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Double construction for monoidal categories

by

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One of the most important mathematical achievements of the last decade has been the theory of quantum groups created by V. Drinfeld, M. Jimbo, and others. Quantum groups provide an algebraic background for various chapters of theoretical physics such as the quantum inverse scattering method, the theory of exactly solvable models of statistical mechanics, the 2-dimensional conformal field theory, the quantum theory of angular momentum, etc. Quantum groups also found remarkable applications in low-dimensional topology.

Quantum groups are defined in terms of what Drinfeld [D1] calls "quasitriangular Hopf algebras" and their construction is based on a general procedure also due to V. Drinfeld assigning to a Hopf algebra A a quasitriangular Hopf algebra $D(A)$ (see [D1] or §5). The Hopf algebra $D(A)$ is called the "quantum double" of A. When considering topological applications, one has to extend the algebra $D(A)$ by a so-called ribbon element (see Reshetikhin and Turaev [RT]). This yields a "ribbon Hopf algebra".

The notions of quasitriangular and ribbon Hopf algebras have purely categorical counterparts that are related to algebras via representation theory. It is well-known that the category of finite-dimensional representations of a Hopf algebra acquires in a canonical way the structure of a monoidal category with duality. Moreover, if the Hopf algebra is quasitriangular, then the category of its finite-dimensional representations is a braided monoidal category in the sense of Joyal and Street [JS1]. The distinctive feature of a braided monoidal category is the presence of a "braiding" which may be viewed as a commutativity law for the tensor product satisfying the Yang–Baxter equation (see [JS1] or §1). If the Hopf algebra is a ribbon algebra, then the category of its finite-dimensional representations is a ribbon category in the sense of Turaev [T1] (such categories are also called tortile categories in [JS1], [JS2]). In addition to a braiding each ribbon category possesses a "twist" which is responsible for the involutivity of the braiding and relates the braiding to duality (see §1).

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The above-mentioned relationships between Hopf algebras and monoidal categories raise the problem of a direct description of the quantum double and its ribbon extension in terms of monoidal categories. Such a description would clarify these two constructions and place them into a most general framework. A categorical interpretation of the quantum double was given by Drinfeld and independently by Joyal–Street [JS2] and Majid [Mj]. They introduced a beautiful and simple "centre construction" producing a braided monoidal category $\mathcal{Z}(\mathcal{C})$ out of any monoidal category $\mathcal{C}$. Unfortunately, the centre construction does not allow to upgrade duality in $\mathcal{C}$ to a duality in $\mathcal{Z}(\mathcal{C})$. It turns out that the duality may be tamed if it is considered simultaneously with the twist. In other words, there is a categorical analogue of the composition of the ribbon extension with the quantum double.

This categorical construction is the main result of this paper. More precisely, we show how to assign a ribbon category $\mathcal{D}(\mathcal{C})$ to an arbitrary monoidal category with duality $\mathcal{C}$. The definition of $\mathcal{D}(\mathcal{C})$ is an elaboration of the definition of the centre $\mathcal{Z}(\mathcal{C})$. When $\mathcal{C}$ is the category of finite-dimensional representations of a finite-dimensional Hopf algebra A, the category $\mathcal{D}(\mathcal{C})$ is shown to be isomorphic to the category of finite-dimensional representations of the ribbon extension of $D(A)$.

In the authors' opinion, one of the most interesting features of this work is the systematic use of elementary ideas of knot theory in the proof of purely categorical results. It is this beautiful blend of algebra and 3-dimensional topology that makes the whole subject so amazing.

Ribbon Hopf algebras were originally invented with topological applications in mind. Namely, any ribbon Hopf algebra A gives rise to a topological invariant of knots and links in the 3-sphere (see [RT]). This invariant is applicable to oriented framed links whose components are labeled with finite-dimensional representations of A. A more general invariant may be derived from an arbitrary ribbon category $\mathcal{D}$ (see [T1]). It applies to oriented framed links in S^3 whose components are labeled with objects of $\mathcal{D}$. In particular, in the rôle of $\mathcal{D}$ we may use the ribbon category $\mathcal{D}(\mathcal{C})$ constructed from an arbitrary monoidal category with duality $\mathcal{C}$. This leads to a link invariant taking values in the semigroup of endomorphisms of the unit object of $\mathcal{C}$. This construction generalizes the famous Jones polynomial of links.

The paper is essentially self-contained. It is organized as follows. The first four sections are concerned with categories. In §1 we recall the definitions of monoidal, braided, and ribbon categories. In §2 we present our main construction and state the main result (Theorem 2.3). In §3 we set up a graphical calculus for monoidal categories. In §4 we use this calculus to prove Theorem 2.3. Finally, in §5 we recall the notions of quasi-triangular and ribbon Hopf algebras and describe the relationship between our categorical construction on the one hand and the ribbon extension and the quantum double for Hopf algebras on the other hand.

1. Definitions

We start by recalling a few definitions and facts on monoidal and ribbon categories. For more details, see [Mc], [JS1], [JS2], [T2].

1.1. Monoidal categories

Let $\mathcal{C}$ be a category and $\otimes$ a covariant functor from $\mathcal{C}\times\mathcal{C}$ to $\mathcal{C}$: for any pair (U,V) of objects of $\mathcal{C}$ there exists an object $U\otimes V$, called the *tensor product* of U and V, and for any pair

$$(f\colon U\to U',\, g\colon V\to V')$$

of morphisms of $\mathcal{C}$, there exists a morphism

$$f\otimes g\colon U\otimes V\to U'\otimes V'.$$

We have $\mathrm{id}_U\otimes\mathrm{id}_V=\mathrm{id}_{U\otimes V}$ for all objects U and V, and

$$(f'\otimes g)\circ(f\otimes g)=(f'\circ f)\otimes(g'\circ g) \tag{1.1a}$$

whenever composition is defined.

An *associativity constraint* is a family of natural isomorphisms

$$a_{U,V,W}\colon (U\otimes V)\otimes W\to U\otimes(V\otimes W)$$

defined for all objects U,V,W in $\mathcal{C}$ and satisfying Mac Lane's pentagonal axiom (see [Mc]).

A *unit* is an object I of $\mathcal{C}$ for which there exist natural isomorphisms

$$l_U\colon U\otimes I\to U \quad\text{and}\quad r_U\colon I\otimes U\to U$$

satisfying three conditions expressing compatibility with the associativity constraint.

A *monoidal category* is a category $\mathcal{C}$ equipped with a functor $\otimes\colon\mathcal{C}\times\mathcal{C}\to\mathcal{C}$, an associativity constraint and a unit I. In the sequel, we shall assume for simplicity that all monoidal categories considered here are strict, i.e., that the isomorphisms $a_{U,V,W}$, l_U, and r_U are all identities in $\mathcal{C}$. Then the pentagon axiom and the compatibility conditions of the unit are automatically satisfied. There is a coherence theorem by Mac Lane [Mc] which allows to replace any monoidal category by a strict one.

1.2. Duality

Let $(\mathcal{C}, \otimes, I)$ be a (strict) monoidal category with tensor product $\otimes$ and unit I as defined above. It is a *monoidal category with left duality* if for each object V of $\mathcal{C}$ there exist an object V^* and morphisms

$$b_V : I \to V \otimes V^* \quad \text{and} \quad d_V : V^* \otimes V \to I$$

in the category $\mathcal{C}$ such that

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V \quad \text{and} \quad (d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}. \tag{1.2a}$$

We define the *transpose* $f^* : W^* \to V^*$ of any morphism $f : V \to W$ in $\mathcal{C}$ by

$$f^* = (d_W \otimes \mathrm{id}_{V^*})(\mathrm{id}_{W^*} \otimes f \otimes \mathrm{id}_{V^*})(\mathrm{id}_{W^*} \otimes b_V). \tag{1.2b}$$

It is easy to check that

$$(\mathrm{id}_V)^* = \mathrm{id}_{V^*} \quad \text{and} \quad (f \circ g)^* = g^* \circ f^*$$

whenever f and g can be composed.

1.3. Braidings

Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A *braiding* in $(\mathcal{C}, \otimes, I)$ consists of a family of natural isomorphisms

$$c_{U,V} : U \otimes V \to V \otimes U$$

defined for all objects U, V of $\mathcal{C}$ such that

$$c_{U,V \otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) \tag{1.3a}$$

and

$$c_{U \otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}) \tag{1.3b}$$

for all U, V, W in $\mathcal{C}$.

A *braided monoidal category* is a monoidal category $(\mathcal{C}, \otimes, I)$ equipped with a braiding.

1.4. Ribbon categories

Let $(\mathcal{C}, \otimes, I)$ be a braided monoidal category with left duality. A *twist* is a family $\theta_V : V \to V$ of natural isomorphisms defined for all objects V in $\mathcal{C}$ such that

$$\theta_{U\otimes V} = (\theta_U \otimes \theta_V) c_{V,U} c_{U,V} \tag{1.4a}$$

and

$$\theta_{V^*} = (\theta_V)^*. \tag{1.4b}$$

A *ribbon category* is a braided monoidal category with left duality and with a twist. Observe that we also have

$$\theta_{U\otimes V} = c_{V,U} c_{U,V} (\theta_U \otimes \theta_V) = c_{V,U} (\theta_V \otimes \theta_U) c_{U,V} \tag{1.4c}$$

because of the naturality of the twist and of the braiding.

Finally in any ribbon category $\mathcal{C}$ we have the following relations for any pair (V, W) of objects of $\mathcal{C}$,

$$\theta_V^{-2} = (d_V \otimes \mathrm{id}_V)(\mathrm{id}_{V^*} \otimes c_{V,V}^{-1})(c_{V,V^*} \otimes \mathrm{id}_V)(b_V \otimes \mathrm{id}_V) \tag{1.4d}$$

and

$$c_{V^*,W} = (d_V \otimes \mathrm{id}_{W\otimes V^*})(\mathrm{id}_{V^*} \otimes c_{V,W}^{-1} \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*\otimes W} \otimes b_V). \tag{1.4e}$$

They can easily be proved using isotopies of framed tangles (see, e.g., [T2]).

2. The main result

Let $(\mathcal{C}, \otimes, I)$ be a strict monoidal category with left duality as defined in §1.2. We now define a new category $\mathcal{D}(\mathcal{C})$ which will eventually turn out to be a ribbon category.

Definition 2.1. An object of $\mathcal{D}(\mathcal{C})$ is a triple $(V, c_{V,-}, \theta_V)$ where

(a) V is an object of $\mathcal{C}$,

(b) $c_{V,-}$ is a family of natural isomorphisms $c_{V,X} : V \otimes X \to X \otimes V$ defined for all objects X in $\mathcal{C}$,

(c) θ_V is an automorphism of V in $\mathcal{C}$, subject to the following relations:

(i) for all objects X, Y in $\mathcal{C}$ we have

$$c_{V,X\otimes Y} = (\mathrm{id}_X \otimes c_{V,Y})(c_{V,X} \otimes \mathrm{id}_Y), \tag{2.1a}$$

(ii) for each object X we have

$$(\mathrm{id}_X \otimes \theta_V) c_{V,X} = c_{V,X} (\theta_V \otimes \mathrm{id}_X), \tag{2.1b}$$

(iii) we have

$$\theta_V^{-2} = (d_V \otimes \mathrm{id}_V)(\mathrm{id}_{V^*} \otimes c_{V,V}^{-1})(c_{V,V^*} \otimes \mathrm{id}_V)(b_V \otimes \mathrm{id}_V). \tag{2.1c}$$

The naturality in condition (b) above means that for any morphism $f: X \to Y$ in $\mathcal{C}$ the square

$$\begin{array}{ccc} V\otimes X & \xrightarrow{c_{V,X}} & X\otimes V \\ {\scriptstyle \mathrm{id}_V\otimes f}\downarrow & & \downarrow{\scriptstyle f\otimes \mathrm{id}_V} \\ V\otimes Y & \xrightarrow{c_{V,Y}} & Y\otimes V \end{array} \tag{2.1d}$$

commutes.

The morphisms in $\mathcal{D}(\mathcal{C})$ are defined as follows.

Definition 2.2. A morphism from $(V, c_{V,-}, \theta_V)$ to $(W, c_{W,-}, \theta_W)$ is a morphism $f: V \to W$ in $\mathcal{C}$ such that for each object X of $\mathcal{C}$ we have

$$(\mathrm{id}_X \otimes f) c_{V,X} = c_{W,X} (f \otimes \mathrm{id}_X) \tag{2.2a}$$

and

$$f\theta_V = \theta_W f. \tag{2.2b}$$

It is clear that the identity id_V is a morphism in $\mathcal{D}(\mathcal{C})$ and that if f, g are composable morphisms in $\mathcal{D}(\mathcal{C})$ then the composition $g \circ f$ in $\mathcal{C}$ is a morphism in $\mathcal{D}(\mathcal{C})$. Consequently, $\mathcal{D}(\mathcal{C})$ is a category in which the identity of $(V, c_{V,-}, \theta_V)$ is id_V.

We now state the first main theorem.

THEOREM 2.3. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category with left duality. Then $\mathcal{D}(\mathcal{C})$ is a ribbon category where*

(i) *the unit is $(I, \mathrm{id}, \mathrm{id}_I)$,*

(ii) *the tensor product of $(V, c_{V,-}, \theta_V)$ and $(W, c_{W,-}, \theta_W)$ is given by*

$$(V, c_{V,-}, \theta_V) \otimes (W, c_{W,-}, \theta_W) = (V \otimes W, c_{V\otimes W,-}, \theta_{V\otimes W})$$

where $c_{V\otimes W,X}: V \otimes W \otimes X \to X \otimes V \otimes W$ is the morphism in $\mathcal{C}$ defined for all objects X in $\mathcal{C}$ by

$$c_{V\otimes W,X} = (c_{V,X} \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes c_{W,X}) \tag{2.3a}$$

and $\theta_{V\otimes W}$ is the automorphism of $V \otimes W$ given by

$$\theta_{V\otimes W} = (\theta_V \otimes \theta_W) c_{W,V} c_{V,W}, \tag{2.3b}$$

(iii) *the triple $(V^*, c_{V^*,-}, \theta_{V^*})$ is left dual to $(V, c_{V,-}, \theta_V)$ where $c_{V^*,X}$ is the morphism from $V^* \otimes X$ to $X \otimes V^*$ defined by*

$$c_{V^*,X} = (d_V \otimes \mathrm{id}_{X\otimes V^*})(\mathrm{id}_{V^*} \otimes c_{V,X}^{-1} \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*\otimes X} \otimes b_V) \tag{2.3c}$$

and θ_{V^*} *is the automorphism*

$$\theta_{V^*} = (\theta_V)^*, \tag{2.3d}$$

(iv) *the braiding is given by*

$$c_{V,W}: (V, c_{V,-}, \theta_V) \otimes (W, c_{W,-}, \theta_W) \to (W, c_{W,-}, \theta_W) \otimes (V, c_{V,-}, \theta_V)$$

and the twist by

$$\theta_V: (V, c_{V,-}, \theta_V) \to (V, c_{V,-}, \theta_V).$$

This theorem is proven in §4. The second main theorem (Theorem 5.4.1) of the paper relating the construction $\mathcal{D}$ with the quantum double is stated in §5.

The $\mathcal{D}$-construction should be compared to the "centre construction" of Drinfeld, Joyal–Street [JS2], and Majid [Mj]. Let us recall that their category $\mathcal{Z}(\mathcal{C})$ is defined as follows for any monoidal category $\mathcal{C}$. Objects of $\mathcal{Z}(\mathcal{C})$ are pairs $(V, c_{V,-})$ where V and $c_{V,-}$ are defined as in Definition 2.1 and satisfy condition (2.1a). Morphisms of $\mathcal{Z}(\mathcal{C})$ are defined as in Definition 2.2 and satisfy condition (2.2a). In contrast to our construction $\mathcal{D}$, the centre construction does not involve duality.

The reader will find in [JS2] a proof that $\mathcal{Z}(\mathcal{C})$ is a braided monoidal category, the tensor product, the unit and the braiding being given as in Theorem 2.3. Note, however, that our proof of Theorem 2.3 is independent of the results of [JS2].

