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On the p-typical curves in Quillen's K-theory

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Contents

Introduction 1
1. The complex $\mathrm{TR}_*(A;p)$ 5
2. Smooth algebras 21
3. Typical curves in K-theory 34
References 52

Introduction

Twenty years ago Bloch, [Bl], introduced the complex $C_*(A;p)$ of p-typical curves in K-theory and outlined its connection to the crystalline cohomology of Berthelot–Grothendieck. However, to prove this connection Bloch restricted his attention to the symbolic part of K-theory, since only this admitted a detailed study at the time. In this paper we evaluate $C_*(A;p)$ in terms of the fixed sets of Bökstedt's topological Hochschild homology. Using this we show that for any smooth algebra A over a perfect field k of positive characteristic, $C_*(A;p)$ is isomorphic to the de Rham–Witt complex of Bloch–Deligne–Illusie. This confirms the outlined relationship between p-typical curves in K-theory and crystalline cohomology in the smooth case. In the singular case, however, we get something new. Indeed, we calculate $C_*(A;p)$ for the ring $k[t]/(t^2)$ of dual numbers over k and show that in contrast to crystalline cohomology, its cohomology groups are finitely generated modules over the Witt ring $W(k)$.

Let A be a ring, by which we shall always mean a commutative ring, and let $K(A)$ denote the algebraic K-theory spectrum of A. More generally, if $I\subset A$ is an ideal, $K(A,I)$ denotes the relative algebraic K-theory, that is, the homotopy theoretical fiber of the map $K(A)\to K(A/I)$. We define the *curves on* $K(A)$ to be the homotopy limit of spectra

$$C(A)=\underset{\overleftarrow{n}}{\operatorname{holim}}\,\Omega K(A[X]/(X^n),(X)).$$

The homotopy groups $C_*(A)=\pi_*C(A)$ are given by Milnor's exact sequence

$$0\to\varprojlim_n{}^{(1)}K_{*+2}(A[X]/(X^n),(X))\to C_*(A)\to\varprojlim_n K_{*+1}(A[X]/(X^n),(X))\to 0,$$

so in particular, $C_0(A)=\mathbf{W}(A)$, the big Witt ring of A. We note that originally Bloch defined the curves on $K(A)$ to be the inverse limit on the right-hand side. So our definition differs from his in that we include a possible $\lim^{(1)}$-term. Furthermore, Bloch defined a pairing on $C(A)$ which makes $C(A)$ a homotopy associative ring spectrum. In particular, this gives a ring homomorphism from $\mathbf{W}(A)$ to the ring of cohomology operations in $C(A)$. So when A is a $\mathbf{Z}_{(p)}$-algebra, the idempotents of $\mathbf{W}(A)$ give a splitting

$$C(A)\simeq\prod_{(k,p)=1}C(A;p),$$

as a product of copies of a spectrum $C(A;p)$, the *p-typical* curves on $K(A)$.

For any ring A, the topological Hochschild homology $\mathrm{TH}(A)$ is an S^1-equivariant spectrum, and there are maps

$$R,F\colon \mathrm{TH}(A)^{C_{p^n}}\to\mathrm{TH}(A)^{C_{p^{n-1}}}$$

of the fixed sets under the cyclic groups of order p^n and p^{n-1}. The map F is the obvious inclusion while the map R, introduced by Bökstedt, Hsiang and Madsen, [BHM], is given by the *cyclotomic* structure of $T(A)$. We write

$$\mathrm{TR}(A;p)=\underset{R}{\operatorname{holim}}\,\mathrm{TH}(A)^{C_{p^n}}.$$

THEOREM A. *Let A be a $\mathbf{Z}/p^j$-algebra. Then $C(A;p)\simeq\mathrm{TR}(A;p)$.*

The proof is based on a recent result of McCarthy, [Mc], which states that the cyclotomic trace of [BHM],

$$\mathrm{trc}\colon K(A)\to\mathrm{TC}(A),$$

from K-theory to a certain topological version of Connes' cyclic homology, induces an equivalence of the relative theories $K(A,I)$ and $\mathrm{TC}(A,I)$ after profinite completion, provided that the ideal $I\subset A$ is nilpotent. For any $\mathbf{F}_p$-algebra A, $\mathrm{TR}(A;p)$ is a generalized Eilenberg–MacLane spectrum, and therefore, determined up to homotopy by its homotopy groups $\mathrm{TR}_*(A;p)$. In higher characteristic, however, this is not likely to be the case.

Let k be a perfect field of characteristic $p>0$ and let $W_n(k)$ be its ring of p-typical Witt vectors of length n. For any k-algebra A we have the de Rham–Witt complex $W_n\Omega^*_A$

of Bloch, Deligne and Illusie, [I]. It is a differential graded algebra over $W_n(k)$ whose restriction modulo p is equal to the de Rham complex Ω^*_A. The restriction, Frobenius and Verschiebung maps of Witt vectors extend to operations

$$R, F\colon W_n\Omega^*_A \to W_{n-1}\Omega^*_A, \quad V\colon W_n\Omega^*_A \to W_{n+1}\Omega^*_A,$$

suitably compatible with the differential structure, and $W_n\Omega^*_A$ may be characterized as the universal example of such a structure. We show in §1 below that topological Hochschild homology provides another example. In particular, there are maps

$$I\colon W_n\Omega^*_A \to \pi_* \operatorname{TH}(A)^{C_{p^{n-1}}}.$$

The differential δ on $\pi_* \operatorname{TH}(A)^{C_{p^{n-1}}}$ is induced from the S^1-action. In the basic case $n=1$, it corresponds to Connes' B-operator under linearization $\pi_* \operatorname{TH}(A) \to \operatorname{HH}_*(A)$. In §2 we prove

THEOREM B. *Suppose that A is a smooth k-algebra. Then the map I extends to an isomorphism*

$$I\colon W_n\Omega^*_A \otimes_{W_n(k)} S_{W_n}\{\sigma_n\} \to \pi_* \operatorname{TH}(A)^{C_{p^{n-1}}}, \quad \deg \sigma_n = 2.$$

Moreover, $F(\sigma_n)=\sigma_{n-1}$, $V(\sigma_n)=p\sigma_{n+1}$ and $R(\sigma_n)=p\lambda_n\sigma_{n-1}$, where λ_n is a unit of $W_n(\mathbf{F}_p)=\mathbf{Z}/p^n$.

The basic case $A=k$ was proved in [HM1]. The bulk of §2 is the explicit calculation of the right-hand side in the case where A is a polynomial algebra. We find that it is abstractly isomorphic to the left-hand side, which is known from [I], and prove that the map I is an isomorphism. The general case follows by a covering argument. If we take the limit over the restriction maps, the extra generator σ_n vanishes, and hence

THEOREM C. *If A is a smooth k-algebra, then $W\Omega^*_A \cong \operatorname{TR}_*(A;p) \cong C_*(A;p)$.*

We note that Bloch proved that the same result holds if $C_*(A;p)$ is replaced by its symbolic part $SC_*(A;p)$, provided that A is local of Krull dimension less than p. The restriction on the dimension was later removed by Kato, [K]. We do not require that A be local.

For any scheme X over $\operatorname{Spec} k$, Berthelot, [Be], has defined its crystalline cohomology,

$$H^*(X/W_n), \quad H^*(X/W) = \varprojlim_n H^*(X/W_n).$$

It is a good cohomology theory when X is proper and smooth. In particular, the cohomology groups are finitely generated $W(k)$-modules. However, if X is either not smooth or not proper, the theory behaves rather pathologically. The main theorem of [I] states that there are natural isomorphisms

$$H^*(X/W_n) \cong \mathbf{H}^*(X; W_n\Omega_X^*), \quad H^*(X/W) \cong \mathbf{H}^*(X; W\Omega_X^*),$$

provided that X be smooth and smooth and proper, respectively. Here the right-hand sides denote the hypercohomology of X with coefficients in the complexes $W_n\Omega_X^*$ and $W\Omega_X^*$ of Zariski sheaves on X. It follows that

$$H^*(X/W) \cong \mathbf{H}^*(X; C_*(-;p))$$

when X is smooth and proper. However, in the non-reduced case we get something new: in §3.5 below we evaluate $C_*(A;p)$ for the ring $k[t]/(t^2)$ of dual numbers over k. The argument is based on [HM1]. Let $m(i,j)$ be the unique natural number such that $p^{m(i,j)-1}j \leqslant 2i+1 < p^{m(i,j)}j$, then

THEOREM D. *Let* $X = \operatorname{Spec} k[t]/(t^2)$. *Then*
(i) $p>2$: $\mathbf{H}^{2i}(X; C_*(-;p)) = \mathbf{H}^{2i+1}(X; C_*(-;p)) = \bigoplus_{1\leqslant j\leqslant 2i+1,\, j \text{ odd}} W(k)/(j, p^{m(i,j)})$,
(ii) $p=2$: $\mathbf{H}^{2i}(X; C_*(-;p)) = \mathbf{H}^{2i-1}(X; C_*(-;p)) = k^{\oplus i}$,
and in both cases $\mathbf{H}^0(X; C_*(-;p)) = W(k)$.

We note that in comparison the crystalline cohomology $H^*(X/W)$ is concentrated in degree 0 and 1 and the latter is infinitely generated as a $W(k)$-module. This suggests that $\mathbf{H}^*(X; C(-;p))$ might be a more well-behaved theory than crystalline cohomology for non-reduced schemes. The proof of Theorem A involves the following result, which is also of interest in its own right.

THEOREM E. *Let A be a $\mathbf{Z}/p^j$-algebra. Then $K_i(A[X]/(X^n),(X))$ is a bounded p-group, i.e. any element is annihilated by p^N for some number N which may depend on i.*

The fact that these groups are p-groups has previously been proved by Weibel, [We], by quite different methods. However, the result that they are bounded is new.

Throughout the paper we shall use the notion of a G-spectrum. The reader is referred to [LMS] for this material. We use the term equivalence to mean a weak homotopy equivalence, i.e. a map of spectra which induces isomorphism of all homotopy groups. A G-equivalence will mean a G-equivariant map which induces an equivalence of H-fixed spectra for all closed subgroups $H \subset G$. Throughout, G will denote the circle group and rings are assumed commutative without further notice.

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1. The complex $\mathbf{TR}_*(A;p)$

1.1. We briefly recall the ring of p-typical Witt vectors associated to a ring A. The standard reference is [Se] but see also G. M. Bergman's lecture in [Mu]. As a set $W(A)= A^{\mathbf{N}_0}$ and the ring structure is determined by the requirement that the ghost map

$$w: W(A) \to A^{\mathbf{N}_0}, \tag{1.1.1}$$

given by the Witt polynomials

$$\begin{aligned} w_0 &= a_0, \\ w_1 &= a_0^p + pa_1, \\ w_2 &= a_0^{p^2} + pa_1^p + p^2 a_2, \\ &\vdots \end{aligned}$$

be a natural transformation of functors from rings to rings. In more detail, we consider the polynomial ring $B=\mathbf{Z}[a_0, a_1, \ldots; b_0, b_1, \ldots]$ and the ring endomorphism $\phi: B \to B$ which raises the variables to the pth power. Then, as one easily verifies, a sequence $(x_0, x_1, \ldots)$ is in the image of the ghost map if and only if $x_n \equiv \phi(x_{n-1}) \pmod{p^n B}$, for all $n \geqslant 0$, and hence the image of w is a subring. In fact, w is the inclusion of a subring. Let s_i and p_i be the unique polynomials such that

$$\begin{aligned} (a_0, a_1, \ldots) + (b_0, b_1, \ldots) &= (s_0, s_1, \ldots), \\ (a_0, a_1, \ldots) \cdot (b_0, b_1, \ldots) &= (p_0, p_1, \ldots) \end{aligned}$$

in this subring structure. Then if A is any ring, we may substitute elements of A for the variables of B and get a sum and product on the set $W(A)$. The ring axioms hold in the special case $A=B$ since $W(B)$ is a subring of $B^{\mathbf{N}_0}$. By functoriality they hold in general. We note that $1=(1,0,0,\ldots) \in W(A)$.

There are natural operators, Frobenius and Verschiebung, on $W(A)$ characterized by the formulas

$$\begin{aligned} F&: W(A) \to W(A), & F(w_0, w_1, \ldots) &= (w_1, w_2, \ldots), \\ V&: W(A) \to W(A), & V(a_0, a_1, \ldots) &= (0, a_0, a_1, \ldots). \end{aligned} \tag{1.1.2}$$

By considering the universal case, one shows that F is well-defined and a ring homomorphism, that V is additive and that the following relations hold:

$$x \cdot V(y) = V(F(x) \cdot y), \quad FV = p, \quad VF = \mathrm{mult}_{V(1)}. \tag{1.1.3}$$

When A is an $\mathbf{F}_p$-algebra, one has in addition $V(1)=p$ and $F=W(\varphi)$, where φ is the Frobenius endomorphism of A. The Teichmüller character $\omega\colon A \to W(A)$ given by $\omega(x)=(x,0,0,\dots)$ is multiplicative but not additive. We will also write $\underline{x}=\omega(x)$.

The additive subgroup $V^n W(A)$ of $W(A)$ is an ideal by (1.1.3) and the quotient

$$W_n(A) = W(A)/V^n W(A)$$

is called the ring of Witt vectors of length n in A. The elements in $W_n(A)$ are in 1-1 correspondence with tuples $(a_0, \dots, a_{n-1})$, with addition and multiplication given by the same polynomials s_i, p_i as in $W(A)$. We note that s_i and p_i only depend on the variables $a_0, \dots, a_i$ and $b_0, \dots, b_i$. Therefore, $W(A)$ is the limit of the $W_n(A)$ over the restriction maps

$$R\colon W_n(A) \to W_{n-1}(A), \quad R(a_0, \dots, a_{n-1}) = (a_0, \dots, a_{n-2}), \tag{1.1.4}$$

and hence complete and separated in the topology defined by the ideals $V^n W(A)$, $n \geqslant 1$. If k is a perfect field of characteristic p, $W(k)$ is the unique complete discrete valuation ring with maximal ideal $VW(k)=VFW(k)=pW(k)$. In particular, $W(\mathbf{F}_p)=\mathbf{Z}_p$.

1.2. This section recalls the de Rham–Witt pro-complex of Bloch–Deligne–Illusie. We shall only give a very brief account of the construction and refer to [I] for details. Suppose that A is an $\mathbf{F}_p$-algebra. The de Rham–Witt pro-complex associated to A is a limit system of commutative DGA's over $\mathbf{Z}$,

$$\begin{tikzcd}
\vdots \arrow[d, "R"] & \vdots \arrow[d, "R"] & \vdots \arrow[d, "R"] & \\
W_3\Omega^0_A \arrow[r, "d"] \arrow[d, "R"] & W_3\Omega^1_A \arrow[r, "d"] \arrow[d, "R"] & W_3\Omega^2_A \arrow[r, "d"] \arrow[d, "R"] & \cdots \\
W_2\Omega^0_A \arrow[r, "d"] \arrow[d, "R"] & W_2\Omega^1_A \arrow[r, "d"] \arrow[d, "R"] & W_2\Omega^2_A \arrow[r, "d"] \arrow[d, "R"] & \cdots \\
W_1\Omega^0_A \arrow[r, "d"] & W_1\Omega^1_A \arrow[r, "d"] & W_1\Omega^2_A \arrow[r, "d"] & \cdots
\end{tikzcd}$$

such that $W_n\Omega^0_A=W_n(A)$ and such that R extends the restriction map (1.1.4). Furthermore, there are additive maps

$$F\colon W_n\Omega^i_A \to W_{n-1}\Omega^i_A, \quad V\colon W_n\Omega^i_A \to W_{n+1}\Omega^i_A$$

which extend the Frobenius and Verschiebung maps (1.1.2), and the following set of relations hold:

$$\begin{gathered} RF = FR, \quad RV = VR, \\ FV = VF = p, \quad FdV = d, \quad Fd\underline{a} = \underline{a}^{p-1}d\underline{a}, \\ F(xy) = F(x)F(y), \quad xV(y) = V(F(x)y). \end{gathered} \tag{1.2.1}$$

Moreover, $W_\bullet\Omega^*_A$ is *universal* with these properties, cf. [I, I.1.3 and I.2.17]. The de Rham–Witt complex of A is defined as the limit

$$W\Omega^*_A = \varprojlim_R W_n\Omega^*_A. \tag{1.2.2}$$

We note a few easy facts. The canonical map

$$\Omega^*_{W_n(A)} \to W_n\Omega^*_A$$

is a surjection for any $n \geqslant 1$, and an isomorphism when $n=1$. Similarly, the maps R: $W_n\Omega^*_A \to W_{n-1}\Omega^*_A$ are surjective for all $n \geqslant 1$. Finally, consider the composite

$$\mathbf{F}\colon W_n\Omega^i_A \xrightarrow{\varphi} W_n\Omega^i_A \xrightarrow{R} W_{n-1}\Omega^i_A,$$

where φ is induced from the Frobenius endomorphism of A. One has $\mathbf{F} = p^iF$. In §2 below we recall the concrete description of the de Rham–Witt complex of a polynomial algebra.

1.3. We shall recall some facts about the topological Hochschild spectrum $T(A)$ and refer the reader to [HM1] where this material is treated in detail, see also [Ma]. We begin by recalling the notion of an equivariant spectrum following [LMS].

Let G be a compact Lie group and let $\mathcal{U}$ be a complete G-universe, that is, a countably-dimensional G-representation with a G-invariant inner product such that every finite-dimensional G-representation is isomorphic to a subrepresentation of $\mathcal{U}$. A *G-prespectrum* t indexed on $\mathcal{U}$ is a collection of G-spaces $\{t(V)\}$, one for every finite-dimensional subrepresentation $V \subset \mathcal{U}$, together with a transitive system of G-maps

$$\sigma\colon t(V) \to \Omega^{W-V}t(W),$$

where $W-V$ denotes the orthogonal complement of $V \subset W$. A map $t \to t'$ of G-prespectra is a collection $\{t(V) \to t'(V)\}$ of G-maps compatible with the structure maps. A *G-spectrum* T indexed on $\mathcal{U}$ is a G-prespectrum where the structure maps are homeomorphisms. We denote by $G\mathcal{PU}$ the category of G-prespectra indexed on $\mathcal{U}$ and by $G\mathcal{SU}$

the full subcategory of G-spectra. The forgetful functor $l\colon G\mathcal{SU}\to G\mathcal{PU}$ has a left adjoint $L\colon G\mathcal{PU}\to G\mathcal{SU}$, called spectrification, given by

$$Lt(V)=\varinjlim_{W\supset V}\Omega^{W-V}t(W),$$

provided that the structure maps are closed inclusions; this can always be arranged.

The smash product of a pointed G-space X and a G-prespectrum t is the G-prespectrum $\{X\wedge t(V)\}$ with the obvious structure maps. For a G-spectrum T, however, we need to spectrify to get a G-spectrum back: we define $X\wedge T=L(X\wedge lT)$. Dually, we define function G-prespectra $F(X;t)$ as $\{F(X,t(V))\}$. This construction preserves G-spectra. We shall often make use of the following duality result: Let $\mathrm{Ad}(G)$ be the adjoint representation and let $H\subset G$ be a *finite* subgroup. Then there is a natural G-equivalence

$$\Sigma^{\mathrm{Ad}(G)}F(G/H_+,T)\simeq_G T\wedge G/H_+, \tag{1.3.1}$$

valid for any G-spectrum T indexed on $\mathcal{U}$; see [LMS, p. 89], or [HM1, §8.1]. Similarly, colimits and homotopy colimits of G-prespectra are defined spacewise, but for G-spectra we again have to spectrify. Limits and homotopy limits, on the other hand, are formed spacewise both for G-prespectra and G-spectra.

