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Theta functions and Siegel–Jacobi forms

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1. Introduction

The main theme of this paper is the computation of rings of Siegel modular forms (of arbitrary level) and rings of Jacobi forms using an algebraic method. In the case of Siegel modular forms we get a generalization of results of Igusa. Our main tool is to introduce other theta functions, which are easier to handle and are more general than the classical one. As a geometric application we give a description of the Shioda surfaces and a compactification (even a projective variety) of the universal abelian variety. This leads to a result about Jacobi forms similar to Igusa's fundamental lemma for modular forms.

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2. Siegel modular forms of higher level

Throughout the paper we will use the same notation as in [R1], [R2]. For general facts we refer to [Ig3], [Kr], [Wi]. So let

$$\begin{aligned}\mathbf{H}_g &= \{\tau \in \mathrm{Mat}_{g\times g}(\mathbf{C}) \mid \tau \text{ symmetric}, \mathrm{Im}(\tau) > 0\},\\ \Gamma_g &= \mathrm{Sp}(2g, \mathbf{Z}),\\ \Gamma_g(n) &= \mathrm{Ker}(\Gamma_g \to \mathrm{Sp}(2g, \mathbf{Z}/n)).\end{aligned}$$

For a subgroup of finite index $\Gamma \subset \Gamma_g$, we denote by $A(\Gamma) = \bigoplus_k [\Gamma, k]$ the ring of modular forms for Γ and by $\mathcal{A}_g(\Gamma) = \mathrm{Proj}(A(\Gamma))$ the corresponding Satake compactification, which contains $\mathbf{H}_g/\Gamma$ as an open dense subset. The open part $\mathbf{H}_g/\Gamma$ is the coarse moduli space for principally polarized abelian varieties with level-Γ structure.

We recall the classical notation for theta functions, i.e.

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau, z) = \sum_{x\in\mathbf{Z}^g} \exp 2\pi i\left(\tfrac{1}{2}\tau\left[x+\tfrac{1}{2}\alpha\right] + \left\langle x+\tfrac{1}{2}\alpha, z+\tfrac{1}{2}\beta\right\rangle\right)$$

with $\tau[x]={}^t x\tau x$ and $\langle x,y\rangle$ the standard scalar product. To simplify notation, we write $e(\cdot)=\exp 2\pi i(\mathrm{Tr}(\cdot))$ for matrices and numbers. The elements α and β are elements in $\mathbf{Q}^g$ with $m\alpha\in\mathbf{Z}^g$ and $m\beta\in\mathbf{Z}^g$ for some fixed positive integer m.

More generally we define for any positive integer m and $a\in(\mathbf{Z}/2m)^g$ the following theta functions (similar to Mumford [Mu])

$$f_a^{(m)}(\tau,z)=\sum_{x\in\mathbf{Z}^g} e\Big(m\tau\Big[x+\frac{a}{2m}\Big]+\Big\langle x+\frac{a}{2m},2mz\Big\rangle\Big).$$

These functions will be our basic object of study. The functions

$$f_a^{(m)}(\tau)=\sum_{x\in\mathbf{Z}^g} e\Big(m\tau\Big[x+\frac{a}{2m}\Big]\Big)$$

are the corresponding theta constants. For $m=1$ we omit the m and call $f_a(\tau,z)=f_a^{(1)}(\tau,z)$ and $f_a=f_a^{(1)}(\tau,0)$. Let $\mathcal{TH}_{g,(r)}^{(m)}$ be the ring generated by the polynomials in $f_a^{(m)}$ whose degree is divisible by r. We have the following equality:

LEMMA 2.1.
$$\mathcal{TH}_{g,(r)}^{(2m^2)}=\mathbf{C}\left[\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \text{ with } m\alpha,m\beta\in\mathbf{Z}^g\right]_{(r)}.$$

Proof.

$$\begin{aligned}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}&=\sum_{x\in\mathbf{Z}^g,p\in(\mathbf{Z}/2m)^g} e\big(\tfrac12\tau\big[2mx+p+\tfrac12\alpha\big]+\langle 2mx+p+\tfrac12\alpha,\tfrac12\beta\rangle\big)\\ &=\sum_{p\in(\mathbf{Z}/2m)^g}\sum_{x\in\mathbf{Z}^g} e\Big(2m^2\tau\Big[x+\frac{2mp+m\alpha}{4m^2}\Big]\Big)e\Big(\frac{\langle 2mp+m\alpha,m\beta\rangle}{4m^2}\Big)\\ &=\sum_{p\in(\mathbf{Z}/2m)^g} e\Big(\frac{\langle 2mp+m\alpha,m\beta\rangle}{4m^2}\Big)f_{2mp+m\alpha}^{(2m^2)}.\end{aligned}$$

This can be written as

$$e(-\tfrac14\langle\alpha,\beta\rangle)\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\sum_{p\in(\mathbf{Z}/2m)^g} e\big(\tfrac12\langle p,\beta\rangle\big)f_{2mp+m\alpha}^{(2m^2)}$$

and proves one inclusion. The other inclusion follows from the fact that the matrix

$$e\big(-\tfrac14\langle\alpha,\beta\rangle\big)_{\alpha,\beta}$$

is invertible. □

Hence the theta constants $f_a^{(m)}$ are a natural generalization of the "classical" thetas for arbitrary m as considered e.g. by Igusa. There are natural maps of Veronese type between the various rings $\mathcal{TH}_{g,(r)}^{(m)}$ for different m. For a $(g\times d)$-matrix $A=(a_1,\dots,a_d)$, we set

$$f_A^{(m)}=\prod_{i=1,\dots,d} f_{a_i}^{(m)}$$

and get the following:

LEMMA 2.2. *Let $T\in \mathrm{Mat}_{d\times d}(\mathbf{Z})$ be a matrix with $TT^t=n$ and let $A=(a_1,\dots,a_d)$ be a $(g\times d)$-matrix with integer entries. Then*

$$f_A^{(m)}=\sum_{C\in \mathbf{Z}^{dg}T/n\mathbf{Z}^{dg}} f^{(nm)}_{AT+2mC}.$$

Proof. Because of $((1/n)T)T^t=1$ we have $\mathbf{Z}^{dg}\subset((1/n)T)\mathbf{Z}^{dg}$ is a sublattice of index $(\sqrt{n})^{dg}$. Hence we get

$$\begin{aligned} f_A^{(m)} &= \sum_{x\in\mathbf{Z}^{dg}} e\left(m\left(x+\frac{A}{2m}\right)^t\tau\left(x+\frac{A}{2m}\right)\right)\\ &= \sum_{x\in\mathbf{Z}^{dg}} e\left(nm\left(\frac{1}{n}T\right)\left(\frac{1}{n}T^t\right)\left(x+\frac{A}{2m}\right)^t\tau\left(x+\frac{A}{2m}\right)\right)\\ &= \sum_{x\in\mathbf{Z}^{dg}} e\left(nm\left(\frac{1}{n}T^t\right)\left(x+\frac{A}{2m}\right)^t\tau\left(x+\frac{A}{2m}\right)\left(\frac{1}{n}T\right)\right) \end{aligned}$$

which proves the above formula. □

A special case is $d=1$, n a square. More interesting are the cases $d=2$ and $n=2$ or a prime with $n\equiv 1$ (4). Then we choose

$$T=\begin{pmatrix}1&1\\-1&1\end{pmatrix}\text{ for } n=2 \quad\text{or}\quad T=\begin{pmatrix}a&b\\-b&a\end{pmatrix}\text{ for } n=a^2+b^2.$$

For $d=3$ we get a special case for

$$n=9\quad\text{and}\quad T=\begin{pmatrix}-1&2&2\\2&-1&2\\2&2&-1\end{pmatrix}$$

which may be generalized for squares of primes n with $n\equiv 3$ (4). The most important special case is the following. Any natural number may be written as $n=a^2+b^2+c^2+d^2$, which may be used to construct

$$T=\begin{pmatrix}a&b&c&d\\b&-a&d&-c\\c&-d&-a&b\\d&c&-b&-a\end{pmatrix}\quad\text{with } TT^t=n.$$

COROLLARY 2.3. *We have natural inclusions of rings* $\mathcal{TH}^{(m)}_{g,(1)} \subset \mathcal{TH}^{(mn^2)}_{g,(1)}$, $\mathcal{TH}^{(m)}_{g,(2)} \subset \mathcal{TH}^{(mn)}_{g,(2)}$ *for* $n=2$ *or* n *a prime with* $n \equiv 1$ (4), $\mathcal{TH}^{(m)}_{g,(4)} \subset \mathcal{TH}^{(mn)}_{g,(4)}$ *for any* n.

Any of these inclusions produces a huge number of relations of Veronese type between the theta constants $f_a^{(m)}$. Moreover we have the trivial relations

$$f_a^{(m)}(\tau, -z) = f_{-a}^{(m)}(\tau, z) \quad \text{and} \quad f_a^{(m)} = f_{-a}^{(m)}.$$

For $m=1$ we get the classical cases. Usually the $f_a^{(1)} = f_a$ are called theta constants of the second kind and the $f_a^{(2)}$ are (linear combinations) of the theta constants of the first kind (with half-integral characteristic). From our point of view one should reverse the classical names.

The $f_a f_b$ and $f_a^{(2)} f_b^{(2)}$ are linearly independent ([R1]) (one has to look carefully at the Fourier expansions). More generally, the $f_a^{(m)} f_b^{(m)}$ are linearly independent for small m. One may consider $\mathcal{TH}^{(m)}_{g,(2)}$ as a polynomial ring in the symbols $f_a^{(m)} f_b^{(m)}$ where a runs over $(\mathbf{Z}/2m)^g / a \sim -a$ divided by a certain ideal of theta relations. The Fourier expansion of $f_a^{(m)}$ is just

$$f_a^{(m)} = \sum_{\substack{T \text{ symmetric, pos. semi-def.} \\ 4mT \text{ even}}} a(T) q_T$$

where $q_T = e\left(\frac{1}{2}\tau T\right)$ and

$$a(T) = \#\{x \in \mathbf{Z}^g \mid 2mT = (2mx+a)(2mx+a)^t\}.$$

It is well known that Γ_g is generated by matrices of the form

$$\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix},$$

where S runs over the symmetric $g \times g$-matrices and $1 \in \mathrm{Gl}(g, \mathbf{Z})$, and the Fourier-transformation matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The modular group acts on the rings $\mathcal{TH}^{(m)}_{g,(r)}$ as follows:

For

$$D_S = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$$

we have

$$D_S(f_a^{(m)})(\tau) = f_a^{(m)}(\tau + S) = e\left(\frac{S[a]}{4m}\right) f_a^{(m)}(\tau).$$

For

$$\sigma = \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix}$$

with $\det(U)=1$,

$$\sigma(f_a^{(m)})(\tau) = f_a^{(m)}(\tau[U]) = f_{Ua}^{(m)}(\tau).$$

The Fourier-transformation matrix $\sigma = J$ acts as follows: Let

$$T_g = e\left(\frac{1}{8}\right)^g \left(\frac{1}{\sqrt{2m}}\right)^g \left(e\left(\frac{\langle a,b\rangle}{2m}\right)\right)_{a,b\in(\mathbf{Z}/2m)^g},$$

a scalar multiple of a square matrix of size $(2m)^g$ in roots of unity. Then

$$\frac{J(f_a^{(m)})}{\sqrt{\det(-\tau)}} = \sum_{b\in(\mathbf{Z}/2m)^g} (T_g)_{a,b} f_b^{(m)}$$

for all $a \in (\mathbf{Z}/2m)^g$. Here, as usual, we set

$$\sqrt{\det(\tau/i)} = e\left(\tfrac{1}{8}\right)^g \sqrt{\det(-\tau)}$$

(compare [Fr, 0.13]). This is a matrix equation, which is independent of the choice of the square root on the 2-ring, i.e. on the ring $\mathcal{TH}_{g,(2)}^{(m)}$.

