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Beurling's Theorem for the Bergman space

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1. Introduction

Many interesting Hilbert space operators can be modelled by natural operations on spaces of functions analytic in the unit disk $\mathbf{D}$. The most basic of these operations is multiplication by the coordinate function z, and in this case the invariant subspaces of the operator correspond to what we call the invariant subspaces of the function space, i.e. those closed subspaces M for which $zM\subset M$. As a matter of terminology, we will call the smallest invariant subspace containing a given set S the *invariant subspace generated by* S, and we will denote it by $[S]$. An invariant subspace generated by a single function will be called *cyclic*.

The best known example in this area is the case where the function space is the Hardy space H^2. This space consists of those functions f analytic in $\mathbf{D}$ for which

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

By means of radial limits, H^2 can be identified with the subspace of $L^2(\partial\mathbf{D})$ of functions f for which

$$\hat{f}(n) = \int_{|z|=1} f(z)\,\bar{z}^n \frac{|dz|}{2\pi} = 0 \quad \text{for } n=-1,-2,\dots.$$

Multiplication by z on H^2 models the unilateral shift $(a_0, a_1, \dots)\mapsto(0, a_0, a_1, \dots)$ on l^2_+, an operator of basic importance in many areas of analysis. A famous classical result of A. Beurling [B] classifies the invariant subspaces of H^2, and thus the invariant subspaces of the unilateral shift. To describe this result we recall that an *inner function* in H^2 is a function $\varphi\in H^2$ whose radial limits have modulus 1 a.e. on $\partial\mathbf{D}$. We will use the notation $M\ominus N=M\cap N^\perp$ for closed subspaces N, M such that $N\subset M$.

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BEURLING'S THEOREM. *Let $M \neq \{0\}$ be an invariant subspace of H^2. Then $M \ominus zM$ is a one-dimensional subspace spanned by an inner function φ, and*

$$M = [\varphi] = [M \ominus zM].$$

For a proof and other background about H^2, see [D], [Garn] and [Koo]. Here we will give the simple proof that any function $\varphi \in M \ominus zM$ of unit norm is inner. To see this, note that $z^n \varphi \perp \varphi$ for $n=1,2,...$, hence

$$\int_{|z|=1} |\varphi(z)|^2 z^n \frac{|dz|}{2\pi} = 0 \quad \text{for } n = 1, 2, \ldots. \tag{1.1}$$

This equation together with its complex conjugate shows that $\widehat{|\varphi|^2}(n)=0$ for all $n \neq 0$. Hence $|\varphi|^2$ is constant a.e. on $\partial \mathbf{D}$, and this constant must be 1 since φ has unit norm.

For his description of the invariant subspaces of a unilateral shift of arbitrary multiplicity, P. Halmos introduced the concept of a *wandering subspace* [Hal]: a subspace N of a Hilbert space is said to be wandering for an operator S if N is orthogonal to $S^n(N)$ for $n=1,2,\ldots$. If M is an invariant subspace of S, then clearly $M \ominus S(M)$ is wandering for S, and we will refer to this subspace as the *wandering subspace of M*. Thus in this terminology Beurling's Theorem can be restated as saying that the invariant subspaces of H^2 are in one-to-one correspondence with the wandering subspaces of M_z, where the correspondence is given by

$$M = [M \ominus zM].$$

Furthermore, all nonzero wandering subspaces are one-dimensional and are spanned by an inner function.

Beurling's Theorem has played an important role in operator theory, function theory and their intersection, function-theoretic operator theory. However, despite the great development in these fields over the past forty years, it is only fairly recently that progress has been made in proving analogues for the other classical Hilbert spaces of analytic functions in $\mathbf{D}$, the Dirichlet space and the Bergman space. In [R], the second named author proved that Beurling's Theorem in the form we have stated it is true in the Dirichlet space. Namely, all invariant subspaces are generated by their wandering subspaces, and the nonzero wandering subspaces are one-dimensional.

In this paper we will be concerned with the *Bergman space* L^2_a, defined to be the space of functions f analytic in $\mathbf{D}$ for which

$$\|f\|^2_{L^2_a} = \iint_{|z|<1} |f(z)|^2 \frac{dA(z)}{\pi} < \infty.$$

It has been known for some time that the invariant subspace lattice of L_a^2 is very complicated indeed. In [ABFP], C. Apostol, H. Bercovici, C. Foias, and C. Pearcy showed that if n is any positive integer or ∞, then there is an invariant subspace M of L_a^2 such that $\dim(M\ominus zM)=n$. They deduced from this that any strict contraction on a Hilbert space is unitarily equivalent to the compression of multiplication by z to a subspace of the form $M\ominus N$, where $N\subset M$ are invariant subspaces of L_a^2. In particular, the invariant subspace conjecture for Hilbert space operators is equivalent to the conjecture that if $N\subset M$ are invariant subspaces of L_a^2 such that $\dim(M\ominus N)\geqslant 2$, then there exists another invariant subspace properly between them. The proof in [ABFP] is quite abstract and applies to many function spaces other than L_a^2. A more concrete construction is in [HRS].

These results show that unlike in the H^2 situation, wandering subspaces may have any dimension. In particular, not every invariant subspace of L_a^2 is cyclic, since it is easy to show that if M is cyclic then $\dim(M\ominus zM)=1$. Nevertheless, the following analogue of Beurling's Theorem is true and is the main result of this paper (Theorem 3.5):

THEOREM. *Let M be an invariant subspace of L_a^2. Then $M=[M\ominus zM]$.*

Thus, as in the Hardy and Dirichlet space cases, invariant subspaces in L_a^2 are in one-to-one correspondence with their wandering subspaces.

This result and its proof have roots in several recent papers. In the following discussion and in the sequel we will use the following definition, which has become fairly standard.

Definition. An *L_a^2-inner function* is a $\varphi\in L_a^2$ of unit norm for which

$$\iint_{|z|<1} |\varphi(z)|^2 z^n \,\frac{dA(z)}{\pi}=0 \quad \text{for } n=1,2,\dots. \tag{1.2}$$

The analogy with (1.1) and hence the reason for the terminology is apparent. Note that this definition is equivalent to the condition that

$$\iint_{|z|<1} |\varphi(z)|^2 u(z)\,\frac{dA(z)}{\pi}=u(0) \tag{1.3}$$

for any bounded harmonic function u.

A big breakthrough in the study of the invariant subspaces of L_a^2 was made by H. Hedenmalm in the papers [Hed1] and [Hed2]. Given an invariant subspace M of L_a^2, he considered the extremal problem

$$\sup\{\operatorname{Re} f(0) : f\in M,\ \|f\|_{L_a^2}\leqslant 1\}. \tag{1.4}$$

(If $f(0)=0$ for all $f\in M$, we replace $\operatorname{Re} f(0)$ by $\operatorname{Re} f^{(n)}(0)$, where n is the smallest integer for which there exists an $f\in M$ such that $f^{(n)}(0)\neq 0$.) It is easy to see that the extremal function φ for this problem is unique and in $M\ominus zM$. We will refer to this function simply as the extremal function for M. By the same argument as the one above for the space H^2, φ is an L^2_a-inner function. Conversely if φ is L^2_a-inner, then φ is a constant multiple of the extremal function for the invariant subspace $M=[\varphi]$. Hedenmalm showed that there exists a unique function $\Phi\in C(\overline{\mathbf{D}})\cap C^\infty(\mathbf{D})$ such that $\Phi\equiv 0$ on $\partial\mathbf{D}$ and $\Delta\Phi=4(|\varphi|^2-1)$ in $\mathbf{D}$. He further showed that $\Phi\geqslant 0$ in $\overline{\mathbf{D}}$ and that

$$\|f\varphi\|^2_{L^2_a}=\|f\|^2_{L^2_a}+\frac{1}{4}\iint_{|z|<1}\Phi(z)\,\Delta|f(z)|^2\,\frac{dA(z)}{\pi} \tag{1.5}$$

for all polynomials f. This shows that φ has the *expansive multiplier property*, i.e. that $\|f\varphi\|_{L^2_a}\geqslant\|f\|_{L^2_a}$ for all polynomials f. Now consider an invariant subspace M determined by a zero set, i.e. let $\{z_n\}$ be a sequence of points in $\mathbf{D}$ and let M consist of those $f\in L^2_a$ which have a zero at every $z\in\mathbf{D}$ of order at least as great as the number of times z appears in the sequence $\{z_n\}$. We assume $M\neq\{0\}$. It is easy to see that $\dim(M\ominus zM)=1$ and hence that $M\ominus zM$ is spanned by the extremal function φ for M. We will refer to φ as the extremal function for the zero set $\{z_n\}$. Hedenmalm used the expansive multiplier property of extremal functions and a limit argument to show that $f/\varphi\in L^2_a$ whenever $f\in M$ and that, in fact, $\|f/\varphi\|_{L^2_a}\leqslant\|f\|_{L^2_a}$, i.e. that φ is a *contractive divisor.*

A different proof of Hedenmalm's results was found by P. L. Duren, D. Khavinson, H. S. Shapiro and the third named author ([DKSS1], [DKSS2]). They showed that the function Φ found by Hedenmalm could be written as

$$\Phi(z)=4\iint_{|w|<1}\Gamma(z,w)\,\Delta|\varphi(w)|^2\,\frac{dA(w)}{\pi},$$

where $\Gamma(z,w)$ is the biharmonic Green function (see §2). The fact that $\Phi\geqslant 0$ now follows from the well-known fact that $\Gamma>0$. Thus (1.5) can be written in the form

$$\|f\varphi\|^2_{L^2_a}=\|f\|^2_{L^2_a}+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta|f(w)|^2\Delta|\varphi(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi} \tag{1.6}$$

for all polynomials f. This approach led to an extension of Hedenmalm's results to the L^p_a-spaces (of which we will have more to say below). The formula (1.6) and generalizations of it are central to the work in the present paper.

The classical inner-outer factorization of H^2 functions was discovered before Beurling's Theorem but is nevertheless closely related. Suppose $f\in H^2$, $f\neq 0$, and let φ be the inner function that generates $[f]$. Since $|\varphi|\equiv 1$ a.e. on $\partial\mathbf{D}$, it is easy to see that $[f]=\varphi H^2$. If we write

$$f=\varphi F, \tag{1.7}$$

then it is immediate that F is *cyclic* in H^2, i.e. that $[F]=H^2$. On the other hand, it is well known that the cyclicity of F is equivalent to the following property:

$$g\in H^2 \text{ and } |g|\leqslant|F| \text{ a.e. on } \partial\mathbf{D} \quad\Rightarrow\quad |g(0)|\leqslant|F(0)|. \tag{1.8}$$

We take (1.8) as the defining property of *outer functions.* Then (1.7) is the classical inner-outer factorization alluded to above.

In seeking to extend these ideas to L^2_a, B. Korenblum was led to the concept of *domination.* Note that if g and h are in H^2, then $|g|\leqslant|h|$ a.e. on $\partial\mathbf{D}$ if and only if $\|fg\|_{H^2}\leqslant\|fh\|_{H^2}$ for all polynomials f. This motivates the following definition.

Definition (Korenblum [Kor]). Let $g,h\in L^2_a$. Then we say that h *dominates* g, in symbols $g\prec h$, if $\|fg\|_{L^2_a}\leqslant\|fh\|_{L^2_a}$ for all polynomials f.

The results of Hedenmalm we have been discussing show that $\varphi\in L^2_a$ is L^2_a-inner if and only if $\|\varphi\|_{L^2_a}=1$ and $1\prec\varphi$.

In analogy with the H^2-case (1.8), Korenblum made the following definition in [Kor].

Definition. An *L^2_a-outer function* is an $F\in L^2_a$ for which

$$g\in L^2_a \text{ and } g\prec F \quad\Rightarrow\quad |g(0)|\leqslant|F(0)|.$$

He showed that cyclic functions are outer and asked whether the converse were true. As a consequence of our main result, we are able to prove this converse and to show that every L^2_a-function can be written as the product of an L^2_a-inner and an L^2_a-outer function, (Propositions 4.6 and 4.8).

As we mentioned above, the invariant subspace lattice of L^2_a is exceedingly rich, and while our results illuminate it, they certainly do not provide the kind of complete description that Beurling's Theorem affords in the H^2-case. The main reason for this is the absence of any kind of structure theory for L^2_a-inner functions and for the spaces of the type $M\ominus zM$ that show up in our work. For instance if M is an invariant subspace of L^2_a such that $\dim(M\ominus zM)=2$ and f,g form an orthonormal basis of $M\ominus zM$, then it is easy to show that

$$\iint_{|z|<1} f(z)\overline{g(z)}\,u(z)\,\frac{dA(z)}{\pi}=0$$

for all bounded harmonic functions u. No such pair of functions is concretely known. Our results point to a need for an investigation of these types of questions.