2.4

We end this section with a universal property of the construction $\mathcal{D}$.

Let $F: \mathcal{C} \to \mathcal{C}'$ be a functor between monoidal categories with left duality. We say that F is a monoidal functor if F preserves the tensor product and the duality, i.e., if we have

$$F(I) = I, \quad F(V \otimes W) = F(V) \otimes F(W), \quad F(V^*) = F(V)^*$$

and

$$F(b_V) = b_{F(V)} \quad \text{and} \quad F(d_V) = d_{F(V)}$$

for all objects V, W in $\mathcal{C}$.

If, moreover, $\mathcal{C}$ and $\mathcal{C}'$ are ribbon categories, then F is said to be a ribbon functor if it is monoidal and preserves the braidings and the twists, i.e., if for all objects V, W of $\mathcal{C}$ we have

$$F(c_{V,W}) = c_{F(V),F(W)} \quad \text{and} \quad F(\theta_V) = \theta_{F(V)}.$$

For any monoidal category $\mathcal{C}$ with left duality, the functor $\Pi: \mathcal{D}(\mathcal{C}) \to \mathcal{C}$ given by

$$\Pi(V, c_{V,-}, \theta_V) = V$$

is a monoidal functor. It is universal in the following sense.

THEOREM 2.5. *Let F be a monoidal functor from a ribbon category $\mathcal{R}$ to a monoidal category $\mathcal{C}$ with left duality. Suppose that F is bijective on objects and surjective on morphisms. Then there exists a unique ribbon functor $\mathcal{D}(F): \mathcal{R} \to \mathcal{D}(\mathcal{C})$ such that $F = \Pi \circ \mathcal{D}(F)$.*

Proof. Let us first prove the existence of $\mathcal{D}(F)$. For any object V of $\mathcal{R}$ we set

$$\mathcal{D}(F)(V) = (F(V), c_{F(V),-}, \theta_{F(V)})$$

where $c_{F(V),-}$ and $\theta_{F(V)}$ are defined for all objects X in $\mathcal{C}$ by

$$c_{F(V),X} = F(c_{V,F^{-1}(X)}) \quad \text{and} \quad \theta_{F(V)} = F(\theta_V).$$

Here $c_{V,-}$ and θ_V are respectively the braiding and the twist in $\mathcal{R}$.

Let us check that $\mathcal{D}(F)(V)$ is an object in $\mathcal{D}(\mathcal{C})$. Relation (2.1a) is satisfied because F is monoidal and we have (1.3a) in $\mathcal{R}$. Relation (2.1b) follows from the fact that the braiding $c_{V,-}$ in $\mathcal{R}$ is natural in V. Relation (2.1c) is a consequence of the corresponding relation (1.4d) in $\mathcal{R}$.

If $f: V \to V'$ is a morphism in $\mathcal{R}$, then set $\mathcal{D}(F)(f) = F(f)$. Relations (2.2a)–(2.2b) are satisfied because of the naturality of the braiding and of the twist in $\mathcal{R}$. This proves that $\mathcal{D}(F)$ is a functor. Clearly, $\Pi \circ \mathcal{D}(F) = F$. Let us now check that $\mathcal{D}(F)$ is a ribbon functor.

It preserves the tensor products because of (1.3b) and the duality because of (1.4e). We have

$$\mathcal{D}(F)(b_V) = F(b_V) = b_{F(V)},$$

which is $b_{\mathcal{D}(F)(V)}$ by definition of the duality in $\mathcal{D}(\mathcal{C})$. Similarly, we have $\mathcal{D}(F)(d_V) = d_{\mathcal{D}(F)(V)}$.

The monoidal functor $\mathcal{D}(F)$ respects braidings and twists. Indeed, we have

$$\mathcal{D}(F)(c_{V,W}) = F(c_{V,W}) = c_{F(V),F(W)},$$

which is the braiding of $\mathcal{D}(\mathcal{C})$. Similarly,

$$\mathcal{D}(F)(\theta_V) = F(\theta_V) = \theta_{F(V)}$$

is the twist in $\mathcal{D}(\mathcal{C})$.

The uniqueness of $\mathcal{D}(F)$ is a consequence of the fact that it preserves braidings and twists. □

Applying Theorem 2.5 to the identity functor of the ribbon category $\mathcal{R}$, we get the following result.

COROLLARY 2.6. *For any ribbon category $\mathcal{R}$ there exists a unique ribbon functor D from $\mathcal{R}$ to $\mathcal{D}(\mathcal{R})$ such that*

$$\Pi \circ D = \mathrm{id}_{\mathcal{R}}.$$

3. Graphical calculus

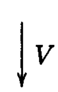
Theorem 2.3 can be proved by purely algebraic formulas. However, because of their complexity, we prefer giving graphical proofs following conventions we describe in this section.

3.1. Representing morphisms in a monoidal category

We discuss a pictorial technique to present morphisms of a monoidal category by planar diagrams. This technique is a kind of geometric calculus which replaces algebraic arguments obscured by their complexity. For further details and references the reader is referred to [JS3], [K], [RT], [T2].

Let $\mathcal{C}$ be a monoidal category. We represent a morphism $f: U \to V$ in $\mathcal{C}$ by a box with two vertical arrows oriented downwards as in Figure 3.1.1. Here U, V are treated as the "colours" of the arrows and f as the "colour" of the box. Such coloured boxes are called coupons. The picture for the composition of $f: U \to V$ and of $g: V \to W$ is obtained by putting the picture of g on top of the picture of f, as showed in Figure 3.1.2. From now on the symbol $\doteq$ displayed in the figures means equality of the corresponding morphisms in $\mathcal{C}$.

The identity of V will be represented by the vertical arrow



directed downwards. The tensor product of two morphisms f and g is represented by boxes placed side by side as in Figure 3.1.3. If we represent a morphism $f: U_1 \otimes \ldots \otimes U_m \to V_1 \otimes \ldots \otimes V_n$ as in Figure 3.1.4, then we have the equality of morphisms of Figure 3.1.5. The pictorial incarnation of the identity

$$f \otimes g = (f \circ \mathrm{id}) \otimes (\mathrm{id} \circ g) = (\mathrm{id} \circ f) \otimes (g \circ \mathrm{id})$$

is in Figure 3.1.6.

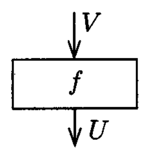


Fig. 3.1.1

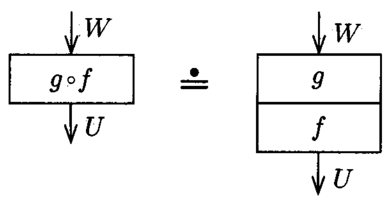


Fig. 3.1.2

Fig. 3.1.3

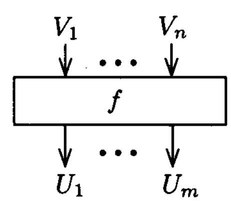


Fig. 3.1.4

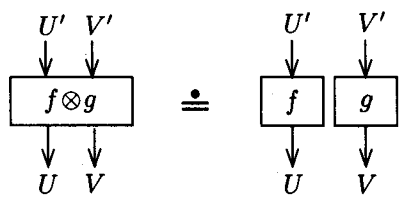


Fig. 3.1.5

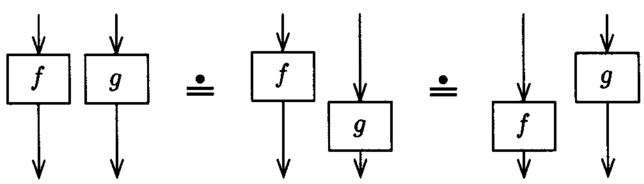
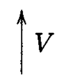


Fig. 3.1.6

3.2. Duality

Suppose in addition that the monoidal category $\mathcal{C}$ is a category with left duality. Then we represent the identity of V^* by the vertical arrow



directed upwards. More generally, we shall use vertical arrows oriented upwards under the convention that the morphism involves not the colour of the arrow, but rather the dual object. For example, any morphism $f: U^* \to V^*$ may be represented in the four ways of Figure 3.2.1.

The morphisms $b_V: I \to V \otimes V^*$ and $d_V: V^* \otimes V \to I$ are respectively represented by the pictures of Figure 3.2.2. The identities (1.2a) between these morphisms have the graphical form given in Figure 3.2.3.

With our convention we can represent the transpose f^* of a morphism $f: V \to W$ as in Figure 3.2.4.

We define a morphism $\lambda_{V,W}: W^* \otimes V^* \to (V \otimes W)^*$ by the formula

$$\lambda_{V,W} = (d_W \otimes \mathrm{id}_{(V\otimes W)^*})(\mathrm{id}_{W^*} \otimes d_V \otimes \mathrm{id}_{W\otimes(V\otimes W)^*})(\mathrm{id}_{W^*\otimes V^*} \otimes b_{V\otimes W}) \tag{3.2a}$$

and a morphism $\lambda^{-1}_{V,W}: (V \otimes W)^* \to W^* \otimes V^*$ by

$$\lambda^{-1}_{V,W} = (d_{V\otimes W} \otimes \mathrm{id}_{W^*\otimes V^*})(\mathrm{id}_{(V\otimes W)^*\otimes V} \otimes b_W \otimes \mathrm{id}_{V^*})(\mathrm{id}_{(V\otimes W)^*} \otimes b_V). \tag{3.2b}$$

The morphisms $\lambda_{V,W}$ and $\lambda^{-1}_{V,W}$ are represented by the pictures in Figure 3.2.5. We invite the reader to use the graphical calculus to give a painless proof of the fact that $\lambda_{V,W}$ is an isomorphism from $W^* \otimes V^*$ onto $(V \otimes W)^*$ with inverse given by $\lambda^{-1}_{V,W}$.

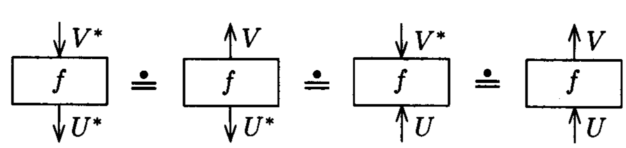


Fig. 3.2.1

Fig. 3.2.2

Fig. 3.2.3

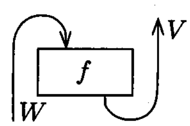


Fig. 3.2.4

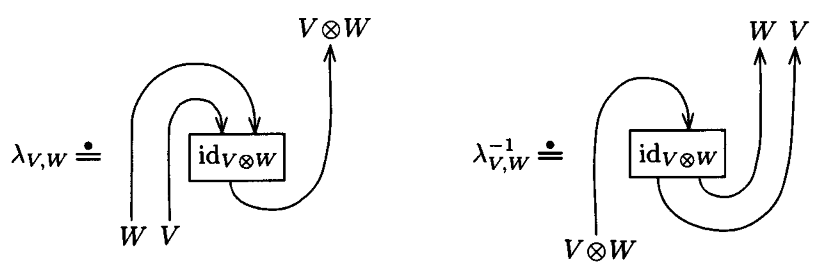


Fig. 3.2.5

3.3. Picturing objects of $\mathcal{D}(\mathcal{C})$

Let $(V, c_{V,-}, \theta_V)$ be an object of $\mathcal{D}(\mathcal{C})$ as defined in §2. By convention we shall represent $c_{V,X}$ and its inverse $c^{-1}_{V,X}$ respectively by the pictures in Figure 3.3.0. Figure 3.3.1 follows from the definitions.

The naturality of $c_{V,-}$ is expressed in the left part of Figure 3.3.2. It implies the naturality of $c^{-1}_{V,-}$ shown in the right part of Figure 3.3.2.

The pictorial transcription of (2.1a) is given in Figure 3.3.3. For (2.1b) see Figure 3.3.4. The oddly-looking relation (2.1c) has the simple pictorial translation drawn in Figure 3.3.5.

The relations (2.2a) and (2.2b) ensuring that a morphism $f: V \to W$ is in $\mathcal{D}(\mathcal{C})$ respectively correspond to the pictures of Figure 3.3.6 and Figure 3.3.7.

Finally, for any object $(V, c_{V,-}, \theta_V)$ of $\mathcal{D}(\mathcal{C})$ and any object X of $\mathcal{C}$ we agree that the pictures of Figure 3.3.8 represent the morphisms $c_{V,X^*}: V \otimes X^* \to X^* \otimes V$ and $c^{-1}_{V,X^*}: X^* \otimes V \to V \otimes X^*$ respectively. The relations shown in Figure 3.3.9 are obvious.

Fig. 3.3.0

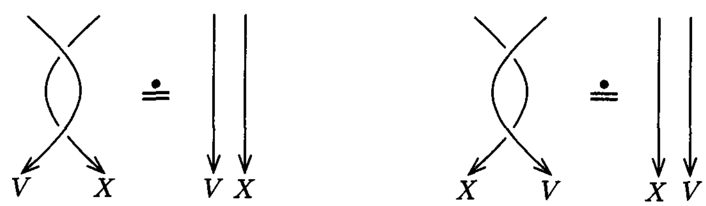


Fig. 3.3.1

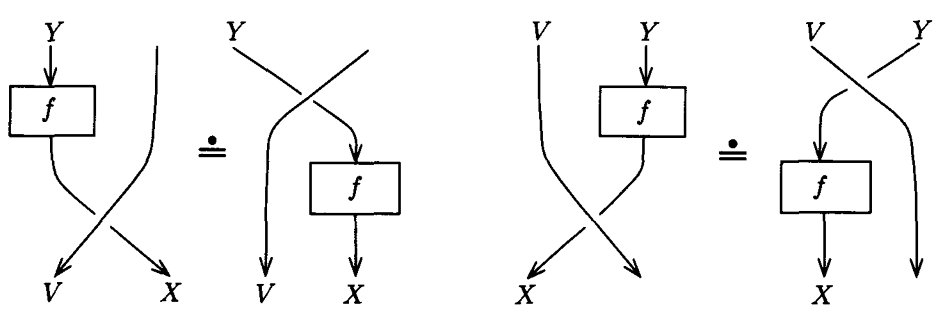


Fig. 3.3.2

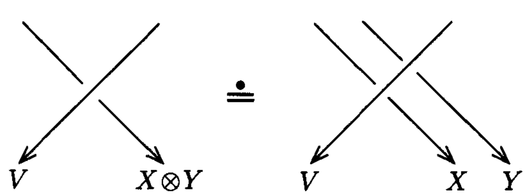


Fig. 3.3.3

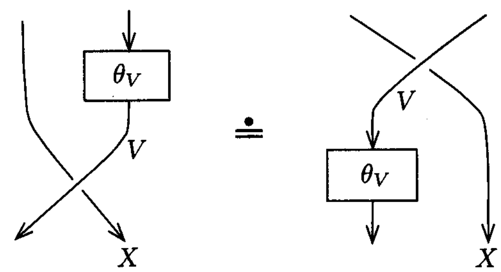


Fig. 3.3.4

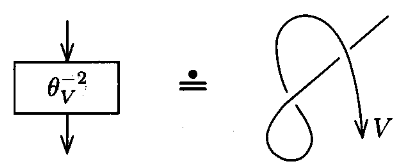


Fig. 3.3.5

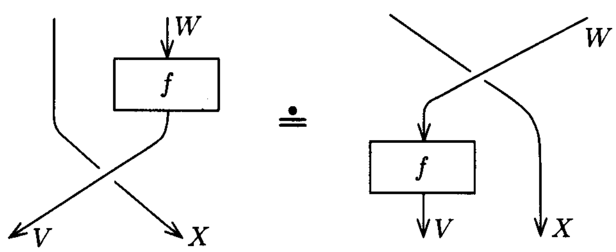


Fig. 3.3.6

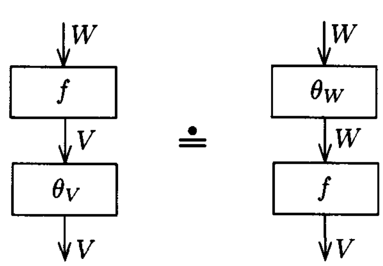


Fig. 3.3.7

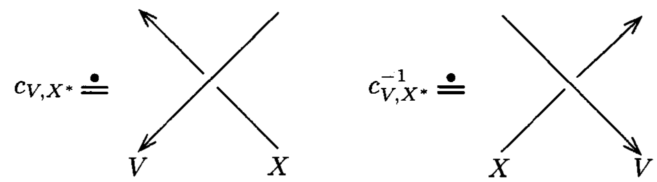


Fig. 3.3.8

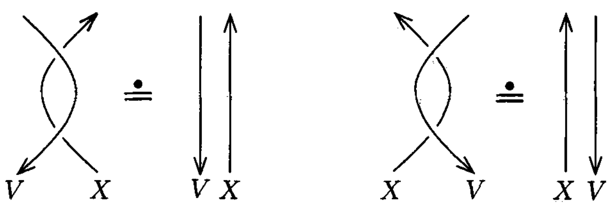


Fig. 3.3.9

4. Proof of Theorem 2.3

Let $(\mathcal{C}, \otimes, I)$ be a monoidal category with left duality. In order to prove Theorem 2.3, we have to show

(i) that $\mathcal{D}(\mathcal{C})$ is a monoidal category, which reduces essentially to check that the triple $(V \otimes W, c_{V\otimes W,-}, \theta_{V\otimes W})$ defined in Theorem 2.3 is an object of $\mathcal{D}(\mathcal{C})$,

(ii) that $\mathcal{D}(\mathcal{C})$ has left duality, which means verifying that the triple $(V^*, c_{V^*,-}, \theta_{V^*})$ of Theorem 2.3 is an object of $\mathcal{D}(\mathcal{C})$ and that b_V and d_V are morphisms of $\mathcal{D}(\mathcal{C})$,

(iii) that $\mathcal{D}(\mathcal{C})$ is a ribbon category, which needs checking that both $c_{V,W}$ and θ_V are morphisms in $\mathcal{D}(\mathcal{C})$.

