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On stability of exterior stationary Navier–Stokes flows

by

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Dedicated to Professor Fumi-Yuki Maeda on his 60th birthday

1. Introduction

In an exterior domain Ω of $\mathbf{R}^n$, $n\geqslant 3$, with smooth boundary S we consider the following stationary problem for the Navier–Stokes equations:

$$\begin{gathered} -\Delta w+w\cdot\nabla w+\nabla p=f \quad (x\in\Omega), \\ \nabla\cdot w=0 \quad (x\in\Omega), \\ w|_S=w^*, \quad w\to 0 \ \ (|x|\to\infty), \end{gathered} \tag{1.1}$$

with a (smooth) prescribed boundary data w^* and a smooth external force f of the form

$$f=(f_1,\dots,f_n), \quad f_j=\sum_{k=1}^{n}\partial_k F_{kj}, \quad \partial_j=\partial/\partial x_j. \tag{1.2}$$

Here $w=(w_1,\dots,w_n)$ and p denote, respectively, unknown velocity and pressure; and

$$\nabla=(\partial_1,\dots,\partial_n), \quad \Delta=\sum_{j=1}^{n}\partial_j^2, \quad \nabla\cdot u=\sum_{j=1}^{n}\partial_j u_j, \quad u\cdot\nabla u=\sum_{j=1}^{n}u_j\partial_j u.$$

By an *exterior domain* we mean a connected open set with compact complement.

As shown in the next section, problem (1.1) possesses a solution w satisfying

$$|w|\leqslant C/|x|, \quad |\nabla w|\leqslant C/|x|^2, \tag{1.3}$$

under appropriate assumptions on given data. In this paper we are interested in the stability property of the solutions w as mentioned above. To be more precise, consider the nonstationary problem

$$\begin{gathered}\frac{\partial v}{\partial t}+v\cdot\nabla v-\Delta v+\nabla q=f \quad (t>0,\ x\in\Omega),\\ \nabla\cdot v=0 \quad (t\geqslant 0,\ x\in\Omega),\\ v|_S=w^*,\quad v\to 0\ \ (|x|\to\infty),\quad v|_{t=0}=v_0.\end{gathered}\tag{1.4}$$

Inserting $v=w+u$, $v_0=w+a$ and $q=p+p'$ into (1.4) we obtain the equations governing the perturbation u:

$$\begin{gathered}\frac{\partial u}{\partial t}-\Delta u+w\cdot\nabla u+u\cdot\nabla w+u\cdot\nabla u+\nabla p'=0 \quad (t>0,\ x\in\Omega),\\ \nabla\cdot u=0 \quad (t\geqslant 0,\ x\in\Omega),\\ u|_S=0,\quad u\to 0\ \ (|x|\to\infty),\quad u|_{t=0}=a.\end{gathered}\tag{1.5}$$

Under some smallness assumptions on w, problem (1.5) was studied by [6], [20]–[22], [31] and [34]. Inspired by the works [20], [21] of Heywood, Masuda [31] treated (1.5) in the case where $n=3$ and $w\to w^\infty\in\mathbf{R}^3$ as $|x|\to\infty$, and deduced an algebraic decay rate in time of L^∞-norm of a weak solution u. Heywood [22] then improved the decay result of [31]. Miyakawa and Sohr [34] also studied (1.5), assuming $n=3$ and $w\to w^\infty\in\mathbf{R}^3$, and proved that any weak solution u satisfying the strong energy inequality goes to 0 in L^2-norm as $t\to\infty$. In [6] the present authors also studied weak solutions and deduced an algebraic decay rate in time of L^2-norm in case $w^\infty=0$ and a logarithmic decay rate in case $w^\infty\neq 0$.

When $n=3$, it is known that a weak solution satisfying the strong energy inequality becomes a strong (i.e. regular) solution after a finite time. Heywood [22] and Masuda [31] used this fact in deducing decay rates of L^∞-norm of weak solutions. However, it is not yet clear whether the L^∞-decay result of [22], [31] is optimal, mainly because of lack of knowledge on the behavior of the derivative ∇w of the stationary solution w. Kozono and Ogawa [25] and Chen [11] have recently treated problem (1.5), assuming that $w\in L^n$ and $\nabla w\in L^{n/2}$ are small enough, and discussed asymptotic behavior of weak solutions ([11]) and strong solutions ([25]) of problem (1.5). However, their assumption: $\nabla w\in L^{n/2}$ seems too restrictive in case $n=3$. In fact, Galdi and Padula [16] shows that when $n=3$, such a stationary flow w exists only in some unrealistic situations; and Kozono and Sohr [27] shows that if $n=3$ and if w is a weak solution of (1.1) constructed as in Leray [29], then ∇w is in $L^{3/2}$ if and only if the total net force exerted to the obstacle $\mathbf{R}^3\setminus\Omega$ by the flow (w,p) and the external force f vanishes:

$$\int_S \nu\cdot(T[w,p]-w^*\otimes w^*+F)\,dS=0,\tag{1.6}$$

where ν is the unit outward normal to S and

$$T[w,p]=(T_{jk}[w,p])_{j,k=1}^{n},\quad T_{jk}[w,p]=-\delta_{jk}p+\partial_j w_k+\partial_k w_j.$$

As shown in §2, the main reason for this difficulty in three dimensions lies in the fact that if $n=3$, then the dual exponent $n'=n/(n-1)$ of n equals $\frac{3}{2}=\frac{1}{2}n$, which is just the exponent of required integrability of ∇w.

To overcome the above-mentioned difficulty in three dimensions, we have to find out the right behavior of ∇w. In §2 we establish our existence result of a stationary flow w, with the aid of the recent results of Deuring and Varnhorn [12] and Wiegner [47] on Schauder estimates for the boundary layer potentials, and the estimate of Novotny and Padula [35] for the volume potentials. Novotny and Padula [35] gave a complete proof of the existence of a stationary flow satisfying (1.3) in a different functional setting, while we had proved similar results for $n\geqslant 4$ and the result

$$|w|\leqslant C/|x|,\quad \nabla w\in L_w^{n/2}\cap L^\infty\quad (n\geqslant 3), \tag{1.3'}$$

where L_w^p denotes the weak L^p spaces. This last result is actually equivalent to estimate (1.3) from the point of view of stability theory, and our results in §§ 3–6 all hold even if we replace (1.3) by (1.3′). However, we shall give in §2 a complete proof of (1.3) within the framework of our functional setting, employing a result (see Lemma 2.2 in §2) from [35], only in order to clarify the situation and to simplify the subsequent presentation. The result of [35] contains additional information, especially on the behavior of the associated pressure, and will be published elsewhere. Estimate (1.3) not only extends the result of Finn to higher space dimensions, but also provides the estimate: $|\nabla w|\leqslant C|x|^{-2}$ in all dimensions $n\geqslant 3$, while Finn [13] deduces only a weaker estimate: $|\nabla w|\leqslant C|x|^{-2}\log|x|$ in case $n=3$. We further show that if $n=3$, $w\in L^3$ and $F\in L^{3/2}$, then our solution w has to satisfy (1.6) provided $\nabla w\in L^{3/2}$. In this sense, the result (1.3) for ∇w seems to be optimal when $n=3$.

After establishing our existence result for problem (1.1), we discuss the existence and asymptotic behavior of solutions of perturbation equation (1.5). We first consider in §3 the linearization of problem (1.5):

$$\begin{gathered}\frac{\partial u}{\partial t}-\Delta u+w\cdot\nabla u+u\cdot\nabla w+\nabla p'=0\quad (t>0,\ x\in\Omega),\\ \nabla\cdot u=0\quad (t\geqslant 0,\ x\in\Omega),\\ u|_S=0,\quad u\to 0\ \ (|x|\to\infty),\quad u|_{t=0}=a,\end{gathered}\tag{1.7}$$

and show that if w is small in an appropriate sense, the correspondence $a\mapsto u(t)$ defines a bounded analytic C_0 semigroup in general L^r spaces of solenoidal vector fields. We then apply simple perturbation arguments to the results of [4], [11], [23] to deduce various decay estimates in general L^r spaces for solutions of (1.7). These decay results will be effectively applied in §§4 and 6 to the study of weak and strong solutions of nonlinear problem (1.5).

§4 is devoted to the study of existence and asymptotic behavior of L^2 weak solutions of (1.5). Based on the results of §3, we show that if w is small, then problem (1.5) possesses for every $a \in L^2$ at least one weak solution whose L^2-norm goes to 0 as $t \to \infty$. We further deduce explicit decay rates for the weak solutions under additional smallness assumptions on w, applying the results of §3. The decay result in §4 improves, in the case $w^\infty = 0$, our previous result obtained in [6]. We prove the result of §4 by employing a spectral method, which was initiated in the case of the Cauchy problem by Schonbek [39], [40] and Wiegner [46] in terms of the Fourier transformation and then systematically developed by the works [3], [4], [5], [6], [24] of the present authors in an operator-theoretic formulation.

§6 studies strong solutions of (1.5) belonging to the weak L^n space: L^n_w. It is now well known in the Navier–Stokes theory that the space L^n is the basic space in which to find a strong solution of the Navier–Stokes equations. In other words, we have so far been able to construct global-in-time strong solutions of the Navier–Stokes equations only when the initial velocities are small in L^n. Since $L^n \subset L^n_w$, our result in §6 contains as a special case the known existence results of global strong solutions. The results in §6 are closely related to the work [25] of Kozono and Ogawa, which establishes the same type of existence results in L^n, local in time for general initial data and global in time for small initial data. Although our results provide only global solutions for small initial data, they not only generalize and improve the global existence results of [25] to weak L^r spaces, but also include decay results for L^∞-norm, which will be derived by employing an idea of Chen [11]. We further show that if $a \in L^n_w$, $\nabla a \in L^{n/2}_w$ and $a|_S = 0$, then the corresponding strong solution u also satisfies

$$u(t) \in L^n_w \quad \text{and} \quad \nabla u(t) \in L^{n/2}_w \quad \text{for all } t \geqslant 0. \tag{1.8}$$

Observe that $|x|^{-1} \in L^n_w$ and $|x|^{-2} \in L^{n/2}_w$, and so our stationary solutions w obtained in §2 always satisfy (1.8).

To find strong solutions satisfying (1.8), we need to examine the fractional powers of the Stokes operator in weak L^r spaces. This is the subject treated in §5. We introduce Lorentz spaces $L^{(r,q)}$, $1 < r < \infty$, $1 \leqslant q \leqslant \infty$, so that $L^r = L^{(r,r)}$ and $L^{(r,\infty)} = L^r_w$, and discuss properties of the Stokes operator in these spaces. In particular, we give a characterization of the domain of (the square root of) the Stokes operator in Lorentz spaces and apply it in §6 to finding global strong solutions of (1.5) satisfying (1.8).

The basic tool for proving our results is the so-called L^p-L^q estimates for the semigroup defined by solutions of problem (1.7). These estimates are first deduced in §3 by applying duality and perturbation arguments to the estimates for the Stokes semigroup as given in [4], [11], [23], and then extended in §6 to estimates in weak L^r spaces through an interpolation argument. We apply these L^p-L^q estimates to the proof of the stability results in §§4 and 6. In particular, it will be shown in §6 that they can be applied to improving, in the case $w^\infty=0$, the L^∞-decay result of [22], [31] for strong solutions of problem (1.5).

As will be shown in §2, when $n\geqslant 4$, problem (1.1) possesses a solution w satisfying

$$\nabla w \in L^r \cap L^\infty \quad \text{for some } n' < r < \tfrac{1}{2}n, \tag{1.9}$$

under appropriate assumptions on given data. It should be noticed here that our basic assumption (1.3) implies

$$\nabla w \in L^{n/2}_w \cap L^\infty,$$

which is apparently weaker than property (1.9). Based on a result of [11], we show in §§3, 4 and 6 that the time-decay properties of solutions of problems (1.5) and (1.7) can be improved if the stationary flows w satisfy (1.9). When $n=3$, we have to assume that

$$\nabla w \in L^r \cap L^\infty \quad \text{for some } 1 < r < \tfrac{3}{2}, \tag{1.10}$$

in order to get similar improvements in time-decay properties for nonstationary problems. However, condition (1.10) automatically implies $\nabla w \in L^{3/2}$ and so it turns out that we are dealing with stationary flows w satisfying the vanishing flux condition (1.6). We do not know if our time-decay rates for weak and strong solutions are optimal. We refer the reader to [40], [41] for the optimality in the case of the Cauchy problem with $w=0$.

When $w=0$, problem (1.5) is just the exterior nonstationary problem for the Navier–Stokes equations. Since the solution $w=0$ obviously satisfies (1.9) or (1.10), the above-mentioned improvements hold also for the weak and strong solutions of the Navier–Stokes equations. We note in particular that an L^∞ decay result given in §6 actually improves the known result of [26] for strong solutions of the Navier–Stokes equations.