An important difference from the non-equivariant theory is the role of the universe. In the discussion above we can replace $\mathcal{U}$ by the trivial universe $\mathcal{U}^G$ and get categories $G\mathcal{PU}^G$ and $G\mathcal{SU}^G$ of G-prespectra and G-spectra, respectively, indexed on $\mathcal{U}^G$. We also call a G-spectrum D indexed on $\mathcal{U}^G$ a naive G-spectrum or a spectrum with a G-action. The forgetful functors $j^*\colon G\mathcal{PU}\to G\mathcal{PU}^G$ and $j^*\colon G\mathcal{SU}\to G\mathcal{SU}^G$ induced from the inclusion $j\colon\mathcal{U}^G\to\mathcal{U}$ have left adjoints $j_*\colon G\mathcal{PU}^G\to G\mathcal{PU}$ and $j_*\colon G\mathcal{SU}^G\to G\mathcal{SU}$. If d is a G-prespectrum indexed on $\mathcal{U}^G$ and $V\subset\mathcal{U}$ is a finite-dimensional subrepresentation, then

$$(j_*d)(V)=S^{V-V^G}\wedge d(V^G),$$

with the evident structure maps, and for spectra, $j_*D=L(j_*lD)$. The unit $\eta\colon D\to j^*j_*D$ and counit $\varepsilon\colon j_*j^*T\to T$ are non-equivariant equivalences but usually not equivariant equivalences. We also note that evidently $j^*F(X,T)\cong F(X,j^*T)$ which by category theory implies that $X\wedge j_*D\cong j_*(X\wedge D)$. For a G-spectrum T indexed on $\mathcal{U}$, we call the spectrum j^*T with its G-action forgotten the underlying non-equivariant spectrum of T and denote it $|T|$. We shall often use the following isomorphism of spectra with a G-action:

$$\xi\colon |T|\wedge G_+\xrightarrow{\mathrm{id}\wedge\Delta}|T|\wedge G_+\wedge G_+\xrightarrow{\mu\wedge\mathrm{id}}j^*T\wedge G_+. \tag{1.3.2}$$

Finally, we recall from [LMS, p. 97], that if T is a G-spectrum indexed on $\mathcal{U}$ and E is a G-space which is *free* in the pointed sense, then the equivariant transfer induces an equivalence

$$\tau: j^*T\wedge_G E \to j^*(\Sigma^{-\operatorname{Ad}(G)}(T\wedge E))^G. \tag{1.3.3}$$

From now on, G will denote the circle group. Finite and hence cyclic subgroups of G are denoted by C or C_r if we want to specify the order. The topological Hochschild spectrum $T(A)$ is a G-spectrum indexed on a complete universe $\mathcal{U}$. We shall mostly be concerned with the associated naive G-spectrum $j^*T(A)$ which we write $\mathrm{TH}(A)$. The fact that $T(A)$ is indexed on a complete universe implies in particular that the obvious inclusion maps

$$F_r\colon \mathrm{TH}(A)^{C_{rs}} \to \mathrm{TH}(A)^{C_s} \tag{1.3.4}$$

are companioned by 'transfer' maps going in the opposite direction,

$$V_r\colon \mathrm{TH}(A)^{C_s} \to \mathrm{TH}(A)^{C_{rs}}. \tag{1.3.5}$$

We call these maps the rth *Frobenius* and *Verschiebung*, respectively.

In addition, $T(A)$ is a cyclotomic spectrum in the sense of [HM1, §2]. This implies in particular the existence of an extra map

$$R_r\colon \mathrm{TH}(A)^{C_{rs}} \to \mathrm{TH}(A)^{C_s},$$

called the rth *restriction*. It has the following equivariance property: Let $C\subset G$ be a subgroup of order r. The rth root defines an isomorphism of groups $\varrho_C\colon G\to G/C$, and we may view the G/C-spectrum $\mathrm{TH}(A)^C$ as a G-spectrum $\varrho_C^*\,\mathrm{TH}(A)^C$ via ϱ_C. Then R_r is a map of G-spectra

$$R_r\colon \varrho_{C_{rs}}^*\mathrm{TH}(A)^{C_{rs}} \to \varrho_{C_s}^*\mathrm{TH}(A)^{C_s}. \tag{1.3.6}$$

We also define G-spectra

$$\mathrm{TR}(A) = \underset{R}{\underleftarrow{\operatorname{holim}}}\, \varrho_C^*\,\mathrm{TH}(A)^C, \quad \mathrm{TR}(A;p) = \underset{R}{\underleftarrow{\operatorname{holim}}}\, \varrho_{C_{p^n}}^*\mathrm{TH}(A)^{C_{p^n}} \tag{1.3.7}$$

and note that F_r and V_r induce (non-equivariant) selfmaps of these which we again denote F_r and V_r. Finally, $\mathrm{TH}(A)$ is a commutative ring spectrum in a strong sense, see [HM1, Proposition 2.7.1]. In particular, for any $C\subset G$, the homotopy groups $\pi_*\mathrm{TH}(A)^C$ form a graded commutative ring. The following relations are proved in op. cit., Lemma 3.3,

$$\begin{aligned}
&(1)\ \ R_r(xy) = R_r(x)R_r(y),\\
&(2)\ \ F_r(xy) = F_r(x)F_r(y),\ \ V_r(F_r(x)y) = xV_r(y),\\
&(3)\ \ F_rV_r = r,\ \ V_rF_r = V_r(1),\\
&(4)\ \ F_rV_s = V_sF_r,\ \text{ if } (r,s)=1,\\
&(5)\ \ R_rF_s = F_sR_r,\ \ R_rV_s = V_sR_r.
\end{aligned} \tag{1.3.8}$$

We will mainly be interested in the fixed point spectra of $\mathrm{TH}(A)$ for the cyclic p-groups. When there is no danger of confusion we shall write R, F and V instead of F_p, R_p and V_p. We shall also write $\pi_* T(A)^C$ instead of $\pi_* \mathrm{TH}(A)^C$. In [HM1, Theorem 3.3], we proved that

$$\pi_0 T(A)^{C_{p^n}} \cong W_{n+1}(A), \tag{1.3.9}$$

such that $\pi_0 R$, $\pi_0 F$ and $\pi_0 V$ corresponds to the restriction, Frobenius and Verschiebung of Witt vectors, respectively. We also recall from op. cit., Theorem 2.2, the cofibration sequence of non-equivariant spectra

$$\mathrm{TH}(A)_{hC_{p^n}} \xrightarrow{N} \mathrm{TH}(A)^{C_{p^n}} \xrightarrow{R} \mathrm{TH}(A)^{C_{p^{n-1}}}, \tag{1.3.10}$$

where $\mathrm{TH}(A)_{hC_{p^n}} = \mathrm{TH}(A) \wedge_{C_{p^n}} EC_{p^n +}$ is the homotopy orbit spectrum. The spectra in this sequence are all $\mathrm{TH}(A)^{C_{p^n}}$-module spectra, and therefore by (1.3.8), the associated homotopy long exact sequence is a sequence of $W_{n+1}(A)$-modules. The skeleton filtration of EC_{p^n} gives rise to a first quadrant homology type spectral sequence

$$E^2 = H_*(C_{p^n}; (F^n)^*(\pi_* T(A))) \implies \pi_* \mathrm{TH}(A)_{hC_{p^n}}. \tag{1.3.11}$$

This is a spectral sequence of $W_{n+1}(A)$-modules in the sense that the E^r-terms are bigraded $W_{n+1}(A)$-modules and the differentials $W_{n+1}(A)$-linear. The homotopy groups $\pi_* T(A)$ are A-modules which we view as $W_{n+1}(A)$-modules $(F^n)^*(\pi_* T(A))$ via the iterated Frobenius $F^n \colon W_{n+1}(A) \to A$. This specifies the $W_{n+1}(A)$-module structure on the E^2-term.

1.4. We shall define a differential on $\pi_* T(A)$, and more generally, on $\pi_* D$ for any spectrum D with a G-action.

We let T be a G-spectrum indexed on $\mathcal{U}$ and consider the Tate construction of [GM],

$$\widehat{\mathbf{H}}(G;T) = [\widetilde{E}G \wedge F(EG_+, T)]^G,$$

where $\widetilde{E}G$ is the unreduced suspension of EG and the smash product on the right is formed in $GS\mathcal{U}$. A map $T \to T'$ of G-spectra which is a non-equivariant equivalence induces an equivalence of Tate constructions. In particular, so does the counit $\varepsilon \colon j_* j^* T \to T$. Let $\mathbf{C}$ be the standard G-representation. We can take EG to be $S(\mathbf{C}^\infty) = \bigcup S(\mathbf{C}^n)$, where $S(V)$ is the unit sphere in V, and then $\widetilde{E}G$ is

$$S^{\mathbf{C}^\infty} = \bigcup_{n \geqslant 1} S^{\mathbf{C}^n},$$

the union of the one-point compactifications. We recall Greenlees' 'filtration' of $\widetilde{E}G$,

$$\dots \to S^{-\mathbf{C}^2} \xrightarrow{i_{-1}} S^{-\mathbf{C}} \xrightarrow{i_0} S^0 \xrightarrow{i_1} S^{\mathbf{C}} \xrightarrow{i_2} S^{\mathbf{C}^2} \to \dots \to S^{\mathbf{C}^\infty}.$$

When we smash with the G-spectrum $F(EG_+,T)$ this makes perfect sense and we obtain a whole plane spectral sequence

$$\widehat{E}^1 = P[s,s^{-1}]\otimes\pi_*(T) \implies \pi_*(\widehat{\mathbf{H}}(G;T)); \quad \deg s = (-1,-1).$$

If T is a G-homotopy associative and commutative G-ring spectrum then this is a multiplicative spectral sequence in the sense that $(\widehat{E}^r, d^r)$ is a differential bigraded algebra, $r\geqslant 1$. This spectral sequence (indexed slightly differently) is treated in great detail in [BM] and [HM2].

Recall that the obvious collapse maps give an isomorphism

$$\pi_*^S(S^1_+)\cong\pi_*^S(S^1)\oplus\pi_*^S(S^0), \tag{1.4.1}$$

and let $\sigma,\eta\in\pi_1^S(G_+)$ denote the generators which reduce to $(\mathrm{id},0)$ and $(0,\eta)$, respectively.

LEMMA 1.4.2. *The differential $d^1_{n,m}\colon \widehat{E}^1_{n,m}\to\widehat{E}^1_{n-1,m}$ is given by the composition*

$$\pi_{m-n}(T)\xrightarrow{\sigma+n\eta}\pi_{m-n+1}(G_+\wedge j^*T)\xrightarrow{\mu}\pi_{m-n+1}(T),$$

where the first map is exterior multiplication by $\sigma+n\eta$ and the second map is induced by the action map.

Proof. We first show that $\widehat{E}^1$ is as claimed. The cofiber of $i_1\colon S^0\to S^{\mathbf{C}}$ may be identified with ΣG_+. Indeed, for any representation $S^V\cong S(\mathbf{R}\oplus V)$ and we have the cofibration sequence

$$S(\mathbf{R})\times D(\mathbf{C})\hookrightarrow S(\mathbf{R})\times D(\mathbf{C})\cup D(\mathbf{R})\times S(\mathbf{C})\to(D(\mathbf{R})\times S(\mathbf{C}))/(S(\mathbf{R})\times S(\mathbf{C})).$$

In general, $i_n=\mathrm{id}_{S^{(n-1)\mathbf{C}}}\wedge i_1$ so the cofiber is $\Sigma G_+\wedge S^{(n-1)\mathbf{C}}$. The map which collapses EG to a point induces an equivariant equivalence

$$\Sigma G_+\wedge S^{(n-1)\mathbf{C}}\wedge F(EG_+,T)\simeq_G\Sigma G_+\wedge S^{(n-1)\mathbf{C}}\wedge T,$$

and hence an equivalence of G-fixed spectra. The transfer equivalence of (1.3.3) gives in the case at hand an equivalence

$$\tau\colon\Sigma^2 G_+\wedge_G j^*(S^{(n-1)\mathbf{C}}\wedge T)\to j^*[\Sigma G_+\wedge S^{(n-1)\mathbf{C}}\wedge T]^G,$$

and finally, the left-hand side is isomorphic to $|\Sigma^2S^{(n-1)\mathbf{C}}\wedge T|=\Sigma^{2n}|T|$, with the isomorphism given by the action map.

In order to evaluate the d^1-differential we consider the following diagram of non-equivariant spectra:

$$\begin{tikzcd}
j^*[\Sigma G_+\wedge S^{(n-1)\mathbf{C}}\wedge T]^G \arrow[r, "\partial"] & j^*[\Sigma S^{(n-1)\mathbf{C}}\wedge T]^G \arrow[r, "\mathrm{pr}"] & j^*[\Sigma^2 G_+\wedge S^{(n-2)\mathbf{C}}\wedge T]^G \\
\Sigma^2 G_+\wedge_G S^{(n-1)\mathbf{C}}\wedge j^*T \arrow[u, "\simeq", "\tau"'] \arrow[r, "\partial"] \arrow[d, "\cong"', "\bar\mu"] & (\Sigma^2 S^{(n-1)\mathbf{C}}\wedge j^*T)/G \arrow[r, "\mathrm{pr}"] & \Sigma^3 G_+\wedge_G S^{(n-2)\mathbf{C}}\wedge j^*T \arrow[u, "\cong", "\tau"'] \arrow[d, "\cong"', "\bar\mu"] \\
\Sigma^2|S^{(n-1)\mathbf{C}}\wedge T| & & \Sigma^3|S^{(n-2)\mathbf{C}}\wedge T|.
\end{tikzcd}$$

If we apply $\pi_{n+m}(-)$, then $d^1_{n,m}$ is the composite of the maps from the lower left-hand corner to the lower right-hand corner. The unit map $S^0\to G_+$ induces a map of non-equivariant spectra

$$\iota\colon \Sigma^2|S^{(n-1)\mathbf{C}}\wedge T|\to \Sigma^2 G_+\wedge S^{(n-1)\mathbf{C}}\wedge j^*T,$$

which composed with the projection onto the orbit spectrum is the inverse of $\bar\mu$. Moreover, the composite

$$\Sigma^2|S^{(n-1)\mathbf{C}}\wedge T|\xrightarrow{\iota}\Sigma^2 G_+\wedge S^{(n-1)\mathbf{C}}\wedge j^*T\xrightarrow{\partial}\Sigma^2 S^{(n-1)\mathbf{C}}\wedge j^*T\xrightarrow{\mathrm{pr}}\Sigma^3 G_+\wedge S^{(n-2)\mathbf{C}}\wedge j^*T$$

represents exterior multiplication by $\sigma\in\pi_1^S(G_+)$. We may write the action map μ as the composite

$$\begin{aligned}G_+\wedge S^{(n-2)\mathbf{C}}\wedge j^*T &\xrightarrow{\xi\wedge 1} G_+\wedge S^{(n-2)\mathbf{C}}\wedge j^*T\xrightarrow{\mathrm{tw}\wedge 1} S^{(n-2)\mathbf{C}}\wedge G_+\wedge j^*T\\ &\xrightarrow{1\wedge\mu} S^{(n-2)\mathbf{C}}\wedge j^*T,\end{aligned}$$

where ξ is given by

$$\xi\colon G_+\wedge S^{(n-2)\mathbf{C}}\xrightarrow{\Delta\wedge 1} G_+\wedge G_+\wedge S^{(n-2)\mathbf{C}}\xrightarrow{1\wedge\mu} G_+\wedge S^{(n-2)\mathbf{C}}.$$

We claim that under the isomorphism in (1.4.1),

$$\xi=\begin{pmatrix}1 & 0\\ (n-2)\eta & 1\end{pmatrix},$$

where the matrix multiplies from the right. It suffices to consider the case $n=3$ where the representation is $\mathbf{C}$. For the case $n>3$ follows by composition and the case $n<3$ follows by smashing with $S^{N\mathbf{C}}$ for some $N>3-n$. Moreover, when $n=3$ we may identify ξ with the composite

$$D(\mathbf{C})\times S(\mathbf{C})\cup S(\mathbf{C})\times D(\mathbf{C})\xrightarrow{\mathrm{pr}}(S(\mathbf{C})\times D(\mathbf{C}))/(S(\mathbf{C})\times S(\mathbf{C}))\xrightarrow{\mu} D(\mathbf{C})/S(\mathbf{C}),$$

and finally, it is well known that this map represents η. $\square$

The lemma leaves us two choices of differentials on $\pi_*(T)$. For the identification $\pi_*(T)\cong\widehat{E}^1_{n,n+*}$ gives, in general, different differentials for n even and n odd. If D is a spectrum with a G-action, we can consider the spectral sequence for $T=j_*D$.

Definition 1.4.3. For a spectrum D with a G-action we let δ be the degree-one operator on $\pi_*(D)$ given by

$$\delta\colon \pi_n(D) \xrightarrow{\sigma} \pi_{n+1}(G_+\wedge D) \xrightarrow{\mu} \pi_{n+1}(D),$$

where the first map is exterior multiplication by $\sigma\in\pi_1^S(G_+)$ and the second map is induced by the G-action.

One might also want to replace σ by η. However, the operator which results is just exterior multiplication by $\eta\in\pi_1^S(S^0)$. We note the formula

$$\delta\delta=\eta\delta=\delta\eta. \tag{1.4.4}$$

In particular, the two choices of differentials coincide and equal δ when η acts trivially on $\pi_*(D)$. For example, this is the case if multiplication by 2 on $\pi_*(D)$ is an isomorphism or if the underlying non-equivariant spectrum of D is a generalized Eilenberg–MacLane spectrum, e.g. $D=\mathrm{TH}(A)$.

For later use we record the effect of the isomorphism ξ of (1.3.2) on homotopy groups.

LEMMA 1.4.5. *Under the decomposition* $\pi_*(T\wedge G_+)\cong\pi_*(T)\oplus\pi_{*-1}(T)$ *induced from* (1.4.1),

$$\xi=\begin{pmatrix}\mathrm{id} & 0\\ \delta & \mathrm{id}\end{pmatrix},$$

where $\delta\colon\pi_{*-1}(T)\to\pi_*(T)$ *is the map from Definition* 1.4.3 *and the matrix multiplies from the right.*

Proof. It is enough to prove that $\xi_{22}\colon\pi_{*-1}(T)\to\pi_{*-1}(T)$ is the identity. Since the unit and counit of the adjunction $j_*\dashv j^*$ are non-equivariant equivalences, we may instead study the following map of G-spectra indexed on $\mathcal{U}$:

$$j_*|T|\wedge G_+ \xrightarrow{\mathrm{id}\wedge\Delta} j_*|T|\wedge G_+\wedge G_+ \xrightarrow{\mu} j_*(j^*T)\wedge G_+ \xrightarrow{\varepsilon\wedge 1} T\wedge G.$$

We consider the diagram

$$\begin{tikzcd}
j^*(j_*|T|\wedge G_+)^G \arrow[r, "\mathrm{id}\wedge\Delta"] & j^*(j_*|T|\wedge G_+\wedge G_+)^G \arrow[r, "(\varepsilon\wedge\mathrm{id})\mu"] & j^*(T\wedge G_+)^G \\
\Sigma|T| \arrow[u, "\simeq"] \arrow[r, "(\mathrm{id},1)"] & \Sigma|T|\wedge G_+ \arrow[u, "\simeq"] \arrow[r, "\mu"] & \Sigma|T|, \arrow[u, "\simeq"]
\end{tikzcd}$$

in which the vertical maps are the transfer equivalences of (1.3.3). Both squares are homotopy commutative by naturality of the transfer, and finally, the composition of the maps in the bottom row is the identity. $\square$

We end this section with a comparison with ordinary Hochschild homology. Recall from [HM1, Proposition 2.4] that the 0th space of $\mathrm{TH}(A)$ is naturally equivalent to Bökstedt's topological Hochschild space $\mathrm{THH}(A)$. This is the realization of a cyclic space with k-simplices

$$\mathrm{THH}(A)_k = \underset{I^{k+1}}{\underrightarrow{\mathrm{holim}}}\, F(S^{i_0}\wedge\ldots\wedge S^{i_k}, A(S^{i_0})\wedge\ldots\wedge A(S^{i_k})),$$

where $A(S^i)$ is an Eilenberg–MacLane space for A concentrated in degree i. The set of components of $\mathrm{THH}(A)_k$ is equal to the iterated tensor product $A^{\otimes(k+1)}$, that is, the k-simplices in the cyclic abelian group $\mathrm{HH}(A)_\bullet$ which defines ordinary Hochschild homology. Moreover, the cyclic structure maps are such that we get a map of cyclic spaces $l_\bullet\colon \mathrm{THH}(A)_\bullet \to \mathrm{HH}(A)_\bullet$ and hence a G-equivariant map of their realizations,

$$l\colon \mathrm{THH}(A)\to\mathrm{HH}(A),$$

called *linearization*. It induces an isomorphism of $\pi_n(-)$ for $n\leqslant 1$.

PROPOSITION 1.4.6. *There is a commutative diagram*

$$\begin{array}{ccc}
\pi_n(T(A)) & \xrightarrow{\ l\ } & \mathrm{HH}_n(A) \\
\big\downarrow{\scriptstyle \delta} & & \big\downarrow{\scriptstyle B} \\
\pi_{n+1}(T(A)) & \xrightarrow{\ l\ } & \mathrm{HH}_{n+1}(A),
\end{array}$$

where l is linearization and B is Connes' operator.