We shall constantly use the graphical notation of §3.

4.1. Preliminaries

Let $(V, c_{V,-}, \theta_V)$ be an object of the category $\mathcal{D}(\mathcal{C})$. As a consequence of the naturality of $c_{V,-}$ and of $c^{-1}_{V,-}$ we have the equalities of morphisms in $\mathcal{C}$ represented in Figures 4.1.1 and 4.1.2 (they show special cases). In particular, we have the equalities depicted in Figure 4.1.3, expressing the exchange between $c_{V,-}$ and the structural duality maps b_V and d_V.

Let us first state a Yang–Baxter-type relation.

LEMMA 4.1.1. *Let* $(V, c_{V,-}, \theta_V)$ *and* $(W, c_{W,-}; \theta_W)$ *be objects of* $\mathcal{D}(\mathcal{C})$. *For each object* X *in* $\mathcal{C}$ *we have*

$$(c_{W,X}\otimes \mathrm{id}_V)(\mathrm{id}_W\otimes c_{V,X})(c_{V,W}\otimes \mathrm{id}_X) = (\mathrm{id}_X\otimes c_{V,W})(c_{V,X}\otimes \mathrm{id}_W)(\mathrm{id}_V\otimes c_{W,X}).$$

The graphical representation of this equality is in Figure 4.1.4. For the proof see Figure 4.1.5 where we use the equalities of Figures 3.3.2 and 3.3.3.

We need a few variants of Lemma 4.1.1. Let us list them. First, replacing X by X^*, Figure 4.1.4 becomes Figure 4.1.6. Taking the inverses in Figure 4.1.4 gives Figure 4.1.7. We shall also need the equality of Figure 4.1.8: it follows from the equalities of Figures 3.3.2 and 3.3.3. Finally, we have the equality in Figure 4.1.9: its proof is given in Figure 4.1.10 and relies on Figure 4.1.6.

LEMMA 4.1.2. *Under the hypothesis of Lemma* 4.1.1, *we have*

$$(\theta_V\otimes\theta_W)c_{W,V}c_{V,W} = c_{W,V}(\theta_W\otimes\theta_V)c_{V,W} = c_{W,V}c_{V,W}(\theta_V\otimes\theta_W).$$

Proof. See Figure 4.1.11. The second and sixth equalities are derived from relation (2.1b), whereas the third and the fifth ones come from the functoriality of $c_{V,-}$ and of $c_{W,-}$. □

LEMMA 4.1.3. *For any object* $(V, c_{V,-}, \theta_V)$ *of* $\mathcal{D}(\mathcal{C})$ *and any pair* (X, Y) *of objects in* $\mathcal{C}$, *we have*

$$c_{V,(X\otimes Y)^*} = (\lambda_{X,Y}\otimes \mathrm{id}_V)c_{V,Y^*\otimes X^*}(\mathrm{id}_V\otimes\lambda_{X,Y}^{-1}). \tag{4.1a}$$

Proof. Applying the functoriality of $c_{V,-}$ to the isomorphism $\lambda_{X,Y}$ of (3.2a), we get

$$c_{V,(X\otimes Y)^*}(\mathrm{id}_V\otimes\lambda_{X,Y}) = (\lambda_{X,Y}\otimes \mathrm{id}_V)c_{V,Y^*\otimes X^*}.$$ □

Using Figures 3.2.5 and 3.3.3, we can represent $c_{V,(X\otimes Y)^*}$ as in Figure 4.1.12.

In order to prove that $\mathcal{D}(\mathcal{C})$ has left duality we need some further preliminary results. Let $(V, c_{V,-}, \theta_V)$ be an object of $\mathcal{D}(\mathcal{C})$. Define morphisms

$$b'_V : I \to V^*\otimes V \quad \text{and} \quad d'_V : V\otimes V^* \to I$$

in $\mathcal{C}$ by the pictures in Figure 4.1.13. By convention we shall represent b'_V and d'_V as in Figure 4.1.14.

LEMMA 4.1.4. *For any object* $(V, c_{V,-}, \theta_V)$ *of* $\mathcal{D}(\mathcal{C})$ *we have*

$$(d'_V \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes b'_V) = \mathrm{id}_V.$$

Proof. Following the above definition we can represent the left-hand side as in Figure 4.1.15. It is enough to show the equality represented in Figure 4.1.16. This follows from the sequence of equalities represented in Figure 4.1.17, the second one resulting from the naturality of $c_{V,-}$ (see Figure 4.1.1). □

Similarly, we have

LEMMA 4.1.5. *For any object* $(V, c_{V,-}, \theta_V)$ *of* $\mathcal{D}(\mathcal{C})$ *we have*

$$(\mathrm{id}_{V^*} \otimes d'_V)(b'_V \otimes \mathrm{id}_{V^*}) = \mathrm{id}_{V^*}.$$

Proof. Let us first prove the equality depicted in Figure 4.1.18. The proof is given in Figure 4.1.19. The first equality is by definition, the second by the naturality of $c^{-1}_{V,-}$ (Figure 4.1.2), and the third one by (2.1c).

Now the proof of Lemma 4.1.5 is in Figure 4.1.20. □

LEMMA 4.1.6. *For any object* $(V, c_{V,-}, \theta_V)$ *of* $\mathcal{D}(\mathcal{C})$ *we have the equality between the endomorphisms of* V^* *depicted in Figure* 4.1.21.

Proof. The equality of Figure 4.1.21 is obtained from the one in Figure 4.1.22 by transposition. In Figure 4.1.22 the first equality results from the naturality of $c^{-1}_{V,-}$ (Figure 4.1.2), the second one from (2.1c), the third one from (2.1b), the fourth one from Figure 3.3.9, the fifth one from (1.2a), and the last one from (2.1c). □