2. Existence of stationary flows

We first collect some basic results on weak L^r spaces, which will be effectively applied throughout the paper. Let $1<r<\infty$. A measurable function f defined on a domain D of $\mathbf{R}^n$ is said to belong to $L^r_w = L^r_w(D)$ if and only if

$$\|f\|^*_{r,w} \equiv \sup_{t>0} t|D(|f|>t)|^{1/r} < +\infty,$$

where $D(|f|>t)=\{x\in D:|f(x)|>t\}$ and $|E|$ denotes Lebesgue measure of a measurable set E. It is easy to see that $L^r\subset L^r_w$, with estimate $\|f\|^*_{r,w}\leqslant\|f\|_r$ for $f\in L^r$, where $\|\cdot\|_r$ denotes L^r-norm, and that $L^r_w(D)\subset L^s_{\mathrm{loc}}(\bar{D})$ for any $1\leqslant s<r$ with continuous injection. Furthermore, f is in L^r_w if and only if

$$\|f\|_{r,w}\equiv\sup_E|E|^{-1+1/r}\int_E|f|\,dx<+\infty,$$

and L^r_w is a Banach space with norm $\|\cdot\|_{r,w}$. Indeed, we have

$$\|f\|^*_{r,w}\leqslant\|f\|_{r,w}\leqslant\frac{r}{r-1}\|f\|^*_{r,w}. \tag{2.1}$$

To show (2.1), suppose $f\in L^r_w$, so that

$$\lambda(t)\equiv|D(|f|>t)|\leqslant(\|f\|^*_{r,w})^r t^{-r}.$$

Then, for any $E\subset D$ with $0<|E|<+\infty$, the function $\lambda_E(t)=|E(|f|>t)|$ satisfies

$$\lambda_E(t)\leqslant\min(\lambda(t),|E|)\leqslant\min((\|f\|^*_{r,w})^r t^{-r},|E|).$$

Letting $\beta=\|f\|^*_{r,w}/|E|^{1/r}$, we thus obtain

$$\begin{aligned}\int_E|f|\,dx&=\int_0^\infty\lambda_E(t)\,dt=\int_0^\beta\lambda_E(t)\,dt+\int_\beta^\infty\lambda_E(t)\,dt\\&\leqslant|E|\int_0^\beta dt+(\|f\|^*_{r,w})^r\int_\beta^\infty t^{-r}\,dt=\frac{r}{r-1}\|f\|^*_{r,w}|E|^{1-1/r},\end{aligned}$$

which implies the second inequality of (2.1). To show the first, we take $E=E_\varrho=D\cap B_\varrho$, where B_ϱ is the ball with radius ϱ centered at the origin. By Chebychev's inequality we get

$$t|E_\varrho(|f|>t)|\leqslant\int_{E_\varrho(|f|>t)}|f|\,dx\leqslant\|f\|_{r,w}|E_\varrho(|f|>t)|^{1-1/r},$$

and therefore $t|E_\varrho(|f|>t)|^{1/r}\leqslant\|f\|_{r,w}$. Letting $\varrho\to\infty$ gives

$$t|D(|f|>t)|^{1/r}\leqslant\|f\|_{r,w},$$

which shows the first inequality of (2.1).

The following lemma is frequently used in this paper.

LEMMA 2.1. (i) (The weak Hölder inequality.) *Let* $1<p\leqslant\infty$, $1<q<\infty$ *and* $1<r<\infty$ *satisfy* $1/r=1/p+1/q$. *If* $f\in L^p_w$ *and* $g\in L^q_w$, *then* $fg\in L^r_w$ *and the estimate*

$$\|fg\|_{r,w}\leqslant C\|f\|_{p,w}\|g\|_{q,w}$$

holds with $C>0$ *depending only on p and q. Here, we understand that* $L^\infty_w=L^\infty$.

(ii) (The weak Young inequality.) *Let* $1<p<\infty$, $1<q<\infty$ *and* $1<r<\infty$ *satisfy* $1+1/r=1/p+1/q$. *If* $f\in L^p_w(\mathbf{R}^n)$ *and* $g\in L^q_w(\mathbf{R}^n)$, *then the convolution* $f*g$ *is in* $L^r_w(\mathbf{R}^n)$ *and satisfies the estimate*

$$\|f*g\|_{r,w,\mathbf{R}^n}\leqslant C\|f\|_{p,w,\mathbf{R}^n}\|g\|_{q,w,\mathbf{R}^n}$$

with $C>0$ *depending only on p, q and n.*

Proof. (i) Suppose first that $1<p<\infty$. We need only deduce the desired inequality in terms of the quasinorms $\|\cdot\|^*$. From Young's inequality:

$$|fg|\leqslant\frac{r}{p}\varepsilon^{p/r}|f|^{p/r}+\frac{r}{q}\varepsilon^{-q/r}|g|^{q/r}$$

for any $\varepsilon>0$, we obtain

$$D(|fg|>t)\subset D(|f|>c_1\varepsilon^{-1}t^{r/p})\cup D(|g|>c_2\varepsilon t^{r/q})$$

with constants c_1 and c_2 depending only on p and q. Direct calculation then gives

$$(\|fg\|^*_{r,w})^r\leqslant C_1\varepsilon^p(\|f\|^*_{p,w})^p+C_2\varepsilon^{-q}(\|g\|^*_{q,w})^q$$

for all $\varepsilon>0$ with C_1 and C_2 depending only on p and q. The result now follows by taking the minimum with respect to $\varepsilon>0$.

When $p=\infty$, direct calculation shows that

$$\int_E|fg|\,dx\leqslant\|f\|_\infty\int_E|g|\,dx\leqslant\|f\|_\infty\|g\|_{q,w}|E|^{1-1/q}$$

which shows the desired result.

(ii) is found in [37, pp. 31–32], so the details are omitted here.

In the rest of this section we establish an existence result for the problem:

$$\begin{aligned}-\Delta w+w\cdot\nabla w+\nabla p=f\quad(x\in\Omega),\\ \nabla\cdot w=0\quad(x\in\Omega),\\ w|_S=w^*,\quad w\to0\ \ (|x|\to\infty),\end{aligned}\tag{2.2}$$

with smooth prescribed boundary data w^* and a smooth external force f of the form

$$f=(f_1,\dots,f_n),\quad f_j=\sum_{k=1}^{n}\partial_k F_{kj}. \tag{2.3}$$

Under appropriate decay conditions on F_{kj} and ∇F_{kj}, we apply the results of [12], [47] and [35] and show in particular the existence of a stationary flow w such that

$$|w|\leqslant C/|x|^{n-2},\quad |\nabla w|\leqslant C/|x|^{n-1}. \tag{2.4}$$

Novotny and Padula [35] gave a complete proof of (2.4) for the first time in case $n=3$ in a different functional setting. In this section we employ their technique of estimating volume potentials and prove (2.4) in all dimensions $n\geqslant 3$.

We formally transform (2.2) into the integral equation

$$w=\Phi(w)\equiv E\cdot(f-w\cdot\nabla w)+\int_S(\varphi\cdot T[E,Q]\cdot\nu+E\cdot h)\,dS, \tag{2.5}$$

where

$$E\cdot(f-w\cdot\nabla w)(x)=\int_\Omega E(x-y)\cdot(f-w\cdot\nabla w)(y)\,dy,$$

and $E=(E_{jk})$ and $Q=(Q_j)$ are the Stokes fundamental solution tensor with components

$$E_{jk}(x)=\frac{1}{2\omega_n}\left(\frac{\delta_{jk}}{n-2}|x|^{2-n}+\frac{x_jx_k}{|x|^n}\right),\quad Q_j(x)=\frac{x_j}{\omega_n|x|^n}.$$

Here, ν is the unit outward normal to S; $T[E,Q]\cdot\nu$ denotes the normal stress corresponding to E and Q ([28], [36]); and ω_n is the surface area of the $(n-1)$-dimensional unit sphere. The function $\varphi=\varphi(w)$ will be defined below as a solution of the boundary integral equation

$$\tfrac{1}{2}\varphi+\int_S\varphi\cdot T[E,Q]\cdot\nu\,dS=g \tag{2.6}$$

with the right-hand side

$$g=w^*-\left(E\cdot(f-w\cdot\nabla w)+\int_S E\cdot h\,dS\right)\Big|_S \tag{2.7}$$

and

$$h=h(w)=\sum_{i=1}^{n(n+1)/2}c_i\psi_i^*,\quad c_i=c_i(w)\in\mathbf{R}. \tag{2.8}$$

Here, $\{\psi_i^*\}$ is any fixed basis of null solutions of the integral equation adjoint to (2.6). As shown in [28], [36], equation (2.6) is solvable for any given continuous g satisfying the relation

$$\int_S g\psi_i^*\,dS=0,\quad i=1,\dots,\tfrac{1}{2}n(n+1). \tag{2.9}$$

Moreover, the matrix $A=(A_{jk})$ with components

$$A_{jk}=\iint_{S\times S}\psi_j^*(\xi)\cdot E(\xi-\eta)\cdot\psi_k^*(\eta)\,dS_\xi\,dS_\eta$$

is nonsingular (see [28], [36]); so the coefficients c_i in (2.8) are uniquely determined by (2.9). To define the function φ in (2.6) uniquely, we also require

$$\int_S \varphi\psi_i^*\,dS=0,\quad i=1,\dots,\tfrac{1}{2}(n+1). \tag{2.10}$$

Recall that (see [28], [36]) the boundary values of the single layer potentials $\int_S E\cdot\psi_i^*\,dS$ form a basis of the null space of integral equation (2.6), which consists of the boundary values of the infinitesimal generators of rigid motions.

Now suppose that the tensor F satisfies the decay condition

$$|F_{jk}|\leqslant C/|x|^{\mu-1},\quad |\nabla F_{jk}|\leqslant C/|x|^{\mu}, \tag{2.11}$$

for some $\mu\geqslant 3$, and introduce the Banach space

$$X_\mu=\{w\in C^1(\bar\Omega)^n:\sup|x|^{\mu-2}|w|+\sup|x|^{\mu-1}|\nabla w|<+\infty\}$$

with norm

$$\|w\|_{X_\mu}=\sup|x|^{\mu-2}|w|+\sup|x|^{\mu-1}|\nabla w|.$$

(Here and in what follows we assume that $0\notin\bar\Omega$.)

Given $w^*\in C^2(S)$, $F\in X_{\mu+1}$ and $w\in X_\mu$ with $w|_S=w^*$, we estimate the function $\Phi(w)$ in the space X_μ, applying the Schauder estimates of [12], [47] for potentials defined by E and Q, and the estimates

$$\int_\Omega |x-y|^{-\alpha}(1+|y|)^{-\beta}\,dy\leqslant\begin{cases} C|x|^{n-\alpha-\beta} & (0<\beta<n),\\ C|x|^{-\alpha} & (\beta>n),\end{cases} \tag{2.12}$$

which hold for $0<\alpha<n$ and $\alpha+\beta>n$. Suppose first that $n>3$ and $3\leqslant\mu<n$, and fix $0<\alpha<1$. By Theorem 1 of [47], the double layer potential belongs to the Hölder space $C^{1+\alpha}(\bar\Omega)$ if the corresponding density is in $C^{1+\alpha}(S)$. This implies that the integral operator in (2.6) defines a bounded linear operator from $C^{1+\alpha}(S)$ into itself; so equation (2.6) is inverted under constraints (2.9)–(2.10) to yield the estimate

$$\|\varphi\|_{C^{1+\alpha}(S)}\leqslant C\|g\|_{C^{1+\alpha}(S)}, \tag{2.13}$$

where $\|\cdot\|_{C^{1+\alpha}}$ is the usual Hölder norm. Since g is defined by (2.7), we get

$$\begin{aligned}\|g\|_{C^{1+\alpha}(S)}&\leqslant\|w^*\|_{C^2(S)}+\|E\cdot(f-w\cdot\nabla w)\|_{C^{1+\alpha}(\bar\Omega)}+\left\|\int_S E\cdot h\,dS\right\|_{C^{1+\alpha}(\bar\Omega)}\\ &\equiv\|w^*\|_{C^2(S)}+I_1+I_2.\end{aligned} \tag{2.14}$$

The last two terms are estimated in the following way. Firstly, (2.11) and (2.12) imply

$$|E\cdot(f-w\cdot\nabla w)|\leqslant C(\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu})|x|^{2-\mu}, \tag{2.15}$$

and

$$|\nabla E\cdot(f-w\cdot\nabla w)|\leqslant C(\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu})|x|^{1-\mu}. \tag{2.16}$$

Moreover, since $|\nabla E|\leqslant C|x-y|^{1-n}$, applying Morrey's inequality [15] implies that the α-Hölder seminorm of $\nabla E\cdot(f-w\cdot\nabla w)$ is estimated as

$$\leqslant C\|f-w\cdot\nabla w\|_r\leqslant C(\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu}), \tag{2.17}$$

where $r>n$ is taken so that $\alpha=1-n/r$. Since $|x|^{-1}\in L^\infty(\Omega)$ because of the assumption $0\notin\bar{\Omega}$, combining (2.15)–(2.17) gives

$$I_1\leqslant C(\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu}). \tag{2.18}$$

Secondly, to estimate I_2 we use the following result of [12], [47]:

$$I_2\leqslant C\|h\|_{C^\alpha(S)}. \tag{2.19}$$

By (2.8), $\|h\|_{C^\alpha(S)}\leqslant C\sum_i|c_i(w)|$, and $c_i(w)$ are determined by constraint (2.9). So, by (2.12) and (2.19) we have

$$\begin{aligned} I_2\leqslant C\|h\|_{C^\alpha(S)} &\leqslant C(\|w^*\|_{C^2(S)}+\sup_\Omega|E\cdot(f-w\cdot\nabla w)|)\\ &\leqslant C\|w^*\|_{C^2(S)}+C(\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu})|x|^{2-\mu}\\ &\leqslant C(\|w^*\|_{C^2(S)}+\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu}), \end{aligned} \tag{2.20}$$

since $|x|^{-1}\in L^\infty(\Omega)$ by assumption $0\notin\bar{\Omega}$. From (2.13), (2.14), (2.18) and (2.20) we obtain

$$\|\varphi\|_{C^{1+\alpha}(S)}\leqslant C(\|w^*\|_{C^2(S)}+\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu}). \tag{2.21}$$

Using (2.21), we can now estimate in X_μ the nonlinear map Φ defined by (2.5). By (2.15) and (2.16) we obtain

$$\|E\cdot(f-w\cdot\nabla w)\|_{X_\mu}\leqslant C(\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu}). \tag{2.22}$$