Hence we get a homomorphism of groups

$$\varrho_{\mathrm{theta},m} \colon \Gamma_g \to \mathrm{Aut}\left(\mathcal{TH}_{g,(2)}^{(m)}\right).$$

By definition $\varrho_{\mathrm{theta},m}$ is a unitary representation. We call it *the theta representation* of index m. The next problem is to describe the kernel of $\varrho_{\mathrm{theta},m}$.

We use the computations of Igusa in [Ig1] as a general reference. Because of the above formulas we get a lot of elements in the kernel. Then it follows from a well-known result of Mennicke ([Me, Satz 10, p. 128]), that the kernel contains $\Gamma_g(4m)$. The group $\Gamma_g(2m)/\Gamma_g(4m)$ is a $\mathbf{F}_2$-vector space of dimension $g(2g+1)$ with the obvious generators. Compare [Ig1, pp. 222–223] for analogous statements. Adapting Igusa's method we get

$$\Gamma_g(2m,4m)/\Gamma_g(4m) = \langle B_{ij}, C_{ij}, A_{ij}, A_i \rangle$$

with the following elements: For $i \neq j$ and $i,j \leqslant g$ we replace the (i,j)-coefficient in 1_g with $2m$ and call the matrix a. Then

$$A_{ij} = \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix}.$$

For $i\neq j$ and $i,j\leqslant g$ we replace the (i,j)- and (j,i)-coefficient in 0_g with $2m$, and call that x. Then

$$B_{ij}=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \quad\text{and}\quad C_{ij}=\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}.$$

The definition of the A_i is more special. Take some fixed matrix

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{Sl}(2,\mathbf{Z}),$$

which is congruent to

$$\begin{pmatrix}1+2m & 0\\ 0 & 1+2m\end{pmatrix} \mod 4m.$$

Then A_i for $i=1,\ldots,g-1$ is given by

$$A_i=\begin{pmatrix}U_i^t & 0\\ 0 & U_i^{-1}\end{pmatrix}$$

with

$$U_i=\begin{pmatrix}a & 0 & b & 0\\ 0 & 1_{i-1} & 0 & 0\\ c & 0 & d & 0\\ 0 & 0 & 0 & 1_{g-i-1}\end{pmatrix},$$

and A_g is given by

$$\begin{pmatrix}a & 0 & b & 0\\ 0 & 1_{g-1} & 0 & 0_{g-1}\\ c & 0 & d & 0\\ 0 & 0_{g-1} & 0 & 1_{g-1}\end{pmatrix}.$$

All elements are contained in $\Gamma_g(2m)$. By the explicit formulas we have the following elements in the kernel (let us denote by $S_\triangle$ the diagonal of a matrix S):

$$\begin{pmatrix}1 & mS\\ 0 & 1\end{pmatrix} \quad \text{with } S_\triangle\equiv 0 \mod 2,$$
$$\begin{pmatrix}1 & 0\\ mS & 1\end{pmatrix} \quad \text{with } S_\triangle\equiv 0 \mod 2, \quad\text{and}$$
$$\begin{pmatrix}{}^tU & 0\\ 0 & U^{-1}\end{pmatrix} \quad \text{with } U\equiv 1 \mod 2m \text{ and } \det(U)=1.$$

We denote for odd m by $\Gamma_g^*(2m,4m)$ the group generated by $\Gamma_g(4m)$ and the elements B_{ij}, C_{ij}, A_{ij} and A_i for $i=1,\ldots,g-1$ and $(-1)^{g+1}$. By definition $\Gamma_g^*(2m,4m)$ is a subgroup of $(\pm)\Gamma_g(2m,4m)$.

THEOREM 2.4. *The kernel of $\varrho_{\mathrm{theta},m}$ is equal to*

$$\ker(\varrho_{\mathrm{theta},m}) = \begin{cases} \Gamma_g^*(2m,4m) & \text{for } m \text{ odd,} \\ \Gamma_g(2m,4m) & \text{for } m \text{ even, } g \text{ odd,} \\ (\pm)\Gamma_g(2m,4m) & \text{for } m \text{ even, } g \text{ even.} \end{cases}$$

Proof. The $\mathbf{F}_2$-vector space $\Gamma_g(2m,4m)/\Gamma_g(4m)$ has dimension $2g^2-g$. All constructed elements above are in the kernel with the only possible exception of the element A_g. Indeed we will show that A_g is in the kernel if and only if m is even. But first we show that the kernel cannot be bigger than $\Gamma_g(2m,4m)$ for odd genus or $(\pm)\Gamma_g(2m,4m)$ for even genus. For that we regard indices. It is well known that

$$[\Gamma_g(2m):\Gamma_g(2mp)] = [\Gamma_g(2m,4m):\Gamma_g(2mp,4mp)],$$
$$[\Gamma_g(2m):\Gamma_g(2m,4m)] = [\Gamma_g(2mp):\Gamma_g(2mp,4mp)] = 2^{2g}.$$

Moreover,

$$\Gamma_g(2m)/\Gamma_g(2mp) = \begin{cases} \mathbf{F}_p^{g(2g+1)} & \text{if } p \mid m, \\ \mathrm{Sp}(2g,\mathbf{F}_p) & \text{else,} \end{cases}$$

is a $\mathbf{F}_p$-vector space in the first case with the obvious generators or a nearly simple group in the second case (after dividing by the centre ± 1 one gets a simple group with some exceptions for genus 1 and $p=2,3$). By using the Veronese map and induction one easily proves that the kernel cannot be bigger than $\Gamma_g(2m,4m)$.

For even m we have to show that the kernel is equal to $(-1)^{g+1}\Gamma_g(2m,4m)$. By the above choice of generators we have reduced the question already to the genus $g=1$ case. Hence we have to show that

$$A_1 = \begin{pmatrix} 1+2m & 2m^2 \\ 4m & 1-2m+4m^2 \end{pmatrix}$$

is acting trivially on $\mathcal{TH}_{g,(2)}^{(m)}$. One easily computes that

$$\begin{pmatrix} 1+2m & 2m^2 \\ 4m & 1-2m+4m^2 \end{pmatrix} = J\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} J\begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} J\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} J\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}.$$

We choose a $2m$th root of unity $x=e(1/2m)$ and get the following explicit formulas:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}(f_a^{(m)}) = x^{a^2} f_a^{(m)} \quad \text{and} \quad \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}(f_a^{(m)}) = i^{a^2} f_a^{(m)}.$$

Hence all together for the action of A_1 we have

$$A_1(f_a^{(m)}) = \left(\frac{1}{2m}\right)^2 \sum_{b,c,d,e\in\mathbf{Z}/2m} x^{b^2-b(a+c)-d^2-d(c+e)}(i)^{a^2-c^2} f_e^{(m)}$$

for the action on $\mathcal{TH}^{(m)}_{g,(2)}$. We have for arbitrary α,

$$\sum_{\substack{c \text{ even}\\ c\in\mathbf{Z}/2m}} x^{\alpha c}+(-i)\sum_{\substack{c \text{ odd}\\ c\in\mathbf{Z}/2m}} x^{\alpha c}=\sum_{k=0,\dots,m-1} e\Big(\frac{\alpha}{m}\Big)^k+(-i)\sum_{k=0,\dots,m-1} e\Big(\frac{\alpha(2k+1)}{2m}\Big)$$
$$=\Big(1+(-i)e\Big(\frac{\alpha}{2m}\Big)\Big)\sum_{k=0,\dots,m-1} e\Big(\frac{\alpha}{m}\Big)^k$$
$$=\begin{cases} m(1-i), & \alpha=0,\\ m(1+i), & \alpha=m,\\ 0, & \text{else.}\end{cases}$$

Hence we get

$$\Big(\frac{1}{2m}\Big)^2\sum_{b,c,d,e\in\mathbf{Z}/2m} x^{b^2-b(a+c)-d^2-d(c+e)}(i)^{a^2-c^2}f^{(m)}_e$$
$$=\Big(\frac{1}{4m}\Big)i^{a^2}\sum_{b,e\in\mathbf{Z}/2m} x^{b^2-ba-d^2-de}f^{(m)}_e\begin{cases}(1-i) & \text{if } -b-d\equiv 0\ (2m),\\ (1+i) & \text{if } -b-d\equiv m\ (2m)\end{cases}$$
$$=\tfrac{1}{2}i^{a^2}((1-i)+(-1)^{m+a}(1+i))f^{(m)}_a=f^{(m)}_a,$$

and therefore the result in the even case.

For odd m one has that

$$A_1=\begin{pmatrix}1+2m & 6m^2+2m\\ 4m & 1-2m+12m^2\end{pmatrix}$$

is acting with $(-i)$ on $\mathcal{TH}^{(m)}_{g,(2)}$. One easily computes that

$$\begin{pmatrix}1+2m & 6m^2+2m\\ 4m & 1-2m+12m^2\end{pmatrix}=J\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}J\begin{pmatrix}1 & -m\\ 0 & 1\end{pmatrix}J\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}J\begin{pmatrix}1 & 3m\\ 0 & 1\end{pmatrix}.$$

We choose a $2m$th root of unity $x=e(1/2m)$ and get the following explicit formula:

$$A_1(f^{(m)}_a)=\Big(\frac{1}{2m}\Big)^2\sum_{b,c,d,e\in\mathbf{Z}/2m} x^{b^2-b(a+c)-d^2-d(c+e)}(-i)^{a^2+c^2}f^{(m)}_e$$
$$=\Big(\frac{1}{4m}\Big)(-i)^{a^2}\sum_{b,e\in\mathbf{Z}/2m} x^{b^2-ba-d^2-de}f^{(m)}_e\begin{cases}(1-i) & \text{if } -b-d\equiv 0\ (2m),\\ (1+i) & \text{if } -b-d\equiv m\ (2m)\end{cases}$$
$$=\tfrac{1}{2}(-i)^{a^2}((1-i)+(-1)^{m+a}(1+i))f^{(m)}_a=(-i)f^{(m)}_a,$$

and hence the result. $\square$

Remark 2.5. For the ring $\mathcal{TH}^{(m)}_{g,(4)}$ the above proof shows that the kernel of the corresponding theta representation is always $(\pm)\Gamma_g(2m,4m)$. That corresponds to the ring of Siegel modular forms of even weight.

Remark 2.6. For index computations with subgroups of Γ_g compare [ST], [GN1], [GN2].

As a corollary we get a generalized version of Igusa's *fundamental lemma* in [Ig2, Theorem 5].

THEOREM 2.7. *The ring of modular forms of even weight*

$$A(\Gamma_g(2m,4m))_{(2)} = \bigoplus_{2|k}[\Gamma_g(2m,4m),k] = \left(\mathcal{TH}^{(m)}_{g,(4)}\right)^N$$

where N denotes the normalization of a ring in its field of fractions.