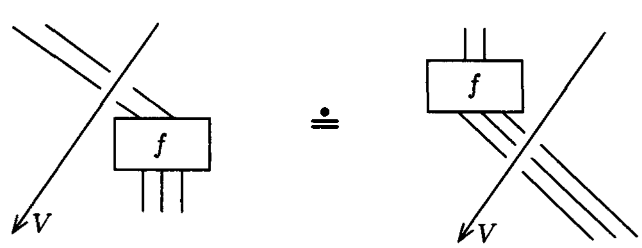


Fig. 4.1.1

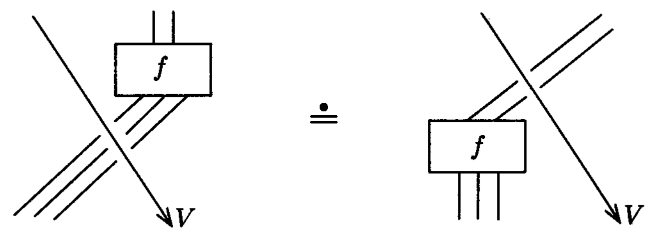


Fig. 4.1.2

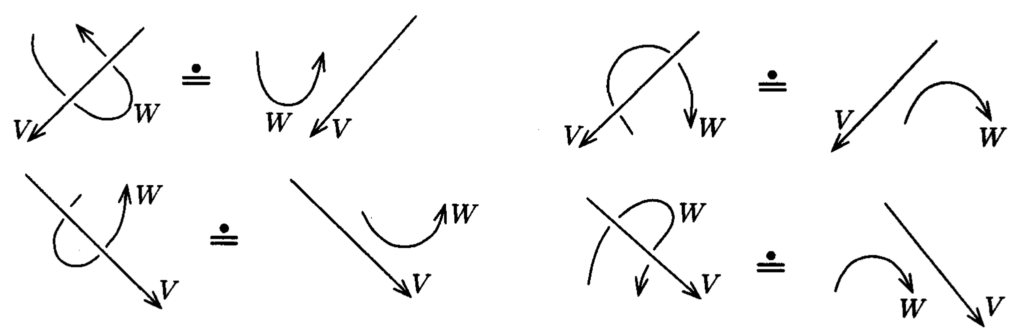


Fig. 4.1.3

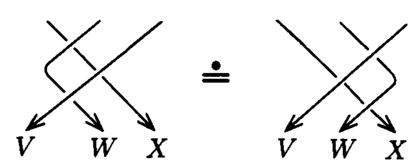


Fig. 4.1.4

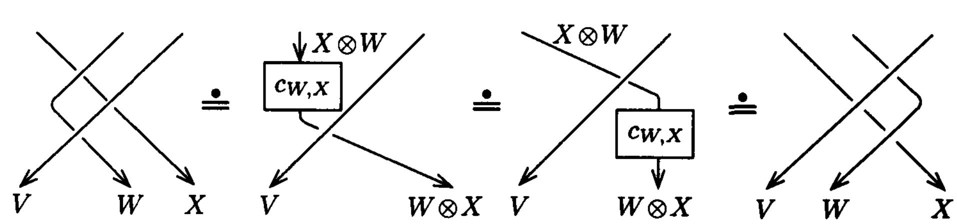


Fig. 4.1.5

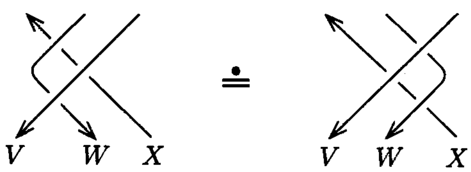


Fig. 4.1.6

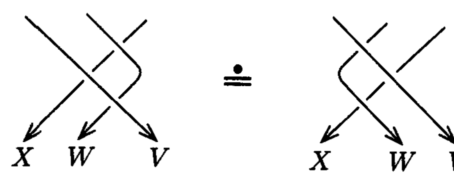


Fig. 4.1.7

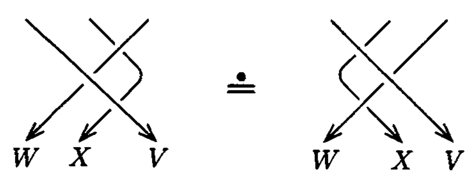


Fig. 4.1.8

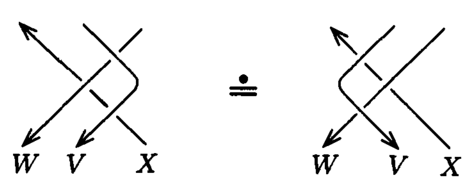


Fig. 4.1.9

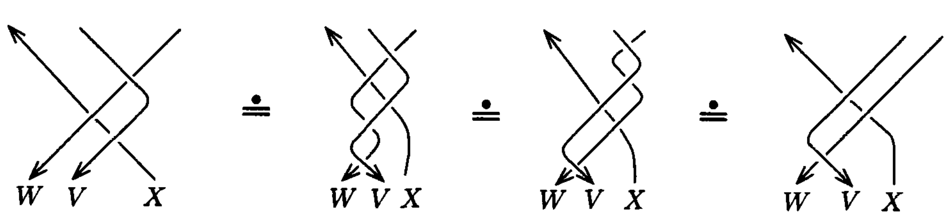


Fig. 4.1.10

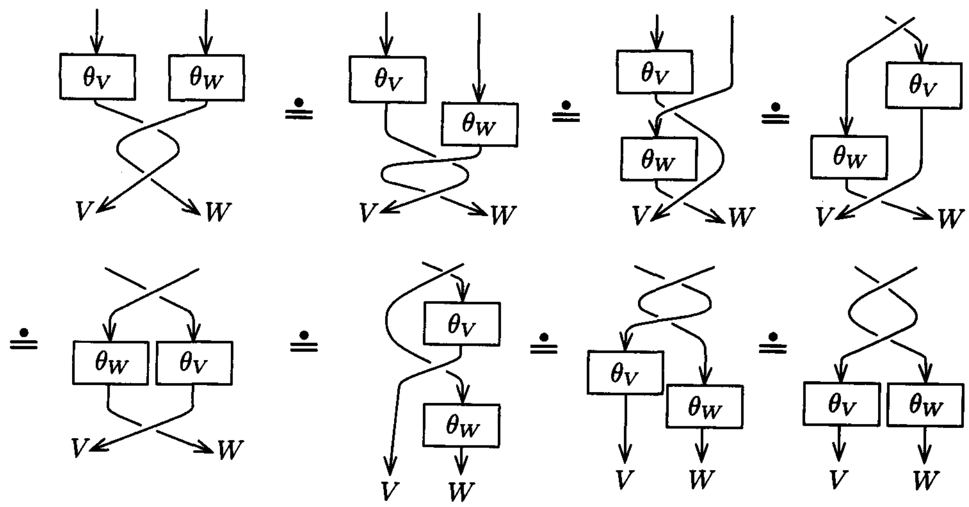


Fig. 4.1.11

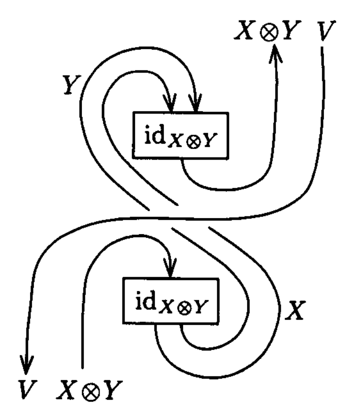


Fig. 4.1.12

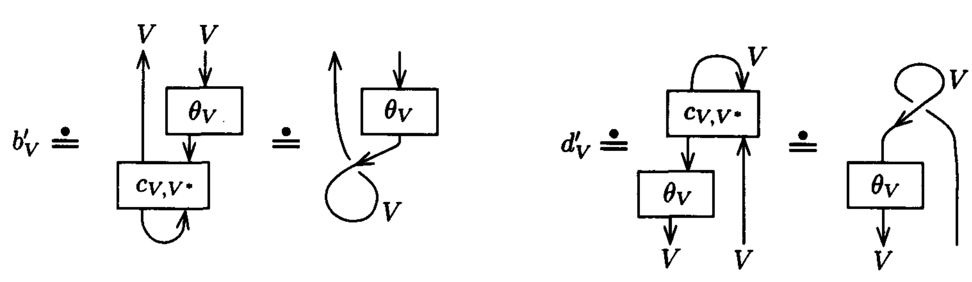


Fig. 4.1.13

Fig. 4.1.14

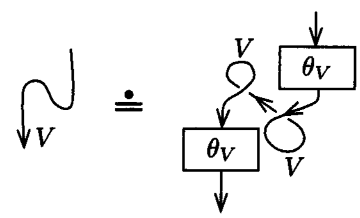


Fig. 4.1.15

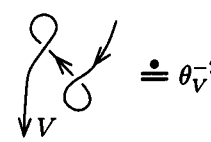


Fig. 4.1.16

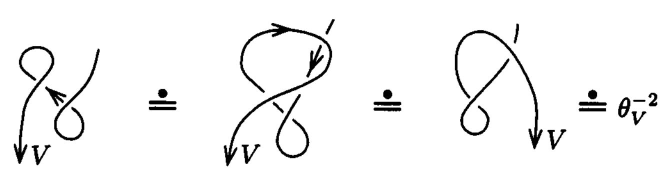


Fig. 4.1.17

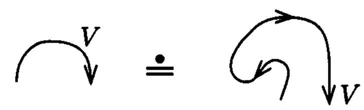


Fig. 4.1.18

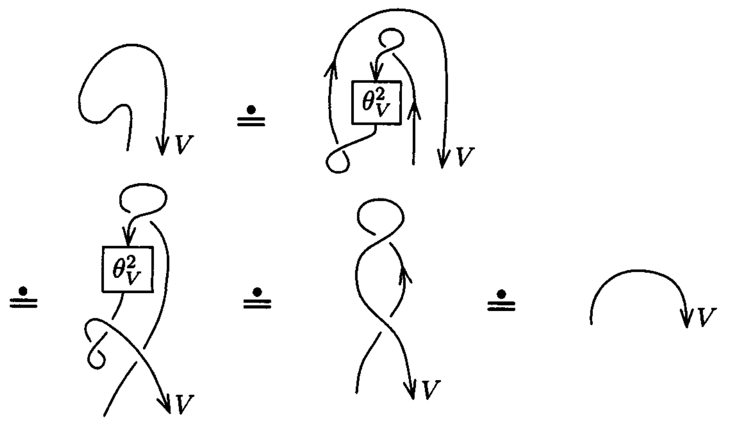


Fig. 4.1.19

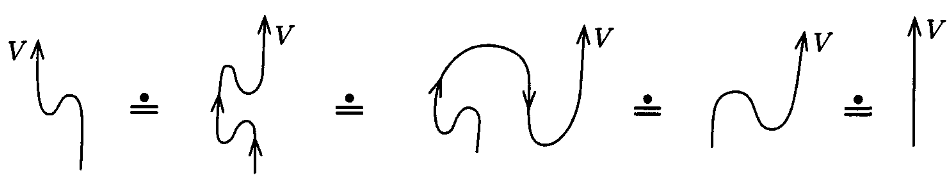


Fig. 4.1.20

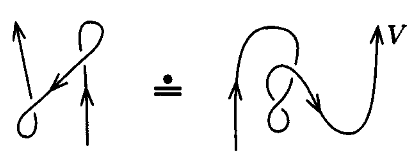


Fig. 4.1.21

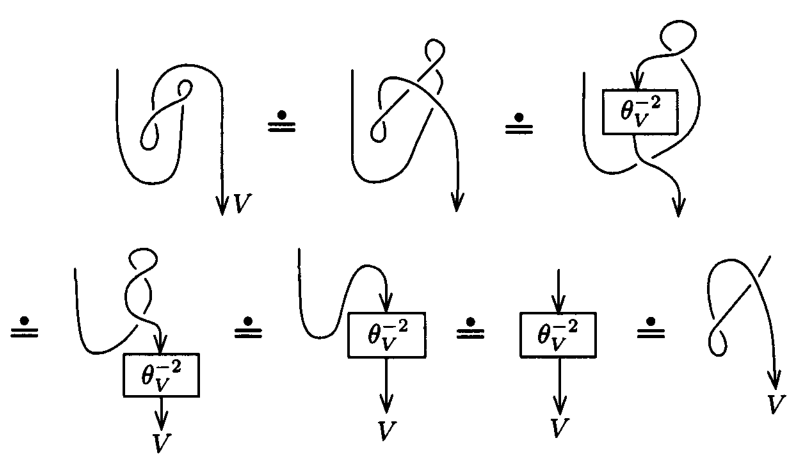


Fig. 4.1.22

4.2. Proof that $\mathcal{D}(\mathcal{C})$ is a monoidal category

We now start the proof of Theorem 2.3. We have the following lemma.

LEMMA 4.2.1. *Let* $(V, c_{V,-}, \theta_V)$ *and* $(W, c_{W,-}, \theta_W)$ *be objects in* $\mathcal{D}(\mathcal{C})$. *Then the triple* $(V \otimes W, c_{V\otimes W,-}, \theta_{V\otimes W})$ *defined in Theorem* 2.3 (ii) *is an object of* $\mathcal{D}(\mathcal{C})$.

The pictorial descriptions of $c_{V\otimes W,-}$ and of $\theta_{V\otimes W}$ are in Figure 4.2.1.

Proof. It follows from the properties of $(V, c_{V,-}, \theta_V)$ and $(W, c_{W,-}, \theta_W)$ that $c_{V\otimes W,X}$ and $\theta_{V\otimes W}$ are isomorphisms in $\mathcal{C}$ and that $c_{V\otimes W,X}$ is natural in X. Let us check graphically relations (2.1a)–(2.1c) of Definition 2.1.

Relation (2.1a): Let X, Y be objects in $\mathcal{C}$. The proof of relation (2.1a) holds in Figure 4.2.2.

Relation (2.1b): See Figure 4.2.3. The first and last equalities are by definition, the third and the fourth ones by (2.1b), the fifth and sixth ones by Lemma 4.1.1.

Relation (2.1c): We have to prove (2.1c) with V replaced by $V \otimes W$. This is done in Figure 4.2.4. The first equality results from Lemma 4.1.3 (Figure 4.1.12), the second one from (1.1a) (Figure 3.1.6), the third one from (1.2a), the fourth one from (2.1a) and the definition (2.3a), the fifth one from the naturality of $c_{V,-}^{-1}$ and $c_{W,-}^{-1}$ (Figure 4.1.3), the sixth one from Figure 4.1.7, the seventh one from Figure 4.1.8, the eighth one from Figure 4.1.6, the ninth and tenth ones from Figure 3.3.1, the eleventh one from the naturality of $c_{W,-}^{-1}$ (Figure 4.1.2), the twelfth one from (2.1c), the thirteenth one from Figure 4.1.8 and from (2.1b), the fourteenth one from the naturality of $c_{W,-}$ (Figure 4.1.1), the fifteenth one from (2.1c), the sixteenth from (2.1b) and the naturality of $c_{W,-}$, and the last one by definition and by Lemma 4.1.2. □

LEMMA 4.2.2. *If f and f' are morphisms in* $\mathcal{D}(\mathcal{C})$, *then so is* $f \otimes f'$.

Proof. We have to check relations (2.2a)–(2.2b) for $f \otimes f'$. Relation (2.2a) is proved in Figure 4.2.5. The second and fourth equalities result from (2.2a).

Relation (2.2b) is proved in Figure 4.2.6 (to be found on p. 28 in §4.3). The second equality results from (2.2b), the third and fifth ones from (2.2a), and the fourth one from the naturality of $c_{V,-}$ and of $c_{V',-}$. □

PROPOSITION 4.2.3. *The category* $\mathcal{D}(\mathcal{C})$ *is a monoidal category.*

Proof. Lemma 4.2.1 and Lemma 4.2.2 show that $\otimes$ is well-defined on the objects and on the morphisms of $\mathcal{D}(\mathcal{C})$. The tensor product is functorial and satisfies all the required axioms because it already does so in the original category $\mathcal{C}$. □

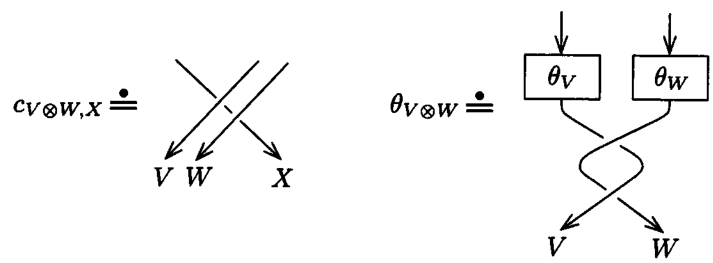


Fig. 4.2.1

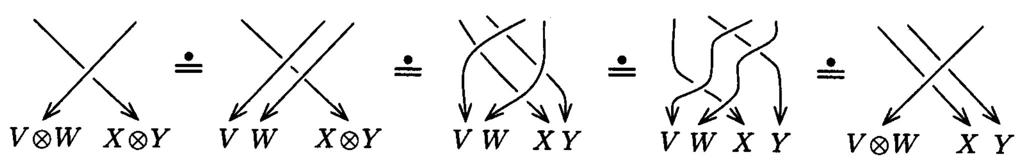


Fig. 4.2.2

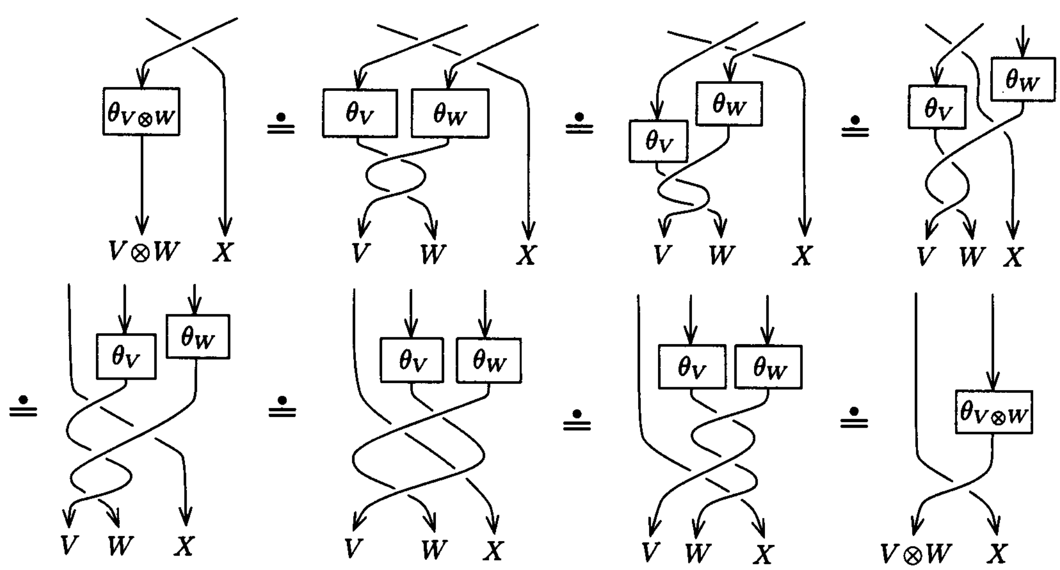


Fig. 4.2.3

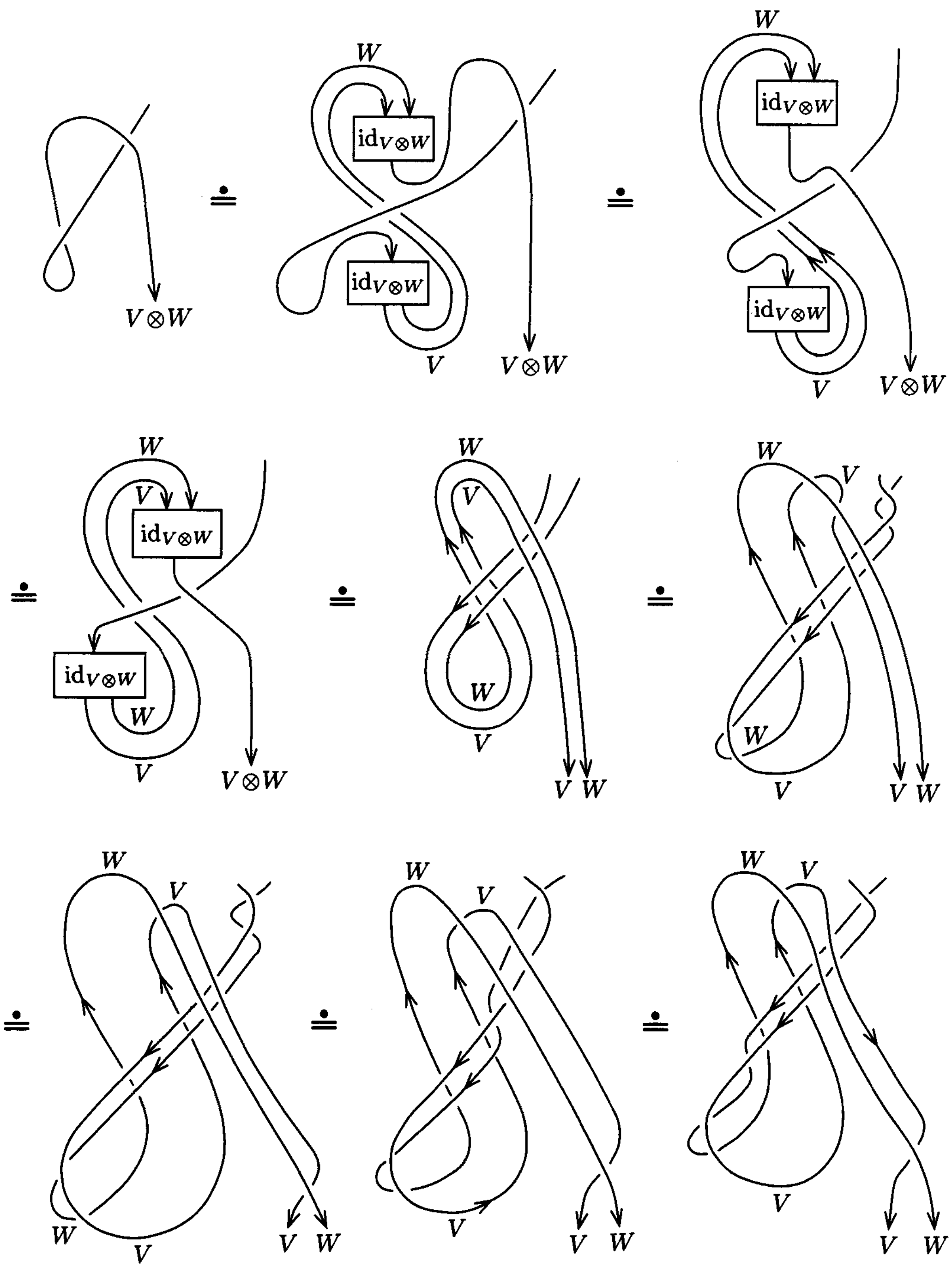


Fig. 4.2.4 (first part)

