w) \leqslant \zeta^t e^{-\tau \varrho(v,w)}$ for all $(v, w) \in DTM$ and all sufficiently large $t > 0$.*

Proof. The corollary follows immediately from Lemma 5.3 with the arguments of Ledrappier (proof of Proposition 3 in [L4]). □

Recall that every leaf of the stable foliation W^s of T^1M is locally diffeomorphic to M. Hence as before, via the lift of the Riemannian metric on M we can define for every $v \in T^1M$ and $\tau \in (0,1)$ a $C^{2,\tau}$-norm $\|\cdot\|_{2,\tau}^v$ for functions on $W^s(v)$.

By abuse of notation denote again by Q_t $(t \geqslant 0)$ the action of $[0, \infty)$ on functions on T^1M which describes the L-diffusion. Then we obtain

LEMMA 5.5. *For sufficiently small $\tau > 0$ there is a number $c_1 = c_1(\tau) > 0$ such that $\sup_v \|Q_t f\|_{2,\tau}^v \leqslant c_1 \sup_v |f(v)|$ for every continuous function f on T^1M and all $t \geqslant 1$.*

Proof. Let $f: T^1M \to \mathbf{R}$ be continuous. Then clearly $\sup_v |Q_t f(v)| \leqslant \sup_v |f| = m$ for all $t \geqslant 0$.

Now for every $v \in T^1M$ the function $f_v: W^s(v) \times [0, \infty) \to \mathbf{R}$, $f_v(z, t) = (Q_t f)(z)$ is a uniformly bounded solution of the parabolic equation $L^v - \partial/\partial t = 0$. Schauder theory for parabolic equations then tells us that for every $t \geqslant 1$ and for $\tau > 0$ sufficiently small (depending on the coefficients of L) the $C^{2,\tau}$-norm of $Q_t f|_{W^s(v)}$ is bounded from above by a constant multiple of m. This shows the lemma. □

For $\tau>0$ define now a norm $\|\cdot\|_\tau$ on the space of continuous functions f on T^1M by $\|f\|_\tau=\sup_v|f(v)|+\sup\{|f(v)-f(w)|e^{\tau\varrho(v,w)}\mid(v,w)\in DTM\}$ and let $\mathcal{H}_\tau$ be the Banach space of functions f on T^1M with $\|f\|_\tau<\infty$.

For a function φ on DTM write moreover

$$\|\varphi\|_0=\sup_{(v,w)}|\varphi(v,w)|,\quad \|\varphi\|_{\tau,1}=\sup\{|\varphi(v,w)-\varphi(v,v)|e^{\tau\varrho(v,w)}\mid(v,w)\in DTM\}$$

and

$$\|\varphi\|_{\tau,2}=\sup\{|\varphi(v,w)-\varphi(w,w)|e^{\tau\varrho(v,w)}\mid(v,w)\in DTM\}.$$

First of all we have

LEMMA 5.6. *Let $\tau_0>0$ be as in Corollary 5.4, let $\tau\leqslant\tau_0$ and let $\zeta=\zeta(\tau)<1$ be as in Corollary 5.4. Then $\|Q_t\varphi\|_{\tau,1}\leqslant\zeta^t\|\varphi\|_{\tau,1}$ for every continuous function φ on DTM with $\|\varphi\|_{\tau,1}<\infty$ and all sufficiently large $t>0$.*

Proof. Let $\varphi: T^1M\to\mathbf{R}$ be such that $\|\varphi\|_{\tau,1}<\infty$ and for $(v,w)\in DTM$ let $b(v,w)=|\varphi(v,w)-\varphi(v,v)|\leqslant e^{-\tau\varrho(v,w)}\|\varphi\|_{\tau,1}$. Corollary 5.4 then shows that

$$|Q_t\varphi(v,w)-Q_t\varphi(v,v)|\leqslant(Q_tb)(v,w)\leqslant\zeta^t\|\varphi\|_{\tau,1}e^{-\tau\varrho(v,w)}$$

for all sufficiently large $t>0$, and from this the lemma immediately follows. □

For a function f on T^1M denote by $\tilde{f}$ its lift to DTM via the second factor projection $R_2: DTM\to T^1M$, i.e. $\tilde{f}(v,w)=f(w)$ for all $(v,w)\in DTM$. Then we have

LEMMA 5.7. *For sufficiently small $\tau>0$ there is a number $c_2=c_2(\tau)>0$ such that $\|Q_t(\widetilde{Q_1f})\|_{\tau,2}\leqslant c_2\sup_v|f(v)|$ for all $f\in\mathcal{H}_\tau$ and all $t\geqslant1$.*

Proof. Let $f\in\mathcal{H}_\tau$ and write $\varphi=Q_1f$. Let $(v,w)\in DTM$ and let $(u,z)\in DT\widetilde{M}$ be a lift of (v,w). The restriction to $W^s(z)$ of the lift of φ to $T^1\widetilde{M}$ then projects to a function $\bar{\varphi}$ on $\widetilde{M}$ which satisfies $\|\bar{\varphi}\|_{2,\tau}\leqslant c_1\sup_v|f(v)|$ where $c_1>0$ is as in Lemma 5.5.

Denote by $\bar{\varphi}_u$ (or $\bar{\varphi}_z$) the solution of the Cauchy problem $L_u-\partial/\partial t=0$ (or $L_z-\partial/\partial t=0$) with initial condition $\bar{\varphi}_u(x,0)=\bar{\varphi}(x)$ (or $\bar{\varphi}_z(x,0)=\bar{\varphi}(x)$). Corollary 4.10 then shows that for sufficiently small $\tau>0$ there is a constant $\chi=\chi(\tau)>0$ such that

$$\begin{aligned}|Q_t\widetilde{\varphi}(v,w)-Q_t\widetilde{\varphi}(w,w)|&=|\bar{\varphi}_u(Pu,t)-\bar{\varphi}_z(Pu,t)|\\&\leqslant\chi e^{-\tau\varrho(v,w)}\|\bar{\varphi}\|_{2,\tau}\leqslant\chi c_1e^{-\tau\varrho(v,w)}\sup_v|f(v)|\end{aligned}$$

for all $t\geqslant0$. From this the lemma follows. □

COROLLARY 5.8. *For sufficiently small $\tau>0$ there is a number $c_3=c_3(\tau)>0$ such that $\|Q_t f\|_\tau \leqslant c_3\|f\|_\tau$ for all $f\in\mathcal{H}_\tau$ and all $t\geqslant 1$.*

Proof. Recall that the fundamental solution of the L-diffusion on T^1M is Hölder continuous; this means that there is a number $\varrho>0$ such that $\|Q_1 f\|_\tau\leqslant\varrho\|f\|_\tau$ for all $f\in\mathcal{H}_\tau$. Write $\varphi=Q_1 f$. From Lemma 5.6 and Lemma 5.7 we then obtain for sufficiently large $t\geqslant 0$ that

$$\begin{aligned}\|Q_{t+1}f\|_\tau &\leqslant \|Q_t\widetilde{\varphi}\|_0+\|Q_t\widetilde{\varphi}\|_{\tau,1}+\|Q_t\widetilde{\varphi}\|_{\tau,2}\\ &\leqslant \|\widetilde{\varphi}\|_0+\zeta^t\|\widetilde{\varphi}\|_{\tau,1}+c_2\|f\|_\tau \leqslant \|\varphi\|_\tau+c_2\|f\|_\tau\leqslant(\varrho+c_2)\|f\|_\tau\end{aligned}$$

from which the corollary follows. □

Since $Q_{s+t}=Q_s\circ Q_t$ for all $s,t>0$ Corollary 5.8 shows that $\{Q_t\,|\,t\geqslant 1\}$ is an equicontinuous family of linear endomorphisms of $\mathcal{H}_\tau$.

As before let now η be the unique harmonic measure for L and let $\mathcal{H}_\tau^0\subset\mathcal{H}_\tau$ be the closed subspace of functions $f\in\mathcal{H}_\tau$ which satisfy $\int f\,d\eta=0$. Clearly $\mathcal{H}_\tau^0$ is invariant under the action of Q_t $(t\geqslant 0)$.

LEMMA 5.9. *For every $\varepsilon>0$ there is a number $k_0(\varepsilon)>0$ such that*

$$\sup_v\left|\frac{1}{k}\sum_{j=1}^{k}(Q_j f)(v)\right|\leqslant\varepsilon\|f\|_\tau$$

for all $f\in\mathcal{H}_\tau^0$ and all $k\geqslant k_0(\varepsilon)$.

Proof. Since Q_j is a linear operator on $\mathcal{H}_\tau^0$ it suffices to show the lemma for all $f\in B=\{\varphi\in\mathcal{H}_\tau^0\,|\,\|\varphi\|_\tau\leqslant 1\}$.

Define a norm $|||\cdot|||$ on the space of functions f on T^1M by

$$|||f|||=\|f\|_\tau+\sup_v\|f\|_{2,\tau}^v.$$

Then $|||\cdot|||$ is a Hölder norm in the usual sense (since the stable foliation is transversal to the vertical foliation of T^1M) and there is a constant $c>0$ such that $|||Q_t f|||\leqslant c$ for all $f\in B$ and all $t\geqslant 1$ by Lemma 5.5 and Corollary 5.8.

For $v\in T^1M$ and $j\geqslant 0$ let $\mu_{v,j}$ be the image of the Dirac mass at v under the time-j-map of the L-diffusion. Then $\mu_{v,j}$ is a Borel probability measure on T^1M. Since η is the unique harmonic measure for L, the measures $(1/k)\sum_{j=0}^{k-1}\mu_{v,j}$ converge as $k\to\infty$ weakly to η (see [Ga]).

By Arzela–Ascoli's theorem the inclusion of $\{Q_1 f \mid f \in B\}$ into the space $C^0(T^1M)$ of continuous functions on M is precompact. Since $\int (Q_1 f)\, d\eta = 0$ for all $f \in B$ this implies that for $\varepsilon > 0$ there is a number $k(v,\varepsilon) > 0$ such that

$$\left| \frac{1}{k} \sum_{j=0}^{k-1} \int (Q_1 f)\, d\mu_{v,j} \right| = \left| \frac{1}{k} \sum_{j=1}^{k} (Q_j f)(v) \right| \leqslant \varepsilon$$

for all $f \in B$ and all $k \geqslant k(v,\varepsilon)$.

The Hölder norm of the functions $w \to (1/k) \sum_{j=1}^{k} (Q_j f)(w)$ is bounded independent of $k \geqslant 1$ and $f \in B$. Thus there is an open neighborhood $U(v,\varepsilon)$ of v in T^1M such that $\left|(1/k) \sum_{j=1}^{k} (Q_j f)(w)\right| \leqslant 2\varepsilon$ for all $w \in U(v,\varepsilon)$ and all $k \geqslant k(v,\varepsilon)$.

Choose now finitely many points $v_1, ..., v_m \in T^1M$ such that the sets $U(v_i,\varepsilon)$ $(i = 1, ..., m)$ cover T^1M. Let $k_0 = \max\{k(v_i,\varepsilon) \mid i = 1, ..., m\}$. It then follows from the above that $\left|(1/k) \sum_{j=1}^{k} (Q_j f)(v)\right| \leqslant 2\varepsilon$ for all $f \in B$ and all $v \in T^1M$, $k \geqslant k_0$. □

Corollary 5.10. *For every $\varepsilon > 0$ there is a number $k_1(\varepsilon) > 0$ such that*

$$\left\| \frac{1}{k} \sum_{j=1}^{k} Q_j f \right\|_\tau \leqslant \varepsilon \|f\|_\tau$$

for all $f \in \mathcal{H}_\tau^0$ and all $k \geqslant k_1(\varepsilon)$.