The paper is organized as follows. After preliminaries in §2, we prove the main result in §3. §4 is devoted to consequences of this result, including the material concerning L^2_a-outer functions and inner-outer factorizations. We also prove an analogue of the contractive divisor property for arbitrary invariant subspaces (Proposition 4.9). In §5 we extend some of these results to the L^p_a spaces. These are the spaces of functions f analytic in $\mathbf{D}$ for which

$$\|f\|^p_{L^p_a} = \iint_{|z|<1} |f(z)|^p \frac{dA(z)}{\pi} < \infty.$$

As is well known, if $1 \leqslant p < \infty$, $\|\cdot\|_{L^p_a}$ makes L^p_a into a Banach space, and if $0<p<1$, $d(f,g)=\|f-g\|^p_{L^p_a}$ makes L^p_a into an F-space. In analogy with the L^2_a-case, we say that $\varphi \in L^p_a$ is an *L^p_a-inner function*, if $\|\varphi\|_{L^p_a}=1$ and

$$\iint_{|z|<1} |\varphi(z)|^p z^n \frac{dA(z)}{\pi} = 0 \quad \text{for } n=1,2,\dots.$$

Furthermore, an *L^p_a-outer function* is a function $F \in L^p_a$ such that $|g(0)| \leqslant |F(0)|$ whenever $g \in L^p_a$ and $\|fg\|_{L^p_a} \leqslant \|fF\|_{L^p_a}$ for all polynomials f. Notice that the concepts of L^p_a-inner and L^p_a-outer functions depend on the index p, $0<p<\infty$. In Proposition 5.1 and Theorem 5.2, we shall prove a structure theorem for cyclic invariant subspaces of L^p_a. In particular, cyclic invariant subspaces are generated by L^p_a-inner functions. As consequences one obtains that the cyclic vectors in L^p_a are the L^p_a-outer functions and that every function in L^p_a can be factored as a product of an L^p_a-inner and an L^p_a-outer function. We shall also see that invariant subspaces that are described by zero sets are always cyclic (see Proposition 5.4 and the remark following it).

Finally, in §6 we prove an interesting inequality with a strong connection to the proof of our main result.

2. Preliminaries

In this section we gather material that will be needed in the proofs of our main results. The first two lemmas record well-known facts and are included here for purposes of reference. The first is an exercise involving Fatou's Lemma and Egoroff's Theorem (see [D, Lemma 1 of §2.3]), and the second consists of standard formulas proven by using Green's Theorem.

LEMMA 2.1. *Suppose that μ is a finite positive measure, $0<p<\infty$, and that f_n, f are measurable functions such that*

$$\overline{\lim} \int |f_n|^p \, d\mu \leqslant \int |f|^p \, d\mu < \infty$$

and

$$f_n \to f \quad a.e.\ [\mu].$$

Then $\int |f-f_n|^p \, d\mu \to 0$. □

LEMMA 2.2. *If v is a C^2 function in $[|w|\leqslant r]$, where $0<r<\infty$, and $|z|<r$ is fixed, then*

$$\text{(a)}\quad \iint_{|w|<r} \frac{1}{2}\log\left|\frac{r(z-w)}{r^2-\bar{w}z}\right| \Delta v(w)\,\frac{dA(w)}{\pi} = -\int_{|w|=r} \frac{r^2-|z|^2}{|z-w|^2}\, v(w)\,\frac{|dw|}{2\pi r} + v(z)$$

and

$$\text{(b)}\quad \iint_{|w|<r} \frac{1}{4}(r^2-|w|^2)\,\Delta v(w)\,\frac{dA(w)}{\pi} = r^2\int_{|w|=r} v(w)\,\frac{|dw|}{2\pi r} - \iint_{|w|<r} v(w)\,\frac{dA(w)}{\pi}. \quad \square$$

The Green function for $\mathbf{D}$ is

$$G(z,w)=\frac{1}{2}\log\left|\frac{z-w}{1-\bar{w}z}\right|,$$

the biharmonic Green function for $\mathbf{D}$ is

$$\Gamma(z,w)=\frac{1}{16}\left[|z-w|^2\log\left|\frac{z-w}{1-\bar{w}z}\right|^2+(1-|z|^2)(1-|w|^2)\right],$$

and the corresponding potentials are

$$G[u](z)=\iint_{|w|<1} G(z,w)\,u(w)\,\frac{dA(w)}{\pi}$$

and

$$\Gamma[u](z)=\iint_{|w|<1} \Gamma(z,w)\,u(w)\,\frac{dA(w)}{\pi}.$$

As is well known (see [Gara, Chapter 7]), for sufficiently nice functions u, $G[u]$ and $\Gamma[u]$ satisfy and are determined by the properties

$$\begin{aligned} G[u]&\in C(\bar{\mathbf{D}})\cap C^2(\mathbf{D}),\\ G[u]&=0 \quad \text{on } \partial\mathbf{D},\\ \Delta G[u]&=u \quad \text{in } \mathbf{D}\end{aligned}$$

and

$$\begin{aligned} \Gamma[u]&\in C^1(\bar{\mathbf{D}})\cap C^4(\mathbf{D}),\\ \Gamma[u]&=\frac{\partial}{\partial n}\Gamma[u]=0 \quad \text{on } \partial\mathbf{D},\\ \Delta^2\Gamma[u]&=u \quad \text{in } \mathbf{D}.\end{aligned}$$

We state a few more facts about Γ in the following lemma. Note that an important consequence of (a) is the well-known fact that $\Gamma(z,w)>0$ for $z,w\in\mathbf{D}$.

LEMMA 2.3. *Let $z,w\in\mathbf{D}$. Then*

(a) $\dfrac{1}{32}\cdot\dfrac{(1-|z|^2)^2(1-|w|^2)^2}{|1-\bar{z}w|^2}\leqslant\Gamma(z,w)\leqslant\dfrac{1}{16}\cdot\dfrac{(1-|z|^2)^2(1-|w|^2)^2}{|1-\bar{z}w|^2}$,

(b) $\dfrac{(1-|w|)^2}{32}(1-|z|^2)^2\leqslant\Gamma(z,w)\leqslant\dfrac{(1+|w|)^2}{16}(1-|z|^2)^2$,

(c) $\Delta_z\Gamma(z,w)=G(z,w)+\dfrac{1}{4}(1-|w|^2)\operatorname{Re}\dfrac{1+\bar{w}z}{1-\bar{w}z}$ *for* $z\neq w$.

Proof. Simple manipulations with the definition of Γ and the identity

$$1-\left|\frac{z-w}{1-\bar{z}w}\right|^2=\frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2}$$

yield the formula

$$\Gamma(z,w)=\frac{1}{16}\cdot\frac{(1-|z|^2)^2(1-|w|^2)^2}{|1-\bar{z}w|^2}f\left(1-\left|\frac{z-w}{1-\bar{z}w}\right|^2\right),$$

where $f(x)=((1-x)\log(1-x)+x)/x^2$. By l'Hopitals rule $f(0^+)=\frac{1}{2}$, and it is not difficult to show that $\frac{1}{2}\leqslant f(x)\leqslant 1$ for $0\leqslant x\leqslant 1$. This proves (a), and (b) follows from (a) and the inequalities

$$\frac{1}{(1+|w|)^2}\leqslant\frac{1}{|1-\bar{z}w|^2}\leqslant\frac{1}{(1-|w|)^2}.$$

The proof of (c) is a straightforward calculation with the differential operators

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y}\right)$$

and the identity

$$\Delta_z=4\frac{\partial}{\partial z}\frac{\partial}{\partial\bar{z}}.$$ □

If f is a function in $\mathbf{D}$ and $0\leqslant s\leqslant 1$, we denote by f_s the dilation of f by s,

$$f_s(z)=f(sz).$$

PROPOSITION 2.4. *Let $0<p<\infty$.*

(a) *If f is analytic in $\mathbf{D}$ and $w\in\mathbf{D}$ is fixed, then*

$$\lim_{s\to 1^-}\iint_{|z|<1}\Gamma(z,w)\Delta|f_s(z)|^p\,\frac{dA(z)}{\pi}=\iint_{|z|<1}\Gamma(z,w)\Delta|f(z)|^p\,\frac{dA(z)}{\pi}.$$

(b) *If f is analytic in* $\mathbf{D}$ *and* $w\in\mathbf{D}$ *is fixed, then*

$$\iint_{|z|<1} \Gamma(z,w)\,\Delta|f(z)|^p\,\frac{dA(z)}{\pi} = \iint_{|z|<1} \Delta_z\Gamma(z,w)\,|f(z)|^p\,\frac{dA(z)}{\pi}.$$

Furthermore, these integrals are finite if and only if $f\in L^p_a$.

(c) *If* φ *is an* L^p_a*-inner function, then*

$$\iint_{|z|<1} \Gamma(z,w)\,\Delta|\varphi(z)|^p\,\frac{dA(z)}{\pi} = \iint_{|z|<1} G(z,w)\,|\varphi(z)|^p\,\frac{dA(z)}{\pi} + \frac{1}{4}(1-|w|^2).$$

Proof. By a change of variable argument,

$$\iint_{|z|<1} (1-|z|^2)^2\Delta|f_s(z)|^p\,\frac{dA(z)}{\pi} = \frac{1}{s^4}\iint_{|z|<s} (s^2-|z|^2)^2\Delta|f(z)|^p\,\frac{dA(z)}{\pi},$$

so by monotone convergence we see that

$$\lim_{s\to 1^-}\iint_{|z|<1} (1-|z|^2)^2\Delta|f_s(z)|^p\,\frac{dA(z)}{\pi} = \iint_{|z|<1} (1-|z|^2)^2\Delta|f(z)|^p\,\frac{dA(z)}{\pi}. \tag{2.1}$$

If $\iint_{|z|<1}(1-|z|^2)^2\Delta|f(z)|^p\,dA(z)/\pi<\infty$, Lemma 2.1 (with $p=1$) shows us that

$$\lim_{s\to 1^-}\iint_{|z|<1} (1-|z|^2)^2\,\big|\Delta|f(z)|^p-\Delta|f_s(z)|^p\big|\,\frac{dA(z)}{\pi} = 0.$$

Together with Lemma 2.3 (b) this proves (a). If $\iint_{|z|<1}(1-|z|^2)^2\Delta|f(z)|^p\,dA(z)/\pi=\infty$, (a) is an immediate consequence of (2.1) and Lemma 2.3 (b).

To prove (b) we first note that $\Gamma(z,w)=\partial\Gamma(z,w)/\partial n_z=0$ for $z\in\partial\mathbf{D}$, by Lemma 2.3 (a). Hence (b), with f replaced by f_s, is an immediate consequence of Green's Theorem (it is easy to show that the singularities at $z=w$ and the zeros of f cause no problem). Now (b) follows from this together with (a), since it is obvious from the form of $\Delta_z\Gamma(z,w)$ given in Lemma 2.3 (c) that

$$\lim_{s\to 1^-}\iint_{|z|<1} \Delta_z\Gamma(z,w)\,|f_s(z)|^p\,\frac{dA(z)}{\pi} = \iint_{|z|<1} \Delta_z\Gamma(z,w)\,|f(z)|^p\,\frac{dA(z)}{\pi}.$$

The subsequent assertion is obvious given Lemma 2.3 (b).