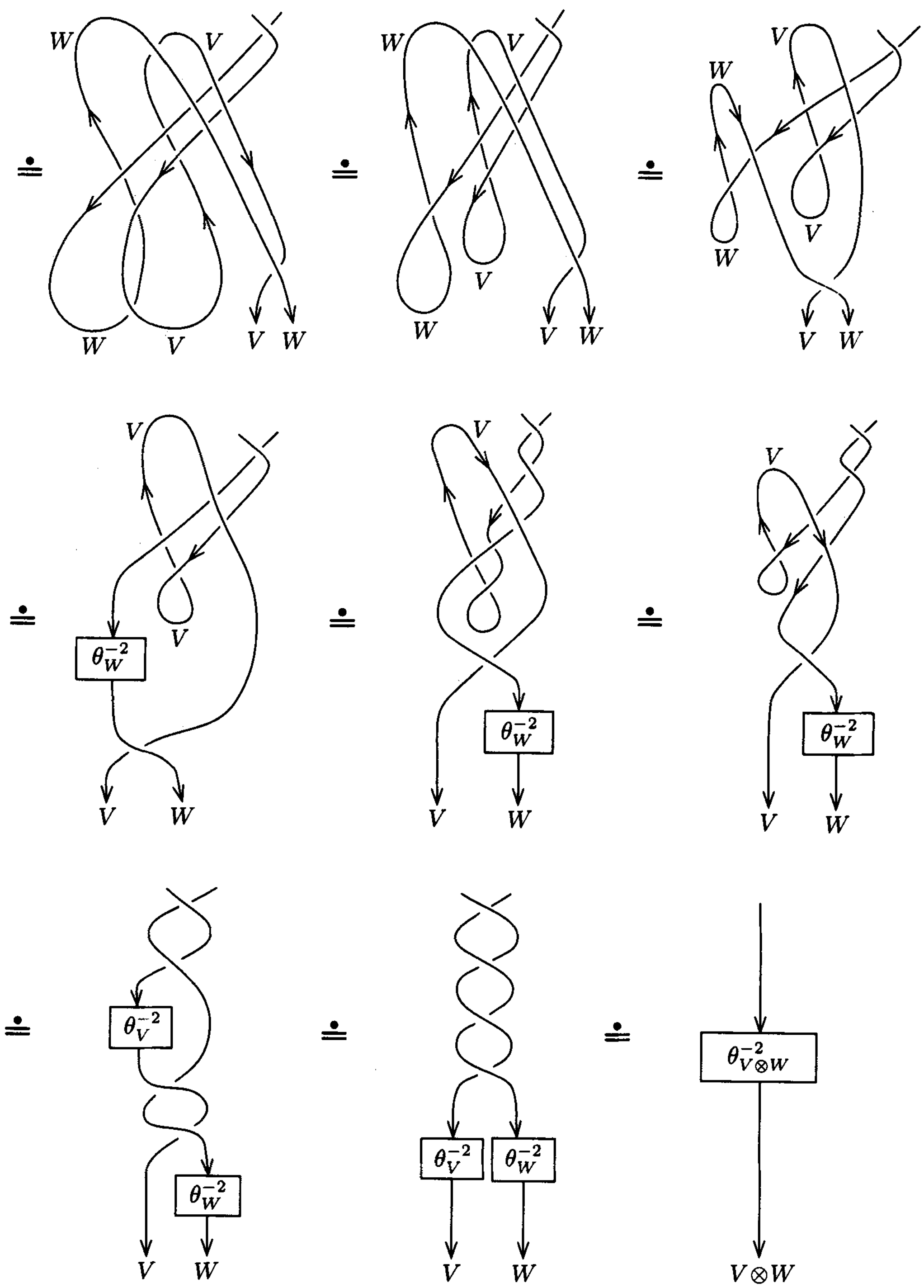


Fig. 4.2.4 (second part)

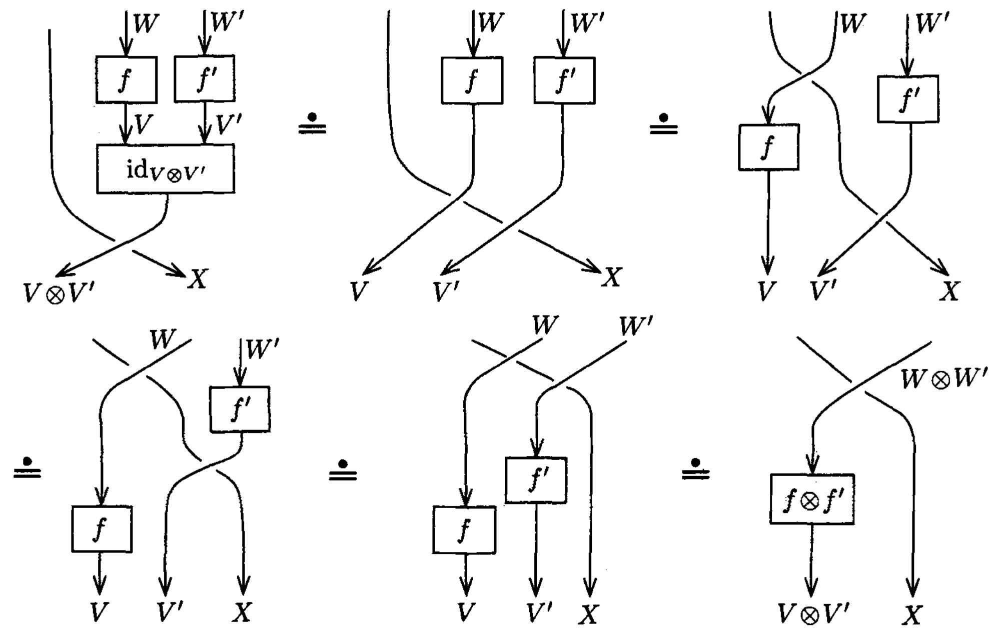


Fig. 4.2.5

4.3. Duality

Let $(V, c_{V,-}, \theta_V)$ be an object of $\mathcal{D}(\mathcal{C})$. In order to prove that $\mathcal{D}(\mathcal{C})$ is a monoidal category with left duality, we have to show that the triple $(V^*, c_{V^*,-}, \theta_V^*)$ defined in Theorem 2.3 (iii) is an object of $\mathcal{D}(\mathcal{C})$ and that b_V and d_V are morphisms of $\mathcal{D}(\mathcal{C})$. Since b_V and d_V satisfy relations (1.2a) in $\mathcal{C}$, they will satisfy them in $\mathcal{D}(\mathcal{C})$. The morphisms $c_{V^*,X}$ and θ_V^* are represented graphically in Figure 4.3.1.

We start with the following preliminary result.

Lemma 4.3.1. *For all X in $\mathcal{C}$, the map $c_{V^*,X}$ is invertible with inverse $c_{V^*,X}^{-1}$ represented in Figure* 4.3.2.

In Figure 4.3.2 we use the conventions of §3 and of §4.1.

Proof. The proof of $c_{V^*,X}^{-1} c_{V^*,X} = \mathrm{id}_{V^*\otimes X}$ is given in Figure 4.3.3. The second and seventh equalities follow by definition, the third one by (2.1b), the fourth one from the naturality of $c_{V,-}$ (Figure 4.1.1), the fifth one from (1.2a), the sixth one from Figure 3.3.9, and the last one from Lemma 4.1.5.

The proof of $c_{V^*,X} c_{V^*,X}^{-1} = \mathrm{id}_{X\otimes V^*}$ is in Figure 4.3.4. The second equality follows by definition of b'_V, the third one by naturality of $c_{V,-}^{-1}$ (Figure 4.1.2), the fourth one from (2.1c), the fifth and seventh ones from (2.1b), the sixth one by definition of d'_V, the eighth one from Figure 3.3.9, and the last one from (1.2a). □

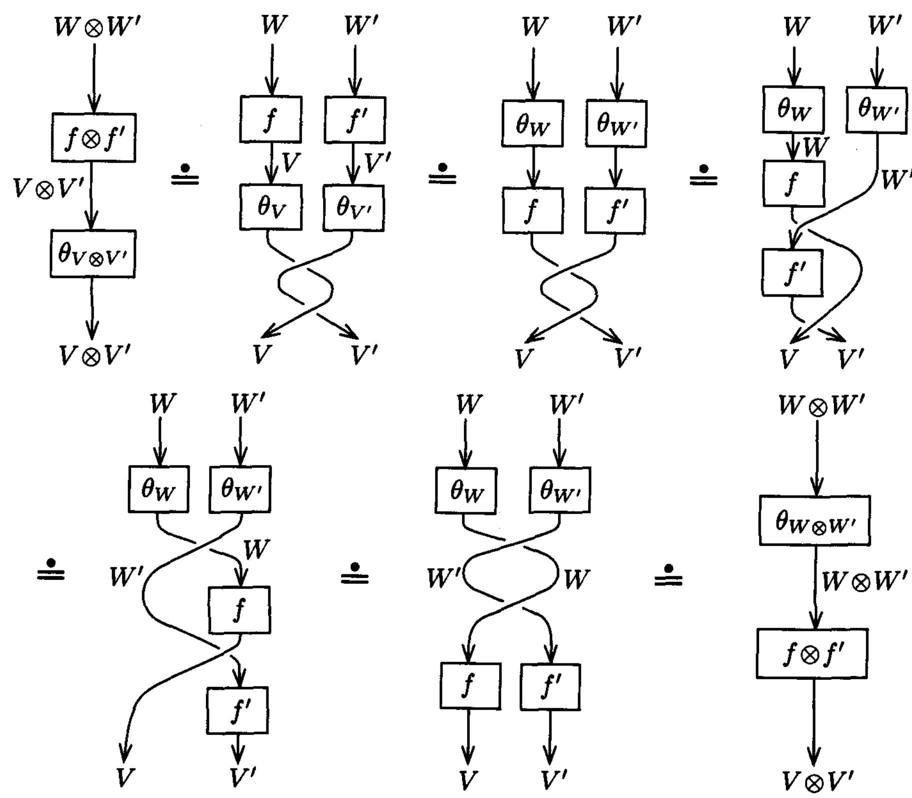


Fig. 4.2.6

It allows us to prove the following lemma.

LEMMA 4.3.2. *The triple* $(V^*, c_{V^*,-}, \theta_{V^*})$ *is an object of* $\mathcal{D}(\mathcal{C})$.

Proof. The maps $c_{V^*,X}$ are invertible by Lemma 4.3.1. They are natural in X. We have to check relations (2.1a)–(2.1c).

Relation (2.1a): We have to prove the equality in Figure 4.3.5. This is done in Figure 4.3.6 where the second equality follows from (2.1a) and the third one from (1.2a).

Relation (2.1b): We have to prove the equality in Figure 4.3.7. This is done in Figure 4.3.8 where the third and fifth equalities follow from (1.2a), and the fourth one from (2.1b).

Relation (2.1c): We have to prove the left equality in Figure 4.3.9. The right one follows from the definition of θ_{V^*}. This is done in Figure 4.3.10 where the second equality follows from Lemma 4.3.1 and from the definition of $c_{V^*,-}$, the third one from Lemma 4.1.5 and from the naturality of $c_{V,-}$, the fourth one from the naturality of $c_{V,-}^{-1}$, the fifth and seventh ones from (1.1a) (Figure 3.1.6), the sixth one from (1.2a), the eighth one from Lemma 4.1.4, the ninth one by definition of b'_V and d'_V, the tenth one from (2.1b) and the naturality of $c^{-1}_{V,-}$, the eleventh and the thirteenth ones from (2.1c), the twelfth one from Lemma 4.1.6. □

The following statement concludes the proof that $\mathcal{D}(\mathcal{C})$ is a monoidal category with left duality.

LEMMA 4.3.3. *The morphisms $b_V: I \to V \otimes V^*$ and $d_V: V^* \otimes V \to I$ are morphisms of $\mathcal{D}(\mathcal{C})$.*

Proof. (a) Let us prove it for b_V. Relation (2.2a) which is

$$c_{V\otimes V^*,X}(b_V \otimes \mathrm{id}_X) = \mathrm{id}_X \otimes b_V$$

is proved graphically in Figure 4.3.11 where the first equality follows by definition and the second one from (1.2a).

Relation (2.2b) reads as $b_V = \theta_{V\otimes V^*} b_V$. It is proved in Figure 4.3.12. There the first and third equalities follow from the definitions, the second one from (2.1c), and the fourth one from (1.2a).

(b) Proof for d_V. Relation (2.2a) reads: $(\mathrm{id}_X \otimes d_V) c_{V^*\otimes V,X} = d_V \otimes \mathrm{id}_X$. The proof is in Figure 4.3.13. The first equality is by definition and the third one follows from (1.2a).

Relation (2.2b) reads: $d_V \theta_{V^*\otimes V} = d_V$. The proof is in Figure 4.3.14. The second equality follows from the naturality of $c^{-1}_{V,-}$, the third one from the naturality of $c_{V,-}$, the fourth one from (2.1c) and the definition of θ_{V^*}, the fifth one from (1.2a).

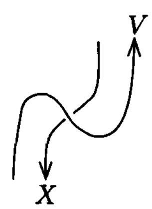
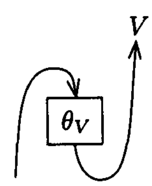
$c_{V^*,X} \doteq$  $\theta_{V^*} \doteq$ 

