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Multiplicities of recurrence sequences

by

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1. An introduction

We will study equations

$$\sum_{i=1}^{r} f_i(m)\alpha_i^m = 0 \tag{1.1}$$

in the variable $m\in\mathbf{Z}$. Here the f_i are nonzero polynomials with complex coefficients of respective degrees k_i ($1\leqslant i\leqslant r$) and we put

$$k_1+\ldots+k_r+r=q. \tag{1.2}$$

We suppose that the α_i are nonzero elements of a number field K with

$$[K:\mathbf{Q}]=d \tag{1.3}$$

and that moreover for each pair i,j with $1\leqslant i<j\leqslant r$,

$$\alpha_i/\alpha_j \text{ is not a root of unity.} \tag{1.4}$$

We prove

THEOREM 1.1. *Assume that we have* (1.2), (1.3), (1.4). *Then equation* (1.1) *has not more than*

$$d^{6q^2}2^{2^{28q!}} \tag{1.5}$$

solutions $m\in\mathbf{Z}$.

Results on equations (1.1) have been derived recently in [14] and shortly afterwards in [12]. However in both papers the bound for the number of solutions is only "semi-uniform", as it depends upon q, d and moreover upon ω, which is defined as the number

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of distinct prime ideal factors occurring in the decomposition of the fractional ideals (α_i) in K. The main feature in (1.5) is that now we avoid this parameter ω and thus we have a completely uniform upper bound for the number of solutions.

As is well known, Theorem 1.1 has consequences for linear recurrence sequences: Let n be a natural number and consider the recurrence relation

$$u_{m+n}=\nu_{n-1}u_{m+n-1}+\nu_{n-2}u_{m+n-2}+\ldots+\nu_0 u_m. \tag{1.6}$$

Here we assume that $\nu_{n-1},\ldots,\nu_0$ are algebraic numbers and that $\nu_0\neq 0$. We call n the order of relation (1.6). We assume moreover that the initial values $u_0,\ldots,u_{n-1}$ of our sequence have $|u_0|+\ldots+|u_{n-1}|>0$. Let

$$G(z)=z^n-\nu_{n-1}z^{n-1}-\ldots-\nu_0=\prod_{i=1}^{r}(z-\alpha_i)^{\varrho_i} \tag{1.7}$$

be the companion polynomial of the recurrence (1.6) with distinct zeros $\alpha_1,\ldots,\alpha_r$ of respective multiplicities ϱ_i $(1\leqslant i\leqslant r)$. It is well known that if $(u_m)_{m\in\mathbf{Z}}$ is a linear recurrence sequence, then there is a minimal n and there are complex numbers $\nu_{n-1},\ldots,\nu_0$ with $\nu_0\neq 0$ such that the sequence satisfies (1.6), but no such relation of order $<n$. Then we have a unique representation

$$u_m=\sum_{i=1}^{r}g_i(m)\alpha_i^m \quad (m\in\mathbf{Z}) \tag{1.8}$$

where the g_i are polynomials of degree ϱ_i-1 $(1\leqslant i\leqslant r)$. (The actual shape of the polynomials g_i will depend also upon the initial values $u_0,\ldots,u_{n-1}$. But this will be of no importance in the sequel.)

For a complex number a, the *a-multiplicity* $U(a)$ of the sequence $(u_m)_{m\in\mathbf{Z}}$ is defined as the number of indices m such that $u_m=a$. Moreover the *multiplicity* of (u_m) is defined as

$$U=\sup_a U(a). \tag{1.9}$$

The Theorem of Skolem–Mahler–Lech says the following: *If $(u_m)_{m\in\mathbf{Z}}$ is a recurrence sequence with infinite 0-multiplicity, then those m for which $u_m=0$ form a finite union of arithmetic progressions plus possibly a finite set.* A particular consequence of this theorem is: *If a recurrence* (1.6) *with companion polynomial* (1.7) *generates a sequence (u_m) with infinite 0-multiplicity, then there exist indices $i\neq j$ such that the quotient α_i/α_j is a root of unity.*

We therefore call the recurrence sequence $\{u_m\}_{m\in\mathbf{Z}}$ *nondegenerate* if for each pair i,j with $1\leqslant i<j\leqslant r$ the ratio α_i/α_j of the roots of the companion polynomial (1.7) is not a root of unity, i.e. if (1.4) is satisfied.

For nondegenerate *binary sequences* (i.e. sequences of order $n=2$) *of rational integers* M. Ward in the thirties conjectured that their multiplicity is bounded by 5. Kubota [6] succeeded in showing that in fact the multiplicity of such sequences does not exceed 4. Beukers [1] even proved that with five exceptions (which he gives explicitly) nondegenerate binary sequences of rational integers have multiplicity at most 3.

In the binary case, if the terms of the sequences $\{u_m\}$ belong to a number field K of degree d, Kubota [7] showed that the multiplicity is bounded by a constant that depends only upon d. Beukers and Tijdeman [3] here established the bound

$$U \leqslant 100 \max\{d, 300\}. \tag{1.10}$$

For *ternary sequences of rational integers* Beukers [2] proved that the 0-multiplicity does not exceed 6.

As for general *nondegenerate sequences of order* n we first remark that there is one very simple case: If the sequence $\{u_m\}$ has a representation (1.8) where the α_i as well as the coefficients of the polynomials g_i are real, an application of Rolle's Theorem implies that $\{u_m\}$ has multiplicity $U \leqslant 2n$ (cf. Pólya–Szegő [11, Aufgabe 75, p. 48]).

Now assume that the roots of the companion polynomial are contained in a number field K of degree d. Then by the results of [12] and [14], the multiplicity of a sequence of order n has $U \leqslant c(n, d, \omega)$, where ω denotes the number of prime ideal factors in the decomposition of the fractional ideals (α_i) in K. On the other hand, the natural extension of Ward's conjecture says that *a nondegenerate sequence of rational numbers of order* n *has multiplicity bounded by a constant that depends only upon* n. Notice that the result of [12] and [14] in that case gives the semi-uniform bound $c(n, \omega)$.

Our Theorem 1.1 now implies the conjecture in general. We get:

THEOREM 1.2. *Let* $(u_m)_{m\in\mathbf{Z}}$ *be a nondegenerate linear recurrence sequence of order* n. *Assume that the characteristic roots* α_i *of the recurrence relation* (*as defined in* (1.7)) *are contained in a number field* K *of degree* d. *Then the zero-multiplicity of the sequence* $(u_m)_{m\in\mathbf{Z}}$ *satisfies*

$$U(0) \leqslant d^{6n^2} 2^{2^{28n!}}. \tag{1.11}$$

As for the multiplicity we have

THEOREM 1.3. *Let the hypotheses be the same as in Theorem* 1.2. *Assume moreover that the sequence* $(u_m)_{m\in\mathbf{Z}}$ *is not periodic. Then its multiplicity satisfies*

$$U \leqslant d^{6(n+1)^2} 2^{2^{28(n+1)!}}. \tag{1.12}$$

For rational sequences these results give:

COROLLARY 1.4. *Let (u_m) be a nondegenerate linear recurrence sequence of rational numbers of order n. Then (u_m) has zero-multiplicity*

$$U(0) \leqslant 2^{2^{29n!}}. \tag{1.13}$$

Moreover we have

COROLLARY 1.5. *Let (u_m) be as in Corollary 1.4. Assume moreover that it is nonperiodic. Then its multiplicity satisfies*

$$U \leqslant 2^{2^{29(n+1)!}}. \tag{1.14}$$

We remark that Corollaries 1.4 and 1.5 remain true for sequences (u_m), whose recurrence relation (1.6) has rational coefficients $\nu_{n-1}, \ldots, \nu_0$.

The method of proof we apply, basically is the method developed in [14]. In [14], the main ingredient is my p-adic generalization [13] of W. M. Schmidt's Subspace Theorem [16] in its quantitative version. In [13], using an integral basis, the Subspace Theorem for number fields was reduced to the case, where the variables are taken in $\mathbf{Z}$. If we apply such a reduction process to equation (1.1), we loose the feature that essentially the variables we have to consider are powers of the α_i.

In this paper, implicitly we give a direct proof of the Subspace Theorem for number fields that avoids this reduction.

The second difference in our current approach is that in [14], we apply a very general version of the Subspace Theorem, where for each absolute value we have linear forms that in principle have no link with each other. However in dealing with (1.1) the situation is much more special, and here we derive a version of the Subspace Theorem with very particular linear forms that is more suitable in the context of (1.1) (Lemma 6.1). The third difference, and this is the crucial part as far as the uniformity of our results is concerned, is that our method now takes care of the fact that in equation (1.1) the different absolute values we have to consider are connected with each other in an intrinsic way (§14). We derive a version of the Subspace Theorem that allows it to exploit this connection in a much better way than in the previous version. It is at this point that we get rid of the parameter ω.

2. Main Lemma

In this section we give the Main Lemma from which the theorems may be derived. We remark that in the Main Lemma we have a hypothesis that is considerably weaker than

(1.4). We may assume that $r>1$, as otherwise (1.1) trivially has not more than k_1 solutions $m\in\mathbf{Z}$. Now for $r\geqslant 2$ we assume that

$$1,\alpha_2/\alpha_1,\dots,\alpha_r/\alpha_1 \text{ generate a number field } K \text{ with } [K:\mathbf{Q}]=d \tag{2.1}$$

and that

$$\text{there exists a pair } i,j \text{ with } 1\leqslant i<j\leqslant r \text{ such that } \alpha_i/\alpha_j \text{ is not a root of unity.} \tag{2.2}$$

In (1.1) we may suppose moreover, that for $r=2$ we have $k_1+k_2>0$, since otherwise we have an equation of type

$$a\alpha_1^m=b\alpha_2^m. \tag{2.3}$$

But by (2.2), (2.3) has at most one solution m. For if (2.3) had two solutions $m_1\neq m_2$ then we would get $\alpha_1^{m_1-m_2}=\alpha_2^{m_1-m_2}$ and α_1/α_2 would be a root of unity which contradicts (2.2). Thus in the sequel we may suppose that

$$r=2 \quad\text{and}\quad k_1+k_2>0 \quad\text{or}\quad r\geqslant 3. \tag{2.4}$$

We may suppose without loss of generality that the polynomials f_i in (1.1) have all coefficients different from zero. In fact we may reach such a situation by shifting the variable m if necessary and considering an equation

$$\sum_{i=1}^{r} f_i^*(m)\alpha_i^m=0 \tag{2.5}$$

with $f_i^*(m)=\alpha_i^{m_0}f_i(m+m_0)$. Therefore writing $f_i(X)=a_{0i}+a_{1i}X+\dots+a_{k_ii}X^{k_i}$, equation (1.1) becomes

$$a_{01}\alpha_1^m+\dots+a_{k_11}m^{k_1}\alpha_1^m+\dots+a_{0r}\alpha_r^m+\dots+a_{k_rr}m^{k_r}\alpha_r^m=0. \tag{2.6}$$

Put

$$q=k_1+\dots+k_r+r \quad\text{and}\quad k=k_1+\dots+k_r. \tag{2.7}$$

We read (2.6) as an equation

$$a_1x_1+\dots+a_qx_q=0 \tag{2.8}$$

with nonzero complex coefficients a_i and we are interested in solutions

$$\mathbf{x}^{(m)}=(x_1^{(m)},\dots,x_q^{(m)})=(\alpha_1^m,\dots,m^{k_1}\alpha_1^m,\dots,\alpha_r^m,\dots,m^{k_r}\alpha_r^m).$$

Notice that our assumption (2.4) implies that $q\geqslant 3$.

MAIN LEMMA 2.1. *Suppose that* (2.1), (2.2) *and* (2.4) *are satisfied. Let T be the $(q-1)$-dimensional linear subspace of $\mathbf{C}^q$ defined by* (2.8). *Then for any finite subset $\mathcal{M}$ of the set of solutions m of equation* (1.1), *there exist proper subspaces $T_1, \ldots, T_{t_1}$ of T with*

$$t_1 \leqslant q \cdot d^{6q} 2^{2^{27q!}} \tag{2.9}$$

and with the following property. There is a subset $\mathcal{M}_1$ of $\mathcal{M}$ of cardinality

$$|\mathcal{M}_1| \geqslant \tfrac{1}{3}|\mathcal{M}| \tag{2.10}$$

such that the points $\mathbf{x}^{(m)}$ with $m \in \mathcal{M}_1$ lie in the union $\bigcup_{i=1}^{t_1} T_i$.

This lemma seems to be weak, but it suffices to deduce Theorem 1.1.

We proceed by induction on r and k. The cases $r=1$ and $r=2$, $k=0$ are already settled. So assume that either $r=2$ and $k>0$ or that $r \geqslant 3$. The induction hypothesis says that Theorem 1.1 is true for parameters r', k' such that either $r'<r$ or $r=r'$ and $k'<k$. Now let $\mathcal{M}$ be any finite subset of the set of solutions of (1.1). By the Main Lemma, at least one third of the elements $m \in \mathcal{M}$ satisfy one out of t_1 relations, each of the shape

$$\sum_{i=1}^{r} h_i^{(j)}(m) \alpha_i^{(m)} \quad (1 \leqslant j \leqslant t_1) \tag{2.11}$$

where the $h_i^{(j)}$ are polynomials with $\deg h_i^{(j)} \leqslant \deg f_i$. However, since the Main Lemma gives *proper* subspaces of T, and since we had normalized such that all coefficients in (2.6) (and hence also in (2.8)) are nonzero, we may suppose without loss of generality that

$$\deg h_r^{(j)} < \deg f_r \quad \text{for each } j \ (1 \leqslant j \leqslant t_1). \tag{2.12}$$

But (2.12) implies that *either* $h_r^{(j)} \equiv 0$. Then (2.11) actually is a nontrivial equation

$$\sum_{i=1}^{r'} h_i^{(j)}(m) \alpha_i^m$$

with $r'<r$.

Or $h_r^{(j)} \not\equiv 0$. Then equation (2.11) is a relation with $r'=r$ but with $k'<k$.

In either case, the induction hypothesis says that each relation (2.11) has not more than

$$d^{6(q-1)^2} 2^{2^{28(q-1)!}}$$

solutions m.

Allowing a factor t_1 for the number of relations (2.11), we see that

$$\tfrac{1}{3}|\mathcal{M}| \leqslant |\mathcal{M}_1| \leqslant t_1 \cdot d^{6(q-1)^2} 2^{2^{28(q-1)!}}$$

and therefore

$$|\mathcal{M}| \leqslant d^{6q^2} 2^{2^{28q!}}. \tag{2.13}$$

As (2.13) is true for any finite subset of the set of solutions of (1.1), Theorem 1.1 follows.

Theorem 1.2 is a simple consequence of Theorem 1.1. The parameter q in Theorem 1.1, in view of (1.7), (1.8) now becomes $\varrho_1+...+\varrho_r=n$ and thus the assertion follows at once from (1.5).

As for the proof of Theorem 1.3, by Theorem 1.2 it suffices to consider equations $u_m=a$ with $a\neq 0$.

Let us first treat the case $r=1$. Then in view of (1.8) we ask for solutions m of an equation

$$g(m)\alpha^m = a \tag{2.14}$$

where g is a polynomial of degree $n-1$. Applying Rolle's Theorem to the function

$$g(x)\bar{g}(x)(a\bar{a})^x - a\bar{a}$$

of the real variable x, we see that for $n\geqslant 2$, (2.14) has not more than $2n-1$ solutions m. There remains the case $n=1$. Then α is not a root of unity, as we suppose that our sequence is not periodic. Since however for $n=1$ the polynomial g is constant, (2.14) cannot have more than 1 solution m. Thus for $r=1$ we have $U(a)\leqslant 2n-1$.

Next suppose that $r>1$. Since (u_m) is nongenerate, we may suppose without loss of generality that α_r is not a root of unity. By (1.8), the equation $u_m=a$ may be written as

$$g_1(m)\alpha_1^m+...+g_r(m)\alpha_r^m - a\cdot 1^m = 0. \tag{2.15}$$

The characteristic roots α_r and 1 in (2.15) guarantee that we may apply the Main Lemma. The subspaces we get may be chosen such that their defining equations do not contain the term $a\cdot 1^m$. So they will be of the shape

$$\sum_{i=1}^{r} g_i^{(j)}(m)\alpha_i^m = 0 \quad (1\leqslant j\leqslant t_1) \tag{2.16}$$

with polynomials $g_i^{(j)}$ having $\deg g_i^{(j)} \leqslant \deg g_i$ and not all identically zero. The number of solutions of (2.16) may be estimated with Theorem 1.2. So, similarly as in the proof of Theorem 1.1, we get for any finite subset $\mathcal{M}$ of the set of solutions m of (2.15) the estimate

$$\tfrac{1}{3}|\mathcal{M}| \leqslant |\mathcal{M}_1| \leqslant t_1 \cdot d^{6n^2} 2^{2^{28n!}}.$$

In the context of (2.15) we have

$$t_1 \leqslant (n+1)d^{6(n+1)}2^{2^{27(n+1)!}}.$$

So we obtain

$$|\mathcal{M}| \leqslant d^{6(n+1)^2}2^{2^{28(n+1)!}}$$

and Theorem 1.3 follows.

For the proof of Corollaries 1.4 and 1.5, it suffices to remark that a sequence of rational numbers u_m satisfies a recurrence relation (1.6) with rational coefficients. Therefore the roots $\alpha_1, \ldots, \alpha_r$ of the companion polynomial (1.7) in that case generate a number field K of degree $[K:\mathbf{Q}] \leqslant n!$. Corollaries 1.4 and 1.5 follow at once from the assertions of Theorems 1.2 and 1.3 respectively with d replaced by $n!$.

In the next section we will further reduce our assertions, to arrive at a formulation that is more suitable for a direct application of the Subspace Theorem.

3. Introducing determinants

With the notation of §2, let $\mathbf{x}^{(m)}$ be a solution of (2.6) (or what is the same of (2.8)). Recall the definition of q in (2.7). It is clear that any q solutions $\mathbf{x}^{(m_1)}, \ldots, \mathbf{x}^{(m_q)}$ of (2.8) are linearly dependent. We conclude that any q solutions $m_1, \ldots, m_q$ of (1.1) yield a solution of the determinant equation

$$\begin{vmatrix} x_1^{(m_1)} & \ldots & x_q^{(m_1)} \\ \vdots & & \vdots \\ x_1^{(m_1)} & \ldots & x_q^{(m_q)} \end{vmatrix} = 0. \tag{3.1}$$

Expanding the determinant in (3.1) and writing

$$N+1 = q! \tag{3.2}$$

we get an equation

$$z_1 + \ldots + z_{N+1} = 0, \tag{3.3}$$

where $\mathbf{z} = (z_1, \ldots, z_{N+1})$ is the vector in $(N+1)$-dimensional space whose components are the summands in the Laplace expansion of the $(q \times q)$-determinant with rows $\mathbf{x}^{(m_1)}, \ldots, \mathbf{x}^{(m_q)}$ in (3.1).

LEMMA 3.1. *Suppose that* $q \geqslant 3$. *Let* U *be the* N*-dimensional linear subspace of* $\mathbf{C}^{N+1}$ *defined by* (3.3). *There exist proper subspaces* $U_1, \ldots, U_{t_2}$ *of* U *with*

$$t_2 \leqslant d^{6q} 2^{2^{27N}} \tag{3.4}$$

and with the following property. Any solution $\mathbf{z}=(z_1, \ldots, z_{N+1})$ *of* (3.3) *arising from solutions* $\mathbf{x}^{(m_1)}, \ldots, \mathbf{x}^{(m_q)}$ *of* (2.8) *such that*

$$m_1 < m_2 < \ldots < m_q; \quad m_i \neq 0; \quad m_1 < 0,\ m_q > 0 \tag{3.5}$$

holds true, is contained in the union

$$\bigcup_{i=1}^{t_2} U_i.$$

Suppose for the moment Lemma 3.1 to be proved. We proceed to deduce the Main Lemma.

So let $\mathcal{M}$ be a finite subset of the set of solutions m of (1.1). We may suppose without loss of generality that

$$|\mathcal{M}| \geqslant 3q^2 \tag{3.6}$$

as otherwise the Main Lemma is trivial.

Now there exists an integer m_0 such that the set $\mathcal{M}^- = \{m \in \mathcal{M} \mid m < m_0\}$ and the set $\mathcal{M}^+ = \{m \in \mathcal{M} \mid m > m_0\}$ each have cardinality $\geqslant \frac{1}{3}|\mathcal{M}|$. By (3.6), using the transformation $m \mapsto m - m_0$ we may suppose that $m_0 = 0$ and that $\mathcal{M}^+ = \{m \in \mathcal{M} \mid m > 0\}$, $\mathcal{M}^- = \{m \in \mathcal{M} \mid m < 0\}$. Moreover m_0 may be chosen such that in the shifted equation (1.1) as given by (2.5) all the coefficients of the polynomials f_i^* are nonzero. Using Lemma 3.1, we now may prove in exactly the same way, as was done in [14, Lemma 4.1] (cf. also [15, Lemma 4] and [15, §11]), that there are $q \cdot t_2$ vectors $(a_1^{(j)}, \ldots, a_q^{(j)})$ $(1 \leqslant j \leqslant q t_2)$ such that for each j the coefficient vector $(a_1, \ldots, a_q)$ in (2.8) and $(a_1^{(j)}, \ldots, a_q^{(j)})$ are nonproportional and such that moreover the following is true:

Either each solution $\mathbf{x}^{(m)}$ of (2.8) with $m \in \mathcal{M}^-$ *or* each solution $\mathbf{x}^{(m)}$ of (2.8) with $m \in \mathcal{M}^+$ satisfies one at least of the equations

$$a_1^{(j)} x_1^{(m)} + \ldots + a_q^{(j)} x_q^{(m)} = 0 \quad (1 \leqslant j \leqslant q t_2). \tag{3.7}$$

The equations (3.7) define the subspaces $T_1, \ldots, T_{t_1}$ in the Main Lemma. We put

$$t_1 = q t_2 \tag{3.8}$$

and the Main Lemma follows.

The deduction of the Main Lemma from Lemma 3.1 is the same as in [14] the deduction of Lemma 4.1 from Theorem 1.4. As the corresponding considerations are given in detail in [14] and [15] we restrict ourselves here to a sketch of the proof.

In fact each of the subspaces U_i of Lemma 3.1 is defined by a linear equation, say

$$b_1^{(i)} z_1 + \dots + b_{N+1}^{(i)} z_{N+1} = 0, \tag{3.9}$$

where the coefficient vector $(b_1^{(i)}, \dots, b_{N+1}^{(i)})$ is not proportional to the coefficient vector $(1, \dots, 1)$ of equation (3.3).

Now as was shown in [14] and [15], (3.9) may be interpreted as a multi-linear form in the vectors $\mathbf{x}^{(m_1)}, \dots, \mathbf{x}^{(m_q)}$ of U making up the components of $(z_1, \dots, z_{N+1})$. Write $F_i(\mathbf{x}^{(m_1)}, \dots, \mathbf{x}^{(m_q)})$ for this q-linear form. As was shown in [14] and [15], F_i does not vanish identically on the $(q-1)$-dimensional subspace T of K^q defined by equation (2.8).

Thus the solutions $\mathbf{x}$ of (2.8) for which $F_i(\mathbf{x}, \mathbf{y}_2, \dots, \mathbf{y}_q)$ vanishes identically in $\mathbf{y}_2, \dots, \mathbf{y}_q \in V$ are contained in a proper subspace of T_i of T. So we may distinguish two cases: either for each $m \in \mathcal{M}^-$ some $F_i(\mathbf{x}^{(m)}, \mathbf{y}_2, \dots, \mathbf{y}_q)$ vanishes identically.

Then (taking into consideration all subspaces T_i) we may conclude that in fact for each $m \in \mathcal{M}^-$ one at least out of t_2 equations of type (3.7) is satisfied and we are done. Otherwise, we may pick $m_1 < 0$ such that none of the t_2 forms F_i vanishes identically in $\mathbf{y}_2, \dots, \mathbf{y}_q \in T$. Then the set of $\mathbf{x} \in T$ such that for some i, $F_i(\mathbf{x}^{(m_1)}, \mathbf{x}, \mathbf{y}_3, \dots, \mathbf{y}_q)$ vanishes identically in $\mathbf{y}_3, \dots, \mathbf{y}_q \in T$ again is contained in the union of proper subspaces T_i of T.

Either there exists a solution m_2 of our original equation such that for each i, $F_i(\mathbf{x}^{(m_1)}, \mathbf{x}^{(m_2)}, \mathbf{y}_3, \dots, \mathbf{y}_q)$ is not identically zero, or we may conclude that $2t_2$ subspaces suffice to cover all solutions.

Finally suppose that $m_1, \dots, m_{q-1}$ are chosen. Then any solution $m > 0$ has

$$F_i(\mathbf{x}^{(m_1)}, \mathbf{x}^{(m_2)}, \dots, \mathbf{x}^{m_{q-1})}, \mathbf{x}^{(m)}) = 0$$

and consequently qt_2 proper subspaces of T suffice to cover the solutions of (2.8) with either $m \in \mathcal{M}^-$ or $m \in \mathcal{M}^+$.

The remainder of the paper concentrates on the proof of Lemma 3.1. The main part will consist in adjusting the machinery of the quantitative p-adic Subspace Theorem.

4. Review of heights

Let K be a number field of degree d. Let $M(K)$ be the set of places of K. We write $M_\infty(K)$ for the set of infinite places and $M_0(K)$ for the set of finite places of K. Throughout the paper S will be a finite subset of $M(K)$ containing $M_\infty(K)$. We shall denote by S_∞ the set of infinite places in S and by S_0 the set of finite places. For every place $v\in M(K)$ we define an absolute value $\|\cdot\|_v$ as follows:

If $v|\infty$ we put $\|x\|_v=|x|_v^{d_v/d}$, where $|\cdot|_v$ denotes the standard absolute value on K_v, the completion of K with respect to v and where d_v is the local degree $[K_v:\mathbf{Q}_v]=d_v$.

If $v|p$, where p is a rational prime number, we normalize $\|\cdot\|_v$ by $\|p\|_v=p^{-d_v/d}$ where again d_v is the local degree.

Given a vector $\mathbf{x}=(x_1,...,x_N)\in K^N$ we put

$$\|\mathbf{x}\|_v=\begin{cases}(|x_1|_v^2+...+|x_N|_v^2)^{d_v/2d} & \text{if } v|\infty,\\ \max\{\|x_1\|_v,...,\|x_N\|_v\} & \text{if } v|p,\end{cases} \tag{4.1}$$

and we define the height

$$H(\mathbf{x})=\prod_{v\in M(K)}\|\mathbf{x}\|_v. \tag{4.2}$$

Moreover for a subset T of $M(K)$ the T-height is defined as

$$H_T(\mathbf{x})=\prod_{v\in T}\|\mathbf{x}\|_v. \tag{4.3}$$

Given an element $x\in K$, we put

$$h(x)=H((1,x)) \tag{4.4}$$

and we define $h_T(x)$ analogously. At some instances we will prefer another height, which takes the maximum norm also for the absolute values $\|\cdot\|_v$ with $v|\infty$. For $v\in M(K)$ write

$$\|\mathbf{x}\|_{1,v}=\max\{\|x_1\|_v,...,\|x_N\|_v\} \tag{4.5}$$

and put

$$H_1(\mathbf{x})=\prod_{v\in M(K)}\|\mathbf{x}\|_{1,v}. \tag{4.6}$$

Define $H_{1,T}(\mathbf{x})$ and $h_{1,T}(x)$ similarly.

It follows at once from (4.1), (4.2), (4.5), (4.6) that

$$N^{-1/2}H(\mathbf{x})\leqslant H_1(\mathbf{x})\leqslant H(\mathbf{x}) \quad\text{and}\quad N^{-1/2}H_T(\mathbf{x})\leqslant H_{1,T}(\mathbf{x})\leqslant H_T(\mathbf{x}). \tag{4.7}$$

Given a polynomial f with coefficients in K, we define the heights $H(f)$, $H_1(f)$ etc. as the heights of the vector of coefficients of f.