To prove (c), plug $f=\varphi$ into (b) and use Lemma 2.3 (c) to obtain

$$\begin{aligned}\iint_{|z|<1} &\Gamma(z,w)\,\Delta|\varphi(z)|^p\,\frac{dA(z)}{\pi}\\ &= \iint_{|z|<1} G(z,w)\,|\varphi(z)|^p\,\frac{dA(z)}{\pi} + \frac{1}{4}(1-|w|^2)\iint_{|z|<1} \operatorname{Re}\frac{1+\bar{w}z}{1-\bar{w}z}\,|\varphi(z)|^p\,\frac{dA(z)}{\pi}\\ &= \iint_{|z|<1} G(z,w)\,|\varphi(z)|^p\,\frac{dA(z)}{\pi} + \frac{1}{4}(1-|w|^2),\end{aligned}$$

since $\operatorname{Re}(1+\bar{w}z)/(1-\bar{w}z)$ is a bounded harmonic function of z whose value at 0 is 1. $\square$

Proposition 2.4 (c) can be written in the form

$$\Gamma[\Delta|\varphi|^p]=G[|\varphi|^p-1]. \tag{2.2}$$

This is implicit in [DKSS2] and can be proved using the methods there. An immediate consequence is the recent result of Khavinson and Shapiro [KS] that

$$0\leqslant G[|\varphi|^p-1](z)\leqslant \tfrac{1}{4}(1-|z|^2).$$

We can also use (2.2) to give an alternate proof of one of the main results in [DKSS2]:

PROPOSITION 2.5 (Duren, Khavinson, Shapiro and Sundberg). *Let φ be an L^p_a-inner function, where $0<p<\infty$, and $v\in C^2(\overline{\mathbf{D}})$. Then*

(a)

$$\begin{aligned}\iint_{|z|<1}&|\varphi(z)|^p v(z)\,\frac{dA(z)}{\pi}\\ &=\iint_{|z|<1} v(z)\,\frac{dA(z)}{\pi}+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta v(w)\,\Delta|\varphi(z)|^p\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}\end{aligned}$$

and

(b) *if in addition v is subharmonic, we have*

$$\iint_{|z|<1}|\varphi(z)|^p v(z)\,\frac{dA(z)}{\pi}\geqslant\iint_{|z|<1} v(z)\,\frac{dA(z)}{\pi}.$$

Proof. We write

$$v=G[\Delta v]+h,$$

where h is a bounded harmonic function in $\mathbf{D}$. By (1.3) and Proposition 2.4 (c),

$$\begin{aligned}\iint_{|z|<1}&|\varphi(z)|^p v(z)\,\frac{dA(z)}{\pi}\\ &=\iint_{|z|<1}|\varphi(z)|^p\iint_{|w|<1}G(z,w)\,\Delta v(w)\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}+h(0)\\ &=\iint_{|w|<1}\left[\iint_{|z|<1}G(z,w)\,|\varphi(z)|^p\,\frac{dA(z)}{\pi}\right]\Delta v(w)\,\frac{dA(w)}{\pi}+v(0)-G[\Delta v](0)\\ &=\iint_{|w|<1}\iint_{|z|<1}\Gamma(z,w)\,\Delta|\varphi(z)|^p\Delta v(w)\,\frac{dA(z)}{\pi}\,\frac{dA(w)}{\pi}\\ &\quad+\left[-\iint_{|w|<1}\frac{1}{4}(1-|w|^2)\,\Delta v(w)\,\frac{dA(w)}{\pi}+v(0)-G[\Delta v](0)\right].\end{aligned}$$

By Lemma 2.2 with $r=1$, $z=0$, we see that the quantity in brackets is

$$\iint_{|w|<1} v(w)\,\frac{dA(w)}{\pi},$$

so (a) is proved.

If v is subharmonic in $\mathbf{D}$, then $\Delta v \geqslant 0$ there, so (b) is a consequence of (a) and the fact that $\Gamma(z,w)>0$. □

The following proposition, although quite simple, is one of the keys to our results.

PROPOSITION 2.6. *If* $v\geqslant 0$ *in* $\mathbf{D}$ *then*

$$\Gamma[s^3 v_s](z) \leqslant 2\Gamma[v](z)$$

for $0\leqslant s\leqslant 1$ *and* $z\in\mathbf{D}$.

Proof. We define

$$\widetilde{\Gamma}(z,w) = \frac{1}{16}\cdot\frac{(1-|z|^2)^2(1-|w|^2)^2}{|1-\bar{z}w|^2}$$

so that by Lemma 2.3 (a),

$$\tfrac{1}{2}\widetilde{\Gamma}(z,w) \leqslant \Gamma(z,w) \leqslant \widetilde{\Gamma}(z,w). \tag{2.3}$$

One sees easily that

$$\begin{aligned}\frac{d}{ds} s\widetilde{\Gamma}\Big(z,\frac{w}{s}\Big) &= \frac{1}{16}\cdot\frac{(1-|z|^2)^2(s^2-|w|^2)^2}{s|s-\bar{z}w|^2}\left[-\frac{1}{s}+\frac{2}{s+|w|}+\frac{2}{s-|w|}-\frac{1}{s-\bar{z}w}-\frac{1}{s-z\bar{w}}\right]\\ &\geqslant \frac{1}{16}\cdot\frac{(1-|z|^2)^2(s^2-|w|^2)^2}{s|s-\bar{z}w|^2}\cdot\frac{s-|w|}{s(s+|w|)}>0\end{aligned}$$

if $|w|<s$. Hence by a change of variable argument

$$\begin{aligned}\iint_{|w|<1}\widetilde{\Gamma}(z,w)\,s^3 v_s(w)\,\frac{dA(w)}{\pi} &= \iint_{|w|<s} s\widetilde{\Gamma}\Big(z,\frac{w}{s}\Big)v(w)\,\frac{dA(w)}{\pi}\\ &\leqslant \iint_{|w|<1}\widetilde{\Gamma}(z,w)\,v(w)\,\frac{dA(w)}{\pi}.\end{aligned}$$

The proposition follows from this and (2.3). □

3. The Wandering Subspace Theorem in the Bergman space

Throughout this section we let M be an invariant subspace of L^2_a, and we denote by T the restriction to M of multiplication by z. We will also denote the L^2_a-norm simply by $\|\cdot\|$.

The objective in this section is to prove our main result, that M is generated by $M\ominus TM$. Since the proof is rather long we will here attempt to provide an overview, considering first the case when $\dim(M\ominus TM)=1$. In this case $M\ominus TM$ is spanned by a single L^2_a-inner function φ. An argument of Hedenmalm's ([Hed1]) shows that if $f\in M$, then f/φ is analytic in $\mathbf{D}$. We can thus define operators $R_s\colon M\to[M\ominus TM]=[\varphi]$ by

$$R_s f=\left(\frac{f}{\varphi}\right)_s\varphi$$

for $0\leqslant s<1$. Obviously $R_s f\to f$ pointwise as $s\to 1^-$. To complete the proof we must get some control over $\|R_s f\|$, and it is here that (1.6) comes into play. If we replace f in (1.6) by respectively f/φ and $(f/\varphi)_s$, we obtain

$$\|f\|^2=\left\|\frac{f}{\varphi}\right\|^2+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta\left|\frac{f}{\varphi}(w)\right|^2\Delta|\varphi(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi} \tag{3.1}$$

and

$$\|R_s f\|^2=\left\|\left(\frac{f}{\varphi}\right)_s\right\|^2+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta\left|\left(\frac{f}{\varphi}\right)_s(w)\right|^2\Delta|\varphi(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}. \tag{3.2}$$

An easy limit argument shows that (3.2) is true. The most difficult part of our proof will be to show that the inequality $\geqslant$ holds in (3.1) for all $f\in M$. This together with Proposition 2.6 will show that for $\frac{1}{2}\leqslant s<1$, the functions of z given by

$$\iint_{|w|<1}\Gamma(z,w)\,\Delta\left|\left(\frac{f}{\varphi}\right)_s(w)\right|^2\frac{dA(w)}{\pi}\,\Delta|\varphi(z)|^2$$

is dominated by the integrable function

$$4\iint_{|w|<1}\Gamma(z,w)\,\Delta\left|\left(\frac{f}{\varphi}\right)(w)\right|^2\frac{dA(w)}{\pi}\,\Delta|\varphi(z)|^2.$$

An application of the Dominated Convergence Theorem then shows that $\lim_{s\to 1^-}\|R_s f\|^2$ exists and is equal to the right-hand side of (3.1). Since this is bounded by $\|f\|^2$, an application of Lemma 2.1 shows that $R_s f\to f$ in L^2_a, completing the proof in the case when $\dim(M\ominus TM)=1$.

Notice that for $\lambda\in\mathbf{D}$, the map $f\mapsto(f/\varphi)(\lambda)\varphi$ defines a skewed projection Q_λ of M onto $M\ominus TM$ with null space $(T-\lambda I)M$, and that R_s can be expressed in terms of these projections by the formula $R_sf(z)=Q_{sz}f(z)$. In Lemma 3.1 we will show that the skewed projections Q_λ also exist when $\dim(M\ominus TM)>1$, allowing us to extend the above discussion to this case. We can define the operators R_s by the same formula as above, and it will easily be seen that $R_sf\to f$ pointwise as $s\to 1^-$.

In Lemma 3.2 we will show that R_s maps M into $[M\ominus TM]$ and that $Q_wR_s=Q_{sw}$ (these facts were trivial in the case when $\dim(M\ominus TM)=1$). The analogues of (3.1) and (3.2) are then seen to be respectively

$$\begin{aligned}\|f\|^2=&\iint_{|\lambda|<1}\|Q_\lambda f\|^2\,\frac{dA(\lambda)}{\pi}\\&+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta_z\Delta_w|Q_wf(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}\|R_sf\|^2=&\iint_{|\lambda|<1}\|Q_{s\lambda} f\|^2\,\frac{dA(\lambda)}{\pi}\\&+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta_z\Delta_w|Q_{sw}f(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}.\end{aligned}\tag{3.4}$$

In Lemma 3.3 we will show that (3.3) holds for all $f\in[M\ominus TM]$; in view of Lemma 3.2 this will show that (3.4) holds for all $f\in M$. The heart of the proof will be Lemma 3.4, where we show that the inequality $\geqslant$ holds in (3.3) for all $f\in M$. Once this is done it will follow that $R_sf\to f$ in L^2_a as we indicated above for the case $\dim(M\ominus TM)=1$. This fact, along with the evident consequences that $M=[M\ominus TM]$ and that (3.3) holds for all $f\in M$, is stated formally as Theorem 3.5.

We will now proceed with the details.

LEMMA 3.1. *For any $\lambda\in\mathbf{D}$, M is the Banach space direct sum of the closed subspaces $M\ominus TM$ and $(T-\lambda I)M$. Furthermore, if Q_λ is the skewed projection operator onto $M\ominus TM$ corresponding to this decomposition of M, then*

$$\|Q_\lambda\|\leqslant C_\lambda,\quad \textit{where } C_\lambda=\frac{\sqrt{2-|\lambda|^2}}{1-|\lambda|^2}.$$

Proof. The case $\lambda=0$ is clear, so we assume $\lambda\neq 0$. We will first consider the case $M=[f]$. Let n_f be the order of the zero of f at 0 and let φ be the extremal function for $[f]$. The argument of Hedenmalm's mentioned above shows that g/φ is analytic in $\mathbf{D}$ for any $g\in[f]$. We are going to refine this argument to show that if $g\in[f]$, then

$$\left|\frac{g}{\varphi}(\lambda)\right|\leqslant C_\lambda\|g\|.\tag{3.5}$$

It will clearly suffice to prove (3.5) under the assumption that $\varphi(\lambda)\neq 0$. It is easy to see that the extremal function associated to the zero set $\{\lambda\}$ is

$$\varphi_\lambda(z)=\left(1-\frac{1}{k_\lambda(\lambda)}\right)^{-1/2}\left(1-\frac{k_\lambda(z)}{k_\lambda(\lambda)}\right),$$

where $k_\lambda(z)=1/(1-\bar{\lambda}z)^2$ is the reproducing kernel for L^2_a, so that

$$\varphi_\lambda(0)=|\lambda|(2-|\lambda|^2)^{1/2}. \tag{3.6}$$

If $g(\lambda)=0$ there is nothing to prove. So assume $g(\lambda)\neq 0$ and set

$$h(z)=\varphi(z)-\frac{\varphi(\lambda)}{\lambda g(\lambda)}zg(z).$$

It is easy to see that $h/\varphi_\lambda\in[f]$. Hence by the extremal property of φ,

$$\left|\frac{h^{(n_f)}(0)}{\varphi_\lambda(0)}\right|\leqslant\varphi^{(n_f)}(0)\left\|\frac{h}{\varphi_\lambda}\right\|. \tag{3.7}$$

Now $h^{(n_f)}(0)=\varphi^{(n_f)}(0)$, and by Hedenmalm's Theorem

$$\left\|\frac{h}{\varphi_\lambda}\right\|\leqslant\|h\|.$$

Since

$$\|h\|^2=1+\left|\frac{\varphi(\lambda)}{\lambda g(\lambda)}\right|^2\iint|z|^2|g(z)|^2\,\frac{dA(z)}{\pi}\leqslant 1+\left|\frac{\varphi(\lambda)}{\lambda g(\lambda)}\right|^2\|g\|^2,$$

we can deduce (3.5) from (3.6) and (3.7). The computation also shows that

$$C_\lambda=\frac{\sqrt{2-|\lambda|^2}}{1-|\lambda|^2}.$$

This shows that if

$$g(z)=\alpha\varphi(z)+(z-\lambda)k(z)$$

with $k\in[f]$, then

$$\|\alpha\varphi\|=|\alpha|=\left|\frac{g}{\varphi}(\lambda)\right|\leqslant C_\lambda\|g\|.$$

Together with the obvious identity

$$g(z)=\frac{g}{\varphi}(\lambda)\varphi(z)+(z-\lambda)\frac{g(z)-(g/\varphi)(\lambda)\varphi(z)}{z-\lambda},$$

this proves the lemma in the case $M=[f]$.