Proof. Recall that a simplicial abelian group $X_\bullet$ determines a (generalized Eilenberg–MacLane) spectrum $\mathbf{X}$ whose 0th space is $|X_\bullet|$. A model for the nth deloop is provided by the realization of the n-fold iterated nerve

$$\mathbf{X}(n) = |N\ldots NX_{\bullet,\ldots,\bullet}|,$$

where the realization may either be formed by realizing the diagonal simplicial set or inductively realizing the simplicial directions one after one; the spaces which result are canonically homeomorphic. From this description it is clear that if $X_\bullet$ is a cyclic abelian group then each $\mathbf{X}(n)$ is a G-space and the spectrum structure maps are G-equivariant. In particular, the G-action gives a map of spectra $\mu\colon G_+\wedge\mathbf{X}\to\mathbf{X}$. We evaluate the induced map on homotopy groups

$$\mu_*\colon H_*(G;\mathbf{Z})\otimes\pi_*(\mathbf{X})\to\pi_*(\mathbf{X}).$$

Let $\mathbf{C}_{\bullet}$ be the standard cyclic model for the circle with n-simplices the cyclic group $\mathbf{C}_n=\{1,\tau_n,\dots,\tau_n^n\}$ and structure maps

$$d_i\tau_n^s=\begin{cases}\tau_{n-1}^s, & \text{if } i+s\leqslant n,\\ \tau_{n-1}^{s-1}, & \text{if } i+s>n,\end{cases}$$
$$s_i\tau_n^s=\begin{cases}\tau_{n+1}^s, & \text{if } i+s\leqslant n,\\ \tau_{n+1}^{s+1}, & \text{if } i+s>n,\end{cases}$$
$$t_n\tau_n^s=\tau_n^{s-1}.$$

Then we may identify the 0th space of the smash product spectrum $G_+\wedge\mathbf{X}$ with the realization of the simplicial abelian group $\mathbf{Z}[\mathbf{C}_{\bullet}]\otimes X_{\bullet}$. We write X_* for the chain complex associated with $X_{\bullet}$ and recall that the Eilenberg–MacLane shuffle map provides an explicit chain homotopy equivalence

$$\theta\colon \mathbf{Z}[\mathbf{C}_{\bullet}]_*\otimes X_*\xrightarrow{\simeq}(\mathbf{Z}[\mathbf{C}_{\bullet}]\otimes X_{\bullet})_*.$$

The forgetful functor from cyclic abelian groups to simplicial abelian groups has a left adjoint which assigns to a simplicial abelian group $Y_{\bullet}$ the cyclic abelian group $FY_{\bullet}$ with n-simplices

$$FY_n=\mathbf{Z}[\mathbf{C}_n]\otimes Y_n$$

and cyclic structure maps

$$d_i(\tau_n^s\otimes y)=\begin{cases}\tau_{n-1}^s\otimes d_{i+s}y, & \text{if } i+s\leqslant n,\\ \tau_{n-1}^{s-1}\otimes d_{i+s}y, & \text{if } i+s>n,\end{cases}$$
$$s_i(\tau_n^s\otimes y)=\begin{cases}\tau_{n+1}^s\otimes s_{i+s}y, & \text{if } i+s\leqslant n,\\ \tau_{n+1}^{s+1}\otimes s_{i+s}y, & \text{if } i+s>n,\end{cases}$$
$$t_n(\tau_n^s\otimes y)=\tau_n^{s-1}\otimes y.$$

Here all indices are to be understood as the principal representatives modulo $n+1$. Although $FY_{\bullet}$ and $\mathbf{Z}[\mathbf{C}_{\bullet}]\otimes Y_{\bullet}$ have the same n-simplices they are not isomorphic as simplicial abelian groups. However, their associated chain complexes are canonically isomorphic, the isomorphism given by

$$h\colon FY_*\to(\mathbf{Z}[\mathbf{C}_{\bullet}]\otimes Y_{\bullet})_*,\quad h(\tau_n^s\otimes y)=(-1)^{ns}\tau_n^{-s}\otimes y.$$

Finally, if $X_{\bullet}$ is a cyclic abelian group, then μ_* is given by the composite

$$\mu_*\colon \mathbf{Z}[\mathbf{C}_{\bullet}]_*\otimes X_*\xrightarrow{h\otimes 1}\mathbf{Z}[\mathbf{C}_{\bullet}]_*\otimes X_*\xrightarrow{\theta}(\mathbf{Z}[\mathbf{C}_{\bullet}]\otimes X_{\bullet})_*\xrightarrow{h^{-1}}FX_*\xrightarrow{\varepsilon}X_*,$$

where $\varepsilon\colon FX_\bullet \to X_\bullet$ is the counit of the adjunction.

When $X_\bullet = \mathrm{HH}(A)_\bullet$ is the cyclic abelian group which defines Hochschild homology of A one readily calculates

$$\mu_*(\tau_1 \otimes (a_0 \otimes \ldots \otimes a_n)) = \sum_{i=0}^{n} (-1)^{ni} 1 \otimes a_i \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}.$$

This is the usual formula for Connes' B-operator, cf. [L]. □

We note that by [HKR], $(\mathrm{HH}_*(A/k), B) \cong (\Omega^*_{A/k}, d)$ when A is a smooth k-algebra.

1.5. Suppose that A is an $\mathbf{F}_p$-algebra. We may take $D = \varrho^*_{C_{p^n}} \mathrm{TH}(A)^{C_{p^n}}$ in Definition 1.4.3 and get a map

$$\delta\colon \pi_* T(A)^{C_{p^n}} \to \pi_{*+1} T(A)^{C_{p^n}}.$$

We know from [HM1, Proposition 5.4] that $\mathrm{TR}(\mathbf{F}_p; p) = H\mathbf{Z}_p$, the Eilenberg–MacLane spectrum for the p-adic integers concentrated in degree zero. Hence $\mathrm{TH}(A)^{C_{p^n}}$ is also an Eilenberg–MacLane spectrum. For it is a module spectrum over $\mathrm{TR}(\mathbf{F}_p; p)$ and any module spectrum over an Eilenberg–MacLane spectrum is again Eilenberg–MacLane. It follows that $\delta \circ \delta = 0$ such that δ makes $\pi_* T(A)^{C_{p^n}}$ a graded commutative DGA over $\mathbf{Z}$. Moreover, (1.3.6) shows that the restriction map R is a map of DGA's. We get a new limit system of DGA's over $\mathbf{Z}$,

$$\begin{tikzcd}
\vdots \arrow[d, "R"] & \vdots \arrow[d, "R"] & \vdots \arrow[d, "R"] & \\
\pi_0 T(A)^{C_{p^2}} \arrow[r, "\delta"] \arrow[d, "R"] & \pi_1 T(A)^{C_{p^2}} \arrow[r, "\delta"] \arrow[d, "R"] & \pi_2 T(A)^{C_{p^2}} \arrow[r, "\delta"] \arrow[d, "R"] & \cdots \\
\pi_0 T(A)^{C_p} \arrow[r, "\delta"] \arrow[d, "R"] & \pi_1 T(A)^{C_p} \arrow[r, "\delta"] \arrow[d, "R"] & \pi_2 T(A)^{C_p} \arrow[r, "\delta"] \arrow[d, "R"] & \cdots \\
\pi_0 T(A) \arrow[r, "\delta"] & \pi_1 T(A) \arrow[r, "\delta"] & \pi_2 T(A) \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

and $\pi_0 T(A)^{C_{p^{n-1}}} = W_n(A)$. It remains to prove two of the relations in (1.2.1).

LEMMA 1.5.1. *Let A be a ring. Then $F_r \delta V_r = \delta + (r-1)\eta$.*

Proof. We consider the basic case $s=1$; the general case is similar. Let us write $T = T(A)$ and $C = C_r$. We have an isomorphism of spectra with a G/C-action

$$j^* T^C \cong F(G/C_+, j^* T)^G,$$

where the (left) G/C-action on the spectrum on the right-hand side is induced from the (right) action by G/C on itself by multiplication. Moreover, the duality equivalence of (1.3.1) gives

$$F(G/C_+, j^*T)^G \simeq \Sigma^{-1} j^*(T\wedge G/C_+)^G,$$

and one verifies readily that the duality map is G/C-equivariant, when we give the spectrum on the left the (left) G/C-action induced from the action by G/C on itself by (left) multiplication. Let ξ denote the composition

$$G/C_+\wedge j^*(T\wedge G_+)^G \xrightarrow{\mathrm{pr}} G/C_+\wedge j^*(T\wedge G/C)^G \xrightarrow{\mu} j^*(T\wedge G/C_+)^G \xrightarrow{\mathrm{pr}^!} j^*(T\wedge G_+)^G,$$

where $\mathrm{pr}^!$ is the equivariant transfer associated with the projection $\mathrm{pr}\colon G\to G/C$. It follows from [HM1, Lemma 8.1] that $F_r\delta V_r$ is equal to

$$\begin{aligned}\pi_*((T\wedge G_+)^G) &\xrightarrow{\sigma} \pi_{*+1}(G_+\wedge(T\wedge G_+)^G) \xrightarrow[\cong]{\varrho_C\wedge 1} \pi_{*+1}(G/C_+\wedge(T\wedge G_+)^G)\\ &\xrightarrow{\xi} \pi_{*+1}((T\wedge G_+)^G).\end{aligned}$$

We claim that ξ is homotopic to the composition

$$G/C_+\wedge j^*(T\wedge G_+)^G \xrightarrow{\tau\wedge 1} G_+\wedge j^*(T\wedge G_+)^G \xrightarrow{\mu} j^*(T\wedge G_+)^G,$$

where $\tau\colon \Sigma^\infty_+ G/C\to\Sigma^\infty_+ G$ is the ordinary non-equivariant transfer of the r-fold covering $G\to G/C$. Granting this for the moment, the fact that under (1.4.1),

$$\tau = \begin{pmatrix} 1 & 0\\ (r-1)\eta & r\end{pmatrix},$$

shows that $F_r dV_r = d+(r-1)\eta$ as stated. To prove the claim we consider the two subgroups of $G\times G$ given by

$$C_1 = \{(x,1)\mid x\in C\},\quad \Delta = \{(x,x^{-1})\mid x\in C\}$$

and note that $\Delta\cap C_1 = 1$ and $\Delta C_1 = C\times C$. Let $|X|$ denote the underlying non-equivariant space of the G-space X. Then we have a G-homotopy commutative diagram of equivariant suspension spectra

$$\begin{tikzcd}
\Sigma^\infty_+(|G|\times G)/C_1 \arrow[r, "\pi^{C\times C}_{C_1}"] \arrow[d, "(\pi_1^{C_1})^!"] & \Sigma^\infty_+(|G|\times G)/(C\times C) \arrow[r, "\mu"] \arrow[d, "(\pi_\Delta^{C\times C})^!"] & \Sigma^\infty_+ G/C \arrow[d, "(\pi_1^C)^!"] \\
\Sigma^\infty_+|G|\times G \arrow[r, "\pi_1^\Delta"] & \Sigma^\infty_+(|G|\times G)/\Delta \arrow[r, "\bar\mu"] & \Sigma^\infty_+ G.
\end{tikzcd} \tag{1.5.2}$$

For in the diagram

$$\begin{tikzcd}
(|G|\times G)/C_1 \arrow[r, "\pi_{C_1}^{C\times C}"] & (|G|\times G)/(C\times C) \arrow[r, "\mu"] & G/C \\
|G|\times G \arrow[u, "\pi_1^{C_1}"] \arrow[r, "\pi_1^{\Delta}"] & (|G|\times G)/\Delta \arrow[u, "\pi_{\Delta}^{C\times C}"] \arrow[r, "\bar{\mu}"] & G \arrow[u, "\pi_1^{C}"]
\end{tikzcd}$$

both squares are pull backs. Moreover, the equivariant transfer $\tau_1^{C_1}$ is G-homotopic to

$$\tau\wedge 1: \Sigma_+^\infty|G/C|\wedge\Sigma_+^\infty G\to\Sigma_+^\infty|G|\wedge\Sigma_+^\infty G$$

by [LMS, IV.5.3]. Finally, we note that in (1.5.2), $\bar{\mu}\circ\pi_1^{\Delta}$ is equal to the multiplication $\mu: |G|\times G\to G$, and now the claim follows from the commutativity of (1.5.2) upon taking G-fixed points. □

We shall need some facts about the multiplicative structure of topological Hochschild which we now recall, see also [HM1, 2.7]. There are G-spaces $\mathrm{THH}(A^{(n)};S^n)$, $n\geqslant 0$, together with a transitive system of G-equivariant maps

$$\mu_{m,n}: \mathrm{THH}(A^{(m)};S^m)\wedge\mathrm{THH}(A^{(n)};S^n)\to\mathrm{THH}(A^{(m+n)};S^{m+n}), \qquad (1.5.3)$$

where G acts diagonally on the left-hand side. When $n\geqslant 1$ there is a natural chain of equivalences

$$\begin{aligned}\mathrm{TH}(A)(0)\leftarrow\mathrm{THH}(A)\to\Omega\,\mathrm{THH}(A^{(1)};S^1)\to\underset{\overrightarrow{I}}{\mathrm{holim}}\,\Omega^m\,\mathrm{THH}(A^{(m)};S^m)\\ \leftarrow\Omega^n\,\mathrm{THH}(A^{(n)};S^n),\end{aligned}$$

where $\mathrm{TH}(A)(0)$ denotes the 0th space of $\mathrm{TH}(A)$. The last two maps are equivalences by the approximation lemma, see [Böl, 1.6] or [Ma, Lemma 2.3.7]. More generally, the induced map of C-fixed sets is an equivalence for every finite subgroup $C\subset G$. In particular, we have a canonical isomorphism of groups

$$\pi_*T(A)^C\cong\pi_*(\Omega^n\,\mathrm{THH}(A^{(n)};S^n)^C), \quad n\geqslant 1,$$

and under this isomorphism the maps $\mu_{*,*}$ make $\pi_*T(A)^C$ a graded commutative ring. When $n=0$ we have

$$\mathrm{THH}(A^{(0)};S^0)=|N_\wedge^{\mathrm{cy}}(A)_\bullet|.$$

The right-hand side is the cyclic bar construction of A considered a pointed monoid under multiplication with $0\in A$ as basepoint, cf. [HM1, §7], and §3.1 below. It is a commutative topological monoid under the product $\mu_{0,0}$. Moreover, there is a canonical G-equivariant map

$$\iota_n\colon \mathrm{THH}(A^{(0)}; S^0) \to \Omega^n\,\mathrm{THH}(A^{(n)}; S^n),$$

for every $n \geqslant 0$, and the diagram

$$\begin{tikzcd}
{|N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}| \wedge |N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}|} \arrow[r, "\iota_m \wedge \iota_m"] \arrow[d, "\mu_{0,0}"] & \Omega\,\mathrm{THH}(A^{(m)}; S^m) \wedge \Omega^n\,\mathrm{THH}(A^{(n)}; S^n) \arrow[d, "\mu_{m,n}"] \\
{|N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}|} \arrow[r, "\iota_{m+n}"] & \Omega^{m+n}\,\mathrm{THH}(A^{(m+n)}; S^{m+n})
\end{tikzcd}$$

commutes. In particular, we obtain a multiplicative map on the level of homotopy groups

$$\iota\colon \pi_*(|N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}|^C) \to \pi_* T(A)^C. \tag{1.5.4}$$

Next, recall from [HM1, §7] that there are equivariant homeomorphisms

$$|N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}| \xrightarrow{\Delta_C} |(\mathrm{sd}_C N^{\mathrm{cy}}_{\wedge}(A)_{\bullet})^C| \xrightarrow{D} \varrho_C^* |N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}|^C,$$

where, we remember, $\varrho_C\colon G \to G/C$ is the root isomorphism. They are multiplicative by naturality and induce multiplicative isomorphisms on homotopy groups. Composing with (1.5.4) we obtain a multiplicative map

$$\omega\colon \pi_*(|N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}|) \to \pi_* T(A)^C. \tag{1.5.5}$$

When $*=0$ this is the Teichmüller map $A \to \mathbf{W}_{\langle r\rangle}(A)$, by [HM1, Addendum 3.3].

LEMMA 1.5.6. *Suppose that A is a ring. Then $F_r\delta\underline{x} = \underline{x}^{r-1} d\underline{x}$ for all $x \in A$.*

Proof. We again restrict ourselves to the case $s=1$ leaving the general case to the reader. It suffices to prove that the diagram

$$\begin{tikzcd}
G_+ \wedge N^{\mathrm{cy}}_{\wedge}(A)_0 \arrow[rr, "D \circ \Delta_C"] \arrow[d, "\mathrm{incl}"] & & G_+ \wedge |N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}|^C \arrow[r, "\varrho_C \wedge \mathrm{id}"] & G/C_+ \wedge |N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}|^C \arrow[d, "\mu"] \\
G_+ \wedge |N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}| \arrow[d, "{(P_{r-1}, \mathrm{id})}"] & & & {|N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}|^C} \arrow[d, "F"] \\
G_+ \wedge |N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}| \wedge |N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}| \arrow[r, "\mu \wedge 1"] & {|N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}| \wedge |N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}|} \arrow[rr, "\mu_{0,0}"] & & {|N^{\mathrm{cy}}_{\wedge}(A)_{\bullet}|}
\end{tikzcd} \tag{1.5.7}$$

is homotopy commutative. Here μ is the action map, P_{r-1} is induced from the $(r-1)$st power map on A and F is the obvious inclusion map. To this end we note that the composition of the maps in the top row and the right-hand column is equal to the composition

$$\begin{aligned}G_+\wedge N^{\mathrm{cy}}_\wedge(A)_0 &\xrightarrow{\mathrm{id}\wedge\mathrm{incl}} G_+\wedge|N^{\mathrm{cy}}_\wedge(A)_\bullet| \xrightarrow{\mathrm{id}\wedge\Delta_C} G_+\wedge|(\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(A)_\bullet)^C|\\ &\xrightarrow{\mu}|(\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(A)_\bullet)^C| \xrightarrow{F} |\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(A)_\bullet| \xrightarrow{D} |N^{\mathrm{cy}}_\wedge(A)_\bullet|.\end{aligned}$$

The (non-simplicial) homeomorphism $D\colon|\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(A)_\bullet|\to|N^{\mathrm{cy}}_\wedge(A)_\bullet|$ is homotopic to the realization of the simplicial map $\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(A)_\bullet\to N^{\mathrm{cy}}_\wedge(A)_\bullet$ which in simplicial degree k is given by the iterated face map $d_0^{(r-1)(k+1)}$. This fact, true for any simplicial space $X_\bullet$, follows from the proof of [BHM, 2.5]. Now let $\mathbf{C}_\bullet$ be the standard cyclic model for the circle as recalled in the proof of Proposition 1.4.6 and consider the diagram of simplicial sets (if there is more than one simplicial direction the diagonal simplicial set is understood),

$$\begin{tikzcd}
\mathbf{C}_\bullet\wedge N^{\mathrm{cy}}_\wedge(A)_0 \arrow[r] \arrow[d] & \mathbf{C}_\bullet\wedge N^{\mathrm{cy}}_\wedge(A)_\bullet \arrow[r, "\mathrm{id}\wedge\Delta_C"] & \mathbf{C}_\bullet\wedge(\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(A)_\bullet)^C \arrow[d, "\mu"] \\
\mathbf{C}_\bullet\wedge N^{\mathrm{cy}}_\wedge(A)_\bullet \arrow[d, "(P_{r-1},\mathrm{id})"] & & (\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(A)_\bullet)^C \arrow[d, "F\circ d_0^{(\bullet+1)(r-1)}"] \\
\mathbf{C}_\bullet\wedge N^{\mathrm{cy}}_\wedge(A)_\bullet\wedge N^{\mathrm{cy}}_\wedge(A)_\bullet \arrow[r, "\mu\wedge\mathrm{id}"] & N^{\mathrm{cy}}_\wedge(A)_\bullet\wedge N^{\mathrm{cy}}_\wedge(A)_\bullet \arrow[r, "\mu_{0,0}"] & N^{\mathrm{cy}}_\wedge(A)_\bullet.
\end{tikzcd}$$

One verifies by inspection that this commutes. Therefore (1.5.7) is homotopy commutative; compare with the proof of Proposition 1.4.6. □

The universal property of the de Rham–Witt pro-complex immediately gives

PROPOSITION 1.5.8. *Suppose that A is an $\mathbf{F}_p$-algebra. Then there is a natural map*

$$I\colon W_n\Omega^*_A\to\pi_*T(A)^{C_{p^{n-1}}},$$

such that $RI=IR$, $FI=IF$, $VI=IV$ and $\delta I=Id$, and such that for $*=0$, I is the isomorphism of (1.3.9). □

The homotopy groups of $\mathrm{TR}(A;p)$, which we denote $\mathrm{TR}_*(A;p)$, are given by Milnor's short exact sequence

$$0\to\varprojlim_R{}^{(1)}\pi_{*+1}T(A)^{C_{p^n}}\to\mathrm{TR}_*(A;p)\to\varprojlim_R\pi_*T(A)^{C_{p^n}}\to 0.$$

Therefore, if the derived limit on the left vanishes, (1.5.5) defines a natural map of complexes

$$I\colon W\Omega^*_A\to\mathrm{TR}_*(A;p),\tag{1.5.9}$$

where the differential on $\mathrm{TR}_*(A;p)$ is given by Definition 1.4.3.