Proof. Let $\varepsilon > 0$ and choose $k_0(\varepsilon/6c_1c_2) = k$ as in Lemma 5.9, where $c_1 > 0$ is as in Lemma 5.5 and $c_2 > 0$ is as in Lemma 5.7. Let $f \in \mathcal{H}_\tau^0$ and write $\varphi = Q_1((1/k) \sum_{j=0}^{k} Q_j f)$. Lemmas 5.5, 5.7 and 5.9 then show that $\|Q_j \tilde{\varphi}\|_{\tau,2} \leqslant \frac{1}{6}\varepsilon \|f\|_\tau$ for all $j \geqslant 1$, and from this we conclude with the arguments in the proof of Corollary 5.8 that $\|Q_j((1/k) \sum_{l=0}^{k} Q_l f)\|_\tau \leqslant \frac{1}{2}\varepsilon \|f\|_\tau$ for all $f \in \mathcal{H}_\tau^0$ and all sufficiently large $j > 1$. Now for $m \geqslant 1$ we have

$$\frac{1}{mk} \sum_{j=1}^{mk} Q_j = \frac{1}{m} \left(\sum_{i=0}^{m-1} Q_{ik} \left(\frac{1}{k} \sum_{j=0}^{k-1} Q_j \right) \right).$$

Since the operator norm of the maps Q_j $(j \geqslant 1)$ is uniformly bounded, from this the corollary immediately follows. □

Corollary 5.11. $(\mathrm{Id} - Q_1)\mathcal{H}_\tau^0$ *is dense in* $\mathcal{H}_\tau^0$.

Proof. The closure in $\mathcal{H}_\tau^0$ of $(\mathrm{Id} - Q_1)\mathcal{H}_\tau^0$ consists of all functions $f \in \mathcal{H}_\tau^0$ which satisfy

$$\lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k} Q_j f = 0$$

in $\mathcal{H}_\tau^0$. Thus the corollary follows from Corollary 5.10. □

COROLLARY 5.12. *The spectral radius of* Q_1 *is strictly smaller than* 1.

Proof. Since the operator norm of Q_k is bounded independent of $k>0$, the spectral radius of Q_1 is not larger than 1. Thus it suffices to show that 1 is contained in the resolvent set for Q_1. By Corollary 5.11 it suffices for this to show that there is a number $\varepsilon>0$ such that $\|(\mathrm{Id}-Q_1)f\|_\tau \geqslant \varepsilon \|f\|_\tau$ for all $f\in\mathcal{H}_\tau^0$.

We argue by contradiction and we assume to the contrary that there is a sequence $\{f_j\}_j\subset\mathcal{H}_\tau^0$ such that $\|f_j\|_\tau=1$ for all $j\geqslant 1$ and $\|f_j-Q_1f_j\|_\tau\to 0$ $(j\to\infty)$. Thus we may assume that $\frac{5}{4}\geqslant\|Q_1f_j\|_\tau\geqslant\frac{3}{4}$ for all $j\geqslant 1$. Now the operator Q_1 is continuous and consequently we also have $\|Q_1(f_j-Q_1f_j)\|_\tau=\|Q_1f_j-Q_2f_j\|_\tau\to 0$ $(j\to\infty)$; in particular, we may assume that $\frac{3}{2}\geqslant\|Q_2f_j\|_\tau\geqslant\frac{1}{2}$ for all $j\geqslant 1$.

Recall that there is a number $c>0$ such that $\|Q_1f_j\|_\tau+\sup_v\|Q_1f_j\|_{2,\tau}^v\leqslant c$ for all $j\geqslant 1$. Thus by the theorem of Arzela–Ascoli we may assume by passing to a subsequence that the functions Q_1f_j converge as $j\to\infty$ in $C^0(T^1M)$ to a continuous function φ. Since $\mathrm{Id}-Q_1$ extends to a continuous operator on $C^0(T^1M)$ we then have $(\mathrm{Id}-Q_1)\varphi=0$. Now $\int(Q_1f_j)\,d\eta=0$ for all $j\geqslant 1$ implies $\int\varphi\,d\eta=0$; moreover $\varphi=Q_1\varphi$ means $L\varphi=0$ and consequently $\varphi=0$.

Consider now the functions Q_2f_j. Since $Q_1f_j\to 0$ in $C^0(T^1M)$ it follows from Lemma 5.7 that $\|Q_k(\widetilde{Q_2f_j})\|_{\tau,2}\to 0$ as $j\to\infty$, uniformly in $k\geqslant 1$.

On the other hand we have $\|Q_k(\widetilde{Q_2f_j})\|_0\to 0$ uniformly in $k\geqslant 1$ as $j\to\infty$ and $\|Q_2f_j\|_\tau\leqslant\frac{3}{2}$ for all $j\geqslant 1$. Thus by Lemma 5.6 there is a number $k\geqslant 1$ and a number $j_0\geqslant 1$ such that $\|Q_kf_j\|_\tau\leqslant\frac{1}{8}$ for all $j\geqslant j_0$.

But also $f_j-Q_kf_j=\sum_{l=0}^{k-1}Q_l((\mathrm{Id}-Q_1)f_j)$, and since $\|(\mathrm{Id}-Q_1)f_j\|_\tau\to 0$ $(j\to\infty)$ we conclude that $\|f_j-Q_kf_j\|_\tau\to 0$, a contradiction to $\|f_j\|_\tau=1$ and $\|Q_kf_j\|_\tau\leqslant\frac{1}{8}$ for all $j\geqslant j_0$. This shows the corollary. □

Now Corollary 5.12 implies that there is a number $k>0$ such that the operator norm of Q_k as a linear endomorphism of $\mathcal{H}_\tau^0$ is strictly smaller than 1. Write now N for the operator on continuous functions on T^1M which associates to f the constant $\int f\,d\eta$. Then we obtain a generalization of Theorem 3 in [L4]:

THEOREM 5.13. *For sufficiently small* $\tau>0$ *there are numbers* $C>0$ *and* $\zeta<1$ *such that* $\|Q_t-N\|_\tau\leqslant C\zeta^t$ *for all* $t>0$.

As in the paper [L4] of Ledrappier we deduce from this the following.

COROLLARY 5.14. *For every function* $f\in\mathcal{H}_\tau^0$ *there is a unique function* $u\in\mathcal{H}_\tau^0$ *such that* $Lu=f$. *The function* u *is of class* C^2 *along the leaves of the stable foliation.*

Recall that there is no continuous non-constant function f on T^1M which satisfies $Lf=0$. However the next corollary implies that the space of non-trivial sections ψ of

T^*W^s with the property that for every $v \in T^1M$ the restriction of ψ to $W^s(v)$ is the differential of an L-harmonic function is infinite-dimensional.

COROLLARY 5.15. *Let Z be a section of T^*W^s of class $C_s^{1,\alpha}$ for some $\alpha>0$. Then there is a function $u \in \mathcal{H}_\tau^0$ such that* $\operatorname{div}(Z+\nabla u)+g(Y, Z+\nabla u)=\int(\operatorname{div}(Z)+g(Y,Z))\,d\eta$.

Corollary 5.15 contrasts sharply the case when $L=\Delta+Y$ admits a self-adjoint harmonic measure η. In this case the vector space of L^2-integrable sections ψ of T^*W^s which restrict to differentials of L-harmonic functions on the leaves of W^s is just the vector space $\mathcal{H}^1$ of harmonic 1-forms in the sense of §2. We then have

PROPOSITION 5.18. *Let η be a self-adjoint harmonic measure for $L=\Delta+Y$ and let $\mathcal{H}^1$ be the space of harmonic sections of T^*W^s over (T^1M, η). Then* $\dim \mathcal{H}^1=1$.

Proof. Clearly $\dim \mathcal{H}^1 \geqslant 1$. So assume to the contrary that there are square-integrable linear independent sections A, E of TW^s which are g-dual to elements of $\mathcal{H}^1$. For every smooth function f on T^1M we then have $\int A(f)\,d\eta=0=\int E(f)\,d\eta$ and hence for all $a, e \in \mathbf{R}$ the measure η is harmonic for the operator $L+aA+eE$.

Let $\overline{X}$ be defined as in §2. If $\int(\operatorname{div}(\overline{X})+g(Y+A,\overline{X}))\,d\eta=0$ then η is a self-adjoint harmonic measure for $L+A$, a contradiction to the fact that the g-gradient of η equals Y. Thus by suitably rescaling A we may assume that $\int g(A,\overline{X})\,d\eta=-1$. Similarly we may adjust E in such a way that $\int(\operatorname{div}(\overline{X})+g(Y+E,\overline{X}))\,d\eta=\int g(E,\overline{X})\,d\eta=1$. Then $\int(\operatorname{div}(\overline{X})+g(Y+A+E,\overline{X}))\,d\eta=0$ and hence η is self-adjoint harmonic for $L+A+E$. Thus $A+E=0$, a contradiction to our assumption that A and E are linearly independent. □

Appendix A

In this appendix we collect some basic properties of solutions of parabolic differential equations on a simply connected Riemannian manifold $(\widetilde{M}, \langle\cdot,\cdot\rangle)$ of bounded negative sectional curvature.

Fix a number $r \in (0,\infty)$ and recall that for every $x \in \widetilde{M}$ the exponential map of $\langle\cdot,\cdot\rangle$ at x maps the Euclidean ball B of radius r about zero diffeomorphically onto the ball $B(x,r)$ of radius r about x in $\widetilde{M}$. These coordinates define for every $j \geqslant 0$ and $\alpha \in (0,1]$ a $C^{j,\alpha}$-norm for functions on $B(x,r)$; we refer to these norms in the sequel.

Let g be a Riemannian metric on $\widetilde{M}$ which is uniformly equivalent to $\langle\cdot,\cdot\rangle$ and such that for some $\alpha \in (0,1)$ the $C^{1,\alpha}$-norm of g on the balls $B(x,r)$ in exponential coordinates is uniformly bounded independent of x. Since the curvature of $\widetilde{M}$ is bounded this is for example true for $g=\langle\cdot,\cdot\rangle$. Let Y be a uniformly bounded continuous section of $T\widetilde{M}$

with uniformly bounded $C^{1,\alpha}$-norm in the exponential coordinates on the balls $B(x,r)$, and let Δ be the Laplacian of g and define $L=\Delta+Y$.

For a C^1-vector field Z on $\widetilde{M}$ let moreover $\operatorname{div}(Z)$ be the divergence of Z with respect to the volume element dx on $\widetilde{M}$ induced by g.

Let $u_0\colon \widetilde{M}\to\mathbf{R}$ be continuous. A continuous function $u\colon \widetilde{M}\times[0,T)\to\mathbf{R}$ ($T>0$) is a *solution of the L-Cauchy problem with initial condition* u_0 if the following is satisfied:

(1) $u|_{\widetilde{M}\times(0,T)}$ is of class C^2 in the space variable, of class C^1 in the time variable.

(2) $Lu-\partial u/\partial t=0$ on $\widetilde{M}\times(0,T)$.

(3) $u(x,0)=u_0(x)$ for all $x\in\widetilde{M}$.

A non-negative measurable map $p\colon \widetilde{M}\times\widetilde{M}\times(0,\infty)\to\mathbf{R}$ is called a *fundamental solution of the L-Cauchy problem* if for every bounded continuous function u_0 on $\widetilde{M}$ the function

$$u(x,t)=\begin{cases}\int_{\widetilde{M}} p(x,y,t)\,u_0(y)\,dy & \text{for } t>0,\\ u_0(x) & \text{for } t=0\end{cases}$$

is a solution of the L-Cauchy problem with initial condition u_0.