LEMMA 4.1. *Suppose that* $M \geqslant 2$. *Let* $\mathbf{x}=(x_1,\ldots,x_M)\in K^M$ *be given with* $x_1\neq 0$. *Then there exists* i *with* $2\leqslant i\leqslant M$ *such that*

$$H_1(\mathbf{x})\leqslant h_1\Big(\frac{x_i}{x_1}\Big)^{M-1}. \tag{4.8}$$

Proof. By the product formula, (4.5) and (4.6),

$$\begin{aligned} H_1(\mathbf{x}) &= \prod_{v\in M(K)} \max_{1\leqslant i\leqslant M} \|x_i\|_v = \prod_{v\in M(K)} \max_{1\leqslant i\leqslant M} \Big\|\frac{x_i}{x_1}\Big\|_v \\ &\leqslant \prod_{i=2}^{M} \prod_{v\in M(K)} \max\Big\{1, \Big\|\frac{x_i}{x_1}\Big\|_v\Big\} = \prod_{i=2}^{M} H_1\Big(\Big(1,\frac{x_i}{x_1}\Big)\Big) = \prod_{i=2}^{M} h_1\Big(\frac{x_i}{x_1}\Big) \end{aligned}$$

and (4.8) follows.

LEMMA 4.2. *Let* f *and* g *be polynomials in* $K[x]$ *with* $\deg f+\deg g=r$. *Then*

$$H_1(f)H_1(g)\leqslant 4^r H_1(fg). \tag{4.9}$$

This is a special instance of Proposition 2.4 of Lang [8, p. 57].

LEMMA 4.3. *Let* K *be a number field of degree* d. *Suppose that* $\alpha\in K^*$ *is not a root of unity. Then*

$$h_1(\alpha)>1+\frac{1}{20d^3}. \tag{4.10}$$

This is a well known consequence of the result of Dobrowolski [5].

LEMMA 4.4. *Let* K *be a number field of degree* $d>1$. *Let* D_K *be the absolute value of the discriminant of* K. *Let* $\alpha_1,\ldots,\alpha_N$ *be elements in* K *such that* $1,\alpha_1,\ldots,\alpha_N$ *generate* K. *Then we have*

$$H((1,\alpha_1,\ldots,\alpha_N))\geqslant D_K^{1/2d(d-1)}. \tag{4.11}$$

Proof. Assertion (4.11) essentially is a special case of Theorem 2 of Silverman [19]. Actually Silverman uses the height $H_1((1,\alpha_1,\ldots,\alpha_N))$ and obtains the lower bound

$$\big(D_K^{1/2d}/\sqrt{d}\big)^{1/(d-1)}. \tag{4.12}$$

But a closer look at the proof in [19, pp. 397–398] shows that at one point Silverman estimates a determinant with Hadamard's inequality, which involves the Euclidean norm of the row vectors of the matrix under consideration. He then replaces the Euclidean norm by the maximum norm. It is at this point, where the term $\sqrt{d}$ in (4.12) originates. If we use the height H instead of H_1, it may be seen at once that we can omit the term $\sqrt{d}$.

5. Encore heights

Let $\alpha_1, \ldots, \alpha_r$ be as in §1. Let K be the number field generated by $1, \alpha_2/\alpha_1, \ldots, \alpha_r/\alpha_1$. By homogeneity, we may in fact suppose that $\alpha_1, \ldots, \alpha_r \in K$ and that $\alpha_1, \ldots, \alpha_r$ generate K. Assume that

$$[K:\mathbf{Q}] = d \tag{5.1}$$

and that moreover at least one of the ratios α_i/α_j $(1 \leqslant i < j \leqslant r)$ is not a root of unity. Without loss of generality, we will suppose throughout that

$$\alpha_r/\alpha_1 \text{ is not a root of unity.} \tag{5.2}$$

Consider the determinant

$$\begin{vmatrix} \alpha_1^{x_1} & x_1\alpha_1^{x_1} & \ldots & x_1^{k_1}\alpha_1^{x_1} & \ldots & \alpha_r^{x_1} & x_1\alpha_r^{x_1} & \ldots & x_1^{k_r}\alpha_r^{x_1} \\ \vdots & & & & & & & & \vdots \\ \alpha_1^{x_q} & x_q\alpha_1^{x_q} & \ldots & x_q^{k_1}\alpha_1^{x_q} & \ldots & \alpha_r^{x_q} & x_q\alpha_r^{x_q} & \ldots & x_q^{k_r}\alpha_r^{x_q} \end{vmatrix} \tag{5.3}$$

where we have

$$k_1 + \ldots + k_r + r = q \quad \text{and} \quad k = \max_{1 \leqslant i \leqslant r} k_i, \tag{5.4}$$

$$x_i \in \mathbf{Z}\backslash\{0\}, \quad x_1 < 0,\, x_q > 0, \quad x_1 < x_2 < \ldots < x_q. \tag{5.5}$$

Write

$$\boldsymbol{\alpha} = (\underbrace{\alpha_1, \ldots, \alpha_1}_{k_1+1}, \underbrace{\alpha_2, \ldots, \alpha_2}_{k_2+1}, \ldots, \underbrace{\alpha_r, \ldots, \alpha_r}_{k_r+1}) = (\beta_1, \ldots, \beta_q) = \boldsymbol{\beta}. \tag{5.6}$$

For a permutation σ of the set $\{1, \ldots, q\}$ we let $\boldsymbol{\beta}_\sigma$ be the vector with components $(\beta_{\sigma(1)}, \beta_{\sigma(2)}, \ldots, \beta_{\sigma(q)})$. Moreover given $\mathbf{x} = (x_1, \ldots, x_q)$ we write

$$\boldsymbol{\beta}^{\mathbf{x}} = \beta_1^{x_1} \ldots \beta_q^{x_q} \tag{5.7}$$

(and accordingly $\boldsymbol{\beta}_\sigma^{\mathbf{x}} = \beta_{\sigma(1)}^{x_1} \ldots \beta_{\sigma(q)}^{x_q}$). With this notation the determinant (5.3) may be written as

$$\sum_{\sigma \in \mathfrak{S}_q} M_\sigma(\mathbf{x}) \boldsymbol{\beta}_\sigma^{\mathbf{x}} \tag{5.8}$$

where $\mathfrak{S}_q$ is the symmetric group and where $M_\sigma(\mathbf{x})$ is a monomial in $x_1, \ldots, x_q$ with coefficient ± 1 and of total degree $\leqslant qk$ (cf. (5.4)).

Throughout the remainder of the paper S will be the set of archimedean absolute values of K together with those nonarchimedean ones $\|\cdot\|_v$ for which

$$\|\alpha_i\|_v \neq 1 \quad \text{for some } i \ (1 \leqslant i \leqslant r).$$

An element $\alpha \in K$ is called an S-integer if $\|\alpha\|_v \leqslant 1$ for each $v \notin S$, it is called an S-unit if $\|\alpha\|_v = 1$ for each $v \notin S$. In particular the elements $\boldsymbol{\beta}_\sigma^{\mathbf{x}}$ are S-units.

LEMMA 5.1. *Suppose that* $\mathbf{x}=(x_1, \ldots, x_q)\in\mathbf{Z}^q$ *has*

$$x_1 \ldots x_q \neq 0, \quad x_1 < x_2 < \ldots < x_q, \quad x_1 < 0,\ x_q > 0. \tag{5.9}$$

Let $\gamma>0$ *be given and assume that*

$$\max\{|x_1|, |x_q|\} > 625 d^6 q^4 \gamma^{-2}. \tag{5.10}$$

Then the point $(\beta_\sigma^{\mathbf{x}})_{\sigma\in\mathfrak{S}_q}$ *satisfies*

$$H_1((\beta_\sigma)_{\sigma\in\mathfrak{S}_q})^\gamma \geqslant \max\{|x_1|, |x_q|\}^{q^2}. \tag{5.11}$$

Proof. Let σ_1 be the permutation that in the Laplace expansion of the determinant (5.3) corresponds to the main diagonal. Let σ_2 be the permutation where again we go along the main diagonal, except that the element in the top left corner is replaced by the element in the bottom left corner and the element in the bottom right corner is replaced by the element in the top right corner. Then, we get

$$H_1((\beta_\sigma^{\mathbf{x}})_{\sigma\in\mathfrak{S}_q}) \geqslant H_1((\beta_{\sigma_1}^{\mathbf{x}}, \beta_{\sigma_2}^{\mathbf{x}})) = H_1\left(\left(1, \frac{\beta_{\sigma_2}^{\mathbf{x}}}{\beta_{\sigma_1}^{\mathbf{x}}}\right)\right).$$

But (5.6) and (5.7) imply that

$$\frac{\beta_{\sigma_2}^{\mathbf{x}}}{\beta_{\sigma_1}^{\mathbf{x}}} = \left(\frac{\alpha_q}{\alpha_1}\right)^{x_1-x_q}.$$

Therefore we get

$$H_1((\beta_\sigma^{\mathbf{x}})_{\sigma\in\mathfrak{S}_q}) \geqslant H_1\left(1, \frac{\alpha_q}{\alpha_1}\right)^{|x_1-x_q|}. \tag{5.12}$$

Write $X=\max\{|x_1|, |x_q|\}$. Using Lemma 4.3 and (5.2), we see that (5.11) will be true if

$$\left(1+\frac{1}{20d^3}\right)^{\gamma X} \geqslant X^{q^2}$$

and this will certainly be satisfied if

$$\gamma X \cdot \frac{1}{25 d^3} \geqslant q^2 X^{1/2},$$

i.e. for $X \geqslant 625 d^6 \gamma^{-2} q^4$ as asserted.

LEMMA 5.2. *Let τ be a permutation of $\{1, \dots, q\}$. Then for any i with $1 \leqslant i \leqslant q$ we have*

$$H_1((\beta_\sigma^{\mathbf{x}})_{\sigma \in \mathfrak{S}_q}) \geqslant H_1\left(\left(1, \frac{\beta_{\tau(i)}}{\beta_i}\right)\right)^{|x_1 - x_q|}. \tag{5.13}$$

Proof. The assertion is clear for $\tau(i)=i$. Otherwise it may be proved with the same argument that led in the proof of the preceding lemma to (5.12). In fact (5.12) is the special case $i=1$, $\tau(1)=q$ of our assertion.

LEMMA 5.3. *Suppose that $\mathbf{x}=(x_1, \dots, x_q) \in \mathbf{Z}^q$ satisfies* (5.9). *Let $\beta \in \{\beta_1, \dots, \beta_q\}$ be given. Let T be a subset of S. Suppose that for each $v \in T$ we are given a subset $\mathcal{T}_v$ of $\mathfrak{S}_q$. Then for any j with $1 \leqslant j \leqslant q$ we have*

$$H_1((\beta_\sigma^{\mathbf{x}})_{\sigma \in \mathfrak{S}_q})^{-q!} \leqslant \prod_{v \in T} \prod_{\tau \in \mathcal{T}_v} \left\| \frac{\beta_{\tau(j)}^{x_j}}{\beta^{x_j}} \right\|_v \leqslant H_1((\beta_\sigma^{\mathbf{x}})_{\sigma \in \mathfrak{S}_q})^{q!}.$$

Proof. Let Π^+ be the product over terms $\|\beta_{\tau(j)}^{x_j}/\beta^{x_j}\|_v \geqslant 1$. It is clear that all such terms contribute to $\prod_{\tau \in \mathfrak{S}_q} H_1((1, \beta_{\tau(j)}^{x_j}/\beta^{x_j}))$. Thus by Lemma 5.2 and by (5.9)

$$\Pi^+ \leqslant H_1((\beta_\sigma^{\mathbf{x}})_{\sigma \in \mathfrak{S}_q})^{q!}$$

and the right hand side of (5.14) follows.

Let Π^- be the product over terms $\|\beta_{\tau(j)}^{x_j}/\beta^{x_j}\|_v < 1$. Since $\prod_{v \in S} \|\beta_{\tau(j)}^{x_j}/\beta^{x_j}\|_v = 1$ and $H_1((1, \beta_{\tau(j)}^{x_j}/\beta^{x_j})) = \prod_{v \in S} \max\{1, \|\beta_{\tau(j)}^{x_j}/\beta^{x_j}\|_v\}$, it follows that

$$\prod_{v \in S} \min\{1, \|\beta_{\tau(j)}^{x_j}/\beta^{x_j}\|_v\} = H_1((1, \beta_{\tau(j)}^{x_j}/\beta^{x_j}))^{-1},$$

and again by Lemma 5.2, we may infer that

$$\Pi^- \geqslant H_1((\beta_\sigma^{\mathbf{x}})_{\sigma \in \mathfrak{S}_q})^{-q!}$$

and the left hand side of (5.14) follows as well.

6. Linear subspaces

Suppose that $N \geqslant 2$. Consider the set of linear forms in $\mathbf{X}=(X_1, \dots, X_N)$ given by

$$\begin{aligned} L_1(\mathbf{X}) &= X_1, \\ &\vdots \\ L_N(\mathbf{X}) &= X_N, \\ L_{N+1}(\mathbf{X}) &= X_1 + \dots + X_N. \end{aligned} \tag{6.1}$$

Assume that K is a number field of degree d. We suppose that for each $v \in S$, we are given a fixed system of N different forms $L_1^{(v)}(\mathbf{X}), \dots, L_N^{(v)}(\mathbf{X})$ out of the $N+1$ forms in (6.1). As there are $N+1$ choices for such a system, we obtain a partition

$$S = S^{(1)} \cup \dots \cup S^{(N+1)}$$

such that we have

$$\begin{aligned} L_1^{(v)}(\mathbf{X}) &= L_1^{(j)}(\mathbf{X}), \\ &\vdots \\ L_N^{(v)}(\mathbf{X}) &= L_N^{(j)}(\mathbf{X}), \end{aligned} \tag{6.2}$$

say, for each $v \in S^{(j)}$, i.e. elements v in a set $S^{(j)}$ give rise to the same set of linear forms.

We suppose moreover that for each $v \in S$ we are given an N-tuple $e_{1v}, \dots, e_{Nv}$ of real numbers such that the following conditions hold true:

$$\sum_{v \in S} \sum_{i=1}^{N} e_{iv} = 0, \tag{6.3}$$

$$\left| \sum_{v \in S'} e_{i(v),v} \right| \leqslant 1 \tag{6.4}$$

for each subset S' of S and any tuple $(i(v))_{v \in S'}$ with $1 \leqslant i(v) \leqslant N$. Moreover we suppose that $0 < \delta < 1$ and that $Q > 1$ has

$$Q > \max\{N^{2/\delta}, D_K^{1/2d}\}. \tag{6.5}$$

For $v \in S_0$ we define real numbers ε_{iv} $(i = 1, \dots, N)$ as follows. Let G_v be the subgroup of the multiplicative group of positive real numbers consisting of values taken by the absolute value $\|\cdot\|_v$ on K^*, i.e.

$$G_v = \{x \mid \exists y \in K^*,\ \|y\|_v = x\}.$$

Given Q, let $\varepsilon_{iv} = \varepsilon_{iv}(Q)$ be such that

$$Q^{\varepsilon_{iv}} \text{ is the largest element in } G_v \text{ having } Q^{\varepsilon_{iv}} \leqslant Q^{e_{iv}}. \tag{6.6}$$

Put

$$\eta = \delta \cdot 2^{-5N} \tag{6.7}$$

and suppose that

$$\prod_{v \in S_0} Q^{e_{iv} - \varepsilon_{iv}} \leqslant Q^{\eta} \quad \text{for } i = 1, \dots, N. \tag{6.8}$$

Now consider the simultaneous inequalities

$$\|L_i^{(v)}(\mathbf{x})\|_v \leqslant Q^{e_{iv} - \delta \cdot d_v / d} \quad (v \in S_\infty,\ i = 1, \dots, N), \tag{6.9}$$

$$\begin{aligned} &\|L_i^{(v)}(\mathbf{x})\|_v \leqslant Q^{e_{iv}} \quad (v \in S_0,\ i = 1, \dots, N), \\ &\|\mathbf{x}\|_v \leqslant 1 \quad (v \notin S). \end{aligned} \tag{6.10}$$

LEMMA 6.1. *Let S be given as above. Suppose that for each $v \in S$ we have linear forms $L_1^{(v)}, \ldots, L_N^{(v)}$ as in* (6.1), (6.2). *Assume that the tuples $(e_{iv})_{v \in S, 1 \leqslant i \leqslant N}$ satisfy* (6.3), (6.4). *Let $0 < \delta < 1$ and suppose that η is as in* (6.7). *Then as Q ranges over values satisfying* (6.5), (6.6), (6.8), *the solutions $\mathbf{x} \in K^N$ of the simultaneous inequalities* (6.9), (6.10) *are contained in the union of proper linear subspaces $U_1, \ldots, U_{t_3}$ of K^N with*

$$t_3 \leqslant 2^{2^{21N}\delta^{-2}}. \tag{6.11}$$

It is clear that Lemma 6.1 is a disguised version of the p-adic generalization of W. M. Schmidt's Subspace Theorem in diophantine approximation. The main saving we get in (6.11) as compared with the earlier version in [13] relies on the fact that our forms are taken from the set in (6.1). A considerable saving also comes from hypothesis (6.4), which at first glance might seem to be only of technical nature.

We will give the proof of Lemma 6.1 in §§ 7–13.

7. Line up of facts from the geometry of numbers

Given k with $1 \leqslant k \leqslant N$ we denote by $C(N,k)$ the set of k-tuples

$$\sigma = \{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant N\}.$$

Write

$$M = \binom{N}{k}. \tag{7.1}$$

For $\sigma = \{i_1 < \ldots < i_k\} \in C(N,k)$ and $v \in S$, we define for our linear forms $L_1^{(v)}, \ldots, L_N^{(v)}$ in (6.2) new forms $L_\sigma^{(v)}(\mathbf{X}^{(k)}) = L_\sigma^{(v)}(X_1, \ldots, X_M)$ by $L_\sigma^{(v)} = L_{i_1}^{(v)} \wedge \ldots \wedge L_{i_k}^{(v)}$.

We remark that in view of the special structure of the matrix of $L_1^{(v)}, \ldots, L_N^{(v)}$ we have:

$$\det((L_\sigma^{(v)})_{\sigma \in C(N,k)}) = \pm 1. \tag{7.2}$$

The coefficient matrix of the forms $L_\sigma^{(v)}$ $(\sigma \in C(N,k))$ contains only entries ± 1 and so does the inverse matrix. In the sequel, to recall this fact, we will speak of a "special" system. To avoid heavy notation, in the sequel we will write $L_1^{(v)}, \ldots, L_M^{(v)}$ instead of $L_\sigma^{(v)}$ $(\sigma \in C(N,k))$. Only in contexts where this origin is of importance we will refer to the notation $L_\sigma^{(v)}$.

Let $K_{\mathbf{A}}$ be the adèle ring of K. Elements of $K_{\mathbf{A}}$ will be written as $\mathbf{x} = (x_v) = (x_v)_{v \in M(K)}$, such that x_v is the v-component of $\mathbf{x}$. We define the Haar measure on $K_{\mathbf{A}}$ in the same way as Bombieri and Vaaler [4]:

If $v\in M_0(K)$ we let β_v denote the Haar measure on K_v normalized so that

$$\beta_v(\mathcal{O}_v)=\|\mathcal{D}_v\|_v^{d/2}, \tag{7.3}$$

where $\mathcal{O}_v$ is the ring of integers of K_v and $\mathcal{D}_v$ is the local different of K at v.

If $v\in M_\infty(K)$ and $K_v=\mathbf{R}$ we let β_v denote the ordinary Lebesgue measure on $\mathbf{R}$. If $v\in M_\infty(K)$ and $K_v=\mathbf{C}$ we let β_v denote the Lebesgue measure on the complex plane multiplied by 2.

Write

$$\beta=\prod_{v\in M(K)}\beta_v. \tag{7.4}$$

Then given our subset S of $M(K)$, β determines a Haar measure on $\prod_{v\in S}K_v\times\prod_{v\notin S}\mathcal{O}_v=\mathcal{K}$, say. We let V be the unique product measure on $\mathcal{K}^M$ determined by β.

Suppose that $v\in M_0(K)$ is given with v/p. Write e_v for the ramification index of v over p and f_v for the residue class degree of v over p. Recall that we have

$$d_v=e_vf_v. \tag{7.5}$$

LEMMA 7.1. *Suppose that $Q>1$ and that $c\in\mathbf{R}$. Assume that $v\in M_0(K)$ has v/p. Let $A=\{x\in K_v\mid \|x\|_v\leqslant Q^c\}$. Then we have*

$$\beta_v(A)=p^{f_v\cdot g}\|\mathcal{D}_v\|_v^{d/2}, \tag{7.6}$$

where g is the largest integer such that $p^{f_vg}\leqslant Q^{cd}$.

Proof. Recall that $\|\cdot\|_v$ is normalized such that $\|p\|_v=p^{-d_v/d}$.

Choose a prime element π of K_v. Then in view of (7.5) our normalization implies that

$$\|\pi\|_v=p^{-f_v/d}. \tag{7.7}$$

Let R be a complete system of representatives of the residue class field. Then the elements $y\in K_v$ may be uniquely expressed in the form

$$y=\sum_{\nu=r}^{\infty}a_\nu\pi^\nu \tag{7.8}$$

where $a_\nu\in R$, r is an integer and $a_r\neq 0$. In view of (7.7), (7.8), it is clear that A consists of the elements y, whose expansion (7.8) starts at an index r with

$$p^{-rf_v/d}\leqslant Q^c. \tag{7.9}$$

Let $r_0 \in \mathbf{Z}$ be the smallest value r for which (7.9) is satisfied.

Suppose first that $r_0 \leqslant 0$. Then A consists of the numbers $x = \sum_{\nu=r_0}^{-1} a_\nu \pi^\nu + \alpha$ with $a_\nu \in R$, $\alpha \in \mathcal{O}_v$. Therefore A consists of the translates of $\mathcal{O}_v$ of the shape $\sum_{\nu=r_0}^{-1} a_\nu \pi^\nu + \mathcal{O}_v$. As R contains p^{f_v} elements, we get $p^{-r_0 f_v}$ such translates. As any two such translates are disjoint, the assertion follows from (7.3) with $g = -r_0$.

Next suppose that $r_0 > 0$. Then the elements of $\mathcal{O}_v$ may be written as $\sum_{\nu=0}^{r_0-1} a_\nu \pi^\nu + \alpha$ with $a_\nu \in R$ and $\alpha \in A$. Thus $\mathcal{O}_v$ is the union of the translates $\sum_{\nu=0}^{r_0-1} a_\nu \pi^\nu + A$ of A, where the a_ν run through R. The number of translates is $p^{r_0 \cdot f_v}$. Therefore

$$p^{r_0 f_v} \beta_v(A) = \beta_v(\mathcal{O}_v) = \|\mathcal{D}_v\|_v^{d/2},$$

and the assertion follows again with $g = -r_0$.

We now fix M as in (7.1) and consider for each $v \in S$ our special system of linear forms $L_1^{(v)}(\mathbf{X}), \ldots, L_M^{(v)}(\mathbf{X})$ in $\mathbf{X} = (X_1, \ldots, X_M)$.

For i with $1 \leqslant i \leqslant M$ and for $v \in S$, let c_{iv} be real numbers with

$$\sum_{v \in S} \sum_{i=1}^{M} c_{iv} = 0, \tag{7.10}$$

$$\left| \sum_{v \in S'} c_{i(v),v} \right| \leqslant 1 \tag{7.11}$$

for any subset S' of S and for each tuple $(i(v))_{v \in S'}$ with $1 \leqslant i(v) \leqslant M$.

Given $Q > 1$, we denote by $\Pi(Q)$ the subset of $\mathcal{K}^M$ defined by the inequalities

$$\begin{aligned} &\|L_i^{(v)}(\mathbf{x})\|_v \leqslant Q^{c_{iv}} \quad (v \in S,\ 1 \leqslant i \leqslant M), \\ &\|\mathbf{x}\|_v \leqslant 1 \quad (v \notin S). \end{aligned} \tag{7.12}$$

We call $\Pi = \Pi(Q)$ a parallelepiped in $\mathcal{K}^M$.

We assume moreover that for $v \in S_0$ and each i $(1 \leqslant i \leqslant M)$ there exists a real number $\gamma_{iv} \leqslant c_{iv}$ such that $Q^{\gamma_{iv}}$ lies in the value group G_v of $\|\cdot\|_v$ and that for some fixed $\eta > 0$ we have

$$\prod_{v \in S_0} Q^{c_{iv} - \gamma_{iv}} \leqslant Q^\eta \quad \text{for each } i \text{ with } 1 \leqslant i \leqslant M. \tag{7.13}$$

LEMMA 7.2. *The volume $V(\Pi)$ of the parallelepiped defined by* (7.12) *satisfies*

$$(Q^{-d\eta} 2^{r_1+r_2} \pi^{r_2} D_K^{-1/2})^M \leqslant V(\Pi) \leqslant (2^{r_1+r_2} \pi^{r_2} D_K^{-1/2})^M, \tag{7.14}$$

where r_1 and r_2 respectively are the number of real and complex places of K and where D_K is the absolute value of the discriminant of K.

Proof. For $v \in M_\infty(K)$ and $K_v = \mathbf{R}$ we have

$$\beta_v(\{x \in K_v \mid \|x\|_v \leqslant Q^c\}) = 2Q^{dc}.$$

For $v\in M_\infty(K)$ and $K_v=\mathbf{C}$ we get

$$\beta_v(\{x\in K_v \mid \|x\|_v\leqslant Q^c\})=2\pi Q^{dc}.$$

For $v\in M_0(K)$, we get by Lemma 7.1,

$$\beta_v(\{x\in K_v \mid \|x\|_v\leqslant Q^c\})=p^{f_v g}\|\mathcal{D}_v\|_v^{d/2}\leqslant Q^{dc}\|\mathcal{D}_v\|_v^{d/2}.$$

Combining these inequalities with (7.12) and (7.10) we obtain

$$\begin{aligned} V(\Pi) &\leqslant 2^{(r_1+r_2)M}\pi^{r_2 M}\prod_{v\in S}\prod_{i=1}^{M} Q^{dc_{iv}}\prod_{v\in M_0(K)}\|\mathcal{D}_v\|_v^{Md/2} \\ &= 2^{(r_1+r_2)M}\pi^{r_2 M} D_K^{-M/2}. \end{aligned}$$

On the other hand (7.13) implies that

$$\begin{aligned} V(\Pi) &\geqslant 2^{(r_1+r_2)M}\pi^{r_2 M}\prod_{v\in S_\infty}\prod_{i=1}^{M} Q^{dc_{iv}}\cdot\prod_{i=1}^{M}\Big(Q^{-d\eta}\prod_{v\in S_0} Q^{dc_{iv}}\Big) D_K^{-M/2} \\ &= 2^{(r_1+r_2)M}\pi^{r_2 M} D_K^{-M/2} Q^{-Md\eta} \end{aligned}$$

and the assertion follows.

Following Bombieri and Vaaler [4], we introduce a scalar multiplication by real numbers on $K_{\mathbf{A}}$: Given $\mathbf{x}\in K_{\mathbf{A}}$ and $\alpha\in\mathbf{R}$ we let $\alpha\mathbf{x}$ be the point $\mathbf{y}\in K_{\mathbf{A}}$ with components

$$\begin{aligned} y_v &= \alpha x_v \quad \text{if } v\in M_\infty(K), \\ y_v &= x_v \quad \text{if } v\in M_0(K). \end{aligned}$$

With this scalar multiplication, we define successive minima of $\Pi(Q)$. For each integer i $(1\leqslant i\leqslant M)$ let

$$\lambda_i=\min\{\lambda>0 \mid \lambda\Pi(Q)\cap K^M \text{contains } i \text{ linearly independent vectors}\}.$$

For each i, we associate with λ_i a vector $\mathbf{g}_i\in K^M$ with $\mathbf{g}_i\in\lambda_i\Pi$ and such that $\mathbf{g}_1,\ldots,\mathbf{g}_i$ are linearly independent.

LEMMA 7.3. *The successive minima of* Π *satisfy the inequality*

$$\frac{2^{dM}\pi^{r_2 M}}{(M!)^{r_1}((2M)!)^{r_2} D_K^{M/2}}\leqslant(\lambda_1\ldots\lambda_M)^d V(\Pi)\leqslant 2^{dM}. \tag{7.15}$$

This follows at once from Theorems 3 and 4 of Bombieri and Vaaler [4].