2. Smooth algebras

2.1. Let k be any ring. We recall that a k-algebra A is said to be *smooth* if it is finitely presented and if every k-algebra map $A \to C/N$ into the quotient of a k-algebra C by a nilpotent ideal N has a lifting to a k-algebra map $A \to C$. If there is at most one such lifting, A is called *unramified.* A k-algebra is *étale* if it is smooth and unramified. Equivalently, A is smooth if there exist relatively prime elements $f_1, \dots, f_s \in A$ such that $A_{f_i} = A[1/f_i]$ is an étale extension of a polynomial algebra in finitely many variables over k,

$$k[X_1, \dots, X_n] \xrightarrow{\text{étale}} A_{f_i}.$$

See for example [L, appendix]. To prove Theorem B we first calculate $\mathrm{TR}_*(A;p)$ when A is a polynomial algebra over $\mathbf{F}_p$. The proof in the general case is a covering argument based on the second characterization of smoothness. So let A denote $\mathbf{F}_p[X_1, \dots, X_n]$. We first recall the description of $W_\bullet \Omega_A^*$ from [I, I.2].

Consider the ring

$$C = \varinjlim_r \mathbf{Q}_p[X_1^{p^{-r}}, \dots, X_n^{p^{-r}}].$$

The formula $dX^k = kX^k d\log X$ shows that any $\omega \in \Omega^m_{C/\mathbf{Q}_p}$ may be written uniquely

$$\omega = \sum_{i_1 < \dots < i_m} a_{i_1, \dots, i_m}(X) d\log X_{i_1} \dots d\log X_{i_m}, \tag{2.1.1}$$

such that each $a_{i_1,\dots,i_m}(X) \in C$ is divisible by $X_{i_1}^{p^{-s}} \dots X_{i_m}^{p^{-s}}$ for some $s \geqslant 0$. We say that ω is *integral* if the $a_{i_1,\dots,i_m}(X)$ have their coefficients in $\mathbf{Z}_p$ and let

$$E^* \subset \Omega^*_{C/\mathbf{Q}_p} \tag{2.1.2}$$

be the sub-DGA of those forms ω such that ω and $d\omega$ are both integral. If we write ω as in (2.1.1), then the formula

$$F(\omega) = \sum_{i_1 < \dots < i_m} a_{i_1, \dots, i_m}(X^p) d\log X_{i_1} \dots d\log X_{i_m}$$

defines an automorphism of $\Omega^*_{C/\mathbf{Q}_p}$ considered as a graded ring. We let $V = pF^{-1}$ and note that F and V restrict to endomorphisms of E^*. Moreover, the Teichmüller map $\omega: \mathbf{F}_p \to \mathbf{Z}_p$ extends to a multiplicative map $\omega: A \to E^0$ specified by $\omega(X_i) = X_i$.

We filter E^* by the DG-ideals

$$\mathrm{Fil}^r E^m = V^r E^m + dV^r E^{m-1}$$

and consider the quotient DGA's

$$E_r^* = E^*/\mathrm{Fil}^r E^*.$$

The operations F and V restricts to $F: E_r^* \to E_{r-1}^*$ and $V: E_r^* \to E_{r+1}^*$, respectively, and we let $R: E_r^* \to E_{r-1}^*$ be the projection. The formulas (1.2.1) are trivially verified, and moreover, one may prove that $E_r^0 \cong W_r(A)$ such that R, F and V correspond to restriction, Frobenius and Verschiebung, respectively.

THEOREM 2.1.3 (Deligne). *The canonical map $W_\bullet\Omega_A^* \to E_\bullet^*$ is an isomorphism.* □

Let $\mathbf{N}[1/p]$ be the monoid of non-negative rationals with denominator a power of p and let

$$K = \mathbf{N}[1/p]^n \tag{2.1.4}$$

be the n-fold product. We grade E^* over K as follows: C is a K-graded ring in an obvious way and we call an m-form ω written as in (2.1.1) homogeneous of degree k if the $a_{i_1,\dots,i_m}(X)$ are so. We note that R and d preserve the grading after k whereas F (or V) multiplies (or divides) the degree by p. Let

$${}_kE^* \subset E^* \tag{2.1.5}$$

denote the subgroup of E^* of homogeneous elements of degree k and let ${}_kE_r^*$ be the image of ${}_kE^*$ in E_r^*. It is proved in [I, I.2.8] that ${}_kE^m$ is a f.g. free $\mathbf{Z}_p$-module and an explicit basis is given. We recall how this basis is defined.

We fix $k \in K$ and let

$$I_m = \{\underline{i} = (i_1, \dots, i_m) \mid 1 \leqslant i_s \leqslant n \text{ and } k_{i_s} \neq 0\}.$$

Reordering if necessary we may assume that $v(k_1) \leqslant \dots \leqslant v(k_n)$, where $v(-)$ denotes the p-adic valuation. For $\underline{i} \in I_m$ fixed we let

$$\begin{aligned} t_0 &= \begin{cases} 1, & \text{if } i_1 = 1, \\ p^{-v(k_1)} X^k_{[1,i_1[}, & \text{if } i_1 > 1 \text{ and } v(k_1) < 0, \\ X^k_{[1,i_1[}, & \text{if } i_1 > 1 \text{ and } v(k_1) \geqslant 0, \end{cases} \\ t_s &= p^{-v(k_{i_s})} X^k_{[i_s,i_{s+1}[}, \quad \text{for } 1 \leqslant s \leqslant m \text{ with } v(k_{i_s}) < 0, \\ u_s &= X^{p^{-v(k_{i_s})}k}_{[i_s,i_{s+1}[}, \quad \text{for } 1 \leqslant s \leqslant m \text{ with } v(k_{i_s}) \geqslant 0, \end{aligned} \tag{2.1.6}$$

where $X_S^k = \prod_{i \in S} X_i^{k_i}$ for $S \subset [1,n]$ and $[i_m, i_{m+1}[= [i_m, n]$. We define

$$e_{\underline{i}}(k) \in {}_kE^m \tag{2.1.7}$$

by the formula

$$e_{\underline{i}}(k)=t_0\prod_{\substack{1\leqslant s\leqslant m\\ v(k_{i_s})<0}} dt_s \prod_{\substack{1\leqslant s\leqslant m\\ v(k_{i_s})\geqslant 0}} u_s^{p^{v(k_{i_s})}-1}du_s.$$

Then by [I, I.2.8],

$$_kE^m=\mathbf{Z}_p\langle e_{\underline{i}}(k)\,|\,\underline{i}\in I_m\rangle. \tag{2.1.8}$$

We also recall the description of $_kE^m_r$ from [I, I.2.12]. Let $s=s(k)=-\min\{v(k_i)\}$ and set

$$v=v(r,k)=\begin{cases} r-s, & \text{if } s>0,\ r>s,\\ 0, & \text{if } s>0,\ r\leqslant s,\\ r, & \text{if } s\leqslant 0.\end{cases}$$

Then

$$_kE^m_r=\mathbf{Z}/p^v\langle e_{\underline{i}}(k)\,|\,\underline{i}\in I_m\rangle. \tag{2.1.9}$$

We note that $_kE^m_r$ is non-zero if and only if $v(r,k)>0$ and $1\leqslant m\leqslant n$. We shall also write $_kW_r\Omega^*_A$ for $_kE^*_r$.

2.2. In this section we evaluate $\pi_*T(A)^{C_{p^d}}$. We recall that, for any monoid Γ, Waldhausen has defined its cyclic bar construction $N^{\mathrm{cy}}_\bullet(\Gamma)$. This is a cyclic set in the sense of Connes, and therefore, the realization carries an action by the circle G, see for example [J] or [L]. Let $\langle X_1,...,X_n\rangle$ be the free abelian monoid on the listed generators. Then we proved in [HM1, Theorem 7.1] that there is a canonical equivalence of G-spectra indexed on $\mathcal{U}$,

$$T(A)\simeq_G T(\mathbf{F}_p)\wedge|N^{\mathrm{cy}}_\bullet(\langle X_1,...,X_n\rangle)|_+, \tag{2.2.1}$$

where the smash product is formed in $G\mathcal{SU}$. Taking cyclic nerve and realization commute with finite products, so we have a G-equivariant homeomorphism

$$|N^{\mathrm{cy}}_\bullet(\langle X_1,...,X_n\rangle)|\cong_G|N^{\mathrm{cy}}_\bullet(\langle X_1\rangle)|\times...\times|N^{\mathrm{cy}}_\bullet(\langle X_n\rangle)|.$$

We now note that as a cyclic set

$$N^{\mathrm{cy}}_\bullet(\langle X\rangle)=\coprod_{l\geqslant 0}N^{\mathrm{cy}}_\bullet(\langle X\rangle;l), \tag{2.2.2}$$

where $N^{\mathrm{cy}}_\bullet(\langle X\rangle;l)$ is the cyclic subset whose k-simplices is the tuples $(X^{i_0},...,X^{i_k})$ with $i_0+...+i_k=l$. In particular, $N^{\mathrm{cy}}_\bullet(\langle X\rangle;0)=*$. We let $S^1(l)$ denote S^1 with $z\in G$ acting as multiplication by z^l.

LEMMA 2.2.3. *$S^1(l)$ is a strong G-deformation retract of $|N_\bullet^{\mathrm{cy}}(\langle X\rangle;l)|$.*

Proof. As a cyclic set $N_\bullet^{\mathrm{cy}}(\langle X\rangle;l)$ is generated by the $(l-1)$-simplex $(X,\dots,X)$. Therefore the realization is a quotient of standard cyclic $(l-1)$-simplex Λ^{l-1}, [J]. In fact,

$$|N_\bullet^{\mathrm{cy}}(\langle X\rangle;l)|=\Lambda^{l-1}/C_l,$$

where the generator of C_l acts as the cyclic operator τ_{l-1}. As a G-space $\Lambda^{l-1}\cong S^1\times\Delta^{l-1}$ with G acting by multiplication in the first variable. Moreover, the homeomorphism may be chosen such that τ_{l-1} acts as

$$\tau_{l-1}(z;u_0,\dots,u_{l-1})=(z\zeta_l^{-1};u_1,\dots,u_{l-1},u_0),$$

where $\zeta_l=\exp(2\pi i/l)$, see [HM1, 7.2]. Hence the inclusion of the barycenter in Δ^{l-1} induces a strong G-equivariant deformation retraction $S^1/C_l\to|N_\bullet^{\mathrm{cy}}(\langle X\rangle;l)|$. Finally, $S^1(l)\cong S^1/C_l$ by the lth root. $\square$

Let $l=(l_1,\dots,l_n)$ be a tuple of non-negative integers and let $(l_{i_1},\dots,l_{i_j})$ be the subtuple of positive integers. We have shown that

$$T(A)\simeq_G\bigvee_{l\in\mathbf{N}^n}T(\mathbf{F}_p)\wedge(S^1(l_{i_1})\times\dots\times S^1(l_{i_j}))_+,$$

and restricting to the trivial universe we get

$$\mathrm{TH}(A)\simeq_G\bigvee_{l\in\mathbf{N}^n}j^*(T(\mathbf{F}_p)\wedge(S^1(l_{i_1})\times\dots\times S^1(l_{i_j}))_+). \tag{2.2.4}$$

The splitting being G-equivariant induces a similar splitting of C_{p^d}-fixed points and the maps F, V and δ preserve this splitting. The restriction R maps a summand l to the summand l/p if p divides l and annihilates the remaining summands. This follows from [HM1, Theorem 7.1]. We write $e=\min\{v(l_i)\}$ and for $d\geqslant 1$ we let $w=w(d,l)=\min\{d,e\}$.

PROPOSITION 2.2.5. *Let l and d be as above. Then*

$$j^*(T(\mathbf{F}_p)\wedge(S^1(l_{i_1})\times\dots\times S^1(l_{i_j}))_+)^{C_{p^d}}\simeq\mathrm{TH}(\mathbf{F}_p)^{C_{p^w}}\wedge(S^1_{\alpha_{i_1}}\times\dots\times S^1_{\alpha_{i_j}})_+,$$

where the α_{i_s} are dummies.

Proof. For any G-spectrum T indexed on $\mathcal{U}$ and any C-trivial G-space X we have an equivalence of G/C-spectra indexed on $\mathcal{U}^C$,

$$(T\wedge X)^C\simeq_{G/C}T^C\wedge X. \tag{2.2.6}$$

In the case at hand this shows that

$$(T(\mathbf{F}_p)\wedge(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+)^C \simeq_{G/C} T(\mathbf{F}_p)^C\wedge(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+,$$

whenever $C\subset C_{p^e}$. The proposition follows when $d\leqslant e$. We consider the remaining case $d>e$ where we write $T=T(\mathbf{F}_p)$ and $C=C_{p^e}$. From (2.2.6) we have

$$j^*(T\wedge(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+)^{C_{p^d}} \simeq j^*(T^C\wedge(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+)^{C_{p^d}/C},$$

where now C_{p^d}/C acts freely on $S^1(l_{i_1})\times\ldots\times S_1(l_{i_j})$. Hence by (1.3.3),

$$j^*(T^C\wedge(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+)^{C_{p^d}/C} \simeq j^*T^C\wedge_{C_{p^d}/C}(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+.$$

Suppose first that $l_i=p^e$ for some i. Then $S^1(l_i)\cong G/C$ by the p^eth root, so (1.3.2) provides a G/C-equivariant isomorphism

$$j^*T^C\wedge(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+ \cong |T^C\wedge(S^1(l_{i_1})\times\ldots\times\widehat{S^1(l_i)}\times\ldots\times S^1(l_{i_j}))_+|\wedge S^1(l_i)_+, \tag{2.2.7}$$

and finally, $S^1(l_i)/(C_{p^d}/C)\cong S^1$ by the p^{d-e}th power map. In the general case, where $l_i=p^e m$ with $(m,p)=1$, we use that the mth power map

$$P_m\colon S^1(p^e)\to S^1(p^e m),\quad P_m(z)=z^m,$$

is G-equivariant with cofiber a Moore space for $\mathbf{Z}/m$. Since $T(\mathbf{F}_p)^C$ is a p-local spectrum, P_m induces an equivalence

$$\begin{aligned}T^C\wedge_{C_{p^d}/C}&(S^1(l_{i_1})\times\ldots\times S^1(l_i)\times\ldots\times S^1(l_{i_j}))_+\\ &\simeq T^C\wedge_{C_{p^d}/C}(S^1(l_{i_1})\times\ldots\times S^1(p^e)\times\ldots\times S^1(l_{i_j}))_+\end{aligned}$$

and we are back to the special case. □

We compare the gradings of (2.1.5) and (2.2.4). Let $k\in K$ and $r\geqslant 1$ and suppose that $v(r,k)>0$. Then $l=p^{r-1}k$ is an n-tuple of non-negative integers. On the other hand we obtain all n-tuples of non-negative integers this way. Therefore, if we write

$${}_k\mathrm{TH}(A)^{C_{p^{r-1}}} = j^*(T(\mathbf{F}_p)\wedge(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+)^{C_{p^{r-1}}}$$

then we may write (2.2.4) as

$$\mathrm{TH}(A)^{C_{p^{r-1}}} \simeq \bigvee_{\substack{k\in K\\ v(r,k)>0}} {}_k\mathrm{TH}(A)^{C_{p^{r-1}}}, \tag{2.2.8}$$

with $v=v(r,k)$ as in (2.1.9). From Proposition 2.2.5 we get

$${}_k\mathrm{TH}(A)^{C_{p^{r-1}}} \simeq \mathrm{TH}(\mathbf{F}_p)^{C_{p^{v-1}}}\wedge(S^1_{\alpha_{i_1}}\times\ldots\times S^1_{\alpha_{i_j}})_+,$$

and hence, the homotopy groups become

$$\pi_*({}_k\mathrm{TH}(A)^{C_{p^{r-1}}}) \cong S_{\mathbf{Z}/p^v}\{\sigma_r\}\otimes\Lambda_{\mathbf{Z}/p^v}\{\iota_{i_1},\ldots,\iota_{i_j}\}, \tag{2.2.9}$$

where $\deg(\sigma_r)=2$ and $\deg(\iota_{i_s})=1$. Let us note that in the grading after k the restriction R and the differential δ preserve the grading whereas F and V multiply and divide by p, respectively.

2.3. We use the isomorphism (1.3.9) and the formulas (2.1.6) and (2.1.7) to define elements

$$g_{\underline{i}}(k) \in \pi_*({}_k\mathrm{TH}(A)^{C_{p^{r-1}}}).$$

Concretely, we assume that $v(k_1) \leqslant \ldots \leqslant v(k_n)$ and let

$$\begin{aligned} a_0 &= \begin{cases} 1, & \text{if } i_1 = 1, \\ p^{-v(k_{i_1})}\underline{X}^k_{[1,i_1[}, & \text{if } i_1 > 1 \text{ and } v(k_1) < 0, \\ \underline{X}^k_{[1,i_1[}, & \text{if } i_1 > 1 \text{ and } v(k_1) \geqslant 0, \end{cases} \\ a_s &= p^{-v(k_{i_s})}\underline{X}^k_{[i_s,i_{s+1}[}, \quad \text{for } 1 \leqslant s \leqslant m \text{ with } v(k_{i_s}) < 0, \\ b_s &= \underline{X}^{p^{-v(k_{i_s})}k}_{[i_s,i_{s+1}[}, \quad \text{for } 1 \leqslant s \leqslant m \text{ with } v(k_{i_s}) \geqslant 0, \end{aligned} \tag{2.3.1}$$

where $\underline{X}^k_S = \prod_{i \in S} \underline{X}_i^{k_i}$ for $S \subset [1,n]$ and $[i_m, i_{m+1}[= [i_j, n]$. Then

$$g_{\underline{i}}(k) \in \pi_m({}_k\mathrm{TH}(A)^{C_{p^{r-1}}}) \tag{2.3.2}$$

is defined by the formula

$$g_{\underline{i}}(k) = a_0 \prod_{\substack{1 \leqslant s \leqslant m \\ v(k_{i_s}) < 0}} \delta a_s \prod_{\substack{1 \leqslant s \leqslant m \\ v(k_{i_s}) \geqslant 0}} b_s^{p^{v(k_{i_s})}-1}\delta b_s.$$

By definition, $I(e_{\underline{i}}(k)) = g_{\underline{i}}(k)$, and the calculations in §2.2 above show that Theorem B will follow in the case of a polynomial algebra over $\mathbf{F}_p$ given

PROPOSITION 2.3.3. *The $g_{\underline{i}}(k)$ generate the subgroup*

$$\pi_*^{(0)}({}_k\mathrm{TH}(A)^{C_{p^{r-1}}}) = \Lambda_{\mathbf{Z}/p^v}\{\iota_{i_1}, \ldots, \iota_{i_j}\}$$

of $\pi_*({}_k\mathrm{TH}(A)^{C_{p^{r-1}}})$.

Proof. We already know that this is true when $v(k_1) \geqslant 0$ and $r=1$. For by Proposition 1.4.6 the composite

$$\Omega^*_A \to \pi_* T(A) \to \mathrm{HH}_*(A)$$

is the isomorphism of [HKR]. The case $v(k_1) \geqslant 0$ and $r > 1$ follows similarly because the iterated restriction

$$R^{r-1}\colon {}_kW_r\Omega^*_A \to {}_k\Omega^*_A$$

maps generators to generators. So we are left with the case $v(k_1) = -s < 0$. Again the iterated restriction $R^{r-s-1}\colon {}_kW_r\Omega^*_A \to {}_kW_{s+1}\Omega^*_A$ maps generators to generators, so we may assume that $r=s+1$. This has the advantage that ${}_kW_{s+1}\Omega^*_A$ is an $\mathbf{F}_p$-vector space, cf. (2.1.9).

We first take $m=1$ in (2.3.2). When $i_1=1$ we have

$$g_{i_1}(k)=\delta a_1=\delta(p^s\underline{X}^k),$$

and $p^s\underline{X}^k$ generates $\pi_0({}_k\mathrm{TH}(A)^{C_{p^s}})\cong\mathbf{F}_p$. Recall that by definition

$$\delta(p^s\underline{X}^k)=(\mu_s)_*(\sigma\otimes p^s\underline{X}^k),$$

where μ_s is the composition

$${}_k\mathrm{TH}(A)^{C_{p^s}}\wedge S^1\xrightarrow{\mathrm{id}\wedge\varrho}{}_k\mathrm{TH}(A)^{C_{p^s}}\wedge(S^1/C_{p^s})_+\xrightarrow{\mu}{}_k\mathrm{TH}(A)^{C_{p^s}}$$

and $\sigma\in\pi_1^S(S^1_+)$ is the generator from (1.4.1). From (2.2.7) we have the G/C_{p^s}-equivariant equivalence

$${}_k\mathrm{TH}(A)^{C_{p^s}}\simeq|T(\mathbf{F}_p)\wedge(S^1(l_2)\times\ldots\times S^1(l_j))_+|\wedge(S^1/C_{p^s})_+.$$

It follows that $g_{i_1}(k)\in\pi_1({}_kT(A)^{C_{p^s}})$ is a generator.