The single and double layer potentials on the right-hand side of (2.5) are estimated as follows: Choose $R>0$ so that $(\mathbf{R}^n\setminus\Omega)\subset\{|x|<R\}$. Then, by (2.21),

$$\begin{aligned} \sup_{|x|>R}|x|^{\mu-2}\left|\int_S\varphi\cdot T[E,Q]\cdot\nu\,dS\right| &\leqslant C_R\|\varphi\|_{\infty,S}\leqslant C_R\|\varphi\|_{C^{1+\alpha}(S)}\\ &\leqslant C(\|w^*\|_{C^2(S)}+\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu}). \end{aligned} \tag{2.23}$$

By Theorem 1 of [47] and (2.21),

$$\begin{aligned} \sup_{\Omega\cap\{|x|<R\}} |x|^{\mu-2}\left|\int_S \varphi\cdot T[E,Q]\cdot\nu\,dS\right| &\leqslant C_R \sup_{\Omega}\left|\int_S \varphi\cdot T[E,Q]\cdot\nu\,dS\right| \\ &\leqslant C_R\left\|\int_S \varphi\cdot T[E,Q]\cdot\nu\,dS\right\|_{C^{1+\alpha}(\Omega)} \\ &\leqslant C\|\varphi\|_{C^{1+\alpha}(S)} \\ &\leqslant C(\|w^*\|_{C^2(S)}+\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu}). \end{aligned} \tag{2.24}$$

In the same way, we can estimate the first derivatives, to obtain the following estimate for the double layer:

$$\left\|\int_S \varphi\cdot T[E,Q]\cdot\nu\,dS\right\|_{X_\mu} \leqslant C(\|w^*\|_{C^2(S)}+\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu}). \tag{2.25}$$

To estimate the single layer potential in (2.5), we apply the result of [12], [47] as used in deducing (2.19), which asserts that the single layer potential is in $C^{1+\alpha}(\bar{\Omega})$ if the corresponding density is in $C^\alpha(S)$. Using this and (2.20), we obtain as above

$$\left\|\int_S E\cdot h\,dS\right\|_{X_\mu} \leqslant C\|h\|_{C^\alpha(S)} \leqslant C(\|w^*\|_{C^2(S)}+\|F\|_{X_{\mu+1}}+\|w\|^2_{X_\mu}). \tag{2.26}$$

By (2.22), (2.25) and (2.26), we conclude that if $3\leqslant\mu<n$, then Φ maps X_μ into itself with the estimates

$$\|\Phi(w)\|_{X_\mu} \leqslant C_0\|w^*\|_{C^2(S)}+C_1\|F\|_{X_{\mu+1}}+C_2\|w\|^2_{X_\mu}$$

and

$$\|\Phi(w_1)-\Phi(w_2)\|_{X_\mu} \leqslant C_3(\|w_1\|_{X_\mu}+\|w_2\|_{X_\mu})\|w_1-w_2\|_{X_\mu}.$$

So, the contraction mapping principle ensures the existence of a unique solution of (2.5) in X_μ provided that w^* and F are sufficiently small in $C^2(S)$ and in $X_{\mu+1}$, respectively.

When $\mu>n>3$, the foregoing calculation, together with (2.12) in case $\beta>n$, applies with slight modification to deduce that Φ maps X_n into itself and there hold the estimates

$$\|\Phi(w)\|_{X_n} \leqslant C_0\|w^*\|_{C^2(S)}+C_1\|F\|_{X_{\mu+1}}+C_2\|w\|^2_{X_n}$$

and

$$\|\Phi(w_1)-\Phi(w_2)\|_{X_n} \leqslant C_3(\|w_1\|_{X_n}+\|w_2\|_{X_n})\|w_1-w_2\|_{X_n}.$$

The existence of a desired solution $w\in X_n$ of (2.5) is again deduced via the contraction mapping principle.

We next prove the existence of w satisfying (2.4), assuming $n\geqslant3$ and $F\in X_{n+1}$. In this case, however, the foregoing estimate for the volume potential does not work for deducing estimate (2.22) with $\mu=n$, and so we are forced to employ another way. It is here, where we appeal to the following lemma, which is due to Novotny and Padula [35].

LEMMA 2.2. *Let $n\geqslant 3$, $F\in X_{n+1}$, $f=(f_1,\dots,f_n)$ and*

$$f_j=\sum_{k=1}^{n}\partial_k F_{kj}.$$

Then

$$\|E\cdot(f-w\cdot\nabla w)\|_{X_n}\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2),$$

with $C>0$ independent of F and $w\in X_n$.

Proof. The proof below is due to [35]. We may assume without loss of generality that $|x|>1$ if $x\in\Omega$. An integration by parts gives

$$E\cdot(f-w\cdot\nabla w)=\int_S E\cdot\nu\cdot(F-w\otimes w)\,dS+(\nabla E)\cdot(F-w\otimes w).$$

Applying the result of [12], we see as in the foregoing paragraph that

$$\begin{aligned}\left|\int_S E\cdot\nu\cdot(F-w\otimes w)\,dS\right| &\leqslant C\|F-w\otimes w\|_{C^\alpha(S)}|x|^{2-n}\\ &\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)|x|^{2-n},\end{aligned}\tag{2.27}$$

while $(\nabla E)\cdot(F-w\otimes w)$ is directly estimated as

$$\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)\int_\Omega |x-y|^{1-n}|y|^{1-n}\,dy\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)|x|^{2-n}.\tag{2.28}$$

To deduce the bound for the derivatives of $E\cdot(f-w\cdot\nabla w)$, we divide the integral into several terms. We may assume that $R=|x|$ is sufficiently large. Let

$$\begin{gathered}\Omega_1=\Omega\cap B_{R/2}(0),\quad \Omega_2=B_1(x),\quad \Omega_3=B_{R/2}(x)\backslash B_1(x),\\ \Omega_4=B_{3R/2}(0)\backslash(B_{R/2}(0)\cup B_{R/2}(x)),\quad \Omega_5=\Omega\backslash B_{3R/2}(0),\end{gathered}$$

where $B_r(y)$ is the open ball with radius r centered at y. We write

$$\begin{aligned}(\nabla E)\cdot(f-w\cdot\nabla w)(x)&=(\nabla E)\cdot(\nabla\cdot(F-w\otimes w))\\ &=\sum_{j=1}^{5}\int_{\Omega_j}(\nabla E)(x-y)(\nabla\cdot(F-w\otimes w))(y)\,dy\equiv\sum_{j=1}^{5}I_j,\end{aligned}$$

and estimate each term separately. First, integrating by parts gives

$$\begin{aligned}|I_1|\leqslant C\int_{\Omega_1}|x-y|^{-n}|(F-w\otimes w)(y)|\,dy+\left|\int_S(\nabla E)(\nu\cdot(F-w\otimes w))\,dS\right|\\ +C\int_{|y|=R/2}|x-y|^{1-n}|(F-w\otimes w)(y)|\,dS_y.\end{aligned}$$

By Theorem 1 of [47], the integral over S is estimated as

$$\leqslant C\|F-w\otimes w\|_{C^\alpha(S)}|x|^{1-n}\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)|x|^{1-n}.$$

On the other hand, since $|y|\leqslant\frac{1}{2}R$ implies $|x-y|\geqslant\frac{1}{2}R$, and since

$$|(F-w\otimes w)(y)|\leqslant C(\|F\|_{X_{n+1}}|y|^{1-n}+\|w\|_{X_n}^2|y|^{4-2n})\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)|y|^{1-n},$$

the integral on $\{|y|=\frac{1}{2}R\}$ is bounded as

$$\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)R^{1-n}=C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)|x|^{1-n},$$

while the integral over Ω_1 is bounded as

$$\begin{aligned}&\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)R^{-n}\int_{\Omega_1}|y|^{1-n}\,dy\\&\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)R^{-n}\int_{|y|<R/2}|y|^{1-n}\,dy\\&=C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)R^{1-n}=C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)|x|^{1-n}.\end{aligned}$$

We thus have

$$|I_1|\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)|x|^{1-n}. \tag{2.29}$$

Secondly, we have

$$\begin{aligned}|I_2|&\leqslant C\int_{|y-x|<1}|x-y|^{1-n}|\nabla(F-w\otimes w)(y)|\,dy\\&\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)\int_{|y-x|<1}|x-y|^{1-n}|y|^{-n}\,dy\\&\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)|x|^{-n}.\end{aligned} \tag{2.30}$$

Third, $y\in\Omega_3$ implies $1\leqslant|x-y|<\frac{1}{2}R$; so $|y|\geqslant R-|y-x|>\frac{1}{2}R$, and

$$\begin{aligned}|I_3|&\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)\int_{\Omega_3}|x-y|^{1-n}|y|^{-n}\,dy\\&\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)R^{-n}\int_{1\leqslant|x-y|<R/2}|x-y|^{1-n}\,dy\\&\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)R^{1-n}=C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)|x|^{1-n}.\end{aligned} \tag{2.31}$$

Fourth, $y\in\Omega_4$ implies $|x-y|\geqslant\frac{1}{2}R$ and $\frac{1}{2}R\leqslant|y|<\frac{3}{2}R$. Thus,

$$\begin{aligned}|I_4|&\leqslant C\int_{\Omega_4}|x-y|^{1-n}|\nabla(F-w\otimes w)(y)|\,dy\\&\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)\int_{\Omega_4}|x-y|^{1-n}|y|^{-n}\,dy\\&\leqslant C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)R^{1-2n}\int_{|y|<3R/2}dy=C(\|F\|_{X_{n+1}}+\|w\|_{X_n}^2)|x|^{1-n}.\end{aligned} \tag{2.32}$$

Finally, if $y \in \Omega_5$, then

$$|x-y| \geqslant |y| - R \geqslant |y| - \tfrac{2}{3}|y| \geqslant \tfrac{1}{3}|y|,$$

and so,

$$\begin{aligned} |I_5| &\leqslant C(\|F\|_{X_{n+1}} + \|w\|_{X_n}^2) \int_{\Omega_5} |x-y|^{1-n} |y|^{-n}\, dy \\ &\leqslant C(\|F\|_{X_{n+1}} + \|w\|_{X_n}^2) \int_{|y|>R/2} |y|^{1-2n}\, dy \\ &= C(\|F\|_{X_{n+1}} + \|w\|_{X_n}^2) |x|^{1-n}. \end{aligned} \tag{2.33}$$

Combining (2.27)–(2.33) gives the desired result. This proves Lemma 2.2.

By Lemma 2.2, we can now apply the contraction mapping principle to the map Φ in X_n and obtain the desired solution w. In conclusion, we have proved

THEOREM 2.3. (i) *Let $n>3$ and $3 \leqslant \mu < n$. If $w^* \in C^2(S)^n$ and $F \in X_{\mu+1}$ are sufficiently small, there exists a unique solution w of* (2.2) *such that*

$$|w| \leqslant C/|x|^{\mu-2}, \quad |\nabla w| \leqslant C/|x|^{\mu-1}.$$

(ii) *Let $n \geqslant 3$ and $n \leqslant \mu < \infty$. If $w^* \in C^2(S)^n$ and $F \in X_{\mu+1}$ are sufficiently small, there exists a unique solution w of* (2.2) *such that*

$$|w| \leqslant C/|x|^{n-2}, \quad |\nabla w| \leqslant C/|x|^{n-1}.$$

Remark. Theorem 2.3 generalizes the three-dimensional existence result of Finn [13], [14] to the case of general space dimensions $n \geqslant 3$ (but, only in the case where $w \to 0$ as $|x| \to \infty$). By the standard regularity result of [1], [9] for the stationary Stokes system, the function w is a classical solution of (2.2). As seen from its proof, Theorem 2.3 is most difficult to prove in case $n=3$; in fact, we had to appeal to the technique of Novotny and Padula [35] in estimating the volume potential. In [35] they deal with the three-dimensional case in a different functional setting and give a complete proof of the existence of solutions with the above-mentioned decay property. They further prove a decay result for the associated pressure. When $n>3$ and $\mu>3$, we have

$$|x|^{2-\mu} \in L_w^{n/(\mu-2)} \cap L^\infty \subset L^n \cap L^\infty \quad \text{and} \quad |x|^{1-\mu} \in L_w^{n/(\mu-1)} \cap L^\infty \subset L^{n/2} \cap L^\infty,$$

so that the solution w given in Theorem 2.3 with $\mu>3$ satisfies

$$w \in L^n \cap L^\infty, \quad |\nabla w| \in L^{n/2} \cap L^\infty. \tag{2.4$'$}$$

Stability of stationary flows w satisfying (2.4′) for $n \geqslant 3$ was discussed in [11], [25]. However, when $n=3$, condition (2.4′) seems to be too restrictive, as easily seen from the following result (note that $n' = \frac{1}{2}n = \frac{3}{2}$ if $n=3$).