LEMMA 7.4. *Assume that we have* (7.13). *Then the successive minima of* Π *in* (7.12) *satisfy*

$$M^{-dM} \leqslant (\lambda_1 \dots \lambda_M)^d \leqslant (Q^{d\eta} D_K^{1/2})^M. \tag{7.16}$$

Proof. This follows from combination of Lemmata 7.2 and 7.3.

LEMMA 7.5. *Let* $(c_v)_{v \in S}$ *be a tuple of real numbers. Let* $Q > 1$ *and suppose that for* $v \in S_0$ *we have real numbers* γ_v *with*

$$\gamma_v \leqslant c_v \quad \textit{and} \quad Q^{\gamma_v} \in G_v \quad (\textit{the value group of } \|\cdot\|_v). \tag{7.17}$$

Then there exists a nonzero element $\alpha \in K$ *satisfying*

$$\|\alpha\|_v \leqslant Q^{c_v - (\sum_{w \in S} c_w) d_v/d + (\sum_{w \in S_0} c_w - \gamma_w) d_v/d} D_K^{d_v/2d^2} \quad \textit{for } v \in S_\infty, \tag{7.18}$$

$$\|\alpha\|_v \leqslant Q^{c_v} \quad \textit{for } v \in S_0, \tag{7.19}$$

$$\|\alpha\|_v \leqslant 1 \quad \textit{for } v \notin S. \tag{7.20}$$

Proof. Consider the parallelepiped Π in $\mathcal{K}$ defined by the inequalities

$$\|\alpha\|_v \leqslant Q^{c_v - (\sum_{w \in S_\infty} c_w + \sum_{w \in S_0} \gamma_w) d_v/d} \quad (v \in S_\infty),$$

$$\|\alpha\|_v \leqslant Q^{\gamma_v} \quad (v \in S_0),$$

$$\|\alpha\|_v \leqslant 1 \quad (v \notin S).$$

It has volume $V = 2^{r_1} (2\pi)^{r_2} D_K^{-1/2}$. Thus by Lemma 7.3 it has first minimum λ_1 with

$$\lambda_1 \leqslant D_K^{1/2d}.$$

But by definition $\lambda_1 \Pi$ contains a point different from 0. The lemma follows since $Q^{\gamma_v} \leqslant Q^{c_v}$ $(v \in S_0)$.

LEMMA 7.6. *Let* $\mathbf{x} \in K^M$, $\mathbf{x} \neq \mathbf{0}$. *Define the real number* c *by*

$$c = \left(\prod_{v \in M_\infty(K)} \max_{1 \leqslant j \leqslant M} \|Q^{-c_{jv} d/d_v} L_j^{(v)}(\mathbf{x})\|_v \right) D_K^{1/2d}. \tag{7.21}$$

Then there exists an algebraic integer $\varkappa \in K^*$ *satisfying for each* $v \in M_\infty(K)$,

$$\|\varkappa\|_v \leqslant \left(\max_{1 \leqslant j \leqslant M} \|Q^{-c_{jv} d/d_v} L_j^{(v)}(\mathbf{x})\|_v \right)^{-1} c^{d_v/d}. \tag{7.22}$$

Proof. As we require $\varkappa$ to be an algebraic integer, it has apart from (7.22) to satisfy the condition

$$\|\varkappa\|_v \leqslant 1 \quad \text{for } v \notin M_\infty(K). \tag{7.23}$$

Now (7.22), (7.23) define a parallelepiped in K of volume $2^{r_1+r_2} \pi^{r_2}$, where r_1 and r_2 denote respectively the number of real and complex embeddings of K. Thus by Lemma 7.3 our parallelepiped has first minimum $\leqslant (2/\pi)^{r_2/d} \leqslant 1$. The assertion follows.

LEMMA 7.7. *Suppose that* $\mathbf{x}\in K^M$ *is a point which satisfies inequalities* (7.12) *for each* $v\notin M_\infty(K)$. *Let* $\mathbf{g}_1,\dots,\mathbf{g}_M$ *be linearly independent points in* K^M *corresponding to the minima* $\lambda_1,\dots,\lambda_M$ *of* $\Pi(Q)$. *Write* S_i *for the subspace of* K^M *generated by* $\mathbf{g}_1,\dots,\mathbf{g}_i$. *Then for* $i=1,\dots,M$ *and for* $\mathbf{x}\notin S_{i-1}$ *we have*

$$\prod_{v\in M_\infty(K)}\max_{1\leqslant j\leqslant M}\|Q^{-c_{jv}d/d_v}L_j^{(v)}(\mathbf{x})\|_v\geqslant D_K^{-1/2d}\lambda_i \quad (1\leqslant i\leqslant M). \tag{7.24}$$

Proof. Suppose $\mathbf{x}\notin S_{i-1}$. Choose the algebraic integer $\varkappa$ according to Lemma 7.6. Then the point $\varkappa\mathbf{x}$ again satisfies inequalities (7.12) for each $v\notin M_\infty(K)$. As $\mathbf{x}\notin S_{i-1}$, also $\varkappa\mathbf{x}\notin S_{i-1}$. Consequently there exists $v_0\in M_\infty(K)$ such that

$$\max_{1\leqslant j\leqslant M}|Q^{-c_{jv_0}\cdot d/d_{v_0}}L_j^{(v_0)}(\varkappa\mathbf{x})|_{v_0}\geqslant\lambda_i. \tag{7.25}$$

On the other hand our choice of $\varkappa$ in (7.22) implies that for each $v\in M_\infty(K)$ we have

$$\max_{1\leqslant j\leqslant M}|Q^{-c_{jv}d/d_v}L_j^{(v)}(\varkappa\mathbf{x})|_v\leqslant D_K^{1/2d}\prod_{w\in M_\infty(K)}\max_{1\leqslant j\leqslant m}\|Q^{-c_{jw}d/d_w}L_j^{(w)}(\mathbf{x})\|_w. \tag{7.26}$$

Combination of (7.25) and (7.26) yields the assertion.

LEMMA 7.8 (Davenport's Lemma). *Suppose that* $Q>1$ *and that* $\zeta>0$. *Let* $\varrho_1,\dots,\varrho_M$ *be real numbers with*

$$\varrho_1\geqslant\varrho_2\geqslant\dots\geqslant\varrho_M>0, \tag{7.27}$$

$$\varrho_i\lambda_i\leqslant Q^\zeta\varrho_{i+1}\lambda_{i+1}\quad \textit{for } i=1,\dots,M-1, \tag{7.28}$$

$$\varrho_1\cdots\varrho_M=1. \tag{7.29}$$

Fix $v_0\in M_\infty(K)$. *Then there exists a permutation* τ *of the set* $\{1,\dots,M\}$ *with the following property*:

Let $\Pi'(Q)$ *be the parallelepiped defined by*

$$\|L_i^{(v_0)}(\mathbf{x})\|_{v_0}\leqslant\varrho_{\tau(i)}^{-1}Q^{c_{iv_0}}\quad(1\leqslant i\leqslant M) \tag{7.30}$$

and for $v\neq v_0$ *as in* (7.12). *Let* $\lambda_1',\dots,\lambda_M'$ *be the successive minima of* $\Pi'(Q)$. *Then we have*

$$D^{-1/2d}2^{-M}Q^{-M\zeta}\varrho_i\lambda_i\leqslant\lambda_i'\leqslant 4^{M^2}Q^{M^2\zeta}D_K^{(2M-1)/2d}\varrho_i\lambda_i. \tag{7.31}$$

Moreover, for $v\in M_\infty(K)$ *define the linear forms* $G_i^{(v)}(\mathbf{X})$ *by*

$$G_i^{(v)}(\mathbf{X})=Q^{-c_{iv}d/d_v}L_i^{(v)}(\mathbf{X})\quad \textit{for } v\neq v_0 \textit{ and } 1\leqslant i\leqslant M \tag{7.32}$$

and by

$$G_i^{(v_0)}(\mathbf{X}) = \varrho_{\tau(i)}^{d/d_{v_0}} Q^{-c_{iv_0}d/d_{v_0}} L_i^{(v_0)}(\mathbf{X}) \quad (1 \leqslant i \leqslant M). \tag{7.33}$$

Then any point $\mathbf{x} \in K^M$ *which satisfies inequalities* (7.12) *for* $v \notin M_\infty(K)$ *but does not lie in the subspace* S_{i-1} *spanned by the points* $\mathbf{g}_1, ..., \mathbf{g}_{i-1}$ *corresponding to* $\lambda_1, ..., \lambda_{i-1}$ *has*

$$\max_{v \in M_\infty(K)} \max_{1 \leqslant j \leqslant M} \{|G_j^{(v)}(\mathbf{x})|_v\} \geqslant 2^{-M} Q^{-M\zeta} D^{-1/2d} \varrho_i \lambda_i. \tag{7.34}$$

Proof. The proof goes along the same lines as in W. M. Schmidt [16, §IV, Theorem 3A]. For $v \in S_\infty$ we define the absolute value $|\cdot|_v$ by $|x|_v = \|x\|_v^{d/d_v}$. In the sequel, when we consider points $\mathbf{x} \in K^M$, we shall whenever necessary tacitly assume that they satisfy inequalities (7.12) for $v \notin S_\infty$.

For i with $1 \leqslant i \leqslant M$ and for $v \in S_\infty$ we write

$$Q^{-c_{iv}d/d_v} L_i^{(v)}(\mathbf{x}) = \boldsymbol{\beta}_i^{(v)} \mathbf{x} = \beta_{i1}^{(v)} x_1 + ... + \beta_{iM}^{(v)} x_M.$$

So, $\boldsymbol{\beta}_i^{(v)}$ is the coefficient vector of the linear form $Q^{-c_{iv}d/d_v} L_i^{(v)}$. Write

$$N(\mathbf{x}) = \prod_{v \in M_\infty(K)} \max_{1 \leqslant j \leqslant M} \{\|\boldsymbol{\beta}_j^{(v)} \mathbf{x}\|_v\}.$$

By Lemma 7.7, any point $\mathbf{x} \notin S_{j-1}$ satisfies

$$N(\mathbf{x}) \geqslant D_K^{-1/2d} \lambda_j. \tag{7.35}$$

To determine the permutation τ, consider the fixed element $v_0 \in S_\infty$. If $\mathbf{x}$ lies in S_i,(1) then the point $(\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}, ..., \boldsymbol{\beta}_M^{(v_0)}\mathbf{x})$ satisfies $M-i$ independent linear equations with coefficients in K_{v_0}. In particular for $\mathbf{x} \in S_{M-1}$ we have

$$a_1 \boldsymbol{\beta}_1^{(v_0)} \mathbf{x} + ... + a_M \boldsymbol{\beta}_M^{(v_0)} \mathbf{x} = 0 \tag{7.36}$$

with certain fixed coefficients $a_1, ..., a_M \in K_{v_0}$, not all equal to zero. Choosing a suitable permutation, we may assume that

$$|a_M|_{v_0} = \max\{|a_1|_{v_0}, ..., |a_M|_{v_0}\}. \tag{7.37}$$

But then (7.36) implies that

$$\boldsymbol{\beta}_M^{(v_0)} \mathbf{x} = -\frac{a_1}{a_M} \boldsymbol{\beta}_1^{(v_0)} \mathbf{x} - ... - \frac{a_{M-1}}{a_M} \boldsymbol{\beta}_{M-1}^{(v_0)} \mathbf{x}.$$

(1) There should be no confusion between the subspaces S_i and the subsets S_0 and S_∞ of our set of places S.

Using (7.37), we obtain

$$|\boldsymbol{\beta}_M^{(v_0)}\mathbf{x}|_{v_0} \leqslant |\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0}+\ldots+|\boldsymbol{\beta}_{M-1}^{(v_0)}\mathbf{x}|_{v_0}$$

which in turn yields

$$|\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0}+\ldots+|\boldsymbol{\beta}_{M-1}^{(v_0)}\mathbf{x}|_{v_0} \geqslant \tfrac{1}{2}(|\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0}+\ldots+|\boldsymbol{\beta}_M^{(v_0)}\mathbf{x}|_{v_0}) \tag{7.38}$$

for each $\mathbf{x}\in S_{M-1}$.

If $\mathbf{x}$ lies in S_{M-2}, it satisfies a second relation which is independent of (7.36) and may be written as

$$b_1\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}+\ldots+b_{M-1}\boldsymbol{\beta}_{M-1}^{(v_0)}\mathbf{x}=0.$$

Again, after choosing a suitable permutation we may suppose that

$$|b_{M-1}|_{v_0}=\max\{|b_1|_{v_0},\ldots,|b_{M-1}|_{v_0}\}.$$

And similarly as above we get

$$|\boldsymbol{\beta}_{M-1}^{(v_0)}\mathbf{x}|_{v_0} \leqslant |\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0}+\ldots+|\boldsymbol{\beta}_{M-2}^{(v_0)}\mathbf{x}|_{v_0}.$$

Together with (7.38) this gives

$$\begin{aligned} |\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0}+\ldots+|\boldsymbol{\beta}_{M-2}^{(v_0)}\mathbf{x}|_{v_0} &\geqslant \tfrac{1}{2}(|\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0}+\ldots+|\boldsymbol{\beta}_{M-1}^{(v_0)}\mathbf{x}|_{v_0}) \\ &\geqslant 2^{-2}(|\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0}+\ldots+|\boldsymbol{\beta}_M^{(v_0)}\mathbf{x}|_{v_0}) \end{aligned} \tag{7.39}$$

for each $\mathbf{x}\in S_{M-2}$. Continuing in this way we obtain for each j with $1\leqslant j\leqslant M-1$ inequalities of type (7.38), (7.39).

So after reordering $\boldsymbol{\beta}_1^{(v_0)},\ldots,\boldsymbol{\beta}_M^{(v_0)}$, we may suppose that

$$|\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0}+\ldots+|\boldsymbol{\beta}_{M-j}^{(v_0)}\mathbf{x}|_{v_0} \geqslant 2^{-j}(|\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0}+\ldots+|\boldsymbol{\beta}_M^{(v_0)}\mathbf{x}|_{v_0}) \tag{7.40}$$

for each j $(1\leqslant j\leqslant M-1)$ and for each $\mathbf{x}\in S_{M-j}$.

Now suppose that $\mathbf{x}\notin S_{i-1}$. Then there exists j with $i\leqslant j\leqslant M$ such that $\mathbf{x}\in S_j$, $\mathbf{x}\notin S_{j-1}$, so that by (7.35) $N(\mathbf{x})\geqslant D_K^{-1/2d}\lambda_j$.

On the other hand using (7.27), (7.40) we obtain

$$\begin{aligned} \max\{|\varrho_1^{d/d_{v_0}}\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0},\ldots,|\varrho_M^{d/d_{v_0}}\boldsymbol{\beta}_M^{(v_0)}\mathbf{x}|_{v_0}\} & \\ &\geqslant \max\{|\varrho_1^{d/d_{v_0}}\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0},\ldots,|\varrho_j^{d/d_{v_0}}\boldsymbol{\beta}_j^{(v_0)}\mathbf{x}|_{v_0}\} \\ &\geqslant \varrho_j^{d/d_{v_0}}\max\{|\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0},\ldots,|\boldsymbol{\beta}_j^{(v_0)}\mathbf{x}|_{v_0}\} \\ &\geqslant \frac{\varrho_j^{d/d_{v_0}}}{j}\{|\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0}+\ldots+|\boldsymbol{\beta}_j^{(v_0)}\mathbf{x}|_{v_0}\} \\ &\geqslant \frac{2^{j-M}}{j}\varrho_j^{d/d_{v_0}}\{|\boldsymbol{\beta}_1^{(v_0)}\mathbf{x}|_{v_0}+\ldots+|\boldsymbol{\beta}_M^{(v_0)}\mathbf{x}|_{v_0}\} \\ &\geqslant 2^{-M}\varrho_j^{d/d_{v_0}}\max_{1\leqslant i\leqslant M}\{|\boldsymbol{\beta}_i^{(v_0)}(\mathbf{x})|_{v_0}\}. \end{aligned} \tag{7.41}$$

Recall the definition of the forms $G_i^{(v)}(\mathbf{X})$ in (7.32), (7.33). We infer from (7.41), (7.35) that

$$\prod_{v\in M_\infty(K)} \max_{1\leqslant i\leqslant M} \|G_i^{(v)}(\mathbf{x})\|_v \geqslant 2^{-M}\varrho_j \prod_{v\in M_\infty(K)} \max_{1\leqslant i\leqslant M} \|\boldsymbol{\beta}_i^{(v)}\mathbf{x}\|_v$$
$$=2^{-M}\varrho_j N(\mathbf{x}) \geqslant D_K^{-1/2d} 2^{-M}\varrho_j\lambda_j.$$

Using (7.28), we may conclude that

$$\max_{v\in M_\infty(K)} \max_{1\leqslant k\leqslant M} |G_k^{(v)}(\mathbf{x})|_v \geqslant D_K^{-1/2d} 2^{-M}\varrho_j\lambda_j \geqslant D_K^{-1/2d} 2^{-M} Q^{-M\zeta}\varrho_i\lambda_i.$$

This is true for each $\mathbf{x}\notin S_{i-1}$ and thus (7.34) is established.

However (7.34) implies in particular that

$$\lambda_i' \geqslant 2^{-M} D_K^{-1/2d} Q^{-M\zeta}\varrho_i\lambda_i \quad (i=1,\dots,M), \tag{7.42}$$

and this is the left hand side of (7.31).

As for the right hand side of (7.31), we first remark that by (7.29), $V(\Pi)=V(\Pi')$. According to Lemma 7.3 we have

$$\frac{2^{dM}\pi^{r_2M}}{(M!)^{r_1}((2M)!)^{r_2}} D_K^{-M/2} \leqslant (\lambda_1 \dots \lambda_M)^d V(\Pi)$$

and

$$(\lambda_1' \dots \lambda_M')^d V(\Pi') \leqslant 2^{dM}.$$

We may infer using (7.29) and (7.42) that

$$\lambda_i'^d \leqslant \frac{2^{dM} V(\Pi')^{-1}}{(\lambda_1' \dots \lambda_{i-1}'\lambda_{i+1}' \dots \lambda_M')^d} \leqslant \frac{2^{dM+dM(M-1)} Q^{dM(M-1)\zeta} D_K^{(M-1)/2}}{(\varrho_1\lambda_1 \dots \varrho_{i-1}\lambda_{i-1}\varrho_{i+1}\lambda_{i+1} \dots \varrho_M\lambda_M)^d} V(\Pi')^{-1}$$
$$\leqslant \frac{2^{dM^2} Q^{dM^2\zeta} \varrho_i^d D_K^{(M-1)/2}}{(\lambda_1 \dots \lambda_{i-1}\lambda_{i+1} \dots \lambda_M)^d} V(\Pi)^{-1}$$
$$\leqslant \frac{2^{dM^2} Q^{dM^2\zeta} (M!)^{r_1}((2M)!)^{r_2}}{2^{dM}\pi^{r_2M}} D_K^{(2M-1)/2} (\varrho_i\lambda_i)^d$$
$$\leqslant 4^{dM^2} D_K^{(2M-1)/2} Q^{dM^2\zeta} (\varrho_i\lambda_i)^d$$

and (7.31) follows.

In analogy with what we said at the beginning of this section, given k with $1\leqslant k\leqslant M$ we denote by $C(M,k)$ the set of k-tuples $\sigma=\{1\leqslant i_1<i_2<\dots<i_k\leqslant M\}$. Put $L=\binom{M}{k}$

and for $\sigma=\{i_1<...<i_k\}$ define the linear forms $M_\sigma^{(v)}(\mathbf{X}^{(k)})=M_\sigma^{(v)}(X_1,...,X_L)$ by $M_\sigma^{(v)}=L_{i_1}^{(v)}\wedge...\wedge L_{i_k}^{(v)}$. If $\Pi(Q)$ is the parallelepiped (7.12), we define $\Pi^{(k)}(Q)$ by

$$\begin{gathered}\|M_\sigma^{(v)}(\mathbf{x}^{(k)})\|_v \leqslant Q^{c_{\sigma v}} \quad (\sigma\in C(M,k),\, v\in S),\\ \|\mathbf{x}^{(k)}\|_v\leqslant 1 \quad (v\notin S).\end{gathered} \tag{7.43}$$

Here $\mathbf{x}^{(k)}$ stands for a vector in K^L and

$$c_{\sigma v}=c_{i_1 v}+...+c_{i_k v}, \tag{7.44}$$

where we suppose that the c_{iv} satisfy (7.10) and (7.11). We call $\Pi^{(k)}$ the kth compound of Π. Let $\lambda_1,...,\lambda_M$ be the successive minima of Π. For $\tau\in C(M,k)$ write

$$\lambda_\tau=\prod_{i\in\tau}\lambda_i.$$

There is an ordering $\tau_1,...,\tau_L$ of the elements of $C(M,k)$ such that

$$\lambda_{\tau_1}\leqslant\lambda_{\tau_2}\leqslant...\leqslant\lambda_{\tau_L}. \tag{7.45}$$

We denote the successive minima of $\Pi^{(k)}$ by $\nu_1,...,\nu_L$.

The following result was proved by Mahler [10] for convex bodies in $\mathbf{R}^M$. Here we give its extension to the space of adèles of K.

LEMMA 7.9 (Mahler [10]). *Suppose that we have* (7.10), (7.11) *and* (7.13). *Then the successive minima of* $\Pi^{(k)}$ *given by* (7.43) *satisfy*

$$Q^{-Lk\eta}D_K^{-Lk/2d}(k!)^{-(L-1)}2^{-L^2}\lambda_{\tau_i}\leqslant\nu_i\leqslant k!\,\lambda_{\tau_i}. \tag{7.46}$$

Proof. Our proof follows the lines of Schmidt [16, §IV, Theorem 7A]. Let $\mathbf{g}_1,...,\mathbf{g}_M$ be independent points in K^M with $\mathbf{g}_i\in\lambda_i\Pi$. Thus

$$\|L_i^{(v)}(\mathbf{g}_j)\|_v\leqslant\lambda_j^{d_v/d}Q^{c_{iv}} \quad (i,j=1,...,M;\ v\in S_\infty), \tag{7.47}$$

$$\|L_i^{(v)}(\mathbf{g}_j)\|_v\leqslant Q^{c_{iv}} \quad (i,j=1,...,M;\ v\in S_0), \tag{7.48}$$

$$\|\mathbf{g}_j\|_v\leqslant 1 \quad (j=1,...,M;\ v\notin S). \tag{7.49}$$

For $\tau=\{j_1<...<j_k\}\in C(M,k)$ write

$$\mathbf{G}_\tau=\mathbf{g}_{j_1}\wedge...\wedge\mathbf{g}_{j_k}. \tag{7.50}$$

Then we may infer from (7.40) that for each $v \in S_0$ and for any $\tau \in C(M,k)$,

$$\|M_\sigma^{(v)}(\mathbf{G}_\tau)\|_v = \|(L_{i_1}^{(v)} \wedge \ldots \wedge L_{i_k}^{(v)})(\mathbf{g}_{j_1} \wedge \ldots \wedge \mathbf{g}_{j_k})\|_v = \left\| \det \begin{pmatrix} L_{i_1}^{(v)}(\mathbf{g}_{j_1}) & \ldots & L_{i_1}^{(v)}(\mathbf{g}_{j_k}) \\ \vdots & & \vdots \\ L_{i_k}^{(v)}(\mathbf{g}_{j_1}) & \ldots & L_{i_k}^{(v)}(\mathbf{g}_{j_k}) \end{pmatrix} \right\|_v \leqslant Q^{c_{\sigma v}} \tag{7.51}$$

for each $\sigma \in C(M,k)$. Moreover, trivially we have

$$\|\mathbf{G}_\tau\|_v \leqslant 1 \quad \text{for each } \tau \in C(M,k) \text{ and for each } v \notin S. \tag{7.52}$$

Now suppose that $v \in S_\infty$. Then by considering the determinant in (7.51) and using (7.47) we get

$$\|M_\sigma^{(v)}(\mathbf{G}_\tau)\|_v \leqslant (k!)^{d_v/d} \lambda_\tau^{d_v/d} Q^{c_{\sigma v}} \quad \text{for each } \sigma \in C(M,k). \tag{7.53}$$

(7.51), (7.52), (7.53) imply that

$$\nu_i \leqslant k!\, \lambda_{\tau_i} \quad (1 \leqslant i \leqslant L), \tag{7.54}$$

which is the right hand side of (7.46).

As for the left hand side in (7.46), Lemma 7.3 says that

$$(\lambda_{\tau_1} \ldots \lambda_{\tau_L})^{-d} \geqslant V(\Pi)^{\binom{M-1}{k-1}} 2^{-dM\binom{M-1}{k-1}} = V(\Pi)^{Lk/M} 2^{-dLk}$$

and

$$(\nu_1 \ldots \nu_L)^d \geqslant V(\Pi^{(k)})^{-1} \cdot \frac{2^{dL} \Pi^{r_2 L}}{(L!)^{r_1}((2L)!)^{r_2} D_K^{L/2}}.$$

Combining these two inequalities we get

$$\prod_{i=1}^{L} \left(\frac{\nu_i}{\lambda_{\tau_i}} \right)^d \geqslant \frac{V(\Pi)^{Lk/M}}{V(\Pi^{(k)})} \cdot \frac{2^{-dL(k-1)} \pi^{r_2 L}}{(L!)^{r_1}((2L)!)^{r_2}} D_K^{-L/2}. \tag{7.55}$$

On the other hand, Lemma 7.2 yields

$$V(\Pi)^{Lk/M} \geqslant Q^{-dLk\eta} 2^{(r_1+r_2)Lk} \pi^{r_2 Lk} D_K^{-Lk/2}, \tag{7.56}$$

and similarly we obtain

$$V(\Pi^{(k)}) \leqslant 2^{(r_1+r_2)L} \pi^{r_2 L} D_K^{-L/2}. \tag{7.57}$$

Combining (7.55), (7.56), (7.57) we may infer that

$$\prod_{i=1}^{L} \left(\frac{\nu_i}{\lambda_{\tau_i}} \right)^d \geqslant Q^{-dLk\eta} \cdot (2^{(r_1+r_2)} \pi^{r_2} D_K^{-1/2})^{L(k-1)} \cdot \frac{2^{-dL(k-1)} \pi^{r_2 L}}{(L!)^{r_1}((2L)!)^{r_2}} D_K^{-L/2}.$$

Together with (7.54) this gives

$$\left(\frac{\nu_i}{\lambda_{\tau_i}}\right)^d \geqslant Q^{-dLk\eta}(k!)^{-(L-1)d}\frac{2^{-r_2L(k-1)}\pi^{r_2Lk}}{(L!)^{r_1}((2L)!)^{r_2}}D_K^{-Lk/2}$$
$$\geqslant Q^{-dLk\eta}(k!)^{-(L-1)d}(L!)^{-d}\cdot 2^{-r_2L}D_K^{-Lk/2}$$

and (7.46) follows.

Of particular interest in our applications will be the case $k=M-1$. Then $L=\binom{M}{k}=M$. $\Pi^{(M-1)}$ essentially is what Mahler calls polar to Π. Then we have

$$\lambda_{\tau_i}=\frac{\lambda_1\ldots\lambda_M}{\lambda_{M+1-i}}. \tag{7.58}$$

In this case Lemma 7.9 implies

LEMMA 7.10. *Let* $\lambda_1,\ldots,\lambda_M$ *be the successive minima of* Π *and* $\lambda_1^*,\ldots,\lambda_M^*$ *be the successive minima of* $\Pi^{(M-1)}$ *(cf. (7.12), (7.43)). Then*

$$Q^{-M^2\eta}D_K^{-M^2/2d}(2M)^{-M^2}\leqslant\lambda_{M+1-i}\lambda_i^*\leqslant M!\,Q^{M\eta}D_K^{M/2d}\quad(i=1,\ldots,M). \tag{7.59}$$

Proof. By Lemma 7.4

$$M^{-M}\leqslant\lambda_1\cdot\ldots\cdot\lambda_M\leqslant Q^{M\eta}D_K^{M/2d}.$$

Combining this with (7.58) and Lemma 7.9 with $L=M$ and $k=M-1$ we get

$$Q^{-M(M-1)\eta}D_K^{-M(M-1)/2d}((M-1)!)^{-(M-1)}2^{-M^2}M^{-M}\leqslant\lambda_{M+1-i}\lambda_i^*$$
$$\leqslant(M-1)!\,Q^{M\eta}D_K^{M/2d}$$

and (7.59) follows.