When $i_1>1$ and $v(k_{i_1})\geqslant0$ we have

$$g_{i_1}(k)=a_0b_1^{p^{v(k_{i_1})}-1}\delta b_1,$$

where $a_0\in\pi_0({}_{k_{[1,i_1[}}\mathrm{TH}(A)^{C_{p^s}})$ and $b_1^{p^{v(k_{i_1})}-1}\delta b_1\in\pi_1({}_{k_{[i_1,n]}}\mathrm{TH}(A)^{C_{p^s}})$ are both generators. We show that $g_{i_1}(k)\in\pi_1({}_k\mathrm{TH}(A)^{C_{p^s}})$ is a generator. Recall that

$$\begin{aligned}{}_{k_{[1,i_1[}}\mathrm{TH}(A)^{C_{p^s}}&=j^*(T(\mathbf{F}_p)\wedge(S^1(l_1)\times\ldots\times\widehat{S^1(l_{i_1})})_+)^{C_{p^s}},\\ {}_{k_{[i_1,n]}}\mathrm{TH}(A)^{C_{p^s}}&=j^*(T(\mathbf{F}_p)\wedge(S^1(l_{i_1})\times\ldots\times S^1(l_j))_+)^{C_{p^s}},\end{aligned}$$

where $l_i=p^sk_i$. Hence $v(l_1)=0$ and as demonstrated in the proof of Proposition 2.2.5 we may assume that $l_1=1$. For notational reasons we shall further assume that $i_1=j=2$. We shall also write $T=T(\mathbf{F}_p)$ and $C=C_{p^s}$. We must evaluate the multiplication map

$$j^*(T\wedge S^1(l_1)_+)^C\wedge j^*(T\wedge S^1(l_2)_+)^C\to j^*(T\wedge(S^1(l_1)\times S^1(l_2))_+)^C \tag{2.3.4}$$

under the equivalences of Proposition 2.2.5. Recall from [LMS] that the equivalence of (1.3.3)

$$\tau\colon j^*T\wedge_C S^1(l_1)_+\xrightarrow{\simeq}j^*(T\wedge S^1(l_1)_+)^C$$

is the adjoint of the equivariant transfer $\tilde{\tau}: j^*T\wedge_C S^1(l_1)_+ \to j^*(T\wedge S^1(l_1)_+)$ and consider the diagram

$$\begin{tikzcd}
j^*(T\wedge S^1(l_1)_+\wedge T\wedge S^1(l_2)_+) \arrow[r, "m"] & j^*(T\wedge(S^1(l_1)\times S^1(l_2))_+) \\
(j^*T\wedge S^1(l_1)_+\wedge j^*T\wedge S^1(l_2)_+)/C \arrow[r, "m"] \arrow[u, "\tilde{\tau}A"] & j^*T\wedge_C(S^1(l_1)\times S^1(l_2))_+ \arrow[u, "\tilde{\tau}"] \\
j^*T\wedge_C S^1(l_1)_+\wedge j^*T^C\wedge S^1(l_2)_+ \arrow[r, "m'"] \arrow[u, "1\wedge F^s"] & j^*T\wedge_C(S^1(l_1)\times S^1(l_2))_+ \arrow[u, equal] \\
|T|\wedge(S^1(l_1)/C)_+\wedge|T^C|\wedge|S^1(l_2)_+| \arrow[r, "m'"] \arrow[u, "\cong"', "\xi_{12}"] & |T|\wedge((S^1(l_1)/C)\times|S^1(l_2)|)_+ \arrow[u, "\cong"', "\xi"] \\
|T|\wedge(S^1(l_1)/C)_+\wedge j^*T^C\wedge S^1(l_2)_+ \arrow[u, "\cong"', "\xi_2^{-1}"] &
\end{tikzcd}$$

where the map m' first includes $|T^C|$ in $|T|$ and then multiplies. The map in the top row induces (2.3.4) on C-fixed sets. Moreover, the composition of the maps in the left- and right-hand columns induces the equivalences of Proposition 2.2.5 which identifies the left- and right-hand sides of (2.3.4), respectively. Now the top square is homotopy commutative by naturality of transfer and the bottom square commutes because the multiplication $m: T\wedge T\to T$ is G-equivariant when G acts diagonally on $T\wedge T$. Therefore, it follows that under the equivalences of Proposition 2.2.5 the multiplication (2.3.4) may be written

$$\begin{aligned}
|T|\wedge(S^1(l_1)/C)_+\wedge j^*T^C\wedge S^1(l_2)_+ &\xrightarrow{\xi_2^{-1}} |T|\wedge(S^1(l_1)/C)_+\wedge|T^C|\wedge|S^1(l_2)_+| \\
&\xrightarrow{1\wedge F^s} |T|\wedge(S^1(l_1)/C)_+\wedge|T|\wedge|S^1(l_2)_+| \\
&\xrightarrow{m} |T|\wedge((S^1(l_1)/C)\times|S^1(l_2)|)_+.
\end{aligned}$$

The element

$$a_0\otimes b_1^{p^{v(k_{i_1})}-1}\delta b_1\in\pi_1(|T|\wedge(S^1(l_1)/C)_+\wedge j^*T^C\wedge S^1(l_2)_+)$$

is fixed by the isomorphism $(\xi_2^{-1})_*$. Moreover,

$$F^s: \pi_1(|T^C|\wedge S^1(l_2)_+)\to\pi_1(|T|\wedge S^1(l_2)_+)$$

maps generators to generators. For $F^s: \pi_0T^C\to\pi_0T$ is the reduction $\mathbf{Z}/p^{s+1}\to\mathbf{F}_p$. Finally, the multiplication $m: T\wedge T\to T$ induces an isomorphism on $\pi_0(-)$, and hence $g_{i_1}(k)\in\pi_1({}_k\mathrm{TH}(A)^{C_{p^s}})$ is a generator.

The case $i_1>1$ and $v(k_{i_1})<0$ requires some extra work. We have

$$g_{i_1}(k)=a_0\delta a_1,$$

and $a_0\in\pi_0({}_{k_{[1,i_1[}}\mathrm{TH}(A)^{C_{p^s}})$ and $\delta a_1\in\pi_1({}_{k_{[i_1,n]}}\mathrm{TH}(A)^{C_{p^s}})$ are both generators. We prove that $g_{i_1}(k)$ is a generator under the same assumptions as above, i.e. $l_1=1$ and $l_2=j=2$. The general case is similar. We again write $T=T(\mathbf{F}_p)$ and $C=C_{p^s}$, and let $C'=C_{p^{v(l_2)}}$ and $\bar{C}=C/C'$. We also let $j_1\colon\mathcal{U}^{C'}\to\mathcal{U}$ and $j_2\colon\mathcal{U}^{C}\to\mathcal{U}^{C'}$ and $j_3\colon\mathcal{U}^{G}\to\mathcal{U}^{C}$ be the inclusions such that $j=j_3\circ j_2\circ j_1\colon\mathcal{U}^G\to\mathcal{U}$. Then the diagram

$$\begin{tikzcd}
j^*(T\wedge S^1(l_1)_+\wedge T\wedge S^1(l_2)_+) \arrow[r, "m"] & j^*(T\wedge(S^1(l_1)\times S^1(l_2))_+) \\
j_3^*j_2^*(j_1^*T\wedge S^1(l_1)_+\wedge j_1^*T\wedge S^1(l_2)_+)/C' \arrow[r, "m"] \arrow[u, "\tau_1^{C'}"] & j_3^*j_2^*(j_1^*T\wedge_{C'}(S^1(l_1)\times S^1(l_2))_+) \arrow[u, "\tau_1^{C'}"'] \\
j_3^*j_2^*(j_1^*T\wedge_{C'}S^1(l_1)_+\wedge j_1^*T^{C'}\wedge S^1(l_2)_+) \arrow[r, "m'"] \arrow[u, "1\wedge F^s"] & j_3^*j_2^*(j_1^*T\wedge_{C'}(S^1(l_1)\times S^1(l_2))_+) \arrow[u, equal] \\
j_3^*(j_2^*j_1^*T\wedge_{C'}S^1(l_1)_+\wedge j_2^*j_1^*T^{C'}\wedge S^1(l_2)_+)/\bar{C} \arrow[r, "m'"] \arrow[u, "\tau_{C'}^{C}"] & j_3^*(j_2^*j_1^*T\wedge_C(S^1(l_1)\times S^1(l_2))_+) \arrow[u, "\tau_{C'}^{C}"']
\end{tikzcd}$$

is homotopy commutative by naturality of transfers. The spectra in the left column are $G\times G$-spectra which we consider G-spectra via the diagonal $\Delta\colon G\to G\times G$. Let us write Δ (or Δ') for the image of C (or C'). Then the transfer $\tau_1^{C'}$ (or $\tau_{C'}^{C}$) of G-spectra is equal to the transfer $\tau_1^{\Delta'}$ (or $\tau_{\Delta'}^{\Delta}$) of $G\times G$-spectra. Moreover,

$$\tau_{C'}^{C}\wedge\tau_{C'}^{C}\simeq\tau_{\Delta'}^{\Delta}\circ\tau_{\Delta}^{C\times C}$$

as maps of $G\times G$-spectra

$$j^*T\wedge_C S^1(l_1)_+\wedge j^*T^{C'}\wedge_{\bar{C}}S^1(l_2)_+\to j_3^*j_2^*(j_1^*T\wedge_{C'}S^1(l_1)_+\wedge j_1^*T^{C'}\wedge S^1(l_2)_+).$$

Finally, we consider the diagram

$$\begin{tikzcd}
{|T|\wedge(S^1(l_1)/C)_+\wedge|T^{C'}|\wedge(S^1(l_2)/\bar{C})_+} \arrow[r, "\xi\wedge\xi"] \arrow[d, "\tau_{\Delta}^{C\times C}"] & j^*T\wedge_C S^1(l_1)_+\wedge j^*T^{C'}\wedge_{\bar{C}}S^1(l_2)_+ \arrow[d, "\tau_{\Delta}^{C\times C}"] \\
(|T|\wedge(S^1(l_1)/C')_+\wedge|T^{C'}|\wedge S^1(l_2)_+)/\bar{C} \arrow[r, "\xi\wedge\xi"] \arrow[d, "m''"] & j_3^*(j_2^*j_1^*T\wedge_{C'}S^1(l_1)_+\wedge j_2^*j_1^*T^{C'}\wedge S^1(l_2)_+)/\bar{C} \arrow[d, "m'"] \\
{|T|\wedge((S^1(l_1)/C)\times|S^1(l_2)|)_+} \arrow[r, "\xi_{12}"] & j_3^*(j_2^*j_1^*T\wedge_C(S^1(l_1)\times S^1(l_2))_+).
\end{tikzcd}$$

Again the top square homotopy commutes by naturality and m'' is defined to make the bottom square commute. Our analysis so far shows that under the equivalences of Proposition 2.2.5 the multiplication map is equal to $m''\circ\tau_\Delta^{C\times C}$. One may argue as in the previous case that m'' maps generators to generators on $\pi_1(-)$. Therefore it will suffice to show that $\tau_\Delta^{C\times C}$ maps the generator $a_0\otimes\delta a_1$ non-trivially. To see this we note that $\tau_\Delta^{C\times C}$ is homotopic to the identity on $|T|\wedge|T^{C'}|$ smashed with the ordinary (non-equivariant) transfer associated to the p^s-fold covering

$$\mathrm{pr}_\Delta^{C\times C}\colon (S^1\times(S^1/C'))/\Delta\to(S^1\times S^1)/(C\times C).$$

The homology of the base is

$$H_*((S^1\times S^1)/(C\times C);\mathbf{F}_p)\cong H_*(S^1/C;\mathbf{F}_p)\otimes H_*(S^1/C;\mathbf{F}_p)\cong\Lambda_{\mathbf{F}_p}\{\iota_1\}\otimes\Lambda_{\mathbf{F}_p}\{\iota_2\},$$

and $\tau_\Delta^{C\times C}(a_0\otimes da_1)$ is non-trivial if and only if the homology transfer of $1\otimes\iota_2$ is non-trivial. We have a pull-back square

$$\begin{tikzcd}
(S^1/C)\times(S^1/C') \arrow[r, "\xi"] \arrow[d, "1\times\mathrm{pr}_{C'}^C"] & (S^1\times(S^1/C'))/\Delta \arrow[d, "\mathrm{pr}_\Delta^{C\times C}"] \\
(S^1/C)\times(S^1/C) \arrow[r, "\xi"] & (S^1\times S^1)/(C\times C),
\end{tikzcd}$$

where ξ is the homeomorphism given by $\xi(z_1,z_2)=(z_1,z_1z_2)$. On homology

$$\mathrm{trf}_{C'}^C(1)=p=0,\quad \mathrm{trf}_{C'}^C(\iota)=\iota,$$

and therefore

$$(1\otimes\mathrm{trf}_{C'}^C)\circ\xi^{-1}(1\otimes\iota_2)=(1\otimes\mathrm{trf}_{C'}^C)(1\otimes\iota_2)=1\otimes\iota_2,$$

which is a generator. This shows that $g_{i_1}(k)$ is a generator.

We can now prove that the $g_{i_1}(k)$ generate

$$\pi_1^{(0)}({}_k\mathrm{TH}(A)^{C_{p^s}})\cong\mathbf{F}_p\langle\iota_1,\dots,\iota_j\rangle.$$

We have seen that the $g_{i_1}(k)$, $i_1=1,\dots,j$, are non-trivial. However we have proved more, namely that

$$\iota_s\in\mathbf{F}_p\langle g_i(k),\dots,g_j(k)\rangle$$

if and only if $i\leqslant s$. This concludes the proof in the case where $m=1$ in (2.3.2).

The general case is similar. Again one proves that the $g_{\underline{i}}(k)$, $\underline{i}\in I_m$, are non-trivial. The argument is only notationally more involved than the argument above. We have from (2.2.9),

$$\pi_m^{(0)}({}_k\mathrm{TH}(A)^{C_{p^s}})\cong\Lambda^m(\mathbf{F}_p\langle\iota_1,\dots,\iota_j\rangle).$$

Finally, one orders I_m lexicographically to see that

$$\pi_m^{(0)}({}_k\mathrm{TH}(A)^{C_{p^s}}) \cong \mathbf{F}_p\langle g_{\underline{i}}(k) \mid \underline{i} \in I_m\rangle$$

as claimed. □

2.4. The proof of Theorem B in the general case is a covering argument based on the fact that in a smooth k-algebra A one can find relatively prime elements $f_1, \ldots, f_s$ such that $A_{f_i} = A[1/f_i]$ is an étale extension of a polynomial algebra in a finite number of variables,

$$k[X_1, \ldots, X_n] \xrightarrow{\text{étale}} A_{f_i}. \tag{2.4.1}$$

We first study étale extensions.

LEMMA 2.4.2. *If $f: A \to B$ is an étale extension of $\mathbf{F}_p$-algebras then*

$$\pi_* T(B) \cong B \otimes_A \pi_* T(A).$$

Proof. Recall the spectral sequence used in [Bö2],

$$E^2(B) = \mathrm{HH}_*(\mathcal{A}_B) \implies \mathcal{A} \otimes T_*(B), \tag{2.4.3}$$

where $\mathcal{A}_B = H_*^{\mathrm{spec}}(HB; \mathbf{F}_p)$ is the mod p spectrum homology of the Eilenberg–MacLane spectrum for B and $\mathcal{A} = \mathcal{A}_{\mathbf{F}_p}$. As an algebra $\mathcal{A}_B \cong B \otimes \mathcal{A}$. For the Hurewitz map induces a multiplicative homomorphism

$$B = \pi_* HB \to H_*^{\mathrm{spec}}(HB; \mathbf{Z}) \to H_*^{\mathrm{spec}}(HB; \mathbf{F}_p) = \mathcal{A}_B,$$

and so does the unit map $\mathbf{F}_p \to B$. The ring homomorphism $B \otimes \mathcal{A} \to \mathcal{A}_B$ is an isomorphism because B as an abelian group is an $\mathbf{F}_p$-vector space and because spectrum homology commutes with sums. When B is an A-algebra we get

$$\mathcal{A}_B \cong B \otimes_A \mathcal{A}_A$$

as $\mathbf{F}_p$-algebras. We have a Künneth formula for Hochschild homology,

$$\mathrm{HH}_*(\mathcal{A}_B) \cong \mathrm{HH}_*(B) \otimes \mathrm{HH}_*(\mathcal{A}),$$

and similarly with A in the place of B, cf. [CE, p. 204]. The lemma may now be deduced from the fact that for étale extensions $\mathrm{HH}_*(B) \cong B \otimes_A \mathrm{HH}_*(A)$. This result seems to have been proved in a varying degree of generality by a number of people. A well-written account is [WG]. In the case at hand we get

$$\mathrm{HH}_*(\mathcal{A}_B) \cong B \otimes_A \mathrm{HH}_*(\mathcal{A}_A).$$

Since (2.4.3) is a spectral sequence of B-algebras, $E^\infty(B) \cong B \otimes_A E^\infty(A)$, and hence $\mathcal{A} \otimes T_*(B) \cong \mathcal{A} \otimes B \otimes_A T_*(A)$. The last step uses that the filtration of the actual homology groups induced from the spectral sequence is finite in each degree. □

PROPOSITION 2.4.4. *If $f: A \to B$ is an étale map of $\mathbf{F}_p$-algebras then the canonical map*

$$W_r(B) \otimes_{W_r(A)} \pi_* T(A)^{C_{p^{r-1}}} \to \pi_* T(B)^{C_{p^{r-1}}}$$

is an isomorphism.

Proof. The proof is by induction on r with the case $r=1$ proved in Lemma 2.4.2. In the induction step we use the long-exact sequences of W_r-modules induced from the cofibration sequence (1.3.10). We get a diagram in which the rows are exact,

$$\begin{tikzcd} \cdots \arrow[r] & \pi_m T(A)_{hC_{p^{r-1}}} \arrow[r, "N"] \arrow[d] & \pi_m T(A)^{C_{p^{r-1}}} \arrow[r, "R"] \arrow[d] & \pi_m T(A)^{C_{p^{r-2}}} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \pi_m T(B)_{hC_{p^{r-1}}} \arrow[r, "N"] & \pi_m T(B)^{C_{p^{r-1}}} \arrow[r, "R"] & \pi_m T(B)^{C_{p^{r-2}}} \arrow[r] & \cdots . \end{tikzcd}$$

For an étale extension of $\mathbf{F}_p$-algebras one has $W_r(B) \otimes_{W_r(A)} W_{r-1}(A) \cong W_{r-1}(B)$, and therefore,

$$W_r(B) \otimes_{W_r(A)} \pi_* T(A)^{C_{p^{r-2}}} \cong W_{r-1}(B) \otimes_{W_{r-1}(A)} \pi_* T(A)^{C_{p^{r-2}}}.$$

Hence, to prove the induction step, we must show that

$$W_r(B) \otimes_{W_r(A)} \pi_* T(A)_{hC_{p^{r-1}}} \to \pi_* T(B)_{hC_{p^{r-1}}}$$

is an isomorphism. This follows from the corresponding statement for the E^∞-terms of (1.3.11), which in turn follows from the statement on E^2. So it suffices to prove that

$$W_r(B) \otimes_{W_r(A)} (F^{r-1})^* \pi_* T(A) \to (F^{r-1})^* \pi_* T(B)$$

is an isomorphism. We have $F = R \circ \varphi_A$ where $\varphi_A: W_r(A) \to W_r(A)$ is the map induced from the Frobenius endomorphism of A. The claim now follows from the identifications

$$\begin{aligned} W_r(B) \otimes_{W_r(A)} (F^{r-1})^* \pi_* T(A) &\cong W_r(B) \otimes_{W_r(A)} A \otimes_A (\varphi_A^{r-1})^* \pi_* T(A) \\ &\cong (R^{r-1})^* B \otimes_A (\varphi_A^{r-1})^* \pi_* T(A) \\ &\cong (R^{r-1})^* B \otimes_A (\varphi_A^{r-1})^* A \otimes_A \pi_* T(A) \\ &\cong (R^{r-1})^* (\varphi_B^{r-1})^* B \otimes_A \pi_* T(A) \\ &\cong (F^{r-1})^* B \otimes_A \pi_* T(A), \end{aligned}$$

where the second and the fourth identifications use that B is étale over A. $\square$

Let k be a perfect field of characteristic p and recall that $\mathrm{HH}_*(k) = k$ concentrated in degree zero. Moreover, $W_r(k) \otimes_{W_r(\mathbf{F}_p)} W_{r-1}(\mathbf{F}_p) \cong W_{r-1}(k)$ so the argument above shows that

$$\pi_* T(k \otimes A)^{C_{p^{r-1}}} \cong W_r(k) \otimes \pi_* T(A)^{C_{p^{r-1}}}, \tag{2.4.5}$$

for any $\mathbf{F}_p$-algebra A.