For the full ring of modular forms we get a partial result:

THEOREM 2.8. *The ring of modular forms for odd genus g and odd m, which contains any prime $p\equiv 3$ (4) with an even power, is*

$$A(\Gamma^*_g(2m,4m)) = \bigoplus_{k}[\Gamma^*_g(2m,4m),k] = \left(\mathcal{TH}^{(m)}_{g,(2)}\right)^N$$

for even m, which contains any prime $p\equiv 3$ (4) with an even power, and arbitrary g we get

$$A(\Gamma_g(2m,4m)) = \bigoplus_{k}[\Gamma_g(2m,4m),k] = \left(\mathcal{TH}^{(m)}_{g,(2)}\right)^N$$

where N denotes the normalization of a ring in its field of fractions.

Proof of the theorems. We may just copy the proof of Igusa in [Ig2] using the results in [R1]. First of all the ring $\mathcal{TH}^{(m)}_{g,(r)}$ is clearly contained in the corresponding ring of modular forms. Moreover for any $\tau\in\mathbf{H}_g$ there are theta constants $f_a^{(m)}$ which do not vanish at τ (compare with Lemma 6 in [Ig2]). This may be done easily by regarding the Fourier expansion. The kernel of $\varrho_{\text{theta},m}$, which keep $\mathcal{TH}^{(m)}_{g,(r)}$ element-wise invariant, is computed in Theorem 2.4. In [R1] it is proved that for odd genus, $A(\Gamma^*_g(2,4))= \left(\mathcal{TH}^{(1)}_{g,(2)}\right)^N$ and $A(\Gamma_g(2,4))_{(2)}=\left(\mathcal{TH}^{(1)}_{g,(4)}\right)^N$. Igusa's *fundamental lemma* in [Ig1] is just $A(\Gamma_g(4,8))=\left(\mathcal{TH}^{(2)}_{g,(2)}\right)^N$. If we apply Galois theory and Corollary 2.3 to this situation we get the result. $\square$

Let us denote by $K(\Gamma)$ the quotient field of $A(\Gamma)$. The field of quotients of degree zero is usually called field of modular functions (of level Γ).

COROLLARY 2.9. *Let Γ be a subgroup of finite index in Γ_g. Choose some*

$$\Gamma_g(2m,4m)\subset\Gamma$$

such that m is even with the property to contain any prime $p\equiv 3\ (4)$ with an even power. Then

$$K(\Gamma)=\mathbf{C}(f_a^{(m)}(\tau)\ \textit{for}\ a\in(\mathbf{Z}/2m)^g)^G,$$
$$A(\Gamma)=\left(\mathbf{C}[f_a^{(m)}(\tau)\ \textit{for}\ a\in(\mathbf{Z}/2m)^g]^G_{(2)}\right)^N$$

where $G=\Gamma/\Gamma_g(2m,4m)$.

Remark 2.10. In genus one we have $\Gamma_1^*(2m,4m)=\Gamma_1(4m)$ by computing the index. (The group $\Gamma_g^*(2m,4m)$ was only defined for odd m.)

3. Jacobi forms

Instead of considering the theta nullwerte one may consider the theta functions $f_a^{(m)}(\tau,z)$. On $\mathbf{H}_g\times\mathbf{C}^g$ we have an action of $\mathbf{Z}^{2g}\rtimes\Gamma_g$ by the following definition:

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\ \text{acts by}\ M(\tau,z)=(M\langle\tau\rangle,{}^t(C\tau+D)^{-1}Z)$$

and

$$\mathbf{Z}^{2g}\ \text{acts by}\ (x,y)(\tau,z)=(\tau,z+\tau x+y).$$

It is easy to see that one gets a group action of the semi direct product

$$\Gamma^{\mathrm{Jac}}=\mathbf{Z}^{2g}\rtimes\Gamma$$

on $\mathbf{H}_g\times\mathbf{C}^g$. This group is called the Jacobi group (Berndt denoted it by $G^0(N)$ in the case of the main congruence subgroup $\Gamma(N)$ [Be1]). After dividing out the action of $\mathbf{Z}^{2g}$ one gets the universal family of principally polarized abelian varieties together with a map to $\mathbf{H}_g$. If one divides the action of Γ^{Jac} for some $\Gamma\subset\Gamma_g(n)$ for some $n\geqslant 3$, one gets the universal abelian variety of level Γ (an abelian scheme).

If one divides by the action of $\Gamma_g(2)^{\mathrm{Jac}}$, one gets the universal Kummer variety. The theta functions $f_a^{(1)}(\tau,z)$ give a map to the Heisenberg quotient $\mathbf{P}(g)=\mathbf{P}^{2^g-1}/N_g$ (see [R4], [R6]). The image of $(\tau,0)$ in $\mathbf{P}(g)$ is the moduli point of the Kummer variety. More generally, for any Γ with $-1\in\Gamma$ one gets with $\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}$ the universal Kummer variety of level Γ.

We start by recalling the action of the modular group and of $\mathbf{Z}^{2g}$ on theta functions. The $f_a^{(m)}(\tau,z)$ are holomorphic functions on $\mathbf{H}_g\times\mathbf{C}^g$ with

$$\begin{pmatrix}1 & S\\ 0 & 1\end{pmatrix}(f_a^{(m)}(\tau,z))=f_a^{(m)}(\tau+S,z)=e\left(\frac{S[a]}{4m}\right)f_a^{(m)}(\tau,z),$$

$$\begin{aligned}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}(f_a^{(m)}(\tau,z))&=f_a^{(m)}(-\tau^{-1},-\tau^{-1}z)\\ &=e(\tau^{-1}[z])^m\sqrt{\det(\tau/2mi)}\sum_{b\in(\mathbf{Z}/2m)^g}e\left(\frac{\langle a,b\rangle}{2m}\right)f_b^{(m)}(\tau,z),\end{aligned}$$

$$f_a^{(m)}(\tau,z+\tau x+y)=e\Big(-\tau[x]-2\langle z,x\rangle\Big)^m f_a^{(m)}(\tau,z).$$

(We remark that -1 is acting trivial on $\mathbf{H}_g$, but *not* on $\mathbf{C}^g$.) We want to restate this action and introduce the Petersson notation: For a holomorphic function $\phi\colon\mathbf{H}_g\times\mathbf{C}^g\to\mathbf{C}$ one defines as for modular forms a group action of the Jacobi group by

$$\begin{aligned}\phi|_{k,m}\begin{pmatrix}A & B\\ C & D\end{pmatrix}&=\det(C\tau+D)^{-k}e(-(C\tau+D)^{-1}C[z])^m\\ &\qquad\times\phi((A\tau+B)(C\tau+D)^{-1},{}^t(C\tau+D)^{-1}z),\end{aligned}$$

$$\phi|_{k,m}(x,y)=e(\tau[x]+2\langle z,x\rangle)^m\phi(\tau,z+\tau x+y).$$

Then

$$f_a^{(m)}|_{1/2,m}J=\sum_{b\in(\mathbf{Z}/2m)^g}(T_g)_{a,b}f_b^{(m)}$$

if one as before explains $\sqrt{\det(\tau/i)}=e\left(\frac{1}{8}\right)^g\sqrt{\det(-\tau)}$. However, on the 2-ring the above formulas are independent of the choice of the square root.

The essential point for our further considerations is that the group action is given by the same formulas for theta functions and theta constants. Hence the results of §2 apply to theta functions, especially Theorem 2.4.

Let us denote by $\mathcal{THET}_{g,(r)}^{(\leqslant m)}$ the ring generated by the polynomials in $f_a^{(n)}(\tau,z)$ and $f_a^{(n)}(\tau)$ for $n\leqslant m$ of degree divisible by r. We get a bigraded ring with

$$\begin{aligned}\operatorname{index}(f_a^{(n)}(\tau,z))&=n,\\ \operatorname{index}(f_a^{(n)}(\tau))&=0,\\ \operatorname{weight}(f_a^{(n)}(\tau,z))&=\tfrac{1}{2},\\ \operatorname{weight}(f_a^{(n)}(\tau))&=\tfrac{1}{2}.\end{aligned}$$

Moreover we get a theta representation respecting the bigraduation

$$\varrho_{\mathrm{theta},m}\colon\Gamma_g\to\operatorname{Aut}\big(\mathcal{THET}_{g,(2)}^{(\leqslant m)}\big).$$

We recall the definition of Jacobi forms. Jacobi forms of weight k, index m and of level Γ (for a subgroup of finite index $\Gamma \subset \Gamma_g$) are holomorphic functions on $\mathbf{H}_g \times \mathbf{C}^g$ which transform "like theta functions" of index m and weight k. More precisely, a holomorphic function $\phi: \mathbf{H}_g \times \mathbf{C}^g \to \mathbf{C}$ is a Jacobi form of weight k, index m and level Γ if and only if

$$\phi|_{k,m}M = \phi \quad \text{for } M \in \Gamma \quad \text{and} \quad \phi|_{k,m}(x,y) = \phi \quad \text{for } (x,y) \in \mathbf{Z}^{2g}.$$

(In genus one one has to add, as usual, a condition for the Fourier expansion. In the case of the full modular group we set $q=e(\tau)$ and $p=e(z)$. Then the Fourier expansion has the form

$$\phi(\tau,z) = \sum_{i,j} c(i,j) q^i p^j$$

and one demands $c(i,j)=0$ for $4im<j^2$. For the case of genus 1 we refer to [EZ]. For higher genus the analogue of this condition is automatically fulfilled by the Koecher principle, see [Koe], [Zi].)

On $\mathbf{H}_g \times \mathbf{C}^g$ one has the theta map

$$\mathrm{Th}^{(m)}: \mathbf{H}_g \times \mathbf{C}^g \to \mathbf{P}^{(2m)^g - 1}$$

given by the $f_a^{(m)}(\tau,z)$. As a projective action the action of the Jacobi group is the same as the natural action given before. Hence the theta representation may be considered as a homomorphism to $\mathrm{PGl}((2m)^g, \mathbf{C})$.

By definition for any $\Gamma \subset \Gamma_g^*(2m,4m)$ the ring $\mathcal{THET}_{g,(2)}^{(\leq m)}$ is contained in the ring of Jacobi forms $\bigoplus_{k,m} J_{k,m}(\Gamma)$. In classical terminology, the last condition for Jacobi forms $\phi|_{k,m}(x,y)=\phi$ (the functional equation) is equivalent to "theta function of order $2m$ for the characteristic $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$" (see [Wi]). It is well known that there is a $\frac{1}{2}((2m)^g+2^g)$-dimensional vector space of even theta functions of that type and a $\frac{1}{2}((2m)^g-2^g)$-dimensional vector space of odd theta functions of that type. Explicitly a basis is given by

$$\begin{cases} f_a^{(m)}(\tau,z) + f_{-a}^{(m)}(\tau,z) & \text{(even functions)}, \\ f_a^{(m)}(\tau,z) - f_{-a}^{(m)}(\tau,z) & \text{(odd functions)}. \end{cases}$$

Theta functions are uniquely determined by their functional equation. Let us denote by

$$\psi_a(\tau,z) = f_a^{(2)}(\tau,z) - f_{-a}^{(2)}(\tau,z), \quad \text{for } a \in (\mathbf{Z}/4)^g,$$

the odd theta functions of order 4 $\left(\text{index 2, weight } \frac{1}{2}\right)$. Then $\psi_a(\tau,z) = -\psi_{-a}(\tau,z) =$

$-\psi_a(\tau,-z)$ and the action of the modular group is given by

$$\begin{pmatrix}1 & S\\ 0 & 1\end{pmatrix}(\psi_a(\tau,z)) = \psi_a(\tau+S,z) = e\big(\tfrac{1}{8}S[a]\big)\psi_a(\tau,z),$$

$$\begin{aligned}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}(\psi_a(\tau,z)) &= \psi_a(-\tau^{-1},-\tau^{-1}z)\\ &= e(\tau^{-1}[z])^2\sqrt{\det(-\tau)}\,e\big(\tfrac{1}{8}\big)^g\big(\tfrac{1}{2}\big)^g\\ &\quad\times\sum_{b\in(\mathbf{Z}/4)^g}\big(e\big(\tfrac{1}{4}\langle a,-b\rangle\big)+e\big(\tfrac{1}{4}\langle a,b\rangle\big)\big)\psi_b,\end{aligned}$$

where in the last formula some cancellation occurs, e.g. $\psi_0(\tau,z)=0$.