LEMMA 7.11. *Suppose* $1\leqslant k\leqslant M$. *Define* $\tau_1,\ldots,\tau_L$ *and points* $\mathbf{G}_\tau$ *as in* (7.45), (7.50) *respectively. Once the span of* $\mathbf{G}_{\tau_1},\ldots,\mathbf{G}_{\tau_{L-1}}$ *in* K^L *is determined, the span of* $\mathbf{g}_1,\ldots,\mathbf{g}_{M-k}$ *in* K^M *is determined.*

This is Lemma 6.4 of Schmidt [17].

LEMMA 7.12. *Let* $L_1,\ldots,L_t$ *be linear forms in* M *variables and with coefficients in the set* $\{-1,0,1\}$. *Suppose that we know that there is a point* $\mathbf{h}\neq\mathbf{0}$ *in* $\mathbf{Q}^M$ *with*

$$L_i(\mathbf{h})=0\quad(i=1,\ldots,t). \tag{7.60}$$

Then there is a point $\mathbf{h}\neq\mathbf{0}$ *in* $\mathbf{Z}^M$ *with* (7.60) *and with*

$$\prod_{v\in S_\infty}\|\mathbf{h}\|_{1,v}\leqslant M^{(M-1)/2}.$$

This is a very special instance of Siegel's Lemma (cf. e.g. Schmidt [18, Lemma 4D, p. 11]).

8. Again geometry of numbers

Let $L_1^{(v)},\dots,L_M^{(v)}$ $(v\in S)$ be a special system of linear forms as introduced at the beginning of §7. Let c_{iv} $(1\leqslant i\leqslant M,\ v\in S)$ be real numbers satisfying (7.10), (7.11). Π is the parallelepiped given by (7.12) and $\mathbf{g}_1,\dots,\mathbf{g}_M$ are linearly independent points in K^M having

$$\mathbf{g}_i\in\lambda_i\Pi. \tag{8.1}$$

For i with $1\leqslant i\leqslant M$ we write

$$\hat{L}_i^{(v)}=L_1^{(v)}\wedge\dots\wedge L_{i-1}^{(v)}\wedge L_{i+1}^{(v)}\wedge\dots\wedge L_M^{(v)}\quad(1\leqslant i\leqslant M,\ v\in S)$$

and

$$\hat{\mathbf{g}}_i=\mathbf{g}_1\wedge\dots\wedge\mathbf{g}_{i-1}\wedge\mathbf{g}_{i+1}\wedge\dots\wedge\mathbf{g}_M\quad(1\leqslant i\leqslant M). \tag{8.2}$$

LEMMA 8.1. *The point* $\hat{\mathbf{g}}_M$ *defined in* (8.2) *has*

$$H_{1,S}(\hat{\mathbf{g}}_M)\leqslant M!\,\lambda_1\dots\lambda_{M-1}Q. \tag{8.3}$$

Proof. Notice that $\hat{\mathbf{g}}_M$ is the same as the point $\mathbf{G}_{\tau_1}$ in (7.50) with $k=M-1$. We get from (7.53),

$$\|\hat{L}_j^{(v)}(\hat{\mathbf{g}}_M)\|_v\leqslant((M-1)!)^{d_v/d}(\lambda_1\dots\lambda_{M-1})^{d_v/d}Q^{c_{1v}+\dots+c_{j-1,v}+c_{j+1,v}+\dots+c_{Mv}} \tag{8.4}$$

for $1\leqslant j\leqslant M$, $v\in S_\infty$, and from (7.51),

$$\|\hat{L}_j^{(v)}(\hat{\mathbf{g}}_M)\|_v\leqslant Q^{c_{1v}+\dots+c_{j-1,v}+c_{j+1,v}+\dots+c_{Mv}}\quad\text{for } 1\leqslant j\leqslant M,\ v\in S_0. \tag{8.5}$$

We may write the ith component $\hat{g}_{iM}$ of $\hat{\mathbf{g}}_M$ as

$$\hat{g}_{iM}=u_{i1}^{(v)}\hat{L}_1^{(v)}(\hat{\mathbf{g}}_M)+\dots+u_{iM}^{(v)}\hat{L}_M^{(v)}(\hat{\mathbf{g}}_M)\quad(1\leqslant i\leqslant M,\ v\in S).$$

Since we consider special systems, here the coefficients $u_{ij}^{(v)}$ are contained in the set $\{-1,0,1\}$.

Thus (8.4) implies

$$\|\hat{\mathbf{g}}_M\|_{1,v}\leqslant(M!\,\lambda_1\dots\lambda_{M-1})^{d_v/d}Q^{\max_{1\leqslant j\leqslant M}c_{1v}+\dots+c_{j-1,v}+c_{j+1,v}+\dots+c_{Mv}}\quad(v\in S_\infty), \tag{8.6}$$

and we infer from (8.5) that

$$\|\hat{\mathbf{g}}_M\|_v\leqslant Q^{\max_{1\leqslant j\leqslant M}c_{1v}+\dots+c_{j-1,v}+c_{j+1,v}+\dots+c_{Mv}}\quad(v\in S_0). \tag{8.7}$$

Combination of (8.6), (8.7) gives with (7.10),

$$H_{1,S}(\hat{\mathbf{g}}_M) \leqslant M!\,\lambda_1 \dots \lambda_{M-1} Q^{-\sum_{v\in S} c_{j(v),v}} \tag{8.8}$$

where for $v \in S$ the subscript $j(v)$ is such that

$$c_{j(v),v} = \min_{1\leqslant j\leqslant M} c_{jv}.$$

The assertion follows, since by (7.11),

$$\left|\sum_{v\in S} c_{j(v),v}\right| \leqslant 1.$$

Recall that with our conventions in §6 and by the discussion at the beginning of §7, the set of forms $L_1^{(v)}, \dots, L_M^{(v)}$ $(v\in S)$ was partitioned into $N+1$ subsets, according to a partition $S = S^{(1)} \cup \dots \cup S^{(N+1)}$ of our set of places. This partition was such that for $v_1, v_2 \in S^{(j)}$ we have

$$L_i^{(v_1)} = L_i^{(v_2)} \quad \text{for each } i\ (1 \leqslant i \leqslant M). \tag{8.9}$$

LEMMA 8.2. *Let $(i(v))_{v\in S}$ be a tuple with*

$$\prod_{v\in S} \|\hat{L}_{i(v)}^{(v)}(\hat{\mathbf{g}}_M)\|_v \neq 0. \tag{8.10}$$

Then we have

$$H_1(\hat{\mathbf{g}}_M) \geqslant M^{-2}\big((\lambda_1 \dots \lambda_{M-1})^{-1} Q^{\sum_{v\in S} c_{i(v),v}}\big)^{1/M(N+1)}. \tag{8.11}$$

Proof. First, we obtain using (8.4), (8.5) together with (7.10),

$$\prod_{v\in S} \|\hat{L}_{i(v)}^{(v)}(\hat{\mathbf{g}}_M)\|_v \leqslant (M-1)!\,\lambda_1 \dots \lambda_{M-1} Q^{-\sum_{v\in S} c_{i(v),v}}. \tag{8.12}$$

Now consider a set $S^{(j)}$. Given i with $1 \leqslant i \leqslant M$ let $S_i^{(j)}$ be the subset of $S^{(j)}$ consisting of those $v \in S^{(j)}$ with $i(v) = i$. So, for $v \in S_i^{(j)}$ we may write $\hat{L}_{i(v)}^{(v)} = \hat{L}_i$.

We obtain (using again the fact that the forms are special)

$$\begin{aligned}
1 &= \prod_{v\in M(K)} \|\hat{L}_i(\mathbf{g}_M)\|_v = \prod_{v\in S_i^{(j)}} \|\hat{L}_{i(v)}^{(v)}(\hat{\mathbf{g}}_M)\|_v \prod_{v\notin S_i^{(j)}} \|\hat{L}_i(\hat{\mathbf{g}}_M)\|_v \\
&\leqslant \prod_{v\in S_i^{(j)}} \|\hat{L}_{i(v)}^{(v)}(\hat{\mathbf{g}}_M)\|_v \Bigg(\prod_{\substack{v\in M_\infty(K)\\ v\notin S_i^{(j)}}} M^{d_v/d}\Bigg) \prod_{v\notin S_i^{(j)}} \|\hat{\mathbf{g}}_M\|_{1,v} \\
&= \prod_{v\in S_i^{(j)}} \|\hat{L}_{i(v)}^{(v)}(\hat{\mathbf{g}}_M)\|_v \Bigg(\prod_{\substack{v\in M_\infty(K)\\ v\notin S_i^{(j)}}} M^{d_v/d}\Bigg) \Bigg(\prod_{v\in S_i^{(j)}} \|\hat{\mathbf{g}}_M\|_{1,v}^{-1}\Bigg) H_1(\hat{\mathbf{g}}_M).
\end{aligned}$$

Taking the product over i with $1 \leqslant i \leqslant M$ and over j with $1 \leqslant j \leqslant M$ we may infer that

$$1 \leqslant \prod_{v \in S} \|\hat{L}_{i(v)}^{(v)}(\hat{\mathbf{g}}_M)\|_v M^{M(N+1)-1} H_{1,S}(\hat{\mathbf{g}}_M)^{-1} H_1(\hat{\mathbf{g}}_M)^{M(N+1)}. \tag{8.13}$$

(8.12) and (8.13) entail

$$H_1(\hat{\mathbf{g}}_M)^{M(N+1)} \geqslant H_{1,S}(\hat{\mathbf{g}}_M) M^{-M(N+1)+1}((M-1)!)^{-1}(\lambda_1 \ldots \lambda_{M-1})^{-1} Q^{\sum_{v \in S} c_{i(v),v}}.$$

But $\hat{\mathbf{g}}_M$ has S-integral components. Thus

$$H_1(\hat{\mathbf{g}}_M)^{M(N+1)} \geqslant M^{-M(N+1)+1}((M-1)!)^{-1}(\lambda_1 \ldots \lambda_{M-1})^{-1} Q^{\sum_{v \in S} c_{i(v),v}}$$

and (8.11) follows.

There is a nonzero linear form $V = V(\mathbf{X}) = v_1 X_1 + \ldots + v_M X_M$ with coefficients $v_i \in K$ that vanishes on $\mathbf{g}_1, \ldots, \mathbf{g}_{M-1}$. This form is determined up to a nonzero factor in K, however its height $H_1(V)$ is uniquely determined.

Let $\mathcal{S}$ be the set of tuples $(j(v))_{v \in S}$ such that

$$\sum_{v \in S} c_{j(v),v} > 0. \tag{8.14}$$

LEMMA 8.3. *Suppose that $\delta > 0$, that*

$$\lambda_{M-1} = \lambda_{M-1}(Q) \leqslant Q^{-\delta} \tag{8.15}$$

and

$$Q^{(M-1)\delta} > M^{4M(N+1)}. \tag{8.16}$$

Assume that there exists a tuple $(j(v))_{v \in S}$ in $\mathcal{S}$ with $\prod_{v \in S} \|\hat{L}_{j(v)}^{(v)}(\mathbf{g}_M)\|_v \neq 0$, i.e. with (8.10). Then we have

$$Q^{\delta/4(N+1)} < H_1(V) < Q. \tag{8.17}$$

Proof. Clearly the vector $\hat{\mathbf{g}}_M = \mathbf{g}_1 \wedge \ldots \wedge \mathbf{g}_{M-1}$ is orthogonal to $\mathbf{g}_1, \ldots, \mathbf{g}_{M-1}$. So it suffices to show that $H_1(\hat{\mathbf{g}}_M)$ satisfies (8.17).

Since $\hat{\mathbf{g}}_M$ has S-integral components, Lemma 8.1 implies that

$$H_1(\hat{\mathbf{g}}_M) \leqslant M! \, \lambda_1 \ldots \lambda_{M-1} Q$$

and the right hand side of (8.17) follows from (8.15), (8.16).

As for the lower bound in (8.17), we apply Lemma 8.2 and get for $(j(v))_{v \in S}$ in $\mathcal{S}$ using (8.15),

$$H_1(\hat{\mathbf{g}}_M) \geqslant M^{-2} Q^{(M-1)\delta/M(N+1)}.$$

In view of (8.16) this entails the lower bound in (8.17).

LEMMA 8.4. *Suppose that we have (7.13) for some $\eta>0$. Assume that*

$$(2M)^{2M}Q^{M\eta}D_K^{M/2d}\lambda_{M-1}\leqslant 1. \tag{8.18}$$

Suppose that there is a point $\mathbf{h}\neq 0$ in K^M with integral components satisfying

$$\prod_{v\in S}\|\hat{L}^{(v)}_{j(v)}(\mathbf{h})\|_v=0 \tag{8.19}$$

for every tuple $(j(v))_{v\in S}$ with $1\leqslant j(v)\leqslant M$ such that

$$(2MQ^\eta D_K^{1/2d})^{2M^2}Q^{\sum_{v\in S}c_{j(v),v}}\prod_{v\in S_\infty}\|\mathbf{h}\|_{1,v}\lambda_{M-1}\geqslant 1 \tag{8.20}$$

is true. Then

$$\mathbf{g}_i\mathbf{h}=0 \quad \textit{for } i=1,\dots,M-1. \tag{8.21}$$

Proof. Let $\mathbf{h}$ be a point with (8.19), (8.20). For $v\in S$ we define the set

$$C_v=\{j\mid 1\leqslant j\leqslant M,\ \hat{L}^{(v)}_j(\mathbf{h})\neq 0\}. \tag{8.22}$$

Since $\hat{L}^{(v)}_1,\dots,\hat{L}^{(v)}_M$ are linearly independent, we have $C_v\neq\varnothing$. Moreover, we define for each $v\in S$ an element $i(v)\in C_v$ by

$$c_{1v}+\dots+c_{i(v)-1,v}+c_{i(v)+1,v}+\dots+c_{Mv}=\min_{j\in C_v}c_{1v}+\dots+c_{j-1,v}+c_{j+1,v}+\dots+c_{Mv}. \tag{8.23}$$

For $v\in S_0$ let γ_{iv} be real numbers as in (7.13), so in particular we have for each pair (i,v), $\gamma_{iv}\leqslant c_{iv}$. Write

$$\begin{aligned} c_v^{(i)}&=c_{1v}+\dots+c_{i-1,v}+c_{i+1,v}+\dots+c_{Mv},\\ \gamma_v^{(i)}&=\gamma_{1v}+\dots+\gamma_{i-1,v}+\gamma_{i+1,v}+\dots+\gamma_{Mv},\\ \gamma_v&=\gamma_v^{(i(v))} \quad\text{and}\quad c_v=c_v^{(i(v))}\end{aligned} \tag{8.24}$$

$(1\leqslant i\leqslant M,\ v\in S_0)$. If α is an S-integer in K with

$$\|\alpha\|_v\leqslant Q^{c_v}, \tag{8.25}$$

then, since $\mathbf{h}$ has integral components and by the special shape of our forms $\hat{L}^{(v)}_i$, we see that

$$\|\hat{L}^{(v)}_i(\alpha\mathbf{h})\|_v\leqslant Q^{c_v}\quad \text{for each } i\ (1\leqslant i\leqslant M) \text{ and for each } v\in S_0.$$

But now the definition of the sets C_v in (8.22) and the definition of $i(v)$ in (8.23) together with $c_v\leqslant c_v^{(i)}$ imply that

$$\|\hat{L}^{(v)}_i(\alpha\mathbf{h})\|_v\leqslant Q^{c_v^{(i)}}\quad \text{for each } i\ (1\leqslant i\leqslant M) \text{ and for each } v\in S_0. \tag{8.26}$$

Moreover $\alpha\mathbf{h}$ has S-integral components, i.e. we have

$$\|\alpha\mathbf{h}\|_v \leqslant 1 \quad \text{for } v \notin S. \tag{8.27}$$

But (8.26), (8.27) mean that for $v \notin S_\infty$ the point $\alpha\mathbf{h}$ satisfies the inequalities (7.43) defining the parallelepiped $\Pi^{(M-1)}(Q)$.

Next suppose that $v \in S_\infty$. Define c_v by

$$Q^{c_v} = \|\mathbf{h}\|_{1,v}^{-1} Q^{c_{1v}+\dots+c_{i(v)-1,v}+c_{i(v)+1,v}+\dots+c_{Mv}}. \tag{8.28}$$

Write

$$c_v^{(i(v))} = c_{1v}+\dots+c_{i(v)-1,v}+c_{i(v)+1,v}+\dots+c_{Mv} \quad (v \in S_\infty). \tag{8.29}$$

We now apply Lemma 7.5. Accordingly there exists a nonzero S-integer $\alpha \in K$ satisfying (8.25) and

$$\|\alpha\|_v \leqslant Q^{c_v} \left(D_K^{1/2d} Q^{-\sum_{w\in S} c_w}\right)^{d_v/d} \cdot Q^{d_v/d(\sum_{w\in S_0}(c_w-\gamma_w))} \quad \text{for each } v \in S_\infty. \tag{8.30}$$

By (7.13) and (8.24) we have

$$\sum_{w\in S_0} (c_w - \gamma_w) < (M-1)\eta. \tag{8.31}$$

Let us study consequences of (8.30). The definition of C_v implies that

$$\|\hat{L}_j^{(v)}(\alpha\mathbf{h})\|_v = 0 \quad \text{for each } j \notin C_v.$$

On the other hand, for $j \in C_v$ we get with (8.28), (8.29) and (8.30), (8.31),

$$\begin{aligned}\|\hat{L}_j^{(v)}(\alpha\mathbf{h})\|_v &\leqslant M^{d_v/d}\|\alpha\|_v\|\mathbf{h}\|_v \\ &< M^{d_v/d} Q^{c_v^{(i(v))}} \left(\left(\prod_{w\in S_\infty} \|\mathbf{h}\|_{1,w}\right) Q^{-\sum_{w\in S} c_w^{(i(w))}}\right)^{d_v/d} \cdot Q^{(M-1)\eta d_v/d} D_K^{(1/2d)(d_v/d)}.\end{aligned}$$

But by (7.10) this equals

$$Q^{c_v^{(i(v))}} \left(M \cdot \left(\prod_{w\in S_\infty} \|\mathbf{h}\|_{1,v}\right) Q^{\sum_{w\in S} c_{i(w),w}} Q^{(M-1)\eta} D^{1/2d}\right)^{d_v/d}. \tag{8.32}$$

The definition of the sets C_v in (8.22) and of the tuple $(i(w))_{w\in S}$ in (8.23) implies that (8.20) is violated with the exponent $\sum_{v\in S} c_{i(v),v}$. We may conclude that the expression in the parenthesis in (8.32) is

$$< (Q^{-M^2\eta} D_K^{-M^2/2d} (2M)^{-M^2} \lambda_{M-1}^{-1})^{d_v/d}. \tag{8.33}$$

We now apply Lemma 7.10. It follows that the term in (8.33) is $\leqslant \lambda_2^{*d_v/d}$. Altogether we get with (8.23) and (8.24),

$$\|\hat{L}_j^{(v)}(\alpha\mathbf{h})\|_v < \lambda_2^{*d_v/d} Q^{c_v^{(i(v))}} \leqslant \lambda_2^{*d_v/d} Q^{c_v^{(j)}} \quad \text{for } v \in S_\infty \text{ and } j \in C_v. \tag{8.34}$$

We infer from (8.34) and the definition of C_v that in fact

$$\|\hat{L}_j^{(v)}(\alpha\mathbf{h})\|_v < \lambda_2^{*d_v/d} Q^{c_v^{(j)}} \quad \text{for each } v \in S_\infty \text{ and for } 1 \leqslant j \leqslant M. \tag{8.35}$$

In view of (8.26), (8.27), (8.35) we see that the point $\alpha\mathbf{h}$ lies in the interior of $\lambda_2^*\Pi^{(M-1)}$.

We now study the point $\hat{\mathbf{g}}_M$. It has S-integral components and moreover it satisfies (8.5). On the other hand, for $v \in S_\infty$ we get from (8.4) and Lemma 7.10 for $1 \leqslant j \leqslant M$,

$$\begin{aligned}\|\hat{L}_j^{(v)}(\hat{\mathbf{g}}_M)\|_v &\leqslant ((M-1)!\,\lambda_1 \ldots \lambda_{M-1})^{d_v/d} Q^{c_v^{(j)}} \\ &\leqslant ((M-1)!\,\lambda_1 \ldots \lambda_{M-2}\lambda_{M-1}^2 \lambda_2^* Q^{M^2\eta} D_K^{M^2/2d} (2M)^{M^2})^{d_v/d} Q^{c_v^{(j)}} \\ &< (\lambda_{M-1}^M (2M)^{2M^2} Q^{M^2\eta} D_K^{M^2/2d})^{d_v/d} \cdot \lambda_2^{*d_v/d} Q^{c_v^{(j)}}.\end{aligned}$$

By (8.18) this implies that

$$\|\hat{L}_j^{(v)}(\hat{\mathbf{g}}_M)\|_v < \lambda_2^{*d_v/d} Q^{c_v^{(j)}} \quad \text{for each } v \in S_\infty \text{ and for } j \text{ with } 1 \leqslant j \leqslant M.$$

Therefore $\hat{\mathbf{g}}_M$ also lies in the interior of $\lambda_2^*\Pi^{(M-1)}$.

Since any two points in K^M that lie in the interior of $\lambda_2^*\Pi^{(M-1)}$ are linearly dependent, it follows that $\alpha\mathbf{h}$ and $\hat{\mathbf{g}}_M$ are proportional, and hence also $\mathbf{h}$ and $\hat{\mathbf{g}}_M$ are proportional. As $\hat{\mathbf{g}}_M$ is orthogonal to $\mathbf{g}_1, \ldots, \mathbf{g}_{M-1}$ the same holds true for $\mathbf{h}$ and (8.21) is proved.

LEMMA 8.5. *Suppose that there is a point* $\mathbf{h} \neq \mathbf{0}$ *in* K^M *with integral components such that*

$$\prod_{v \in S} \|\hat{L}_{j(v)}^{(v)}(\mathbf{h})\|_v = 0 \quad \textit{for each tuple } (j(v))_{v \in S} \textit{ in } \mathcal{S}. \tag{8.36}$$

In fact assume that $\mathbf{h}$ *is a point with this property where moreover* $\prod_{v \in S_\infty} \|\mathbf{h}\|_{1,v}$ *is minimal. Suppose that* $\delta > 0$ *and assume that* (7.13) *is satisfied for a value* $\eta > 0$ *with*

$$\eta \leqslant \delta / 4M^2. \tag{8.37}$$

Suppose moreover that

$$\lambda_{M-1} \leqslant Q^{-\delta} \tag{8.38}$$

and that

$$Q^\delta > (2M)^{6M^2} D_K^{2M^2/d}. \tag{8.39}$$

Then (8.21) *is true, i.e. we have*

$$\mathbf{g}_1\mathbf{h}=\ldots=\mathbf{g}_{M-1}\mathbf{h}=0.$$

Proof. It will suffice to prove that with a minimal $\mathbf{h}$ the hypotheses (8.18), (8.19), (8.20) of Lemma 8.4 are satisfied.

As for (8.18), we get with (8.37), (8.38), (8.39),

$$(2M)^{2M}Q^{M\eta}D_K^{M/2d}\lambda_{M-1}\leqslant(2M)^{2M}Q^{\delta/4M}D_K^{M/2d}Q^{-\delta}\leqslant(2M)^{2M}D_K^{M/2d}Q^{-\delta/2}<1$$

So (8.18) holds true.

We next check (8.19), (8.20). Lemma 7.12 says that if there is a point $\mathbf{h}\neq\mathbf{0}$ with (8.19) at all, then in fact there exists such an $\mathbf{h}$ with components in $\mathbf{Z}$ and with

$$\prod_{v\in S_\infty}\|\mathbf{h}\|_{1,v}\leqslant M^{(M-1)/2}.$$

With such $\mathbf{h}$, tuples $(j(v))_{v\in S}$ having (8.20) in view of (8.38) satisfy

$$M^{(M-1)/2}(2MQ^{\eta}D_K^{1/2d})^{2M^2}Q^{-\delta}Q^{\sum_{v\in S}c_{j(v),v}}\geqslant 1. \tag{8.40}$$

However we infer from (8.37) and (8.39) that

$$\begin{aligned}M^{(M-1)/2}(2MQ^{\eta}D_K^{1/2d})^{2M^2}Q^{-\delta}&<(2M)^{3M^2}\cdot Q^{\delta/2}D_K^{M^2/d}Q^{-\delta}\\&=(2M)^{3M^2}D_K^{M^2/d}Q^{-\delta/2}<1.\end{aligned}$$

Therefore (8.40) implies that

$$Q^{\sum_{v\in S}c_{j(v),v}}>1,$$

i.e. that $(j(v))_{v\in S}$ lies in our set $\mathcal{S}$. But then (8.36) guarantees that (8.19) of Lemma 8.4 is satisfied. The assertion follows from Lemma 8.4.

LEMMA 8.6. *Suppose that the numbers* c_{iv} $(v\in S,\ 1\leqslant i\leqslant M)$ *satisfy* (7.10) *and* (7.11). *Let* $1>\delta>0$ *be given. Consider the inequalities*

$$\|L_i^{(v)}(\mathbf{x})\|_v\leqslant Q^{c_{iv}-\delta d_v/d}\quad(v\in S_\infty,\ 1\leqslant i\leqslant M), \tag{8.41}$$

$$\|L_i^{(v)}(\mathbf{x})\|_v\leqslant Q^{c_{iv}}\quad(v\in S_0,\ 1\leqslant i\leqslant M), \tag{8.42}$$

$$\|\mathbf{x}\|_v\leqslant 1\quad(v\notin S). \tag{8.43}$$

Let Q_0 *be a quantity with*

$$Q_0^{\delta}\geqslant M^2. \tag{8.44}$$

Suppose that $E>1$.

Then as Q runs through the range

$$Q_0 < Q \leqslant Q_0^E \tag{8.45}$$

the solutions $\mathbf{x}\in K^M$ of (8.41), (8.42), (8.43) *are contained in the union of not more than*

$$1+\frac{4}{\delta}\log E \tag{8.46}$$

proper subspaces of K^M.