We turn to the general case. Suppose that $h \in M \ominus TM$, $f \in M$ and

$$g(z) = h(z) + (z-\lambda) f(z).$$

It is easy to see that

$$P_{[f]} h \in [f] \ominus T[f],$$

so by what we have already proved,

$$\|P_{[f]} h\| \leqslant C_\lambda \|P_{[f]} g\|.$$

Since $P_{[f]^\perp} h = P_{[f]^\perp} g$, this implies that

$$\|h\| \leqslant C_\lambda \|g\|.$$

Hence the subspaces $M \ominus TM$ and $(T-\lambda I)M$ are at a positive angle, and the projection of their sum onto the first summand has norm at most C_λ. To complete the proof, we must show that their sum is all of M. To see this, suppose that

$$g \in M \ominus ((M \ominus TM) + (T-\lambda I)M) = TM \ominus (T-\lambda I)M.$$

Write $g = Tf$ with $f \in M$. By what we have already proved, $[f] = ([f] \ominus T[f]) + (T-\lambda I)[f]$, so

$$T[f] \cap ([f] \ominus (T-\lambda I)[f]) = [f] \ominus (([f] \ominus T[f]) + (T-\lambda I)[f]) = \{0\}.$$

It is easy to see that g is contained in the subspace on the left. Hence $g=0$ and we are done. □

Remark. As noted above, we are eventually going to show that (3.3) holds for all $f \in M$. From this it is easily deduced that in fact $\|Q_\lambda\| \leqslant 1/(1-|\lambda|^2)$.

Standard methods show that Q_λ is analytic in λ. We can get an explicit formula for Q_λ in terms of the operator

$$L = (T^*T)^{-1} T^*.$$

Notice that we could also define L by the formulas

$$L = 0 \quad \text{on } M \ominus TM$$

and

$$LT = I.$$

If $g \in M \ominus TM$ and $h \in M$, then it is easy to calculate that

$$(I - \lambda L)(g + Th) = g + (T-\lambda I)h.$$

Lemma 3.1 thus implies that $(I-\lambda L)^{-1}$ exists for all $\lambda\in\mathbf{D}$, and we see that

$$Q_\lambda = P_{M\ominus TM}(I-\lambda L)^{-1} = (I-TL)(I-\lambda L)^{-1}. \tag{3.8}$$

Thus

$$Q_\lambda = \sum_{n=0}^{\infty} \lambda^n A_n, \tag{3.9}$$

where $A_n = P_{M\ominus TM} L^n$ is a map from M to $M\ominus TM$.

We now define for $f\in M$ and $0\leqslant s<1$,

$$R_s f(z) = Q_{sz} f(z). \tag{3.10}$$

The definition of Q_λ makes it obvious in particular that $f(z)-Q_\lambda f(z)$ is zero when $z=\lambda$. Hence $Q_z f(z)=f(z)$, so it is obvious from the continuity of the map $\lambda\mapsto Q_\lambda$ that $R_s f(z)\to f(z)$ as $s\to 1^-$, for any $z\in\mathbf{D}$. In the next lemma other important properties of the operators Q_λ and R_s are studied.

LEMMA 3.2. *$R_s f\in[M\ominus TM]$ for any $f\in M$. Furthermore, $Q_w R_s = Q_{sw}$.*

Proof. From (3.9) we see that

$$R_s = \sum_{n=0}^{\infty} s^n T^n A_n, \quad 0\leqslant s<1. \tag{3.11}$$

This series in fact converges in norm. To see this note that as a consequence of the convergence of (3.9),

$$\sum_{n=0}^{\infty} s^n \|T^n A_n\| \leqslant \sum_{n=0}^{\infty} s^n \|A_n\| < \infty$$

for all $0\leqslant s<1$. Since A_n maps M into $M\ominus TM$ it is now clear that R_s maps M into $[M\ominus TM]$.

To prove the remaining assertion we note that if $n\geqslant k$ then

$$A_n T^k = P_{M\ominus TM} L^{n-k} L^k T^k = P_{M\ominus TM} L^{n-k},$$

and if $n<k$ then

$$A_n T^k = P_{M\ominus TM} L^n T^n T^{k-n} = P_{M\ominus TM} T^{k-n} = 0.$$

Combined with the obvious fact that $LP_{M\ominus TM}=0$, these formulas show that $A_n T^k A_k = 0$ if $k\neq n$, and $A_n T^n A_n = A_n$. Hence $A_n R_s = s^n A_n$ by the norm convergence of (3.11). Combined with (3.9) this shows that $Q_w R_s = Q_{sw}$. □

Although our proof depends on a study of the operators Q_λ rather than R_s, it is nevertheless interesting to note a connection between R_s and operators arising in classical approximation theory. If f is in M we can decompose f as a sum of an element of $M\ominus TM$ and a "remainder term" by the formula $f=Pf+TLf$, where $P=P_{M\ominus TM}$. Repeating this for Lf, we obtain $f=Pf+T(PLf+TL^2f)=Pf+TPLf+T^2L^2f$. Continuing this process we get the formal series

$$f=Pf+TPLf+T^2PL^2f+T^3PL^3f+\dots,$$

each term of which is in $[M\ominus TM]$. We see that comparison with (3.11) and the definition of A_n shows that the functions R_sf are Abel means of this formal series:

$$R_sf=Pf+sTPf+s^2T^2PL^2f+s^3T^3PL^3f+\dots.$$

In our next result the important formula (1.6) is generalized.

LEMMA 3.3. *If $f\in[M\ominus TM]$ then*

$$\begin{aligned}\|f\|^2=&\iint_{|\lambda|<1}\|Q_\lambda f\|^2\,\frac{dA(\lambda)}{\pi}\\&+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta_z\Delta_w|Q_wf(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}.\end{aligned}\tag{3.12}$$

Proof. First suppose $f(z)=\sum_{n=0}^N z^n\varphi_n(z)$ with $\varphi_n\in M\ominus TM$. If $\varphi\in M\ominus TM$ then clearly

$$\iint_{|z|<1}|\varphi(z)|^2z^n\,\frac{dA(z)}{\pi}=\langle T^n\varphi|\varphi\rangle=0\quad\text{for } n=1,2,\dots.$$

Hence if $\varphi\neq0$, then $\varphi/\|\varphi\|$ is inner. Thus Proposition 2.5 tells us that

$$\begin{aligned}\iint_{|z|<1}|\varphi(z)|^2v(z)\,\frac{dA(z)}{\pi}=&\iint_{|z|<1}v(z)\,\frac{dA(z)}{\pi}\,\|\varphi\|^2\\&+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta v(w)\,\Delta|\varphi(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}\end{aligned}\tag{3.13}$$

for all $\varphi\in M\ominus TM$ and $v\in C^2(\bar{\mathbf{D}})$. We polarize (3.13) and set $v(z)=z^m\bar z^n$ to get

$$\begin{aligned}\iint_{|z|<1}\varphi_m(z)\overline{\varphi_n(z)}\,z^m\bar z^n\,\frac{dA(z)}{\pi}=&\iint_{|z|<1}z^m\bar z^n\,\frac{dA(z)}{\pi}\,\langle\varphi_m|\varphi_n\rangle\\&+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta(w^m\bar w^n)\,\Delta(\varphi_m(z)\overline{\varphi_n(z)})\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}.\end{aligned}$$

Summing up over m, n we prove (3.12) for f as above by using the obvious fact that

$$Q_\lambda f(z) = \sum_{n=0}^{N} \lambda^n \varphi_n(z).$$

To prove the general case, we introduce the temporary notation of $\|\cdot\|_*^2$ for the right-hand side of (3.12). The fact that

$$\Delta_z \Delta_w |Q_w f(z)|^2 = 16 \left| \frac{\partial}{\partial z} \frac{\partial}{\partial w} Q_w f(z) \right|^2$$

shows that $\|\cdot\|_*$ is a norm. Now let $f \in [M \ominus TM]$ and f_n be functions of the form we have treated such that $f_n \to f$ in L^2_a.

Since $\|f_m - f_n\|_* = \|f_m - f_n\|$ by what we have already shown, Fatou's Lemma shows that

$$\|f - f_n\|_*^2 \leqslant \varliminf_{m\to\infty} \|f_m - f_n\|_*^2 = \varliminf_{m\to\infty} \|f_m - f_n\|^2 = \|f - f_n\|^2.$$

Hence $\|f\| = \lim_{n\to\infty} \|f_n\| = \lim_{n\to\infty} \|f_n\|_* = \|f\|_*$, so we are done. $\square$

Our main result will now follow fairly easily from Proposition 2.6 and the following.

LEMMA 3.4. *If $f \in M$, then*

$$\begin{aligned} \|f\|^2 \geqslant & \iint_{|\lambda|<1} \|Q_\lambda f\|^2 \frac{dA(\lambda)}{\pi} \\ & + \iint_{|z|<1} \iint_{|w|<1} \Gamma(z,w) \Delta_z \Delta_w |Q_w f(z)|^2 \frac{dA(w)}{\pi} \frac{dA(z)}{\pi}. \end{aligned} \tag{3.14}$$

Proof. Our first objective will be to verify the following formula, for all $f \in M$ and $0 < r < 1$.

$$\|f\|^2 = \int_{|\lambda|=r} \|Q_\lambda f\|^2 \frac{|d\lambda|}{2\pi r} + \iint_{|z|<1} \int_{|\lambda|=r} (|z|^2 - r^2) \left| \frac{f(z) - Q_\lambda f(z)}{z-\lambda} \right|^2 \frac{|d\lambda|}{2\pi r} \frac{dA(z)}{\pi}. \tag{3.15}$$

The ideas behind this formula and its application came from the work in [AR].

The proof of (3.15) is obtained by integrating

$$\left| \frac{zf(z) - \lambda Q_\lambda f(z)}{z-\lambda} \right|^2 - r^2 \left| \frac{f(z) - Q_\lambda f(z)}{z-\lambda} \right|^2 \tag{3.16}$$

over $|\lambda| = r$ and $|z| < 1$. We first observe that

$$\begin{aligned} \left| \frac{zf(z) - \lambda Q_\lambda f(z)}{z-\lambda} \right|^2 &= \left| f(z) + \lambda \frac{f(z) - Q_\lambda f(z)}{z-\lambda} \right|^2 \\ &= |f(z)|^2 + 2 \operatorname{Re} \overline{f(z)}\, \lambda \frac{f(z) - Q_\lambda f(z)}{z-\lambda} + |\lambda|^2 \left| \frac{f(z) - Q_\lambda f(z)}{z-\lambda} \right|^2. \end{aligned}$$

The integral of the middle term of this last expression over $|\lambda|=r$ is 0, since $\lambda\mapsto(f(z)-Q_\lambda f(z))/(z-\lambda)$ is analytic in $\mathbf{D}$ (remember that $Q_z f(z)=f(z)$), and it follows that

$$|f(z)|^2=\int_{|\lambda|=r}\left|\frac{zf(z)-\lambda Q_\lambda f(z)}{z-\lambda}\right|^2-r^2\left|\frac{f(z)-Q_\lambda f(z)}{z-\lambda}\right|^2\frac{|d\lambda|}{2\pi r}. \tag{3.17}$$

We use a similar idea in integrating (3.16) over $|z|<1$. Here the key observation is that $f-Q_\lambda f\in(T-\lambda I)M$ by the definition of Q_λ; hence $z\mapsto(f(z)-Q_\lambda f(z))/(z-\lambda)$ is in M. Since $Q_\lambda f\in M\ominus TM$, this means that

$$\iint_{|z|<1}\overline{Q_\lambda f(z)}\,z\frac{f(z)-Q_\lambda f(z)}{z-\lambda}\frac{dA(z)}{\pi}=0. \tag{3.18}$$

Now write

$$\begin{aligned}\left|\frac{zf(z)-\lambda Q_\lambda f(z)}{z-\lambda}\right|^2&=\left|Q_\lambda f(z)+z\frac{Q_\lambda f(z)-f(z)}{\lambda-z}\right|^2\\&=|Q_\lambda f(z)|^2+2\operatorname{Re}\overline{Q_\lambda f(z)}\,z\frac{Q_\lambda f(z)-f(z)}{\lambda-z}+|z|^2\left|\frac{Q_\lambda f(z)-f(z)}{\lambda-z}\right|^2.\end{aligned}$$

Combined with (3.18) this shows that

$$\begin{aligned}\iint_{|z|<1}&\left(\left|\frac{zf(z)-\lambda Q_\lambda f(z)}{z-\lambda}\right|^2-|\lambda|^2\left|\frac{f(z)-Q_\lambda f(z)}{z-\lambda}\right|^2\right)\frac{dA(z)}{\pi}\\&=\|Q_\lambda f\|^2+\iint_{|z|<1}(|z|^2-|\lambda|^2)\left|\frac{f(z)-Q_\lambda f(z)}{z-\lambda}\right|^2\frac{dA(z)}{\pi}.\end{aligned} \tag{3.19}$$

Equation (3.15) is now established by combining (3.17) and (3.19).