Remark 3.1. If $-1=J^2\in\Gamma$ the invariance condition for a Jacobi form implies that $\phi(\tau,-z)=(-1)^k\phi(\tau,z)$. Hence for such Γ we have $J_{k,1}(\Gamma)=0$ for odd k. (Compare with Theorem 2.2 in [EZ] for the genus one case.) We call a Jacobi form *even* or *odd* if and only if it is even or odd as a function in z. Hence if $-1=J^2\in\Gamma$ then even weight implies evenness as a function. The above consideration implies that

$$\begin{aligned}\dim_{K(\Gamma)}(J^{\text{even}}_{*,m}(\Gamma)\otimes K(\Gamma)) &= 2^{g-1}(m^g+1),\\ \dim_{K(\Gamma)}(J^{\text{odd}}_{*,m}(\Gamma)\otimes K(\Gamma)) &= 2^{g-1}(m^g-1).\end{aligned}$$

It seems to be that one needs infinitely many theta functions $f_a^{(m)}(\tau,z)$ for a fixed genus and varying index and weight. However, this is not the case.

We start with the following consequence of the theta addition formula.

LEMMA 3.2.

$$f_a^{(m)}(\tau,z)f_b^{(m)}(\tau,z) = \sum_{q\in\mathbf{F}_2^g} f^{(2m)}_{a+b+2mq}(\tau,z)f^{(2m)}_{a-b+2mq}(\tau).$$

Proof. The left-hand side is given by

$$\sum_{x,y\in\mathbf{Z}^g} e\left(m\tau\left[x+\frac{a}{2m}\right]+m\tau\left[y+\frac{b}{2m}\right]+\left\langle x+\frac{a}{2m},2mz\right\rangle+\left\langle y+\frac{b}{2m},2mz\right\rangle\right).$$

We choose the following "Ansatz" ([Wi, p. 9]): $x+y=2p+q$, $x-y=2r-q$ for $p,r\in\mathbf{Z}^g$ and $q\in\mathbf{F}_2^g=\{0,1\}^g$. Then the sum becomes

$$\sum_{\substack{p,r\in\mathbf{Z}^g\\ q\in\mathbf{F}_2^g}} e\left(2m\tau\left[p+\frac{q}{2}+\frac{a+b}{4m}\right]+2m\tau\left[r-\frac{q}{2}+\frac{a-b}{4m}\right]+\left\langle p+\frac{q}{2}+\frac{a+b}{4m},4mz\right\rangle\right),$$

and hence the result. □

The case of the theta nullwerte is just a special case of Lemma 2.2. An essential point for our reduction process is the following:

COROLLARY 3.3. *For any $d\in\mathbf{F}_2^g$ we have*

$$\sum_{c\in\mathbf{F}_2^g} e\big(\tfrac{1}{2}\langle c,d\rangle\big) f^{(m)}_{a+mc}(\tau,z) f^{(m)}_{b-mc}(\tau,z)$$
$$=\Big(\sum_{q\in\mathbf{F}_2^g} e\big(\tfrac{1}{2}\langle q,d\rangle\big) f^{(2m)}_{a+b+2mq}(\tau,z)\Big)\Big(\sum_{p\in\mathbf{F}_2^g} e\big(\tfrac{1}{2}\langle p,d\rangle\big) f^{(2m)}_{a-b+2mp}(\tau)\Big).$$

Proof. Choose a and b in Lemma 3.2 as $a+mc$ and $b-mc$, multiply the equation by $e\big(\frac{1}{2}\langle c,d\rangle\big)$. Summation over $c\in\mathbf{F}_2^g$ gives

$$\sum_{c\in\mathbf{F}_2^g} e\big(\tfrac{1}{2}\langle c,d\rangle\big) f^{(m)}_{a+mc}(\tau,z) f^{(m)}_{b-mc}(\tau,z)$$
$$= \sum_{c,q\in\mathbf{F}_2^g} e\big(\tfrac{1}{2}\langle c,d\rangle\big) f^{(2m)}_{a+b+2mq}(\tau,z) f^{(2m)}_{a-b+2mq+2mc}(\tau,0)$$
$$= \sum_{c,q\in\mathbf{F}_2^g} e\big(\tfrac{1}{2}\langle q,d\rangle\big) f^{(2m)}_{a+b+2mq}(\tau,z) e\big(\tfrac{1}{2}\langle q+c,d\rangle\big) f^{(2m)}_{a-b+2m(q+c)}(\tau,0),$$

and hence the result. □

Remark 3.4. This formula is given e.g. in [Mu, Vol. I, pp. 222–223].

We observe that the matrix

$$e\big(\tfrac{1}{2}\langle q,d\rangle\big)_{q,d\in\mathbf{F}_2^g}$$

in Corollary 3.3 is invertible. We want to show that a function like $f^{(2m)}_a(\tau,z)$ may be written as a linear combination of $f^{(m)}_a(\tau,z) f^{(m)}_b(\tau,z)$ with coefficients in the field $K(\Gamma)$ (of some level), e.g. in the field $\mathbf{C}(f^{(2m)}_a(\tau))$. For 2-power-index we have to decide whether or not it is possible to invert

$$\sum_{p\in\mathbf{F}_2^g} e\big(\tfrac{1}{2}\langle p,d\rangle\big) f^{(2m)}_{a-b+2mp}(\tau)$$

in the quotient field of modular forms. We may further reduce to the case $a-b\in\{0,1\}^g$. Hence it remains to show that for arbitrary $c,d\in\{0,1\}^g$,

$$\sum_{p\in\mathbf{F}_2^g} e\big(\tfrac{1}{2}\langle p,d\rangle\big) f^{(2m)}_{c+2mp}(\tau)\neq 0.$$

However, theta constants are linearly independent with the only relation $f^{(m)}_a(\tau)= f^{(m)}_{-a}(\tau)$. Hence cancellation can only occur for $c+2mp=-c \bmod 4m$, hence for $m=1$.

And indeed, for the final step $m=2$ to $m=1$ the argument must break down, because there are no odd Jacobi functions which may be written as polynomials in $f_a^{(1)}(\tau,z)$ which are even functions. If we denote by $K(A)$ the quotient field of a ring A, we have proved so far that

$$K(\mathcal{THET}_{g,(2)}^{(\leqslant m)}) = K(\mathcal{TH}^{(m)})K(\mathcal{THET}^{(\leqslant 2)})_{(2)}.$$

But we can do better. Our main result is the following:

THEOREM 3.5. *The ring of even Jacobi forms is given for odd genus by*

$$\bigoplus_{k,m} J_{k,m}^{\text{even}}(\Gamma) = \left(\left(K(\Gamma\cap\Gamma_g^*(2,4))[f_a^{(1)}(\tau,z) \text{ for } a\in \mathbf{F}_2^g]\right)_{(2)}^{\Gamma}\right)^{\text{Hol}},$$

where the index Hol *indicates taking the subring of holomorphic functions (with an additional cusp condition in genus $g=1$).*

COROLLARY 3.6. *In the case of the full modular group we get for odd genus*

$$\bigoplus_{k,m} J_{2k,m}(\Gamma_g) = \left(\left(\mathbf{C}(f_a(\tau))[f_a(\tau,z)]\right)^{H_g}\right)^{\text{Hol}},$$

where H_g is the finite group constructed in [R1], *a central extension of $\Gamma_g/\Gamma_g^*(2,4)$.*

As in the case of modular forms we get the following result for the full ring and arbitrary genus.

THEOREM 3.7. *We have*

$$\bigoplus_{k,m} J_{k,m}(\Gamma) = \left(\left(K(\Gamma\cap\Gamma_g(4,8))[f_a(\tau,z),\psi_b(\tau,z)]\right)_{(2)}^{\Gamma}\right)^{\text{Hol}},$$

where Hol *as before denotes the subring of holomorphic functions (with an additional cusp condition in genus $g=1$).*

Proof of the theorems. We start with the observation that for index m we can find other generators for the vector space of theta functions than $f_a^{(m)}(\tau,z)+f_{-a}^{(m)}(\tau,z)$ for the even functions and $f_a^{(m)}(\tau,z)-f_{-a}^{(m)}(\tau,z)$ for the odd functions. We start with the elliptic case, i.e. with genus 1. It is easy to check that the $m+1$ even functions

$$f_0(\tau,z)^i f_1(\tau,z)^{m-i}, \quad \text{for } i=0,\dots,m,$$

and the $m-1$ odd functions

$$\psi_1 f_0(\tau,z)^i f_1(\tau,z)^{m-i-2}, \quad \text{for } i=0,\dots,m-2,$$

are linearly independent, hence a basis of this vector space. Hence for even functions in genus one we are already done by the following argument: If

$$P=\sum_{i=0,\ldots,m} P_i(\tau)f_0(\tau,z)^i f_1(\tau,z)^{m-i}$$

is given and P is a Jacobi form of level Γ, the P_i can be chosen in $K(\Gamma\cap\Gamma_1^*(2,4))$. For odd functions this argument is easily modified. To generalize this type of argument, we observe that for higher genus one easily finds polynomials of index (degree) m in f_a and ψ_a (of index 2!), such that the restriction to the diagonal

$$\tau=\begin{pmatrix} * & 0\\ 0 & * \end{pmatrix}$$

decomposes into a product of elliptic theta functions in $\tau_{1,1},\ldots,\tau_{g,g}$ as above. More precisely, we have for $a=(a_1,\ldots,a_g)^t$,

$$f_a^{(m)}\left(\begin{pmatrix} \tau_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \tau_g \end{pmatrix},\begin{pmatrix} z_1\\ \vdots\\ z_g \end{pmatrix}\right)=\prod_i f_{a_i}^{(m)}(\tau_i,z_i).$$

We may write the special basis for the elliptic case as above as polynomials in $f_a^{(2)}(\tau_i,z_i)$ and $f_a^{(1)}(\tau_i,z_i)$. Hence, using the just mentioned rule we find a vector subspace of dimension $(2m)^g$ in polynomials of $f_a^{(2)}(\tau,z)$ and $f_a^{(1)}(\tau,z)$ for such special τ. Any such function is even or odd if it contains an even or odd number of odd factors. Hence there are

$$\sum_{i=0,\ldots,g/2}\binom{g}{2i}(m+1)^{g-2i}(m-1)^{2i}=2^{g-1}(m^g+1)$$

even and

$$\sum_{i=0,\ldots,(g-1)/2}\binom{g}{2i+1}(m+1)^{g-2i-1}(m-1)^{2i+1}=2^{g-1}(m^g-1)$$

odd functions among them. Hence for generic τ (which can always be assumed), a Jacobi form is a polynomial in $f_a^{(2)}(\tau,z)$ and $f_a^{(1)}(\tau,z)$ with coefficients in the field $K(\Gamma\cap\Gamma_g(4,8))$.