Proof. Consider an interval of type

$$Q_0 < Q \leqslant Q_0^{1+\delta/2} \tag{8.47}$$

and let $Q_1, \dots, Q_M$ be any values of Q in (8.47). For $j=1,\dots,M$ let $\mathbf{x}_1,\dots,\mathbf{x}_M$ be solutions of (8.41)–(8.43) with $Q=Q_1,\dots,Q=Q_M$ respectively. So we have

$$\begin{aligned}
&\|L_i^{(v)}(\mathbf{x}_j)\|_v \leqslant Q_j^{c_{iv}-\delta d_v/d} \quad (v\in S_\infty,\ 1\leqslant i,j\leqslant M),\\
&\|L_i^{(v)}(\mathbf{x}_j)\|_v \leqslant Q_j^{c_{iv}} \quad (v\in S_0,\ 1\leqslant i,j\leqslant M),\\
&\qquad \|\mathbf{x}_j\|_v \leqslant 1 \quad (v\notin S).
\end{aligned}$$

Since our forms are special, we get for $v\in S_\infty$,

$$\begin{aligned}
\|\det(\mathbf{x}_1,\dots,\mathbf{x}_M)\|_v &= \|\det(L_i^{(v)}(\mathbf{x}_j))\|_v\\
&\leqslant M!^{d_v/d} \max_{j_1,\dots,j_M} Q_{j_1}^{c_{1v}-\delta d_v/d}\cdots Q_{j_M}^{c_{Mv}-\delta d_v/d}
\end{aligned} \tag{8.48}$$

where the maximum is taken over permutations $j_1,\dots,j_M$ of $1,\dots,M$. Similarly for $v\in S_0$ we obtain

$$\|\det(\mathbf{x}_1,\dots,\mathbf{x}_M)\|_v \leqslant \max_{j_1,\dots,j_M} Q_{j_1}^{c_{1v}}\cdots Q_{j_M}^{c_{Mv}} \tag{8.49}$$

and finally

$$\|\det(\mathbf{x}_1,\dots,\mathbf{x}_M)\|_v \leqslant 1 \quad \text{for } v\notin S. \tag{8.50}$$

For $v\in S$ define the ordered tuple $c_{1(v),v},\dots,c_{M(v),v}$ such that the maximum in (8.48) or (8.49) is

$$Q_1^{c_{1(v),v}-\delta d_v/d}\cdots Q_M^{c_{M(v),v}-\delta d_v/d} \quad \text{or} \quad Q_1^{c_{1(v),v}}\cdots Q_M^{c_{M(v),v}}$$

respectively. Combining (8.48), (8.49), (8.50) we get with (8.47) and (7.10),

$$\begin{aligned}
\prod_{v\in M(K)} \|\det(\mathbf{x}_1,\dots,\mathbf{x}_M)\|_v &\leqslant M!\prod_{i=1}^{M} Q_i^{(\sum_{v\in S} c_{i(v),v})-\delta}\\
&\leqslant M!\,Q_0^{(\sum_{v\in S}\sum_{i=1}^{M} c_{i,v})-M\delta+(\Sigma_+)^{\delta/2}}\\
&= M!\,Q_0^{-M\delta+(\Sigma_+)^{\delta/2}}
\end{aligned} \tag{8.51}$$

where Σ_+ is the sum over terms $(\sum_{v\in S} c_{i(v),v})-\delta$ with $(\sum_{v\in S} c_{i(v),v})-\delta>0$. By (7.11) we see that $\Sigma_+\leqslant M$. Therefore we infer from (8.51) and (8.44) that

$$\prod_{v\in M(K)} \|\det(\mathbf{x}_1,\dots,\mathbf{x}_M)\|_v \leqslant M!\, Q_0^{-M\delta/2} < 1.$$

So in fact by the product formula $\det(\mathbf{x}_1,\dots,\mathbf{x}_M)=0$. We may conclude that solutions $\mathbf{x}$ corresponding to values Q in an interval of type (8.47) contribute only a single subspace of dimension $M-1$. But the interval (8.45) may be covered by

$$1+\frac{\log E}{\log(1+\delta/2)} < 1+\frac{4}{\delta}\log E$$

intervals of type (8.47). This proves (8.46).

COROLLARY 8.7. *Let $\Pi(Q)$ be given by* (7.12). *Assume that the parameters c_{iv} satisfy* (7.10), (7.11). *Let $S(Q)$ be the subspace spanned by $\mathbf{g}_1=\mathbf{g}_1(Q),\dots,\mathbf{g}_{M-1}=\mathbf{g}_{M-1}(Q)$. Suppose that $\delta>0$. Then the values Q with*

$$\lambda_{M-1}=\lambda_{M-1}(Q)\leqslant Q^{-\delta} \tag{8.52}$$

and lying in an interval

$$Q_0<Q\leqslant Q_0^E \tag{8.53}$$

where $E>1$ and where

$$Q_0^\delta>M^2 \tag{8.54}$$

give rise to not more than

$$1+\frac{4}{\delta}\log E$$

distinct subspaces $S(Q)$.

Proof. This is an immediate consequence of Lemma 8.6. It suffices to remark that (8.52) implies that $\mathbf{g}_1,\dots,\mathbf{g}_{M-1}$ satisfy inequalities (8.41).

9. Bounds for the index

We denote by $\mathcal{R}$ the ring of polynomials

$$P(X_{11},\dots,X_{1M};X_{21},\dots,X_{2M};X_{m1},\dots,X_{mM})$$

in mM variables and with coefficients in K. Given an m-tuple

$$\mathbf{r}=(r_1,\dots,r_m) \tag{9.1}$$

of natural numbers, $\mathcal{R}'$ will denote the set of polynomials in $\mathcal{R}$ which are homogeneous of degree r_h in the block of variables $X_{h1}, \ldots, X_{hM}$ $(1 \leqslant h \leqslant m)$. We write $\mathcal{I}$ for mM-tuples of nonnegative integers

$$\mathcal{I} = (i_{11}, \ldots, i_{1M}; \ldots i_{m1}, \ldots, i_{mM}),$$

and by $(\mathcal{I}/\mathbf{r})$ we denote the expression

$$(\mathcal{I}/\mathbf{r}) = \sum_{h=1}^{m} \frac{i_{h1} + \ldots + i_{hM}}{r_h}.$$

We write

$$P^{\mathcal{I}} = \frac{1}{i_{11}! \ldots i_{mM}!} \cdot \frac{\partial^{i_{11} + \ldots + i_{mM}}}{\partial X_{11}^{i_{11}} \ldots \partial X_{mM}^{i_{mM}}} P.$$

Given $\mathbf{r}$ as in (9.1), put

$$r = r_1 + \ldots + r_m. \tag{9.2}$$

It is easily seen that for $P \in \mathcal{R}'$ we have

$$H_1(P^{\mathcal{I}}) \leqslant 2^r H_1(P), \tag{9.3}$$

in fact we get

$$\begin{aligned} H_{1,v}(P^{\mathcal{I}}) &\leqslant 2^{r d_v / d} H_{1,v}(P) \quad \text{if } v \mid \infty, \\ H_{1,v}(P^{\mathcal{I}}) &\leqslant H_{1,v}(P) \quad \text{if } v \nmid \infty. \end{aligned} \tag{9.4}$$

Let $L_1, \ldots, L_m$ be nonzero linear forms with coefficients in K. Assume that L_h is a form in the variables $X_{h1}, \ldots, X_{hM}$ $(h=1, \ldots, m)$, so that

$$L_h = \alpha_{h1} X_{h1} + \ldots + \alpha_{hM} X_{hM} \quad (h = 1, \ldots, m).$$

The *index* of a polynomial $P \in \mathcal{R}$ with respect to $(L_1, \ldots, L_m; \mathbf{r})$ is defined as follows: *when* $P \equiv 0$, *set* $\operatorname{Ind} P = \infty$.

When $P \not\equiv 0$, *the index is the least value of* c *such that there is an* $\mathcal{I}$ *with* $(\mathcal{I}/\mathbf{r}) = c$ *and such that* $P^{\mathcal{I}}$ *is not identically zero on the subspace* T *of* K^{mM} *defined by the equations* $L_1 = \ldots = L_m = 0$.

Given a linear form

$$L = \alpha_1 X_1 + \ldots + \alpha_M X_M$$

we make m forms out of it by setting

$$L_{[h]} = \alpha_1 X_{h1} + \ldots + \alpha_M X_{hM} \quad (h = 1, \ldots, m).$$

The *index with respect to* $(L; \mathbf{r})$ is then defined as the index with respect to

$$(L_{[1]}, \ldots, L_{[m]}; \mathbf{r}).$$

LEMMA 9.1 (Index Theorem). *Suppose that $L_1, \ldots, L_t$ are nonzero linear forms in M variables and with coefficients in $\{-1,0,1\}$. Suppose that $\varepsilon > 0$ and that*

$$m > 4\varepsilon^{-2}\log(2t). \tag{9.5}$$

Then given $\mathbf{r}=(r_1, \ldots, r_m)$, there exists a nonzero polynomial $P \in \mathcal{R}'$, in fact with coefficients in $\mathbf{Z}$ having

(i) $\operatorname{Ind} P \geqslant (1/M-\varepsilon)m$ *with respect to* $(L_i; \mathbf{r})$ $(i=1, \ldots, t)$,

(ii) $H_1(P) < 2^{mM}(3M)^r$.

This is a very special version of the Index Theorem as proved in Schmidt [17, §9]. It suffices to remark that our forms L_i have rational coefficients and $H(L_i) \leqslant M^{1/2}$.

Now suppose that we are given $N+1$ systems of linear forms

$$L_1^{(j)}(\mathbf{X}), \ldots, L_M^{(j)}(\mathbf{X}) \quad (1 \leqslant j \leqslant N+1)$$

in $\mathbf{X}=(X_1, \ldots, X_M)$. We assume that for each j the system $L_1^{(j)}, \ldots, L_M^{(j)}$ is special as defined in §7.

Remember that this implies in particular that if we express the variables X_i in terms of $L_1^{(j)}(\mathbf{X}), \ldots, L_M^{(j)}(\mathbf{X})$ as

$$X_i = \eta_{i1}^{(j)} L_1^{(j)}(\mathbf{X}) + \ldots + \eta_{iM}^{(j)} L_M^{(j)}(\mathbf{X}) \tag{9.6}$$

then the $\eta_{ik}^{(j)}$ lie in $\{-1,0,1\}$. With this assumption, Schmidt's Polynomial Theorem ([17, §9]) may be quoted as follows.

Let P be the polynomial of the Index Theorem, and suppose that it holds with respect to the special forms $L_1^{(1)}(\mathbf{X}), \ldots, L_M^{(N+1)}(\mathbf{X})$, i.e. suppose that we have

$$t = (N+1)M. \tag{9.7}$$

Given an mM-tuple $\mathcal{I}$, for each j $(1 \leqslant j \leqslant N+1)$ we may write $P^{\mathcal{I}}$ uniquely as

$$P^{\mathcal{I}} = \sum_{j_{11}, \ldots, j_{mM}} d_{(j)}^{\mathcal{I}}(j_{11}, \ldots, j_{mM}) L_{1[1]}^{(j)j_{11}} \cdots L_{M[1]}^{(j)j_{1M}} \cdots L_{1[m]}^{(j)j_{m1}} \cdots L_{M[m]}^{(j)j_{mM}}, \tag{9.8}$$

and here the summation may be restricted to $j_{h1} + \ldots + j_{hM} \leqslant r_h$ $(h=1, \ldots, m)$.

LEMMA 9.2 (Polynomial Theorem). *Suppose that for each j $(1 \leqslant j \leqslant N+1)$ the forms $L_1^{(j)}, \ldots, L_M^{(j)}$ are a special system. Then the following assertions hold true:*

(i) *When $(\mathcal{I}/\mathbf{r}) \leqslant 2\varepsilon m$, then $d_{(j)}^{\mathcal{I}}(j_{11}, \ldots, j_{mM})=0$ for each j $(1 \leqslant j \leqslant N+1)$ unless*

$$\left|\sum_{h=1}^{m} \frac{j_{hk}}{r_h} - \frac{m}{M}\right| \leqslant 3mM\varepsilon \quad (1 \leqslant k \leqslant M).$$

(ii) *Each coefficient $d_{(j)}^{\mathcal{I}}(j_{11}, \ldots, j_{mM})$ lies in $\mathbf{Z}$ and has standard absolute value*

$$|d_{(j)}^{\mathcal{I}}(j_{11}, \ldots, j_{mM})| \leqslant 2^{mM}(6M^2)^r.$$

This follows at once from Schmidt's proof in [17, pp. 156–157] upon noting that with our special system the $\eta_{ik}^{(j)}$ in (9.6) lie in $\{-1,0,1\}$ and upon using the bound for $H_1(P)$ from Lemma 9.1.

The polynomial constructed in Lemmata 9.1 and 9.2 fits well into our special systems from §§ 6, 7 and 8. In fact recall that in §8, we study systems of forms, where in view of (8.9) we really do have just $M(N+1)$ forms. So in the sequel we suppose that $\varepsilon>0$. We apply Lemma 9.1 with

$$t=(N+1)M \tag{9.9}$$

and with

$$m>4\varepsilon^{-2}\log(2(N+1)M). \tag{9.10}$$

For $v\in S$, the system $L_1^{(v)},\dots,L_M^{(v)}$ will be just one out of the systems $L_1^{(j)},\dots,L_M^{(j)}$ we considered in the polynomial theorem. As in §7, c_{iv} will be real numbers satisfying (7.10) and (7.11), i.e.

$$\sum_{v\in S}\sum_{i=1}^{M} c_{iv}=0, \tag{9.11}$$

$$\left|\sum_{v\in S'} c_{i(v),v}\right|\leqslant 1 \tag{9.12}$$

for each subset S' of S and for any tuple $(i(v))_{v\in S'}$ with $1\leqslant i(v)\leqslant M$. Given $Q>1$, let $\Pi(Q)$ be the parallelepiped (7.12), i.e.

$$\begin{gathered}\|L_i^{(v)}(\mathbf{x})\|_v\leqslant Q^{c_{iv}}\quad (v\in S,\ 1\leqslant i\leqslant M),\\ \|\mathbf{x}\|_v\leqslant 1\quad (v\notin S).\end{gathered} \tag{9.13}$$

We have minima $\lambda_1=\lambda_1(Q),\dots,\lambda_M=\lambda_M(Q)$ and we have certain points $\mathbf{g}_1=\mathbf{g}_1(Q)$, $\dots,\mathbf{g}_M=\mathbf{g}_M(Q)$ corresponding to the minima as in §§ 7 and 8. Let $V=V(Q)$ be the linear form with coefficients in K (determined up to a nonzero factor) that vanishes on $\mathbf{g}_1,\dots,\mathbf{g}_{M-1}$.

If $V=v_1X_1+\dots+v_MX_M$, we write

$$V_{[h]}=v_1X_{h1}+\dots+v_MX_{hM}\quad (h=1,\dots,m).$$

LEMMA 9.3. *Suppose that* $0<\delta<1$ *and that*

$$0<\varepsilon\leqslant\delta/15M^2. \tag{9.14}$$

Let $Q_1,\dots,Q_m$ *be real numbers* >1 *satisfying*

$$r_1\log Q_1\leqslant r_h\log Q_h\leqslant(1+\varepsilon)r_1\log Q_1\quad (h=1,\dots,m), \tag{9.15}$$

$$\lambda_{M-1}(Q_h)\leqslant Q_h^{-\delta} \tag{9.16}$$

and

$$Q_h^{\delta} > 2^{30M}\varepsilon^{-5}. \tag{9.17}$$

Then P has index $\geqslant m\varepsilon$ with respect to $(V_{[1]}(Q_1), \dots, V_{[m]}(Q_m); \mathbf{r})$.

Proof. We proceed similarly as Schmidt in [17, Lemma 10.1]. So let us go through the proof in [17] to check the appropriate changes to be made.

It suffices to show that

$$P^{\mathcal{I}}(\mathbf{u}_1, \dots, \mathbf{u}_m) = 0 \tag{9.18}$$

whenever $\mathcal{I}/\mathbf{r} < 2\varepsilon m$ and $\mathbf{u}_h$ $(1 \leqslant h \leqslant m)$ lies in the grid of points

$$\mathbf{u} = u_1 \mathbf{g}_1(Q_h) + \dots + u_{M-1}\mathbf{g}_{M-1}(Q_h)$$

where the u_i are rational integers having

$$1 \leqslant u_i \leqslant [\varepsilon^{-1}] + 1. \tag{9.19}$$

To prove (9.18) we show that

$$\prod_{v \in M(K)} \|P^{\mathcal{I}}(\mathbf{u}_1, \dots, \mathbf{u}_m)\|_v < 1.$$

As $P^{\mathcal{I}}$ has rational integral coefficients and since the points $\mathbf{g}_1, \dots, \mathbf{g}_{M-1}$ have S-integral components, we get at once

$$\prod_{v \notin S} \|P^{\mathcal{I}}(\mathbf{u}_1, \dots, \mathbf{u}_m)\|_v \leqslant 1.$$

Thus, it suffices to show that

$$\prod_{v \in S} \|P^{\mathcal{I}}(\mathbf{u}_1, \dots, \mathbf{u}_m)\|_v < 1. \tag{9.20}$$

To do so, we use the representation (9.8) of $P^{\mathcal{I}}$, where we recall that the superscript (j) stands for the numbering of the system $L_1^{(j)}, \dots, L_M^{(j)}$. Thus in the current context it will be convenient to replace it by v with $v \in S$.

As is shown in [17], the coefficients $d_v^{\mathcal{I}}(j_{11}, \dots, j_{mM})$ in (9.8) vanish unless

$$\left| \left(\sum_{h=1}^{m} j_{hk} \log Q_h \right) - r_1 \log Q_1 \frac{m}{M} \right| < r_1 \log Q_1 \tfrac{7}{2} M m\varepsilon.$$

(It is at this stage that (9.15) is needed.)

With our hypothesis (9.14) on ε this implies that

$$r_1 \frac{m}{M} \log Q_1 \left(1-\tfrac{1}{4}\delta\right) < \sum_{h=1}^{m} j_{hk} \log Q_h < r_1 \frac{m}{M} \log Q_1 \left(1+\tfrac{1}{4}\delta\right) \tag{9.21}$$

for each k ($1 \leqslant k \leqslant M$). Suppose first that $v \in S_\infty$.

Consider a point $\mathbf{u}_h$ in the grid. Then (9.13), (9.16), (9.19) imply that

$$\|L_k^{(v)}(\mathbf{u}_h)\|_v < (M(\varepsilon^{-1}+1)\lambda_{M-1}(Q_h))^{d_v/d} Q_h^{c_{kv}} \leqslant \left(\frac{2M}{\varepsilon}\right)^{d_v/d} Q_h^{c_{kv}-\delta d_v/d}.$$

Therefore we get for exponents $j_{11}, \ldots, j_{mM}$ with (9.21),

$$\begin{aligned} \|L_k^{(v)}(\mathbf{u}_1)^{j_{1k}} \ldots L_k^{(v)}(\mathbf{u}_m)^{j_{mk}}\|_v &< \left(\frac{2M}{\varepsilon}\right)^{(j_{1k}+\ldots+j_{mk})d_v/d} \\ &\quad \times Q_1^{r_1(m/M)(c_{kv}-\delta d_v/d)+r_1(m/M)(\delta/4)|c_{kv}-\delta d_v/d|}. \end{aligned} \tag{9.22}$$

Combining (9.22) with the estimate for $d_v^{\mathcal{I}}(j_{11}, \ldots, j_{mM})$ of part (ii) of Lemma 9.2, we see that for $v \in S_\infty$ each nonzero summand in (9.8) has $\|\cdot\|_v$-modulus

$$< \left(2^{mM}(6M^2)^r \left(\frac{2M}{\varepsilon}\right)^r\right)^{d_v/d} Q_1^{r_1(m/M)(c_{1v}+\ldots+c_{Mv}-M\delta d_v/d)+r_1(m/M)(\delta/4)\sum_{k=1}^{M}|c_{kv}-\delta d_v/d|}.$$

The number of summands in (9.8) is $\leqslant 2^{Mr}$, so that

$$\begin{aligned} \|P^{\mathcal{I}}(\mathbf{u}_1, \ldots, \mathbf{u}_m)\|_v &< \left(2^{mM+Mr}(6M^2)^r \left(\frac{2M}{\varepsilon}\right)^r\right)^{d_v/d} \\ &\quad \times Q_1^{r_1(m/M)(c_{1v}+\ldots+c_{Mv}-M\delta d_v/d)+r_1(m/M)(\delta/4)\sum_{k=1}^{M}|c_{kv}-\delta d_v/d|}. \end{aligned} \tag{9.23}$$

Next we consider $v \in S_0$. Here we get for exponents $j_{11}, \ldots, j_{mM}$ such that the corresponding coefficient in (9.8) does not vanish in analogy with (9.22), again using (9.13) and (9.21),

$$\|L_k^{(v)}(\mathbf{u}_1)^{j_{1k}} \ldots L_k^{(v)}(\mathbf{u}_m)^{j_{mk}}\|_v \leqslant Q_1^{r_1 m c_{kv}/M+(r_1 m\delta/4M)|c_{kv}|}.$$

Since the coefficients of $P^{\mathcal{I}}$ lie in $\mathbf{Z}$ this implies

$$\|P^{\mathcal{I}}(\mathbf{u}_1, \ldots, \mathbf{u}_m)\|_v \leqslant Q_1^{r_1(m/M)(c_{1v}+\ldots+c_{Mv})+r_1(m/M)(\delta/4)\sum_{k=1}^{M}|c_{kv}|}. \tag{9.24}$$

Combination of (9.23) and (9.24) will give the desired estimate (9.20).

To do the details let us first consider the exponent of Q_1 in this estimate. As for the main term we get

$$r_1\frac{m}{M}\left(\sum_{v\in S}\sum_{k=1}^{M}c_{kv}-M\delta\sum_{v\in S_\infty}\frac{d_v}{d}\right)=-r_1m\delta$$

by (9.11). As for the error term, we obtain

$$r_1\frac{m}{M}\cdot\frac{\delta}{4}\left(\sum_{v\in S_\infty}\sum_{k=1}^{M}\left|c_{kv}-\delta\frac{d_v}{d}\right|+\sum_{v\in S_0}\sum_{k=1}^{M}|c_{kv}|\right).$$

Write Σ_+ for the terms in the parenthesis where $c_{kv}-\delta d_v/d$ or c_{kv} are nonnegative. By (9.12) the contribution of Σ_+ to the parenthesis does not exceed M.

Similarly, write Σ_- for the terms in the parenthesis where $c_{kv}-\delta d_v/d$ or c_{kv} are negative. Again by (9.12) and since $\delta<1$ these terms give a contribution not exceeding $2M$.

Altogether, we see that the error term in the exponent may be estimated by

$$r_1\frac{m}{M}\cdot\frac{\delta}{4}(M+2M)\leqslant\tfrac{3}{4}r_1m\delta.$$

Now combination of (9.23) and (9.24) gives with (9.15),

$$\begin{aligned}\prod_{v\in S}\|P^{\mathcal{I}}(\mathbf{u}_1,\ldots,\mathbf{u}_m)\|_v &< 2^{mM+Mr}(6M^2)^r\left(\frac{2M}{\varepsilon}\right)^r\cdot Q_1^{-r_1m\delta/4}\\ &<(2^{6M}\varepsilon^{-1})^rQ_1^{-r_1m\delta/4}\leqslant\prod_{h=1}^{m}(2^{6M}\varepsilon^{-1})^{r_h}Q_h^{-r_h\delta/4(1+\varepsilon)}\\ &<\prod_{h=1}^{m}(2^{6M}\varepsilon^{-1}Q_h^{-\delta/5})^{r_h}<1\end{aligned}$$

by (9.15) and (9.17), and thus (9.20) follows.

10. A variant of Roth's Lemma

In our application we need a version of Roth's Lemma for number fields and in homogenized form. Such a variant has been derived by W. M. Schmidt [16, §VI, Theorem 10B] with respect to linear forms with rational integral coefficients.

The classical Roth's Lemma has been extended to number fields by LeVeque [9] as well as by Lang [8], however they use slightly different normalizations. In our context we prefer the normalization as chosen in Schmidt [16].

So we first give an extension to number fields of the classical Roth's Lemma in the notation of Schmidt [16, §VI, Theorem 10A].

Let $P(X_1, \dots, X_m)$ be a polynomial with coefficients in K. Given nonnegative integers $i_1, \dots, i_m$ we write

$$P_{i_1,\dots,i_m} = \frac{1}{i_1! \dots i_m!} \cdot \frac{\partial^{i_1+\dots+i_m}}{\partial X_1^{i_1} \dots \partial X_m^{i_m}} P.$$

Let $r_1, \dots, r_m$ be positive integers, and let $(\xi_1, \dots, \xi_m)$ be an arbitrary point in K^m. We define the *Roth Index* of P with respect to $(\xi_1, \dots, \xi_m; r_1, \dots, r_m)$ as follows: If $P \equiv 0$, we put R-Ind $P = +\infty$. If $P \not\equiv 0$, R-Ind P is the smallest value of

$$\frac{i_1}{r_1} + \dots + \frac{i_m}{r_m}$$

for which $P_{i_1,\dots,i_m}(\xi_1, \dots, \xi_m) \neq 0$.

LEMMA 10.1 (Roth's Lemma for number fields). *Suppose that*

$$0 < \vartheta < \tfrac{1}{12}. \tag{10.1}$$

Let m be a fixed natural number. Put

$$\omega = \omega(m, \vartheta) = 12 \cdot 2^{-m} \left(\tfrac{1}{12}\vartheta\right)^{2^{m-1}}. \tag{10.2}$$

Let $r_1, \dots, r_m$ be positive integers with

$$\omega r_h \geqslant r_{h+1} \quad (1 \leqslant h < m). \tag{10.3}$$

Suppose that $0 < \gamma \leqslant 1$ and let $\xi_1, \dots, \xi_m$ be elements of K with

$$h_1(\xi_h)^{r_h} \geqslant h_1(\xi_1)^{\gamma r_1} \quad (1 \leqslant h \leqslant m), \tag{10.4}$$

$$h_1(\xi_h)^{\omega\gamma} \geqslant 2^{3m} \quad (1 \leqslant h \leqslant m). \tag{10.5}$$

Further, suppose that $P(X_1, \dots, X_m) \not\equiv 0$ is a polynomial of degree $\leqslant r_h$ in X_h $(1 \leqslant h \leqslant m)$ with coefficients in K and with

$$H_1(P) \leqslant h_1(\xi_1)^{\omega\gamma r_1}. \tag{10.6}$$

Then P has R-Ind $\leqslant \vartheta$ *with respect to $(\xi_1, \dots, \xi_m; r_1, \dots, r_m)$.*

Proof. We proceed by induction on m.

If $m=1$, we may write

$$P(X) = (X-\xi_1)^l M(X) \tag{10.7}$$

where $M(X)$ has coefficients in K and $M(\xi_1) \neq 0$. Write $f(X) = X - \xi_1$. Then our definition of the height H_1 in §4 implies that

$$H_1^l(f) \leqslant H_1(f^l).$$

On the other hand we get

$$H_1(f) = h_1(\xi_1).$$

Therefore (10.7) in conjunction with Lemma 4.2 implies

$$h_1(\xi_1)^l H_1(M) \leqslant 4^{r_1} H_1(P).$$

Combining this with (10.5), (10.6) and using $H_1(M) \geqslant 1$, we obtain

$$h_1(\xi_1)^l \leqslant h_1(\xi_1)^{2\omega\gamma r_1}$$

and therefore

$$\frac{l}{r_1} \leqslant 2\omega = \vartheta.$$

But l/r_1 is the Roth Index of P with respect to $(\xi_1; r_1)$ and the lemma is true with $m=1$.