THEOREM 2.4. *Let $n \geqslant 3$ and let w be any solution of* (2.2) *given in Theorem* 2.3 (ii). *If $F \in L^{n'}$ and $\nabla w \in L^{n'}$, then we have*

$$\int_S \nu\cdot(T[w,p]-w^*\otimes w^*+F)\,dS=0, \tag{2.34}$$

where

$$T[w,p]=(T_{jk}[w,p])_{j,k=1}^n,\quad T_{jk}[w,p]=-\delta_{jk}p+\partial_j w_k+\partial_k w_j.$$

Proof. First observe that the equation in (2.2) is written as

$$\nabla\cdot(T[w,p]-w\otimes w+F)=0. \tag{2.35}$$

Multiplying (2.35) by $\varphi \in C_0^\infty(\Omega)^n$ such that $\nabla\cdot\varphi=0$ and then integrating by parts gives

$$2\langle\varepsilon(w),\varepsilon(\varphi)\rangle=-\langle F-w\otimes w,\nabla\varphi\rangle, \tag{2.36}$$

where

$$\varepsilon(w)=(\varepsilon_{jk}(w))_{j,k=1}^n,\quad \varepsilon_{jk}(w)=\tfrac{1}{2}(\partial_j w_k+\partial_k w_j).$$

Since $\nabla w \in L^{n'}$, it follows from the Sobolev-type inequality as given in [8], [19], [34] that $w \in L^{n/(n-2)}$. Thus, $w \in L^{n/(n-2)} \cap L^\infty \subset L^n$, and we conclude that $w\otimes w \in L^{n'}$. Hence, both sides of (2.36) are continuous in φ with respect to the norm $\|\nabla\varphi\|_n$. We fix an arbitrary $c \in \mathbf{R}^n$ with $c \neq 0$, take $\psi \in C^\infty(\mathbf{R}^n)$ so that $\psi(x)=1$ if $|x| \geqslant R$ and $\psi(x)=0$ if $|x| \leqslant \frac{1}{2}R$ for some large R so that $(\mathbf{R}^n\setminus\Omega) \subset \{|x|<\frac{1}{8}R\}$. Consider the function

$$\varphi_c=c-S_D(c\cdot\nabla\psi)$$

where S_D is the operator introduced in [4, Proposition 3.3] with respect to the bounded domain $D=\{\frac{1}{4}R<|x|<2R\}$. Then $\varphi_c \in C^\infty(\bar\Omega)$, $\nabla\cdot\varphi_c=0$, and $\varphi_c(x)=c$ $(|x| \geqslant 3R)$, $\varphi_c(x)=0$ $(|x| \leqslant \frac{1}{4}R)$. As shown in [7], [17], φ_c is approximated in the norm $\|\nabla\varphi\|_n$ by functions φ_k in $C_0^\infty(\Omega)^n$ with $\nabla\cdot\varphi_k=0$; so (2.36) with $\varphi=\varphi_k$ implies

$$2\langle\varepsilon(w),\varepsilon(\varphi_c)\rangle=-\langle F-w\otimes w,\nabla\varphi_c\rangle. \tag{2.37}$$

We next multiply (2.35) by φ_c and integrate over $\Omega\cap\{|x|<4R\}$ to get

$$\int_{\Omega\cap\{|x|<4R\}}\nabla\cdot(w\otimes w)\cdot\varphi_c\,dx=\int_{\Omega\cap\{|x|<4R\}}\nabla\cdot(T[w,p]+F)\cdot\varphi_c\,dx.$$

Since $\nabla\varphi_c=0$ for $|x| \geqslant 4R$, integration by parts gives

$$\int_{\Omega\cap\{|x|<4R\}}\nabla\cdot(w\otimes w)\cdot\varphi_c\,dx=\int_{|x|=4R}\nu\cdot(w\otimes w)\cdot c\,dS-\langle w\otimes w,\nabla\varphi_c\rangle,$$

and

$$\int_{\Omega\cap\{|x|<4R\}} \nabla\cdot(T[w,p]+F)\cdot\varphi_c\,dx = \int_{|x|=4R} \nu\cdot(T[w,p]+F)\cdot c\,dS - 2\langle\varepsilon(w),\varepsilon(\varphi_c)\rangle - \langle F,\nabla\varphi_c\rangle.$$

Thus, in view of (2.37) we have

$$\int_{|x|=4R} \nu\cdot(T[w,p]+F)\cdot c\,dS = \int_{|x|=4R} \nu\cdot(w\otimes w)\cdot c\,dS.$$

But, c was arbitrary; so

$$\int_{|x|=4R} \nu\cdot(T[w,p]-w\otimes w+F)\,dS = 0. \tag{2.38}$$

Integrating (2.35) over $\Omega\cap B_{4R}(0)$, applying the divergence theorem and taking (2.38) into account, we obtain (2.34). The proof is complete.

The following result examines the integrability property of ∇w when w satisfies the assumption of Theorem 2.4.

THEOREM 2.5. *Let $n\geqslant 3$ and let w be the stationary flow given in Theorem* 2.3 *with $F\in X_{\mu+1}$ for some $\mu\geqslant n+1$.*

(i) *Suppose that $n\geqslant 4$, $w\in L^{n/(n-2)}$ and $\nabla w\in L^{n'}$. Then $\nabla w\in L^r$ for all $1<r\leqslant\infty$.*

(ii) *If $n=3$, $\nabla w\in L^q$ for some $1<q<3/2$ and $w\in L^{q^*}$ with $1/q^*=1/q-1/3$, then $\nabla w\in L^r$ for all $1<r\leqslant\infty$.*

Proof. By the assumptions on w and ∇w and by Theorem 2.4, the function w satisfies the vanishing total flux condition (2.34) together with an associated pressure p. To examine the integrability property of p we write $w=w_1+w_2$ and $p=p_1+p_2$, where (w_1,p_1) solves

$$-\Delta w_1+\nabla p_1=0 \quad \text{in } \Omega,$$
$$\nabla\cdot w_1=0 \quad \text{in } \Omega,$$
$$w_1|_S=w^*, \qquad \lim_{|x|\to\infty} w_1=0,$$

and (w_2,p_2) satisfies

$$-\Delta w_2+\nabla p_2=\nabla\cdot(F-w\otimes w) \quad \text{in } \Omega,$$
$$\nabla\cdot w_2=0 \quad \text{in } \Omega,$$
$$w_2|_S=0, \qquad \lim_{|x|\to\infty} w_2=0.$$

As is well known ([28], [36]), the pair of functions (w_1, p_1) can be taken to satisfy

$$|p_1(x)| = O(|x|^{1-n}), \quad |\nabla p_1(x)| = O(|x|^{-n}),$$
$$|w_1(x)| = O(|x|^{2-n}), \quad |\nabla w_1(x)| = O(|x|^{1-n}),$$

as $|x| \to \infty$. So we see in particular that $\nabla w_1 \in L^s$ and $p_1 \in L^s$ for all $n' < s < n$. Thus, $\nabla w_2 = \nabla w - \nabla w_1 \in L^s$ for all $n' < s < n$. On the other hand, the assumption implies that $F - w \otimes w \in L^s$ for all $n' < s < n$. Hence, it follows from [4, Theorem 3.5 (ii)] that p_2 can be taken to satisfy $p_2 \in L^s$ for all $n' < s < n$. We thus conclude that $p \in L^s$ for all $n' < s < n$. Using this integrability property of the function p, we can apply the standard argument of the potential theory to obtain the representation

$$\begin{aligned} w(x) &= E \cdot (\nabla \cdot (F - w \otimes w)) + \int_S E \cdot \nu \cdot T[w, p]\, dS + \int_S w^* \cdot T[E, Q] \cdot \nu\, dS \\ &= (\nabla E) \cdot (F - w \otimes w) + \int_S E \cdot \nu \cdot (T[w, p] - w^* \otimes w^* + F)\, dS + \int_S w^* \cdot T[E, Q] \cdot \nu\, dS. \end{aligned}$$

The derivatives of the last term are $O(|x|^{-n})$ as $|x| \to \infty$, so belong to L^r for all $1 < r \leqslant \infty$. By using (2.34), the single layer potential is rewritten as

$$\int_S \widetilde{E}(x, y) \cdot \nu \cdot (T[w, p] - w^* \otimes w^* + F)(y)\, dS_y$$

where

$$\widetilde{E}(x, y) = E(x - y) - E(x) = \int_0^1 \frac{d}{d\theta} E(x - \theta y)\, d\theta.$$

Since

$$|(\nabla_x \widetilde{E})(x, y)| \leqslant C|x|^{-n} \quad \text{for large } |x| \text{ and } y \in S,$$

with $C > 0$ independent of x and y, the derivatives of the single layer potential are in L^r for all $1 < r \leqslant \infty$. It thus suffices to discuss the behavior of the function $(\nabla^2 E) \cdot (F - w \otimes w)$. Suppose first that $n \geqslant 4$. Then, since $w \in L^\infty$, the assumption $w \in L^{n/(n-2)}$ implies that $w \otimes w \in L^r$ for all $1 < r \leqslant \infty$. Since $F \in L^r$ for all $1 < r \leqslant \infty$ by assumption, and since $\nabla^2 E$ is a Calderon–Zygmund kernel [43], it follows that $(\nabla^2 E) \cdot (F - w \otimes w) \in L^r$ for all $1 < r < \infty$. We thus obtain $\nabla w \in L^r$ for all $1 < r < \infty$. But, since we already know that $\nabla w \in L^\infty$, this shows the result in case $n \geqslant 4$. Suppose next $n = 3$. In this case,

$$w \otimes w \in L^{q_1} \cap L^\infty \quad \text{with } q_1 = \frac{q^*}{2} = \frac{3q}{2(3 - q)} < q,$$

and so $(\nabla^2 E) \cdot (F - w \otimes w) \in L^{q_1}$. Hence, $\nabla w \in L^{q_1}$, and therefore $w \in L^{q_1^*}$ with $1/q_1^* = 1/q_1 - 1/3$, by the Sobolev-type inequality as given in [8], [19], [34]. Hence,

$$w \otimes w \in L^{q_2} \cap L^\infty \quad \text{with } q_2 = \frac{3q_1}{2(3 - q_1)} < q_1,$$

and so $(\nabla^2 E)\cdot(F-w\otimes w)\in L^{q_2}$. Repeating these processes, we see that

$$(\nabla^2 E)\cdot(F-w\otimes w)\in L^{q_{j+1}}, \quad q_{j+1}=\frac{3q_j}{2(3-q_j)}<q_j<q<\frac{3}{2} \quad (j=1,2,\dots), \tag{2.39}$$

and that the sequence q_j defined recursively by (2.39) tends to zero as $j\to\infty$. This eventually shows that $\nabla w\in L^r$ for all $1<r<\infty$, and the proof is complete.

Remark. When $n=3$, we do not know if $w\in L^3\cap L^\infty$ and $\nabla w\in L^{3/2}\cap L^\infty$ together imply that $\nabla w\in L^r$ for all $1<r\leqslant\infty$. Indeed, in this case the above proof of Theorem 2.5 merely implies that $\nabla w\in L^{3/2}$. Chen [11] discusses L^2 stability of stationary flows w satisfying the assumption of Theorem 2.5 (ii). However, the existence of such flows is guaranteed only in a very restrictive case; see [16].

3. Analysis of the linearized operators

In this section we fix a stationary flow w satisfying

$$|w|\leqslant C/|x|, \quad |\nabla w|\leqslant C/|x|^2 \tag{3.1}$$

and discuss large time behavior of solutions of linear problem (1.7). Observe that since $n-2\geqslant 1$ and $\frac{1}{2}n\geqslant n'$ when $n\geqslant 3$, the stationary flows w obtained in Theorem 2.3 all satisfy condition (3.1). In case $n\geqslant 4$, we have $n'<\frac{1}{2}n$; so Theorem 2.3 ensures existence of a stationary flow w satisfying

$$|\nabla w|\in L^q\cap L^\infty \quad \text{for some } q \text{ with } n'<q<\tfrac{1}{2}n. \tag{3.1'}$$

Employing a result of Chen [11], we further show that under condition (3.1′) the solutions of (1.7) behave better than in the case of condition (3.1).

The following notation is adopted: Let L^r_σ, $1<r<\infty$, be the L^r-closure of the set $C^\infty_{0,\sigma}(\Omega)$ of compactly supported smooth solenoidal vector fields in Ω. Then we have the Helmholtz decomposition [33], [42]:

$$L^r(\Omega)^n=L^r_\sigma\oplus G^r,$$

where L^r_σ and G^r are characterized as

$$L^r_\sigma=\{u\in L^r(\Omega)^n\colon \nabla\cdot u=0, u\cdot\nu|_S=0\}, \quad G^r=\{\nabla p\in L^r(\Omega)^n\colon p\in L^r_{\mathrm{loc}}(\bar\Omega)\}.$$

We denote by $P=P_r$ the associated bounded projector onto L^r_σ. As shown in [33],

$$(L^r_\sigma)^*=L^{r'}_\sigma, \quad (G^r)^*=G^{r'}, \quad P^*_r=P_{r'} \quad (1/r'=1-1/r),$$

where $(L^r_\sigma)^*$ and $(G^r)^*$ mean the dual spaces and P^*_r the dual operator. We next introduce the Stokes operator in L^r_σ

$$A = A_r = -P_r \Delta$$

and the operator

$$L = A + B, \quad Bu = B_r u = P_r(w \cdot \nabla u + u \cdot \nabla w) \tag{3.2}$$

with

$$D(L) = D(A) = \{u \in W^{2,r}(\Omega)^n : u|_S = 0, \nabla \cdot u = 0\}.$$

It is also shown in [33] that

$$A^*_r = A_{r'}, \quad 1/r' = 1 - 1/r.$$

We examine properties of the operator L and its adjoint L^*, and apply them to the study of solutions of (1.7). The results will then be applied in §§ 4 and 6 in discussing existence and asymptotic behavior of weak and strong solutions of problem (1.5).

We begin with the proof of

LEMMA 3.1. *Let $n \geqslant 3$ and let Ω be a smooth exterior domain in $\mathbf{R}^n$.*

(i) *For $1 < r < n$ there is a number $C = C_r > 0$ such that we have the estimate*

$$\|w \cdot \nabla u\|_r \leqslant C \|w\| \cdot \|\nabla^2 u\|_r, \quad u \in D(A_r), \tag{3.3}$$

where $\|\cdot\|_r$ is the L^r-norm and

$$\|w\| = \sup(|x| \cdot |w(x)|).$$

(ii) *For $1 < r < \frac{1}{2}n$ there is a number $C = C_r > 0$ such that we have the estimate*

$$\|u \cdot \nabla w\|_r \leqslant C \|\nabla w\| \cdot \|\nabla^2 u\|_r, \quad u \in D(A_r), \tag{3.4}$$

where

$$\|\nabla w\| = \sup(|x|^2 |\nabla w(x)|).$$

(iii) *For $1 < r < \frac{1}{2}n$ we have*

$$\|Bu\|_r, \|B^* u\|_r \leqslant C_r(\|w\| + \|\nabla w\|)\|\nabla^2 u\|_r, \quad u \in D(A_r), \tag{3.5}$$

where B^* is the adjoint to B.