It follows from (3.15) that

$$\|f\|^2\geqslant\int_{|\lambda|=r}\|Q_\lambda f\|^2\frac{|d\lambda|}{2\pi r}-\iint_{|z|<r}\int_{|\lambda|=r}\frac{r^2-|z|^2}{|\lambda-z|^2}|f(z)-Q_\lambda f(z)|^2\frac{|d\lambda|}{2\pi r}\frac{dA(z)}{\pi}. \tag{3.20}$$

By Lemma 2.2 (a) with $v(w)=|Q_w f(z)-f(z)|^2$ and the observation $\Delta_w|Q_w f(z)-f(z)|^2=\Delta_w|Q_w f(z)|^2$,

$$\begin{aligned}-\int_{|w|=r}&\frac{r^2-|z|^2}{|z-w|^2}|Q_w f(z)-f(z)|^2\frac{|dw|}{2\pi r}\\&=\iint_{|w|<r}\frac{1}{2}\log\left|\frac{r(z-w)}{r^2-\bar{w}z}\right|\Delta_w|Q_w f(z)|^2\frac{dA(w)}{\pi},\end{aligned} \tag{3.21}$$

and by Lemma 2.2 (b) with $v(w)=|Q_w f(z)|^2$,

$$r^2\int_{|w|=r}|Q_wf(z)|^2\,\frac{|dw|}{2\pi r}=\iint_{|w|<r}|Q_wf(z)|^2\,\frac{dA(w)}{\pi}+\iint_{|w|<r}\frac{1}{4}(r^2-|w|^2)\,\Delta_w|Q_wf(z)|^2\,\frac{dA(w)}{\pi}. \tag{3.22}$$

Using (3.21) and (3.22), we can deduce from (3.20) that

$$\|f\|^2\geqslant\iint_{|w|<r}\|Q_wf\|^2\,\frac{dA(w)}{\pi}+\iint_{|w|<r}\left[\frac{1}{4}(r^2-|w|^2)\iint_{|z|<1}\Delta_w|Q_wf(z)|^2\,\frac{dA(z)}{\pi}+\iint_{|z|<r}\frac{1}{2}\log\left|\frac{r(z-w)}{r^2-\bar{w}z}\right|\Delta_w|Q_wf(z)|^2\,\frac{dA(z)}{\pi}\right]\frac{dA(w)}{\pi}. \tag{3.23}$$

Denote by $\phi(r,w)$ the quantity in brackets in the second integral of the righthand side of (3.23). Here $0<r\leqslant1$ and $|w|<r$. We claim that $\phi(r,w)\geqslant0$. To see this, substitute $z=r\zeta$ to write

$$\iint_{|z|<r}\frac{1}{2}\log\left|\frac{r(z-w)}{r^2-\bar{w}z}\right|\Delta_w|Q_wf(z)|^2\,\frac{dA(z)}{\pi}=r^2\iint_{|\zeta|<1}\frac{1}{2}\log\left|\frac{\zeta-w/r}{1-(\bar{w}/r)\zeta}\right|\Delta_w|Q_wf(r\zeta)|^2\,\frac{dA(\zeta)}{\pi}=\Psi(r),$$

where

$$\Psi(\eta)=r^2\iint_{|\zeta|<1}\frac{1}{2}\log\left|\frac{\zeta-w/r}{1-(\bar{w}/r)\zeta}\right|\Delta_w|Q_wf(\eta\zeta)|^2\,\frac{dA(\zeta)}{\pi}.$$

The function Ψ is clearly continuous in $\overline{\mathbf{D}}$ and superharmonic in $\mathbf{D}$, so

$$\Psi(r)\geqslant\min_{|\eta|=1}\Psi(\eta).$$

If $|\eta|=1$ we substitute $z=\eta\zeta$ to get

$$\begin{aligned}\Psi(\eta)&=r^2\iint_{|z|<1}\frac{1}{2}\log\left|\frac{z-\eta w/r}{1-(\bar{\eta}\bar{w}/r)z}\right|\Delta_w|Q_wf(z)|^2\,\frac{dA(z)}{\pi}\\&=r^2\iint_{|z|<1}G\Big(z,\frac{\eta w}{r}\Big)\Delta_w|Q_wf(z)|^2\,\frac{dA(z)}{\pi}.\end{aligned} \tag{3.24}$$

The function of z given by $2\partial Q_wf(z)/\partial w$ is in $M\ominus TM$ and hence is a multiple of an inner function. We can thus apply Proposition 2.4 (c) to the expression (3.24) to conclude that

$$\begin{aligned}\Psi(\eta)&\geqslant-r^2\cdot\frac{1}{4}\left(1-\left|\frac{\eta w}{r}\right|^2\right)\iint_{|z|<1}\Delta_w|Q_wf(z)|^2\,\frac{dA(z)}{\pi}\\&=-\frac{1}{4}(r^2-|w|^2)\iint_{|z|<1}\Delta_w|Q_wf(z)|^2\,\frac{dA(z)}{\pi},\end{aligned}$$

which proves the claim. Another application of Proposition 2.4 (c) shows us that

$$\lim_{r\to 1^-}\phi(r,w)=\phi(1,w)=\iint_{|z|<1}\Gamma(z,w)\,\Delta_z\Delta_w|Q_wf(z)|^2\,\frac{dA(z)}{\pi}. \tag{3.25}$$

Since $\phi(r,w)\geqslant 0$ we can apply Fatou's Lemma in (3.23) as we let $r\to 1^-$. By (3.25) we obtain (3.14), completing the proof. □

We are now ready to state and prove our main result.

THEOREM 3.5. *If $f\in M$ then $R_sf\to f$ in L^2_a as $s\to 1^-$, and*

$$\|f\|^2=\iint_{|\lambda|<1}\|Q_\lambda f\|^2\,\frac{dA(\lambda)}{\pi}+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta_z\Delta_w|Q_wf(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}.$$

As a consequence, $M=[M\ominus TM]$.

Proof. Since $R_sf\in[M\ominus TM]$ we see by Lemma 3.3 that

$$\begin{aligned}\|R_sf\|^2=&\iint_{|\lambda|<1}\|Q_{s\lambda}f\|^2\,\frac{dA(\lambda)}{\pi}\\&+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta_z\Delta_w|Q_{sw}f(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}.\end{aligned}\tag{3.26}$$

We now apply Proposition 2.6 with $v(w)=\Delta_z\Delta_w|Q_wf(z)|^2$ to obtain

$$\begin{aligned}&\iint_{|w|<1}s\Gamma(z,w)\,\Delta_z\Delta_w|Q_{sw}f(z)|^2\,\frac{dA(w)}{\pi}\\&\qquad\leqslant 2\iint_{|w|<1}\Gamma(z,w)\,\Delta_z\Delta_w|Q_wf(z)|^2\,\frac{dA(w)}{\pi}\quad\forall z\in\mathbf{D}.\end{aligned}\tag{3.27}$$

By Proposition 2.4 (a) with $f(w)$ replaced by $2\partial Q_wf(z)/\partial z$, we have for $z\in\mathbf{D}$

$$\begin{aligned}&\lim_{s\to 1^-}\iint_{|w|<1}s\Gamma(z,w)\,\Delta_z\Delta_w|Q_{sw}f(z)|^2\,\frac{dA(w)}{\pi}\\&\qquad=\iint_{|w|<1}\Gamma(z,w)\,\Delta_z\Delta_w|Q_wf(z)|^2\,\frac{dA(w)}{\pi}.\end{aligned}\tag{3.28}$$

In view of Lemma 3.4, (3.27) and (3.28) we can apply the Dominated Convergence Theorem to the last integral in (3.26). Together with the elementary fact that

$$\lim_{s\to 1^-}\iint_{|\lambda|<1}\|Q_{s\lambda}f\|^2\,\frac{dA(\lambda)}{\pi}=\iint_{|\lambda|<1}\|Q_\lambda f\|^2\,\frac{dA(\lambda)}{\pi}$$

this implies that $\lim_{s\to 1^-}\|R_sf\|^2$ exists and is bounded by $\|f\|^2$. We have already noted that $R_sf(z)\to f(z)$ for all $z\in\mathbf{D}$, thus by Lemma 2.1, $R_s(f)\to f$ in L^2_a, and so the proof is complete. □

4. Some consequences and further results

We continue to use the notational conventions of the previous section.

If N is a closed subspace of the invariant subspace M, then it is easy to see that $P_{M\ominus TM}(N)=P_{M\ominus TM}([N])$. Thus if $[N]=M$, then $P_{M\ominus TM}(N)=M\ominus TM$, so Theorem 3.5 shows the following:

PROPOSITION 4.1. *M is generated by* $\dim(M\ominus TM)$ *elements, and no smaller set can generate M.*

It is of interest to compare this result with a result of Domingo Herrero [Her]. In his terminology, Proposition 4.1 says that T is $\dim(M\ominus TM)$-*multicyclic.* On the other hand, it is not difficult to show directly that $T-\lambda I$ is semi-Fredholm of index $-\dim(M\ominus TM)$ for all $\lambda\in\mathbf{D}$, and of course $T-\lambda I$ is invertible if $|\lambda|>1$. Herrero's Theorem implies that an operator with these properties is at least in the *norm closure* of the set of the $\dim(M\ominus TM)$-multicyclic operators.

Our results specialized to the case $\dim(M\ominus zM)=1$ yield some interesting new facts.

PROPOSITION 4.2. *If* $\dim(M\ominus TM)=1$ *and φ is the extremal function for M, then* $M=[\varphi]$. □

The next two propositions are special cases of this.

PROPOSITION 4.3. *If $M=[f]$ and φ is the extremal function for M, then $M=[\varphi]$.* □

PROPOSITION 4.4. *If M is the invariant subspace given by a zero set and φ is the extremal function for M, then $M=[\varphi]$.* □

It is of interest to isolate a part of the work of §3 to the case $\dim(M\ominus TM)=1$. For an inner function φ, Hedenmalm defines (with different notation)

$$\mathcal{A}^2(\varphi)=\left\{f\in L^2_a:\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta|f(w)|^2\Delta|\varphi(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}<\infty\right\}.$$

Because of our Proposition 2.4 (b), if φ is not the constant 1 we can drop the requirement that $f\in L^2_a$.

Definition. Let φ be a nonconstant L^2_a-inner function. Then $\mathcal{A}^2(\varphi)$ is the space of analytic functions f in $\mathbf{D}$ for which

$$\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta|f(w)|^2\Delta|\varphi(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}<\infty,$$

supplied with the norm

$$\|f\|^2_{\mathcal{A}^2(\varphi)}=\|f\|^2_{L^2_a}+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta|f(w)|^2\Delta|\varphi(z)|^2\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}.$$

If φ is constant, then $A^2(\varphi)=L^2_a$.

The following result highlights the importance of this space.

PROPOSITION 4.5. *Suppose that φ is L_a^2-inner. Then*

$$[\varphi]=\varphi\cdot\mathcal{A}^2(\varphi)$$

with equality of norms. Moreover, if $f\in\mathcal{A}^2(\varphi)$ then

$$f_s\varphi\to f\varphi \quad \textit{in } L_a^2.$$

Proof. By the statement "with equality of norms" we mean that $\|f\varphi\|_{L_a^2}=\|f\|_{\mathcal{A}^2(\varphi)}$. The inclusion $[\varphi]\subset\varphi\cdot\mathcal{A}^2(\varphi)$ and the equality of norms is due to Hedenmalm; it is a restatement of (1.5).