The final step for even functions to come from polynomials in index-2 functions to polynomials in index-1-theta functions works with the help of Lemma 3.2 and Corollary 3.3 because of the counting identity

$$\begin{aligned}\dim\left(\left\{\text{even thetas of order 4 for }\begin{bmatrix}0\\0\end{bmatrix}\right\}\right)&=\tfrac{1}{2}(4^g+2^g)\\ &=2^{g-1}(2^g+1)=\binom{2^g+1}{2}\\ &=\dim(\{\text{mixed products in }f_a^{(1)}\}).\qquad\square\end{aligned}$$

Remark 3.8. For any even $k>g+2$ and arbitrary index there exist Jacobi–Eisenstein series [Zi]. Hence in the same way as in [EZ, Theorem 8.5, p. 99], one proves that the ring of Jacobi forms is not a finitely generated ring. Nevertheless it is possible to approximate the ring with finitely generated rings of theta functions.

Remark 3.9. The theorems indicate that for higher level (e.g. $\Gamma\subset\Gamma_g(4,8)$) the structure of the ring of Jacobi forms becomes easier. Up to some denominator from $A(\Gamma)$ Jacobi forms are polynomials with coefficients in $A(\Gamma)$ in some fixed theta functions and the zeros of the denominator has to be a zero of the numerator.

4. Jacobi forms in genus one

We want to indicate the connection with the results of [EZ]. For the even part we fix the notation

$$\begin{aligned} A&=f_0(\tau)=\sum_{x\in\mathbf{Z}} q^{x^2},\\ B&=f_0(\tau,z)=\sum_{x\in\mathbf{Z}} q^{x^2}p^{2x},\\ C&=f_1(\tau)=q^{1/4}\sum_{x\in\mathbf{Z}} q^{x(x+1)},\\ D&=f_1(\tau,z)=q^{1/4}\sum_{x\in\mathbf{Z}} q^{x(x+1)}p^{2x+1}. \end{aligned}$$

Then from Theorem 3.5 we get

$$\bigoplus_{k,m} J^{\mathrm{even}}_{k,m}(\Gamma_1(4))=\mathbf{C}(A,C)[B,D]^{\mathrm{Hol}}_{(2)}.$$

Because of the identity

$$\tfrac{1}{2}((2m)^1+2^1)=m+1=\binom{m+1}{1}$$

for the dimension of the space of even theta function of index m and the dimension of the space of polynomials of degree m in (B,D), there cannot be any algebraic relation between B and D. Hence the coefficients of a Jacobi form in the field $\mathbf{C}(A,C)$ cannot have a common pole on $\mathbf{H}_1$. Hence any common denominator has to be a power of the elliptic unit

$$\begin{aligned} \Delta&=\left(\tfrac{1}{12}\right)^3(E_4^3-E_6^2)=\tfrac{1}{16}(AC(A^4-C^4))^4\\ &=(2\pi i)^{-12}(g_2^3-27g_3^2)=\sum_{n\geqslant 1}\tau(n)q^n=q-24q^2+252q^3\ldots \end{aligned}$$

(the normalized discriminant). Clearly we have

$$R_{(2)} = \mathbf{C}[A,B,C,D]_{(2)} \subset \bigoplus_{k,m} J^{\text{even}}_{k,m}(\Gamma_1(4)).$$

The finite group $G = H_1$ is given by its generators

$$M = \tfrac{1}{2}(1+i)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

It is a group generated by (pseudo-)reflections of order 96, if one regards G as subgroup of $\mathrm{Gl}(2,\mathbf{C})$. Let us denote by $\chi(g) = \det(g)$ the determinant regarded as a $\mathrm{Gl}(2,\mathbf{C})$-matrix. We may consider χ as a character on Γ_1 with values in the fourth roots of unity given by

$$\chi(J) = -i \quad \text{and} \quad \chi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = i.$$

We refer to [R1] for further information on computations of invariant rings under the action of a finite group. Let us denote:

$$\begin{aligned}
(i) &= A^i + C^i, \\
(i,j) &= A^i C^j + C^i A^j \quad \text{for } i \neq j, \\
(i,i) &= A^i C^i, \\
E_4 &= (8) + 14(4,4), \\
E_6 &= (12) - 33(8,4).
\end{aligned}$$

Then $A_1(\Gamma_1) = \mathbf{C}[A,C]^G = \mathbf{C}[E_4, E_6]$.

Remark 4.1. To get examples of modular forms one usually writes down Eisenstein series. It is easy to express them as symmetric polynomials in A and C, for example:

$$\begin{aligned}
E_8 &= (16) + 28(12,4) + 198(8,8), \\
E_{10} &= (20) - 19(16,4) - 494(12,8), \\
E_{12} &= (24) + \frac{2022}{691}(20,4) + \frac{516381}{691}(16,8) + \frac{1792148}{691}(12,12), \\
E_{14} &= (28) - 5(24,4) - 759(20,8) - 7429(16,12), \\
E_{16} &= (32) - \frac{13448}{3617}(28,4) + \frac{2108060}{3617}(24,8) + \frac{50891848}{3617}(20,12) + \frac{131063558}{3617}(16,16), \\
E_{18} &= (36) - \frac{199197}{43867}(32,4) - \frac{14343876}{43867}(28,8) - \frac{854608020}{43867}(24,12) - \frac{4880628198}{43867}(20,16).
\end{aligned}$$

This computation is done just by using the Hecke eigenform property. This property has to be checked only for one Hecke operator, e.g. for $T(2)$ [El]. Hence Eisenstein series are uniquely determined by the "Ansatz"

$$E_n = \sum_{4|i} a_i(2n-i,i)$$

with $a_0=1$, $M(E_n)=E_n$ and the Fourier condition $E_n=1+b_1q+(1+2^{n-1})b_1q^2 \mod q^3$, an easy condition in the truncated power series ring $\mathbf{C}[q]/q^3$ ($A\equiv 1+2q$ and $C^4\equiv 16q$ is the only necessary information).

The finite group G acts on the polynomial ring $R=\mathbf{C}[A,B,C,D]$ by

$$M=\tfrac{1}{2}(1+i)\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix} \quad\text{and}\quad E=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&i&0\\0&0&0&i\end{pmatrix}.$$

We have proved

THEOREM 4.2. *The ring of even Jacobi forms is given by*

$$\bigoplus_{k,m} J^{\mathrm{even}}_{k,m}(\Gamma_1) = \Big\{ f=\sum_{i,j} c(i,j)q^ip^j \text{ such that } f\in R^G[\Delta^{-1}] \text{ and } c(i,j)=0 \text{ for } 4im<j^2 \Big\}.$$

Remark 4.3. One may interpret the considerations in [EZ, p. 109] as follows: For any even Jacobi form $\phi\in J_{*,*}(\Gamma_1)$ we have

$$\min\{l \mid \Delta^l\phi\in R^G\}\leqslant \mathrm{index}(\phi)$$

where the bound seems to be not very precise. Hence for any index and weight one gets an easy algorithm to compute the vector space of Jacobi forms. The ring of invariants $R^G=\mathbf{C}[A,B,C,D]^G\subset\bigoplus_{k,m}J_{2k,m}(\Gamma_1)$ consists of Jacobi forms with $2\cdot\mathrm{weight}(f)\geqslant\mathrm{index}(f)$. Nevertheless for small index we have equality for dimension reasons as we will see next.

One easily computes the Poincaré series of the invariant ring R^G as a bigraded ring.

$$\begin{aligned}\Phi_{R^G}(\lambda,\mu)&=\sum_{k,m\geqslant 0}\dim_{\mathbf{C}} R^G_{k,m}\lambda^k\mu^m\\ &=\frac{1}{96}\sum_{\sigma\in G}\frac{1}{\det(1-\lambda^{1/2}\sigma)\det(1-\lambda^{1/2}\mu\sigma)}\end{aligned}$$

$$
\begin{aligned}
&=[\lambda^{16}\mu^{16} \\
&\quad+\lambda^{14}(\mu^{11}+\mu^{12}+\mu^{13}+\mu^{14}+\mu^{15}+\mu^{16}+\mu^{17}) \\
&\quad+\lambda^{12}(\mu^{6}+\mu^{7}+\mu^{8}+2\mu^{9}+2\mu^{10}+2\mu^{11}+3\mu^{12}+2\mu^{13}+2\mu^{14} \\
&\qquad\quad+2\mu^{15}+\mu^{16}+\mu^{17}+\mu^{18}) \\
&\quad+\lambda^{10}(\mu^{5}+\mu^{6}+2\mu^{7}+2\mu^{8}+2\mu^{9}+3\mu^{10}+2\mu^{11}+2\mu^{12}+2\mu^{13}+\mu^{14}+\mu^{15}) \\
&\quad+\lambda^{8}(\mu^{2}+\mu^{3}+\mu^{4}+2\mu^{5}+2\mu^{6}+2\mu^{7}+3\mu^{8}+2\mu^{9}+2\mu^{10} \\
&\qquad\quad+2\mu^{11}+\mu^{12}+\mu^{13}+\mu^{14}) \\
&\quad+\lambda^{6}(\mu+\mu^{2}+2\mu^{3}+2\mu^{4}+2\mu^{5}+2\mu^{6}+2\mu^{7}+2\mu^{8}+2\mu^{9}+\mu^{10}+\mu^{11}) \\
&\quad+\lambda^{4}(\mu+\mu^{2}+\mu^{3}+2\mu^{4}+\mu^{5}+\mu^{6}+\mu^{7}) \\
&\quad+1]/(\lambda^{4}-1)(\lambda^{6}-1)(\lambda^{4}\mu^{8}-1)(\lambda^{6}\mu^{12}-1).
\end{aligned}
$$

The denominator corresponds to the algebraically independent generators

$$
\begin{aligned}
\phi_{4,0}&=E_4=A^8+C^8+14A^4C^4, \\
\phi_{6,0}&=E_6=A^{12}+C^{12}-33A^4C^4(A^4+C^4), \\
\phi_{4,8}&=B^8+D^8+14B^4D^4, \\
\phi_{6,12}&=B^{12}+D^{12}-33B^4D^4(B^4+D^4).
\end{aligned}
$$

As an algebra over the ring generated by these four polynomials, the invariant ring R^G is a finite module and the numerator indicates in which weight and index one finds the other generators, e.g.

$$
\begin{aligned}
\phi_{4,1}&=A^7B+C^7D+7(A^3BC^4+C^3DA^4), \\
\phi_{6,1}&=A^{11}B+C^{11}D-22(A^7BC^4+C^7DA^4)-11(A^8C^3D+C^8A^3B).
\end{aligned}
$$

(These Jacobi forms are denoted by $E_{4,1}$ and $E_{6,1}$ in [EZ].)