We first construct a fundamental solution of the L-Cauchy problem in a probabilistic way. Namely, recall from Corollary 6.2 of [IW] that the operator L induces a unique diffusion on $\widetilde{M}$. This diffusion is a stochastic process which can be described as follows: Compactify $\widetilde{M}$ by adding a point ζ at infinity; $\overline{M}=\widetilde{M}\cup\{\zeta\}$ is naturally a topological space. Let $\Omega_+(\widetilde{M})$ be the set of all continuous maps $\omega\colon[0,\infty)\to\overline{M}$ with $\omega(t)=\zeta$ for all $t\geqslant\inf\{s\geqslant 0\mid \omega(s)=\zeta\}=\zeta(\omega)$.

Denote by $\mathcal{B}$ (or $\mathcal{B}_t$) the σ-algebra on $\Omega_+(\widetilde{M})$ generated by the Borel cylinder sets (or the Borel cylinder sets up to time t) (compare [IW, p. 189]). The L-diffusion is then determined by the unique family $\{P_x\}_{x\in\widetilde{M}}$ of probability measures on $(\Omega_+(\widetilde{M}),\mathcal{B})$ with the following properties:

(i) $P_x\{\omega\mid\omega(0)=x\}=1$ for all $x\in\widetilde{M}$.

(ii) $f(\omega(t))-f(\omega(0))-\int_0^t(Lf)(\omega(s))\,ds$ is a $(P_x,\mathcal{B}_t)$-martingale for every smooth function f on $\widetilde{M}$ with compact support and every $x\in\widetilde{M}$.

Let $x_0\in\widetilde{M}$ and let B be an open ball of radius $r\in(0,\infty)$ about x_0 in $\widetilde{M}$. Then there is a unique fundamental solution q_B of the equation $L-\partial/\partial t=0$ on $B\times(0,\infty)$ vanishing on the boundary ∂B of B ([LSU, Chapter IV]).

Let $B_1,B_2,\ldots$ be an exhaustion of $\widetilde{M}$ by open balls such that $\overline{B}_j\subset B_{j+1}$ and $\bigcup_{j=1}^\infty B_j=\widetilde{M}$. Define

$$q_i(x,y,t)=\begin{cases}q_{B_i}(x,y,t) & \text{for } x,y\in B_i,\\ 0 & \text{otherwise.}\end{cases}$$

By the maximum principle for parabolic differential equations ([PW, §III]) we have $q_i\geqslant 0$ and $q_{i+1}\geqslant q_i$ for all $i>0$. Define $p(x,y,t)=\sup_i q_i(x,y,t)$.

LEMMA A.1. *For every $x\in\widetilde{M}$ and every Borel set $A\subset\widetilde{M}$, $t>0$ we have*

$$P_x\{\omega \mid \omega(t)\in A\}=\int_A p(x,y,t)\,dy.$$

Proof. For every $t>0$ and every $i>0$ the function q_i induces an operator Q_t^i on $L^2(B_i)$ by

$$(Q_t^i f)(x)=\int q_i(x,y,t)\,f(y)\,dy.$$

If $f\colon B_i\to\mathbf{R}$ is a continuous function vanishing near ∂B_i, then the function $u\colon(x,t)\to(Q_t^i f)(x)$ is a solution of the equation $L-\partial/\partial t=0$ on $B_i\times(0,\infty)$ which satisfies

$$\lim_{t\to 0} u(x,t)=f(x).$$

Since such a solution is unique ([LSU, Chapter IV]) we have in particular

$$q_i(x,y,t+s)=\int_{B_i} q_i(x,z,t)\,q_i(z,y,s)\,dz$$

for all $x,y\in B_i$, $t,s>0$. It follows from the maximal principle for parabolic differential equations ([PW, §III]) that $q_i(x,y,t)>0$ for all $x,y\in B_i$, $t>0$ and also $\int q_i(x,y,t)\,dy\leqslant 1$.

Compactify B_i by adding a point β at infinity and define $\Omega_+(B_i)$ as before. We then obtain a Markovian system of probability measures $\{\widetilde{P}_x^i\}_{x\in B_i}$ on $\Omega_+(B_i)$ by defining $\widetilde{P}_x^i\{\omega|\omega(t)\in A\}=\int_A q_i(x,y,t)\,dy$. The measures $\{\widetilde{P}_x^i\}_{x\in\widetilde{M}}$ then describe the unique L-diffusion on B_i ([IW, Chapter V, §3]). For a path $\omega\in\Omega_+(\widetilde{M})$ with $\omega(0)=x\in B_i$ and $t>0$ let $\tau_i=\inf\{s\geqslant 0|\omega(s)\in\widetilde{M}-B_i\}$ and $t\wedge\tau_i(\omega)=\inf\{t,\tau_i(\omega)\}$. Then τ_i is a stopping time for $(\Omega_+(\widetilde{M}),\mathcal{B})$ and consequently

$$f(\omega(t\wedge\tau_i(\omega)))-f(\omega(0))-\int_0^{t\wedge\tau_i(\omega)}(Lf)(\omega(s))\,ds$$

is a $(P_x,\mathcal{B})$-martingale for every $x\in B_i$ and every smooth function f with compact support in B_i.

Let $\{P_x^i\}_{x\in B_i}$ be the unique family of probability measures on $\Omega(\widetilde{M})$ which is defined by

$$P_x^i\{\omega \mid \omega(t)\in A\}=P_x\{\omega \mid \omega(t)\in A,\, t\leqslant\tau_i(\omega)\}$$

where $x\in B_i$, $t>0$ and $A\subset B_i$ is a Borel set. By the above consideration these measures describe the L-diffusion on B_i. Thus $P_x^i=\widetilde{P}_x^i$ for all $x\in B_i$ and $i>0$. Since on the other hand clearly

$$P_x\{\omega \mid \omega(t)\in A\}=\sup P_x^i\{\omega \mid \omega(t)\in A\}$$

we obtain

$$P_x\{\omega \mid \omega(t)\in A\} = \sup_i \int_A q_i(x,y,t)\,dy = \int_A p(x,y,t)\,dy$$

by Lebesgue's theorem of monotone convergence. This shows the lemma. □

Remark. As an increasing limit of continuous functions the function

$$p\colon \widetilde{M}\times\widetilde{M}\times(0,\infty)\to(0,\infty)$$

is measurable and lower semi-continuous.

Next we conclude that p has the required properties:

LEMMA A.2. *The function p is a fundamental solution of the L-Cauchy problem with the following properties*:

(i) $p(x,y,t)>0$ *for all* $x,y\in\widetilde{M}$ *and all* $t>0$.

(ii) $p(x,y,t+s)=\int_{\widetilde{M}} p(x,z,t)p(z,y,s)\,dz$ *for all* $x,y\in\widetilde{M}$ *and all* $s,t>0$.

(iii) *If* $u\colon\widetilde{M}\times[0,T)\to\mathbf{R}$ *is a bounded solution of the L-Cauchy problem then* $u(x,t)=\int p(x,y,t)u(y,0)\,dy$ *for all* $x\in\widetilde{M}$ *and all* $t>0$; *in particular,* $\int p(x,y,t)\,dy=1$ *and the L-diffusion is conservative.*

Proof. Let f be a continuous function on $\widetilde{M}$ with compact support contained in some ball B_i. Then $f\in L^2(B_j)$ for all $j>i$ and consequently by Lebesgue's theorem of monotone convergence and the fact that $\int q_i(x,y,t)\,dy<1$ for all $x\in\widetilde{M}$ we have

$$u_j(x,t)=\int q_j(x,y,t)f(y)\,dy\to u(x,t)=\int p(x,y,t)f(y)\,dy\quad(j\to\infty).$$

For $j>i$ the function u_j on $B_j\times(0,\infty)$ is a solution of the parabolic equation $L-\partial/\partial t=0$ which is uniformly bounded in absolute value, independent of $j>0$, $t>0$. Since L is uniformly elliptic on $B(x,r)$ with C^α-coefficients of uniformly bounded C^α-norm we may apply Schauder theory for parabolic equations (see [LSU]) to conclude that for every $t>0$ the $C^{2,\alpha}$-norm of the functions $z\to u_j(z,t)$ on compact subsets of B_i $(j>i)$ is uniformly bounded. Thus the functions u_j converge uniformly on compact subsets of $\widetilde{M}$ to a solution of the equation $L-\partial/\partial t=0$. In other words, the function

$$(x,t)\to u(x,t)=\int p(x,y,t)f(y)\,dy$$

is a solution of the L-Cauchy problem.

To determine its initial condition, let $x\in B_i$ and let U be an open neighborhood of x in B_i. For $j>i$ we then have

$$1\leqslant\lim_{t\to0}\int_U q_j(x,y,t)\,dy\leqslant\limsup_{t\to0}\int_U p(x,y,t)\,dy.$$

But $\int p(x,y,t)\,dy\leqslant 1$ for all $t>0$ and consequently $\limsup_{t\to 0}\int_{\widetilde{M}-U} p(x,y,t)\,dy=0$. Since U was an arbitrary neighborhood of x it follows that

$$\lim_{t\to 0}\int p(x,y,t)\,f(y)\,dy=f(x)$$

and consequently p is a fundamental solution of the L-Cauchy problem. Property (ii) for p is an immediate consequence of the corresponding properties of the functions q_i.

For the verification of (iii) we use the arguments in the proof of Theorem 2.2 of [Dod]. Namely, let $u:\widetilde{M}\times[0,T)\to\mathbf{R}$ be a bounded solution of the L-Cauchy problem and define $\bar{u}(x,t)=\int p(x,y,t)\,u(y,0)\,dy$ for $x\in\widetilde{M}$, $t>0$ and $\bar{u}(x,0)=u(x,0)$. We have to show that $u=\bar{u}$. Assume for simplicity that $u(x,0)\geqslant 0$ for all $x\in\widetilde{M}$. Choose a non-decreasing function φ of class C^2 on $(0,\infty)$ such that $\varphi(s)=0$ for $s\in\left(0,\tfrac{1}{2}\right)$ and $\varphi(s)=s$ for $s\geqslant 1$. Let $x_0\in\widetilde{M}$ and for $x\in\widetilde{M}$ define $r(x)=\mathrm{dist}(x_0,x)$ (where dist is the distance induced by $\langle\cdot,\cdot\rangle$) and $\varrho(x)=\varphi\circ r(x)$.

Let $\bar{\Delta}$ be the Laplacian on $\widetilde{M}$ of the metric $\langle\cdot,\cdot\rangle$. Since $\widetilde{M}$ has bounded geometry there is then a number $\bar{c}>0$ such that

$$\bar{\Delta}(\varrho)(x)\leqslant\varphi''(r(x))+\bar{c}\varphi'(r(x))$$

for all $x\in\widetilde{M}$ (see [Dod]). But g is uniformly equivalent to $\langle\cdot,\cdot\rangle$, and of uniformly bounded $C^{1,\alpha}$-norm (in exponential coordinates); moreover the vector field Y is uniformly bounded and hence by the choice of φ we conclude that $L\varrho\leqslant K$ for some constant $K>0$.

Let

$$N=\sup\{|(u-\bar{u})(x,t)|\mid(u,t)\in\widetilde{M}\times[0,T)\},$$

let $R>0$ be a large positive constant and choose $i>0$ sufficiently large that $B(x_0,2R)\subset B_i$.