For the opposite inclusion, assume $f\in\mathcal{A}^2(\varphi)$. By Proposition 2.6,

$$\iint_{|w|<1}\Gamma(z,w)\,\Delta|f_s(w)|^2\,\frac{dA(w)}{\pi}\leqslant 4\iint_{|w|<1}\Gamma(z,w)\,\Delta|f(w)|^2\,\frac{dA(w)}{\pi}$$

for $\frac{1}{2}\leqslant s<1$, and by Proposition 2.4 (a),

$$\lim_{s\to1^-}\iint_{|w|<1}\Gamma(z,w)\,\Delta|f_s(w)|^2\,\frac{dA(w)}{\pi}=\iint_{|w|<1}\Gamma(z,w)\,\Delta|f(w)|^2\,\frac{dA(w)}{\pi}.$$

Obviously $f_s\varphi\in[\varphi]$, so $\|f_s\|_{\mathcal{A}^2(\varphi)}=\|f_s\varphi\|$. We can thus apply dominated convergence to show that $\|f_s\varphi\|\to\|f\|_{\mathcal{A}^2(\varphi)}$. In particular, $\|f_s\varphi\|$ is bounded for $\frac{1}{2}\leqslant s<1$, so it is easy to see that $f\varphi$ is in the weak closure of $\{f_s\varphi\}_{1/2\leqslant s<1}$. Since the weak closure of a subspace is the same as the strong closure, this shows that $f\varphi\in[\varphi]$. This in turn shows that $\|f\varphi\|=\|f\|_{\mathcal{A}^2(\varphi)}$; hence $\|f_s\varphi\|\to\|f\varphi\|$. By Lemma 2.1, $f_s\varphi\to f\varphi$ in L_a^2. □

We note that the results we have been discussing answer all conjectures in [Hed1] in the affirmative.

We turn to the study of outer functions and inner-outer factorizations. We first show that Korenblum's conjecture ([Kor, p. 106]) is true.

PROPOSITION 4.6. *L_a^2-outer functions are cyclic in L_a^2.*

Proof. Suppose that F is L_a^2-outer and that φ is the extremal function for $[F]$. Since $qF\in[F]=[\varphi]$ for any polynomial q, Proposition 4.5 applies to show that $\|qF\|\geqslant\|qF/\varphi\|$. Thus $F/\varphi\prec F$, so by the definition of outer function,

$$\left|\frac{F}{\varphi}(0)\right|\leqslant|F(0)|. \tag{4.1}$$

Since $\varphi\in[F]$, $(F/\varphi)(0)\neq0$, so (4.1) implies that $F(0)\neq0$ and then that $|\varphi(0)|\geqslant1$. Since $\|\varphi\|=1$, we must have $\varphi=1$, so F is cyclic. □

Thus the cyclic functions are exactly the outer functions. This allows us to show that the outer functions enjoy a much stronger property than their defining property.

PROPOSITION 4.7. *Suppose that F is outer and $g \prec F$. Then $|g(\lambda)| \leqslant |F(\lambda)|$ for all $\lambda \in \mathbf{D}$.*

Proof. Let $\varphi_\lambda(z) = (z+\lambda)/(1+\bar{\lambda}z)$ be a disk automorphism sending 0 to λ. The results we have been discussing show that $F \circ \varphi_\lambda$ is outer, and clearly $g \circ \varphi_\lambda \prec F \circ \varphi_\lambda$. Thus $|g(\lambda)| = |g \circ \varphi_\lambda(0)| \leqslant |F \circ \varphi_\lambda(0)| = |F(\lambda)|$. □

Finally we prove an analogue of the classical H^2-inner-outer factorization.

PROPOSITION 4.8. *Suppose $f \in L^2_a$. Then f has a factorization*

$$f = \varphi F$$

where φ is L^2_a-inner and F is L^2_a-outer. Furthermore,

$$F \prec f$$

and

$$|F(0)| = \max\{|g(0)| : g \prec f\}.$$

Proof. Let φ be the extremal function of $[f]$. We have already mentioned in the proof of Proposition 4.6 that

$$F = \frac{f}{\varphi} \prec f.$$

Thus if q_n are polynomials such that $q_n f \to \varphi$, then $\{q_n F\}$ must be a Cauchy sequence in L^2_a. Hence $q_n F \to 1$ so F is cyclic, hence outer. Finally, if $g \prec f$ the same reasoning shows that $q_n g \to (\varphi/f)g$ in L^2_a, so $\|(\varphi/f)g\| \leqslant 1$. Thus $|g(0)| \leqslant |(f/\varphi)(0)| = |F(0)|$. □

It is natural to ask whether the factorization in Proposition 4.8 is unique. H. Hedenmalm has shown us the following argument which shows that a function $f \in L^2_a$ may have distinct L^2_a-inner and L^2_a-outer factorizations. Indeed, a construction of Borichev and Hedenmalm [BH] makes it possible to find a nonconstant L^2_a-inner function φ such that

$$|\varphi(z)| \geqslant c(1-|z|)^2 \quad \text{for all } z \in \mathbf{D}$$

(see (4.4) of [BH]). Then for small $\varepsilon > 0$ and certain $\delta > 0$, we have $\varphi^{-\varepsilon}, \varphi^{1-\varepsilon} \in L^{2+\delta}_a$. Hence it follows from a result of H. S. Shapiro [S1], [S2] that $\varphi^{-\varepsilon}$ and $\varphi^{1-\varepsilon}$ are cyclic in L^2_a. Thus $\varphi^{1-\varepsilon} = \varphi \cdot \varphi^{-\varepsilon}$, i.e. an L^2_a-outer function can be written as a product of a nonconstant L^2_a-inner function and an L^2_a-outer function.

The last theorem in this section can be regarded as the analogue of the contractive divisor property for arbitrary invariant subspaces.

PROPOSITION 4.9. *Let M be an invariant subspace of L_a^2 with* $\dim(M\ominus TM)=N$ *(finite or infinite).*

If $\{\varphi_n\}_{n=1}^N$ is an orthonormal basis for $M\ominus TM$, then for each $f\in M$ there is a sequence of functions $\{f_n\}_{n=1}^N\subseteq L_a^2$ such that

$$f(z)=\sum_{n=1}^N f_n(z)\varphi_n(z)$$

for each $z\in\mathbf{D}$, and

$$\sum_{n=1}^N \|f_n\|_{L_a^2}^2 \leqslant \|f\|_{L_a^2}^2.$$

Proof. We define

$$f_n(z)=\langle Q_z f,\varphi_n\rangle_{L_a^2},\quad z\in\mathbf{D}.$$

Then for each $z\in\mathbf{D}$,

$$Q_z f=\sum_{n=1}^N f_n(z)\varphi_n,$$

where the sum converges in the norm of L_a^2. Thus,

$$f(z)=(Q_z f)(z)=\sum_{n=1}^N f_n(z)\varphi_n(z)$$

for each $z\in\mathbf{D}$. Furthermore, from Theorem 3.5 we see that

$$\sum_{n=1}^N \|f_n\|_{L_a^2}^2=\iint_{|\lambda|<1}\sum_{n=1}^N |\langle Q_\lambda f,\varphi_n\rangle|^2\,\frac{dA(\lambda)}{\pi}$$
$$=\iint_{|\lambda|<1}\|Q_\lambda f\|^2\,\frac{dA(\lambda)}{\pi}\leqslant\|f\|^2. \qquad \square$$

Two remarks regarding Proposition 4.9 in the case $N>1$: First, if

$$f(z)=\sum_{n=1}^N f_n(z)\varphi_n(z)$$

as above, then it is not clear whether any individual summands are in the Bergman space. Secondly, there may be many ways to write

$$f(z)=\sum_{n=1}^N g_n(z)\varphi_n(z)$$

even with the condition $\sum_{n=1}^{N}\|g_n\|_{L_a^2}^2 \leqslant \|f\|_{L_a^2}^2$. Indeed, suppose that $N=2$ and $f\in M$, $f(z)=f_1(z)\varphi_1(z)+f_2(z)\varphi_2(z)$ as in the proposition with $\|f_1\|_{L_a^2}^2+\|f_2\|_{L_a^2}^2<\|f\|_{L_a^2}^2$. Then for small $\varepsilon\in\mathbf{C}$, the functions $g_1=f_1+\varepsilon\varphi_2$, $g_2=f_2-\varepsilon\varphi_1$ would satisfy the same conditions. Technically, one could circumvent this problem by formulating a "vector analogue" of Proposition 4.5 for arbitrary invariant subspaces. One would use the formula of Theorem 3.5 to define the relevant space of vector-valued functions. We omit the details.

5. The case $p\neq 2$

Many of the results of the previous section hold in L_a^p for $0<p<\infty$. For an invariant subspace M of L_a^p, we consider the extremal problem

$$\sup\{\operatorname{Re} f^{(n)}(0) : f\in M,\ \|f\|_{L_a^p}\leqslant 1\}, \tag{5.1}$$

where n is the smallest integer for which there exists an $f\in M$ such that $f^{(n)}(0)\neq 0$. It can be shown (see [DKSS2]) that if an extremal function φ for (5.1) exists, then it satisfies $\|\varphi\|_{L_a^p}=1$ and

$$\iint_{|z|<1} |\varphi(z)|^p z^n \,\frac{dA(z)}{\pi} = 0 \quad \text{for } n=1,2,\dots,$$

i.e. is an L_a^p-inner function.

If φ is an L_a^p-inner function, we define $\mathcal{A}^p(\varphi)$ analogously to the case $p=2$ in §4. Proposition 4.5 remains true, but we must alter the proof. Furthermore, notice that from the definition it is not clear that $A^p(\varphi)$ is a metric space (or a normed space in the case $p\geqslant 1$), but that this will follow from the next proposition.

PROPOSITION 5.1. *If φ is L_a^p-inner then*

$$[\varphi]=\varphi\cdot\mathcal{A}^p(\varphi)$$

with equality of norms. Moreover, if $f\in\mathcal{A}^p(\varphi)$ then $f_s\varphi\to f\varphi$ in L_a^p as $s\to 1^-$.

Proof. If f is a polynomial then we know that

$$\|f\varphi\|_{L_a^p}=\|f\|_{\mathcal{A}^p(\varphi)}. \tag{5.2}$$

This was proven in [Hed1] (it is Proposition 2.5 (a) of the present paper with $v=|f|^p$). In the case $p=2$ it is easy to take limits to prove that (5.2) is true whenever $f\varphi\in[\varphi]$; this follows from the inequality

$$\bigl|\Delta|f|^2-\Delta|g|^2\bigr| \leqslant 2\Delta|f-g|^2,$$

which in turn follows from the fact that $\Delta|f|^2=4|f'|^2$. This inequality fails for $p\neq 2$, so we need another approach. Let $f\varphi\in[\varphi]$ and suppose that (f_n) is a sequence of polynomials such that $f_n\varphi\to f\varphi$ in L^p_a. We apply (5.2) with f replaced respectively by f_n and f_s to get

$$\|f_n\varphi\|^p_{L^p_a}=\|f_n\|^p_{L^p_a}+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta|f_n(w)|^p\,\Delta|\varphi(z)|^p\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi} \tag{5.3}$$

and

$$\|f_s\varphi\|^p_{L^p_a}=\|f_s\|^p_{L^p_a}+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta|f_s(w)|^p\,\Delta|\varphi(z)|^p\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}. \tag{5.4}$$

Since $\|f_n\varphi\|^p_{L^p_a}\to\|f\varphi\|^p_{L^p_a}$ as $n\to\infty$, we can apply Fatou's Lemma to (5.3) to see that

$$\|f\varphi\|^p_{L^p_a}\geqslant\|f\|^p_{A^p(\varphi)}, \tag{5.5}$$

which shows that $f\in A^p(\varphi)$. By Proposition 2.6,

$$\iint_{|w|<1}\Gamma(z,w)\,\Delta|f_s(w)|^p\,\frac{dA(w)}{\pi}\leqslant 4\iint_{|w|<1}\Gamma(z,w)\,\Delta|f(w)|^p\,\frac{dA(w)}{\pi}$$

if $\frac{1}{2}\leqslant s<1$, so by Proposition 2.4 (a) we can apply dominated convergence to (5.4) together with Fatou's Lemma to conclude that

$$\|f\varphi\|^p_{L^p_a}\leqslant\lim_{s\to 1^-}\|f_s\varphi\|^p_{L^p_a}=\|f\|^p_{A^p(\varphi)}. \tag{5.6}$$

Combining (5.5) and (5.6) we see that $\|f\varphi\|^p_{L^p_a}=\|f\|^p_{A^p(\varphi)}$ and that $\|f_s\varphi\|^p_{L^p_a}\to\|f\varphi\|^p_{L^p_a}$. By Lemma 2.1, $f_s\varphi\to f\varphi$ in L^p_a.