For odd Jacobi forms we have to start from level $\Gamma_1(4,8)$. We fix the notation

$$
\begin{aligned}
\psi_1&=f_1^{(2)}(\tau,z)-f_3^{(2)}(\tau,z)=q^{1/8}\sum_{x\in\mathbf{Z}}q^{2x^2+x}p^{4x+1}-q^{2x^2-x}p^{4x-1}, \\
X&=f_1^{(2)}(\tau)=q^{1/8}\sum_{x\in\mathbf{Z}}q^{2x^2+x}, \\
Y&=f_0^{(2)}(\tau)-f_2^{(2)}(\tau)=\sum_{x\in\mathbf{Z}}q^{2x^2}-q^{1/2}\sum_{x\in\mathbf{Z}}q^{2x^2+2x}, \\
Z&=f_0^{(2)}(\tau)+f_2^{(2)}(\tau)=\sum_{x\in\mathbf{Z}}q^{2x^2}+q^{1/2}\sum_{x\in\mathbf{Z}}q^{2x^2+2x}.
\end{aligned}
$$

We have the relations

$$X^2 = \tfrac{1}{2}AC, \tag{1}$$

$$Y^2 = A^2 - C^2, \tag{2}$$

$$Z^2 = A^2 + C^2, \tag{3}$$

$$BD = X(f_1^{(2)}(\tau,z) + f_3^{(2)}(\tau,z)), \tag{4}$$

$$B^2 - D^2 = Y(f_0^{(2)}(\tau,z) - f_2^{(2)}(\tau,z)), \tag{5}$$

$$B^2 + D^2 = Z(f_0^{(2)}(\tau,z) + f_2^{(2)}(\tau,z)) \tag{6}$$

from Lemma 3.2 and Corollary 3.3. However, we need furthermore the following relations:

$$f_1^{(2)}(\tau,z)f_3^{(2)}(\tau,z) = f_4^{(4)}(\tau,z)f_2^{(4)}(\tau,0) + f_0^{(4)}(\tau,z)f_2^{(4)}(\tau,0), \tag{7}$$

$$f_2^{(2)}(\tau,z)^2 = f_4^{(4)}(\tau,z)f_0^{(4)}(\tau,0) + f_0^{(4)}(\tau,z)f_4^{(4)}(\tau,0), \tag{8}$$

$$f_0^{(2)}(\tau,z)^2 = f_0^{(4)}(\tau,z)f_0^{(4)}(\tau,0) + f_4^{(4)}(\tau,z)f_4^{(4)}(\tau,0), \tag{9}$$

$$X^2 = f_2^{(4)}(\tau,0)f_0^{(4)}(\tau,0) + f_2^{(4)}(\tau,0)f_4^{(4)}(\tau,0), \tag{10}$$

$$f_0^{(2)}(\tau,0)f_2^{(2)}(\tau,0) = 2f_2^{(4)}(\tau,0)^2. \tag{11}$$

From (8) and (9) we get

$$f_2^{(2)}(\tau,z)^2 + f_0^{(2)}(\tau,z)^2 = (f_0^{(4)}(\tau,z) + f_4^{(4)}(\tau,z))(f_0^{(4)}(\tau) + f_4^{(4)}(\tau)),$$

hence with (7),

$$f_1^{(2)}(\tau,z)f_3^{(2)}(\tau,z) = \frac{f_2^{(2)}(\tau,z)^2 + f_0^{(2)}(\tau,z)^2}{f_0^{(4)}(\tau) + f_4^{(4)}(\tau)} f_2^{(4)}(\tau).$$

Multiplying numerator and denominator with $f_2^{(4)}$ and using (11) and (10) yields

$$f_1^{(2)}(\tau,z)f_3^{(2)}(\tau,z) = (f_2^{(2)}(\tau,z)^2 + f_0^{(2)}(\tau,z)^2)\,\frac{\frac{1}{2}f_2^{(2)}(\tau)f_0^{(2)}(\tau)}{X^2},$$

which, using (5), (6), (1), (2) and (3), gives:

$$f_1^{(2)}(\tau,z)f_3^{(2)}(\tau,z) = \frac{1}{4}\cdot\frac{C^2}{AC}\left(\frac{(B^2+D^2)^2}{A^2+C^2} + \frac{(B^2-D^2)^2}{A^2-C^2}\right).$$

Hence we get finally

$$\begin{aligned}
\psi_1^2 &= (f_1^{(2)}(\tau,z) - f_3^{(2)}(\tau,z))^2\\
&= (f_1^{(2)}(\tau,z) + f_3^{(2)}(\tau,z))^2 - 4f_1^{(2)}(\tau,z)f_3^{(2)}(\tau,z)\\
&= \frac{B^2D^2}{\frac{1}{2}AC} - \frac{C^2}{AC}\left(\frac{(B^2+D^2)^2}{A^2+C^2} + \frac{(B^2-D^2)^2}{A^2-C^2}\right)
\end{aligned}$$

and after a short calculation

$$\psi_1^2\cdot\tfrac{1}{2}AC(A^4-C^4)=(BC+AD)(BC-AD)(AB+CD)(CD-AB).$$

For the action of the modular group one computes

$$J(\psi_1)=(i)e\left(\tfrac{1}{8}\right)\psi_1,\quad J(X)=e\left(\tfrac{1}{8}\right)\cdot\tfrac{1}{2}Y,\quad J(Y)=2e\left(\tfrac{1}{8}\right)X,\quad J(Z)=e\left(\tfrac{1}{8}\right)Z,$$

and

$$D=\begin{pmatrix}1&1\\0&1\end{pmatrix}$$

acts with

$$D(\psi_1)=e\left(\tfrac{1}{8}\right)\psi_1,\quad D(X)=e\left(\tfrac{1}{8}\right)X,\quad D(Y)=Z,\quad D(Z)=Y.$$

We have to study the action of $\Gamma(4)/\Gamma(4,8)=\mathbf{F}_2^2$, an abelian group generated by

$$\begin{pmatrix}1&4\\0&1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1&0\\4&1\end{pmatrix}.$$

But we only know the theta representation on the 2-ring, hence we have furthermore to study the action of M^4 on $\mathbf{C}[X,Y,Z]$, which acts by -1. Hence we have actually a group generated by reflections of order 8, isomorphic to $\mathbf{F}_2^3$, which induces on the 2-ring the action above. The action of $\mathbf{F}_2^3$ on ψ_1 is just by the determinant. For a group generated by reflections the canonical module is just the module over the invariant ring, generated by an element belonging to the character $\det^{-1}$, in our case the element XYZ (see [S2, Theorem 7.1]). Hence one gets with

$$\psi=\psi_1XYZ$$

the following theorem.

THEOREM 4.4. *The ring of Jacobi forms of level* $\Gamma_1(4)$ *is given by*

$$\bigoplus_{k,m} J_{k,m}(\Gamma_1(4))=\mathbf{C}[A,B,C,D,\psi,\Delta^{-1}]_{(2)}^{\mathrm{Hol}}$$

with a condition about the Fourier expansion for the cusps. The group G is acting with χ on ψ. The function ψ is an odd function (regarded as a function in z) of weight 2 and index 2. It holds

$$\psi^2=(BC+AD)(BC-AD)(AB+CD)(CD-AB).$$

Remark 4.5. It would be quite natural to define Jacobi forms of character χ by $\phi|_{k,m}M=\chi(M)\phi$ for $M\in\Gamma^{\mathrm{Jac}}$ and for an arbitrary character χ.

We recall $R=\mathbf{C}[A,B,C,D]$. As in the even case one easily computes the Poincaré series of the invariant ring $R[\psi]^G$ as a bigraded ring. We observe that

$$f\psi+g\in R[\psi]^G \iff f\in R^G_{\chi^{-1}} \text{ and } g\in R^G.$$

Hence we get

$$\begin{aligned}
\Phi_{R[\psi]^G}(\lambda,\mu) &= \sum_{k,m\geqslant 0} \dim_{\mathbf{C}} R[\psi]^G_{k,m}\lambda^k\mu^m \\
&= \frac{1}{96}\sum_{\sigma\in G}\frac{1}{\det(1-\lambda^{1/2}\sigma)\det(1-\lambda^{1/2}\mu\sigma)} \\
&\quad +\lambda^2\mu^2\frac{1}{96}\sum_{\sigma\in G}\frac{1}{\det(1-\lambda^{1/2}\sigma)\det(1-\lambda^{1/2}\mu\sigma)(\chi(\sigma)^{-1})} \\
&= [\lambda^{17}\mu^{17} \\
&\quad +\lambda^{16}\mu^{16} \\
&\quad +\lambda^{15}(\mu^{12}+\mu^{13}+\mu^{14}+\mu^{15}+\mu^{16}+\mu^{17}+\mu^{18}) \\
&\quad +\lambda^{14}(\mu^{11}+\mu^{12}+\mu^{13}+\mu^{14}+\mu^{15}+\mu^{16}+\mu^{17}) \\
&\quad +\lambda^{13}(\mu^{7}+\mu^{8}+\mu^{9}+2\mu^{10}+2\mu^{11}+2\mu^{12}+3\mu^{13}+2\mu^{14}+2\mu^{15}+2\mu^{16} \\
&\qquad +\mu^{17}+\mu^{18}+\mu^{19}) \\
&\quad +\lambda^{12}(\mu^{6}+\mu^{7}+\mu^{8}+2\mu^{9}+2\mu^{10}+2\mu^{11}+3\mu^{12}+2\mu^{13}+2\mu^{14}+2\mu^{15} \\
&\qquad +\mu^{16}+\mu^{17}+\mu^{18}) \\
&\quad +\lambda^{11}(\mu^{2}+\mu^{3}+\mu^{4}+\mu^{5}+2\mu^{6}+2\mu^{7}+3\mu^{8}+3\mu^{9}+3\mu^{10}+4\mu^{11}+3\mu^{12} \\
&\qquad +3\mu^{13}+3\mu^{14}+2\mu^{15}+2\mu^{16}+\mu^{17}+\mu^{18}+\mu^{19}+\mu^{20}) \\
&\quad +\lambda^{10}(\mu^{5}+\mu^{6}+2\mu^{7}+2\mu^{8}+2\mu^{9}+3\mu^{10}+2\mu^{11}+2\mu^{12}+2\mu^{13}+\mu^{14}+\mu^{15}) \\
&\quad +\lambda^{9}(\mu^{3}+\mu^{4}+\mu^{5}+2\mu^{6}+2\mu^{7}+2\mu^{8}+3\mu^{9}+2\mu^{10}+2\mu^{11}+2\mu^{12}+\mu^{13} \\
&\qquad +\mu^{14}+\mu^{15}) \\
&\quad +\lambda^{8}(\mu^{2}+\mu^{3}+\mu^{4}+2\mu^{5}+2\mu^{6}+2\mu^{7}+3\mu^{8}+2\mu^{9}+2\mu^{10}+2\mu^{11}+\mu^{12} \\
&\qquad +\mu^{13}+\mu^{14}) \\
&\quad +\lambda^{7}(\mu^{4}+\mu^{5}+\mu^{6}+\mu^{7}+\mu^{8}+\mu^{9}+\mu^{10}) \\
&\quad +\lambda^{6}(\mu+\mu^{2}+2\mu^{3}+2\mu^{4}+2\mu^{5}+2\mu^{6}+2\mu^{7}+2\mu^{8}+2\mu^{9}+\mu^{10}+\mu^{11}) \\
&\quad +\lambda^{5}\mu^{5}
\end{aligned}$$

$$+\lambda^4(\mu+\mu^2+\mu^3+2\mu^4+\mu^5+\mu^6+\mu^7)$$
$$+1]/(\lambda^4-1)(\lambda^6-1)(\lambda^4\mu^8-1)(\lambda^6\mu^{12}-1).$$

As in the even case the denominator corresponds to the four algebraically independent generators $\phi_{4,0}, \phi_{6,0}, \phi_{4,8}, \phi_{6,12}$. As an algebra over the ring generated by these four polynomials the invariant ring $R[\psi]^G$ is a finite module and the numerator indicates in which weight and index one finds the other generators, e.g.