Proof. We first recall the Sobolev-type inequalities due to [8], [19], [34]:

$$\|u\|_{nr/(n-2r)} \leqslant C_1 \|\nabla u\|_{nr/(n-r)} \leqslant C_2 \|\nabla^2 u\|_r, \quad u \in W^{2,r}(\Omega), \tag{3.6}$$

for $1<r<\frac{1}{2}n$, $n\geqslant 3$, and

$$\|u\|_{nr/(n-r)} \leqslant C \|\nabla u\|_r, \quad u \in W^{1,r}(\Omega), \tag{3.6'}$$

for $1<r<n$, $n\geqslant 2$. Consider now the Riesz potentials

$$T_j g(x) = \int |x-y|^{j-n} g(y)\, dy, \quad j=1,2,\ n\geqslant 3.$$

Since $|x|^{-1} \in L^n_w$ and $|x|^{1-n} \in L^{n'}_w$, Lemma 2.1 gives

$$\left\| \frac{T_1 g}{|x|} \right\|_{r,w,\mathbf{R}^n} \leqslant C_r \|T_1 g\|_{nr/(n-r),w,\mathbf{R}^n} \leqslant C_r \|g\|_{r,w,\mathbf{R}^n} \tag{3.7}$$

for $1<r<n$. Similarly, we have

$$\left\| \frac{T_2 g}{|x|^2} \right\|_{r,w,\mathbf{R}^n} \leqslant C_r \|T_2 g\|_{nr/(n-2r),w,\mathbf{R}^n} \leqslant C_r \|g\|_{r,w,\mathbf{R}^n} \tag{3.8}$$

for $1<r<\frac{1}{2}n$. Applying the Marcinkiewicz interpolation theorem [2], [43], [45] to each of (3.7) and (3.8), we obtain

$$\left\| \frac{T_1 g}{|x|} \right\|_{r,\mathbf{R}^n} \leqslant C_r \|g\|_{r,\mathbf{R}^n} \quad (1<r<n) \tag{3.7'}$$

and

$$\left\| \frac{T_2 g}{|x|^2} \right\|_{r,\mathbf{R}^n} \leqslant C_r \|g\|_{r,\mathbf{R}^n} \quad \left(1<r<\tfrac{1}{2}n\right). \tag{3.8'}$$

Since

$$|u(x)| \leqslant C_n T_j(|\nabla^j u|)(x), \quad j=1,2,$$

for $u \in C_0^\infty(\mathbf{R}^n)$, it follows from (3.7′) and (3.8′) that

$$\left\| \frac{u}{|x|} \right\|_{r,\mathbf{R}^n} \leqslant C_r \|\nabla u\|_{r,\mathbf{R}^n}, \quad u \in W^{1,r}(\mathbf{R}^n), \tag{3.7''}$$

for $1<r<n$, and

$$\left\| \frac{u}{|x|^2} \right\|_{r,\mathbf{R}^n} \leqslant C_r \|\nabla^2 u\|_{r,\mathbf{R}^n}, \quad u \in W^{2,r}(\mathbf{R}^n), \tag{3.8''}$$

for $1<r<\frac{1}{2}n$. We next show that

$$\left\|\frac{u}{|x|}\right\|_r \leqslant C_r\|\nabla u\|_r, \quad u\in W^{1,r}(\Omega), \tag{3.9}$$

for $1<r<n$, and

$$\left\|\frac{u}{|x|^2}\right\|_r \leqslant C_r\|\nabla^2 u\|_r, \quad u\in W^{2,r}(\Omega), \tag{3.10}$$

for $1<r<\frac{1}{2}n$. Indeed, choose a function $\psi\in C^\infty(\mathbf{R}^n)$ so that

$$0\leqslant\psi\leqslant 1, \quad \psi(x)=0 \ \ (|x|\leqslant R), \quad \psi(x)=1 \ \ (|x|\geqslant 2R),$$

for some fixed $R>0$ satisfying $(\mathbf{R}^n\backslash\Omega)\subset\{|x|<R\}$. Then estimates (3.7″), (3.6′) and Hölder's inequality together yield, with $1/r^*=1/r-1/n$,

$$\begin{aligned}\left\|\frac{u\psi}{|x|}\right\|_r &\leqslant C\|\nabla(u\psi)\|_r \leqslant C(\|\psi\nabla u\|_r+\|u\nabla\psi\|_r)\\ &\leqslant C(\|\nabla u\|_r+\|u\|_{r^*})\leqslant C\|\nabla u\|_r.\end{aligned} \tag{3.11}$$

On the other hand, since we assume $0\notin\bar\Omega$, it follows that $|x|^{-1}\in L^\infty(\Omega)$; thus, (3.6′) and Hölder's inequality together yield

$$\left\|\frac{(1-\psi)u}{|x|}\right\|_r \leqslant C\|u\|_{r,\Omega\cap\{|x|<2R\}}\leqslant C\|u\|_{r^*}\leqslant C\|\nabla u\|_r. \tag{3.12}$$

Combining (3.11) and (3.12) gives (3.9). To prove (3.10), we apply (3.6) and (3.8″) to get

$$\begin{aligned}\left\|\frac{u\psi}{|x|^2}\right\|_r &\leqslant C\|\nabla^2(u\psi)\|_r\\ &\leqslant C(\|\psi\nabla^2 u\|_r+\|\nabla\psi\nabla u\|_r+\|u\nabla^2\psi\|_r)\\ &\leqslant C(\|\nabla^2 u\|_r+\|\nabla u\|_{r^*}+\|u\|_{r^{**}})\leqslant C\|\nabla^2 u\|_r\end{aligned} \tag{3.13}$$

and

$$\left\|\frac{(1-\psi)u}{|x|^2}\right\|_r \leqslant C\|u\|_{r,\Omega\cap\{|x|<2R\}}\leqslant C\|u\|_{r^{**}}\leqslant C\|\nabla^2 u\|_r, \tag{3.14}$$

where $1/r^{**}=1/r-2/n$. Estimate (3.10) follows by combining (3.13) and (3.14).

Now, estimate (3.3) is obtained from (3.9) as

$$\|w\cdot\nabla u\|_r\leqslant\|w\|\cdot\left\|\frac{\nabla u}{|x|}\right\|_r\leqslant C_r\|w\|\cdot\|\nabla^2 u\|_r \quad (1<r<n),$$

and estimate (3.4) is similarly obtained from (3.10). Finally, estimates (3.5) are deduced from (3.3), (3.4) and the boundedness of the projector P_r. The proof is complete.

PROPOSITION 3.2. *Let L be the operator defined in* (3.2) *and L^* its formal adjoint. For $1<r<\frac{1}{2}n$ and $0<\omega<\frac{1}{2}\pi$, there is a number $\eta=\eta(r,\omega)>0$ such that if*

$$\|w\|+\|\nabla w\|<\eta,$$

then we have the estimates

$$\|(\lambda+L)^{-1}u\|_r \leqslant C_r\|u\|_r/|\lambda|, \quad \|(\lambda+L^*)^{-1}u\|_r \leqslant C_r\|u\|_r/|\lambda|, \tag{3.15}$$

and

$$\|\nabla^2(\lambda+L)^{-1}u\|_r \leqslant C_r\|u\|_r, \quad \|\nabla^2(\lambda+L^*)^{-1}u\|_r \leqslant C_r\|u\|_r, \tag{3.16}$$

for all $u\in L^r_\sigma$ and $\lambda\in\mathbf{C}\backslash 0$ with $|\arg\lambda|\leqslant\pi-\omega$.

Proof. Due to the equality

$$\lambda+L=\lambda+A+B=(I+B(\lambda+A)^{-1})(\lambda+A) \quad \text{on } D(A),$$

we can formally write

$$(\lambda+L)^{-1}u=(\lambda+A)^{-1}\sum_{j=0}^{\infty}(-B(\lambda+A)^{-1})^j u. \tag{3.17}$$

From the estimate $\|\nabla^2(\lambda+A)^{-1}u\|_r\leqslant C_r\|u\|_r$ for $1<r<\frac{1}{2}n$ (see [8]) and from (3.5) we get

$$\|(B(\lambda+A)^{-1})^j u\|_r \leqslant [C'(\|w\|+\|\nabla w\|)]^j\|u\|_r,$$

so the right-hand side of (3.17) converges provided $C'(\|w\|+\|\nabla w\|)<1$. Estimates (3.15) and (3.16) are then deduced from the estimates

$$\|(\lambda+A)^{-1}\|\leqslant C/|\lambda|, \quad \|\nabla^2(\lambda+A)^{-1}\|\leqslant C,$$

as established in [8]. The case of the adjoint operator L^* is treated similarly. The proof is complete.

COROLLARY 3.3. (i) *For each $1<r<\infty$ and $0<\omega<\frac{1}{2}\pi$ there is a number $\eta=\eta(r,\omega)>0$ so that if*

$$\|w\|+\|\nabla w\|<\eta$$

then $(\lambda+L)^{-1}$ and $(\lambda+L^*)^{-1}$ exist as bounded operators in L^r_σ for $\lambda\in\mathbf{C}\backslash 0$ with $|\arg\lambda|\leqslant\pi-\omega$ and satisfy the estimates

$$\|(\lambda+L)^{-1}\|\leqslant C/|\lambda|, \quad \|(\lambda+L^*)^{-1}\|\leqslant C/|\lambda|.$$

(ii) *If* $1<r<\frac{1}{2}n$ *and*

$$\|w\|+\|\nabla w\|<\eta,$$

then the operators $\nabla(\lambda+L)^{-1}$ *and* $\nabla(\lambda+L^*)^{-1}$ *exist as bounded operators from* L^r_σ *to* L^r *satisfying*

$$\|\nabla(\lambda+L)^{-1}\|\leqslant C/|\lambda|^{1/2}, \quad \|\nabla(\lambda+L^*)^{-1}\|\leqslant C/|\lambda|^{1/2}.$$

Proof. (i) We see by duality that (3.15) holds for $1<r<\frac{1}{2}n$ and $n/(n-2)<r<\infty$. Since the spaces L^r_σ form a complex interpolation family, we obtain (3.15) for all $1<r<\infty$ by interpolating between the above two cases. (ii) is a consequence of (3.15), (3.16) and the estimate

$$|\lambda|^{1/2}\|\nabla u\|_r\leqslant C(|\lambda|\cdot\|u\|_r)^{1/2}\|\nabla^2 u\|_r^{1/2}.$$

The proof is complete.

In view of the standard theory of analytic semigroups in a Banach space as given for instance in [37] and [44], Corollary 3.3 (i) asserts that the operators $-L$ and $-L^*$ generate in L^r_σ, $1<r<\infty$, bounded analytic semigroups, which we denote by $\{e^{-tL}\}_{t\geqslant 0}$ and $\{e^{-tL^*}\}_{t\geqslant 0}$, respectively, through the Dunford integrals

$$e^{-tL}=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda+L)^{-1}\,d\lambda, \quad e^{-tL^*}=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda+L^*)^{-1}\,d\lambda.$$

Here, the path Γ of integration in the complex plane is taken in the form

$$\Gamma=\Gamma_+\cup\Gamma_0\cup\Gamma_-,$$

where

$$\Gamma_\pm=\{re^{\pm i\omega}:1/t\leqslant r<+\infty\}, \quad \Gamma_0=\{t^{-1}e^{i\theta}:-\omega\leqslant\theta\leqslant\omega\},$$

for an arbitrarily fixed ω with $\frac{1}{2}\pi<\omega<\pi$. Furthermore, these integrals, together with Corollary 3.3 (ii), yield the estimates

$$\|\nabla e^{-tL}u\|_r\leqslant C_r t^{-1/2}\|u\|_r, \quad \|\nabla e^{-tL^*}u\|_r\leqslant C_r t^{-1/2}\|u\|_r, \tag{3.18}$$

for $1<r<\frac{1}{2}n$. Estimates of the form (3.18) are indispensable for our purposes and so are desirable to be extended to the case of more general r. This extension is given by the following

PROPOSITION 3.4. *Let $n'<r<n$ and $0<\omega<\frac{1}{2}\pi$. Then there exists a number $\eta=\eta(r,\omega)>0$ such that if*

$$\|w\|<\eta,$$

then we have the estimates (as bounded operators from L^r_σ to L^r)

$$\|\nabla(\lambda+L)^{-1}\|\leqslant C/|\lambda|^{1/2}, \quad \|\nabla(\lambda+L^*)^{-1}\|\leqslant C/|\lambda|^{1/2},$$

for all $\lambda\in\mathbf{C}\backslash 0$ with $|\arg\lambda|\leqslant\pi-\omega$.

Proof. This time, we write

$$\nabla(\lambda+L)^{-1}=\nabla\sum_{j=0}^{\infty}(-(\lambda+A)^{-1}B)^j(\lambda+A)^{-1} \tag{3.19}$$

and

$$\nabla(\lambda+L^*)^{-1}=\nabla\sum_{j=0}^{\infty}(-(\lambda+A)^{-1}B^*)^j(\lambda+A)^{-1} \tag{3.20}$$

and discuss convergence of the right-hand sides. To do so, we need the following

LEMMA 3.5. *If $n'<r<n$, we have the estimates*

$$\|\nabla(\lambda+A)^{-1}Bu\|_r\leqslant C\|w\|\cdot\|\nabla u\|_r, \quad u\in D(A_r), \tag{3.21}$$

and

$$\|\nabla(\lambda+A)^{-1}B^*u\|_r\leqslant C\|w\|\cdot\|\nabla u\|_r, \quad u\in D(A_r), \tag{3.22}$$

uniformly for $\lambda\in\mathbf{C}\backslash 0$ with $|\arg\lambda|\leqslant\pi-\omega$.