Fig. 4.3.1

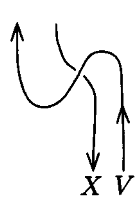
$c^{-1}_{V^*,X} \doteq$ 

Fig. 4.3.2

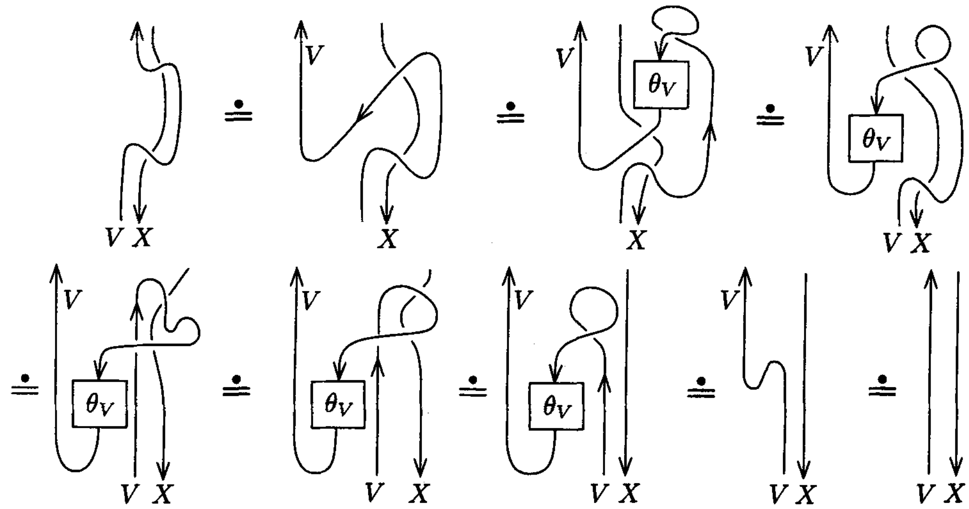


Fig. 4.3.3

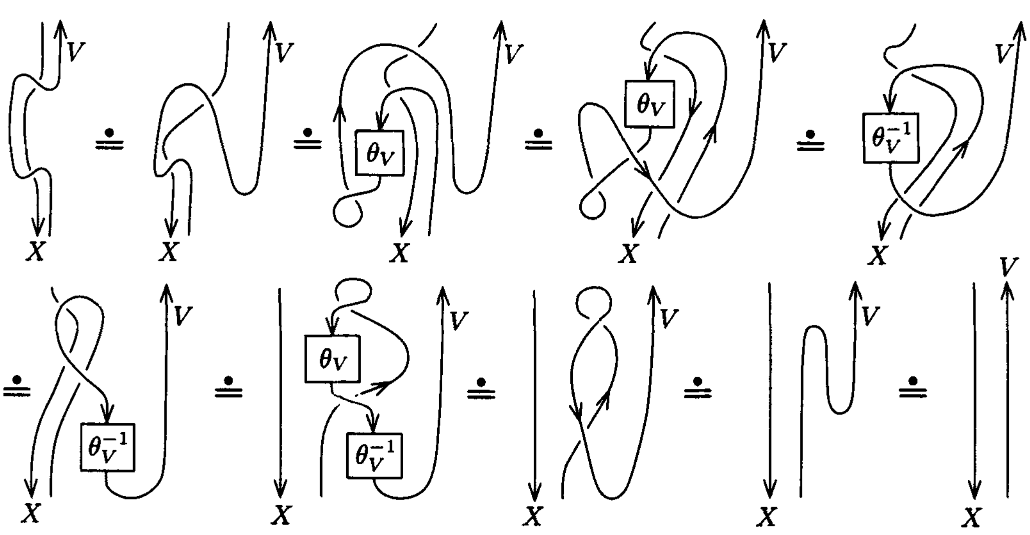


Fig. 4.3.4

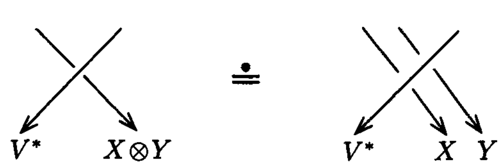


Fig. 4.3.5

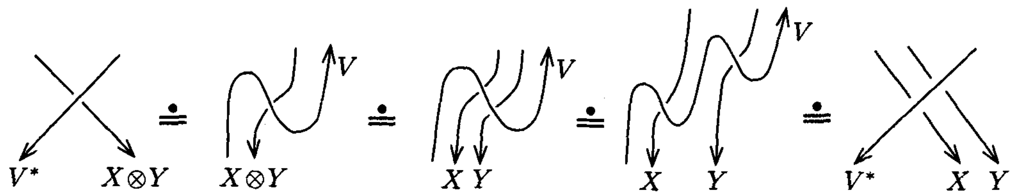


Fig. 4.3.6

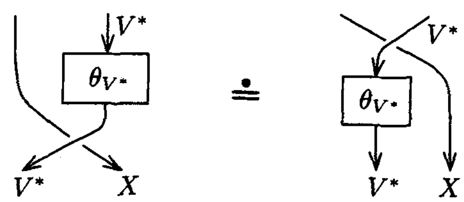


Fig. 4.3.7

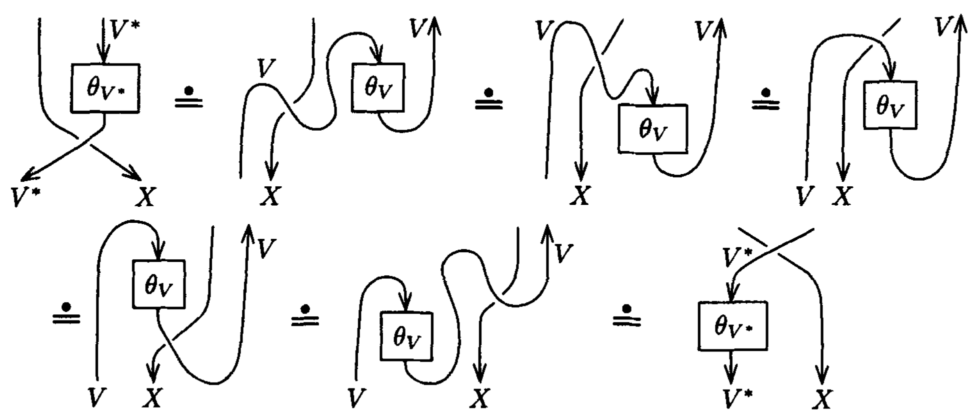


Fig. 4.3.8

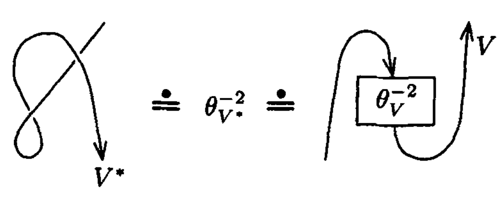


Fig. 4.3.9

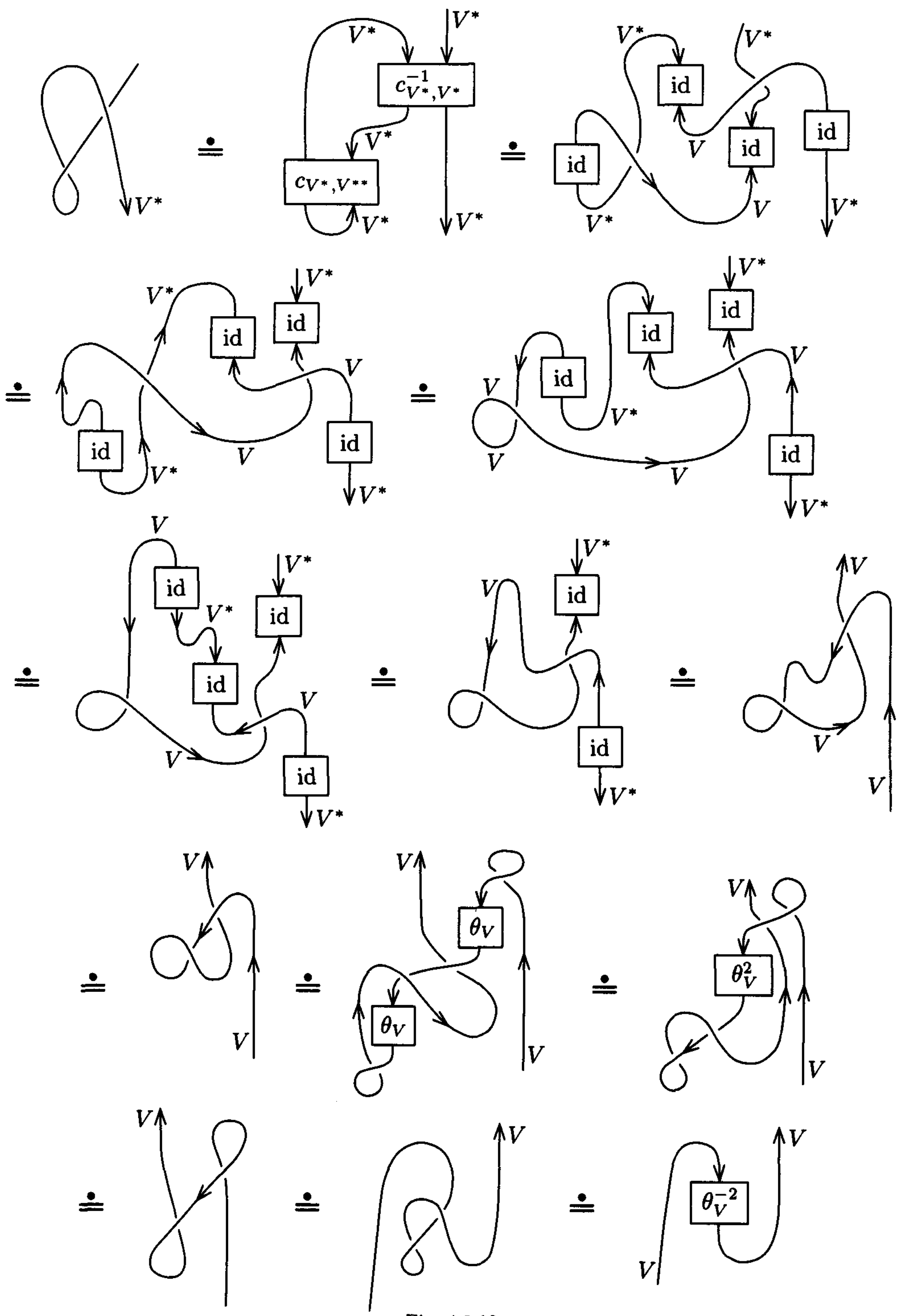


Fig. 4.3.10

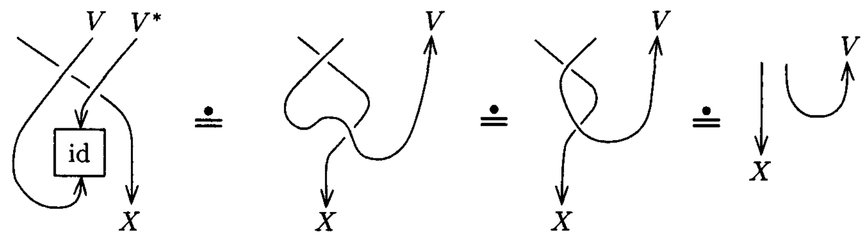


Fig. 4.3.11

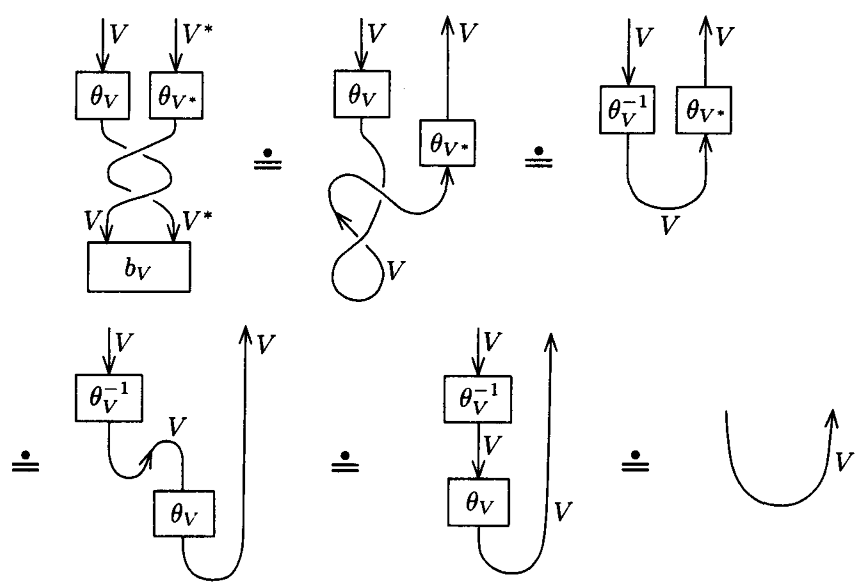


Fig. 4.3.12

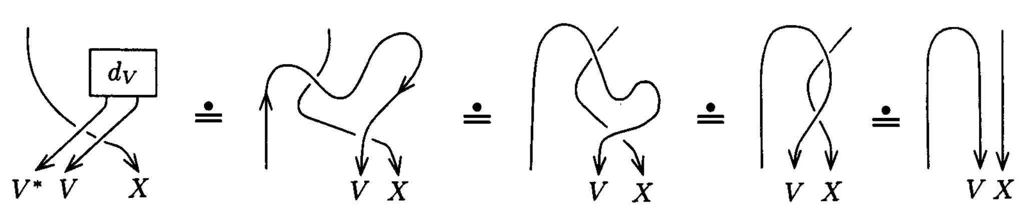


Fig. 4.3.13

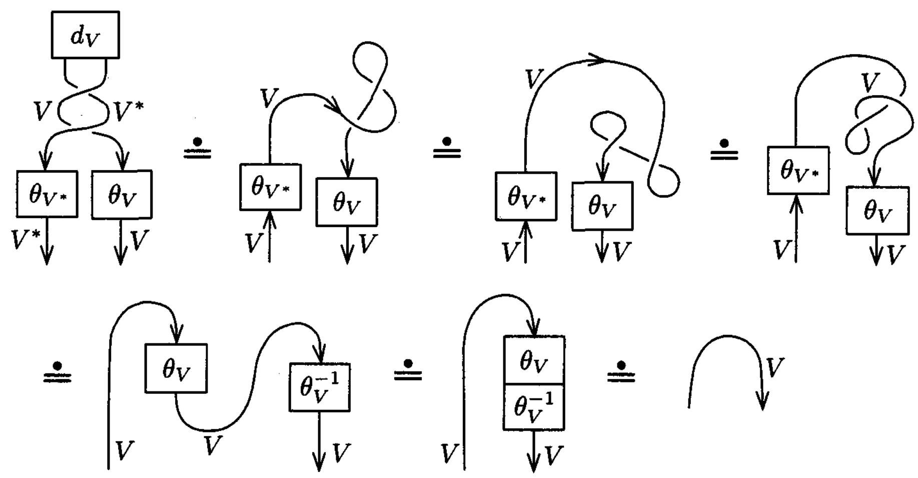


Fig. 4.3.14

4.4. Proof that $\mathcal{D}(\mathcal{C})$ is a ribbon category

Let $(V, c_{V,-}, \theta_V)$ and $(W, c_{W,-}, \theta_W)$ be objects in $\mathcal{D}(\mathcal{C})$.

LEMMA 4.4.1. *The morphism $c_{V,W}$ is a morphism in $\mathcal{D}(\mathcal{C})$.*

Proof. We have to check relations (2.2a) and (2.2b). For (2.2a), see Figure 4.4.1 where the middle equality uses Lemma 4.1.1. For (2.2b), see Figure 4.4.2 which uses Lemma 4.1.2. □

PROPOSITION 4.4.2. *The monoidal category $\mathcal{D}(\mathcal{C})$ is braided with braidings $c_{V,W}$.*

Proof. The morphism $c_{V,W}$ is invertible by definition and it is natural with respect to all morphisms of $\mathcal{C}$, hence to those belonging to $\mathcal{D}(\mathcal{C})$. In order for $c_{V,W}$ to qualify as a braiding, it has to satisfy both relations (1.3a) and (1.3b). Now the first one follows from the hypothesis (2.1a) and the other one by definition from (2.3a). □

We now show that $\mathcal{D}(\mathcal{C})$ has a twist. Let $(V, c_{V,-}, \theta_V)$ be an object of $\mathcal{D}(\mathcal{C})$.

LEMMA 4.4.3. *θ_V is a morphism in $\mathcal{D}(\mathcal{C})$.*

Proof. Relation (2.2a) for θ_V is (2.1b) whereas relation (2.2b) is obvious. □

End of proof of Theorem 2.3. The morphisms θ_V satisfy relations (1.4a) and (1.4b) by definition. So θ_V qualifies as a twist in $\mathcal{D}(\mathcal{C})$. Consequently, the latter is a ribbon category. □

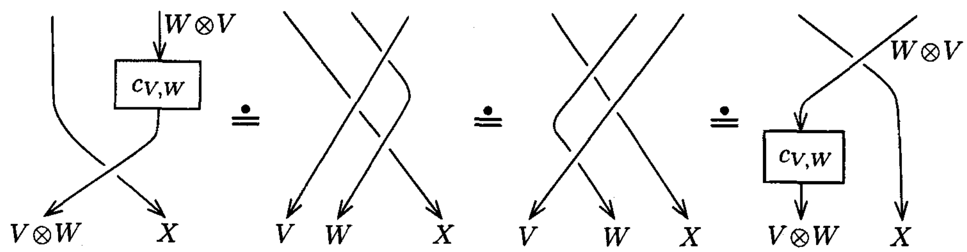


Fig. 4.4.1

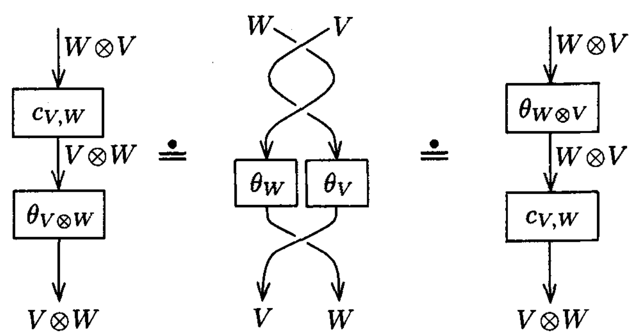


Fig. 4.4.2

5. Application to Hopf algebras

5.1. Categories of modules

Let $A=(A,\varphi,\eta,\Delta,\varepsilon,S)$ be a Hopf algebra over a field k. Here $\varphi\colon A\otimes A\to A$ is the multiplication, $\eta\colon k\to A$ the unit, $\Delta\colon A\to A\otimes A$ the comultiplication, $\varepsilon\colon A\to k$ the counit and $S\colon A\to A$ the antipode. We henceforth assume that S is an isomorphism.

It is well-known that the category $A-\mathrm{Mod}$ of left A-modules is a monoidal category, the tensor product $V\otimes W$ of two A-modules is $V\otimes W=V\otimes_k W$, equipped with the A-action given by

$$a(v\otimes w)=\Delta(a)(v\otimes w)=\sum_{(a)} a'v\otimes a''w \tag{5.1a}$$

for $a\in A$, $v\in V$ and $w\in W$. Here we use the Heynemann–Sweedler convention which expresses the comultiplication of an element a in A as

$$\Delta(a)=\sum_{(a)} a'\otimes a''.$$

Under this convention we have

$$(\Delta\otimes\mathrm{id}_A)\Delta(a)=(\mathrm{id}_A\otimes\Delta)\Delta(a)=\sum_{(a)} a'\otimes a''\otimes a'''.$$

The unit in $A-\text{Mod}$ is the trivial A-module $I=k$ on which A acts by

$$a1=\varepsilon(a) \tag{5.1b}$$

for all $a\in A$.