Next suppose that $m>1$ and the lemma to be shown for $m-1$. We follow closely the exposition in Schmidt [16, pp. 142–148]. We consider decompositions

$$P(X_1, \ldots, X_m) = \sum_{j=1}^{k} \phi_j(X_1, \ldots, X_{m-1}) \psi_j(X_m) \tag{10.8}$$

where $\phi_1, \ldots, \phi_k$ and $\psi_1, \ldots, \psi_k$ are polynomials with coefficients in K. We choose such a decomposition with k minimal. Then

$$k \leqslant r_m + 1, \tag{10.9}$$

and $\phi_1, \ldots, \phi_k$ as well as $\psi_1, \ldots, \psi_k$ are linearly independent over K. Writing

$$\Delta_i' = \frac{1}{i_1! \ldots i_{m-1}!} \cdot \frac{\partial^{i_1 + \ldots + i_{m-1}}}{\partial X_1^{i_1} \ldots \partial X_{m-1}^{i_{m-1}}} \quad (1 \leqslant i \leqslant k)$$

where $i_1 + \ldots + i_{m-1} \leqslant i-1 \leqslant k-1 \leqslant r_m$ (by (10.9)), it follows as in [16] that there exist operators Δ_i' such that

$$W(X_1, \ldots, X_m) = \det\left(\frac{1}{(j-1)!} \cdot \frac{\partial^{j-1}}{\partial X_m^{j-1}} \Delta_i' P\right)_{1 \leqslant i,j \leqslant k}$$

satisfies

$$W(X_1,\dots,X_m)=V(X_1,\dots,X_{m-1})U(X_m)\not\equiv 0,$$

where

$$U(X_m)=\det\left(\frac{1}{(i-1)!}\cdot\frac{\partial^{i-1}}{\partial X_m^{i-1}}\psi_j(X_m)\right)_{1\leqslant i,j\leqslant k}$$

and

$$V(X_1,\dots,X_{m-1})=\det(\Delta_i'\phi_j)_{1\leqslant i,j\leqslant k}.$$

Notice that the entries in the determinant defining W are of the shape $P_{i_1,\dots,i_{m-1},j-1}$. It follows that the coefficients of the entries are sums of coefficients of P multiplied with certain binomial coefficients. In fact (9.4) implies that

$$H_{1,v}(P_{i_1,\dots,i_{m-1},j-1})\leqslant 2^{(r_1+\dots+r_m)d_v/d}H_{1,v}(P)\quad \text{for } v\in M_\infty(K)$$

and moreover

$$H_{1,v}(P_{i_1,\dots,i_{m-1},j-1})\leqslant H_{1,v}(P)\quad \text{for } v\in M_0(K).$$

The number of terms in $P_{i_1,\dots,i_{m-1},j-1}$ is

$$\leqslant (r_1+1)\dots(r_m+1)\leqslant 2^{r_1+\dots+r_m}.$$

The number of summands in the expansion of W is $k!\leqslant k^{k-1}\leqslant k^{r_m}\leqslant 2^{kr_m}$. We may infer that

$$\begin{aligned}H_{1,v}(W)&\leqslant (2^{kr_m}2^{2(r_1+\dots+r_m)k})^{d_v/d}H_{1,v}(P)^k\\&\leqslant (2^{3mr_1d_v/d}H_{1,v}(P))^k\quad \text{for } v\in M_\infty(K),\end{aligned}\tag{10.10}$$

where we have used (10.2), (10.3). Moreover

$$H_{1,v}(W)\leqslant H_{1,v}(P)^k\quad \text{for } v\in M_0(K).\tag{10.11}$$

Combining (10.10), (10.11) with (10.6) and (10.5) we get

$$H_1(W)\leqslant (2^{3mr_1}h_1(\xi_1)^{\omega\gamma r_1})^k\leqslant h_1(\xi_1)^{2\omega\gamma r_1k}.$$

Now, since the variables in V and U are separated, we obtain

$$H_1(V)H_1(U)=H_1(W).$$

We may infer that

$$\begin{aligned}H_1(V)&\leqslant h_1(\xi_1)^{2\omega\gamma r_1k},\\H_1(U)&\leqslant h_1(\xi_1)^{2\omega\gamma r_1k}.\end{aligned}\tag{10.12}$$

To estimate the index of W with respect to $(\xi_1, \dots, \xi_m; r_1, \dots, r_m)$ we apply the induction hypothesis to V. More precisely we apply the assertion of Lemma 10.1 with m replaced by $m-1$, with $r_1, \dots, r_m$ replaced by $kr_1, \dots, kr_{m-1}$, with ϑ replaced by $\frac{1}{12}\vartheta^2$ and with $P(X_1, \dots, X_m)$ replaced by $V(X_1, \dots, X_{m-1})$. Notice that $\omega(m-1, \frac{1}{12}\vartheta^2) = 2\omega(m, \vartheta)$. Now (10.3) and (10.5) are satisfied with $\omega(m, \vartheta)$, hence they are also satisfied for $\omega(m-1, \frac{1}{12}\vartheta^2)$ and with m replaced by $m-1$ on the right hand side of (10.5). It is clear that with our new parameters (10.1) and (10.4) are satisfied. The analogue of (10.6) holds by (10.12). The conclusion is that $V(X_1, \dots, X_{m-1})$ has Roth-Index $\leqslant \frac{1}{12}\vartheta^2$ with respect to $(\xi_1, \dots, \xi_{m-1}; kr_1, \dots, kr_{m-1})$, hence it has Roth-Index $\leqslant \frac{1}{12}k\vartheta^2$ with respect to $(\xi_1, \dots, \xi_{m-1}; r_1, \dots, r_{m-1})$. It follows at once that $V(X_1, \dots, X_{m-1})$, considered as a polynomial in $X_1, \dots, X_m$, has Roth-Index $\leqslant \frac{1}{12}k\vartheta^2$ with respect to $(\xi_1, \dots, \xi_m; r_1, \dots, r_m)$.

Since $\omega(1, \frac{1}{12}\vartheta^2) \geqslant 2\omega(m, \vartheta)$, it follows from (10.12) that U satisfies the hypotheses of Roth's Lemma with $m=1$, ϑ replaced by $\frac{1}{12}\vartheta^2$ and with r_m replaced by kr_m. Applying the case $m=1$, which was already established, we see that U has Roth-Index $\leqslant \frac{1}{12}k\vartheta^2$ with respect to $(\xi_1, \dots, \xi_m; r_1, \dots, r_m)$. Altogether we may conclude as in Schmidt [16, p. 146] that W has Roth-Index $\leqslant \frac{1}{6}k\vartheta^2$ with respect to $(\xi_1, \dots, \xi_m; r_1, \dots, r_m)$.

The remainder of the proof now is verbatim the same as in Schmidt [16, pp. 146–148]. We have only given the details of the first part to ensure that the arguments in [16], where special properties of the rational integers are used, carry over to the more general situation of number fields.

Using Lemma 10.1, we may derive an upper bound for the index of a polynomial $P = P(X_{11}, \dots, X_{1M}; \dots; X_{m1}, \dots, X_{mM})$ in our ring $\mathcal{R}$ as defined in §8. We obtain:

LEMMA 10.2 (Roth's Linear Forms Lemma for number fields). *Suppose that*

$$0 < \vartheta < \tfrac{1}{12}. \tag{10.13}$$

Let m be a positive integer and put

$$\omega = \omega(m, \vartheta) = 12 \cdot 2^{-m} \left(\tfrac{1}{12}\vartheta\right)^{2^{m-1}}. \tag{10.14}$$

Let $r_1, \dots, r_m$ be positive integers with

$$\omega r_h \geqslant r_{h+1} \quad (1 \leqslant h < m). \tag{10.15}$$

Suppose that $M \geqslant 2$ is an integer.

Let $V_1, \dots, V_m$ be nonzero linear forms in M variables with coefficients in K. Suppose that $0 < \Gamma \leqslant M-1$, that

$$H_1(V_h)^{r_h} \geqslant H_1(V_1)^{r_1 \Gamma} \quad (2 \leqslant h \leqslant m) \tag{10.16}$$

and that

$$H_1(V_h)^{\omega\Gamma} \geqslant 2^{3m(M-1)^2} \quad (1 \leqslant h \leqslant m). \tag{10.17}$$

Let $P \in \mathcal{R}$ *be a nonzero polynomial that is homogeneous in* $X_{h1}, \ldots, X_{hM}$ *of degree* r_h $(1 \leqslant h \leqslant m)$. *Suppose moreover that*

$$H_1(P)^{(M-1)^2} \leqslant H_1(V_1)^{\omega r_1 \Gamma}. \tag{10.18}$$

Then the index of P *(in the sense of* §8*) with respect to* $(V_1, \ldots, V_m; r_1, \ldots, r_m)$ *is*

$$\leqslant \vartheta.$$

Proof. This may be derived in exactly the same way as Theorem 10B in Schmidt [16, p. 191] is deduced from Theorem 10A in [16, p. 141]. Schmidt studies a polynomial P with rational integral coefficients and uses the height $\lceil P \rceil$, the maximum absolute value of the coefficients of P. Now, if P has relatively prime coefficients (which we may assume), then $\lceil P \rceil = H_1(P)$. A similar remark applies to the linear forms $V_1, \ldots, V_m$ under consideration.

In the proof in [16] the following fact is used: Given a linear form $V = v_1 X_1 + \ldots + v_M X_M$, assume that $|v_1| = \lceil V \rceil$. Then there exists an i with $2 \leqslant i \leqslant M$ such that $\lceil V \rceil \leqslant h_1(v_i/v_1)^{M-1} \leqslant \lceil V \rceil^{M-1}$. The analogous inequality in our context is

$$H_1(V) \leqslant h_1(v_i/v_1)^{M-1} \leqslant H_1(V)^{M-1},$$

and in view of Lemma 4.1 we can guarantee such an inequality. As otherwise the proof of [16] has not to be changed, we omit the details here.

11. Caring for the penultimate minimum

LEMMA 11.1. *Suppose that* $0 < \delta < 1$ *and that*

$$m > 900 M^4 \delta^{-2} \log(2(N+1)M). \tag{11.1}$$

Put

$$E = \tfrac{1}{6} 2^m (180)^{2^{m-1}}. \tag{11.2}$$

Let $\Pi(Q)$ *be the parallelepiped* (7.12) *with parameters* c_{iv} $(v \in S,\ 1 \leqslant i \leqslant M)$ *satisfying* (7.10), (7.11). *Suppose that there is no point* $\mathbf{h} \neq \mathbf{0}$ *in* K^M *with* (8.36) *for every tuple* $(j(v))_{v \in S}$ *in* $\mathcal{S}$ (cf. (8.14)).

Then the numbers Q *with*

$$\lambda_{M-1}(Q) < Q^{-\delta} \tag{11.3}$$

and

$$Q^{\delta^2} > 2^{24M^3(N+1)^2mE} \tag{11.4}$$

are contained in the union of $m-1$ *intervals of the type*

$$Q_h < Q \leqslant Q_h^E \quad (h=1, \ldots, m-1). \tag{11.5}$$

Proof. Suppose the lemma were false.

Let Q_1 be the infimum of values Q having (11.3) and (11.4). Then Q with (11.3), (11.4) have $Q > Q_1$.

If all the values Q with (11.3), (11.4) were in the interval $Q_1 < Q \leqslant Q_1^E$, the lemma would be true. So there are $Q > Q_1^E$ with (11.3). Let Q_2 be their infimum and so forth. We find in this way values $Q_1, \ldots, Q_m$ with

$$\lambda_{M-1}(Q_h) \leqslant Q_h^{-\delta} \quad (h=1, \ldots, m) \tag{11.6}$$

and

$$Q_{h+1} \geqslant Q_h^E \quad (h=1, \ldots, m-1). \tag{11.7}$$

We want to apply Lemma 9.3. Put $\varepsilon = \delta/15M^2$ and choose r_1 so large that

$$r_1 > \varepsilon^{-1} \frac{\log Q_m}{\log Q_1}.$$

For $h=2, \ldots, m$ we put

$$r_h = r_1 \frac{\log Q_1}{\log Q_h} + 1.$$

Then we get

$$r_1 \log Q_1 \leqslant r_h \log Q_h \leqslant r_1 \log Q_1 + \log Q_h < (1+\varepsilon) r_1 \log Q_1 \tag{11.8}$$

and thus (9.15) is satisfied. Hypothesis (9.16) is the same as (11.6).

With our value of ε, hypothesis (9.17) is satisfied if $Q_h^\delta > 2^{30M} \cdot 15^5 \cdot M^{10}/\delta^5$, but this is amply guaranteed for by (11.4), (11.2), (11.1). We apply Lemma 9.3 to the polynomial of the Index and Polynomial Theorem (Lemmata 9.1, 9.2) with $t=(N+1)M$, i.e. (9.9). In Lemma 9.1 we need (9.5), i.e.

$$m > 4\varepsilon^{-2} \log(2(N+1)M).$$

With our value of ε this becomes

$$m > 900 M^4 \delta^{-2} \log(2(N+1)M)$$

and hypothesis (11.1) takes care of this condition. Thus all hypotheses of Lemma 9.3 are satisfied. We may conclude that P has index

$$\operatorname{ind} P \geqslant m\varepsilon \tag{11.9}$$

with respect to $(V_{[1]}(Q_1), \ldots, V_{[m]}(Q_m); \mathbf{r})$. To get an upper bound for the index, we apply Lemma 10.2 with $\vartheta = \frac{1}{15}$.

Now, with ω given by (10.14), the parameter E in (11.2) has $E = 2/\omega$.

We infer from (11.8) and (11.7) that

$$\omega r_h \geqslant \omega \frac{r_{h+1} \log Q_{h+1}}{(1+\varepsilon)\log Q_h} = \frac{2}{E} \cdot \frac{r_{h+1} \log Q_{h+1}}{(1+\varepsilon)\log Q_h} \geqslant r_{h+1} \quad (h = 1, \ldots, m-1).$$

So (10.15) is satisfied.

As there is no $\mathbf{h} \neq \mathbf{0}$ in K^M with (8.36), we may apply Lemma 8.3. We get

$$Q_h^\Gamma < H_1(V_h) < Q_h \quad (h = 1, \ldots, m) \tag{11.10}$$

with

$$\Gamma = \frac{\delta}{4(N+1)} \tag{11.11}$$

provided that (8.16) holds true, i.e. provided

$$Q^{(M-1)\delta} > M^{4M(N+1)}.$$

Again (11.4) takes care of this.

Now we obtain with (11.10) and (11.8)

$$H_1(V_h)^{r_h} > Q_h^{r_h \Gamma} \geqslant Q_1^{r_1 \Gamma} > H_1(V_1)^{r_1 \Gamma} \quad (h = 1, \ldots, m).$$

Thus (10.16) is true.

As for (10.17), we infer from (11.10), (11.11) and since $E = 2/\omega$ that

$$H_1(V_h)^{\omega\Gamma} > Q_h^{\omega\Gamma^2} = Q_h^{2\Gamma^2/E} = Q_h^{\delta^2/8(N+1)^2 E} > 2^{2m(M-1)^2},$$

the last inequality by (11.4). So (10.17) is satisfied as well.

We still have to check hypothesis (10.18). By Lemma 9.1, our polynomial P has

$$H_1(P) < 2^{mM}(3M)^r < 2^{3Mmr_1}.$$

Combining this with (11.4), (11.10), (11.11) we get

$$\begin{aligned} H_1(P)^{(M-1)^2} &< 2^{3Mmr_1(M-1)^2} < 2^{3M^3 m r_1} < Q_1^{(\delta^2/8(N+1)^2 E) r_1} \\ &= Q_1^{(\delta^2/16(N+1)^2) 2 r_1/E} = Q_1^{\omega r_1 \Gamma^2} < H_1(V_1)^{\omega r_1 \Gamma} \end{aligned}$$

which is the desired (10.18).

The conclusion of Roth's Lemma 10.2 is that the index of P with respect to $(V_{[1]}(Q_1), \ldots, V_{[m]}(Q_m); \mathbf{r})$ satisfies

$$\operatorname{ind} P \leqslant \vartheta = \frac{1}{15} < m\frac{\delta}{15M^2} = m\varepsilon \tag{11.12}$$

(the last inequality by (11.1)).

However (11.9) and (11.12) contradict each other and thus the lemma follows.

LEMMA 11.2. *Let δ, m, E be as in Lemma* 11.1.

Let $\Pi(Q)$ be the parallelepiped (7.12) *with parameters c_{iv} ($v \in S$, $1 \leqslant i \leqslant M$) satisfying* (7.10), (7.11). *Given Q, let $S(Q)$ be the subspace spanned by $\mathbf{g}_1 = \mathbf{g}_1(Q), \ldots, \mathbf{g}_{M-1} = \mathbf{g}_{M-1}(Q)$.*

Let

$$0 \leqslant \eta \leqslant \frac{\delta}{4M^2}. \tag{11.13}$$

Then as Q ranges over values with (7.13), (11.3), (11.4) *and moreover*

$$Q^\delta > D_K^{4M^2/d}, \tag{11.14}$$

$S(Q)$ ranges over less than

$$m\left(1 + \frac{4}{\delta}\log E\right) \tag{11.15}$$

distinct $(M-1)$-dimensional subspaces of K^M.

Proof. We distinguish two alternatives. Suppose first that there exists a nonzero point $\mathbf{h} \in K^M$ having (8.36) for each tuple $(j(v))_{v \in S}$ in the set $\mathcal{S}$ defined by (8.14). We want to apply Lemma 8.5.

So let $\mathbf{h} \neq \mathbf{0}$ be a point with integral components in K satisfying (8.36) and having minimal $\prod_{v \in S_\infty} \|\mathbf{h}\|_{1,v}$.

Hypothesis (8.37) is (11.3), (8.38) is satisfied by (11.3). Moreover we infer from (11.4) that

$$Q^{\delta/2} > Q^{\delta^2/2} > 2^{12M^3} > (2M)^{6M^2},$$

and from (11.14) we get

$$Q^{\delta/2} > D_K^{2M^2/d}.$$

Together this gives

$$Q^\delta > (2M)^{6M^2} D_K^{2M^2/d},$$

so hypothesis (8.39) holds true as well. By Lemma 8.5, $S(Q)$ consists of points $\mathbf{x}$ with $\mathbf{hx} = 0$, i.e. one single subspace suffices.

If there is no such point $\mathbf{h}$, the hypotheses of Lemma 11.1 are satisfied.

In fact, we may combine Lemma 11.1 with Corollary 8.7. Notice that (11.4) amply guarantees (8.54). Therefore, as Q runs through an interval (11.5), $S(Q)$ will run through a set of subspaces of cardinality $\leqslant 1+(4/\delta)\log E$. Summation over h in $1\leqslant h<m$ gives the assertion.

12. Connections between two adjacent minima

LEMMA 12.1. *Suppose that* $0<\delta<10M$ *and that*

$$m>90\,000\,M^6\delta^{-2}\log(2(N+1)M). \tag{12.1}$$

Let E *be given by* (11.2). *Let* $\Pi(Q)$ *be the parallelepiped* (7.12) *with parameters* c_{iv} $(v\in S,$ $1\leqslant i\leqslant M)$ *satisfying* (7.10), (7.11). *Suppose that*

$$0\leqslant\eta\leqslant\frac{\delta}{8M^3}. \tag{12.2}$$

Then for values Q *with* (7.13),

$$Q^\delta>D_K^{8M^3/d}, \tag{12.3}$$

$$Q^{\delta^2/20M^2}>2^{24M^3(N+1)^2mE} \tag{12.4}$$

and

$$\lambda_{M-1}(Q)<Q^{-\delta}\lambda_M(Q). \tag{12.5}$$

$S(Q)$ *is among not more than*

$$2^{6M^2}\delta^{-M}m\left(1+\frac{40M}{\delta}\log E\right) \tag{12.6}$$

subspaces of K^M.

Proof. For $v\in S$ define

$$c_v=\min\{c_{1v},\ldots,c_{Mv}\}. \tag{12.7}$$

For $v\in S_0$ let $\gamma_v=\gamma_v(Q)\leqslant c_v$ be largest such that Q^{γ_v} lies in the value group of $\|\cdot\|_v$. By (7.13) we have

$$\prod_{v\in S_0}Q^{c_v-\gamma_v}\leqslant Q^\eta.$$

By Lemma 7.5, there exists a nonzero S-integer $\alpha\in K$ with

$$\|\alpha\|_v \leqslant Q^{c_v}(Q^{(-\sum_{w\in S}c_w)+\eta}D_K^{1/2d})^{d_v/d} \quad (v\in S_\infty),$$
$$\|\alpha\|_v \leqslant Q^{c_v} \quad (v\in S_0).$$

Since our forms $L_1^{(v)},\dots,L_M^{(v)}$ are a special system in the sense of §7, we obtain with the canonical basis vectors $\mathbf{e}_1,\dots,\mathbf{e}_M$ in view of (12.7),

$$\begin{aligned}\|L_i^{(v)}(\alpha\mathbf{e}_j)\|_v &\leqslant Q^{c_v}(Q^{(-\sum_{w\in S}c_w)+\eta}D_k^{1/2d})^{d_v/d}\\ &\leqslant Q^{c_{iv}}(Q^{(-\sum_{w\in S}c_w)+\eta}D_K^{1/2d})^{d_v/d} \quad (v\in S_\infty,\ 1\leqslant i,j\leqslant M),\end{aligned} \tag{12.8}$$

$$\|L_i^{(v)}(\alpha\mathbf{e}_j)\|_v \leqslant Q^{c_v}\leqslant Q^{c_{iv}} \quad (v\in S_0,\ 1\leqslant i,j\leqslant M). \tag{12.9}$$

We infer from (12.8), (12.9) using (7.11), (12.2), (12.3) that

$$\lambda_M(Q)\leqslant Q^{(-\sum_{w\in S}c_w)+\eta}D_K^{1/2d}<Q^2. \tag{12.10}$$

Again, since we are considering special systems $L_1^{(v)},\dots,L_M^{(v)}$, any point $\mathbf{x}\in K^M$ has a representation

$$x_i=\eta_{i1}^{(v)}L_1^{(v)}(\mathbf{x})+\dots+\eta_{iM}^{(v)}L_M^{(v)}(\mathbf{x}) \quad (v\in S,\ 1\leqslant i\leqslant M)$$

with coefficients $\eta_{ij}^{(v)}$ in $\{-1,0,1\}$. It follows that any point $\mathbf{x}\in K^M$ satisfies

$$\|\mathbf{x}\|_v\leqslant M^{d_v/d}\max_i\|L_i^{(v)}(\mathbf{x})\|_v \quad (v\in S_\infty), \tag{12.11}$$

$$\|\mathbf{x}\|_v\leqslant \max_i\|L_i^{(v)}(\mathbf{x})\|_v \quad (v\in S_0). \tag{12.12}$$

Let $\mathbf{g}_1$ be an S-integral point in K^M corresponding to the first minimum $\lambda_1(Q)$ of $\Pi(Q)$ and assume that $(i(v))_{v\in S}$ is a tuple where the maxima for $\mathbf{g}_1$ in (12.11), (12.12) are assumed. Then we get with (12.11), (12.12) and (7.11),

$$\lambda_1(Q)\geqslant Q^{-\sum_{v\in S}c_{i(v),v}}\prod_{v\in S}\|L_{i(v)}^{(v)}(\mathbf{g}_1)\|_v\geqslant Q^{-\sum_{v\in S}c_{i(v),v}}M^{-1}H_S(\mathbf{g}_1)\geqslant Q^{-1}M^{-1}.$$

In view of (12.4), this implies

$$\lambda_1(Q)\geqslant Q^{-2}. \tag{12.13}$$

Define the parameter R by

$$R=[16M^3\delta^{-1}]+1 \tag{12.14}$$

and divide the interval $[-2,2]$ into $4R$ subintervals, each of length $1/R$ and of type

$$-2+\frac{l}{R}\leqslant x<-2+\frac{l+1}{R}\quad (0\leqslant l<4R). \tag{12.15}$$

We partition the set of Q under consideration into

$$\leqslant (4R)^M \tag{12.16}$$

subsets as follows: Q and Q' will belong to the same subset if for each i $(1\leqslant i\leqslant M)$ the minima $\lambda_i(Q)$ and $\lambda_i(Q')$ satisfy

$$Q^{-2+l_i/R}\leqslant \lambda_i(Q)<Q^{-2+(l_i+1)/R}\quad\text{and}\quad (Q')^{-2+l_i/R}\leqslant \lambda_i(Q')<(Q')^{-2+(l_i+1)/R},$$

i.e. if $\log\lambda_i(Q)/\log Q$ and $\log\lambda_i(Q')/\log Q'$ lie in the same subinterval (12.15).

We now restrict ourselves to values Q that belong to the same subset.

If $\lambda_i=\lambda_i(Q)$ has

$$Q^{-2+l_i/R}\leqslant \lambda_i<Q^{-2+(l_i+1)/R} \tag{12.17}$$

write

$$\Lambda_i=\Lambda_i(Q)=Q^{-2+l_i/R}. \tag{12.18}$$

Put

$$\varrho_0=(\Lambda_1\dots\Lambda_{M-2}\Lambda_{M-1}^2)^{1/M}$$

and

$$\varrho_1=\varrho_0/\Lambda_1,\quad \dots,\quad \varrho_{M-1}=\varrho_0/\Lambda_{M-1}\quad\text{but}\quad \varrho_M=\varrho_{M-1}=\varrho_0/\Lambda_{M-1}.$$

It is clear that relations (7.27), (7.29) of Lemma 7.8 are satisfied. Moreover our construction in (12.17), (12.18) is such that (7.28) is true with

$$\zeta=\frac{1}{R}<\frac{\delta}{16M^3}. \tag{12.19}$$

We fix $v_0\in M_\infty(K)$. Then by Lemma 7.8, given Q, there is a permutation τ of $\{1,\dots,M\}$ such that the successive minima λ_i' of the parallelepiped $\Pi'=\Pi'(Q)$ given by

$$\|L_i^{(v_0)}(\mathbf{x})\|_{v_0}\leqslant \varrho_{\tau(i)}^{-1}Q^{c_{iv_0}}\quad (1\leqslant i\leqslant M) \tag{12.20}$$

and otherwise (i.e. for all other places v) as in (7.12) satisfy

$$D_K^{-1/2d}2^{-M}Q^{-M\zeta}\varrho_i\lambda_i\leqslant \lambda_i'\leqslant 4^{M^2}Q^{M^2\zeta}D_K^{(2M-1)/2d}\varrho_i\lambda_i\quad (1\leqslant i\leqslant M). \tag{12.21}$$

Now the permutation still may depend upon Q. We therefore partition the set of Q under consideration into subsets, such that elements Q and Q' in the same subset give rise to the same permutation τ. As there are $M!$ permutations of $\{1, \ldots, M\}$, the number of possible subsets does not exceed

$$M!. \tag{12.22}$$

In the sequel we restrict ourselves moreover to elements Q in one such subset. Since we are studying values Q with (7.13), we may apply Lemma 7.4.

Accordingly we have

$$M^{-M} \leqslant \lambda_1 \ldots \lambda_M \leqslant (Q^\eta D_K^{1/2d})^M \tag{12.23}$$

and the same inequality is true for the product $\lambda_1' \ldots \lambda_M'$.

Now by (12.17), (12.18), (12.23), (12.5),

$$\begin{aligned} \varrho_0 &= (\Lambda_1 \ldots \Lambda_{M-1} \Lambda_{M-1}^2)^{1/M} \leqslant (\lambda_1 \ldots \lambda_{M-2} \lambda_{M-1}^2)^{1/M} \\ &= \left(\lambda_1 \ldots \lambda_M \left(\frac{\lambda_{M-1}}{\lambda_M}\right)\right)^{1/M} < Q^\eta D_K^{1/2d} Q^{-\delta/M}. \end{aligned}$$

Therefore (12.21) entails with (12.17), (12.18), (12.19),

$$\begin{aligned} \lambda_{M-1}' &\leqslant 4^{M^2} Q^{M^2\zeta} D_K^{(2M-1)/2d} \varrho_{M-1} \lambda_{M-1} \\ &= 4^{M^2} Q^{M^2\zeta} D_K^{(2M-1)/2d} \frac{\varrho_0}{\Lambda_{M-1}} \lambda_{M-1} \\ &< 4^{M^2} Q^{M^2\zeta} D_K^{(2M-1)/2d} Q^\eta D_K^{1/2d} Q^{-\delta/M+\zeta} \\ &= 4^{M^2} Q^{(M^2+1)\zeta+\eta} D_K^{M/d} Q^{-\delta/M} < Q^{-\delta/2M}, \end{aligned} \tag{12.24}$$

the last inequality by (12.2), (12.3), (12.4), (12.19).

Moreover we get using (12.10), (12.13) and the definition of the Λ_i in (12.18) that

$$\varrho_1 = \Lambda_1^{-1} \varrho_0 \leqslant Q^4, \quad \varrho_M = \Lambda_{M-1}^{-1} \varrho_0 \geqslant Q^{-4},$$

and thus we have

$$Q^{-4} \leqslant \varrho_M \leqslant \varrho_{M-1} \leqslant \ldots \leqslant \varrho_1 \leqslant Q^4. \tag{12.25}$$

Notice that our construction is such that there exist *fixed* real numbers say $f_1, \ldots, f_M$ such that for each Q in our subset we may write

$$\varrho_i = \varrho_i(Q) = Q^{f_i} \quad (i = 1, \ldots, M). \tag{12.26}$$

Therefore the parallelepiped $\Pi'(Q)$ we obtain in applying Davenport's Lemma 7.8 again is a parallelepiped of type (7.12), defined with fixed parameters, say c_{iv}' ($v \in S$, $1 \leqslant i \leqslant M$). Since $\varrho_1 \ldots \varrho_M = 1$ we have

$$\sum_{v \in S} \sum_{i=1}^{M} c_{iv}' = 0. \tag{12.27}$$

On the other hand (12.25) and (12.26) imply that

$$\left|\sum_{v\in S'} c'_{i(v),v}\right| \leqslant 5. \tag{12.28}$$

for any subset S' of S and for any tuple $(i(v))_{v\in S'}$ with $1\leqslant i(v)\leqslant M$.