Admitting Lemma 3.5 for a moment, we continue the proof of Proposition 3.4. From estimates (3.21) and (3.22) it follows that if $\|w\|$ is sufficiently small, then

$$\|\nabla(\lambda+L)^{-1}u\|_r\leqslant\left(\sum_{j=0}^{\infty}(C\|w\|)^j\right)\|\nabla(\lambda+A)^{-1}u\|_r\leqslant C\|\nabla(\lambda+A)^{-1}u\|_r, \tag{3.23}$$

and

$$\|\nabla(\lambda+L^*)^{-1}u\|_r\leqslant\left(\sum_{j=0}^{\infty}(C\|w\|)^j\right)\|\nabla(\lambda+A)^{-1}u\|_r\leqslant C\|\nabla(\lambda+A)^{-1}u\|_r. \tag{3.24}$$

On the other hand, since $r<n$, we see by Theorem 4.4 of [4] that

$$\begin{aligned}\|\nabla(\lambda+A)^{-1}u\|_r&\leqslant C\|A^{1/2}(\lambda+A)^{-1}u\|_r\\&\leqslant C\|A(\lambda+A)^{-1}u\|_r^{1/2}\|(\lambda+A)^{-1}u\|_r^{1/2}\leqslant C\|u\|_r/|\lambda|^{1/2},\end{aligned}$$

and the desired result follows.

Proof of Lemma 3.5. We apply a duality argument. In what follows $\langle\cdot,\cdot\rangle$ denotes the duality pairing between various Banach spaces. For $\varphi\in C_0^\infty(\Omega)$ we have

$$\begin{aligned}|\langle\nabla(\lambda+A)^{-1}Bu,\varphi\rangle| &= |\langle Bu,(\lambda+A)^{-1}P(\nabla\varphi)\rangle|\\ &\leqslant |\langle w\cdot\nabla u,(\lambda+A)^{-1}P(\nabla\varphi)\rangle|+|\langle u\cdot\nabla w,(\lambda+A)^{-1}P(\nabla\varphi)\rangle|\\ &= |\langle u,w\cdot\nabla(\lambda+A)^{-1}P(\nabla\varphi)\rangle|+|\langle w,u\cdot\nabla(\lambda+A)^{-1}P(\nabla\varphi)\rangle|\\ &\leqslant 2\|w\|\cdot\||x|^{-1}u\|_r\|\nabla(\lambda+A)^{-1}P(\nabla\varphi)\|_{r'}\\ &\leqslant C\|w\|\cdot\|\nabla u\|_r\|\nabla(\lambda+A)^{-1}P(\nabla\varphi)\|_{r'}\end{aligned}$$

and

$$\begin{aligned}|\langle\nabla(\lambda+A)^{-1}B^*u,\varphi\rangle| &= |\langle B^*u,(\lambda+A)^{-1}P(\nabla\varphi)\rangle| = |\langle u,B(\lambda+A)^{-1}P(\nabla\varphi)\rangle|\\ &\leqslant |\langle u,w\cdot\nabla(\lambda+A)^{-1}P(\nabla\varphi)\rangle|+|\langle u,(\lambda+A)^{-1}P(\nabla\varphi)\cdot\nabla w\rangle|\\ &= |\langle u,w\cdot\nabla(\lambda+A)^{-1}P(\nabla\varphi)\rangle|+|\langle(\lambda+A)^{-1}P(\nabla\varphi)\cdot\nabla u,w\rangle|\\ &\leqslant C\|w\|\cdot\|\nabla u\|_r\|\nabla(\lambda+A)^{-1}P(\nabla\varphi)\|_{r'}.\end{aligned}$$

Estimates (3.21) and (3.22) are thus deduced from

LEMMA 3.6. *If $n'<r<n$, then*

$$\|\nabla(\lambda+A)^{-1}P(\nabla\varphi)\|_r\leqslant C\|\varphi\|_r,\quad \varphi\in C_0^\infty(\Omega),$$

uniformly in $\lambda\in\mathbf{C}\setminus 0$ with $|\arg\lambda|\leqslant\pi-\omega$.

Proof. In view of the relation

$$\nabla(\lambda+A)^{-1}P(\nabla\varphi)=\nabla(\lambda+A)^{-1/2}(\lambda+A)^{-1/2}P(\nabla\varphi)$$

and the fact that $(\lambda+A)^{-1/2}P\nabla$ is just the dual to $\nabla(\lambda+A)^{-1/2}$, it suffices to show that if $n'<r<n$, then the estimate

$$\|\nabla(\lambda+A)^{-1/2}u\|_r\leqslant C\|u\|_r \tag{3.25}$$

holds uniformly in $\lambda\neq 0$ with $|\arg\lambda|\leqslant\pi-\omega$. Since $r<n$, by Theorem 4.4 of [4] we have

$$\|\nabla(\lambda+A)^{-1/2}u\|_r\leqslant C\|A^{1/2}(\lambda+A)^{-1/2}u\|_r. \tag{3.26}$$

Hence we get (3.25) for $\lambda>0$ (see [30]). For general λ, we write $\lambda=|\lambda|e^{i\theta}$ with $0<|\theta|\leqslant\pi-\omega$. Then, as shown below, the standard evaluation of the Dunford integral shows that the fractional powers $(e^{-i\theta}A)^{1/2}$ and $(|\lambda|+e^{-i\theta}A)^{-1/2}$ are well defined and satisfy the relation

$$A^{1/2}=e^{i\theta/2}(e^{-i\theta}A)^{1/2},\quad (\lambda+A)^{-1/2}=e^{-i\theta/2}(|\lambda|+e^{-i\theta}A)^{-1/2}. \tag{3.27}$$

It thus follows from (3.26) and (3.27) that (see [30])

$$\|\nabla(\lambda+A)^{-1/2}u\|_r\leqslant C\|(e^{-i\theta}A)^{1/2}(|\lambda|+e^{-i\theta}A)^{-1/2}u\|_r\leqslant C_\theta\|u\|_r\leqslant C_\omega\|u\|_r,$$

which proves Lemma 3.6.

There remains to establish (3.27). We shall show below more generally that if $\lambda=|\lambda|e^{i\theta}$, $|\theta|\leqslant\pi-\omega$ and $0<\alpha<1$, then the fractional powers $(e^{-i\theta}A)^\alpha$ and $(|\lambda|+e^{-i\theta}A)^{-\alpha}$ are well defined and satisfy

$$A^\alpha=e^{i\alpha\theta}(e^{-i\theta}A)^\alpha,\quad (\lambda+A)^{-\alpha}=e^{-i\alpha\theta}(|\lambda|+e^{-i\theta}A)^{-\alpha}. \tag{3.27$'$}$$

In what follows we assume without loss of generality that $\theta>0$, and write $\|\cdot\|_r=\|\cdot\|$. Since $\|(\mu-e^{-i\theta}A)^{-1}\|\leqslant C_\theta/|\mu|$ for $\mu\neq0$ in a conic neighborhood of the negative real axis, the fractional power $(e^{-i\theta}A)^\alpha$ is defined as

$$(e^{-i\theta}A)^\alpha u=\frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1}(t+e^{-i\theta}A)^{-1}(e^{-i\theta}A)u\,dt,$$

for $u\in D(e^{-i\theta}A)=D(A)$ (see [30]). By the change of the variable $\mu=te^{i\theta}$, we have

$$(e^{-i\theta}A)^\alpha u=\frac{\sin\pi\alpha}{\pi}e^{-i\alpha\theta}\int_\Gamma\mu^{\alpha-1}(\mu+A)^{-1}Au\,d\mu, \tag{3.28}$$

where the path of integration

$$\Gamma=\{te^{i\theta}:0<t<+\infty\}$$

is oriented in the direction from $t=0$ to $t=+\infty$. Now fix $0<\varepsilon<R$. Since $\sigma(-A)\subset\mathbf{R}_-$, where $\sigma(-A)$ is the spectrum of $-A$, the integrand of (3.28) is analytic in the closed region surrounded by the closed curve

$$C=\{\varepsilon\leqslant t\leqslant R\}\cup C_R\cup\Gamma_{\varepsilon,R}\cup C_\varepsilon,$$

with

$$\Gamma_{\varepsilon,R}=\{te^{i\theta}:\varepsilon\leqslant t\leqslant R\}$$

and

$$C_R=\{Re^{i\varphi}:0\leqslant\varphi\leqslant\theta\},\quad C_\varepsilon=\{\varepsilon e^{i\varphi}:\theta\geqslant\varphi\geqslant0\}.$$

By the Cauchy integral theorem in the complex function theory, we have

$$\begin{aligned}\int_{\Gamma_{\varepsilon,R}} \mu^{\alpha-1}(\mu+A)^{-1}Au\,d\mu = \int_\varepsilon^R t^{\alpha-1}(t+A)^{-1}Au\,dt \\ +\int_{C_R} \mu^{\alpha-1}(\mu+A)^{-1}Au\,d\mu \\ +\int_{C_\varepsilon} \mu^{\alpha-1}(\mu+A)^{-1}Au\,d\mu.\end{aligned} \tag{3.29}$$

The norm of the integral over C_R is estimated as

$$\leqslant R^{\alpha-1}\int_0^\theta \|(Re^{i\varphi}+A)^{-1}\|\cdot\|Au\|R\,d\varphi \leqslant C\theta R^{\alpha-1}\|Au\| \to 0,$$

while the norm of the integral over C_ε is estimated as

$$\leqslant \varepsilon^{\alpha-1}\int_0^\theta \|A(\varepsilon e^{i\varphi}+A)^{-1}u\|\varepsilon\,d\varphi \leqslant C\theta\varepsilon^\alpha\|u\| \to 0.$$

Hence, (3.29) gives

$$\int_\Gamma \mu^{\alpha-1}(\mu+A)^{-1}Au\,d\mu = \int_0^\infty t^{\alpha-1}(t+A)^{-1}Au\,dt$$

for $u\in D(A)$. Combining this with (3.28) gives the first assertion of (3.27′). To show the second assertion of (3.27′), we use the representation (see [44])

$$(\lambda+A)^{-\alpha}u = \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(t+\lambda+A)^{-1}u\,dt \tag{3.30}$$

for all $u\in L^r_\sigma$. Since

$$(t+\lambda+A)^{-1} = e^{-i\theta}(te^{-i\theta}+|\lambda|+e^{-i\theta}A)^{-1},$$

the change of the variable $\mu=te^{-i\theta}$ applied to (3.30) yields

$$(\lambda+A)^{-\alpha}u = e^{-i\alpha\theta}\frac{\sin\pi\alpha}{\pi}\int_\Gamma \mu^{-\alpha}(\mu+|\lambda|+e^{-i\theta}A)^{-1}u\,d\mu, \tag{3.31}$$

where the path of integration

$$\Gamma = \{te^{-i\theta} : 0 < t < +\infty\}$$

is oriented from $t=0$ to $t=+\infty$. Since $\sigma(-|\lambda|-e^{-i\theta}A) = -|\lambda|-e^{-i\theta}\sigma(A)$, the integrand of (3.31) is analytic in the closed region surrounded by the closed curve

$$C = \Gamma_{R,\varepsilon}\cup C_R\cup C_\varepsilon\cup\{R\geqslant t\geqslant\varepsilon\},$$

with

$$\Gamma_{R,\varepsilon}=\{te^{-\theta}:\varepsilon\leqslant t\leqslant R\}$$

and

$$C_R=\{Re^{i\varphi}:-\theta\leqslant\varphi\leqslant 0\},\quad C_\varepsilon=\{\varepsilon e^{i\varphi}:0\geqslant\varphi\geqslant-\theta\}.$$

The integral over C_R is estimated as

$$\leqslant R^{-\alpha}\int_0^\theta \|(Re^{i\varphi}+|\lambda|+e^{-i\theta}A)^{-1}\|\cdot\|u\|R\,d\varphi\leqslant C_{|\lambda|}\theta R^{-\alpha}\|u\|\to 0$$

as $R\to\infty$, while the integral over C_ε is estimated as

$$\leqslant \varepsilon^{-\alpha}\int_0^\theta \|(\varepsilon e^{i\varphi}+|\lambda|+e^{-i\theta}A)^{-1}\|\cdot\|u\|\varepsilon\,d\varphi\leqslant C_{|\lambda|}\theta \varepsilon^{1-\alpha}\|u\|\to 0$$

as $\varepsilon\to 0$, since in this case the resolvent is bounded as $\varepsilon\to 0$. So, by the Cauchy integral theorem,

$$\int_\Gamma \mu^{-\alpha}(\mu+|\lambda|+e^{-i\theta}A)^{-1}u\,d\mu=\int_0^\infty t^{-\alpha}(t+|\lambda|+e^{-i\theta}A)^{-1}u\,dt.$$

This, together with (3.31), yields the result.

COROLLARY 3.7. *For each $1<r<n$ there is a number $\eta=\eta(r)>0$ such that if*

$$\|w\|+\|\nabla w\|<\eta,$$

then we have the estimates (3.18).

Proof. The result immediately follows from Corollary 3.3 (ii), Proposition 3.4 and complex interpolation. The proof is complete.

The above proof of Proposition 3.4 implies the following result for the resolvents of $-L$ and $-L^*$, which is interesting when compared with Proposition 3.2 in that no assumptions on ∇w are needed.

PROPOSITION 3.8. *Let $n'<r<n$ and $0<\omega<\frac{1}{2}\pi$. Then there exists a number $\eta=\eta(r,\omega)$ such that if*

$$\|w\|<\eta,$$

then we have (as bounded operators in L^r_σ)

$$\|(\lambda+L)^{-1}\|\leqslant C/|\lambda|,\quad \|(\lambda+L^*)^{-1}\|\leqslant C/|\lambda|,$$

for all $\lambda\in\mathbf{C}\setminus 0$ with $|\arg\lambda|\leqslant\pi-\omega$.