The category $A-\text{Mod}_f$ of left A-modules that are *finite-dimensional* over k is a monoidal subcategory of $A-\text{Mod}$. The category $A-\text{Mod}_f$ has left duality: if V is a left A-module, then V^* is the dual vector space of V over k with left A-action given by

$$\langle af,v\rangle=\langle f,S(a)v\rangle \tag{5.1c}$$

for $a\in A$, $v\in V$, and $f\in V^*$. The maps b_V and d_V are given by

$$b_V(1)=\sum_i v_i\otimes v^i \quad \text{and} \quad d_V(v^i\otimes v_j)=\langle v^i,v_j\rangle=\delta_{ij} \text{ (Kronecker symbol)} \tag{5.1d}$$

where $\{v_i\}_i$ is any basis of V and $\{v^i\}_i$ is the dual basis in V^*.

5.2. Quasitriangular Hopf algebras and the quantum double

According to Drinfeld [D1], a Hopf algebra A is *quasitriangular* if the monoidal category $A-\text{Mod}$ is braided or, equivalently, if there exists an invertible element R in $A\otimes A$, called the *universal R-matrix* of A, such that

$$\Delta^{\text{op}}(a)=R\Delta(a)R^{-1} \tag{5.2a}$$

for all $a\in A$ (here Δ^{op} is the opposite comultiplication) and

$$(\Delta\otimes\text{id}_A)(R)=R_{13}R_{23} \quad \text{and} \quad (\text{id}_A\otimes\Delta)(R)=R_{13}R_{12}. \tag{5.2b}$$

The equivalence between both definitions of quasitriangularity goes as follows. If $c_{V,W}$ denotes the braiding in $A-\text{Mod}$, then R is given by

$$R=(12)(c_{A,A}(1\otimes 1)) \tag{5.2c}$$

where (12) denotes the flip in $A\otimes A$. Conversely, given the universal R-matrix R, then the braiding in $A-\text{Mod}$ is given for all A-modules V,W by

$$c_{V,W}(v\otimes w)=(12)(R(v\otimes w)) \tag{5.2d}$$

where $v\in V$ and $w\in W$.

Now suppose that A is finite-dimensional over k with a basis $\{a_i\}_i$ and dual basis $\{a^i\}_i$. Drinfeld [D1] has defined a quasitriangular Hopf algebra $D(A)$, called the quantum double of A. It is constructed as follows. As a vector space $D(A)$ is identified with $A^*\otimes A$. For simplicity we shall denote an element $f\otimes a$ of $A^*\otimes A$ by fa. With this convention the multiplication of $D(A)$ is determined by the fact that the natural embeddings of A and A^* in $D(A)$ are morphisms of algebras and by the relation

$$af = \sum_{(a)} f(S^{-1}(a''')?a')a'' \tag{5.2e}$$

in $D(A)$ where $a\in A$ and $f\in A^*$. Here $f(S^{-1}(a''')?a')$ is the linear form on A determined by $\langle f(S^{-1}(a''')?a'), x\rangle = \langle f, S^{-1}(a''')xa'\rangle$. The comultiplication of $D(A)$ extends the comultiplication Δ of A and the comultiplication Δ of A^* defined by

$$\langle \Delta(f), a_1\otimes a_2\rangle = \langle f, a_2a_1\rangle \tag{5.2f}$$

for $f\in A^*$ and $a_1, a_2\in A$. The main property of the Hopf algebra $D(A)$ is that it is quasitriangular with universal R-matrix given by

$$R = \sum_i a_i\otimes a^i \in D(A)\otimes D(A). \tag{5.2g}$$

The element R is invertible. By [D2, Proposition 3.1], its inverse R^{-1} is given by

$$R^{-1} = \sum_i a_i\otimes(a^i\circ S) = \sum_i S(a_i)\otimes a^i. \tag{5.2h}$$

We borrow from [Y] the following concept (also called quantum Yang–Baxter module in [R]).

Definition 5.2.1. Under the previous hypotheses, a crossed A-bimodule is a k-vector space V equipped with linear maps

$$\varphi_V: A\otimes V\to V \quad \text{and} \quad \Delta_V: V\to V\otimes A$$

such that

(i) the map φ_V (resp. Δ_V) turns V into a left A-module (resp. into a right A-comodule) and

(ii) the diagram

$$\begin{array}{ccccccc}
A\otimes V & \xrightarrow{\Delta\otimes\Delta_V} & A\otimes A\otimes V\otimes A & \xrightarrow{\mathrm{id}_A\otimes(12)\otimes\mathrm{id}_A} & A\otimes V\otimes A\otimes A & \xrightarrow{\varphi_V\otimes\varphi} & V\otimes A \\
\downarrow{\scriptstyle \Delta\otimes\mathrm{id}_V} & & & & & & \uparrow{\scriptstyle \mathrm{id}_V\otimes\varphi} \\
A\otimes A\otimes V & \xrightarrow{\mathrm{id}_A\otimes\varphi_V} & A\otimes V & \xrightarrow{(12)} & V\otimes A & \xrightarrow{\Delta_V\otimes\mathrm{id}_A} & V\otimes A\otimes A
\end{array}$$

commutes.

If for $a \in A$ and $v \in V$ we write $\varphi_V(a \otimes v) = av$ and

$$\Delta_V(v) = \sum_{(v)} v_V \otimes v_A \in V \otimes A,$$

then the commutativity of the diagram in the previous definition is equivalent to

$$\sum_{(a)(v)} a' v_V \otimes a'' v_A = \sum_{(a)(v)} (a'' v)_V \otimes (a'' v)_A a' \tag{5.2i}$$

for all $a \in A$ and $v \in V$.

The crossed A-bimodules form a category in which a morphism is a linear map commuting with the actions and the coactions. We relate crossed A-bimodules with the quantum double $D(A)$.

PROPOSITION 5.2.2. *If A is a finite-dimensional Hopf algebra, then the category $D(A)-\mathrm{Mod}$ is equivalent to the category of crossed A-bimodules.*

Proof (taken from [K, IX.5]). (a) Let V be a left module over $D(A)$. Let us show that V can be endowed with a crossed bimodule structure. By definition of $D(A)$, the space V is a left A-module as well as a left A^*-module such that for any $a \in A$, $f \in A^*$, and $v \in V$ we have

$$a(fv) = \sum_{(a)} f(S^{-1}(a''')?a')(a''v). \tag{5.2j}$$

Given a basis $\{a_i\}_i$ of A and its dual basis $\{a^i\}_i$, we use the left action of A^* on V to define a map $\Delta_V : V \to V \otimes A$ by

$$\Delta_V(v) = \sum_i a^i v \otimes a_i. \tag{5.2k}$$

Let us show that this defines a right coaction of A on V. We have to check that Δ_V is coassociative and counitary. Rather than verify this directly, we observe that Δ_V is the transpose of the (unitary, associative) right action $V^* \otimes A^* \to V^*$ of A^* on the dual vector space V^* given by

$$\langle \alpha f, v \rangle = \langle \alpha, fv \rangle$$

for $\alpha \in V^*$, $v \in V$, and $f \in A^*$. Indeed, we have

$$\begin{aligned} \langle \alpha \otimes f, \Delta_V(v) \rangle &= \sum_i \alpha(a^i v) f(a_i) \\ &= \left\langle \alpha, \left(\sum_i f(a_i) a^i \right) v \right\rangle \\ &= \langle \alpha, fv \rangle = \langle \alpha f, v \rangle. \end{aligned}$$

Incidentally, it proves that Δ_V is independent of the chosen basis of A.

In order to complete the proof that V is a crossed A-bimodule, we have to check relation (5.2i) using (5.2j) and (5.2k). Let $a \in A$, $v \in V$, and $f \in A^*$. Then

$$\begin{aligned}
(\mathrm{id}\otimes f)\Big(\sum_{(a)(v)} a' v_V \otimes a'' v_A\Big) &= (\mathrm{id}\otimes f)\Big(\sum_{(a),i} a'(a^i v)\otimes a'' a_i\Big) \\
&= \sum_{(a),i} a'(a^i v) f(a'' a_i) = \sum_{(a)(f),i} f'(a_i) f''(a'') a'(a^i v) \\
&= \sum_{(a)(f)} f''(a'') a'\Big(\Big(\sum_i f'(a_i) a^i\Big) v\Big) = \sum_{(a)(f)} f''(a'') a'(f' v) \\
&= \sum_{(a)(f)} f''(a'''') f'(S^{-1}(a''')?a')(a'' v) \\
&= \sum_{(a)} f(a'''' S^{-1}(a''')?a')(a'' v) \\
&= \sum_{(a)} \varepsilon(a''') f(?a')(a'' v) = \sum_{(a)} f(?a')(a'' v) \\
&= \sum_{(a)(f)} f'(a') f''(a'' v) = \sum_{(a)(f),i} a^i(a'' v) f''(a_i) f'(a') \\
&= \sum_{(a),i} a^i(a'' v) f(a_i a') = (\mathrm{id}\otimes f)\Big(\sum_{(a),i} a^i(a'' v)\otimes a_i a'\Big) \\
&= (\mathrm{id}\otimes f)\Big(\sum_{(a)(v)} (a'' v)_V \otimes (a'' v)_A a'\Big).
\end{aligned}$$

This implies (5.2i). In the previous series of equalities, we used the comultiplication on A^*, the fact that S^{-1} is a skew antipode, that ε is a counit, relations (5.2j)–(5.2k) and the fact that $f = \sum_i f(a_i) a^i$.

(b) Conversely, let V be a crossed A-bimodule. We now show that V can be given a $D(A)$-module structure. Observe that if $(V, \Delta_V : V \to V \otimes A)$ is a right A-comodule, then V becomes a left module over the dual algebra A^* by

$$A^* \otimes V \xrightarrow{\mathrm{id}_A \otimes \Delta_V} A^* \otimes V \otimes A \xrightarrow{(23)} A^* \otimes A \otimes V \xrightarrow{\mathrm{ev}_A} V$$

where ev_A is the evaluation map. In other words a linear form $f \in A^*$ acts on an element $v \in V$ by

$$f \cdot v = \sum_{(v)} \langle f, v_A \rangle v_V. \qquad (5.2l)$$

In view of this observation, we see that a crossed bimodule has a left A-action as well as a left A^*-action. In order to prove V is a $D(A)$-module, it is enough to check relation

(5.2j). We have

$$\sum_{(a)} f(S^{-1}(a''')?a')\cdot(a''v) = \sum_{(a)(v)} \langle f, S^{-1}(a''')(a''v)_A a'\rangle (a''v)_V$$
$$= \sum_{(a)(v)} \langle f, S^{-1}(a''')a''v_A\rangle a'v_V = \sum_{(a)(v)} \varepsilon(a'')\langle f, v_A\rangle a'v_V$$
$$= \sum_{(v)} \langle f, v_A\rangle a v_V = a(f\cdot v).$$

The second equality is a consequence of (5.2i). The third one follows from the fact that S^{-1} is a skew-antipode.

Now it is easy to conclude. □

5.3. Ribbon algebras

Let D be a quasitriangular Hopf algebra with universal R-matrix

$$R = \sum_i s_i \otimes t_i \in D\otimes D.$$

Set

$$u = \sum_i S(t_i)s_i. \tag{5.3a}$$

In [D2] it is shown that u is an invertible element of D with inverse

$$u^{-1} = \sum_i t_i S^2(s_i) = \sum_i S^{-2}(t_i)s_i, \tag{5.3b}$$

that $uS(u)=S(u)u$ is central in D, and that we have the following relations:

$$\varepsilon(u)=1 \quad \text{and} \quad \Delta(u) = (R_{21}R)^{-1}(u\otimes u). \tag{5.3c}$$

Moreover, the square of the antipode is given for any x in D by

$$S^2(x) = uxu^{-1}. \tag{5.3d}$$

A quasitriangular Hopf algebra D is a *ribbon algebra* in the sense of Reshetikhin–Turaev [RT] if there exists a central element θ in D satisfying the following relations:

$$\theta^2 = uS(u), \quad S(\theta)=\theta, \quad \varepsilon(\theta)=1, \quad \text{and} \quad \Delta(\theta) = (R_{21}R)^{-1}(\theta\otimes\theta). \tag{5.3e}$$

The main property of a ribbon algebra D is that the braided monoidal category with left duality $D-\mathrm{Mod}_f$ is a ribbon category in the sense of §1.4 with twist θ_V given on any D-module V as the multiplication by the central element θ.

The following ribbon algebra $D(\theta)$ has been associated by [RT] to any quasitriangular Hopf algebra D. As an algebra, $D(\theta)$ is the quotient of the polynomial algebra $D[\theta]$ by the two-sided ideal generated by $\theta^2 - uS(u)$. We still denote by θ the class in $D(\theta)$ of the indeterminate θ. The Hopf algebra structure on $D(\theta)$ is uniquely determined by the requirements that the natural inclusion of D into $D(\theta)$ is a Hopf algebra map and that

$$\Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta), \quad \varepsilon(\theta) = 1, \quad \text{and} \quad S(\theta) = \theta.$$

The following proposition characterizes $D(\theta)$-modules.

PROPOSITION 5.3.1. *Under the previous hypotheses, the category of left $D(\theta)$-modules is equivalent to the category whose objects are pairs (V, θ_V) where V is a left D-module and θ_V is a D-linear automorphism of V such that for all v in V we have*

$$\theta_V^2(v) = uS(u)v, \tag{5.3f}$$

and whose morphisms $(V, \theta_V) \to (W, \theta_W)$ are the D-linear f maps from V to W such that

$$f\theta_V = \theta_W f. \tag{5.3g}$$

Proof. (a) On any $D(\theta)$-module V we define θ_V as the multiplication by θ. Since θ is central and invertible in $D(\theta)$, the map θ_V is a D-linear automorphism satisfying relation (5.3f). If $f: V \to W$ is $D(\theta)$-linear, then f commutes with θ, hence it satisfies relation (5.3g).

(b) Conversely, let (V, θ_V) be a pair as in the proposition. We give V a $D(\theta)$-module structure by setting

$$\theta v = \theta_V(v).$$

This makes sense in view of relation (5.3f). The rest follows easily.

5.4. Determining the category $\mathcal{D}(A-\mathrm{Mod}_f)$

We state the main result of §5.

THEOREM 5.4.1. *Let A be a finite-dimensional Hopf algebra with an invertible antipode. Then*

(i) $\mathcal{Z}(A-\mathrm{Mod})$ *and* $D(A)-\mathrm{Mod}$ *are equivalent braided monoidal categories,*

(ii) $\mathcal{Z}(A-\mathrm{Mod}_f)$ *and* $D(A)-\mathrm{Mod}_f$ *are equivalent braided monoidal categories, and*

(iii) $\mathcal{D}(A-\mathrm{Mod}_f)$ *and* $D(A)(\theta)-\mathrm{Mod}_f$ *are equivalent ribbon categories.*

The $\mathcal{Z}$-construction was recalled after the statement of Theorem 2.3. According to [Mj], part (i) is due to Drinfeld (unpublished). The rest of this section is devoted to the proof of this theorem. We first relate $\mathcal{Z}(A-\mathrm{Mod})$ and $D(A)-\mathrm{Mod}$. Let us start with two lemmas.

LEMMA 5.4.2. *Let $(V, c_{V,-})$ be an object of $\mathcal{Z}(A-\mathrm{Mod})$ and Δ_V the linear map from V to $V\otimes A$ defined for all $v\in V$ by $\Delta_V(v)=c_{V,A}^{-1}(1\otimes v)$. Then along with the given left A-module structure on V, the map Δ_V endows V with the structure of a crossed A-bimodule.*

Proof. Let $\Delta_V: V\to V\otimes A$ be defined as above. By convention we write for any $v\in V$

$$\Delta_V(v)=\sum_{(v)} v_V\otimes v_A \in V\otimes A. \tag{5.4a}$$

We call Δ_V the coaction of A on V.