We turn now to the proof of the inclusion $\varphi\cdot A^p(\varphi)\subset[\varphi]$. Suppose $f\in A^p(\varphi)$. We can use Proposition 2.6 and Proposition 2.4 (a) as we did in the proof of Proposition 4.5 to show that

$$\|f_s\varphi\|_{L^p_a}\leqslant 4^{1/p}\|f\|_{A^p(\varphi)}\quad\text{for }\tfrac{1}{2}\leqslant s<1. \tag{5.7}$$

Now if $p\geqslant 1$, L^p_a is a Banach space and we can show from (5.7) that $f\varphi\in[\varphi]$ as we did in the proof of Proposition 4.5. Unfortunately this does not work if $p<1$, and we need a more complicated argument in this case.

Let $\{z_n\}$ be the zero set of f, and for $s\geqslant 1$ and $n=1,2,\ldots$, let χ_{ns},ψ_{ns} be the extremal functions for the zero sets $\{(1/s)z_1,\ldots,(1/s)z_n\}$, $\{(1/s)z_{n+1},(1/s)z_{n+2},\ldots\}$, respectively. By this we mean e.g. that ψ_{ns} is the extremal function for the problem (5.1), where M is the invariant subspace of L^p_a determined by the zero set $\{(1/s)z_{n+1},(1/s)z_{n+2},\ldots\}$. The existence and uniqueness of such extremals are shown in [DKSS2]. We claim that for each $z\in\mathbf{D}$ and $n=1,2,3,\dots$,

$$\chi_{ns}(z)\to\chi_{n1}(z) \tag{5.8}$$

and

$$\psi_{ns}(z)\to\psi_{n1}(z)\quad\text{as } s\to 1^-. \tag{5.9}$$

To see this, suppose that ψ_{ns_j} is a subsequence and f is a function such that

$$\lim_{j\to\infty}\psi_{ns_j}(z)=f(z)\quad\text{for all } z\in\mathbf{D}.$$

Clearly $\|f\|_{L^p_a}\leqslant 1$ and f vanishes on $\{z_{n+1},z_{n+2},\dots\}$, so by the definition of ψ_{n1}, $\psi_{n1}(0)\geqslant f(0)$. On the other hand, $(\psi_{n1})_{s_j}$ (the ordinary dilation of ψ_{n1} by s_j) vanishes on $\{(1/s_j)z_{n+1},(1/s_j)z_{n+2},\dots\}$ and $\|(\psi_{n1})_{s_j}\|_{L^p_a}\leqslant\|\psi_{n1}\|_{L^p_a}=1$, so by the definition of ψ_{ns_j}, $\psi_{ns_j}(0)\geqslant(\psi_{n1})_{s_j}(0)=\psi_{n1}(0)\geqslant f(0)$. Since $\lim_{j\to\infty}\psi_{ns_j}(0)=f(0)$ we see that $\psi_{n1}(0)=f(0)$, so by the uniqueness of the extremal functions associated to zero sets we see that $f=\psi_{n1}$. We have thus shown that any pointwise convergent subsequence ψ_{ns_j} of ψ_{ns} converges to ψ_{n1}, and, by a standard argument, this proves (5.9). The same proof shows (5.8).

We now use (5.7) together with the contractive divisor property of ψ_{ns} and χ_{ns} ([DKSS1], [DKSS2]) to see that

$$\left\|\frac{f_s}{\chi_{ns}\psi_{ns}}\varphi\right\|_{L^p_a}\leqslant 4^{1/p}\|f\|_{A^p(\varphi)}=C. \tag{5.10}$$

Since $f_s/\chi_{ns}\psi_{ns}$ has no zeros in $\mathbf{D}$ we can write (5.10) as

$$\left\|\left(\frac{f_s}{\chi_{ns}\psi_{ns}}\right)^{p/2}\right\|_{L^2_a(|\varphi|^p)}\leqslant C. \tag{5.11}$$

Since χ_{ns} and ψ_{ns} are extremal functions associated with finite zero sets, they are analytic in a neighborhood of $\overline{\mathbf{D}}$ and their moduli are bounded below by 1 on $\partial\mathbf{D}$ ([DKSS2]). We can thus argue from (5.11), (5.8) and (5.9), as in the proof of Proposition 4.5, to show that there are polynomials q_k such that $q_k\to(f/\chi_{n1}\psi_{n1})^{p/2}$ in $L^2(|\varphi|^p)$. We apply Proposition 2.5 (a) with v respectively equal to $|q_k|^2$ and $(f/\chi_{n1}\psi_{n1})_s$ to get

$$\begin{aligned}\iint_{|z|<1}|q_k(z)|^2|\varphi(z)|^p\,\frac{dA(z)}{\pi}&=\iint_{|z|<1}|q_k(z)|^2\,\frac{dA(z)}{\pi}\\&\quad+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\Delta|q_k(w)|^2\Delta|\varphi(z)|^p\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}\left\|\left(\frac{f}{\chi_{n1}\psi_{n1}}\right)_s\varphi\right\|_{L_a^p}^p &= \left\|\left(\frac{f}{\chi_{n1}\psi_{n1}}\right)_s\varphi\right\|_{L_a^p}^p \\ &+\iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta\left|\left(\frac{f}{\chi_{n1}\psi_{n1}}\right)_s(w)\right|^p\Delta|\varphi(z)|^p\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}.\end{aligned}\tag{5.13}$$

We let $k\to\infty$ in (5.12) and $s\to 1^-$ in (5.13) and argue as we did with (5.3) and (5.4) to see first that

$$\left\|\frac{f}{\chi_{n1}\psi_{n1}}\varphi\right\|_{L_a^p}^p \geqslant \left\|\frac{f}{\chi_{n1}\psi_{n1}}\right\|_{A^p(\varphi)},\tag{5.14}$$

and then that

$$\left\|\frac{f}{\chi_{n1}\psi_{n1}}\varphi\right\|_{L_a^p}^p \leqslant \lim_{s\to 1^-}\left\|\left(\frac{f}{\chi_{n1}\psi_{n1}}\right)_s\varphi\right\|_{L_a^p}^p = \left\|\frac{f}{\chi_{n1}\psi_{n1}}\right\|_{A^p(\varphi)}^p.\tag{5.15}$$

Combining (5.14) and (5.15), we see that

$$\lim_{s\to 1^-}\left\|\left(\frac{f}{\chi_{n1}\psi_{n1}}\right)_s\varphi\right\|_{L_a^p}^p = \left\|\frac{f}{\chi_{n1}\psi_{n1}}\varphi\right\|_{L_a^p}^p,$$

so by Lemma 2.1,

$$\left(\frac{f}{\chi_{n1}\psi_{n1}}\right)_s\varphi \to \frac{f}{\chi_{n1}\psi_{n1}}\varphi.$$

This shows that $f\varphi/\chi_{n1}\psi_{n1}\in[\varphi]$. Multiplication by the bounded function χ_{n1} (it is the extremal function associated to a finite zero set) shows that $(f/\psi_{n1})\varphi\in[\varphi]$. Now by the contractive divisor property of ψ_{n1},

$$\left\|\frac{f}{\psi_{n1}}\varphi\right\|_{L_a^p} \leqslant \|f\varphi\|_{L_a^p},$$

and it is shown in [DKSS2] as a consequence of the contractive divisor property that $\lim_{n\to\infty}\psi_{n1}(z)=1$ for all $z\in\mathbf{D}$. Hence by Lemma 2.1, $(f/\psi_{n1})\varphi\to f\varphi$ in L_a^p, which shows that $f\varphi\in[\varphi]$. □

This will allow us to prove the p-analogue of Proposition 4.3. Before doing this we mention a technical point. Let M be an invariant subspace of L_a^p. If $p\geqslant 1$, the existence and uniqueness of an extremal function for (5.1), which we will refer to simply as an extremal function for M, can be proved easily (see [DKSS1]), but if $0<p<1$ neither the existence nor the uniqueness are known in general. For cyclic invariant subspaces we can prove this, and this is part of our next result.

THEOREM 5.2. *Suppose that M is a cyclic invariant subspace of L^p_a. Then there exists a unique extremal function φ for M, and $M=[\varphi]=\varphi A^p(\varphi)$.*

Proof. By hypothesis, $M=[f]$ for some f. Consider the following two extremal problems:

(a) $\sup\{\operatorname{Re} g(0): gf\in M \text{ and } \|gf\|_{L^p_a}=1\}$,

(b) $\sup\{\operatorname{Re} h(0): h\in \mathcal{P}^2(|f|^p) \text{ and } \|h\|_{L^2(|f|^p)}=1\}$.

Here $\mathcal{P}^2(|f|^p)$ is the closure of the polynomials in the weighted Hilbert space $L^2(|f|^p)$. By elementary Hilbert space considerations, there is a unique extremal function h_0 for (b), and if h_n is any maximizing sequence then $h_n\to h_0$ in $L^2(|f|^p)$. Let h_n be a sequence of polynomials approaching h_0, and let φ_n be the L^2_a-extremal function corresponding to the zero set of h_n. By the results of [DKSS2], since the zero set of h_n is finite, we have that φ_n is analytic in a neighborhood of $\overline{\mathbf{D}}$, $|\varphi_n|\geqslant 1$ on $\partial\mathbf{D}$, and h_n/φ_n is analytic in a neighborhood of $\overline{\mathbf{D}}$. Proposition 2.5 and an easy limit argument now shows that

$$\begin{aligned}\iint_{|z|<1} |h_n(z)|^2 |f(z)|^p \frac{dA(z)}{\pi} &= \iint_{|z|<1} \left|\frac{h_n}{\varphi_n}(z)\right|^2 |f(z)|^p |\varphi_n(z)|^2 \frac{dA(z)}{\pi} \\ &\geqslant \iint_{|z|<1} \left|\frac{h_n(z)}{\varphi_n(z)}\right|^2 |f(z)|^p \frac{dA(z)}{\pi}.\end{aligned}$$

Since $|(h_n/\varphi_n)(0)|\geqslant |h_n(0)|$, this shows that we can assume that h_n has no zeros in $\mathbf{D}$. The same argument (with L^p_a-extremal functions and the results in [DKSS1], [DKSS2]) shows that a maximizing sequence for (a) can be assumed to consist of polynomials with no zeros in $\mathbf{D}$. If g_n is such a sequence, then $h_n=g_n^{p/2}$ is a maximizing sequence for (b). Hence $h_n\to h_0$ in $L^2(|f|^p)$, so $g_n\to g_0=h_0^{2/p}$ in $L^p(|f|^p)$ by Lemma 2.1. This shows that $\varphi=g_0 f$ is the unique extremal function for M. Furthermore, the extremal property of h_0 implies that

$$\iint_{|z|<1} z^n\, \overline{h_0(z)}\, |f(z)|^p \frac{dA(z)}{\pi} = 0 \quad \text{for } n=1,2,\ldots.$$

With h_n as above we see that

$$\iint_{|z|<1} \left| h_n(z) \left(\frac{f}{\varphi}\right)^{p/2}(z)\right|^2 |\varphi(z)|^p \frac{dA(z)}{\pi} = \iint_{|z|<1} |h_n(z)|^2 |f(z)|^p \frac{dA(z)}{\pi} = 1$$

and $h_n(z)(f/\varphi)^{p/2}(z)\to h_0(z)(f/\varphi)^{p/2}(z)=1$ for all $z\in\mathbf{D}$, so by Lemma 2.1,

$$h_n\left(\frac{f}{\varphi}\right)^{p/2} \to 1 \quad \text{in } L^2(|\varphi|^p). \tag{5.17}$$

We also see that

$$\iint_{|z|<1} z^n \left(\frac{f}{\varphi}\right)^{p/2}(z)\,|\varphi(z)|^p \,\frac{dA(z)}{\pi} = \iint_{|z|<1} z^n\, \overline{h_0(z)}\,|f(z)|^p \,\frac{dA(z)}{\pi} = 0 \quad \text{for } n=1,2,\ldots. \tag{5.18}$$