Proof. Since $n'<r<n$ if and only if $n'<r'<n$, it suffices only to deduce the estimate for operator L. As in the proof of Proposition 3.4, we write

$$(\lambda+L)^{-1}=\sum_{j=0}^{\infty}(-(\lambda+A)^{-1}B)^j(\lambda+A)^{-1}. \tag{3.32}$$

In addition to Lemma 3.5, we here also apply

LEMMA 3.9. *If $n'<r<n$, we have the estimate*

$$\|(\lambda+A)^{-1}Bu\|_r \leqslant C|\lambda|^{-1/2}\|w\|\cdot\|\nabla u\|_r, \quad u\in D(A_r), \tag{3.33}$$

for all $\lambda\in\mathbf{C}\setminus 0$ with $|\arg\lambda|\leqslant\pi-\omega$.

We continue the proof of Proposition 3.8, admitting Lemma 3.9 for a moment. From Lemmas 3.5 and 3.9 we see that if $\|w\|$ is small enough, then

$$\begin{aligned}\|(\lambda+L)^{-1}u\|_r &\leqslant C|\lambda|^{-1}\|u\|_r + C\|w\|\cdot|\lambda|^{-1/2}\left(\sum_{j=1}^{\infty}(C'\|w\|)^{j-1}\right)\|\nabla(\lambda+A)^{-1}u\|_r \\ &\leqslant C|\lambda|^{-1}\|u\|_r,\end{aligned}$$

which proves Proposition 3.8.

Proof of Lemma 3.9. As in the proof of Lemma 3.5, we apply a duality argument. For $\varphi\in C_{0,\sigma}^{\infty}(\Omega)$ we have

$$\begin{aligned}|\langle(\lambda+A)^{-1}Bu,\varphi\rangle| &= |\langle Bu,(\lambda+A)^{-1}\varphi\rangle| \\ &\leqslant |\langle w\cdot\nabla u,(\lambda+A)^{-1}\varphi\rangle| + |\langle w, u\cdot\nabla(\lambda+A)^{-1}\varphi\rangle| \\ &\leqslant \|w\|\cdot\|\nabla u\|_r\||x|^{-1}(\lambda+A)^{-1}\varphi\|_{r'} + \|w\|\cdot\||x|^{-1}u\|_r\|\nabla(\lambda+A)^{-1}\varphi\|_{r'} \\ &\leqslant C\|w\|\cdot\|\nabla u\|_r\|\nabla(\lambda+A)^{-1}\varphi\|_{r'} \leqslant C|\lambda|^{-1/2}\|w\|\cdot\|\nabla u\|_r\|\varphi\|_{r'},\end{aligned}$$

which shows Lemma 3.9.

Corollary 3.3, Propositions 3.4 and 3.8 together imply

THEOREM 3.10. (i) *Let $n\geqslant 3$ and $1<r<\infty$. Then there is a number $\lambda=\lambda(n,r)>0$ such that if*

$$\|w\|+\|\nabla w\|<\lambda,$$

then $\{e^{-tL}\}_{t\geqslant 0}$ and $\{e^{-tL^*}\}_{t\geqslant 0}$ are bounded analytic C_0 semigroups on L^r_σ.

(ii) *Let $n\geqslant 3$ and $n'<r<n$. Then there is a number $\eta=\eta(n,r)>0$ such that if*

$$\|w\|<\eta,$$

then $\{e^{-tL}\}_{t\geqslant 0}$ and $\{e^{-tL^*}\}_{t\geqslant 0}$ are bounded analytic C_0 semigroups on L^r_σ and satisfy (3.18).

(iii) *Let $n\geqslant 3$. For $1<q\leqslant r<\infty$, there is a number $\lambda'=\lambda'(n,r,q)>0$ such that if*

$$\|w\|+\|\nabla w\|<\lambda',$$

then we have the estimates

$$\|e^{-tL}a\|_r,\ \|e^{-tL^*}a\|_r \leqslant Mt^{-(n/q-n/r)/2}\|a\|_q. \tag{3.34}$$

(iv) *Let* $n\geqslant 3$. *For* $1<q\leqslant r<n$, *there is a number* $\eta'=\eta'(n,r,q)>0$ *such that if*

$$\|w\|+\|\nabla w\|<\eta',$$

then we have the estimates

$$\|\nabla e^{-tL}a\|_r,\ \|\nabla e^{-tL^*}a\|_r \leqslant Mt^{-1/2-(n/q-n/r)/2}\|a\|_q. \tag{3.35}$$

(v) *Under the assumptions of* (i) *and* (iii) *with* $q<r$, *we have*

$$\lim_{t\to\infty}\|e^{-tL}a\|_r=0, \quad \lim_{t\to\infty}\|e^{-tL^*}a\|_r=0,$$

for all $a\in L^r_\sigma$.

Estimates (3.34) and (3.35) are deduced from (3.18) via the Sobolev inequality. When $w=0$, Iwashita [23] shows that (3.35) holds for $1<q\leqslant r\leqslant n$. Assertion (v) follows from (3.34), the boundedness of the semigroups in L^r_σ and the fact that $C^\infty_{0,\sigma}(\Omega)$ is dense in L^r_σ.

Remark. Kozono and Ogawa [25] prove (3.18) for $1<r<\frac{1}{2}n$, or for $1<r\leqslant 2$ and $n=3,4$, assuming that

$$w\in L^n(\Omega)\cap L^\infty(\Omega), \quad \nabla w\in L^{n/2}(\Omega)\cap L^\infty(\Omega) \tag{3.36}$$

and that $\|w\|_n+\|\nabla w\|_{n/2}$ is small, depending on r. Obviously, our results improve those of [25].

When $n\geqslant 4$, we have $n'<\frac{1}{2}n$. Thus, the stationary flow w obtained for instance in Theorem 2.3 (ii) satisfies

$$\nabla w\in L^r \quad \text{for all } n'<r<\tfrac{1}{2}n.$$

Using this kind of condition, we can improve Theorem 3.10. Our subsequent arguments of this section are based on the following result, which is due to Chen [11].

PROPOSITION 3.11. *Let* $n\geqslant 3$. *Then we have the estimate*

$$\|e^{-tA}a\|_\infty \leqslant M_r t^{-n/2r}\|a\|_r \quad (1<r\leqslant 2n).$$

Proposition 3.11 is proved in [11, Appendix] for $n=3$; but the proof applies in all dimensions $n\geqslant 3$.

PROPOSITION 3.12. (i) *Let $n\geqslant 4$ and suppose $\nabla w\in L^q_w\cap L^\infty$ for some $n'\leqslant q<\frac{1}{2}n$. Then we have the estimates*

$$\|e^{-tA}Bu\|_\infty+\|\nabla e^{-tA}Bu\|_n\leqslant Ct^{-n/2p}(1+t)^{-n/2r+n/2p} \times(\|\nabla w\|_{q,w}+\|\nabla w\|_\infty)(\|u\|_\infty+\|\nabla u\|_n) \tag{3.37}$$

and

$$\|e^{-tA}B^*u\|_\infty+\|\nabla e^{-tA}B^*u\|_n\leqslant Ct^{-n/2p}(1+t)^{-n/2r+n/2p} \times(\|\nabla w\|_{q,w}+\|\nabla w\|_\infty)(\|u\|_\infty+\|\nabla u\|_n), \tag{3.37'}$$

for $u\in C^\infty_{0,\sigma}(\Omega)$ with $q<r<\frac{1}{2}n<p<n$.

When $p=n$, estimates of the form (3.37) and (3.37′) hold with $\|\nabla w\|_{q,w}$ replaced by $\|w\|+\|\nabla w\|_{q,w}$.

(ii) *Let $n=3$ and suppose that $\nabla w\in L^q\cap L^\infty$ for some $1<q<\frac{3}{2}$. Then we have*

$$\|e^{-tA}Bu\|_\infty+\|\nabla e^{-tA}Bu\|_3\leqslant Ct^{-3/2p}(1+t)^{-3/2r+3/2p} \times(\|\nabla w\|_q+\|\nabla w\|_\infty)(\|u\|_\infty+\|\nabla u\|_3) \tag{3.38}$$

and

$$\|e^{-tA}B^*u\|_\infty+\|\nabla e^{-tA}B^*u\|_3\leqslant Ct^{-3/2p}(1+t)^{-3/2r+3/2p} \times(\|\nabla w\|_q+\|\nabla w\|_\infty)(\|u\|_\infty+\|\nabla u\|_3), \tag{3.38'}$$

for $u\in C^\infty_{0,\sigma}(\Omega)$ with $q\leqslant r<\frac{3}{2}<p<3$.

When $p=3$, estimates of the form (3.38) and (3.38′) hold with $\|\nabla w\|_q$ replaced by $\|w\|+\|\nabla w\|_q$.

Proof. We here prove only (i); statement (ii) is proved similarly. For $s=p,r$, Proposition 3.11 yields

$$\begin{aligned}\|e^{-tA}Bu\|_\infty &\leqslant Ct^{-n/2s}(\|w\cdot\nabla u\|_s+\|u\cdot\nabla w\|_s)\\ &\leqslant Ct^{-n/2s}(\|w\|_{ns/(n-s)}\|\nabla u\|_n+\|u\|_\infty\|\nabla w\|_s)\\ &\leqslant Ct^{-n/2s}\|\nabla w\|_s(\|u\|_\infty+\|\nabla u\|_n)\\ &\leqslant Ct^{-n/2s}(\|\nabla w\|_{q,w}+\|\nabla w\|_\infty)(\|u\|_\infty+\|\nabla u\|_n)\end{aligned}$$

and, similarly by [23],

$$\begin{aligned}\|\nabla e^{-tA}Bu\|_n &\leqslant C^{-n/2s}(\|w\cdot\nabla u\|_s+\|u\cdot\nabla w\|_s)\\ &\leqslant Ct^{-n/2s}(\|\nabla w\|_{q,w}+\|\nabla w\|_\infty)(\|u\|_\infty+\|\nabla u\|_n).\end{aligned}$$

Since $n/2p<1<n/2r$, we obtain (3.37). When $p=n$, we apply the estimate

$$\|w\nabla u\|_n\leqslant\|w\|_\infty\|\nabla u\|_n\leqslant C\|w\|\cdot\|\nabla u\|_n$$

in the above calculations and obtain (3.37) with $p=n$. Estimate (3.37′) is deduced in the same way. The proof is complete.

THEOREM 3.13. (i) *Let $n\geqslant 4$ and let w satisfy the assumption of Proposition* 3.12 (i). *Then we have the estimates*

$$\|e^{-tL}a\|_\infty,\ \|e^{-tL^*}a\|_\infty \leqslant M_s t^{-n/2s}\|a\|_s \quad (1<s\leqslant n) \tag{3.39}$$

and

$$\|\nabla e^{-tL}a\|_n,\ \|\nabla e^{-tL^*}a\|_n \leqslant M_s t^{-n/2s}\|a\|_s \quad (1<s\leqslant n), \tag{3.39'}$$

provided $\|w\|+\|\nabla w\|_{q,w}+\|\nabla w\|_\infty$ is sufficiently small, depending on s.

(ii) *Let $n=3$ and let w satisfy the assumption of Proposition* 3.12 (ii). *Then we have*

$$\|e^{-tL}a\|_\infty,\ \|e^{-tL^*}a\|_\infty \leqslant M_s t^{-3/2s}\|a\|_s \quad (1<s\leqslant 3) \tag{3.40}$$

and

$$\|\nabla e^{-tL}a\|_3,\ \|\nabla e^{-tL^*}a\|_3 \leqslant M_s t^{-3/2s}\|a\|_s \quad (1<s\leqslant 3), \tag{3.40'}$$

provided $\|w\|+\|\nabla w\|_q+\|\nabla w\|_\infty$ is sufficiently small, depending on s.