The naturality of $c_{V,-}$, hence of $c_{V,-}^{-1}$, allows us to express $c_{V,X}^{-1}$ in terms of the coaction Δ_V for any A-module X. Indeed, given x in X and $\bar{x}: A\to X$ the unique A-linear map sending 1 to x, we have the following commutative square:

$$\begin{array}{ccc} A\otimes V & \xrightarrow{c_{V,A}^{-1}} & V\otimes A \\ {\scriptstyle \bar{x}\otimes \mathrm{id}_V}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_V\otimes \bar{x}} \\ X\otimes V & \xrightarrow{c_{V,X}^{-1}} & V\otimes X. \end{array}$$

It implies that for any $v\in V$ and $x\in X$ we have

$$c_{V,X}^{-1}(x\otimes v)=\Delta_V(v)(1\otimes x)=\sum_{(v)} v_V\otimes v_A x. \tag{5.4b}$$

Let us show that the coaction Δ_V is coassociative. By (2.1a) we have

$$\begin{aligned} c_{V,X\otimes Y}^{-1}(x\otimes y\otimes v) &= \sum_{(v)} v_V\otimes (v_A)'x\otimes (v_A)''y \\ &= (c_{V,X}^{-1}\otimes \mathrm{id}_Y)(\mathrm{id}_X\otimes c_{V,Y}^{-1})(x\otimes y\otimes v)=\sum_{(v)} (v_V)_V\otimes (v_V)_A x\otimes v_A y. \end{aligned}$$

Setting $X=Y=A$ and $x=y=1$ implies

$$\sum_{(v)} v_V\otimes (v_A)'\otimes (v_A)''=\sum_{(v)} (v_V)_V\otimes (v_V)_A\otimes v_A,$$

which expresses the coassociativity of Δ_V.

We also have $c_{V,k}=\mathrm{id}_V$ because $k=I$ is the unit in the tensor category of k-modules. This implies

$$c_{V,k}^{-1}(1\otimes v)=\sum_{(v)} \varepsilon(v_A)v_V=v$$

for all $v \in V$. This means that the coaction Δ_V is counitary. So far we have proved that the coaction Δ_V equips V with a structure of right A-comodule.

Let us express the fact that $c_{V,X}$ is A-linear. For $a \in A$, $v \in V$, and $x \in X$ we have

$$c_{V,X}^{-1}(a(x \otimes v)) = a c_{V,X}^{-1}(x \otimes v).$$

Replacing $c_{V,X}^{-1}$ by its expression in Δ_V, we get

$$\Delta(a)\Delta_V(v)(1 \otimes x) = \Big(\sum_{(a)} \Delta_V(a''v)(1 \otimes a')\Big)(1 \otimes x).$$

Setting $X = A$ and $x = 1$ we have

$$\sum_{(a)(v)} a'v_V \otimes a''v_A = \sum_{(a)(v)} (a''v)_V \otimes (a''v)_A a',$$

which is relation (5.2i).

By Proposition 5.2.2, we know that V is a left $D(A)$-module. Let $R = \sum_i a_i \otimes a^i$ be the universal R-matrix of $D(A)$. Let us express the braiding in the braided monoidal category $\mathcal{Z}(A-\mathrm{Mod})$ in terms of R.

LEMMA 5.4.3. *Under the previous hypotheses, if* $(V, c_{V,-})$ *is an object of* $\mathcal{Z}(A-\mathrm{Mod})$ *and* X *is an* A*-module, then the braiding* $c_{V,X}$ *is determined by*

$$c_{V,X}^{-1}(x \otimes v) = (12)(R(x \otimes v))$$

for all $x \in X$ *and* $v \in V$.

Proof. By relations (5.4b) and (5.2l) we have

$$\begin{aligned} c_{V,X}^{-1}(x \otimes v) &= \sum_{(v)} v_V \otimes v_A x = \sum_{(v),i} \langle a^i, v_A \rangle v_V \otimes a_i x \\ &= \sum_{(v),i} a^i \cdot v \otimes a_i x = (12)(R(x \otimes v)). \end{aligned}$$

Proof of part (i) *of Theorem* 5.4.1. It will serve as a model for the proof of part (iii).

(1) We first define a functor F from $\mathcal{Z}(A-\mathrm{Mod})$ to $D(A)-\mathrm{Mod}$. Let $(V, c_{V,-})$ be an object of $\mathcal{Z}(A-\mathrm{Mod})$. By Lemma 5.4.2 and Proposition 5.2.2, the vector space $F(V, c_{V,-}) = V$ is a left $D(A)$-module. If f is a map in $\mathcal{Z}(A-\mathrm{Mod})$, then (2.2a) shows that f is a map of A-comodules, hence of A^*-modules. Consequently f is $D(A)$-linear. This defines F as a faithful functor.

(2) Let us show that F preserves the tensor products. The tensor product of $(V, c_{V,-})$ and of $(W, c_{W,-})$ is $(V\otimes W, c_{V\otimes W,-})$ where $c_{V\otimes W,-}$ is determined by

$$c^{-1}_{V\otimes W,A} = (\mathrm{id}_V \otimes c^{-1}_{W,A})(c^{-1}_{V,A}\otimes \mathrm{id}_W).$$

Therefore the coaction on $V\otimes W$ is given by

$$\Delta_{V\otimes W}(v\otimes w) = \sum_{(v)(w)} v_V \otimes w_W \otimes w_A v_A.$$

By (5.2l) the action of a linear form f on a tensor $v\otimes w$ in $V\otimes W$ is expressed as

$$f\cdot(v\otimes w) = \sum_{(v)(w)} \langle f, w_A v_A\rangle v_V \otimes w_W,$$

which, by definition of the comultiplication Δ of A^* (see (5.2f)), is equal to

$$\sum_{(v)(w)} \langle \Delta(f), v_A \otimes w_A\rangle v_V \otimes w_W = \Delta(f)\cdot(v\otimes w).$$

Therefore the $D(A)$-action on $V\otimes W$ is given by

$$(af)(v\otimes w) = \Delta(a)(\Delta(f)\cdot(v\otimes w)) = \Delta(af)(v\otimes w),$$

which is exactly the action given by the comultiplication in the quantum double $D(A)$.

(3) By definition of the braiding in $\mathcal{Z}(A-\mathrm{Mod})$, Lemma 5.4.3 can be reinterpreted as

$$F(c^{-1}_{V,W})(w\otimes v) = (12)(R(w\otimes v)),$$

which is the braiding in the category of $D(A)$-modules. Thus F intertwines the braiding of $\mathcal{Z}(A-\mathrm{Mod})$ and the opposite braiding of $D(A)-\mathrm{Mod}$.

(4) Suppose that V is a left $D(A)$-module. For any A-module X define $c_{V,X}$ by

$$c^{-1}_{V,X}(x\otimes v) = (12)(R(x\otimes v))$$

where $v\in V$ and $x\in X$. This is a well-defined natural isomorphism since R is invertible. Let us prove that it is A-linear. For $a\in A$ we have

$$\begin{aligned} c^{-1}_{V,X}(a(x\otimes v)) &= (12)(R\Delta(a)(x\otimes v)) = (12)(\Delta^{\mathrm{op}}(a)R(x\otimes v)) \\ &= \Delta(a)(12)(R(x\otimes v)) = ac^{-1}_{V,X}(x\otimes v) \end{aligned}$$

in view of relation (5.2a).

We have to check relation (2.1a), namely

$$c^{-1}_{V,X\otimes Y}(x\otimes y\otimes v)=(c^{-1}_{V,X}\otimes \mathrm{id}_Y)((\mathrm{id}_X\otimes c^{-1}_{V,Y})(x\otimes y\otimes v)).$$

The left-hand side is equal to

$$(13)((\Delta\otimes \mathrm{id}_A)(R)(x\otimes y\otimes v))$$

whereas the right-hand side is equal to

$$(13)(R_{13}R_{23}(x\otimes y\otimes v)).$$

Both are equal in view of (5.2b). This construction defines an object $G(V)=(V,c_{V,-})$ in $\mathcal{Z}(A-\mathrm{Mod})$.

Let $f\colon V\to W$ be a map of $D(A)$-modules. We have to check that $G(f)=f$ is a morphism in $\mathcal{Z}(A-\mathrm{Mod})$. First, it is A-linear since it is $D(A)$-linear. Next, we have to check relation (2.2a). Now

$$c^{-1}_{V,X}(\mathrm{id}_X\otimes f)(x\otimes v)=(12)(R(x\otimes f(x)))=(12)((\mathrm{id}_X\otimes f)(R)(x\otimes v))=(f\otimes \mathrm{id}_X)c^{-1}_{V,X}.$$

(5) Clearly, $FG=\mathrm{id}$ whereas $GF=\mathrm{id}$ follows from Lemma 5.4.3. This shows the equivalence of $\mathcal{Z}(A-\mathrm{Mod})$ and of $D(A)-\mathrm{Mod}$. This ends the proof of part (i).

Part (ii) is proved similarly. Before we prove part (iii) of Theorem 5.4.1, we need two more technical results.

Let A be a finite-dimensional Hopf algebra. As recalled above, its quantum double $D(A)$ is quasitriangular with universal R-matrix described by (5.2g). The first result concerns the element $u\in D(A)$ defined by (5.3a).

LEMMA 5.4.4. *We have*

$$u^{-1}=\sum_i S^{-1}(a^i)S(a_i).$$

Proof. According to (5.2g) and to (5.3b) we have

$$u^{-1}=\sum_i a^i\otimes S^2(a_i)$$

after identification of $D(A)$ with $A^*\otimes A$. On the other hand, using the same identification, we see that the right-hand side of the identity in Lemma 5.4.4 is equal to

$$u'=\sum_i S^{-1}(a^i)\otimes S(a_i).$$

Let us evaluate u^{-1} and u' on $a\otimes f$ where a belongs to A and f to A^*. An immediate computation shows that

$$\begin{aligned}\langle u^{-1}, a\otimes f\rangle &= \sum_i \langle a^i, a\rangle\langle f, S^2(a_i)\rangle = \langle f, S^2(a)\rangle \\ &= \sum_i \langle a^i, S(a)\rangle\langle f, S(a_i)\rangle \\ &= \sum_i \langle S^{-1}(a^i), a\rangle\langle f, S(a_i)\rangle = \langle u', a\otimes f\rangle.\end{aligned}$$

The fourth equality results from the fact that the antipode on A^* is the transpose of S^{-1}.

The next result deals with an arbitrary object $(V, c_{V,-}, \theta_V)$ of the ribbon category $\mathcal{D}(A-\mathrm{Mod}_f)$. From Lemma 5.4.2 and Proposition 5.2.2 we know that V has the structure of a module over $D(A)$. Let us determine the square of θ_V.

LEMMA 5.4.5. *For any v in V we have*

$$\theta_V^2(v) = uS(u)v.$$

Proof. Recall from relations (5.2g) and (5.2h) and from Lemma 5.4.3 that

$$c_{V,X}(v\otimes x) = \sum_j S(a_j)x\otimes a^j v \quad\text{and}\quad c_{V,X}^{-1}(x\otimes v) = \sum_j a^j v\otimes a_j x.$$

Now if $\{v_i\}_i$ is a basis of V and $\{v^i\}_i$ is the dual basis of V^*, then from relation (2.1c), namely from

$$\theta_V^{-2} = (d_V\otimes \mathrm{id}_V)(\mathrm{id}_{V^*}\otimes c_{V,V}^{-1})(c_{V,V^*}\otimes \mathrm{id}_V)(b_V\otimes \mathrm{id}_V),$$

we derive the following expression:

$$\theta_V^{-2}(v) = \sum_{i,j,k} \langle S(a_j)v^i, a^k v\rangle\, a_k a^j v_i.$$

This can be rewritten as

$$\theta_V^{-2}(v) = \sum_{i,j,k} \langle v^i, S^2(a_j)a^k v\rangle a_k a^j v_i = \sum_{j,k} a_k a^j S^2(a_j)a^k v = \Big(\sum_k a_k u^{-1} a^k\Big)v$$

by (5.3b). Finally, from (5.3d) we get

$$\sum_k a_k u^{-1}a^k = u^{-1}\Big(\sum_k u a_k u^{-1} a^k\Big) = u^{-1}\Big(\sum_k S^2(a_k)a^k\Big).$$

Now the latter is equal to

$$u^{-1}S\Big(\sum_k S^{-1}(a^k)S(a_k)\Big),$$

which, according to Lemma 5.4.4, equals $u^{-1}S(u^{-1})=(S(u)u)^{-1}$.

Summing up, we see that θ_V^{-2} is the multiplication by the inverse of $S(u)u=uS(u)$, which proves the lemma.

Proof of part (iii) *of Theorem* 5.4.1. (1) We first define a functor F from $\mathcal{D}(A-\mathrm{Mod}_f)$ to $D(A)(\theta)-\mathrm{Mod}_f$. By part (i) we know that if $(V, c_{V,-}, \theta_V)$ is an object of $\mathcal{D}(A-\mathrm{Mod}_f)$, then V is a $D(A)$-module. Relation (2.1b) shows that the A-linear isomorphism θ_V is a map of A-comodules; hence it is $D(A)$-linear. Lemma 5.4.5 and Proposition 5.3.1 imply that V is actually a $D(A)(\theta)$-module. If f is a morphism in $\mathcal{D}(A-\mathrm{Mod}_f)$, then it is $D(A)$-linear again by the results of part (i). We invoke (2.2b) and Proposition 5.3.1 to deduce that f is $D(A)(\theta)$-linear.

(2) Suppose that V is a finite-dimensional left $D(A)(\theta)$-module. We define $c_{V,-}$ as in part (4) of the proof of part (i) and θ_V as the multiplication by θ. We have to check that $(V, c_{V,-}, \theta_V)$ is an object of $\mathcal{D}(A-\mathrm{Mod}_f)$.

We have already showed that $c_{V,X}$ is an A-linear natural isomorphism and that θ_V is an A-linear automorphism. Also relation (2.1a) has been checked in the proof of part (i).

Relation (2.1b) is equivalent to the fact that θ_V is $D(A)$-linear. The latter is implied by the centrality of θ in $D(A)(\theta)$.

Relation (2.1c) is a consequence of Lemmas 5.4.3 and 5.4.5.

This defines a map G from the finite-dimensional $D(A)(\theta)$-modules to the objects of $\mathcal{D}(A-\mathrm{Mod}_f)$. This map extends easily to a functor inverse to F.

It remains to check that F preserves the ribbon category structures.

(3) For the monoidal structure we refer to part (2) of the proof of part (i) and to the relation $\Delta(\theta)=(R_{21}R)^{-1}(\theta\otimes\theta)$ of (5.3e).

(4) We have to check that the duality coincides on $\mathcal{D}(A-\mathrm{Mod}_f)$ and $D(A)(\theta)-\mathrm{Mod}_f$. Let $(V, c_{V,-}, \theta_V)$ be an object of $\mathcal{D}(A-\mathrm{Mod}_f)$ and $(V^*, c_{V^*,-}, \theta_V^*)$ its left dual. By Lemma 5.4.3, the corresponding coaction Δ_{V^*} is given for any $\alpha\in V^*$ by

$$\Delta_{V^*}(\alpha) = c_{V^*,A}^{-1}(1\otimes\alpha) = (12)(R(1\otimes\alpha)) = \sum_i a^i\alpha\otimes a_i = \sum_i \alpha(S(a^i)?)\otimes a_i.$$

By (5.2l) the action of a linear form $f\in A^*$ on $\alpha\in V^*$ is given by

$$f\alpha = \sum_i f(a_i)\alpha(S(a^i)?) = \alpha(S(f)?),$$

which turns out to coincide with the action of $f\in D(A)$ on α.

(5) The compatibility of the braidings has been checked in the proof of part (i). As for the twists, it results from Theorem 2.3 (iv) and the definition of $D(A)(\theta)$.

This concludes part (iii) of Theorem 5.4.1.

Remark 5.4.6. The natural embeddings $A \subset D(A) \subset D(A)(\theta)$ of Hopf algebras induce a monoidal functor $D(A)(\theta) - \mathrm{Mod}_f \to A - \mathrm{Mod}_f$. It is easy to check that it corresponds to the functor

$$\Pi: \mathcal{D}(A - \mathrm{Mod}_f) \to A - \mathrm{Mod}_f$$

of §2.4 under the equivalence of Theorem 5.4.1 (iii).

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