$$\phi_{5,5} = \psi(BC-AD)^3,$$
$$\phi_{11,2} = \psi(AC(A^4-C^4))^3.$$

THEOREM 4.6. *The ring of Jacobi forms is given by*

$$\bigoplus_{k,m} J_{k,m}(\Gamma_1)$$
$$= \Big\{ f = \sum_{i,j} c(i,j) q^i p^j \text{ such that } f \in (R[\psi])^G[\Delta^{-1}] \text{ and } c(i,j) = 0 \text{ for } 4im < j^2 \Big\}.$$

Guided by the dimension formula it is easy to find all the other generators of $(R[\psi])^G$. Let us denote $M(m) = \{f \in (R[\psi])^G, \operatorname{index}(f) = m\}$, a free $A(\Gamma_1)$-submodule of $\bigoplus_k J_{k,m}(\Gamma_1)$. As an application of the dimension formula we give the following table for the generators of $M(m)$.

index	even weight of generators as $A(\Gamma_1)$-module	odd weight of generators as $A(\Gamma_1)$-module
1	4, 6	
2	4, 6, 8	11
3	4, 6, 6, 8	9, 11
4	4, 4, 6, 6, 8	7, 9, 11
5	4, 6, 6, 8, 8, 10	5, 7, 9, 11
6	4, 6, 6, 8, 8, 10, 12	7, 9, 9, 11, 11
7	4, 6, 6, 8, 8, 10, 10, 12	7, 9, 9, 11, 11, 13
8	4, 6, 6, 8, 8, 8, 10, 10, 12	7, 9, 9, 11, 11, 11, 13

Let us denote furthermore

$$j(k,m) = \begin{cases} \dim\big(\bigoplus_{l=k}^{k+2m}[\Gamma_1, l]\big) - \sum_{v=0}^{m} \lceil v^2/4m \rceil & \text{for } k \text{ even, } k \geqslant 4, \\ \dim\big(\bigoplus_{l=k+1}^{k+2m-3}[\Gamma_1, l]\big) - \sum_{v=1}^{m-1} \lceil v^2/4m \rceil & \text{for } k \text{ odd, } k \geqslant 3, \\ \sum_{v=0}^{m} \max(\dim([\Gamma_1, 2v+2]) - \lceil v^2/4m \rceil, 0) & \text{for } k = 2, \\ 0 & \text{for } k = 1, \\ \dim([\Gamma_1, k]) & \text{for } m = 0. \end{cases}$$

Then we may collect [EZ, Theorem 9.1] and the discussion on p. 121 by $\dim(J_{k,m}(\Gamma_1))\leqslant j(k,m)$ and equality for $k\neq 2$.

COROLLARY 4.7. *For a Jacobi form* $\phi\in J_{k,m}(\Gamma_1)$ *with* $\operatorname{index}(\phi)\leqslant 6$ *we have*

$$\phi\in(R[\psi])^G.$$

Proof. A brute force calculation shows $\dim(M(m)_k)=j(k,m)$. Hence we get equality of $M(m)$ and $\bigoplus_k J_{k,m}(\Gamma_1)$. □

The above dimension formula generalizes the table in [EZ, p. 109] for odd Jacobi forms. The left part of the first two lines (even Jacobi forms of index one and two) corresponds to Theorems 3.5 and 8.2 in [EZ].

Remark 4.8. Some classical functions (Weierstraß $\wp$-function and its derivative) are given by

$$\wp(\tau,z)=\tfrac{1}{12}(2\pi i)^2\frac{BC(C^4-5A^4)-AD(A^4-5C^4)}{BC-AD},$$
$$\wp'(\tau,z)=\tfrac{1}{2}(2\pi i)^3\psi\frac{AC(A^4-C^4)}{(BC-AD)^2},$$

and if we fix the notation

$$P(\tau,z)=\frac{BC(C^4-5A^4)-AD(A^4-5C^4)}{BC-AD}\quad\text{and}\quad Q(\tau,z)=\psi\frac{AC(A^4-C^4)}{(BC-AD)^2}$$

we get the Weierstraß equation

$$4\cdot 27Q^2=P^3-3E_4P+2E_6$$

as an identity in the ring $\mathbf{Z}[[q]]((p))$ with the $(q-p)$-expansion map. All the polynomials

$$\psi,\quad BC-AD,\quad AC(A^4-C^4),\quad BC(C^4-5A^4)-AD(A^4-5C^4)$$

are Jacobi forms of character χ. If one weakens the Fourier condition to $c(i,j)=0$ for $i<0$, one gets a weak Jacobi form in the notation of Eichler and Zagier. The generators of the ring of weak Jacobi forms as an $A(\Gamma_1)$-algebra (see [EZ, Theorem 9.4]) are up to some constant given by

$$\phi_{-1,2}(\tau,z)=\frac{\psi}{AC(A^4-C^4)},$$
$$\phi_{-2,1}(\tau,z)=\frac{BC-AD}{AC(A^4-C^4)},$$
$$\phi_{0,1}(\tau,z)=\frac{BC(C^4-5A^4)-AD(A^4-5C^4)}{AC(A^4-C^4)}.$$

5. Jacobi forms and geometry

Rings of Jacobi forms are not finitely generated, hence there is no easy connection to geometry. The rings

$$B_g(\Gamma) = \left(\mathcal{TH}_g^{(m)}[f_a(\tau,z),\psi_b(\tau,z)]\right)_{(2)}^{G}\Big)^N,$$

however, have a geometric interpretation. (As before Γ is a subgroup of finite index in Γ_g and one chooses some $\Gamma_g(2m,4m)\subset\Gamma$ such that m is even with the property to contain any prime $p\equiv 3$ (4) with an even power and set $G=\Gamma/\Gamma_g(2m,4m)$.) We regard $B_g(\Gamma)$ as a bigraded ring with the natural grading given by weight and index. We assume that $\Gamma\subset\Gamma_g(n)$ for some $n\geqslant 3$ to exclude elliptic fixpoints.

We define (quite analogous to the Proj of a graded ring, see [Ha, II.2]) for a bigraded ring $B_{*,*}$ the scheme $X=\mathrm{BiProj}(B_{*,*})$. (We do not assume that $B_{*,*}$ is generated by elements of a fixed bidegree.) As a set we take the bihomogeneous prime ideals, which do not contain all of $B_{+,+}$. The open subset $D_+(f)=\{\mathfrak{p}\in X \text{ such that } f\notin\mathfrak{p}\}$ is equipped with the structure $\mathrm{Spec}(B_{(f)})$, where $B_{(f)}$ is the subring of elements of bidegree $(0,0)$ in the localized ring B_f.

Let us denote $\mathcal{B}_g(\Gamma)=\mathrm{BiProj}(B_g(\Gamma))$ together with the natural morphism $\pi\colon\mathcal{B}_g(\Gamma)\to\mathcal{A}_g(\Gamma)$ to the Satake–Baily–Borel compactification and the zero section $\varepsilon\colon\mathcal{A}_g(\Gamma)\hookrightarrow\mathcal{B}_g(\Gamma)$ induced by the Φ-operator. (We call the map $\phi(\tau,z)\mapsto\phi(\tau,0)$ the Φ-operator. If one considers Siegel modular forms of genus $g+1$ and its Fourier–Jacobi expansion, one gets the Siegel Φ-operator.) By definition $\mathcal{B}_g(\Gamma)$ is a projective variety, hence compact as a complex space. We call $\mathcal{B}_g(\Gamma)$ *the theta compactification* of the universal abelian variety $\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}$ of level $\Gamma\subset\Gamma_g(4)$. The fibre over $\tau\in\mathbf{H}_g$ is isomorphic to the abelian variety

$$A_\tau=\mathbf{C}^g/\tau\mathbf{Z}^g+\mathbf{Z}^g\cong\mathrm{BiProj}(B_g(\Gamma))\times_{\mathcal{A}_g(\Gamma)}\mathrm{Spec}(\mathbf{C})\cong\mathrm{Proj}(\mathbf{C}[f_a(\tau,z),\psi_b(\tau,z)])$$

(one looses one grading). This holds, because the theta functions $f_a^{(2)}(\tau,z)$ define an embedding in a projective space (Lefschetz) and $\mathbf{C}[f_a(\tau,z),\psi_b(\tau,z)]$ is just a homogeneous coordinate ring under this embedding.

In the elliptic case (for $\Gamma\subset\Gamma_1(4)$) we get an elliptic normal surface $\mathrm{BiProj}(B_1(\Gamma))$, which contains the universal elliptic curve of level Γ as an open smooth subset and singularities at most in the fibres over the cusps in $\mathcal{A}_1(\Gamma)$. The morphism π is flat ([Ha, III.9.7]), hence the dimension and arithmetic genus of the fibres are constant 1 ([Ha, III.9.10]).

In the case $\Gamma=\Gamma_1(4)$ we have $\mathrm{BiProj}(B_1(\Gamma_1(4)))=\mathrm{BiProj}(\mathbf{C}[A,B,C,D,\psi]/(\psi^2=\ldots)$ together with the morphism $\mathcal{B}_1(\Gamma_1(4))\to\mathcal{A}_1(4)\cong\mathbf{P}^1$ given by the inclusion of the graded ring $\mathbf{C}[A,C]$ in the bigraded ring $\mathbf{C}[A,B,C,D,\psi]/(\psi^2=\ldots)$. The zero section is given by

$\Phi(A)=\Phi(B)=A$ and $\Phi(C)=\Phi(D)=C$ and $\Phi(\psi)=0$. The fibre of the morphism over $\tau\in\mathbf{H}_1/\Gamma_1(4)\subset\mathbf{P}^1\cong\mathcal{A}_1(4)$ is just the elliptic curve $E_\tau=\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$ embedded in the weighted $\mathbf{P}^2$ as a quartic. (This is nonsingular! Indeed, if we compute the discriminant of

$$(BC+AD)(BC-AD)(AB+CD)(CD-AB)=B^2D^2(C^4+A^4)-A^2C^2(B^4+D^4)$$

regarded as an homogeneous polynomial in B,D over $\mathbf{C}(A,C)$, we get $A^2C^2\Delta$. Hence the affine part is non-singular. But we are not in a $\mathbf{P}^2$ with weights $(1,1,1)$. We have index$(\psi)=2$, so we may regard the curve $(\sqrt{\psi})^4=B^2D^2(C^4+A^4)-A^2C^2(B^4+D^4)$ as a non-singular curve of genus 3 in a $\mathbf{P}^2$ with usual weights $(1,1,1)$ which is a cyclic covering of the fibre E_τ.) Hence we get a very concrete nearly non-singular model, which after desingularization in the fibres over the cusps leads to the Shioda surface $B(4)$, a $K3$-surface with Picard number 20, $p_g=p_a=\dim([\Gamma_1(4),3]_0)=1$ (there is the cusp form $AC(A^4-C^4)$). The fibre over the cusp over infinity is given by

$$\mathrm{BiProj}(\mathbf{C}[A,B,C,D,\psi])\times_{\mathrm{Proj}(\mathbf{C}[A,C])}\mathrm{Spec}(\mathbf{C})=\mathrm{Proj}(\mathbf{C}[\psi,B,D]/(\psi^2-B^2D^2)).$$

For more details we refer to Shioda [Sh]. This may easily be generalized to $\mathcal{B}_1(\Gamma)$, where the equation $(\psi_1XYZ)^2=B^2D^2(C^4+A^4)-A^2C^2(B^4+D^4)$ has to be interpreted after a certain Veronese map (Lemma 2.3). We get a tower of theta compactifications of the universal elliptic curve of level Γ together with morphisms to $\mathcal{A}_1(\Gamma)$. For another description of the field of meromorphic functions we refer to [B2].