Therefore, we may apply Lemma 11.2 to be parallelepiped Π' with Q replaced by Q^5. Then (12.27), (12.28) are the analogues of (7.10), (7.11). As the parameters c_{iv} for $v\in S_0$ are the same in the definition of Π and Π', if we replace Q by Q^5 the c_{iv} $(v\in S_0)$ are replaced by $\frac{1}{5}c_{iv}$.

Therefore (7.13) is true with η replaced by $\frac{1}{5}\eta$. By (12.24), the analogue of (11.3) is satisfied with δ replaced by $\delta/10M$. With these changes (11.13) becomes (12.2), (11.4) becomes (12.3), (11.1) becomes (12.1), (11.4) becomes (12.4).

The conclusion is that for values Q in a fixed subset, we get less than

$$m\left(1+\frac{40M}{\delta}\log E\right) \tag{12.29}$$

subspaces $S'(Q)$ corresponding to the parallelepipeds $\Pi'(Q)$. By (12.16) and (12.22) together with (12.14) we see that the total number of subsets is bounded by

$$(4R)^M\cdot(M!) < 2^{5M^2}\delta^{-M}(M!) < 2^{6M^2}\delta^{-M}.$$

Combining this with (12.29) we may infer that altogether we have less than

$$2^{6M^2}\delta^{-M}m\left(1+\frac{40M}{\delta}\log E\right)$$

subspaces $S'(Q)$ corresponding to the parallelepipeds $\Pi'(Q)$.

To prove Lemma 12.1, it therefore suffices to show that for each Q we have the identity $S(Q)=S'(Q)$. Recall the definition of the forms $G_i^{(v)}(\mathbf{X})$ in (7.32), (7.33). By (7.34) any point $\mathbf{x}\in K^M$ that satisfies inequalities (7.12) for $v\notin S_\infty$ but does not lie in the subspace $S_{M-1}(Q)$ generated by $\mathbf{g}_1(Q),\ldots,\mathbf{g}_{M-1}(Q)$ has

$$\max_{v\in S_\infty}\{\max\{|G_1^{(v)}(\mathbf{x})|_v,\ldots,|G_M^{(v)}(\mathbf{x})|_v\}\} \geqslant 2^{-M}Q^{-M\zeta}D_K^{-1/2d}\varrho_M\lambda_M.$$

But by (12.21) and the analogue of (12.23) for $\lambda'_1,\ldots,\lambda'_M$ we get using (12.19) and (7.31),

$$\begin{aligned} 2^{-M}Q^{-M\zeta}D_K^{-1/2d}\varrho_M\lambda_M &\geqslant 2^{-M}Q^{-M\zeta}4^{-M^2}Q^{-M^2\zeta}D_K^{-1/2d}D_K^{-(2M-1)/2d}\lambda'_M \\ &\geqslant 2^{-M-2M^2}Q^{-2M^2\zeta}D_K^{-M/d}(\lambda'_1\ldots\lambda'_M)^{1/M} \\ &\geqslant 2^{-3M^2}Q^{-2M^2\zeta}D_K^{-M/d} > Q^{-\delta/16M}\cdot Q^{-\delta/8M}\cdot Q^{-\delta/16M} \\ &= Q^{-\delta/4M} \end{aligned}$$

(the last inequality by (12.4), (12.19), (12.3)).

It follows that such points $\mathbf{x}$ do not lie in $Q^{-\delta/4M}\Pi'(Q)$. On the other hand by (12.24) we have $\lambda'_{M-1}<Q^{-\delta/2M}$. Therefore the points $\mathbf{g}'_1(Q),\ldots,\mathbf{g}'_{M-1}(Q)$ that generate $S'_{M-1}(Q)$ cannot lie outside $S_{M-1}(Q)$ and thus $S'_{M-1}(Q)=S_{M-1}(Q)$ as desired.

LEMMA 12.2. *Let* $\Pi(Q)$ *be the parallelepiped* (7.12) *with parameters* c_{iv} ($v\in S$, $1\leqslant i\leqslant M$) *satisfying* (7.10), (7.11). *Suppose that* $1\leqslant l<M$ *and let* $S_l=S_l(Q)$ *be the subspace spanned by* $\mathbf{g}_1,\dots,\mathbf{g}_l$. *Put*

$$L=\binom{M}{l}.$$

Suppose that

$$0<\delta<20L \tag{12.30}$$

and that

$$0\leqslant\eta\leqslant\frac{\delta}{16L^4}. \tag{12.31}$$

Let

$$m>360\,000L^6M^2\delta^{-2}\log(2(N+1)L) \tag{12.32}$$

and put

$$E=\tfrac{1}{6}2^m(180)^{2^{m-1}}. \tag{12.33}$$

Then for values Q *having* (7.13),

$$Q^\delta>D_K^{16L^4/d}, \tag{12.34}$$

$$Q^{\delta^2}>2^{11L^5M^2(N+1)^2mE} \tag{12.35}$$

and

$$\lambda_l(Q)<Q^{-\delta}\lambda_{l+1}(Q), \tag{12.36}$$

the subspaces $S_l(Q)$ *run through a collection of not more than*

$$2^{6L^2}(2M)^L\delta^{-L}m\left(1+\frac{80LM}{\delta}\log E\right) \tag{12.37}$$

l*-dimensional subspaces of* K^M.

Proof. Put $k=M-l$. Recall from §7 that $C(M,k)$ is the set of k-tuples $\sigma=\{i_1<\dots<i_k\}$ of integers in $1\leqslant i\leqslant M$ and define $L_\sigma^{(v)}$ and $c_{\sigma v}$ as in §7. We apply Lemma 7.9 to the parallelepiped $\Pi^{(k)}(Q)$.

Denote its successive minima by $\nu_1,\dots,\nu_L$. It is clear that in Lemma 7.9 we may take

$$\tau_L=\{M-k+1,M-k+2,\dots,M\}=\{l+1,l+2,\dots,M\},$$
$$\tau_{L-1}=\{M-k,M-k+2,\dots,M\}=\{l,l+2,\dots,M\}.$$

By Lemma 7.9, we have

$$\nu_{L-1} \leqslant k!\,\lambda_{\tau_{L-1}} \tag{12.38}$$

and

$$\lambda_{\tau_L} \leqslant \nu_L Q^{Lk\eta} D_K^{Lk/2d} (k!)^{L-1} 2^{L^2}. \tag{12.39}$$

On the other hand we get from (12.36)

$$\lambda_{\tau_{L-1}} < Q^{-\delta} \lambda_{\tau_L}. \tag{12.40}$$

Combining (12.38), (12.39), (12.40) we may infer that

$$\nu_{L-1} \leqslant \nu_L D_K^{Lk/2d} (k!)^L Q^{Lk\eta} 2^{L^2} Q^{-\delta}$$

and therefore by (12.31), (12.34), (12.35),

$$\nu_{L-1} < Q^{-\delta/2} \nu_L. \tag{12.41}$$

Our new exponents $c_{\sigma v}$ have by (7.11),

$$\left| \sum_{v \in S'} c_{\sigma_v, v} \right| \leqslant k \tag{12.42}$$

for any subset S' of S and any tuple $(\sigma_v)_{v \in S'}$ with $\sigma_v \in C(M, k)$. Our goal is to apply Lemma 12.1. To guarantee the analogue of (7.11), in view of (12.42) we have to replace Q by Q^k. To get the analogue of (12.5), by (12.41), δ has to be replaced by $\delta/2k$, whereas η in (7.13) remains unchanged. M becomes L. The hypotheses in Lemma 12.2 are the hypotheses of Lemma 12.1 with this change of parameters, where at several instances we have sharpened the hypotheses slightly to clean up the situation.

The conclusion is that the subspaces $S^{(k)}(Q)$ spanned by the first $L-1$ minimal points of $\Pi^{(k)}(Q)$ are contained in the union of not more than (12.37) $(L-1)$-dimensional subspaces of K^L.

Let $\mathbf{g}_1, \ldots, \mathbf{g}_M$ be independent points with $\mathbf{g}_i \in \lambda_i \Pi$ $(i=1, \ldots, M)$. By (7.51), (7.52), (7.53) the points $\mathbf{G}_{\tau_1}, \ldots, \mathbf{G}_{\tau_{L-1}}$ lie in $k!\,\lambda_{\tau_{L-1}} \Pi^{(k)}$. But with the same argument that gave us (12.41) from (12.38), (12.39), (12.40) we obtain

$$k!\,\lambda_{\tau_{L-1}} < Q^{-\delta/2} \nu_{\tau_L}.$$

Therefore $\mathbf{G}_{\tau_1}, \ldots, \mathbf{G}_{\tau_{L-1}}$ span $S^{(k)}$. Hence there are not more than (12.37) possibilities for the span of $\mathbf{G}_{\tau_1}, \ldots, \mathbf{G}_{\tau_{L-1}}$. By Lemma 7.11 there are not more than (12.37) possibilities for the span of $\mathbf{g}_1, \ldots, \mathbf{g}_{M-k}$, i.e. for the span of $\mathbf{g}_1, \ldots, \mathbf{g}_l$ and hence for S_l.

13. Inning of Lemma 6.1

Consider the parallelepiped $\Pi(Q)$ given by

$$\begin{aligned} &\|L_i^{(v)}(\mathbf{x})\|_v \leqslant Q^{e_{iv}} \quad (v \in S,\ 1 \leqslant i \leqslant N), \\ &\|\mathbf{x}\|_v \leqslant 1 \quad (v \notin S). \end{aligned} \tag{13.1}$$

By Lemma 7.4, using (6.3) and (6.8), we get for the successive minima $\lambda_1, ..., \lambda_N$ the inequality

$$N^{-N} \leqslant \lambda_1 \dots \lambda_N. \tag{13.2}$$

In particular this implies that

$$\lambda_N \geqslant N^{-1}. \tag{13.3}$$

So by (6.5), the solutions $\mathbf{x}$ of (6.9), (6.10) are contained in the subspace $S_{N-1}(Q)$ spanned by $\mathbf{g}_1(Q), ..., \mathbf{g}_{N-1}(Q)$.

Given a solution $\mathbf{x}$ of (6.9), (6.10) let s be minimal, such that $\mathbf{x}$ lies in the subspace S_s spanned by $\mathbf{g}_1, ..., \mathbf{g}_s$. Then $1 \leqslant s \leqslant N-1$ and by (6.9), $\lambda_s \leqslant Q^{-\delta}$. By (13.3) there is an l with $s \leqslant l \leqslant N-1$ such that $\lambda_l \leqslant Q^{-\delta/(N-1)} N^{1/(N-1)} \lambda_{l+1}$. Suppose for the moment that

$$Q^{\delta} > N^N. \tag{13.4}$$

Then we get

$$\lambda_l < Q^{-\delta/N} \lambda_{l+1}. \tag{13.5}$$

Our goal is to apply Lemma 12.2 with M, L, δ respectively replaced by N, $\binom{N}{l} \leqslant 2^{N-1}$, δ/N. Then by (13.5), the analogue of (12.36) is satisfied. As for (12.31), we need

$$\eta \leqslant \frac{\delta}{16N\binom{N}{l}^4}$$

and this is certainly true if we require $\eta \leqslant \delta 2^{-5N}$, i.e. (6.7).

We next choose m as in (12.32). It will suffice to pick m with

$$m > 2^{19} \cdot 2^{6N-6} \cdot N^2 \cdot N^2 \delta^{-2} \log((N+1)2^N)$$

and this in turn will certainly be satisfied if

$$m \geqslant 2^{22+7N} \delta^{-2}. \tag{13.6}$$

Since $\delta < 1$, we may choose such an m with

$$m < 2^{23+7N} \delta^{-2}. \tag{13.7}$$

We next choose E according to (12.33), i.e.

$$E=\tfrac{1}{6}2^m(180)^{2^{m-1}}. \tag{13.8}$$

From (13.7), (13.6) we get

$$\log E\leqslant m+\tfrac{9}{4}\cdot 2^m<2^{2^{24+7N}\delta^{-2}}. \tag{13.9}$$

Suppose for the moment that the analogues of (12.34) and (12.35) are satisfied. In fact suppose that

$$Q^{\delta/N}>D_K^{2^{4N}/d}, \tag{13.10}$$

$$Q^{\delta^2/N^2}>2^{11\cdot 2^{5N-5}N^2(N+1)^2mE}. \tag{13.11}$$

Then Lemma 12.2 is applicable. By (12.37) we may conclude that the subspaces $S_l(Q)$ are contained in a collection of not more than

$$2^{6\cdot 2^{2N-2}}(2N)^{2^{N-1}}\delta^{-2^{N-1}}2^{23+7N}\delta^{-2}2^7 2^{N-1}\cdot N\delta^{-1}2^{2^{24+7N}\delta^{-2}}$$

subspaces of dimension l. Summing over l with $1\leqslant l\leqslant N-1$, we finally see that for values Q with (13.4), (13.10), (13.11) the solutions $\mathbf{x}$ of (6.9), (6.10) are contained in a collection of not more than

$$N\cdot 2^{6\cdot 2^{2N-2}}(2N)^{2^{N-1}}\delta^{-2^{N-1}}2^{23+7N}\delta^{-2}2^7 2^{N-1}\cdot N\delta^{-1}2^{2^{24+7N}\delta^{-2}}$$

proper subspaces of K^N.

And so with a crude estimate we see that

$$2^{2^{20N}\delta^{-2}} \tag{13.12}$$

proper subspaces will suffice.

There remains the range of Q, where (13.4), (13.10) or (13.11) is violated. We treat such small values of Q with Lemma 8.6. We apply Lemma 8.6 with M replaced by N.

In Lemma 8.6 the hypothesis (8.44) then is

$$Q_0\geqslant N^{2/\delta}>2,$$

since $\delta<1$. By (8.46), values Q with

$$N^{2/\delta}<Q\leqslant 2^{11\cdot 2^{5N-5}N^2(N+1)^2mEN^2\delta^{-2}} \tag{13.13}$$

give rise to not more than

$$1+4\delta^{-1}\log(11\cdot 2^{5N-5}N^2(N+1)^2mEN^2\delta^{-2})$$

proper subspaces. But by (13.7), (13.9) this number is bounded by

$$2^{2^{20N}\delta^{-2}}. \qquad (13.14)$$

Moreover, the range

$$\max\{N^{2/\delta}, D_K^{1/2d}\} < Q \leqslant D_K^{cN\cdot 2^{4N}/\delta\cdot d} \qquad (13.15)$$

by (8.46) gives not more than

$$1+\frac{4}{\delta}\cdot\log\frac{2N\cdot 2^{4N}}{\delta} \qquad (13.16)$$

proper subspaces. Notice that the ranges (13.13) and (13.15) take care of values Q, where one of (13.4), (13.10), (13.11) is violated. Combining (13.12), (13.14), (13.16), we see that the solutions $\mathbf{x}$ of (6.9), (6.10) with $Q>\max\{N^{2/\delta}, D_K^{1/2d}\}$ are contained in the union of not more than

$$2^{2^{21N}\delta^{-2}}$$

proper subspaces of K^N, and this is the assertion of Lemma 6.1.

14. Not yet the last section

We resume the notation of §§ 2 and 3. Recall that we want to study the equation

$$\det\begin{pmatrix} \alpha_1^{x_1} & x_1\alpha_1^{x_1} & \dots & x_1^{k_1}\alpha_1^{x_1} & \dots & \alpha_r^{x_1} & x_1\alpha_r^{x_1} & \dots & x_1^{k_r}\alpha_r^{x_1} \\ \vdots & & & & & & & & \vdots \\ \alpha_1^{x_q} & x_q\alpha_1^{x_q} & \dots & x_q^{k_1}\alpha_1^{x_q} & \dots & \alpha_r^{x_q} & x_q\alpha_r^{x_q} & \dots & x_q^{k_r}\alpha_r^{x_q} \end{pmatrix} = 0. \qquad (14.1)$$

As in §5 we write

$$\boldsymbol{\alpha} = (\underbrace{\alpha_1,\dots,\alpha_1}_{k_1+1},\underbrace{\alpha_2,\dots,\alpha_2}_{k_2+1},\dots,\underbrace{\alpha_r,\dots,\alpha_r}_{k_r+1}) = (\beta_1,\dots,\beta_q) = \boldsymbol{\beta}$$

with $q=k_1+\dots+k_r+r$. Given $\mathbf{x}=(x_1,\dots,x_q)$ we had $\boldsymbol{\beta}^{\mathbf{x}}=\beta_1^{x_1}\dots\beta_q^{x_q}$ and for a permutation σ from the symmetric group $\mathfrak{S}_q$ we write $\boldsymbol{\beta}_\sigma^{\mathbf{x}}=\beta_{\sigma(1)}^{x_1}\dots\beta_{\sigma(q)}^{x_q}$. Then equation (14.1) may be written as

$$\sum_{\sigma\in\mathfrak{S}_q} M_\sigma(\mathbf{x})\boldsymbol{\beta}_\sigma^{\mathbf{x}} = 0 \qquad (14.2)$$

where $M_\sigma(\mathbf{x})$ is a monomial in $x_1,\dots,x_q$ of degree $\leqslant q^2$.

Our goal is to manipulate equation (14.2) such that we are in a position to apply Lemma 6.1. The main difficulty consists in the problem to show that equation (14.1) leads to a finite number of systems of simultaneous inequalities of the type considered in §6. In principle such a reduction may be easily achieved, the point however is that we want the number of such systems bounded *independently of the cardinality of the set S.* The following considerations are undertaken with this aim.

Given $v\in S$, we define

$$\gamma_{jv} = \log \|\beta_j\|_v \quad (1 \leqslant j \leqslant q). \tag{14.3}$$

Then we get

$$\log \|\boldsymbol{\beta}_\sigma^{\mathbf{x}}\|_v = x_1\gamma_{\sigma(1),v} + \ldots + x_q\gamma_{\sigma(q),v}.$$

Recall that in Lemma 3.1 we are interested in solutions $\mathbf{x}=(x_1, \ldots, x_q)$ of (14.1) with

$$x_1 < x_2 < \ldots < x_q. \tag{14.4}$$

For each $v\in S$ let $\sigma_v \in \mathfrak{S}_q$ be a permutation such that

$$\gamma_{\sigma_v(1),v} \leqslant \gamma_{\sigma_v(2),v} \leqslant \ldots \leqslant \gamma_{\sigma_v(q),v}. \tag{14.5}$$

In view of (14.4) and (14.5) we have for each $v\in S$ and each permutation $\sigma \in \mathfrak{S}_q$,

$$x_1\gamma_{\sigma(1),v} + \ldots + x_q\gamma_{\sigma(q),v} \leqslant x_1\gamma_{\sigma_v(1),v} + \ldots + x_q\gamma_{\sigma_v(q),v}. \tag{14.6}$$

Write

$$N+1 = q!.$$

Let $\sigma_1, \ldots, \sigma_{N+1}$ be an ordering of the elements $\sigma \in \mathfrak{S}_q$. Using the definition of our set S we obtain

$$\prod_{v\in S} \|\boldsymbol{\beta}_{\sigma_1}^{\mathbf{x}} \cdots \boldsymbol{\beta}_{\sigma_{N+1}}^{\mathbf{x}}\|_v = 1,$$

and thus dividing by $\prod_{v\in S} \|\boldsymbol{\beta}_{\sigma_v}^{\mathbf{x}}\|_v$ we get by (14.6),

$$\prod_{v\in S}{}' \|\boldsymbol{\beta}_{\sigma_1}^{\mathbf{x}} \cdots \boldsymbol{\beta}_{\sigma_{N+1}}^{\mathbf{x}}\|_v = H_1((\boldsymbol{\beta}_{\sigma_1}^{\mathbf{x}}, \ldots, \boldsymbol{\beta}_{\sigma_{N+1}}^{\mathbf{x}}))^{-1}, \tag{14.7}$$

where the dash in (14.7) indicates that for each v the factor $\boldsymbol{\beta}_{\sigma_v}^{\mathbf{x}}$ is omitted. For technical reasons we prefer to write (14.7) as

$$\prod_{v\in S}{}' \left\| \frac{\boldsymbol{\beta}_{\sigma_1}^{\mathbf{x}}}{\boldsymbol{\beta}_{\sigma_1}^{\mathbf{x}}} \cdot \frac{\boldsymbol{\beta}_{\sigma_2}^{\mathbf{x}}}{\boldsymbol{\beta}_{\sigma_1}^{\mathbf{x}}} \cdot \ldots \cdot \frac{\boldsymbol{\beta}_{\sigma_{N+1}}^{\mathbf{x}}}{\boldsymbol{\beta}_{\sigma_1}^{\mathbf{x}}} \right\|_v = H_1\left(\left(1, \frac{\boldsymbol{\beta}_{\sigma_2}^{\mathbf{x}}}{\boldsymbol{\beta}_{\sigma_1}^{\mathbf{x}}}, \ldots, \frac{\boldsymbol{\beta}_{\sigma_{N+1}}^{\mathbf{x}}}{\boldsymbol{\beta}_{\sigma_1}^{\mathbf{x}}}\right)\right)^{-1}. \tag{14.8}$$

As in §4 write $S=S_\infty\cup S_0$. For $1\leqslant j\leqslant q$ we define sets $T_{\infty j}$ and T_{0j} as follows:

$$T_{\infty j}=\{(\sigma,j,v)\mid v\in S_\infty,\ \sigma\in\mathfrak{S}_q,\ \sigma\neq\sigma_v\},$$
$$T_{0j}=\{(\sigma,j,v)\mid v\in S_0,\ \sigma\in\mathfrak{S}_q,\ \sigma\neq\sigma_v\}.$$

We divide T_{0j} into two disjoint subsets T_{0j}^+ and T_{0j}^-.

The triple $(\sigma,j,v)\in T_{0j}$ will be in T_{0j}^+ precisely if

$$\gamma_{\sigma(j),v}-\gamma_{\sigma_1(j),v}=\Gamma_{\sigma,j,v}\geqslant 0, \tag{14.9}$$

i.e. if

$$\left\|\frac{\beta_{\sigma(j)}}{\beta_{\sigma_1(j)}}\right\|\geqslant 1.$$

Similarly T_{0j}^- consists of the triples $(\sigma,j,v)\in T_{0j}$ with

$$\gamma_{\sigma(j),v}-\gamma_{\sigma_1(j),v}=\Gamma_{\sigma,j,v}<0. \tag{14.10}$$

We partition $T_{\infty j}$ into two disjoint subsets $T_{\infty j}^+$ and $T_{\infty j}^-$ as follows:

The triple $(\sigma,j,v)\in T_{\infty j}$ will be in $T_{\infty j}^+$ precisely if

$$\gamma_{\sigma(j),v}-\gamma_{\sigma_1(j),v}-\frac{1}{qN}\cdot\frac{d_v}{d}\Biggl(\sum_{w\in S}\sum_{\substack{\sigma\in\mathfrak{S}_q\\ \sigma\neq\sigma_w}}\gamma_{\sigma(j),w}-\gamma_{\sigma_1(j),w}\Biggr)=\Gamma_{\sigma,j,v}\geqslant 0, \tag{14.11}$$

i.e. if

$$\left\|\frac{\beta_{\sigma(j)}}{\beta_{\sigma_1(j)}}\right\|_v\Bigg/\Biggl({\prod_{w\in S}}'\left\|\frac{\beta_{\sigma_1(j)}}{\beta_{\sigma_1(j)}}\cdot\frac{\beta_{\sigma_2(j)}}{\beta_{\sigma_1(j)}}\cdots\frac{\beta_{\sigma_{N+1}(j)}}{\beta_{\sigma_1(j)}}\right\|_w\Biggr)^{(d_v/d)(1/qN)}\geqslant 1.$$

Similarly $T_{\infty j}^-$ contains the triples $(\sigma,j,v)\in T_{\infty j}$ with

$$\gamma_{\sigma(j),v}-\gamma_{\sigma_1(j),v}-\frac{1}{qN}\cdot\frac{d_v}{d}\Biggl(\sum_{w\in S}\sum_{\substack{\sigma\in\mathfrak{S}_q\\ \sigma\neq\sigma_w}}\gamma_{\sigma(j),w}-\gamma_{\sigma_1(j),w}\Biggr)=\Gamma_{\sigma,j,v}<0. \tag{14.12}$$

Given a solution $\mathbf{x}=(x_1,\dots,x_q)$ of (14.1) we write for $j=1,\dots,q$,

$$\begin{aligned}
B_{0j}^+(\mathbf{x})&=\prod_{\substack{\sigma,v\\(\sigma,j,v)\in T_{0j}^+}}\left\|\frac{\beta_{\sigma(j)}^{x_j}}{\beta_{\sigma_1(j)}^{x_j}}\right\|_v,\\
B_{0j}^-(\mathbf{x})&=\prod_{\substack{\sigma,v\\(\sigma,j,v)\in T_{0j}^-}}\left\|\frac{\beta_{\sigma(j)}^{x_j}}{\beta_{\sigma_1(j)}^{x_j}}\right\|_v,\\
B_{\infty j}^+(\mathbf{x})&=\prod_{\substack{\sigma,v\\(\sigma,j,v)\in T_{\infty j}^+}}\left(\left\|\frac{\beta_{\sigma(j)}^{x_j}}{\beta_{\sigma_1(j)}^{x_j}}\right\|_v\Bigg/\Biggl({\prod_{w\in S}}'\left\|\frac{\beta_{\sigma_1(j)}^{x_j}}{\beta_{\sigma_1(j)}^{x_j}}\cdot\frac{\beta_{\sigma_2(j)}^{x_j}}{\beta_{\sigma_1(j)}^{x_j}}\cdots\frac{\beta_{\sigma_{N+1}(j)}^{x_j}}{\beta_{\sigma_1(j)}^{x_j}}\right\|_w\Biggr)^{(d_v/d)(1/qN)}\right),\\
B_{\infty j}^-(\mathbf{x})&=\prod_{\substack{\sigma,v\\(\sigma,j,v)\in T_{\infty j}^-}}\left(\left\|\frac{\beta_{\sigma(j)}^{x_j}}{\beta_{\sigma_1(j)}^{x_j}}\right\|_v\Bigg/\Biggl(\prod_{w\in S}\left\|\frac{\beta_{\sigma_1(j)}^{x_j}}{\beta_{\sigma_1(j)}^{x_j}}\cdot\frac{\beta_{\sigma_2(j)}^{x_j}}{\beta_{\sigma_1(j)}^{x_j}}\cdots\frac{\beta_{\sigma_{N+1}(j)}^{x_j}}{\beta_{\sigma_1(j)}^{x_j}}\right\|_w\Biggr)^{(d_v/d)(1/qN)}\right).
\end{aligned} \tag{14.13}$$

Notice that our definition (14.13) is such that

$$\prod_{j=1}^{q} B_{0j}^{+}(\mathbf{x}) B_{0j}^{-}(\mathbf{x}) B_{\infty j}^{+}(\mathbf{x}) B_{\infty j}^{-}(\mathbf{x}) = 1. \tag{14.14}$$

Given $\mathbf{x}$ define Q by

$$Q = H_1((\beta_{\sigma_1}^{\mathbf{x}}, \ldots, \beta_{\sigma_{N+1}}^{\mathbf{x}}))^{16qN}. \tag{14.15}$$

Write

$$B_{0j}^{+}(\mathbf{x}) = Q^{\eta_{0j}^{+}}, \quad B_{0j}^{-}(\mathbf{x}) = Q^{\eta_{0j}^{-}}, \quad B_{\infty j}^{+}(\mathbf{x}) = Q^{\eta_{\infty j}^{+}}, \quad B_{\infty j}^{-}(\mathbf{x}) = Q^{\eta_{\infty j}^{-}}. \tag{14.16}$$

Our definitions (14.9)–(14.13) imply that

$$\begin{aligned}
\eta_{0j}^{+} \log Q &= x_j \sum_{\substack{\sigma, v \\ (\sigma,j,v) \in T_{0j}^{+}}} \Gamma_{\sigma,j,v} = x_j \Gamma_{0j}^{+}, \quad \text{say}, \\
\eta_{0j}^{-} \log Q &= x_j \sum_{\substack{\sigma, v \\ (\sigma,j,v) \in T_{0j}^{-}}} \Gamma_{\sigma,j,v} = x_j \Gamma_{0j}^{-}, \\
\eta_{\infty j}^{+} \log Q &= x_j \sum_{\substack{\sigma, v \\ (\sigma,j,v) \in T_{\infty j}^{+}}} \Gamma_{\sigma,j,v} = x_j \Gamma_{\infty j}^{+}, \\
\eta_{\infty j}^{-} \log Q &= x_j \sum_{\substack{\sigma, v \\ (\sigma,j,v) \in T_{\infty j}^{-}}} \Gamma_{\sigma,j,v} = x_j \Gamma_{\infty j}^{-}.
\end{aligned} \tag{14.17}$$

(Relations (14.17) simultaneously define the quantities Γ_{0j}^{+}, Γ_{0j}^{-}, $\Gamma_{\infty j}^{+}$, $\Gamma_{\infty j}^{-}$ $(1 \leqslant j \leqslant q)$.)