LEMMA 2.4.6. *Let $f_1, \dots, f_s \in A$ be relatively prime elements. Then the complex*

$$\bigotimes_{W_r(A)} (W_r(A) \to W_r(A_{f_i})),$$

where i runs through $1, \dots, s$, is acyclic.

Proof. Suppose first that $r=1$ and note that since A_{f_i} is flat over A,

$$H^*\big(\bigotimes_A (A \to A_{f_i})\big) = H^s\big(\bigotimes_A (A \to A_{f_i})\big) = \bigotimes_A (A_{f_i}/A).$$

We write $1 = x_1 f_1 + \dots + x_s f_s$ and note the formula

$$\frac{a_1}{f_1^{i_1}} \otimes_A \dots \otimes_A \frac{a_s}{f_s^{i_s}} = \sum_{j=1}^{s} \frac{a_1}{f_1^{i_1}} \otimes_A \dots \otimes_A \frac{x_j a_j}{f_j^{i_j - 1}} \otimes_A \dots \otimes_A \frac{a_s}{f_s^{i_s}},$$

from which the case $r=1$ follows by induction. In the general case, note that $W_r(A_f) = W_r(A)_{\underline{f}}$ so that it suffices to show that $\underline{f}_1, \dots, \underline{f}_s$ are relatively prime. But

$$\underline{x}_1 \underline{f}_1 + \dots + \underline{x}_s \underline{f}_s \in 1 + V W_r(A) \subset W_r(A)^{\times},$$

and the claim follows. □

Proof of Theorem B. Let us write the map in the statement as $X_r(A) \to Y_r(A)$ and let A_{f_i} be as in (2.4.1). We know from (2.3.3) and (2.4.5) that the theorem holds when $A = k[X_1, \dots, X_n]$. Moreover, if $f\colon A \to B$ is an étale extension then the horizontal maps in the square

$$\begin{tikzcd} W_r(B) \otimes_{W_r(A)} X_r(A) \arrow[r, "\cong"] \arrow[d] & X_r(B) \arrow[d] \\ W_r(B) \otimes_{W_r(A)} Y_r(A) \arrow[r, "\cong"] & Y_r(B). \end{tikzcd}$$

are isomorphisms by [I, I.1.14] and by (2.4.4). Therefore, the theorem holds for étale extensions of polynomial algebras over k and hence for A_{f_i}, $i = 1, \dots, s$. Now recall that $A_f \otimes_A A_g = A_{fg} = (A_f)_g$ which is étale over A_f. It follows that the chain map

$$\bigotimes_{W_r(A)} (X_r(A) \to X_r(A_{f_i})) \to \bigotimes_{W_r(A)} (X_r(A) \to X_r(A_{f_i}))$$

is an isomorphism in degrees $\geqslant 1$. We prove that both complexes are acyclic: write

$$\bigotimes_{W_r(A)} (X_r(A) \to X_r(A_{f_i})) \cong \big(\bigotimes_{W_r(A)} (W_r(A) \to W_r(A_{f_i}))\big) \otimes_{W_r(A)} X_r(A)$$

and use that by Lemma 2.4.6,

$$\bigotimes_{W_r(A)} (W_r(A) \to W_r(A_{f_i}))$$

is an acyclic complex of flat $W_r(A)$-modules. In view of Proposition 2.4.4 the same argument works with Y in place of X and the claim follows. □

COROLLARY 2.4.7. *Let A be as above. Then the map of* (1.5.9)

$$I: W\Omega_A^* \to \mathrm{TR}_*(A)$$

is well-defined and a natural isomorphism of complexes.

Proof. We recall the Milnor short exact sequence

$$0 \to \varprojlim_R{}^{(1)} \pi_{*+1} T(A)^{C_{p^s}} \to \mathrm{TR}_*(A) \to \varprojlim_R \pi_* T(A)^{C_{p^s}} \to 0$$

where by Theorem B,

$$\pi_* T(A)^{C_{p^{r-1}}} \cong W_r\Omega_A^* \otimes S_{\mathbf{F}_p}\{\sigma_r\}$$

with $\deg \sigma_r = 2$. In addition, $R(\sigma_r) = p\sigma_{r-1}$ and therefore the summand corresponding to the augmentation ideal of $S_{\mathbf{F}_p}\{\sigma_r\}$ does not contribute to the limit or the derived limit. The remaining part is the de Rham–Witt pro-complex $W_\bullet\Omega_A^*$ which has limit $W\Omega_A^*$ and satisfies the Mittag-Leffler condition, cf. (1.2.2). □

3. Typical curves in K-theory

3.1. Let A be a ring and let $K(A)$ denote Quillen's algebraic K-theory spectrum of A. We write $\widetilde{K}(A[X]/(X^n))$ for the homotopy fiber of the map

$$K(A[X]/(X^n)) \to K(A); \quad X \mapsto 0,$$

and recall from the introduction that the *curves on* $K(A)$ is the spectrum

$$C(A) = \operatorname*{\underleftarrow{holim}}_n \Omega\widetilde{K}(A[X]/(X^n)). \tag{3.1.1}$$

To evaluate $C(A)$ we use the cyclotomic trace of [BHM]. This is a map of spectra

$$\mathrm{trc}: K(A) \to \mathrm{TC}(A), \tag{3.1.2}$$

which is natural in the ring A. The target is the topological cyclic homology of A. As recalled in §1.3, we have two maps $R_r, F_r: \mathrm{TH}(A)^{C_m} \to \mathrm{TH}(A)^{C_n}$, whenever $m = rn$, and $\mathrm{TR}(A)$ was defined as the homotopy limits over the restriction maps. The Frobenius maps induce self maps $F_r: \mathrm{TR}(A) \to \mathrm{TR}(A)$, one for each positive integer, and $F_rF_s = F_{rs}$. Hence these maps specify an action by the multiplicative monoid of positive integers on $\mathrm{TR}(A)$ and one defines $\mathrm{TC}(A)$ as the homotopy fixed set,

$$\mathrm{TC}(A) = \mathrm{TR}(A)^{h\{F_r \mid r \in \mathbf{N}\}}. \tag{3.1.3}$$

Similarly, restricting attention to p-groups, the self maps $F_{p^s}\colon \mathrm{TR}(A;p)\to\mathrm{TR}(A;p)$ defines an action by the additive monoid of non-negative integers on $\mathrm{TR}(A;p)$ and

$$\mathrm{TC}(A;p)=\mathrm{TR}(A;p)^{h\{F_{p^s}\mid s\in\mathbf{N}_0\}}.$$

In this case, the homotopy fixed point spectrum is naturally equivalent to the homotopy fiber of the self map F_p-1 on $\mathrm{TR}(A;p)$. We note that in the definition of $\mathrm{TC}(A)$ and $\mathrm{TC}(A;p)$ one may interchange the role of the maps R_r and F_r. The resulting spectra are canonically homeomorphic. Finally, we recall from [HM1, Theorem 4.10], that after p-completion

$$\mathrm{TC}(A)^{\wedge}_{p}\simeq\mathrm{TC}(A;p)^{\wedge}_{p}. \tag{3.1.4}$$

If A is a $\mathbf{Z}/p^j$-algebra, then $\mathrm{TC}(A)$ is already p-complete. Indeed, any spectrum whose homotopy groups are bounded p-groups is p-complete, and hence [HM1, Addendum 3.3], shows that $\mathrm{TH}(A)^{C_n}$ is p-complete. Finally, a homotopy limit of p-complete spectra is again p-complete.

Next, we recall from [HM1, §7] that a pointed monoid is a monoid Π in the symmetric monoidal category of pointed spaces and smash product, or equivalently, a (topological) monoid Π with a base point $0\in\Pi$ such that the multiplication factors over the smash product

$$\mu\colon \Pi\wedge\Pi\to\Pi.$$

A pointed monoid has a cyclic bar construction $N^{\mathrm{cy}}_{\wedge}(\Pi)$, which is a cyclic space in the sense of Connes with k-simplices

$$N^{\mathrm{cy}}_{\wedge,k}(\Pi)=\Pi^{\wedge(k+1)}$$

and structure maps

$$\begin{aligned}
d_i(\pi_0\wedge\ldots\wedge\pi_k)&=\begin{cases}\pi_0\wedge\ldots\wedge\pi_i\pi_{i+1}\wedge\ldots\wedge\pi_k, & 0\leqslant i<k,\\ \pi_k\pi_0\wedge\pi_1\wedge\ldots\wedge\pi_{k-1}, & i=k,\end{cases}\\
s_i(\pi_0\wedge\ldots\wedge\pi_k)&=\pi_0\wedge\ldots\wedge\pi_i\wedge 1\wedge\pi_{i+1}\wedge\ldots\wedge\pi_k,\quad 0\leqslant i\leqslant k,\\
\tau_k(\pi_0\wedge\ldots\wedge\pi_k)&=\pi_k\wedge\pi_0\wedge\ldots\wedge\pi_{k-1}.
\end{aligned}$$

The realization has a continuous action by the circle group G.

Consider the pointed monoid $\Pi_n=\{0,1,X,\ldots,X^{n-1}\}$ with 0 as the base point and $X^n=0$. We also allow $n=\infty$. We proved in [HM1, Theorem 7.1] that there is an equivalence of G-spectra indexed on $\mathcal{U}$,

$$T(A[X]/(X^n))\simeq_G T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n)|, \tag{3.1.5}$$

where the smash product is formed in $G\mathcal{SU}$. In analogy with (2.2.2), the cyclic bar construction splits as a wedge of cyclic sets,

$$N_\wedge^{\mathrm{cy}}(\Pi_n) \cong \bigvee_{m=0}^{\infty} N_\wedge^{\mathrm{cy}}(\Pi_n; m),$$

where the summand $N_\wedge^{\mathrm{cy}}(\Pi_n; m)$ consists of 0 and the k-simplices $X^{i_0}\wedge \dots \wedge X^{i_k}$ with $i_0+\dots+i_k=m$. The realization splits accordingly, and this splitting is G-equivariant.

LEMMA 3.1.6. (i) *Let $l(m)$ be the least integer greater than $m/(n-1)$. Then $|N_\wedge^{\mathrm{cy}}(\Pi_n; m)|$ is at least $(l(m)-2)$-connected.*

(ii) *Whenever $m<n$, $N_\wedge^{\mathrm{cy}}(\Pi_n; m)=N_\wedge^{\mathrm{cy}}(\Pi_\infty; m)$.*

(iii) *If $m>0$, then S^1/C_{m+} is a strong G-deformation retract of $|N_\wedge^{\mathrm{cy}}(\Pi_\infty; m)|$, and $|N_\wedge^{\mathrm{cy}}(\Pi_\infty; 0)|=S^0$.*

(iv) *Let $P_n\colon \Pi_\infty \to \Pi_\infty$ be the map of pointed monoids given by $P_n(X)=X^n$. Then the diagram*

$$\begin{CD} S^1/C_{m+} @>>> |N_\wedge^{\mathrm{cy}}(\Pi_\infty; m)| \\ @V{\mathrm{pr}}VV @VV{P_{n*}}V \\ S^1/C_{mn+} @>>> |N_\wedge^{\mathrm{cy}}(\Pi_\infty; mn)| \end{CD}$$

is G-homotopy commutative.

Proof. (i) The k-simplices in $N_\wedge^{\mathrm{cy}}(\Pi_n; m)$ has the form

$$X^{i_0}\wedge \dots \wedge X^{i_k},$$

with $i_0+\dots+i_k=m$. Therefore, if $(k+1)(n-1)<m$, there is only one k-simplex 0 so the k-skeleton of the realization is a point.

(iii) As a pointed monoid $\Pi_\infty=\langle X\rangle_+$, where $\langle X\rangle$ is the free abelian monoid on one generator X. The claim follows from Lemma 2.2.3. $\square$

Under the equivalence of (3.1.5) the restriction map R_r annihilates the summands where m is not divisible by r and induces maps

$$R_{r,m}\colon j^*(T(A)\wedge|N_\wedge^{\mathrm{cy}}(\Pi_n; m)|)^{C_{rs}} \to j^*(T(A)\wedge|N_\wedge^{\mathrm{cy}}(\Pi_n; m/r)|)^{C_s} \tag{3.1.7}$$

of the remaining summands. This follows from the fact that (3.1.5) is an equivalence of cyclotomic spectra, compare [HM1, 7.1 and 8.2]. The Frobenius map preserves the splitting and induces maps

$$F_{r,m}\colon j^*(T(A)\wedge|N_\wedge^{\mathrm{cy}}(\Pi_n; m)|)^{C_{rs}} \to j^*(T(A)\wedge|N_\wedge^{\mathrm{cy}}(\Pi_n; m)|)^{C_s}.$$

THEOREM 3.1.8. *If A is a $\mathbf{Z}/p^j$-algebra then $\widetilde{\mathrm{TC}}_i(A[X]/(X^n))$ is a bounded p-group, i.e. any element is annihilated by p^N for some $N \geqslant 0$ which may depend on i.*

Proof. We recalled in (3.1.4) that the group in question is isomorphic to the group $\widetilde{\mathrm{TC}}_i(A[X]/(X^n);p)$. The topological Hochschild spectrum is given by (3.1.5) as the wedge sum

$$\widetilde{T}(A[X]/(X^n)) \simeq \bigvee_{m=1}^{\infty} T(A) \wedge |N_{\wedge}^{\mathrm{cy}}(\Pi_n;m)|$$

and the splitting being equivariant implies a corresponding splitting of the C_{p^r}-fixed set,

$$\widetilde{\mathrm{TH}}(A[X]/(X^n))^{C_{p^r}} \simeq \bigvee_{m=1}^{\infty} j^*(T(A) \wedge |N_{\wedge}^{\mathrm{cy}}(\Pi_n;m)|)^{C_{p^r}}.$$

The restriction map of (3.1.7) fits into a cofibration sequence

$$\begin{aligned} j^*(T(A) \wedge |N_{\wedge}^{\mathrm{cy}}(\Pi_n;m)|)_{hC_{p^r}} &\xrightarrow{N_{p,m}} j^*(T(A) \wedge |N_{\wedge}^{\mathrm{cy}}(\Pi_n;m)|)^{C_{p^r}} \\ &\xrightarrow{R_{p,m}} j^*(T(A) \wedge |N_{\wedge}^{\mathrm{cy}}(\Pi_n;m/p)|)^{C_{p^{r-1}}}. \end{aligned}$$

Indeed, this follows from [HM1, Proposition 2.1] and the equivalences

$$\begin{aligned} \varrho_{C_p}^{\#} \Phi^{C_p}(T(A) \wedge |N_{\wedge}^{\mathrm{cy}}(\Pi_n;m)|)) &\simeq_G \varrho_{C_p}^{\#} \Phi^{C_p} T(A) \wedge \varrho_{C_p}^{*} |N_{\wedge}^{\mathrm{cy}}(\Pi_n;m)|^{C_p} \\ &\simeq_G T(A) \wedge |N_{\wedge}^{\mathrm{cy}}(\Pi_n;m/p)|. \end{aligned}$$

In particular, when p does not divide m, the map $N_{p,m}$ is an equivalence. Now recall that taking homotopy orbits preserves connectivity. Hence the connectivity statement of (3.1.7) implies that $R_{p,m}$ is an $(l(m)-1)$-connected map, and if we write $m = p^s k$, with $(k,p)=1$, then the obvious induction argument shows that $j^*(T(A) \wedge |N_{\wedge}^{\mathrm{cy}}(\Pi_n;m)|)^{C_{p^r}}$ is an $(l(p^{s-r}k)-1)$-connected spectrum. Hence we may replace the wedge sums above by the corresponding products,

$$\widetilde{\mathrm{TH}}(A[X]/(X^n))^{C_{p^r}} \simeq \prod_{m=1}^{\infty} j^*(T(A) \wedge |N_{\wedge}^{\mathrm{cy}}(\Pi_n;m)|)^{C_{p^r}}.$$

Let us write $\mathrm{TF}(A[X]/(X^n);m;p)$ for the homotopy limit over the Frobenius maps $F_{p,m}$ of the mth factor in this product decomposition. We then have

$$\widetilde{\mathrm{TF}}(A[X]/(X^n);p) \simeq \prod_{m=1}^{\infty} \mathrm{TF}(A[X]/(X^n);m;p)$$

and hence

$$\widetilde{\mathrm{TC}}(A[X]/(X^n);p) \simeq \prod_{(k,p)=1} \underset{\overleftarrow{R}}{\mathrm{holim}}\, \mathrm{TF}(A[X]/(X^n);p^s k;p).$$

Moreover, the restriction map

$$R_{p,m}\colon \mathrm{TF}(A[X]/(X^n);m;p)\to \mathrm{TF}(A[X]/(X^n);m/p;p)$$

is $(l(m)-1)$-connected, and therefore, it suffices to show that the homotopy groups of $\mathrm{TF}(A[X]/(X^n);m;p)$ are bounded p-groups for every $m\geqslant 1$.

We fix $m=p^s k$ and consider $\pi_i(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)^{C_{p^r}}$ for $r\geqslant s$. We have the following tower of fibrations:

$$\begin{tikzcd}
j^*(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)_{hC_{p^r}} \arrow[r, "N_{p,m}"] & j^*(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)^{C_{p^r}} \arrow[d, "R_{p,m}"] \\
j^*(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m/p)|)_{hC_{p^{r-1}}} \arrow[r, "N_{p,m/p}"] & j^*(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m/p)|)^{C_{p^{r-1}}} \arrow[d, "R_{p,m/p}"] \\
 & \vdots \arrow[d, "R_{p,pk}"] \\
j^*(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;k)|)_{hC_{p^{r-s}}} \arrow[r, "N_{p,k}"] & j^*(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;k)|)^{C_{p^{r-s}}} \arrow[d, "R_{p,k}"] \\
 & *.
\end{tikzcd}$$

The spectral sequence of (1.3.11),

$$E^2_{i,j}=H_i(C_{p^r};\pi_j(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)) \implies \pi_{i+j}(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)_{hC_{p^r}},$$

is concentrated in the first quadrant above the line $y=l(m)-2$. Since A is a $\mathbf{Z}/p^j$-algebra the E^2-term, and hence also the E^∞-term, is a $\mathbf{Z}/p^j$-module. Hence $\pi_i(T(A)\wedge N^{\mathrm{cy}}_\wedge(\Pi_n;m))_{hC_{p^r}}$ is a p-group and every element has exponent less than or equal to $j(i-l(m)+3)$. Finally, the tower of fibrations above shows that $\pi_i(T(A)\wedge N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)^{C_{p^r}}$ is a p-group and the exponents of the elements are bounded by

$$\sum_{t=0}^{s} j(i-l(p^t k)+2)\leqslant j\left((s+1)(i+2)+\frac{k}{n-1}\cdot\frac{p^s-1}{p-1}\right).$$

This bound is independent of $r\geqslant s$ and the proposition follows. □

We let A be a $\mathbf{Z}/p^j$-algebra and consider the arithmetic square for $\widetilde{K}(A[X]/(X^n))$, i.e. the homotopy cartesian square

$$\begin{tikzcd}
\widetilde{K}(A[X]/(X^n)) \arrow[r] \arrow[d] & \widetilde{K}(A[X]/(X^n))^\wedge \arrow[d] \\
\widetilde{K}(A[X]/(X^n))\mathbf{Q} \arrow[r] & (\widetilde{K}(A[X]/(X^n))^\wedge)\mathbf{Q}.
\end{tikzcd}$$

Here the decoration $(-)^{\wedge}$ indicates profinite completion, i.e. the product of the p-completions, where p ranges over the primes. The rationalization is contractible by the main theorem of [G]. Indeed,

$$\widetilde{K}_*(A[X]/(X^n))\otimes\mathbf{Q}\cong\widetilde{\mathrm{HC}}_{*-1}(A[X]/(X^n))\otimes\mathbf{Q}$$

and it is easily seen that the cyclic homology of a $\mathbf{Z}/p^j$-algebra is rationally trivial. Similarly, a recent result of R. McCarthy, [Mc], recalled as Theorem A in [HM1], shows that the cyclotomic trace induces an equivalence

$$\widetilde{K}(A[X]/(X^n))^{\wedge}\simeq\widetilde{\mathrm{TC}}(A[X]/(X^n))^{\wedge}.$$

The right-hand side is rationally trivial by Theorem 3.1.8 and therefore the spectrum in the lower right-hand corner of the arithmetic square vanishes. Hence we have

THEOREM 3.1.9. *If A is a $\mathbf{Z}/p^j$-algebra, then*

$$\widetilde{K}(A[X]/(X^n))\simeq\widetilde{\mathrm{TC}}(A[X]/(X^n)). \qquad \square$$

As a corollary of Theorems 3.1.8 and 3.1.9 we get Theorem E of the introduction. We note that, based on work of Jan Stienstra, Chuck Weibel has shown that $\widetilde{K}_i(A[X]/(X^n))$ is a p-group, [We]. However, the fact that it is bounded is new. The bound may very well tend to infinity with i. Indeed, the calculation of the groups $\widetilde{K}_i(\mathbf{F}_p[X]/(X^2))$ in [HM1] shows that as i tends to infinity there are elements of arbitrarily large exponent.