Let us denote by

$$\omega(\Gamma)=\det(\Omega^1_{\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}|\mathbf{H}_g/\Gamma})$$

the Hodge line bundle on $\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}$ and consider the diagonal morphism

$$\Delta\colon\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}\to\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}\times_{\mathbf{H}_g/\Gamma}\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}.$$

Take the Poincaré bundle $\mathcal{P}_g$ on $\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}\times_{\mathbf{H}_g/\Gamma}\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}$ and set $\mathcal{L}=\Delta^*\mathcal{P}_g$. Then one obtains in a canonical way an invertible sheaf on $\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}$ giving twice the principal polarization. We extend $\omega^{\otimes a}\otimes\mathcal{L}^{\otimes b}$ as a line bundle to $\mathcal{B}_g(\Gamma)$ (take the projective closure of the corresponding divisor) and denote this line bundle by $\mathcal{O}_X(a,b)$. Then for $Y=\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}\subset X=\mathrm{BiProj}(B(\Gamma))$ we have

$$\mathcal{F}=\mathcal{O}_X(a,b)|_Y=\omega^{\otimes a}\otimes\mathcal{L}^{\otimes b}.$$

A geometric reformulation (for $g\geqslant2$) of the definition of Jacobi forms is the following:

$$J_{k,m}(\Gamma)=H^0(\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}},\omega^{\otimes k}\otimes\mathcal{L}^{\otimes m})$$

(in terms of automorphy factors $\mathcal{L}$ corresponds to $e(\tau[x]+2\langle z,x\rangle)$ when restricted to the fibres of π in the definition of Jacobi forms). A geometric way to get back Siegel modular forms from $\mathcal{A}_g(\Gamma)$ is

$$A(\Gamma)=\bigoplus_k H^0(\mathcal{A}_g(\Gamma),\varepsilon^*\omega(\Gamma)^{\otimes k}).$$

This equality holds for $g\geqslant 2$ because the codimension of the boundary in the Satake compactification is $\geqslant 2$. Hence any section of $\varepsilon^*\omega(\Gamma)^{\otimes k}$ over $\mathbf{H}_g/\Gamma$ extends to a section over $\mathcal{A}_g(\Gamma)$. This is a geometric analogue of the Koecher principle. But also the case $g=1$ is included, because the Fourier condition is nothing but to demand the property of extension.

THEOREM 5.1. *For $g\geqslant 2$ and $X=\mathcal{B}_g(\Gamma)$ we have*

$$J_{k,m}(\Gamma)=H^0(X,\mathcal{O}_X(k,m)).$$

In other words, any section extends to the boundary.

Proof. The proof uses the structure of the boundary of the Satake compactification (compare [Ch, Appendix III] and [Ig4]). It turns out that the scheme structure is complicated but with the reduced structure the boundary is not too bad. We may argue analytically because $\mathcal{B}_g(\Gamma)$ is a projective variety, hence a compact analytic space. There is a stratification (for $\Gamma=\Gamma_g(2m,4m)$) by sets isomorphic to $\mathcal{A}_{g-1}(\Gamma)\hookrightarrow\mathcal{A}_g(\Gamma)$, and the inclusion is induced by the Siegel Φ-operator (for different cusps we have different Φ-operators, which are related by conjugation under the finite group $\Gamma_g/\Gamma_g(2m,4m)$ similar to the case $m=1$ in [R4]). Using the same argument as above for the Satake case, it remains to show that the dimension of the fibre of π cannot go up for points in the Satake boundary. Hence inductively it remains to show the following property of the Siegel Φ-operator. Let $\phi\in J_{k,m}(\Gamma)$. Then

$$\lim_{\mathrm{Im}(\omega)\to\infty}\phi\left(\begin{pmatrix}\tau & \sigma\\ \sigma^t & \omega\end{pmatrix},z\right)$$

depends only on τ and z. This property is known for Siegel modular forms. Hence using Theorem 3.7 it remains to show this property for $f_a^{(m)}(\tau,z)$, which is easy. Hence the transcendence degree of the fibre is g, which implies that all fibres have dimension g. $\square$

Remark 5.2. For $k,m>0$ the bundle $\omega^{\otimes k}\otimes\mathcal{L}^{\otimes m}$ is ample on $\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\mathrm{Jac}}$ (see [K1, Theorem 2.12]).

Remark 5.3. From the geometric point of view the cusp condition in genus $g=1$ should be chosen such that the equality $J_{k,m}(\Gamma)=H^0(\mathcal{B}_g(\Gamma),\mathcal{O}(k,m))$ remains true. We have not touched this problem. We refer to [K2] for further information how to extend Jacobi cusp forms to the desingularization of $\mathcal{B}_1(\Gamma)$, i.e. the Shioda surfaces.

We continue to describe natural properties of the construction $X = \mathrm{BiProj}(B_{*,*})$. Recall $Y = \mathbf{H}_g \times \mathbf{C}^g/\Gamma^{\mathrm{Jac}} \subset X = \mathrm{BiProj}(B(\Gamma))$ and denote for fixed $a, b > 0$ the sheaf $\mathcal{F} = \mathcal{O}_X(a,b)|_Y = \omega^{\otimes a} \otimes \mathcal{L}^{\otimes b}$ and

$$R(Y,\mathcal{F}) = \bigoplus_{i \geqslant 0} H^0(Y, \mathcal{F}^i) = \bigoplus_{i \geqslant 0} H^0(Y, \mathcal{O}_Y(ai, bi))$$

(considered as an $H^0(Y, \mathcal{O}_Y)$-module). Using the same proof as in [Ha, II.5.14] we have the isomorphism

$$R(Y,\mathcal{F})_{(f)} \cong \mathcal{F}(D_+(f) \cap Y)$$

for any global section $f \in H^0(Y, \mathcal{F})$. We can cover Y with sets of type $D_+(f) \cap Y$, where f runs over a set of generators of $B(\Gamma)$. Because of the ampleness f may as well run over a set of generators of $R(Y,\mathcal{F})$. The sheaf of rings $\bigoplus_{i \geqslant 0} \mathcal{F}^i$ is a sheaf of integrally closed domains (because X is normal). Let us denote (recall $G = \Gamma/\Gamma_g(2m, 4m)$)

$$S = \left(\mathcal{TH}_g^{(m)}[f_a(\tau, z)]\right)_{(2)}^G\Big)^N \subset S' = \bigoplus_{2k \geqslant m} J_{k,m}^{\mathrm{even}}(\Gamma) \subset K(S).$$

We will show that S' is integral over S (hence both rings are equal).

Let $s' \in S'$ be bihomogeneous of bidegree (k, m). Let f_i be a set of generators of S. Then S' is a ring, containing S, and contained in the intersection $\bigcap S_{f_i}$ of the localizations of S at the elements f_i.

(To give a section $t \in H^0(Y, \mathcal{O}_Y(a,b))$ is the same as giving, for each i, sections $t_i \in \mathcal{O}_Y(a,b)(D_+(f_i))$ which agree on the intersections $D_+(f_i f_j)$. Now t_i is just a bihomogeneous element of bidegree (a,b) in the localization S_{f_i}, and its restriction to $D_+(f_i f_j)$ is just the image of that element in $S_{f_i f_j}$. Summing over all multiples of (a,b), we see that $R(Y,\mathcal{F})$ can be identified with the set of tuples (t_i), where for each i the $t_i \in S_{f_i}$, and for each i, j, the images of t_i and t_j in $S_{f_i f_j}$ are the same. Now the f_i are not zero divisors in S, so the localization maps $S \to S_{f_i}$ and $S_{f_i} \to S_{f_i f_j}$ are all injective, and these rings are all subrings of $K(S)$.)

Since $s' \in S_{f_i}$ for each i, we can find an integer n such that $f_i^n s' \in S$. Choose one n that works for all i. Since the f_i generate S, the monomials in the f_i generate $S_{k,m}$ for any bidegree. So by taking a large bidegree b, we may assume that for $y \in S_{\geqslant b} = \bigoplus_{d \geqslant b} S_d$. Hence $ys' \in S_{\geqslant b}$. Take some $f \in S$. Now it follows inductively that for any integer $x \geqslant 1$ we have $(s')^x \in (1/f)S$. It follows by a well-known criterion for integral dependence that s' is integral over S.

We summarize the above by

THEOREM 5.4. *For $g\geqslant 2$ we have that*

$$\bigoplus_{2k\geqslant m} J^{\text{even}}_{k,m}(\Gamma)=B(\Gamma)^{\text{even}}$$

is a finitely generated ring.

To generalize this result, we introduce the following definitions. For a subgroup Γ of finite index in Γ_g containing some $\Gamma_g(2r,4r)$ with the property: r is even and contains any prime $p\equiv 3\ (4)$ with an even power, we define

$$G_r=\Gamma/\Gamma_g(2r,4r),$$
$$B^{\leqslant r}(\Gamma)=\left(\left(\mathcal{THET}^{(\leqslant r)}_{g,(2)}\right)^N\right)^{G_r},$$
$$\mathcal{B}^{\leqslant r}(\Gamma)=\mathrm{BiProj}(B^{\leqslant r}(\Gamma)),$$
$$\text{relative index}(f)=\frac{\text{index}(f)}{2\,\text{weight}(f)}.$$

For $r_1\leqslant r_2$ and $\Gamma_1\supset\Gamma_2$ we have $B^{\leqslant r_1}(\Gamma_1)\subset B^{\leqslant r_2}(\Gamma_2)$ and $B(\Gamma)\subset B^{\leqslant r}(\Gamma)$. These canonical inclusions induce morphisms

$$\mathcal{B}^{\leqslant r_2}(\Gamma_2)\to\mathcal{B}^{\leqslant r_1}(\Gamma_1)\quad\text{and}\quad\mathcal{B}^{\leqslant r}(\Gamma)\to\mathcal{B}(\Gamma)$$

where the second one is birational (an isomorphism when restricted to $Y=\mathbf{H}_g\times\mathbf{C}^g/\Gamma^{\text{Jac}}$).

With the method of proof as above we get the following result for the ring $\bigoplus_{2kr\geqslant m} J_{k,m}(\Gamma)$ of Jacobi forms of relative index $\leqslant r$ and level Γ quite analogous to Igusa's *fundamental lemma.*

THEOREM 5.5. *For $g\geqslant 2$ the ring of Jacobi forms of relative index $\leqslant r$ and level Γ is equal to $B^{\leqslant r}(\Gamma)$, and hence finitely generated.*

The theorem enables us to investigate the "unknown" ring $\bigoplus_{2kr\geqslant m} J_{k,m}(\Gamma)$ as a normalisation of a ring of invariants of a finite group acting on $\mathcal{THET}^{(\leqslant r)}_{g,(2)}$. Now it is true that $\mathcal{THET}^{(\leqslant r)}_{g,(2)}$ is not a "known" ring. However it is constructed explicitly and the problem is now of an algebraic nature. We propose the following problem:

Investigate whether the bigraded ring $\mathcal{THET}^{(\leqslant r)}_{g,(2)}$ is integrally closed or not for varying genus $g\geqslant 2$ and relative index r.

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