For $j>i$ let $\chi_j\colon B_j\to[0,1]$ be a continuous function with compact support which satisfies $\chi_j(x)=1$ for $x\in B_{j-1}$. Define a bounded function $u_j\colon B_j\times[0,\infty)\to\mathbf{R}$ by

$$u_j(x,t)=\int q_j(x,y,t)\,\chi_j(y)\,u(y,0)\,dy$$

for $t>0$ and $u_j(x,0)=\chi_j(x)u(x,0)$. Then $u_j\to\bar{u}$ pointwise on $B(x_0,R)\times[0,\infty)$.

Let $\varepsilon>0$, let $x\in\bar{B}(x_0,R)$ and let $t\in[0,T]$. There is a number $j(x,t)>i$ such that $|\bar{u}(x,t)-u_j(x,t)|<\tfrac{1}{2}\varepsilon$ for all $j\geqslant j(x,t)$. Then $|u_j(x,t)-u(x,t)|<N+\tfrac{1}{2}\varepsilon$ and hence by continuity of u_j and u there is a neighborhood $U(x,t)$ of (x,t) in $\widetilde{M}\times[0,T]$ such that $|u_{j(x,t)}(y,s)-u(y,s)|<N+\varepsilon$ for all $(y,s)\in U(x,t)$. Now for $(y,s)\in U(x,t)$ the sequence of numbers $a_j=u_j(y,s)$ is monotonically increasing and consequently for every $j\geqslant j(x,t)$ we have

$$|a_j-u(y,s)|\leqslant\max\{|a_{j(x,t)}-u(y,s)|,|\bar{u}(y,s)-u(y,s)|\}<N+\varepsilon.$$

But this means that $|u_j(y,s)-u(y,s)|<N+\varepsilon$ for all $(y,s)\in U(x,t)$ and all $j\geqslant j(x,t)$. By the compactness of $\bar{B}(x_0,R)\times[0,T]$ there is then a number $j(\varepsilon)>0$ such that $|u_j(x,t)-u(x,t)|<\varepsilon+N$ for all $(x,t)\in\bar{B}(x_0,R)\times[0,T]$ and all $j\geqslant j(\varepsilon)$.

Let $j\geqslant j(\varepsilon)$ and define

$$\nu(x,t)=u(x,t)-u_j(x,t)-\frac{N+\varepsilon}{R}(\varrho+Kt).$$

Then $\nu\leqslant 0$ on

$$B(x_0,R)\times\{0\}\cup\partial B(x_0,R)\times[0,T)$$

and consequently (see [Dod])

$$|u(x,t)-u_j(x,t)|\leqslant\frac{N+\varepsilon}{R}(\varrho(x)+Kt)$$

for all $(x,t)\in B(x_0,R)\times[0,T)$ by the maximum principle. Since $\varepsilon>0$ and $j\geqslant j(\varepsilon)$ was arbitrary this implies

$$|u(x,t)-\bar{u}(x,t)|\leqslant\frac{N}{R}(\varrho(x)+K(t)).$$

Now $R>0$ was arbitrary as well and hence $u=\bar{u}$ follows (compare [Dod]). This finishes the proof of the lemma. □

Remark. (iii) shows in particular that we have $u(x)=\int p(x,y,t)u(y)\,dy$ for every bounded function u on $\widetilde{M}$ which satisfies $Lu=0$.

LEMMA A.3. *For every $x\in\widetilde{M}$ and $t>0$ the functions $z\to p(x,z,t)$ and $z\to p(z,x,t)$ are of class $C^{2,\alpha}$ with $C^{2,\alpha}$-norm on the balls $B(y,r)$ bounded independent of y.*

Proof (compare [Ch, p. 197]. Recall that $\check{p}(x,y,t)=p(y,x,t)$ is a fundamental solution for the equation $L^*-\partial/\partial t=0$ where $L^*u=\Delta u-\operatorname{div}(uY)$ is the formal adjoint of the operator L. Now if u is any smooth function on $\widetilde{M}$ with compact support then we have

$$\frac{\partial}{\partial t}\int p(x,y,t)u(x)\,dx=\int(L_xp)(x,y,t)u(x)\,dx=\int p(x,y,t)(L^*u)(x)\,dx$$

for all $y\in\widetilde{M}$. From this we conclude that

$$\frac{\partial}{\partial t}\int p(x,y,t)\,dx=-\int p(x,y,t)\operatorname{div}(Y)(x)\,dx\leqslant\varkappa\int p(x,y,t)\,dx$$

where $\varkappa=\sup_{z\in\widetilde{M}}|\operatorname{div}Y(z)|<\infty$. This implies that $\int p(x,y,t)\,dx\leqslant e^{\varkappa t}$ for all $t\geqslant 0$.

Let now f be a smooth function on $\widetilde{M}$ with compact support and for $x\in\widetilde{M}$ and $t>0$ define $u(x,t)=\int p(x,y,t)f(y)\,dy$. The Cauchy–Schwarz inequality for the measure $p(x,y,t)\,dy$ yields $u^2(x,t)\leqslant\int p(x,y,t)f^2(y)\,dy$ and hence

$$\begin{aligned}\int_{\widetilde{M}}u^2(x,t)\,dx&\leqslant\iint p(x,y,t)f^2(y)\,dy\,dx\\&=\int f^2(y)\left(\int p(x,y,t)\,dx\right)dy\leqslant e^{\varkappa t}\int f^2(y)\,dy.\end{aligned}$$

Thus for every $t \geqslant 0$ the L^2-norm of $u(\cdot, t)$ does not exceed $e^{\varkappa t}$ times the L^2-norm of f. Using Schauder theory for parabolic equations with Hölder-continuous coefficients (see [LSU]) we conclude that for every $t>0$ there is a constant $c(t)>0$ such that

$$\sup_{x \in \widetilde{M}} |u(x,t)| \leqslant c(t) \cdot \|f\|_{L^2}.$$

But $u(x,t)$ equals the L^2-scalar product of f with $p(x, \cdot, t)$. Since f was an arbitrary function with compact support it follows that the L^2-norm of $p(x, \cdot, t)$ does not exceed $c(t)$; in particular, the sequence of functions $\{q_j(x, \cdot, t)\}_{j>0}$ from above is bounded in $L^2(\widetilde{M})$.

The functions $q_j(x, \cdot, t)$ are solutions of the equation $L - \partial/\partial t = 0$. Therefore, using Schauder theory for parabolic equations we conclude that the $C^{2,\alpha}$-norm of $q_j(x, \cdot, t)$ on $B(y,r)$ (in exponential coordinates) is bounded independent of $x, y \in \widetilde{M}$ and $j>0$. Then the functions $q_j(x, \cdot, t)$ converge as $j \to \infty$ uniformly on compact sets to $p(x, \cdot, t)$. Moreover $p(x, \cdot, t)$ satisfies the properties stated in the lemma.

Similarly, for a smooth function f on $\widetilde{M}$ define $\breve{u}(y,t) = \int p(x,y,t) f(x)\, dx$. Since $\int p(x,y,t)\, dy = 1$ for all $t>0$ we obtain from the above argument that the L^2-norm of $\breve{u}(\cdot, t)$ does not exceed $e^{2\varkappa t}$ times the L^2-norm of f for all $t>0$. The functions $q_j(\cdot, y, t)$ are solutions of the equation $L^* - \partial/\partial t = 0$. Therefore we obtain as above that the functions $q_j(\cdot, y, t)$ converge uniformly on compact sets to $p(\cdot, y, t)$, and that moreover $p(\cdot, y, t)$ satisfies the properties claimed in the lemma. □

Remark. The proof of the above lemma shows that $p(x, \cdot, t)$ is square integrable for $x \in \widetilde{M}$, $t>0$ with L^2-norm bounded from above by a constant $c(t)$ which only depends on t and C^α-bounds for the coefficients of L in exponential coordinates.

We assume now that $\widetilde{M}$ is the universal covering of a compact manifold M and we consider families of differential operators on $\widetilde{M}$ which are projections of the lift to $T^1\widetilde{M}$ of a differential operator L on the unit tangent bundle T^1M of M with Hölder-continuous coefficients which is subordinate to the stable foliation.

Let g be a positive semi-definite bilinear form on $T^1\widetilde{M}$ of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$ whose restriction to TW^s is positive definite. Let Y be a section of TW^s of class $C_s^{1,\alpha}$ and write $L = \Delta + Y$ where Δ is the leafwise Laplacian subordinate to W^s which is induced by g. For every $v \in T^1\widetilde{M}$ the restriction of L to $W^s(v) \sim \widetilde{M}$ then projects to a second-order uniformly elliptic operator L_v on $\widetilde{M}$ with Hölder-continuous coefficients.

Recall from the beginning of this appendix the definition of the $C^{2,\alpha}$ norms $\|f\|_{2,\alpha}$ for functions f on $\widetilde{M}$ ($\alpha>0$).

Recall from [GH] and the introduction the definition of the Gromov product on $\partial\widetilde{M}$.

Namely, for $x\in\widetilde{M}$ and $\xi,\eta\in\partial\widetilde{M}$ define

$$(\xi|\eta)_x=\lim_{\substack{y\to\xi\\ z\to\eta}}\tfrac{1}{2}(\operatorname{dist}(x,y)+\operatorname{dist}(x,z)-\operatorname{dist}(y,z)).$$

For the proof of the following lemma compare [Dod]:

LEMMA A.4. *For every $\delta>0$ there is a number $\beta=\beta(\delta)>0$ and a number $c=c(\delta)>0$ with the following properties: Let $f:\widetilde{M}\to\mathbf{R}$ be a function with $\|f\|_{2,\alpha}<\infty$. For $v\in T^1\widetilde{M}$ denote by f_v the solution of the parabolic equation $(L_v-\partial/\partial t)f_v=0$ with $f_v(x,0)=f(x)$ for $x\in\widetilde{M}$. Then $|(f_v-f_w)(x,t)|\leqslant c\|f\|_{2,\alpha}e^{\delta t}e^{-\beta(\pi(v)|\pi(w))_x}$ for $v,w\in T^1\widetilde{M}$ and all $(x,t)\in\widetilde{M}\times[0,\infty)$.*

Proof. Let $x_0\in\widetilde{M}$ be arbitrarily fixed. As in the proof of Lemma A.2 choose a non-decreasing function φ of class C^∞ on $(0,\infty)$ such that $\varphi(s)=0$ for $s\in\left(0,\frac{1}{2}\right]$ and $\varphi(s)=s$ for $s\geqslant1$. Define $\varrho(x)=\varphi(\operatorname{dist}(x_0,x))$. Then there is a number $k>0$ such that $|L_z\varrho|\leqslant k$ for all $z\in T^1\widetilde{M}$.

Let $v,w\in T^1\widetilde{M}$ and let p_v (or p_w) be the fundamental solution of the equation $L_v-\partial/\partial t=0$ (or $L_w-\partial/\partial t=0$). Let f be a function on $\widetilde{M}$ with $\|f\|_{2,\alpha}<\infty$ and define

$$f_v(x,t)=\int p_v(x,y,t)f(y)\,dy\quad\text{and}\quad f_w(x,t)=\int p_w(x,y,t)f(y)\,dy.$$

Since $\int p_v(x,y,t)\,dy=1=\int p_w(x,y,t)\,dy$ for all $x\in\widetilde{M}$ and all $t>0$, the C^0-norm of the functions $f_v^t:x\to f_v(x,t)$ and $f_w^t:x\to f_w(x,t)$ is bounded from above by $\|f\|_0$ independent of $t>0$. Using Schauder theory for parabolic equations (see [Fr, pp. 64–65]) we deduce that there is a number $a>0$ not depending on v such that

$$\|f_v^t\|_{2,\alpha}\leqslant a\|f\|_{2,\alpha}$$

for all $t>0$.