THEOREM 3.1.10. *Let A be a $\mathbf{Z}/p^j$-algebra. Then there is a natural equivalence*

$$C(A)\simeq\mathrm{TR}(A)$$

and the spectra are p-complete.

Proof. We have $C(A)\simeq\underleftarrow{\mathrm{holim}}\,\Omega\widetilde{\mathrm{TC}}(A[X]/(X^n))$ by Theorem 3.1.9, and since homotopy limits commute,

$$C(A)\simeq\big(\underset{F}{\underleftarrow{\mathrm{holim}}}\big(\underset{n}{\underleftarrow{\mathrm{holim}}}\,\Omega\widetilde{T}(A[X]/(X^n))^{C_r}\big)\big)^{h\{R_r|r\in\mathbf{N}\}}.$$

The connectivity estimate in the proof of Theorem 3.1.8 shows that

$$\widetilde{\mathrm{TH}}(A[X]/(X^n))^{C_r}\simeq\prod_{m=1}^{\infty}j^*(T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)|)^{C_r},$$

and since homotopy limits commute with products we may take the homotopy limit over n one factor at the time. But for a fixed m the limit system is constant equal to $(T(A)\wedge S^1/C_{m+})^{C_r}$ when $n>m$ by Lemma 3.1.6, and hence

$$\underset{\underset{n}{\longleftarrow}}{\operatorname{holim}}\, \widetilde{\mathrm{TH}}(A[X]/(X^n))^{C_r} \simeq \prod_{m\geqslant 1} j^*(T(A)\wedge S^1/C_{m+})^{C_r}.$$

Since the Frobenius maps F_s preserve the splitting after m, [HM1, Lemma 8.2] shows that

$$\underset{\underset{F}{\longleftarrow}}{\operatorname{holim}}\big(\underset{\underset{n}{\longleftarrow}}{\operatorname{holim}}\, \widetilde{\mathrm{TH}}(A[X]/(X^n))^{C_r}\big) \simeq \prod_{m\geqslant 1} (T(A)\wedge S^1/C_{m+})^{G} \simeq \prod_{m\geqslant 1} \Sigma\, \mathrm{TH}(A)^{C_m}. \quad (3.1.11)$$

The restriction map R_r induces maps $R_{r,m}$ from a factor m to the factor m/r and annihilates the factors where r does not divide m, compare (3.1.7). Moreover, it follows from [HM1, Theorem 7.1] that

$$R_{r,m} = \Sigma R_r \colon \Sigma\, \mathrm{TH}(A)^{C_m} \to \Sigma\, \mathrm{TH}(A)^{C_{m/r}},$$

where R_r is the restriction map associated with $T(A)$. Hence

$$\underset{\underset{n}{\longleftarrow}}{\operatorname{holim}}\, \widetilde{\mathrm{TC}}(A[X]/(X^n)) \simeq \Sigma\bigg(\prod_{m=1}^{\infty} \mathrm{TH}(A)^{C_m}\bigg)^{h\{R_r \mid r\in \mathbf{N}\}},$$

the homotopy fixed set of the multiplicative monoid of natural numbers acting through the restriction maps, see also [HM1, 4.1]. But this is homeomorphic to $\Sigma\,\mathrm{TR}(A)$, and so $C(A)\simeq \Omega\Sigma\,\mathrm{TR}(A)\simeq \mathrm{TR}(A)$. □

3.2. The spectrum $\mathrm{TR}(A)$ on the right-hand side in Theorem 3.1.10 is a ring spectrum in a very strong sense: it is a commutative functor with smash product defined on spheres, cf. [HM1, Proposition 2.7]. On the other hand Bloch defines a pairing on $C(A)$, which makes $C(A)$ a homotopy associative ring spectrum. We recall how this is defined.

For any A-algebra R we consider the exact category $\mathbf{Nil}(R)$ whose objects are pairs (M,α) where M is a f.g. projective R-module and α is a nilpotent endomorphism of M. It comes with an obvious ascending filtration given by the exponent of α. We also write $\mathbf{P}(R)$ for the category of f.g. projective R-modules. If $A_m = A[X]/(X^m)$ and $m>n$ we have a biexact functor

$$\mathrm{Fil}_n\, \mathbf{Nil}(R)\times \mathbf{P}(A_m) \to \mathbf{P}(R), \quad ((M,\alpha),N)\mapsto M_\alpha \otimes_{A_m} N,$$

where M_α denotes M considered as an A_m-module with X acting through α. It induces a natural (weak) map of K-theory spaces

$$K(\mathrm{Fil}_n\, \mathbf{Nil}(R))\wedge K(A_m) \to K(R),$$

cf. [Wa], and since the bifunctors above are compatible as m and n varies we get in turn

$$K(\mathbf{Nil}(R))\wedge \underset{\overleftarrow{m}}{\operatorname{holim}} K(A_m)\to K(R). \tag{3.2.1}$$

Here we have also used the natural equivalence $K(\mathbf{Nil}(R))\simeq \underset{\longrightarrow}{\operatorname{holim}} K(\operatorname{Fil}_n \mathbf{Nil}(R))$. Next, we recall the localization sequence

$$K(\mathbf{H})\to K(R[t])\to K(R[t,t^{-1}]),$$

where $\mathbf{H}$ is the category of finitely generated t-torsion $R[t]$-modules of projective dimension $\leqslant 1$. It is isomorphic as a category to $\mathbf{Nil}(R)$. For any M in $\mathbf{H}$ is projective as an R-module. Hence the boundary map of the localization sequence provides a map

$$\Omega K(R[t,t^{-1}])\to K(\mathbf{Nil}(R)). \tag{3.2.2}$$

Now let $R{=}A_k$ and let $A_k\to A_k[t,t^{-1}]$ be the map which takes X to Xt^{-1}. We may compose the induced map on K-theory with the maps (3.2.1) and (3.2.2) to get a map

$$\Omega K(A_k)\wedge \underset{\overleftarrow{m}}{\operatorname{holim}} K(A_m)\to K(A_k)$$

and hence $\Omega K(A_k)\wedge \operatorname{holim} \Omega K(A_m)\to\Omega K(A_k)$. Since these are strictly compatible as k varies we get

$$\big(\underset{\overleftarrow{k}}{\operatorname{holim}}\, \Omega K(A_k)\big)\wedge\big(\underset{\overleftarrow{m}}{\operatorname{holim}}\, \Omega K(A_m)\big)\to \underset{\overleftarrow{k}}{\operatorname{holim}}\, \Omega K(A_k),$$

and finally, this lifts to a pairing

$$C(A)\wedge C(A)\to C(A). \tag{3.2.3}$$

One may show that this pairing makes $C(A)$ a homotopy associative ring spectrum. Hence we get a ring homomorphism from $C_*(A)$, with the induced ring structure, to the ring $[C(A),C(A)]_*$ of cohomology operations in the spectrum $C(A)$. When A is a $\mathbf{Z}_{(p)}$-algebra, there is an idempotent splitting of the big Witt ring $\mathbf{W}(A)=C_0(A)$ into a product indexed by the natural numbers prime to p of copies of the p-typical Witt ring $W(A)$, and the ring homomorphism above gives a corresponding set of idempotents in the ring of degree-zero cohomology operations in $C(A)$. Hence we get the splitting

$$C(A)\simeq \prod_{(k,p)=1} C(A;p) \tag{3.2.4}$$

of the curves on $K(A)$ as a product of copies of its p-typical part $C(A;p)$.

The homotopy groups $\mathrm{TR}_*(A)=\pi_*\mathrm{TR}(A)$ are given by Milnor's sequence

$$0 \to \varprojlim_n{}^{(1)} \pi_{i+1} T(A)^{C_n} \to \mathrm{TR}_i(A) \to \varprojlim_n \pi_i T(A)^{C_n} \to 0,$$

so in particular,

$$\mathrm{TR}_0(A) = \mathbf{W}(A). \tag{3.2.5}$$

Indeed, the limit on the right is equal to $\mathbf{W}(A)$ by [HM1, Addendum 3.3], and the proof of op. cit., Proposition 3.3 shows that the derived limit on the left vanishes.

LEMMA 3.2.6. *The isomorphism $C_*(A)\cong\mathrm{TR}_*(A)$ given by Theorem 3.1.10 is $\mathbf{W}(A)$-linear.*

Proof. We recall that any element in $\mathbf{W}(A)$ may be written uniquely as an infinite sum

$$x = \sum_{n=1}^{\infty} V^n(\omega(a_n)),$$

where $\omega: A \to \mathbf{W}(A)$ is the Teichmüller character, [Mu]. Therefore, to show that a map is $\mathbf{W}(A)$-linear it suffices to show that it commutes with the Verschiebung maps and with multiplication by $\omega(a)$ for all $a \in A$. In the case at hand, the Verschiebung V^n on $C_*(A)$ is induced from the A-algebra map $v_n: A_m \to A_{mn}$, $v_n(X)=X^n$, and similarly, multiplication by $\omega(a)$ is induced from the A-algebra map $c_a: A_m \to A_m$, $c_a(X)=aX$. We claim that the same holds for $\mathrm{TR}_*(A)$. Given this the lemma follows from the naturality of the equivalence in Theorem 3.1.10. We recall from (3.1.11) the equivalence

$$\operatorname*{holim}_{\overleftarrow{F,k}} \widetilde{\mathrm{TH}}(A[X]/(X^k))^{C_r} \simeq \prod_{m\geqslant 1} j^*(T(A)\wedge S^1/C_{m+})^G \simeq \prod_{m\geqslant 1} \Sigma\,\mathrm{TH}(A)^{C_m},$$

from which we get

$$\operatorname*{holim}_{\overleftarrow{k}} \widetilde{\mathrm{TC}}(A[X]/(X^k)) \simeq \Sigma\,\mathrm{TR}(A)$$

upon taking homotopy fixed sets for the action of the restriction maps. It follows from Lemma 3.1.6 that v_n maps the factor m to the factor mn by the map

$$v_{n,m}: j^*(T(A)\wedge S^1/C_{m+})^G \to j^*(T(A)\wedge S^1/C_{mn+})^G$$

induced from the projection $\pi_n^{mn}: S^1/C_m \to S^1/C_{mn}$. By [HM1, Lemma 8.1] the diagram

$$\begin{tikzcd}
\Sigma\,\mathrm{TH}(A)^{C_m} \arrow[r, "\simeq"] \arrow[d, "\Sigma V_n"] & j^*(T(A)\wedge S^1/C_{m++})^G \arrow[d, "(\pi_n^{mn})_*"] \\
\Sigma\,\mathrm{TH}(A)^{C_{mn}} \arrow[r, "\simeq"] & j^*(T(A)\wedge S^1/C_{mn+})^G
\end{tikzcd}$$

is homotopy commutative, and hence the claim for V_n follows.

The case of c_a is more involved. Consider the map

$$\tilde{c}\colon \underset{F,k}{\underleftarrow{\operatorname{holim}}}\, \widetilde{\mathrm{TH}}(A[X]/(X^k))^{C_r}\wedge A \to \underset{F,k}{\underleftarrow{\operatorname{holim}}}\, \widetilde{\mathrm{TH}}(A[X]/(X^k))^{C_r},$$

which is adjoint to the map which takes $a\in A$ to the map induced from c_a. Obviously, $\tilde{c}$ preserves the splitting of (3.1.11) and we shall prove that on the factor m it is given by the composite

$$\Sigma\,\mathrm{TH}(A)^{C_m}\wedge A \xrightarrow{\widetilde{\omega}} \Sigma\,\mathrm{TH}(A)^{C_m} \xrightarrow{\xi_{22}} \Sigma\,\mathrm{TH}(A)^{C_m}.$$

Here $\widetilde{\omega}$ is adjoint to the map which takes $a\in A$ to multiplication by $\omega(a)$ and ξ_{22} corresponds under the equivalence

$$\Sigma T(A)^{C_m} \simeq (T(A)^{C_m}\wedge S^1/C_{m+})^{G/C_m}$$

to the map induced from the map ξ of (1.3.2). Since we already proved that ξ_{22} is homotopic to the identity in Lemma 1.4.5, it follows that c_a induces multiplication by $\omega(a)$.

To verify the offered description of c we will need to recall precisely how the equivalence (3.1.5) is obtained. For details we refer to [HM1, §2 and §7]. We shall use the same notation as used there. For any ring B and any G-space Y one has a G-equivariant and associative pairing

$$\nu_{1,0}\colon \mathrm{THH}(B;Y)\wedge|N^{\mathrm{cy}}_{\wedge}(B)_{\bullet}| \to \mathrm{THH}(B;Y),$$

where B is viewed as a pointed monoid in the multiplicative structure. When $B= A[X]/(X^n)$ the subset $\Pi_n\subset B$ is a sub-pointed monoid and $A\subset B$ is a subring, so the pairing induces a map

$$f(Y)\colon \mathrm{THH}(A;Y)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n)_{\bullet}| \to \mathrm{THH}(B;Y).$$

Now let $V\subset\mathcal{U}$ be a G-representation and let S^V denote its one-point compactification. We get in particular

$$f(V)\colon t(A)(V)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n)_{\bullet}| \to t(B)(V),$$

and as V varies these form a map of G-prespectra indexed on $\mathcal{U}$. The equivalence of (3.1.5) is the induced map of associated G-spectra.

Next, we note the pairing of cyclic sets

$$\theta_{\bullet}\colon A\wedge N^{\mathrm{cy}}_{\wedge}(\Pi_\infty)_{\bullet} \to N^{\mathrm{cy}}_{\wedge}(A)_{\bullet},$$

which is adjoint to the map which takes $a\in A$ to the map induced from $i_a\colon \Pi_\infty\to A$, $X\mapsto a$. The wedge summand $N^{\mathrm{cy}}_\wedge(\Pi_\infty;m)$ is equal to $N^{\mathrm{cy}}_\wedge(\Pi_n;m)$, provided that $m<n$, which we henceforth assume. It follows immediately from the definitions that the diagram

$$\begin{tikzcd}
\mathrm{THH}(A;Y)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)_\bullet|\wedge A \arrow[r, "f(Y)\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\mathrm{tw}"] & \mathrm{THH}(B;Y)\wedge A \arrow[ddd, "\tilde{c}"] \\
\mathrm{THH}(A;Y)\wedge A\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)_\bullet| \arrow[d, "\mathrm{id}\wedge(\theta,\mathrm{id})"] & \\
\mathrm{THH}(A;Y)\wedge|N^{\mathrm{cy}}_\wedge(A)_\bullet|\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)_\bullet| \arrow[d, "\nu_{1,0}\wedge\mathrm{id}"] & \\
\mathrm{THH}(A;Y)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)_\bullet| \arrow[r, "f(Y)"] & \mathrm{THH}(B;Y)
\end{tikzcd}$$

commutes, and since the maps are equivariant the corresponding diagram of C_m-fixed points is also commutative. Let us write $C=C_m$. We consider the simplicial (but not cyclic) map

$$\psi_\bullet\colon A\to N^{\mathrm{cy}}_\wedge(A)_\bullet \xrightarrow{\Delta_C} (\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(A)_\bullet)^C$$

and note that $\tilde{\omega}$ is induced from

$$\mathrm{THH}(A;Y)^C\wedge A\xrightarrow{\mathrm{id}\wedge D\circ\psi}\mathrm{THH}(A;Y)^C\wedge|N^{\mathrm{cy}}_\wedge(A)_\bullet|^C\xrightarrow{\nu_{1,0}}\mathrm{THH}(A;Y)^C,$$

where D is the equivariant (non-simplicial) homeomorphism of [BHM, §1],

$$D\colon|(\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(A)_\bullet)^C|\to\varrho_C^*|N^{\mathrm{cy}}(A)_\bullet|^C.$$

The simplical map $\psi_\bullet$ induces a cyclic map

$$F\psi_\bullet\colon FA_\bullet\to(\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(A)_\bullet)^C,$$

where $FA_\bullet$ is the free cyclic set generated by the (constant) simplicial set A, cf. [H, §3]. One verifies easily that

$$FA_\bullet\cong A\wedge(\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(\Pi_n;m)_\bullet)^C,$$

such that $F\psi_\bullet=(\mathrm{sd}_C\,\theta_\bullet)^C$. Moreover, $|FA_\bullet|\cong A\wedge G_+$ and the realization of $F\psi_\bullet$ is equal to the composition

$$A\wedge G_+\xrightarrow{\psi\wedge\mathrm{id}}|(\mathrm{sd}_C N^{\mathrm{cy}}_\wedge(A)_\bullet)^C|\wedge G_+\xrightarrow{\mu}|(\mathrm{sd}_C N^{\mathrm{cy}}(A)_\bullet)^C|,$$

where μ is the action. Now the claimed description of $\tilde{c}$ follows since $\nu_{1,0}$ is G-equivariant. □

3.3. In this section we give a splitting of the spectrum $\mathrm{TR}(A)$ into copies of its 'p-typical' part $\mathrm{TR}(A;p)$ and prove Theorem A of the introduction.

PROPOSITION 3.3.1. *Let A be a $\mathbf{Z}_{(p)}$-algebra. Then there is a natural equivalence*

$$\mathrm{TR}(A) \simeq \prod_{(k,p)=1} \mathrm{TR}(A;p),$$

where the product ranges over the natural numbers prime to p.