Let $\mathcal{N}$ be the closure in $L^2(|\varphi|^p)$ of the set of polynomial multiples of $(f/\varphi)^{p/2}$. By (5.17) and (5.18), $1\in\mathcal{N}\ominus z\mathcal{N}$ and as a consequence

$$\iint_{|z|<1} z\,\frac{(f/\varphi)^{p/2}(z)-(f/\varphi)^{p/2}(\lambda)}{z-\lambda}\,|\varphi(z)|^p\,\frac{dA(z)}{\pi} = 0 \quad \forall\lambda\in\mathbf{D}. \tag{5.19}$$

Consider formula (3.18) and its role in the proof of Lemma 3.4. If one replaces $Q_\lambda f(z)$ by $(f/\varphi)(\lambda)\varphi(z)$ one obtains

$$\overline{\frac{f}{\varphi}(\lambda)}\iint_{|z|<1} z\,\frac{(f/\varphi)(z)-(f/\varphi)(\lambda)}{z-\lambda}\,|\varphi(z)|^2\,\frac{dA(z)}{\pi} = 0 \quad \forall\lambda\in\mathbf{D},$$

which makes the connection between (5.19) and (3.18) apparent. We can now use (5.19) in exactly the same way we used (3.18) to show that

$$\|f\|^p_{L^p_a} \geqslant \int_{|\lambda|=r}\left|\frac{f}{\varphi}(\lambda)\right|^p \frac{|d\lambda|}{2\pi r} - \iint_{|z|<r}\int_{|\lambda|=r}(r^2-|z|^2)\left|\frac{(f/\varphi)^{p/2}(z)-(f/\varphi)^{p/2}(\lambda)}{z-\lambda}\right|^2 |\varphi(z)|^p\,\frac{|d\lambda|}{2\pi r}\,\frac{dA(z)}{\pi}.$$

It then follows as before that

$$\|f\|^p_{L^p_a} \geqslant \int_{|\lambda|=r}\left|\frac{f}{\varphi}(\lambda)\right|^p \frac{|d\lambda|}{2\pi r} + \iint_{|z|<r}\iint_{|w|<r}\frac{1}{2}\log\left|\frac{r(z-w)}{r^2-\bar{z}w}\right|\Delta\left|\frac{f}{\varphi}(w)\right|^p \Delta|\varphi(z)|^p\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi},$$

and then that

$$\|f\|^p_{L^p_a} \geqslant \left\|\frac{f}{\varphi}\right\|^p_{L^p_a} + \iint_{|z|<1}\iint_{|w|<1}\Gamma(z,w)\,\Delta\left|\frac{f}{\varphi}(w)\right|^p \Delta|\varphi(z)|^p\,\frac{dA(w)}{\pi}\,\frac{dA(z)}{\pi}.$$

Now an application of Proposition 5.1 to the function f/φ shows that $f\in[\varphi]$, so we are done. $\square$

An immediate consequence of Theorem 5.2 is that Propositions 4.6, 4.7 and 4.8 are true in L^p_a. In particular, we have

PROPOSITION 5.3. *Let* $0<p<\infty$. *A function* $f\in L^p_a$ *is* L^p_a*-outer if and only if* f *is cyclic in* L^p_a. *Furthermore, any* $f\in L^p_a$ *has a factorization*

$$f=\varphi F,$$

where φ *is* L^p_a*-inner and* F *is* L^p_a*-outer.* □

We can also deduce the truth in L^p_a of Proposition 4.4 from the following result, which is of interest in its own right.

PROPOSITION 5.4. *If* (M_n) *is a decreasing sequence of cyclic invariant subspaces of* L^p_a *then* $\bigcap_n M_n$ *is cyclic. Moreover, if* $\bigcap_n M_n\neq\{0\}$ *and* φ_n *is the extremal function for* M_n *then* φ_n *converges in* L^p_a *to the extremal function for* $\bigcap_n M_n$.

As we mentioned above, Proposition 5.4 implies that zero set based invariant subspaces are cyclic because it is known that invariant subspaces defined by finitely many zeros are cyclic. Similarly, one shows that invariant subspaces of $\varkappa$-Beurling type of L^p_a, $0<p<\infty$, are cyclic (see [HKZ], especially the proof of Theorem 4.1).

Proof. If (φ_{n_k}) is a subsequence of (φ_n) that converges uniformly on compact sets to 0, then by Theorem 5.2 and Proposition 5.1, every $f\in\bigcap_n M_n$ can be written in the form $f=\varphi_{n_k}h_{n_k}$ with $h_{n_k}\in L^p_a$ and $\|h_{n_k}\|_{L^p_a}\leqslant\|f\|_{L^p_a}$. Hence $f=0$, so $\bigcap_n M_n=\{0\}$. Assume that $\bigcap_n M_n\neq\{0\}$ and let (φ_{n_k}) be a subsequence of (φ_n) that converges uniformly on compact sets to a function $\varphi\neq 0$. For each index j and $n_k>j$ we have $\varphi_{n_k}\in M_j$. It follows from Proposition 5.1 and Theorem 5.2 that $\varphi_{n_k}/\varphi_j\in A^p(\varphi_j)$ so by Fatou's Lemma $\varphi/\varphi_j\in A^p(\varphi_j)$, hence $\varphi\in[\varphi_j]=M_j$. Hence $\varphi\in\bigcap_n M_n$, and clearly $\|\varphi\|_{L^p_a}\leqslant 1$. It now follows easily that φ is in fact extremal for $\bigcap_n M_n$, and so by Lemma 2.1, φ_{n_k} converges to φ in L^p_a. Suppose that $f\in\bigcap_n M_n$. By Proposition 5.1 and Theorem 5.2, $f/\varphi_{n_k}\in A^p(\varphi_{n_k})$ $\forall k$, so by Fatou's Lemma $f/\varphi\in A^p(\varphi)$. Hence $\bigcap_n M_n=\varphi\cdot A^p(\varphi)=[\varphi]$, so φ is the unique extremal function for $\bigcap_n M_n$ and $\varphi_n\to\varphi$ in L^p_a. □

Our methods do not seem to be sufficient to establish the p-analogues of Propositions 4.1 and 4.2. We state these analogues as a conjecture.

Conjecture. If M is an invariant subspace of L^p_a, then M is generated by $\dim M/zM$ functions.

6. An inequality

Proposition 2.6 played an important role in the proof our main result by justifying the use of the Dominated Convergence Theorem at a crucial point. It can also be used together with Proposition 5.1 to show that if φ is an L^p_a-inner function, then

$$s\|h_s\varphi\|^p_{L^p_a} \leqslant 2\|h\varphi\|^p_{L^p_a}$$

if $h\varphi\in[\varphi]$ and $s<1$. This says that dilation is a bounded operator in the space $\mathcal{P}^p(|\varphi|^p)$, the closure of the polynomials in the weighted space $L^p(|\varphi|^p)$. Of course in the classical spaces of analytic functions dilation has a bound of 1, and it is of interest that this is also true in our situation if $p=2$ (or any even integer):

$$\|h_s\varphi\|_{L^2_a} \leqslant \|h\varphi\|_{L^2_a}$$

if $h\varphi\in[\varphi]$. This follows from the next proposition.

PROPOSITION 6.1. *If h is analytic in $\mathbf{D}$, then*

$$\Gamma[\Delta|h_s|^2](z)$$

is an increasing function of s, for $0\leqslant s\leqslant 1$ and $z\in\mathbf{D}$ fixed.

Proof. By Proposition 2.4 (a), the function in question is continuous in $0\leqslant s\leqslant 1$, so all we need show is that its derivative with respect to s is nonnegative. We remark here that the method used to prove Proposition 2.6 will not work here, since a calculation shows that

$$\frac{d}{ds}s\Gamma(z,w/s)$$

is not nonnegative throughout $|z|<1$, $|w|<s$.

By an easy approximation argument we may assume that h is a polynomial, say $h(z)=\sum_{n=0}^N a_nz^n$. The function

$$\Gamma[\Delta|h|^2](z)$$

satisfies and is determined by the properties

$$\Delta^2\Gamma[\Delta|h|^2]=\Delta|h|^2 \quad \text{in } \mathbf{D}, \tag{6.2}$$

$$\Gamma[\Delta|h|^2]=\frac{\partial}{\partial n}\Gamma[\Delta|h|^2]=0 \quad \text{on } \partial\mathbf{D}. \tag{6.3}$$

Set

$$H(z)=\frac{1}{2}\sum_{n=0}^N \frac{1}{n+1}a_nz^{n+1},$$

so that $\Delta|H|^2=4|H'|^2=|h|^2$ and

$$|H(re^{i\theta})|^2=\frac{1}{4}\sum_{m,n=0}^N \frac{1}{(m+1)(n+1)}a_m\bar{a}_nr^{m+n+2}e^{i(m-n)\theta}.$$

Define

$$\Phi(re^{i\theta}) = \frac{1}{4}\sum_{m,n=0}^{\infty}\frac{1}{(m+1)(n+1)}a_m\bar{a}_n \\ \times [r^{m+n+2} - (m\wedge n+1)r^{|m-n|+2} + (m\wedge n)r^{|m-n|}]e^{i(m-n)\theta}$$

(here $m\wedge n = \min(m,n)$; we will also use the notation $m\vee n = \max(m,n)$). We see that

$$\Delta^2\Phi = \Delta^2|H|^2 = \Delta|h|^2$$

and

$$\Phi = \frac{\partial}{\partial n}\Phi = 0 \quad \text{on } \partial\mathbf{D}.$$

Hence $\Phi = \Gamma[\Delta|h|^2]$. Algebraic manipulations now yield the expression

$$\Gamma[\Delta|h|^2](re^{i\theta}) = \frac{1}{4}\sum_{m,n=1}^{N}\frac{1}{(m+1)(n+1)}a_m\bar{a}_n(1-r^2)^2 r^{|m-n|} \\ \times [m\wedge n + (m\wedge n-1)r^2 + \ldots + r^{2m\wedge n-2}]e^{i(m-n)\theta}. \tag{6.4}$$

If we replace a_n by $s^n a_n$ in (6.4) and differentiate with respect to s, we obtain

$$\frac{d}{ds}\Gamma[\Delta|h_s|^2](re^{i\theta}) = \frac{(1-r^2)^2}{4s}\sum_{m,n=1}^{N}\left(\frac{1}{m+1}e^{im\theta}s^m a_m\right)\left(\frac{1}{n+1}e^{in\theta}s^n a_n\right) \\ \times r^{|m-n|}(m+n)[m\wedge n + (m\wedge n-1)r^2 + \ldots + r^{2m\wedge n-2}].$$

We must show that this is always nonnegative, i.e. that the numbers

$$b_{mn} = r^{|m-n|}(m+n)[m\wedge n + (m\wedge n-1)r^2 + \ldots + r^{2m\wedge n-2}]$$

are the coordinates of a positive-semidefinite matrix for $0\leqslant r<1$. We can show that

$$\det(r^{|m-n|})_{1\leqslant m,n\leqslant N} = (1-r^2)^{N-1}$$

by expanding on the last column and using induction on N (notice that if the last column of this matrix is crossed out, the $(N-1)$st row of the resulting matrix is r times its Nth row, so the expansion on the last column has only two terms). Hence $(r^{|m-n|})$ is a positive-definite matrix, so by the Schur Product Theorem ([M]) all we need show is that the matrix (c_{mn}), where

$$c_{mn} = (m+n)[m\wedge n + (m\wedge n-1)r^2 + \ldots + r^{2m\wedge n-2}],$$

is positive-definite. Now fix k, $0\leqslant k\leqslant N-1$. We will show that the coefficient of r^{2k} in (c_{mn}) is a positive-semidefinite matrix. The coordinates of this coefficient are 0 if $m\wedge n\leqslant k$, and

$$\begin{aligned}(m+n)(m\wedge n-k) &= (m\wedge n+m\vee n)(m\wedge n-k)\\ &= (m\wedge n+k)(m\wedge n-k)+(m\vee n-k)(m\wedge n-k)\\ &= (m\wedge n+k)(m\wedge n-k)+(m-k)(n-k)\end{aligned}$$

if $m\wedge n\geqslant k+1$. The second term of this last expression obviously represents a nonnegative-definite matrix. The proof will be completed by another application of the Schur Product Theorem once we show that

$$(m\wedge n+k)_{m,n\geqslant k+1}$$

and

$$(m\wedge n-k)_{m,n\geqslant k+1}$$

are both positive-definite matrices. To see this we compute that

$$\det(m\wedge n+k)_{k+1\leqslant m,n\leqslant N}=1+2k$$

and

$$\det(m\wedge n-k)_{k+1\leqslant m,n\leqslant N}=1$$

by subtracting each row from the one below it, starting from the next to the last row. □

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