By our assumptions on L there are numbers $b>0$, $\beta>0$ such that $|(L_v-L_w)u(x)|\leqslant b\|u\|_{2,\alpha}e^{-\beta(\pi(v)|\pi(w))_x}$ for all functions u on $\widetilde{M}$ with $\|u\|_{2,\alpha}<\infty$ and all $v,w\in T^1\widetilde{M}$.

Let $\delta>0$. By eventually decreasing β we may moreover assume that the function $\psi:x\to e^{-\beta(\pi(v)|\pi(w))_x}$ satisfies $|L_w\psi|\leqslant\frac{1}{2}\delta\psi$, independent of v and w. Let now $N=2\|f\|_0$ and let $c=2ab$. For $R\geqslant1$, $x\in\widetilde{M}$ and $s\geqslant0$ define

$$\nu(x,s)=(f_w-f_v)(x,s)-\frac{N}{R}(\varrho+Ks)(x)-c\|f\|_{2,\alpha}e^{\delta s}\psi.$$

Since

$$\left|\left(L_w-\frac{\partial}{\partial t}\right)(f_w-f_v)(x,t)\right|=|(L_v-L_w)f_v^tx|\leqslant\tfrac{1}{2}c\psi(x)$$

by the choice of c and the above estimates we have $(L_w - \partial/\partial t)\nu \geqslant 0$ and moreover $\nu \leqslant 0$ on $B(x_0, R) \times \{0\} \cup \partial B(x_0, R) \times [0, t]$. The maximum principle then implies that $\nu \leqslant 0$ on $B(x_0, R) \times [0, t]$, and since $R>0$ was arbitrary we obtain

$$(f_w - f_v)(x, s) \leqslant c \|f\|_{2,\alpha} e^{\delta s} e^{-\beta(\pi(v)|\pi(w))_x} \quad \text{for all } (x, s) \in \widetilde{M} \times (0, \infty).$$

Similarly we obtain an estimate for $f_v - f_w$, and from this the lemma follows. □

Denote by p_v the fundamental solution of the parabolic equation $L_v - \partial/\partial t = 0$. From the above estimates we then obtain

COROLLARY A.5. *There are numbers $a>0$, $b>0$ such that*

$$|p_v(x, y, t) - p_w(x, y, t)| \leqslant e^{at} [e^{-b(\pi(v)|\pi(w))_x} + e^{-b(\pi(v)|\pi(w))_y}]$$

for all $v, w \in T^1\widetilde{M}$ and all $t \geqslant 2$.

Proof. Let $v, w \in T^1\widetilde{M}$, $z \in \widetilde{M}$ and for $t>0$ define a function f_t^z on $\widetilde{M}$ by $f_t^z(y) = p_v(y, z, t)$. Lemma A.3 and its proof shows that there is a constant $c_1>0$ not depending on z such that $\|f_{1/2}^z\|_0 \leqslant c_1$. Now for $t>\frac{1}{2}$ we have $f_t^z(y) = \int p_v(y, u, t-\frac{1}{2}) p_v(u, z, \frac{1}{2})\, du$, and since $\int p_v(y, u, t-\frac{1}{2})\, du = 1$ for all $t>\frac{1}{2}$ this means that $\|f_t^z\|_0 \leqslant c_2$ for all $t \geqslant \frac{1}{2}$ and all $z \in \widetilde{M}$. Schauder theory for parabolic equations then shows that there is a constant $c_2>0$ such that $\|f_t^z\|_{2,\alpha} \leqslant c_2$ for all $t \geqslant 1$ and all $z \in \widetilde{M}$.

Let now $t \geqslant 1$, and for $x \in \widetilde{M}$ and $s>0$ define

$$u_v(x, s) = \int p_v(x, y, s) f_t^z(y)\, dy \quad \text{and} \quad u_w(x, s) = \int p_w(x, y, s) f_t^z(y)\, dy.$$

By Lemma A.4 there are then numbers $a, b, c>0$ such that

$$|(u_v - u_w)(x, s)| \leqslant c e^{as} e^{-b(\pi(v)|\pi(w))_x}$$

for all $(x, t) \in \widetilde{M} \times (0, \infty)$.

On the other hand, for $x \in \widetilde{M}$ and $s>0$ write $g_s^x(y) = p_w(x, y, s)$. The above arguments then show that there is a constant $c_3>0$ such that $\|g_s^x\|_{2,\alpha} \leqslant c_3$ for all $x \in \widetilde{M}$ and all $s \geqslant 1$. Another application of the arguments in Lemma A.4 for the operators L_v^*, L_w^* which are formally adjoint to L_v, L_w shows that $|u_w(x, s) - p_w(x, z, s+t)| \leqslant c e^{as} e^{-b(\pi(v)|\pi(w))_z}$ for all $x \in \widetilde{M}$ and all $s \geqslant 0$ (where we might have to adjust the constants a, b, c from above). Together this just means that

$$|p_v(x, z, 2t) - p_w(x, z, 2t)| \leqslant c e^{at} [e^{-b(\pi(v)|\pi(w))_x} + e^{-b(\pi(v)|\pi(w))_z}]$$

for all $t \geqslant 1$. □

Recall from the introduction the definition of the set $\widetilde{D}\subset T^1\widetilde{M}\times T^1\widetilde{M}$ and let p: $\widetilde{D}\times(0,\infty)\to(0,\infty)$ be the function whose restriction to $\{v\}\times W^s(v)\times(0,\infty)$ just equals the solution of the $L|_{W^s(v)}$-Cauchy problem with initial condition the Dirac mass at v. As an immediate consequence of Corollary A.5 we obtain

COROLLARY A.6. *The function* $p: \widetilde{D}\times(0,\infty)\to(0,\infty)$ *is locally Hölder continuous.*

Appendix B

This appendix is devoted to the investigation of operators L on T^1M with Hölder-continuous coefficients which are weakly coercive. Our general assumption will be that M is a compact Riemannian manifold of negative sectional curvature and g is a positive semi-definite bilinear form on T^1M of class $C^{1,\alpha}$ for some $\alpha\in(0,1]$ whose restriction to TW^s is positive definite. Let Y be a section of TW^s of class $C_s^{1,\alpha}$ and let χ be a function on T^1M of class C^α. Write $L=\Delta+Y+\chi$ where as before Δ is the leafwise Laplacian subordinate to W^s which is induced by g. The operator L lifts to an operator on $T^1\widetilde{M}$ which we denote again by the same symbol. For every $v\in T^1\widetilde{M}$ the restriction of L to $W^s(v)\sim\widetilde{M}$ then projects to a second-order uniformly elliptic operator L_v on $\widetilde{M}$ with Hölder-continuous coefficients.

For a section Z of TW^s of class C_s^1 denote by $\operatorname{div}(Z)$ the function on T^1M whose value at $v\in T^1M$ equals the divergence at v of the restriction of Z to the Riemannian manifold $(W^s(v),g)$. Write $L^*=\Delta-Y+(\chi-\operatorname{div} Y)$. For every $v\in T^1\widetilde{M}$ the operator L_v^* is then formally adjoint to L_v with respect to the projection of $g|_{W^s(v)}$ to $\widetilde{M}$.

We call L *weakly coercive* if for every $v\in T^1\widetilde{M}$ the operator L_v is weakly coercive in the sense of Ancona ([An]). To clarify this notion we observe first of all

LEMMA B.1. *The following are equivalent*:

(1) *L is weakly coercive.*

(2) *There is $v\in T^1\widetilde{M}$ such that L_v is weakly coercive.*

(3) *There is $v\in T^1\widetilde{M}$ such that L_v^* is weakly coercive.*

Proof. Since (1) obviously implies (2), assume that there is some $v\in T^1\widetilde{M}$ such that L_v is weakly coercive. We have to show that for every $w\in T^1\widetilde{M}$ the operator L_w is weakly coercive. For this choose a number $\delta>0$ such that there is a positive $(L_v+\delta)$-harmonic function φ on $\widetilde{M}\sim W^s(v)$. Let $p\in\widetilde{M}$ and let $w\in T_p^1\widetilde{M}$ be arbitrary. Choose a sequence $\{\Psi_i\}_i\subset\pi_1(M)$ such that $\Psi_i(\pi(v))\to\pi(w)$ in $\partial\widetilde{M}$. Let $w_i\in T_p^1\widetilde{M}$ be such that $\pi(w_i)=\Psi_i(\pi(v))$ and define $\varphi_i=\varphi\circ\Psi_i^{-1}/\varphi(\Psi_i^{-1}(p))$. Then φ_i is a positive $(L_{w_i}+\delta)$-harmonic function on $\widetilde{M}$ which is normalized to be 1 at p. Since the coefficients of the operators L_{w_i} are uniformly Hölder continuous we may assume by passing to a subsequence that the funcions φ_i converge uniformly on compact subsets of $\widetilde{M}$ to a function φ. But $L_{w_i}+\delta \to L_w+\delta$ and hence necessarily $(L_w+\delta)(\varphi)=0$. In other words, L_w is weakly coercive and (1) and (2) are equivalent.

On the other hand, if L_v is weakly coercive for some $v\in T^1\widetilde{M}$ then there is $\delta>0$ such that $L_v+\delta$ admits a Green function G on $\widetilde{M}$. Then $G^*(x,y)=G(y,x)$ is a Green function for $L_v^*+\delta$ on $\widetilde{M}$ and hence L_v^* is weakly coercive as well. This shows that (2) and (3) are equivalent and finishes the proof of the lemma. □

We assume from now on that L is weakly coercive. Recall from the introduction the definition of the set $\widetilde{D}\subset T^1\widetilde{M}\times T^1\widetilde{M}$. Let $K:\widetilde{D}\times\partial\widetilde{M}\to(0,\infty)$ (or $K^*:\widetilde{D}\times\partial\widetilde{M}\to(0,\infty)$) be the function whose restriction to $W^s(v)\times W^s(v)\times\partial\widetilde{M}$ equals the Martin kernel of the operator $L|_{W^s(v)}$ (or $L^*|_{W^s(v)}$) and define $K_\infty:\widetilde{D}\to(0,\infty)$ (or $K_\infty^*:\widetilde{D}\to(0,\infty)$) by $K_\infty(v,w)=K(v,w,\pi(v))$ (or $K_\infty^*(v,w)=K^*(v,w,\pi(v))$). We want to show that K_∞ is Hölder continuous.

Choose $\delta>0$ sufficiently small that for every $v\in T^1\widetilde{M}$ the operator $L_v+3\delta$ on $\widetilde{M}\sim W^s(v)$ is weakly coercive. As in the introduction, for $v\in T^1\widetilde{M}$ and $\alpha\in(0,\pi)$ let $C(v,\alpha)$ be the open cone of angle α and direction v in $(\widetilde{M},\langle\cdot,\cdot\rangle)$.

For $v\in T^1\widetilde{M}$ and $w\in W^s(v)$ define $\varphi_v(Pw)=K_\infty(v,w)$. Then φ_v is a minimal positive L_v-harmonic function on $\widetilde{M}$ with pole at $\pi(v)$. Similarly let ψ_v (or η_v) be the unique positive minimal $(L_v+2\delta)$-harmonic function (or positive minimal $(L_v-2\delta)$-harmonic function) on $\widetilde{M}$ with pole at $\pi(v)$ which is normalized by $\psi_v(Pv)=1$ (or $\eta_v(Pv)=1$).