Proof. Let $\mathcal{F}$ be the category of natural numbers ordered after division and let $\mathcal{F}_p$ and $\mathcal{F}_{p'}$ be the full subcategories of natural numbers which are a power of p and prime to p, respectively. Then $\mathcal{F}=\mathcal{F}_p\times\mathcal{F}_{p'}$, and hence

$$\mathrm{TR}(A) = \underset{\mathcal{F}}{\operatorname{holim}} \mathrm{TH}(A)^{C_n} = \underset{\mathcal{F}_{p'}}{\operatorname{holim}} \big(\underset{\mathcal{F}_p}{\operatorname{holim}} \mathrm{TH}(A)^{C_{p^s l}}\big),$$

where, we remember, the limit runs over the restriction maps. We claim that when A is a $\mathbf{Z}_{(p)}$-algebra, the map

$$\prod R_{l/k}F_k\colon \mathrm{TH}(A)^{C_n} \to \prod_{k|l} \mathrm{TH}(A)^{C_{p^s}} \tag{3.3.2}$$

is an equivalence. Given this we get

$$\underset{\mathcal{F}_p}{\operatorname{holim}} \mathrm{TH}(A)^{C_{p^s l}} \simeq \prod_{k|l} \mathrm{TR}(A;p)$$

from which the proposition follows. To prove the claim suppose first that $l=q^t$ where q is a prime. From [HM1, Proposition 2.3] we have a cofibration sequence

$$(\varrho^*_{C_{p^s}} \mathrm{TH}(A)^{C_{p^s}})_{hC_{q^t}} \xrightarrow{N} \mathrm{TH}(A)^{C_{p^s q^t}} \xrightarrow{R_q} \mathrm{TH}(A)^{C_{p^s q^{t-1}}},$$

where, if $t=1$, the spectrum on the right is a trivial spectrum. Moreover, we have a homotopy commutative diagram

$$\begin{array}{ccccc}
(\varrho^*_{C_{p^s}} \mathrm{TH}(A)^{C_{p^s}})_{hC_{q^t}} & \xrightarrow{N} & \mathrm{TH}(A)^{C_{p^s q^t}} & \xrightarrow{R_q} & \mathrm{TH}(A)^{C_{p^s q^{t-1}}} \\
\downarrow{\scriptstyle \mathrm{trf}} & & \downarrow{\scriptstyle F_q} & & \downarrow{\scriptstyle F_q} \\
(\varrho^*_{C_{p^s}} \mathrm{TH}(A)^{C_{p^s}})_{hC_{q^{t-1}}} & \xrightarrow{N} & \mathrm{TH}(A)^{C_{p^s q^{t-1}}} & \xrightarrow{R_q} & \mathrm{TH}(A)^{C_{p^s q^{t-2}}},
\end{array}$$

where trf is the transfer of the projection going in the opposite direction. It is an equivalence because $\mathrm{TH}(A)^{C_{p^s}}$ is a ring spectrum and $1/q\in W_{s+1}(A)=\pi_0 T(A)^{C_{p^s}}$, and hence the right-hand square is homotopy cartesian. Now the obvious induction argument proves (3.3.2) in the case where $l=q^t$. The general case follows by a further induction over the prime divisors in l. ☐

ADDENDUM 3.3.3. *On homotopy groups the map V_dF_d multiplies by d on the factors k with $d|k$ and annihilates the remaining factors.*

Proof. Let T be a G-spectrum indexed on $\mathcal{U}$ and let $C\subset G$ be a finite subgroup. We recall from [HM1, §2] that there is G-spectrum $\varrho_C^{\#}T^C$ indexed on $\mathcal{U}$ such that $j^*(\varrho_C^{\#}T^C)=\varrho_C^*(j^*T)^C$. In particular, if

$$\mathbf{TR}(A;p)=\underset{R_p}{\underleftarrow{\operatorname{holim}}}\,\varrho_{C_{p^s}}^{\#}T(A)^{C_{p^s}}$$

then $j^*\mathbf{TR}(A;p)=\mathrm{TR}(A;p)$. It follows that we have Verschiebung maps $V_d\colon \mathrm{TR}(A;p)\to \mathrm{TR}(A;p)$. Now, from (3.3.2) we have the (non-equivariant) equivalence

$$\prod R_{l/k}F_k\colon \mathrm{TR}(A;p)^{C_l}\to\prod_{k|l}\mathrm{TR}(A;p). \tag{3.3.4}$$

Evidently, $F_d\colon \mathrm{TR}(A;p)^{C_l}\to\mathrm{TR}(A;p)^{C_{l/d}}$ corresponds to the projection which sends a factor k divisible by d to the factor k/d and annihilates the remaining factors. We shall next show that when k is not divisible by d, the composite

$$\mathrm{TR}(A;p)^{C_{l/d}}\xrightarrow{V_d}\mathrm{TR}(A;p)^{C_l}\xrightarrow{R_{l/k}F_k}\mathrm{TR}(A;p)$$

is null homotopic. We note that there exists a prime q which both divides d and l/k. We may assume that q does not divide l/d. Indeed, if $l/d=q^tl'/d$ with $(l'/d,q)=1$ then $R_{l/k}F_kV_d\simeq R_{l'/k}F_kV_dR_{q^t}$. Therefore the spectrum $\varrho_{C_{p^sl/d}}^{\#}T(A)^{C_{p^sl/d}}$ is q-cyclotomic by [HM1, Proposition 2.3], so we have a cofibration sequence

$$(\varrho_{C_{p^sl/d}}^{*}\mathrm{TH}(A)^{C_{p^sl/d}})_{hC_{q^r}}\xrightarrow{N}\mathrm{TH}(A)^{C_{p^sq^rl/d}}\xrightarrow{R_q}\mathrm{TH}(A)^{C_{p^sq^{r-1}l/d}}.$$

We take the homotopy limit over R_p and obtain the cofibration sequence

$$(\varrho_{C_{l/d}}^{*}\mathrm{TR}(A;p)^{C_{l/d}})_{hC_{q^r}}\xrightarrow{N}\mathrm{TR}(A;p)^{C_{q^rl/d}}\xrightarrow{R_q}\mathrm{TR}(A;p)^{C_{q^{r-1}l/d}}.$$

The point of this is that [HM1, Lemma 3.2] shows that V_{q^r} factors through N such that $R_qV_{q^r}$ is null homotopic. Since $R_{l/k}F_kV_d\simeq F_kR_{l/k}V_d\simeq R_{l/qk}V_{d/q^r}R_qV_{q^r}$ the claim follows.

Next recall that F_dV_d induces multiplication by d on homotopy groups. It follows that on homotopy groups $V_d\colon \mathrm{TR}(A;p)^{C_{l/d}}\to\mathrm{TR}(A;p)^{C_l}$ maps the factor k/d to the factor k by multiplication by d. Finally,

$$\mathrm{TR}(A)=\underset{\mathcal{F}_{p'}}{\underleftarrow{\operatorname{holim}}}\,\mathrm{TR}(A;p)^{C_l}$$

and the limit system on the right induces a Mittag-Leffler system on the level of homotopy groups. Therefore we get the same description of F_d and V_d on $\mathrm{TR}_*(A)$. $\square$

Recall that under the idempotent decomposition of $\mathbf{W}(A)$, preceding (3.2.4), the projection onto factors k divisible by d is given by $(1/d)V_dF_d$. Hence the projection onto the factor k is given by

$$\mathrm{pr}_k = \prod_{(d,p)=1} \left(\frac{1}{k}V_kF_k - \frac{1}{dk}V_{dk}F_{dk}\right).$$

It follows that the product decompositions of $\mathrm{TR}_*(A)$ induced from Proposition 3.3.1 and from the idempotents in $\mathbf{W}(A)$ are equal. This proves Theorem A of the introduction.

3.4. The differential defined by Bloch, [Bl], on the symbolic part of $SC_*(A)\subset C_*(A)$ for certain rings was extended to a degree-one operator on all of $C_*(A)$ and all rings by Stienstra, [St]. The basis of the construction is a map

$$K(\mathbf{Z}[t,t^{-1}]) \to K(\mathbf{Nil}(\mathbf{Z}[t]/(t^n))) \tag{3.4.1}$$

which we now descibe. In the localization sequence

$$K(\mathbf{H}) \to K(\mathbf{Z}[t,y]) \to K(\mathbf{Z}[t,y,(1-ty)^{-1}])$$

the right-hand map is split by the map induced from the ring homomorphism mapping t and y to zero, and moreover, the resolution and devissage theorems show that $K(\mathbf{H})\simeq K(\mathbf{Z}[t,t^{-1}])$. In particular, we get a map

$$K(\mathbf{Z}[t,t^{-1}]) \to \Omega K(\mathbf{Z}[t,y,(1-ty)^{-1}]).$$

We compose this with the map

$$\Omega K(\mathbf{Z}[t,y,(1-ty)^{-1}]) \to \Omega K(\mathbf{Z}[t,u,u^{-1}]/(t^n))$$

given by the ring homomorphism which maps t to t and y to u^{-1}, and finally, we compose with (3.2.2) to get the required map.

We may combine (3.4.1), the map on K-theory induced from $\mathbf{Z}[t]/(t^n)\to A[t]/(t^n)$ and the pairing (3.2.1) to get a pairing

$$K(\mathbf{Z}[t,t^{-1}])\wedge \underset{\underset{m}{\longleftarrow}}{\mathrm{holim}}\, K(A_m) \to K(A_n).$$

For varying n these are compatible, so we get a pairing into the homotopy limit of the spaces on the right. Finally, this factors to

$$K(\mathbf{Z}[t,t^{-1}])\wedge C(A) \to C(A). \tag{3.4.2}$$

We have $K_1(\mathbf{Z}[t,t^{-1}])\cong\mathbf{Z}\oplus\mathbf{Z}/2$ generated by t and -1. Let us note that t and -1 correspond to σ and η under the isomorphism $\pi_1^S(S^1_+)\cong K_1(\mathbf{Z}[t,t^{-1}])$ given by the unit of the spectrum $K(\mathbf{Z})$ and the assembly map, compare (1.4.1). Now Stienstra defines a degree-one map

$$d\colon C_*(A)\to C_{*+1}(A) \tag{3.4.3}$$

as the pairing with t. However, little is known about this map except that it extends Bloch's differential on the symbolic part and Stienstra's differential on $\widetilde{K}_*(\mathbf{End}(A))$. We leave it as an open question whether $\delta=d$ under the isomorphism of Theorem 3.1.10.

3.5. In this section we evaluate the complex of p-typical curves for the ring $k[\varepsilon]$ of dual numbers over the perfect field k and prove Theorem D of the introduction.

Let $W_s\subset\mathbf{R}[C_s]$ denote the maximal complex subrepresentation and let S^{W_s} be the one-point compactification. We shall write $T(k)_{W_s}$ for the smash product G-spectrum $T(k)\wedge S^{W_s}$. Recall from [HM1, 8.2] that there is a cofibration sequence of G-spectra

$$\bigvee_{r\geqslant 1} T(k)_{W_{2r}}\wedge S^1/C_{r+}\xrightarrow{\mathrm{sq}} T(k)\vee\bigvee_{s\geqslant 1} T(k)_{W_s}\wedge S^1/C_{s+}\to T(k[\varepsilon]), \tag{3.5.1}$$

where the map sq takes the summand r to the summand $s=2r$ by the map which is the identity on the first smash factor and the projection on the second. In fact, the spectra in (3.5.1) are cyclotomic in the sense of [HM1, 2.2], and the maps preserve the cyclotomic structure. It follows that we may apply the construction $\mathrm{TR}(-;p)$ and still have a cofibration sequence of G-spectra,

$$\mathrm{TR}\Big(\bigvee_{r\geqslant 1} T(k)_{W_{2r}}\wedge S^1/C_{r+};p\Big)\xrightarrow{\mathrm{sq}}\mathrm{TR}\Big(T(k)\vee\bigvee_{s\geqslant 1} T(k)_{W_s}\wedge S^1/C_{s+};p\Big)\to\mathrm{TR}(k[\varepsilon];p). \tag{3.5.2}$$

We evaluate the map sq on homotopy groups. As in the smooth case the homotopy groups turns out to be rather big, so we first introduce some notation.

Consider the differential graded algebra

$$E^*_\sigma=W\Omega^*_{k[X]}\otimes_{W(k)}S_{W(k)}\{\sigma\}, \tag{3.5.3}$$

which is the tensor product of the de Rham–Witt complex for $k[X]$ and a polynomial algebra over $W(k)$ in one generator σ of degree 2 (with zero differential). Based on Theorem 2.1.3 we have the following alternate description: every element $\omega\in E^m_\sigma$ may be written uniquely as either

$$\omega=\sum_{j\in\mathbf{N}[1/p]} a_{ij}\sigma^i X^j \quad\text{or}\quad \omega=\sum_{j\in\mathbf{N}[1/p]-0} b_{ij}\sigma^i X^j d\log X,$$

depending on whether m is even or odd. The coefficients $a_{ij}, b_{ij} \in W(k)$ and are subject to the requirement that $\operatorname{den}(j) | a_{ij}$ and that for every N, $v_p(a_{ij})$ and $v_p(\operatorname{den}(j) b_{ij})$ are $\geqslant N$ for all but finitely many j. Here $v_p(a)$ denotes the p-adic valuation of a.

We next define DG-ideals $I_\sigma, J_\sigma \subset E_\sigma$. For $i \in \mathbf{N}$ and $j \in \mathbf{N}[1/p] - 0$ let

$$\varepsilon(j) = \begin{cases} 1, & \text{if } \operatorname{num}(j) \text{ is even or } p=2, \\ 0, & \text{else,} \end{cases}$$

and let $m(i,j)$ be the unique integer such that $p^{m(i,j)-1} j \leqslant 2i+1+\varepsilon(j) < p^{m(i,j)} j$. We also set $m(i,0)=1$ for $i>0$ and $m(0,0)=0$. We define

$$I_\sigma^m = \{\omega \in E_\sigma^m \,|\, v_p(a_{ij}) \geqslant m(i,j) \text{ or } v_p(\operatorname{den}(j) b_{ij}) \geqslant m(i,j), \text{ for all } j \in \mathbf{N}[1/p]\}$$

and

$$J_\sigma^m = \{\omega \in E_\sigma^m \,|\, v_p(a_{ih}) \geqslant m(i,2h) \text{ or } v_p(\operatorname{den}(h) b_{ih}) \geqslant m(i,2h), \text{ for all } h \in \mathbf{N}[1/p]\}.$$

One may check that this defines DG-ideals of E_σ^*. However, it would be desirable to have a more conceptual description. Also let $\widetilde{E}_\sigma^* \subset E_\sigma^*$ be the augmentation ideal, i.e. the series with $a_{i,0}=0$, and let $\tilde{J}_\sigma^* = \widetilde{E}_\sigma^* \cap J_\sigma^*$.

PROPOSITION 3.5.5. *Let k be a perfect field of positive characteristic p. Then the map* sq *in* (3.5.2) *is given on homotopy groups as*

$$\mathrm{sq}_*\colon \widetilde{E}_\sigma^* / \tilde{J}_\sigma^* \to E_\sigma^* / I_\sigma^*,$$

where sq_* *is the DG-map given by* $\mathrm{sq}_*(X) = X^2$.

Proof. We refer the reader to [HM1, §2] for the notion of a cyclotomic spectrum. For any cyclotomic spectrum T,

$$\mathrm{TR}(T;p) = \underset{R}{\underleftarrow{\operatorname{holim}}}\, \varrho_{C_{p^n}}^* (j^* T)^{C_{p^n}}.$$

So we must evaluate the C_{p^n}-fixed points of the spectra in (3.5.1). Let us write $v = v(n,s) = \min\{n, v_p(s)\}$. Then we have from [HM1, §8] that

$$\varrho_{C_{p^n}}^{\#} (T(k)_{W_s} \wedge S^1/C_{s+})^{C_{p^n}} \simeq_G \varrho_{C_{p^v}}^{\#} T(k)_{W_s}^{C_{p^v}} \wedge S^1/C_{s/p^v +}. \tag{3.5.6}$$

The cyclotomic structure on the spectra in (3.5.1) is given by the remark preceding [HM1, Theorem 7.1]. We recall that the map R takes a summand with s (or r) divisible by p to the summand s/p by a map

$$R^{(s)}\colon \varrho_{C_{p^v}}^{\#} T(k)_{W_s}^{C_{p^v}} \wedge S^1/C_{s/p^v +} \to \varrho_{C_{p^v}}^{\#} T(k)_{W_{s/p}}^{C_{p^{v-1}}} \wedge S^1/C_{s/p^v +}$$

and annihilates the remaining summands. Moreover, $R^{(s)}=R_{W_s}\wedge\mathrm{id}$ and there is a cofibration sequence of non-equivariant spectra

$$j^*(T(k)_{W_s})_{hC_{p^v}} \xrightarrow{N} j^*(T(k)_{W_s})^{C_{p^v}} \xrightarrow{R_{W_s}} j^*(T(k)_{W_{s/p}})^{C_{p^{v-1}}},$$

see [HM1, §2.2]. It is convenient to reindex the wedge sums in (3.5.1) such that R preserves the index. So let $j=s/p^n$ and $h=r/p^n$, respectively. We may then rewrite (3.5.6) as

$$\varrho^{\#}_{C_{p^n}}(T(k)_{W_s}\wedge S^1/C_{s+})^{C_{p^n}} \simeq_G \varrho^{\#}_{C_{p^n/\mathrm{den}(j)}} T(k)^{C_{p^n/\mathrm{den}(j)}}_{W_{p^nj}} \wedge S^1/C_{\mathrm{num}(j)+},$$

and similarly we have

$$\varrho^{\#}_{C_{p^n}}(T(k)_{W_{2r}}\wedge S^1/C_{r+})^{C_{p^n}} \simeq_G \varrho^{\#}_{C_{p^n/\mathrm{den}(h)}} T(k)^{C_{p^n/\mathrm{den}(h)}}_{W_{p^n2h}} \wedge S^1/C_{\mathrm{num}(h)+}.$$

With this indexing the map sq in (3.5.2) takes the summand h to the summand $j=2h$ by a map

$$\mathrm{sq}^{(h)}\colon \varrho^{\#}_{C_{p^n/\mathrm{den}(h)}} T(k)^{C_{p^n/\mathrm{den}(h)}}_{W_{p^n2h}} \wedge S^1/C_{\mathrm{num}(h)+} \to \varrho^{\#}_{C_{p^n/\mathrm{den}(j)}} T(k)^{C_{p^n/\mathrm{den}(j)}}_{W_{p^nj}} \wedge S^1/C_{\mathrm{num}(j)+},$$

and [HM1, Lemma 8.1], shows that

$$\mathrm{sq}^{(h)} = \begin{cases} \mathrm{id}\wedge\mathrm{pr}, & \text{if } p \text{ is odd, or } p=2 \text{ and } \mathrm{den}(h)=1, \\ V_2\wedge\mathrm{id}, & \text{if } p=2 \text{ and } \mathrm{den}(h)>1. \end{cases} \tag{3.5.7}$$

It remains to take wedges over j and h and then homotopy limit over R. However, we have already shown in the proof of Theorem 3.1.8 that the wedge sums in (3.5.1) and (3.5.2) may be replaced by the corresponding products. Therefore, we may instead first take the homotopy limit over R and then wedges over h and j. The canonical map

$$\pi_{2i} \underset{R}{\underleftarrow{\mathrm{holim}}}\, T(k)^{C_{p^n/\mathrm{den}(j)}}_{W_{p^nj}} \to \pi_{2i} T(k)^{C_{p^{m-1}/\mathrm{den}(j)}}_{W_{p^{m-1}j}}$$

is an isomorphism provided that $2i<\dim W_{p^mj}$. Indeed, this follows from the cofibration sequence above. On the other hand, [HM1, Proposition 9.1] shows that

$$\pi_{2i} T(k)^{C_{p^{m-1}/\mathrm{den}(j)}}_{W_{p^{m-1}j}} \cong W(k)/(p^m/\mathrm{den}(j)) \cong \mathrm{den}(j)W(k)/p^mW(k),$$

when $\dim W_{p^{m-1}j}\leqslant 2i$. Similarly, we have an isomorphism

$$\pi_{2i} \underset{R}{\underleftarrow{\mathrm{holim}}}\, T(k)^{C_{p^n/\mathrm{den}(h)}}_{W_{p^n2h}} \to \pi_{2i} T(k)^{C_{p^{m-1}/\mathrm{den}(h)}}_{W_{p^{m-1}2h}},$$

whenever $2i<\dim W_{p^m 2h}$, and

$$\pi_{2i}T(k)^{C_{p^{m-1}/\operatorname{den}(h)}}_{W_{p^{m-1}2h}} \cong W(k)/(p^m/\operatorname{den}(h)) \cong \operatorname{den}(h)W(k)/p^m W(k),$$

if $\dim W_{p^{m-1}2h} \leqslant 2i$. Finally, $\dim W_s = s-1$, if s is odd, and $\dim W_s = s-2$, if s is even. This shows that the homotopy groups of the middle and left-hand term in (3.5.2) are as claimed in the statement of the proposition. The map sq is given by (3.5.7) since pr induces the identity in even dimensions and multiplication by 2 in odd dimensions, respectively, while V_2 corresponds to the map V_2 on Witt vectors, cf. [HM1, Proposition 9.1]. □

When p is odd, sq_* is injective and so we have

COROLLARY 3.5.8. *If* $\operatorname{char} k$ *is odd, then there is a short exact sequence of complexes*

$$0 \to \widetilde{E}^*_\sigma/\tilde{J}^*_\sigma \xrightarrow{\mathrm{sq}_*} E^*_\sigma/I^*_\sigma \to C_*(k[\varepsilon];p) \to 0. \qquad \square$$

Proof of Theorem D. First note that if $\operatorname{den}(j)>1$ then the differential induces an isomorphism

$$\operatorname{den}(j)W(k)/p^{m(i,j)}W(k)\langle\sigma^i X^j\rangle \xrightarrow{d} W(k)/(p^{m(i,j)}/\operatorname{den}(j))W(k)\langle\sigma^i X^j d\log X\rangle,$$

so we may disregard the summands in E^*_σ/I^*_σ and $\widetilde{E}^*_\sigma/\tilde{J}^*_\sigma$ generated by $\sigma^i X^j$ or $\sigma^i X^j d\log X$ with $\operatorname{den}(j)>1$. So we only have to consider the summands where j is a natural number.

Suppose now that p is odd. Then sq_* maps $\widetilde{E}^*_\sigma/\tilde{J}^*_\sigma$ onto the summands in E^*_σ/I^*_σ with $j>0$ and even. So we only get a contribution in the cohomology from the summands $j=0$ and $j\geqslant 1$, odd. The differential on such a summand is given by

$$d\colon W(k)/p^{m(i,j)}W(k)\langle\sigma^i X^j\rangle \to W(k)/p^{m(i,j)}\langle\sigma^i X^j d\log X\rangle, \quad d(\sigma^i X^j) = j\sigma^i X^j d\log X.$$

This finishes the proof for p odd. The proof for $p=2$ is similar. □

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