Our definition of Q in (14.15) implies together with (14.13), (14.14) and Lemma 5.3 that

$$\eta_{0j}^{+},\ \eta_{0j}^{-},\ \eta_{\infty j}^{+},\ \eta_{\infty j}^{-} \in \left[-\frac{1}{8q}, \frac{1}{8q}\right] \quad (1 \leqslant j \leqslant q). \tag{14.18}$$

Define the natural number ν by

$$\nu = 2^4 \cdot 2^{7N} \cdot d. \tag{14.19}$$

Moreover for $j = 1, \ldots, q$ we define integers l_{0j}^{+} and l_{0j}^{-} to be least such that

$$\eta_{0j}^{+} \leqslant -\frac{1}{8q} + l_{0j}^{+} \frac{1}{\nu}, \quad \eta_{0j}^{-} \leqslant -\frac{1}{8q} + l_{0j}^{-} \frac{1}{\nu}.$$

By (14.18) we have $0 < l_{0j}^{+}, l_{0j}^{-} \leqslant \nu/4q$.

Write

$$\xi_{0j}^{+} = -\frac{1}{8q} + l_{0j}^{+} \frac{1}{\nu}, \quad \xi_{0j}^{-} = -\frac{1}{8q} + l_{0j}^{-} \frac{1}{\nu}. \tag{14.20}$$

Then obviously by our definition we have

$$\xi_{0j}^{+}-\frac{1}{\nu}<\eta_{0j}^{+}\leqslant\xi_{0j}^{+},\quad \xi_{0j}^{-}-\frac{1}{\nu}<\eta_{0j}^{-}\leqslant\xi_{0j}^{-}. \tag{14.21}$$

Write

$$\xi_{\infty j}^{+}=-\frac{1}{8q}+l_{\infty j}^{+}\frac{1}{\nu},\quad \xi_{\infty j}^{-}=-\frac{1}{8q}+l_{\infty j}^{-}\frac{1}{\nu}.$$

We claim that it is possible to pick integers $l_{\infty j}^{+}$, $l_{\infty j}^{-}$, ..., $l_{\infty q}^{+}$, $l_{\infty q}^{-}$ such that

$$l_{\infty j}^{+},\ l_{\infty j}^{-}\in\left[-1,\frac{\nu}{4q}+1\right]\quad(1\leqslant j\leqslant q), \tag{14.22}$$

$$|\eta_{\infty j}^{+}-\xi_{\infty j}^{+}|<\frac{2}{\nu},\quad |\eta_{\infty j}^{-}-\xi_{\infty j}^{-}|<\frac{2}{\nu}\quad(1\leqslant j\leqslant q) \tag{14.23}$$

and such that moreover

$$\begin{aligned}&|(\eta_{01}^{+}+\eta_{01}^{-}+\eta_{\infty 1}^{+}+\eta_{\infty 1}^{-}-\xi_{01}^{+}+\xi_{01}^{-}-\xi_{\infty 1}^{+}-\xi_{\infty 1}^{-})+\dots\\&\qquad+(\eta_{0j}^{+}+\eta_{0j}^{-}+\eta_{\infty j}^{+}+\eta_{\infty j}^{-}-\xi_{0j}^{+}-\xi_{0j}^{-}-\xi_{\infty j}^{+}-\xi_{\infty j}^{-})|<\frac{1}{\nu}\quad(1\leqslant j\leqslant q).\end{aligned} \tag{14.24}$$

In fact, choose $l_{\infty 1}^{+}$ such that $\eta_{\infty 1}^{+}-\xi_{\infty 1}^{+}\geqslant 0$ and such that

$$|\eta_{01}^{+}+\eta_{\infty 1}^{+}-\xi_{01}^{+}-\xi_{\infty 1}^{+}|<\frac{1}{\nu}.$$

Such a choice is possible with

$$|\eta_{\infty 1}^{+}-\xi_{\infty 1}^{+}|<\frac{1}{\nu}.$$

Now, if $\eta_{01}^{+}+\eta_{\infty 1}^{+}+\eta_{01}^{-}-\xi_{01}^{+}-\xi_{\infty 1}^{+}-\xi_{01}^{-}$ is $\geqslant 0$ (or <0 respectively) pick $l_{\infty 1}^{-}$ such that $\eta_{\infty 1}^{-}-\xi_{\infty 1}^{-}\leqslant 0$ (or $\geqslant 0$ respectively) and such that moreover

$$|\eta_{01}^{+}+\eta_{\infty 1}^{+}+\eta_{01}^{-}+\eta_{\infty 1}^{-}-\xi_{01}^{+}-\xi_{\infty 1}^{+}-\xi_{01}^{-}-\xi_{\infty 1}^{-}|<\frac{1}{\nu}.$$

Such a choice is possible with $|\eta_{\infty 1}^{-}-\xi_{\infty 1}^{-}|<2/\nu$. And so forth. Then (14.22), (14.23), (14.24) are satisfied. We may infer from (14.14) and (14.17) that

$$\sum_{j=1}^{q}(\eta_{0j}^{+}+\eta_{0j}^{-}+\eta_{\infty j}^{+}+\eta_{\infty j}^{-})=0.$$

Consequently (14.24) for $j=q$ implies that

$$\sum_{j=1}^{q}(\xi_{0j}^{+}+\xi_{0j}^{-}+\xi_{\infty j}^{+}+\xi_{\infty j}^{-})=0. \tag{14.25}$$

Our goal is to construct with the parameters ξ^+_{0j}, ξ^-_{0j}, $\xi^+_{\infty j}$, $\xi^-_{\infty j}$ exponent systems $e_{\sigma v}$ ($\sigma\in\mathfrak{S}_q$, $\sigma\neq\sigma_v$, $v\in S$) that are suitable for an application of Lemma 6.1.

For this purpose let us first review the construction we have performed so far. We first discuss the somewhat special rôle played in our context by elements η^+_{0j}, η^-_{0j}, $\eta^+_{\infty j}$, $\eta^-_{\infty j}$ equal to zero. Then clearly all factors contributing to the corresponding term $B^+_{0j}(\mathbf{x})$, $B^-_{0j}(\mathbf{x})$, $B^+_{\infty j}(\mathbf{x})$ or $B^-_{\infty j}(\mathbf{x})$ in (14.13) are equal to 1.

If e.g. $\eta^+_{0j}=0$, in the sequel we may therefore use the convention that then

$$\frac{\Gamma_{\sigma,j,v}}{\Gamma^+_{0j}}=\xi^+_{0j}=0 \quad \text{for each triple } (\sigma,j,v)\in T^+_{0j}, \tag{14.26}$$

and similarly for $\eta^+_{\infty j}$. Then in all cases, using (14.3), (14.9)–(14.12), (14.13), (14.17) we obtain

$$\frac{\Gamma_{\sigma,j,v}}{\Gamma^+_{0j}}\eta^+_{0j}\log Q=x_j\Gamma_{\sigma,j,v}\leqslant\frac{\Gamma_{\sigma,j,v}}{\Gamma^+_{0j}}\xi^+_{0j}\log Q \quad ((\sigma,j,v)\in T^+_{0j}) \tag{14.27}$$

and

$$0\leqslant\frac{\Gamma_{\sigma,j,v}}{\Gamma^+_{0j}}\xi^+_{0j}-\frac{\Gamma_{\sigma,j,v}}{\Gamma^+_{0j}}\eta^+_{0j}\leqslant\frac{\Gamma_{\sigma,j,v}}{\Gamma^+_{0j}}\cdot\frac{1}{\nu}.$$

We get similar formulas for T^-_{0j}.

Moreover, for $(\sigma,j,v)\in T^+_{\infty j}$ we have

$$\begin{gathered}\frac{\Gamma_{\sigma,j,v}}{\Gamma^+_{\infty j}}\eta^+_{\infty j}\log Q=x_j\Gamma_{\sigma,j,v},\\ \left|\frac{\Gamma_{\sigma,j,v}}{\Gamma^+_{\infty j}}\eta^+_{\infty j}-\frac{\Gamma_{\sigma,j,v}}{\Gamma^+_{\infty j}}\xi^+_{\infty j}\right|<\frac{2}{\nu} \quad ((\sigma,j,v)\in T^+_{\infty j})\end{gathered} \tag{14.28}$$

and similarly for $T^-_{\infty j}$.

Given a $4q$-tuple $(\xi^+_{01},\xi^-_{01},\xi^+_{\infty 1},\xi^-_{\infty 1},\dots,\xi^+_{0q},\xi^-_{0q},\xi^+_{\infty q},\xi^-_{\infty q})$ we construct for each $v\in S$ an N-tuple $(e_{\sigma v})$, where $\sigma\neq\sigma_v$ runs through $\mathfrak{S}_q$, as follows:

For $v\in S_0$ and for $\sigma\neq\sigma_v$ we put

$$e_{\sigma v}=\sum_{\substack{j=1\\(\sigma,j,v)\in T^+_{0j}}}^{q}\frac{\Gamma_{\sigma,j,v}}{\Gamma^+_{0j}}\xi^+_{0j}+\sum_{\substack{j=1\\(\sigma,j,v)\in T^-_{0j}}}^{q}\frac{\Gamma_{\sigma,j,v}}{\Gamma^-_{0j}}\xi^-_{0j}. \tag{14.29}$$

Similarly for $v\in S_\infty$ and $\sigma\neq\sigma_v$ we put

$$e_{\sigma v}=\sum_{\substack{j=1\\(\sigma,j,v)\in T^+_{\infty j}}}^{q}\frac{\Gamma_{\sigma,j,v}}{\Gamma^+_{\infty j}}\xi^+_{\infty j}+\sum_{\substack{j=1\\(\sigma,j,v)\in T^-_{\infty j}}}^{q}\frac{\Gamma_{\sigma,j,v}}{\Gamma^-_{\infty j}}\xi^-_{\infty j}. \tag{14.30}$$

As by (14.18)–(14.22) we have not more than

$$\nu^{4q}\leqslant 2^{29qN}d^{4q} \tag{14.31}$$

tuples $(\xi^+_{01},\dots,\xi^-_{\infty j})$, the numbers of tuples $(e_{\sigma v})$ ($v\in S$, $\sigma\in\mathfrak{S}_q\backslash\{\sigma_v\}$) in (14.29), (14.30) does not exceed the bound in (14.31) either.

LEMMA 14.1. *The tuples $(e_{\sigma v})$, where for each $v\in S$, σ runs through $\mathfrak{S}_q\setminus\{\sigma_v\}$ satisfy the relations*

$$\sum_{v\in S}\sum_{\sigma\neq\sigma_v} e_{\sigma v}=0, \tag{14.32}$$

$$\left|\sum_{v\in S'} e_{\tau(v),v}\right|\leqslant 1 \tag{14.33}$$

for each subset S' of S and for each choice of elements $\tau(v)\in\mathfrak{S}_q$ with $\tau(v)\neq\sigma_v$.

Proof. Using (14.17), (14.29), (14.30), we see that

$$\begin{aligned}\sum_{v\in S}\sum_{\substack{\sigma\in\mathfrak{S}_q\\ \sigma\neq\sigma_v}} e_{\sigma v}&=\sum_{j=1}^{q}\left(\frac{\Gamma_{0j}^{+}}{\Gamma_{0j}^{+}}\xi_{0j}^{+}+\frac{\Gamma_{0j}^{-}}{\Gamma_{0j}^{-}}\xi_{0j}^{-}+\frac{\Gamma_{\infty j}^{+}}{\Gamma_{\infty j}^{+}}\xi_{\infty j}^{+}+\frac{\Gamma_{\infty j}^{-}}{\Gamma_{\infty j}^{-}}\xi_{\infty j}^{-}\right)\\ &=\sum_{j=1}^{q}(\xi_{0j}^{+}+\xi_{0j}^{-}+\xi_{\infty j}^{+}+\xi_{\infty j}^{-})=0,\end{aligned}$$

the last equation by (14.25). This proves (14.32).

As for (14.33), we remark that our construction in (14.9)–(14.12), (14.17) and (14.29), (14.30) is such that e.g. the coefficients $\Gamma_{\sigma,j,v}/\Gamma_{0j}^{+}$ of ξ_{0j}^{+} in (14.29) are all non-negative. Moreover we have $\sum_{\sigma,v,(\sigma,j,v)\in T_{0j}^{+}}\Gamma_{\sigma,j,v}/\Gamma_{0j}^{+}=1$ by (14.17).

Similar remarks apply to each of ξ_{0j}^{-}, $\xi_{\infty j}^{+}$, $\xi_{\infty j}^{-}$ $(j=1,\dots,q)$.

We may infer that

$$\left|\sum_{v\in S'} e_{\tau(v),v}\right|\leqslant\sum_{j=1}^{q}(|\xi_{0j}^{+}|+|\xi_{0j}^{-}|+|\xi_{\infty j}^{+}|+|\xi_{\infty j}^{-}|).$$

In view of (14.18), (14.20), (14.21), (14.22) we have $\max\{|\xi_{0j}^{+}|,|\xi_{0j}^{-}|,|\xi_{\infty j}^{+}|,|\xi_{\infty j}^{-}|\}\leqslant 1/4q$ and (14.33) follows.

We now introduce linear forms

$$\begin{aligned}L_1(Y_1,\dots,Y_N)&=Y_1,\\ &\ \vdots\\ L_N(Y_1,\dots,Y_N)&=Y_N,\\ L_{N+1}(Y_1,\dots,Y_N)&=Y_1+\dots+Y_N.\end{aligned}\tag{14.34}$$

Notice that any N among the linear forms in (14.34) are a special system in the sense of §7. We get with (14.2) for any solution $\mathbf{x}=(x_1,\dots,x_q)$ of (14.1),

$$L_i(M_{\sigma_1}(\mathbf{x})\boldsymbol{\beta}_{\sigma_1}^{\mathbf{x}},\dots,M_{\sigma_N}(\mathbf{x})\boldsymbol{\beta}_{\sigma_N}^{\mathbf{x}})=M_{\sigma_i}(\mathbf{x})\boldsymbol{\beta}_{\sigma_i}^{\mathbf{x}}\quad(i=1,\dots,N) \tag{14.35}$$

and

$$L_{N+1}(M_{\sigma_1}(\mathbf{x})\beta^{\mathbf{x}}_{\sigma_1},\dots,M_{\sigma_N}(\mathbf{x})\beta^{\mathbf{x}}_{\sigma_N}) = -M_{\sigma_{N+1}}(\mathbf{x})\beta^{\mathbf{x}}_{\sigma_{N+1}}. \tag{14.36}$$

In view of (14.35), (14.36) there is a bijection between the linear forms $L_1,\dots,L_{N+1}$ and the permutations $\sigma_1,\dots,\sigma_{N+1}$ in $\mathfrak{S}_q$.

Given $v\in S$, assume that under this bijection the form L_{i_v} corresponds to the permutation σ_v. Relabel the remaining N forms from (14.34) as $L^{(v)}_\sigma$ $(\sigma\in\mathfrak{S}_q, \sigma\neq\sigma_v)$. Put

$$\delta = \frac{1}{32qN^2}. \tag{14.37}$$

Given a solution $\mathbf{x}=(x_1,\dots,x_q)$ of (14.1), define Q by (14.15). For $v\in S_0$ and given real numbers Q and e, we define the real number $\varepsilon_v=\varepsilon_v(Q,e)$ by the following property: Q^{ε_v} is the largest value $\leqslant Q^e$ in the value group of the absolute value $\|\cdot\|_v$.

LEMMA 14.2. *For each solution* $\mathbf{x}=(x_1,\dots,x_q)$ *of* (14.1) *satisfying*

$$x_1<x_2<\dots<x_q,\quad x_i\neq 0,\quad x_1<0,\ x_q>0, \tag{14.38}$$

$$|x_1-x_q|>20\,000\,d^6q^4N^2, \tag{14.39}$$

there exists a tuple $(e_{\sigma v})$ *from* (14.29), (14.30) *with the following properties:*

The point $(1/\beta^{\mathbf{x}}_{\sigma_1})(M_{\sigma_1}(\mathbf{x})\beta^{\mathbf{x}}_{\sigma_1},\dots,M_{\sigma_N}(\mathbf{x})\beta^{\mathbf{x}}_{\sigma_N})=\mathbf{y}$, *say, satisfies the simultaneous inequalities*

$$\begin{aligned}&\|L^{(v)}_\sigma(\mathbf{y})\|_v\leqslant Q^{e_{\sigma v}-\delta d_v/d}\quad (v\in S_\infty,\ \sigma\in\mathfrak{S}_q,\ \sigma\neq\sigma_v),\\ &\|L^{(v)}_\sigma(\mathbf{y})\|_v\leqslant Q^{e_{\sigma v}}\quad (v\in S_0,\ \sigma\in\mathfrak{S}_q,\ \sigma\neq\sigma_v),\\ &\|\mathbf{y}\|_v\leqslant 1\quad (v\notin S)\end{aligned} \tag{14.40}$$

with δ *as in* (14.37) *and* Q *as in* (14.15). *Moreover points* $\mathbf{x}$ *with* (14.38), (14.39) *have*

$$Q>\max\{N^{2/\delta}, D_K^{1/2d}\}. \tag{14.41}$$

Finally, in (14.40) *we have only to consider tuples* $(e_{\sigma v})$ *and values* Q *such that*

$$\left|\sum_{v\in S_0} e_{\tau(v),v}-\varepsilon_{\tau(v),v}\right|\leqslant \delta\cdot 2^{-5N} \tag{14.42}$$

for each tuple $(\tau(v))_{v\in S_0}$ *with* $\tau(v)\in\mathfrak{S}_q$ *where* $\varepsilon_{\tau(v),v}$ *is defined in analogy with* ε_v *above.*

Proof. If $\mathbf{x}$ is a solution (14.1), then by (14.16) it determines uniquely a tuple η^+_{0j}, η^-_{0j}, $\eta^+_{\infty j}$, $\eta^-_{\infty j}$. Given η^+_{0j}, η^-_{0j}, $\eta^+_{\infty j}$, $\eta^-_{\infty j}$ we choose ξ^+_{0j}, ξ^-_{0j}, $\xi^+_{\infty j}$, $\xi^-_{\infty j}$ $(1\leqslant j\leqslant q)$ satisfying (14.21), (14.23).

Since the factors $\Gamma_{\sigma,j,v}/\Gamma^{+}_{0j}$ and $\Gamma_{\sigma,j,v}/\Gamma^{-}_{0j}$ in (14.29) are nonnegative and since by (14.17),

$$\sum_{\substack{\sigma,v\\(\sigma,j,v)\in T^{+}_{0j}}} \frac{\Gamma_{\sigma,j,v}}{\Gamma^{+}_{0j}}=1 \quad\text{and}\quad \sum_{\substack{\sigma,v\\(\sigma,j,v)\in T^{-}_{0j}}} \frac{\Gamma_{\sigma,j,v}}{\Gamma^{-}_{0j}}=1$$

for $j=1,\dots,q$, we may infer from (14.13), (14.29) and (14.21) that for any pair σ,v with $v\in S_0$, $\sigma\in\mathfrak{S}_q$, $\sigma\neq\sigma_v$ we have

$$\left\|\frac{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma}}{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma_1}}\right\|_v \leqslant Q^{e_{\sigma v}}. \tag{14.43}$$

Moreover, with (14.27) we see that for any tuple $(\tau(v))_{v\in S_0}$, where $\tau(v)\in\mathfrak{S}_q$, $\tau(v)\neq\sigma_v$, we have

$$\prod_{v\in S_0} Q^{e_{\tau(v),v}}\left\|\frac{\boldsymbol{\beta}^{\mathbf{x}}_{\tau(v)}}{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma_1}}\right\|_v^{-1} \leqslant Q^{2q/\nu}. \tag{14.44}$$

With ν as in (14.19) assertion (14.42) follows from (14.37), (14.44). Moreover, since the monomial $M_\sigma(\mathbf{x})$ is a rational integer, (14.43) implies that

$$\left\|M_\sigma(\mathbf{x})\frac{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma}}{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma_1}}\right\|_v \leqslant Q^{e_{\sigma v}}.$$

By (14.35), (14.36) and the definition of the forms $L^{(v)}_{\sigma}$, this gives the inequalities

$$\|L^{(v)}_{\sigma}(\mathbf{y})\|_v \leqslant Q^{e_{\sigma v}} \quad (v\in S_0,\ \sigma\in\mathfrak{S}_q,\ \sigma\neq\sigma_v)$$

in (14.40). The inequalities for $v\notin S$ are trivially satisfied.

Now suppose that $v\in S_\infty$. Write

$$A=\prod_{w\in S}{}'\left\|\frac{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma_1}}{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma_1}}\cdot\frac{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma_2}}{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma_1}}\cdots\frac{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma_{N+1}}}{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma_1}}\right\|_w,$$

where the dash at the product indicates that for each w, we leave out the term $\boldsymbol{\beta}^{\mathbf{x}}_{\sigma}/\boldsymbol{\beta}^{\mathbf{x}}_{\sigma_1}$ with $\sigma=\sigma_w$. Using (14.13), (14.23) and (14.30) we obtain with (14.28),

$$\left\|\frac{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma}}{\boldsymbol{\beta}^{\mathbf{x}}_{\sigma_1}}\right\|_v \leqslant A^{(d_v/d)(1/N)}Q^{e_{\sigma v}+4q/\nu} \quad (v\in S_\infty,\ \sigma\in\mathfrak{S}_q,\ \sigma\neq\sigma_v). \tag{14.45}$$

However by (14.7) and (14.15) we have

$$A^{1/N}=Q^{-1/16qN^2}. \tag{14.46}$$

On the other hand, the monomials $M_\sigma(\mathbf{x})$ have total degree $\leqslant q^2$. Thus

$$\|M_\sigma(\mathbf{x})\|_v \leqslant \max\{|x_1|,|x_q|\}^{q^2 d_v/d} \leqslant \max\{|x_1|,|x_q|\}^{2q^2/d} \quad (v\in S_\infty).$$

With our value of Q from (14.15), we may apply Lemma 5.1 with $\gamma=1/4N$ to conclude that for solutions $\mathbf{x}$ with (14.38) and (14.39) we have

$$\|M_\sigma(\mathbf{x})\|_v \leqslant Q^{(1/64qN^2)(d_v/d)}. \tag{14.47}$$

Combining (14.45), (14.46), (14.47) and

$$\frac{4q}{\nu} < \frac{1}{64qdN^2}$$

we get

$$\left\| M_\sigma(\mathbf{x}) \frac{\beta_\sigma^{\mathbf{x}}}{\beta_{\sigma_1}^{\mathbf{x}}} \right\|_v \leqslant Q^{e_{\sigma v}-(1/32qN^2)(d_v/d)} \quad (v \in S_\infty,\ \sigma \in \mathfrak{S}_q,\ \sigma \neq \sigma_v), \tag{14.48}$$

and with δ defined in (14.37), this is assertion (14.40) for $v\in S_\infty$.

As for (14.41), with our value Q in (14.15) and with δ in (14.37), the first requirement is

$$H_1((\beta_{\sigma_1}^{\mathbf{x}}, \dots, \beta_{\sigma_{N+1}}^{\mathbf{x}})) > N^{4N}. \tag{14.49}$$

Arguing as in the proof of Lemma 5.1 and using Lemma 4.3 we see that

$$H_1((\beta_{\sigma_1}^{\mathbf{x}}, \dots, \beta_{\sigma_{N+1}}^{\mathbf{x}})) \geqslant H_1\left(\left(1, \frac{\beta_q}{\beta_1}\right)\right)^{|x_1-x_q|} \geqslant \left(1+\frac{1}{20d^3}\right)^{|x_1-x_q|}.$$

Consequently, (14.49) will be certainly true if

$$|x_1-x_q| \cdot \frac{1}{25d^3} > 4N \log N. \tag{14.50}$$

But in view of (14.39), this is amply satisfied. The second requirement in (14.41) is

$$H_1((\beta_{\sigma_1}^{\mathbf{x}}, \dots, \beta_{\sigma_{N+1}}^{\mathbf{x}}))^{16qN} > D_K^{1/2d}. \tag{14.51}$$

We distinguish two cases:

If $D_K^{1/2d(d-1)}<q$, then arguing as above, we see that (14.51) will be true if

$$H_1\left(\left(1, \frac{\beta_q}{\beta_1}\right)\right)^{|x_1-x_q|16qN} > q^{d-1}$$

and as above, this will be satisfied if $|x_1-x_q|16qN>25d^3(d-1)\log q$.

But (14.39) guarantees much more.

So assume that $q<D_K^{1/2d(d-1)}$. Hypothesis (2.1) and Lemma 4.4 imply that

$$H(\beta_1, \dots, \beta_q) \geqslant D_K^{1/2d(d-1)},$$

thus by (4.7),

$$q^{1/2}H_1(\beta_1,...,\beta_q)\geqslant D_K^{1/2d(d-1)}$$

and therefore

$$H_1(\beta_1,...,\beta_q)>D_K^{1/4(d-1)d}. \tag{14.52}$$

On the other hand, by Lemma 4.1 we may assume that

$$H_1\left(\left(1,\frac{\beta_q}{\beta_1}\right)\right)^{q-1}\geqslant H_1(\beta_1,...,\beta_q)$$

and we get

$$H_1((\beta_{\sigma_1}^{\mathbf{x}},...,\beta_{\sigma_{N+1}}^{\mathbf{x}})^{16qN}\geqslant H_1\left(\left(1,\frac{\beta_q}{\beta_1}\right)\right)^{|x_1-x_q|16Nq}\geqslant H_1(\beta_1,...,\beta_q)^{|x_1-x_q|16N}.$$

In conjunction with (14.39) and (14.52) this yields (14.51) and thus (14.41) is proved.

15. Indeed the end of the proof

Combining Lemmata 14.1, 14.2 with Lemma 6.1, we are in a position, to prove Lemma 3.1. We first treat the small solutions:

There are less than

$$(2^{16}d^6q^4N^2)^q \tag{15.1}$$

solutions $(m_1,...,m_q)$ of (3.1) with (3.5) and

$$\max\{|m_1|,|m_q|\}\leqslant 20\,000\,d^6q^4N^2. \tag{15.2}$$

For all other solutions (14.38) and (14.39) are fulfilled.

By Lemma 14.1 and by (14.42) of Lemma 14.2, we get with

$$\delta=\frac{1}{32qN^2}$$

certain tuples (e_{iv}) ($v\in S$, $i=1,...,N$) that satisfy hypotheses (6.3), (6.4) and (6.7), (6.8) of Lemma 6.1. By (14.31) the number of such tuples does not exceed

$$2^{29qN}d^{4q}. \tag{15.3}$$

Hypothesis (6.5) of Lemma 6.1 is satisfied by (14.41). The conclusion is that solutions $(m_1,...,m_q)$ where (15.2) is violated satisfy (6.9), (6.10) for a suitable set of our tuples (e_{iv}).

By Lemma 6.1, each single tuple (e_{iv}) gives rise to not more than

$$2^{2^{21N}1024q^2N^4}$$

proper subspaces of U.

Introducing the factor from (15.3) for the number of tuples and the summand from (15.1) for the small solutions, we finally see that the number of subspaces to cover all solutions of (3.1) and (3.5) does not exceed

$$2^{16q}d^{6q}q^{4q}N^{2q}+2^{29qN}d^{4q}2^{2^{21N}1024q^2N^4}<d^{6q}\cdot 2^{2^{27N}}=t_2.$$

This proves Lemma 3.1 and hence the theorems.

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