Let again dist be the distance on $\widetilde{M}$ induced by $\langle\cdot,\cdot\rangle$ and write $x=Pv$. Since the operators $L_v-2\delta$, L_v and $L_v+2\delta$ are weakly coercive, there are constants $C_0\geqslant 1$ and $\beta_1>\beta_2>0$ such that

$$\begin{aligned}C_0^{-1}e^{-\beta_1\operatorname{dist}(x,y)}&\leqslant\min\{\varphi_v(y)/\psi_v(y),\eta_v(y)/\varphi_v(y)\}\\&\leqslant\max\{\varphi_v(y)/\psi_v(y),\eta_v(y)/\varphi_v(y)\}\leqslant C_0e^{-\beta_2\operatorname{dist}(x,y)}\end{aligned}$$

for all $y\in C\left(-v,\frac{1}{2}\pi\right)$ (see [An]).

Recall that for every smooth function f on $\widetilde{M}$ we have

$$\varphi_v^{-1}L_v(\varphi_v f)=\Delta(f)+Y(f)+2\nabla\log\varphi_v(f)$$

and hence since L_v is weakly coercive the same is true for $\Delta+Y+2\nabla\log\varphi_v$. For $\varepsilon>0$ denote by $\sigma_{v,\varepsilon}$ the unique minimal positive $(\Delta+Y+2\nabla\log\varphi_v-\varepsilon)$-harmonic function on $\widetilde{M}$ with pole at $\pi(v)$ which is normalized to be 1 at Pv. Notice that $\sigma_{v,0}\equiv 1$ since φ_v is minimal. Then we have

LEMMA B.2. *For every* $\varepsilon \in (0,1]$ *there is a number* $t(\varepsilon)>0$ *such that for every* $v \in T^1\widetilde{M}$ *the following is satisfied*:

(i) *The function* $\psi_v^{\sigma_{v,\varepsilon}} \varphi_v^{1-\sigma_{v,\varepsilon}}$ *is* $(L_v - \delta\sigma_{v,\varepsilon})$*-subharmonic on* $C(\Phi^{t(\varepsilon)}(-v), \frac{1}{2}\pi)$.

(ii) *The function* $\eta_v^{\sigma_{v,\varepsilon}} \varphi_v^{1-\sigma_{v,\varepsilon}}$ *is* $(L_v + \delta\sigma_{v,\varepsilon})$*-superharmonic on* $C(\Phi^{t(\varepsilon)}(-v), \frac{1}{2}\pi)$.

Proof. Fix a number $\varepsilon>0$ and for $v\in T^1\widetilde{M}$ arbitrarily fixed write simply φ (or ψ, η, σ) instead of φ_v (or $\psi_v, \eta_v, \sigma_{v,\varepsilon}$). The lemma now follows from the above estimates for the functions φ, ψ, η and a simple computation.

Let as before g be a positive semi-definite bilinear form on T^1M inducing Δ and for $v\in T^1\widetilde{M}$ and a smooth function α on $\widetilde{M}$ denote by $\nabla\alpha$ the $g|_{W^s(v)}$-gradient of α (here we identify again $W^s(v)$ with $\widetilde{M}$). Let $\|\cdot\|$ be the norm on $T\widetilde{M}$ induced by $g|_{W^s(v)}$ and write simply Δ instead of Δ_v and Y instead of Y_v, χ instead of χ_v. Let α, β be positive functions of class C^2 on $\widetilde{M}$. By the definition of φ, ψ we then have:

$$\Delta(\log\psi)+Y(\log\psi)=\psi^{-1}(\Delta(\psi)+Y(\psi))-\|\nabla\log\psi\|^2=-2\delta-\|\nabla\log\psi\|^2-\chi, \tag{1}$$

$$\Delta(\log\varphi)+Y(\log\varphi)=-\|\nabla\log\varphi\|^2-\chi, \tag{2}$$

$$\begin{aligned}\Delta(\psi^\alpha)+Y(\psi^\alpha)+\alpha\chi\psi^\alpha &= \psi^\alpha[\Delta(\alpha\log\psi)+Y(\alpha\log\psi)+\alpha\chi+\|\nabla(\alpha\log\psi)\|^2]\\ &= \psi^\alpha[(\log\psi)(\Delta(\alpha)+Y(\alpha))+2g(\nabla\alpha,\nabla\log\psi)-2\delta\alpha\\ &\qquad -\alpha\|\nabla\log\psi\|^2+\|(\log\psi)\nabla\alpha+\alpha\nabla\log\psi\|^2]\\ &= \psi^\alpha\alpha[-2\delta-\|\nabla\log\psi\|^2+(\log\psi)\alpha^{-1}(\Delta(\alpha)+Y(\alpha))\\ &\qquad +2g(\nabla\log\alpha,\nabla\log\psi)+\alpha\|(\log\psi)\nabla\log\alpha+\nabla\log\psi\|^2],\end{aligned} \tag{3}$$

$$\begin{aligned}2g(\nabla(\psi^\alpha),\nabla(\varphi^{1-\beta})) &= 2\psi^\alpha\varphi^{1-\beta}g(\nabla(\alpha\log\psi),\nabla((1-\beta)\log\varphi))\\ &= 2\psi^\alpha\varphi^{1-\beta}\alpha[g(\nabla\log\psi,\nabla\log\varphi)+(\log\psi)g(\nabla\log\alpha,\nabla\log\varphi)\\ &\qquad -\beta g(\nabla\log\psi+(\log\psi)\nabla\log\alpha,\nabla\log\varphi+(\log\varphi)\nabla\log\beta)],\end{aligned} \tag{4}$$

$$\begin{aligned}&\Delta(\varphi^{1-\beta})+Y(\varphi^{1-\beta})+(1-\beta)\chi\varphi^{1-\beta}\\ &\quad=\varphi^{1-\beta}[\Delta((1-\beta)\log\varphi)+Y((1-\beta)\log\varphi)+(1-\beta)\chi+\|\nabla((1-\beta)\log\varphi)\|^2]\\ &\quad=\varphi^{1-\beta}[(\beta-1)\|\nabla\log\varphi\|^2-(\log\varphi)(\Delta(\beta)+Y(\beta))\\ &\qquad -2g(\nabla\beta,\nabla\log\varphi)+\|\nabla\log\varphi-(\beta\nabla\log\varphi+(\log\varphi)\nabla\beta)\|^2]\\ &\quad=\varphi^{1-\beta}\beta[-\|\nabla\log\varphi\|^2-(\log\varphi)\beta^{-1}(\Delta(\beta)+Y(\beta))-2g(\nabla\log\beta,\nabla\log\varphi)\\ &\qquad -2(\log\varphi)g(\nabla\log\varphi,\nabla\log\beta)+\beta\|\nabla\log\varphi+(\log\varphi)\nabla\log\beta\|^2].\end{aligned} \tag{5}$$

Now let $\beta=\alpha$. Then we obtain from the above computations

$$\begin{aligned}\Delta(\psi^{\alpha}\varphi^{1-\alpha})+Y(\psi^{\alpha}\varphi^{1-\alpha})+\chi\psi^{\alpha}\varphi^{1-\alpha}&=\varphi^{1-\alpha}\Delta(\psi^{\alpha})\\&\quad+2g(\nabla\psi^{\alpha},\nabla\varphi^{1-\alpha})+\psi^{\alpha}\Delta(\varphi^{1-\alpha})\\&\quad+\varphi^{1-\alpha}Y(\psi^{\alpha})+\psi^{\alpha}Y(\varphi^{1-\alpha})+\chi\psi^{\alpha}\varphi^{1-\alpha}\\&=\psi^{\alpha}\varphi^{1-\alpha}\alpha[-2\delta-\|\nabla\log\psi-\nabla\log\varphi\|^{2}\\&\quad+2g(\nabla\log\alpha,\nabla\log\psi-\nabla\log\varphi)\\&\quad+\alpha^{-1}(\Delta(\alpha)+Y(\alpha))(\log\psi-\log\varphi)\\&\quad+2g(\nabla\log\alpha,\nabla\log\varphi)(\log\psi-\log\varphi)+\alpha R]\end{aligned}\tag{6}$$

where

$$R=\|(\log\psi-\log\varphi)\nabla\log\alpha+\nabla\log\psi-\nabla\log\varphi\|^{2}.$$

Recall that the geometry of $\widetilde{M}$ is bounded and that the operator Δ is uniformly elliptic with respect to $\langle\cdot,\cdot\rangle$, with uniformly bounded coefficients. This implies that there is a number $\xi\geqslant 1$ such that

$$\sup\{(\|\nabla\log\varphi\|+\|\nabla\log\psi\|+\|\nabla\log\eta\|+\|\nabla\log\sigma\|)(y)\mid y\in\widetilde{M}\}\leqslant\xi$$

(see [GT]).

Since

$$\log C_0+\beta_1\operatorname{dist}(x,y)\geqslant\log\psi(y)-\log\varphi(y)>\beta_2\operatorname{dist}(x,y)-\log C_0$$

for all $y\in C\left(-v,\frac{1}{2}\pi\right)$ by the above estimates there is a number $\tau(\varepsilon)>0$ such that

$$(\log\psi-\log\varphi)(y)\geqslant\frac{6\xi^{2}+3\delta}{\varepsilon}$$

for all $y\in C\left(\Phi^{\tau(\varepsilon)}(-v),\frac{1}{2}\pi\right)$. On the other hand we have $\sigma(y)\leqslant ce^{-\beta_3\operatorname{dist}(x,y)}$ for $y\in C\left(-v,\frac{1}{2}\pi\right)$ with some $\beta_3>0$, $c>0$ and hence we can find a number $t(\varepsilon)\geqslant\tau(\varepsilon)$ such that $|\sigma R|(y)\leqslant\frac{1}{2}\delta$ for all $y\in C\left(\Phi^{t(\varepsilon)}(-v),\frac{1}{2}\pi\right)$, where the function $R\colon\widetilde{M}\to\mathbf{R}$ is defined as in (6) above.

Let now $\alpha=\sigma$. Since

$$\sigma^{-1}(\Delta(\sigma)+Y(\sigma))+2g(\nabla\log\sigma,\nabla\log\varphi)=\varepsilon$$

we obtain

$$\begin{aligned}&\Delta(\psi^{\sigma}\varphi^{1-\sigma})+Y(\psi^{\sigma}\varphi^{1-\sigma})+\chi\psi^{\sigma}\varphi^{1-\sigma}\\&\quad=\psi^{\sigma}\varphi^{1-\sigma}\sigma[-2\delta+2g(\nabla\log\sigma,\nabla\log\psi-\nabla\log\varphi)-\|\nabla\log\psi-\nabla\log\varphi\|^{2}\\&\qquad\qquad+\varepsilon(\log\psi-\log\varphi)+\sigma R].\end{aligned}$$

Together with the above estimates this shows that the function $\psi^\sigma\varphi^{1-\sigma}$ is indeed $(L_v-\delta\sigma)$-subharmonic on $C\big(\Phi^{t(\varepsilon)}(-v),\frac{1}{2}\pi\big)$ which is (i) of the lemma.

The same computations and estimates can also be applied to the functions

$$\eta_v^{\sigma_v}\varphi_v^{1-\sigma_v}\quad (v\in T^1\widetilde{M})$$

and yield (ii) above. □

For $y\in\widetilde{M}$ and $v\in T^1\widetilde{M}$ define $\pi_v(y)=W^s(v)\cup P^{-1}(y)$. We use now Lemma B.2 to compare the function φ_v $(v\in T^1\widetilde{M})$ on $C\big(-v,\frac{1}{2}\pi\big)$ with certain L_w-harmonic functions on $C\big(-v,\frac{1}{2}\pi\big)$ provided that $w\in T^1\widetilde{M}$ is close enough to v.

COROLLARY B.3. *There are numbers $\alpha,\nu>0$ with the following properties: Let $v\in T^1\widetilde{M}$, $w\in T^1_{Pv}\widetilde{M}$ with $\angle(v,w)<\nu$ and let f be the unique L_w-harmonic function on $C\big(-v,\frac{1}{2}\pi\big)$ which coincides with φ_v on $\partial C\big(-v,\frac{1}{2}\pi\big)$. Then*

$$(1-\angle(v,w)^\alpha)\varphi_v(x)\leqslant f(x)\leqslant(1+\angle(v,w)^\alpha)\varphi_v(x)$$

for all $x\in C\big(-v,\frac{1}{2}\pi\big)$.

Proof. Let $\nu_1>0$ be sufficiently small that $\pi(w)\notin\partial C\big(-v,\frac{3}{4}\pi\big)\cap\partial\widetilde{M}$ for all $v\in T^1\widetilde{M}$ and all $w\in T^1_{Pv}\widetilde{M}$ with $\angle(v,w)<\nu_1$. Since asymptotic geodesics approach with an exponential speed and since the stable foliation of $T^1\widetilde{M}$ is Hölder continuous there are numbers $a_1>0$, $\varkappa_1>0$, $\alpha_1>0$ such that

$$\angle(\pi_v(y),\pi_w(y))\leqslant a_1e^{-\varkappa_1\operatorname{dist}(Pv,y)}(\angle(v,w))^{\alpha_1}$$

for all $v\in T^1\widetilde{M}$, all $w\in T^1_{Pv}\widetilde{M}$ with $\angle(v,w)<\nu_1$ and all $y\in C\big(-v,\frac{1}{2}\pi\big)$.

For $y\in\widetilde{M}$ and $r>0$ let $B(y,r)$ be the ball of radius r about y in $(\widetilde{M},\langle\cdot,\cdot\rangle)$. Since the geometry of $\widetilde{M}$ is bounded, exponential coordinates centered at y on the ball $B(y,1)$ induce a C^2-norm for functions on $B\big(y,\frac{1}{2}\big)$ with the property that for every $z\in T^1\widetilde{M}$ and every $\varepsilon\in[-2\delta,2\delta]$ the C^2-norm on $B\big(y,\frac{1}{2}\big)$ of every positive $(L_z+\varepsilon)$-harmonic function β on $B(y,1)$ is bounded from above by a constant multiple of $\beta(y)$.

For $\varepsilon\in[0,1]$ and $z\in T^1\widetilde{M}$ write $u_{z,\varepsilon}=\psi_z^{\sigma_{z,\varepsilon}}\varphi_z^{-\sigma_{z,\varepsilon}}$. Fix $v\in T^1\widetilde{M}$ and write $x=Pv$. By the above estimates there are then numbers $a_2,\varkappa_2,\alpha_2>0$ not depending on v and z,ε such that for every $\varepsilon\in[0,1]$, all $z\in W^s(v)$, every $w\in T^1_x\widetilde{M}$ with $\angle(v,w)<\nu_1$ and all $y\in C\big(-v,\frac{1}{2}\pi\big)$ we have

$$|(L_v-L_w)u_{z,\varepsilon}\varphi_v|(y)\leqslant a_2e^{-\varkappa_2\operatorname{dist}(x,y)}(\angle(v,w))^{\alpha_2}u_{z,\varepsilon}\varphi_v(y).$$

Following Ancona, the functions $\sigma_{z,\varepsilon}$ were defined in such a way that we can find a number $\varepsilon>0$ such that

$$c_1e^{-\varkappa_2\operatorname{dist}(Pz,y)/2}\leqslant\sigma_{z,\varepsilon}(y)\leqslant c_1^{-1}e^{-2\varkappa_3\operatorname{dist}(Pz,y)}$$

for some $c_1>0$, $\varkappa_3\in\left(0,\frac{1}{2}\varkappa_2\right)$ and all $y\in C\left(-z,\frac{1}{2}\pi\right)$. This implies in particular that there is a number $r_0>0$ such that $\delta\sigma_{z,\varepsilon}(y)\geqslant a_2 e^{-\varkappa_2 \operatorname{dist}(Pz,y)}$ and

$$-e^{-\varkappa_3 \operatorname{dist}(Pz,y)}\leqslant \log u_{z,\varepsilon}(y)\leqslant e^{-\varkappa_3 \operatorname{dist}(Pz,y)}$$

for all $y\in C\left(\varphi^{r_0}(-z),\frac{1}{2}\pi\right)$, where $z\in T^1\widetilde{M}$ is arbitrary.

Let now $t(\varepsilon)>0$ be as in Lemma B.2 and define $\tau=\max\{t(\varepsilon),r_0\}$ and

$$\nu=\min\{\nu_1,(a_2^{-1}e^{-\tau\varkappa_2})^{1/\alpha_2}\}>0.$$

Let $w\in T^1_{Pv}\widetilde{M}$ with $\chi=\angle(v,w)<\nu$ and define $s=s(\chi)=(-\log a_2-\alpha_2\log\chi)/\varkappa_2\geqslant\tau$ and $z=\Phi^s v$.

For $y\in C\left(-v,\frac{1}{2}\pi\right)$ we then have

$$\begin{aligned}L_w(u_{z,\varepsilon}\varphi_v)(y)&\geqslant (L_v-a_2e^{-\varkappa_2(\operatorname{dist}(Pv,y))+\tau)})u_{z,\varepsilon}\varphi_v(y)\\&\geqslant(\delta\sigma_{z,\varepsilon}(y)-a_2e^{-\varkappa_2\operatorname{dist}(Pz,y)})(u_{z,\varepsilon}\varphi_v)(y)\geqslant 0,\end{aligned}$$

i.e. the function $u_{z,\varepsilon}\varphi_v$ is L_w-subharmonic on $C\left(-v,\frac{1}{2}\pi\right)$. With

$$\varrho(\chi)=e^{-\varkappa_3 s}=a_2^{\varkappa_3/\varkappa_2}\chi^{\varkappa_3\alpha_2/\varkappa_2}$$

it follows moreover that $e^{-\varrho(\chi)}u_{z,\varepsilon}\varphi_v\leqslant\varphi_v$ on $C\left(-v,\frac{1}{2}\pi\right)$.

Let now f be the unique L_w-harmonic function on $C\left(-v,\frac{1}{2}\pi\right)$ which coincides with φ_v on $\partial C\left(-v,\frac{1}{2}\pi\right)$. Then $e^{-\varrho(\chi)}u_{z,\varepsilon}\varphi_v-f$ is L_w-subharmonic on $C\left(-v,\frac{1}{2}\pi\right)$ and $\leqslant 0$ on $\partial C\left(-v,\frac{1}{2}\pi\right)$ and hence by the maximum principle $f\geqslant e^{-\varrho(\chi)}u_{z,\varepsilon}\varphi_v\geqslant e^{-2\varrho(\chi)}\varphi_v$ on $C\left(-v,\frac{1}{2}\pi\right)$. On the other hand, by the definition of $\varrho(\chi)$ there is a number $\alpha>0$ such that $e^{-2\varrho(\chi)}\geqslant 1-\chi^\alpha$ for all $\chi<\nu$ and consequently $f\geqslant(1-\angle(v,w)^\alpha)\varphi_v$. This yields the first inequality in the corollary; the second one follows in exactly the same way by comparing with the $(\Delta_v-\delta\sigma_{z,\varepsilon})$-superharmonic functions $\eta_z^{\sigma_{z,\varepsilon}}\varphi_z^{1-\sigma_{z,\varepsilon}}$ on $C\left(-v,\frac{1}{2}\pi\right)$. □

Ancona showed in [An] that there is a number $c>0$ such that for all $v,w\in T^1\widetilde{M}$ and all positive L_v-harmonic functions f,u on $C\left(w,\frac{1}{2}\pi\right)$ which vanish on $\partial C\left(w,\frac{1}{2}\pi\right)\cap\partial\widetilde{M}$ we have

$$\frac{f(x)}{u(x)}\leqslant c\frac{f(P\Phi^1w)}{u(P\Phi^1w)}\quad\text{for all } x\in C\left(\Phi^1w,\tfrac{1}{2}\pi\right).$$

As a corollary of the above considerations we obtain a similar Harnack inequality for L_v- and L_w-harmonic functions. For this let $\nu>0$, $\alpha>0$ be as in Corollary B.3 and define $\bar{c}=(1+\nu^\alpha)c^2$. Then we have

COROLLARY B.4. *Let $v\in T^1\widetilde{M}, w\in T^1_{Pv}\widetilde{M}$ with $\angle(v,w)<\nu$ and let f (or u) be a positive L_v-harmonic function (or a positive L_w-harmonic function) which is defined on $C(-v,\frac{1}{2}\pi)$ and vanishes on $\partial C(-v,\frac{1}{2}\pi)\cap\partial\widetilde{M}$. Then*

$$\bar{c}^{-1}\frac{f(P\Phi^1(-v))}{u(P(\Phi^1(-v))}\leqslant\frac{f(x)}{u(x)}\leqslant\bar{c}\frac{f(P\Phi^1(-v))}{u(P\Phi^1(-v))}$$

for all $x\in C(\Phi^1(-v),\frac{1}{2}\pi)$.

Corollary B.3 can now be combined with the arguments of Anderson–Schoen (in the proof of Theorem 6.2 of [AS]) to show

COROLLARY B.5. *There is a number $\beta>0$ such that*

$$1-\angle(v,w)^\beta\leqslant\frac{\varphi_v(x)}{\varphi_w(x)}\leqslant 1+\angle(v,w)^\beta$$

for all $v\in T^1\widetilde{M}$, $w\in T^1_{Pv}\widetilde{M}$ with $\angle(v,w)<\nu$ and all $x\in C(-v,\frac{1}{2}\pi)$.

Proof. Let $c>0$ be the constant as above (whose existence is due to Ancona) and define $\chi=(c-1)/(c+1)<1$. Let $w,z\in T^1\widetilde{M}$ and let u, f be positive L_w-harmonic functions on $C(z,\frac{1}{2}\pi)$. By the arguments in the proof of Theorem 6.2 of [AS] we then have

$$\frac{u(x)}{f(x)}-\frac{u(y)}{f(y)}\leqslant\chi^s c\frac{u(\Phi^s z)}{f(\Phi^s z)}$$

for all $x,y\in C(\Phi^{s+1}z,\frac{1}{2}\pi)$ and all $s\geqslant 0$.

Let $v\in T^1\widetilde{M}$, $x=Pv$ and let $w\in T^1_x\widetilde{M}$ be such that $\angle(v,w)<\nu$ where $\nu>0$ is as in Corollary B.3. Recall that there is a number $\varkappa>0$ such that

$$\angle(\Phi^t v,\pi_w(P\Phi^t v))\leqslant e^{\varkappa t}\angle(v,w)$$

for all $t\geqslant 0$ where $\pi_w\colon M\to W^s(w)$ is defined as before. Define

$$s=s(\angle(v,w))=\frac{\log\nu-\log\angle(v,w)}{2\varkappa}$$

and let $\bar{v}=\Phi^s v$, $z=\pi_w(P\Phi^s v)$.

Let f_z be the unique L_z-harmonic function on $C(-\bar{v},\frac{1}{2}\pi)$ which coincides with $\varphi_{\bar{v}}$ on $\partial C(-\bar{v},\frac{1}{2}\pi)$. Since $\angle(\bar{v},z)\leqslant\nu^{1/2}\angle(v,w)^{1/2}$ we then have

$$1-\nu^{\alpha/2}\angle(v,w)^{\alpha/2}\leqslant\frac{\varphi_{\bar{v}}(y)}{f_z(y)}\leqslant 1+\nu^{\alpha/2}\angle(v,w)^{\alpha/2}$$

for all $y\in C(-\bar v,\frac12\pi)$ where $\alpha>0$ as in Corollary B.3. Moreover the Harnack inequality for $\varphi_{\bar v}$ together with the Harnack inequality at infinity of Ancona shows that there is a number $c_1>0$ such that

$$c_1^{-1}\leqslant\frac{\varphi_{\bar v}(y)}{\varphi_z(y)}\leqslant c_1\quad\text{for all } y\in C\big(\Phi^1(-\bar v),\tfrac12\pi\big).$$

By the above estimates, for $y,\bar y\in C(-v,\frac12\pi)$ we then obtain

$$\begin{aligned}\frac{\varphi_v(y)}{\varphi_w(y)}-\frac{\varphi_v(\bar y)}{\varphi_w(\bar y)}&=\frac{\varphi_z(x)}{\varphi_{\bar v}(x)}\left[\frac{\varphi_{\bar v}(\bar y)}{\varphi_z(y)}-\frac{\varphi_{\bar v}(\bar y)}{\varphi_z(\bar y)}\right]\\&\leqslant c_1(1+\nu^{\alpha/2}\angle(v,w)^{\alpha/2})\left|\frac{f_z(y)}{\varphi_z(y)}-\frac{f_z(\bar y)}{\varphi_z(\bar y)}\right|\\&\quad+c_1\left|\frac{f_z(\bar y)-\varphi_{\bar v}(\bar y)}{\varphi_z(\bar y)}\right|+c_1\nu^{\alpha/2}\angle(v,w)^{\alpha/2}\frac{f_z(\bar y)}{\varphi_z(\bar y)}.\end{aligned}$$

But

$$\left|\frac{f_z(y)}{\varphi_z(y)}-\frac{f_z(\bar y)}{\varphi_z(\bar y)}\right|\leqslant 2\chi^{s-1}cc_1$$

by the above estimate,

$$|f_z(\bar y)-\varphi_v(\bar y)|\leqslant\nu^{\alpha/2}\angle(v,w)^{\alpha/2}c_1\varphi_z(\bar y)$$

by Corollary B.3 and

$$\log\chi^{s-1}=\left[\frac{\log\nu-\log\angle(v,w)}{2\varkappa}-1\right]\log\chi$$

and consequently there is a number $\beta>0$ such that

$$\frac{\varphi_v(y)}{\varphi_w(y)}-\frac{\varphi_v(\bar y)}{\varphi_w(\bar y)}\leqslant\angle(v,w)^\beta$$

for all $y,\bar y\in C(-v,\frac12\pi)$. In particular, by choosing $\bar y=x$ (or $y=x$) in the above inequality we obtain

$$1-\angle(v,w)^\beta\leqslant\frac{\varphi_v(y)}{\varphi_w(y)}\leqslant 1+\angle(v,w)^\beta$$

for all $y\in C(-v,\frac12\pi)$. But this is just the assertion of the corollary. □

As a consequence of Corollary B.5 we obtain

COROLLARY B.6. *The function* $K_\infty: D \to (0,\infty)$ *is Hölder continuous.*

Proof. By the results of Ancona ([An]) and Anderson–Schoen ([AS]), for every fixed $v \in T^1\widetilde{M}$ the Martin kernel $K_v: \widetilde{M} \times \widetilde{M} \times \partial\widetilde{M} \to (0,\infty)$ of L_v is uniformly Hölder continuous. Since $K_\infty(v,w) = K_v(Pv, Pw, \pi(v))$ we thus only have to show that for every $(y,z) \in \widetilde{M} \times \widetilde{M}$ the assignment $v \to K_v(y,z,\pi(v))$ is Hölder continuous.

For this let $y, z \in \widetilde{M}$ and let $v \in T^1\widetilde{M}$. Let $\gamma: [0,\infty) \to \widetilde{M}$ be the geodesic ray in $\widetilde{M}$ which satisfies $\gamma(0) = y$ and $\gamma(\infty) = \pi(v)$. Since the angle at $\gamma(t)$ of the geodesic triangle in $(\widetilde{M}, \langle\cdot,\cdot\rangle)$ with vertices $y, z, \gamma(t)$ converges to zero as $t \to \infty$ (see [HI]) there is $t_0 \geqslant 0$ such that $z \in C(-\gamma'(t_0), \frac{1}{2}\pi)$. By Corollary B.5 the maps $w \to K_w(\gamma(t_0), z, \pi(w))$ and

$$w \to K_w(y, \gamma(t_0), \pi(w)) = (K_w(\gamma(t_0), y, \pi(w)))^{-1}$$

are Hölder continuous near v and hence the same is true for the assignment

$$w \to K_w(y, z, \pi(w)) = K_w(y, \gamma(t_0), \pi(w))\, K_w(\gamma(t_0), z, \pi(w)).$$

This shows the corollary. □

As another consequence of Corollary B.5 we also obtain

COROLLARY B.7. *The function*

$$v \to \frac{d}{dt} K_\infty(v, \Phi^t v)\Big|_{t=0}$$

is Hölder continuous on $T^1\widetilde{M}$.

Proof. For $v \in T^1\widetilde{M}$ let again $K_v: \widetilde{M} \times \widetilde{M} \times \partial\widetilde{M} \to (0,\infty)$ be the Martin kernel of L_v. Then for every fixed $v \in T^1\widetilde{M}$ the assignment $w \to dK_v(Pw, P\Phi^t w, \pi(w))/dt|_{t=0}$ is Hölder continuous (Lemma 3.2 of [H1]) and hence we only have to show that for every $v \in T^1\widetilde{M}$ the assignment

$$w \in T^1_{Pv}\widetilde{M} \to \frac{d}{dt} K_w(Pv, P\Phi^t v, \pi(w))\Big|_{t=0} = \frac{d}{dt} \varphi_w(P\Phi^t v)\Big|_{t=0}$$

is Hölder continuous at v.

For this recall from Corollary B.5 and the estimates in the proof of Corollary B.3 that there is a number $\chi > 0$ such that for every $v \in T^1\widetilde{M}$, every $w \in T^1_{Pv}\widetilde{M}$ with $\angle(v,w) < \nu$ and every $y \in \widetilde{M}$ which is contained in the ball $B(Pv,1)$ of radius 1 about Pv in $(\widetilde{M}, \langle\cdot,\cdot\rangle)$ we have $|L_v \varphi_w(y)| < \angle(v,w)^\chi$ and $|\varphi_v - \varphi_w|(y) < \angle(v,w)^\chi$. Let $\varkappa = \angle(v,w)^\chi$ and recall that there is a number $c_0 > 0$ not depending on v such that $c_0^{-1} \leqslant \varphi_v(y) \leqslant c_0$ for all $y \in B(Pv,1)$. Define $\bar{\varphi} = (1 + 2c_0\varkappa)\varphi_v - \varphi_w$. Then $\varkappa \leqslant \bar{\varphi} \leqslant (1 + 2c_0^2)\varkappa$ and $|L_v \bar{\varphi}| < \varkappa$ on $B(Pv,1)$ which means that there is a continuous function $\varrho\colon B(Pv,1)\to[-1,1]$ such that $(L_v+\varrho)\bar\varphi=0$. By our assumption on the coefficients of L_v we then necessarily have

$$\left|\frac{d}{dt}\log\bar\varphi(P\Phi^t v)\Big|_{t=0}\right|\leqslant c_1,\quad \left|\frac{d}{dt}\varphi_v(P\Phi^t v)\Big|_{t=0}\right|\leqslant c_1$$

for some $c_1>0$ not depending on v, w and hence

$$\begin{aligned}\left|\frac{d}{dt}(\varphi_v-\varphi_w)(P\Phi^t v)\Big|_{t=0}\right| &\leqslant \left|\frac{d}{dt}\bar\varphi_v(P\Phi^t v)\Big|_{t=0}\right|+2c_0\varkappa\left|\frac{d}{dt}\varphi_v(P\Phi^t v)\Big|_{t=0}\right|\\ &\leqslant c_1\varkappa(1+2c_0+2c_0^2).\end{aligned}$$

This shows the corollary. □

We conclude this appendix with some remarks about the relation between the operator L and the operator L^* which is leafwise formally adjoint to L. For this recall that K_v^* denotes the Martin kernel of the operator L_v^* which is formally adjoint to L_v. To explain the relation between K_v and K_v^* assume for the moment that for every $v\in T^1\widetilde M$ the vector field $Y_v=Y|_{W^s(v)}$ on $W^s(v)\sim\widetilde M$ is the g-gradient of the logarithm of a function f_v on $\widetilde M$ which we assume to be normalized in such a way that $f_v(Pv)=1$. Then we have

LEMMA B.8. *$K_v^*(Pv,y,\xi)=f_v(y)K_v(Pv,y,\xi)$ for all $v\in T^1\widetilde M$, $\xi\in\partial\widetilde M$ and $y\in\widetilde M$.*

Proof. For a smooth function $\bar\varphi$ on $W^s(v)\sim\widetilde M$ we have

$$L_v^*(\bar\varphi)=\Delta_v(\bar\varphi)-\operatorname{div}(\bar\varphi Y_v)+\bar\varphi\chi_v.$$

Now if φ is any positive L_v-harmonic function on $W^s(v)\sim\widetilde M$ then

$$\begin{aligned}L_v^*(\varphi f_v)&=f_v\Delta_v(\varphi)+2g(\nabla\varphi,\nabla f_v)+\varphi\Delta_v(f_v)-\operatorname{div}(\varphi\nabla f_v)+\varphi\chi_v\\&=f_v(\Delta_v(\varphi)+Y_v(\varphi)+\varphi\chi_v)=0\end{aligned}$$

and hence the assignment $\varphi\to\varphi f_v$ maps the space of positive L_v-harmonic functions on $\widetilde M$ to the space of positive L_v^*-harmonic functions. From this the lemma immediately follows. □

Assume now again that L is an arbitrary weakly coercive operator on T^1M with Hölder-continuous coefficients. Then we have

LEMMA B.9. *There is a number $c_0>0$ such that*

$$c_0^{-1}\leqslant K_v(Pw,P\Phi^t w,\pi(w))K_v^*(Pw,P\Phi^t w,\pi(-w))\leqslant c_0$$

for all $v,w\in T^1\widetilde M$ and all $t\geqslant 0$.

Proof (compare Lemma 3.10 and Corollary 3.11 of [H1]). For $v\in T^1\widetilde M$ let $G_v\colon \widetilde M\times\widetilde M\to(0,\infty)$ be the Green function of the operator L_v. For fixed $x\in\widetilde M$ the function $y\to G_v(y,x)$ is positive and L_v-harmonic on $\widetilde M-\{x\}$ and its values on the distance sphere of radius 1 about x are bounded from above and below by a positive constant not depending on v and x. The Harnack inequality at infinity of Ancona ([An]) as quoted in the text preceding Corollary B.4 then shows that there is a number $\tilde{c}>0$ such that $\tilde{c}^{-1} \leqslant K_v(P\Phi^t w, Pw, \pi(w))/G_v(Pw, P\Phi^t w) \leqslant \tilde{c}$ for all $v, w \in T^1\widetilde{M}$ and all $t \geqslant 1$.

Now $G_v^*(x,y)=G_v(y,x)$ is the Green function of the formal adjoint L_v^* of L_v. Hence another application of the Harnack inequality at infinity for positive L_v^*-harmonic functions on $\widetilde{M}$ shows that $\tilde{c}^{-1} \leqslant K_v^*(Pw, P\Phi^t w, \pi(-w))/G_v(Pw, P\Phi^t w) \leqslant \tilde{c}$. Together this shows the lemma. □

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