Proof. (i) We deduce (3.39) and (3.39′) only for the semigroup $\{e^{-tL}\}_{t\geqslant 0}$; the case of the dual semigroup is treated similarly. Furthermore, in view of Theorem 3.10, it suffices to assume $\frac{1}{2}n<s\leqslant n$. Now, the function $v(t)=e^{-tL}a$ satisfies

$$v(t)=e^{-tA}a-\int_0^t e^{-(t-\tau)A}Bv(\tau)\,d\tau.$$

We take $q<r<\frac{1}{2}n<p<n$, set

$$V(t)=\sup_{0<\tau<t}\tau^{n/2s}(\|v\|_\infty+\|\nabla v\|_n)(\tau)$$

and apply Propositions 3.11 and 3.12, to obtain

$$\begin{aligned}\|v(t)\|_\infty &\leqslant C_1 t^{-n/2s}\|a\|_s\\ &\quad +C_2\int_0^t (t-\tau)^{-n/2p}(t-\tau+1)^{-n/2r+n/2p}(\|v\|_\infty+\|\nabla v\|_n)(\tau)\,d\tau\\ &\leqslant C_1 t^{-n/2s}\|a\|_s\\ &\quad +C_2 V(t)\int_0^t (t-\tau)^{-n/2p}(t-\tau+1)^{-n/2r+n/2p}\tau^{-n/2s}\,d\tau\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}\|\nabla v(t)\|_n &\leqslant C_1' t^{-n/2s}\|a\|_s\\ &\quad +C_2'\int_0^t (t-\tau)^{-n/2p}(t-\tau+1)^{-n/2r+n/2p}(\|v\|_\infty+\|\nabla v\|_n)(\tau)\,d\tau\\ &\leqslant C_1' t^{-n/2s}\|a\|_s\\ &\quad +C_2' V(t)\int_0^t (t-\tau)^{-n/2p}(t-\tau+1)^{-n/2r+n/2p}\tau^{-n/2s}\,d\tau,\end{aligned} \tag{3.42}$$

where C_2 and C_2' are constant multiples of $\|\nabla w\|_{q,w}+\|\nabla w\|_\infty$. Consider the integral

$$I=\int_0^t (t-\tau)^{-n/2p}(t-\tau+1)^{-n/2r+n/2p}\tau^{-n/2s}\,d\tau=\int_0^{t/2}+\int_{t/2}^t\equiv I_1+I_2.$$

When $t\geqslant 1$, the change of the variable $\tau=t\sigma$ yields

$$I_1=t^{1-n/2r-n/2s}\int_0^{1/2}(1-\sigma)^{-n/2p}(1-\sigma+t^{-1})^{-n/2r+n/2p}\sigma^{-n/2s}\,d\sigma.$$

By our choice of p, r and s, we get

$$I_1\leqslant Ct^{1-n/2r-n/2s}\int_0^{1/2}(1-\sigma)^{-n/2p}\sigma^{-n/2s}\,d\sigma\leqslant Ct^{-n/2s}. \tag{3.43}$$

When $0<t<1$, we apply $1-\sigma+t^{-1}>t^{-1}$ to obtain

$$\begin{aligned}I_1&\leqslant t^{1-n/2r-n/2s}\cdot t^{n/2r-n/2p}\int_0^{1/2}(1-\sigma)^{-n/2p}\sigma^{-n/2s}\,d\sigma\\&=Ct^{1-n/2p-n/2s}\leqslant Ct^{-n/2s}.\end{aligned} \tag{3.44}$$

On the other hand, we easily see that

$$\begin{aligned}I_2&\leqslant Ct^{-n/2s}\int_0^{t/2}\tau^{-n/2p}(\tau+1)^{-n/2r+n/2p}\,d\tau\\&\leqslant Ct^{-n/2s}\int_0^{\infty}\tau^{-n/2p}(\tau+1)^{-n/2r+n/2p}\,d\tau=Ct^{-n/2s}.\end{aligned} \tag{3.45}$$

Combining (3.41)-(3.42) with (3.43)-(3.45) we obtain

$$\|v(t)\|_\infty+\|\nabla v(t)\|_n\leqslant Ct^{-n/2s}(\|a\|_s+C'V(t))$$

and therefore

$$V(t)\leqslant C(\|a\|_s+C'V(t)),$$

where C' is a constant multiple of $\|w\|+\|\nabla w\|_{q,w}+\|\nabla w\|_\infty$. Taking w sufficiently small, we get $V(t)\leqslant C\|a\|_s$. This proves (i). Assertion (ii) is proved similarly, so the details are omitted.

COROLLARY 3.14. *Under the assumption of Theorem* 3.13, *we have*

$$\|e^{-tL}a\|_2\leqslant Mt^{-n/4}\|a\|_1,\quad a\in L^2_\sigma\cap L^1. \tag{3.46}$$

Proof. We apply a duality argument. Given $\varphi\in C^\infty_{0,\sigma}(\Omega)$, Theorem 3.13 implies

$$|\langle e^{-tL}a,\varphi\rangle|=|\langle a,e^{-tL^*}\varphi\rangle|\leqslant\|a\|_1\|e^{-tL^*}\varphi\|_\infty\leqslant Ct^{-n/4}\|a\|_1\|\varphi\|_2.$$

Since $C_{0,\sigma}^{\infty}(\Omega)$ is dense in L_σ^2, we see that $e^{-tL}a\in(L_\sigma^2)^*=L_\sigma^2$ and (3.46) holds. The proof is complete.

Remark. Proposition 3.12, Theorem 3.13 and Corollary 3.14 are essentially due to Chen [11]. He deduced similar estimates, assuming that $n=3$ and

$$\nabla w\in L^r\cap L^p \quad \text{for some } 1<r<\tfrac{3}{2}<p\leqslant 2. \tag{3.47}$$

We note that if $n=3$ the assumption of Proposition 3.12 or Theorem 3.13 implies $\nabla w\in L^{3/2}$ and so by Theorem 2.4 the total net force exerted to the obstacle by the flow w vanishes identically. Thus, in case $n=3$ the stationary flows w with this property exist in a very restrictive situation (see [16]).

4. Stability in L^2

We first define the notion of weak solution of perturbation equation (1.5). The definition is due to Masuda [32]. Given $a\in L_\sigma^2$, a weakly continuous function $u:[0,\infty)\to L_\sigma^2$ is called a weak solution of (1.5) with initial velocity a if

$$u\in L^\infty(0,\infty;L_\sigma^2), \quad \nabla u\in L_{\text{loc}}^2([0,\infty);L^2(\Omega)), \quad u(0)=a,$$

and the identity

$$\langle u(t),\varphi(t)\rangle-\langle u(s),\varphi(s)\rangle+\int_s^t\langle\nabla u,\nabla\varphi\rangle\,d\tau$$
$$=\int_s^t\langle u,\varphi'\rangle\,d\tau-\int_s^t\langle w\otimes u+u\otimes w+u\otimes u,\nabla\varphi\rangle\,d\tau$$

holds for all $0\leqslant s\leqslant t$ and all $\varphi\in C([0,\infty);L_\sigma^2\cap W_0^{1,2}(\Omega)\cap L^n(\Omega))\cap C^1([0,\infty);L_\sigma^2)$. Condition $\varphi\in L^n(\Omega)$ is needed for the nonlinear term to make sense when $n\geqslant 5$.

In this section we prove the following

THEOREM 4.1. *Let w be a stationary flow with the property that*

$$\|w\|+\|\nabla w\|<+\infty.$$

There is a constant C_n with $0<C_n\leqslant\frac{1}{2}(n-2)$ such that if

$$\|w\|<C_n,$$

then w is stable in the following sense.

(i) *For each $a\in L_\sigma^2$, problem (1.5) possesses at least one weak solution u defined for all $t\geqslant 0$ such that*

$$\lim_{t\to\infty}\|u(t)\|_2=0.$$

(ii) *For each* $0<\delta<\frac{1}{4}$ *there is a positive number* $\eta=\eta(\delta)$ *such that if*

$$\|w\| < \min(C_n, \eta),$$

and if the initial perturbation $a\in L^2_\sigma$ *satisfies*

$$\|e^{-tL}a\|_2 = O(t^{-\alpha}) \quad \text{as } t\to\infty$$

for some $\alpha>0$, *then, as* $t\to\infty$,

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & (\alpha \leqslant \frac{1}{4}n-\delta), \\ O(t^{\delta-n/4}) & (\alpha \geqslant \frac{1}{4}n-\delta). \end{cases}$$

(iii) *Suppose in addition that* $\nabla w\in L^q_w\cap L^\infty$ *for some* $n'\leqslant q<\frac{1}{2}n$ *in case* $n\geqslant 4$, *and that* $\nabla w\in L^q\cap L^\infty$ *for some* $1<q<\frac{3}{2}$ *in case* $n=3$. *There exists* $\mu=\mu(n)>0$ *so that if*

$$\|w\|+\|\nabla w\|_{q,w}+\|\nabla w\|_\infty \leqslant \mu \quad (n\geqslant 4),$$

or if

$$\|w\|+\|\nabla w\|_q+\|\nabla w\|_\infty \leqslant \mu \quad (n=3),$$

then the following result holds: *Let* $a\in L^2_\sigma$ *satisfy*

$$\|e^{-tL}a\|_2 = O(t^{-\alpha}) \quad \text{as } t\to\infty$$

for some $\alpha>0$. *Then, as* $t\to\infty$,

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & (\alpha < \frac{1}{4}n), \\ O(t^{-n/4}) & (\alpha \geqslant \frac{1}{4}n). \end{cases}$$

Here, the number $\mu=\mu(n)>0$ *is taken so that the semigroup* $\{e^{-tL^*}\}_{t\geqslant 0}$ *satisfies estimates of Theorem* 3.13 *with* $s=\frac{2}{3}n$.

Remark. Theorem 4.1 improves our previous results obtained in [6], in which is shown among others that if $n=3$, $\|w\|<\frac{1}{2}$ and if $\|e^{-tL}a\|_2=O(t^{-\alpha})$, then

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & (\alpha < \frac{1}{2}), \\ O(t^{\varepsilon-1/2}) & (\alpha \geqslant \frac{1}{2}), \end{cases}$$

where $0<\varepsilon<\frac{1}{2}$ is arbitrary. When $n\geqslant 3$ and $w=0$, we proved in [5] that

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & (\alpha < \frac{1}{4}n), \\ O(t^{-n/4}) & (\alpha \geqslant \frac{1}{4}n), \end{cases}$$

provided $\|e^{-tA}a\|_2=O(t^{-\alpha})$.

We notice that in the case of Theorem 4.1 (iii), there exists an initial perturbation a satisfying $\|e^{-tL}a\|_2=O(t^{-n/4})$, as shown in Corollary 3.14. It should also be noticed that no assumption is imposed in statements (i) and (ii) on the size of ∇w.

Given $a\in L^2_\sigma$, we construct approximate solutions u_k of problem (1.5), solving the integral equation

$$u_k(t)=e^{-tL}a_k-\int_0^t e^{-(t-\tau)L}P(\bar{u}_k\cdot\nabla)u_k(\tau)\,d\tau. \tag{4.1}$$

Here, $a_k=(I+k^{-1}L_2)^{-1}a$ and $\bar{u}_k$ is the standard (spatial) mollification of the zero-extension of u_k. As shown below in the proof of Theorem 4.1, L_2 is a regularly accretive operator [44] in L^2_σ provided $\|w\|<\frac{1}{2}(n-2)$, and so, in this case, $\|a_k\|_2\leqslant\|a\|_2$. For any fixed k, integral equation (4.1) is easily solved globally in time by applying the contraction mapping principle, and the convergence of (a subsequence of) u_k to a weak solution is proved in the standard manner; see [34] for the details.

In what follows we write simply $u=u_k$ and deduce decay estimates for u which are uniform in approximation. The desired decay results are then obtained for the constructed weak solution through passage to the limit $k\to\infty$. Let

$$A_2=\int_0^\infty \lambda\,dE_\lambda$$

be the spectral decomposition for the positive self-adjoint operator A_2. As in our previous works [3]–[6], the key estimate for the nonlinear term is given by the following

LEMMA 4.2. (i) *For each* $0<\delta<\frac{1}{4}$ *there exists a number* $\eta=\eta(\delta)>0$ *such that if*

$$\|w\|<\eta,$$

then we have the estimate

$$\|E_\lambda e^{-tL}P(\bar{u}\cdot\nabla u)\|_2\leqslant Ct^{-1/2}\lambda^{n/4-1/2-\delta}\|u\|_2^{1-2\delta}\|\nabla u\|_2^{1+2\delta}.$$

(ii) *Suppose* w *satisfies the assumption of Theorem* 4.1 (iii) *with the same number* $\mu=\mu(n)>0$ *as given there. Then we have the estimate*

$$\|E_\lambda e^{-tL}P(\bar{u}\cdot\nabla u)\|_2\leqslant Ct^{-3/4}\lambda^{(n-3)/4}\|u\|_2\|\nabla u\|_2.$$

Proof. (i) We set $s=n/(1+2\delta)$. Then, Theorem 3.10 (ii) implies, for $\varphi\in C^\infty_{0,\sigma}(\Omega)$,

$$\begin{aligned}|\langle E_\lambda e^{-tL}P(\bar{u}\cdot\nabla)u,\varphi\rangle|&=|\langle \bar{u}\otimes u,\nabla e^{-tL^*}E_\lambda\varphi\rangle|\\&\leqslant\|\bar{u}\otimes u\|_{s'}\|\nabla e^{-tL^*}E_\lambda\varphi\|_s\leqslant Ct^{-1/2}\|u\|^2_{2s'}\|E_\lambda\varphi\|_s.\end{aligned}$$

Since $n'<s<n$ and

$$1/s = 1/n + 2\delta/n = 1/2 - (n/2 - 1 - 2\delta)/n$$

with $\frac{1}{2}n - 1 - 2\delta \geqslant \frac{1}{2} - 2\delta > 0$ if $0<\delta<\frac{1}{4}$, by Corollary 4.5 of [4] we have

$$\|E_\lambda \varphi\|_s \leqslant C\|A^{n/4-1/2-\delta}E_\lambda\varphi\|_2 \leqslant C\lambda^{n/4-1/2-\delta}\|\varphi\|_2.$$

On the other hand, since $\|A^{1/2}u\|_2 = \|\nabla u\|_2$ and since

$$1/2s' = 1/2 - 1/2s = 1/2 - (1/2+\delta)/n,$$

Corollary 4.5 of [4] gives

$$\|u\|_{2s'} \leqslant C\|A^{1/4+\delta/2}u\|_2 \leqslant C\|u\|_2^{1/2-\delta}\|A^{1/2}u\|_2^{1/2+\delta} = C\|u\|_2^{1/2-\delta}\|\nabla u\|^{1/2+\delta},$$

which implies the result.

(ii) For $\varphi \in C_{0,\sigma}^\infty(\Omega)$ we have

$$|\langle E_\lambda e^{-tL}P(\bar{u}\cdot\nabla u), \varphi\rangle| = |\langle \bar{u}\otimes u, \nabla e^{-tL^*}E_\lambda\varphi\rangle| \leqslant \|u\|_{2n'}\|\nabla e^{-tL^*}E_\lambda\varphi\|_n.$$

The Sobolev embedding yields $\|u\|_{2n'} \leqslant C(\|u\|_2\|\nabla u\|_2)^{1/2}$. On the other hand, applying Theorem 3.13 with $s=\frac{2}{3}n$ and Corollary 4.5 of [4] yields

$$\begin{aligned}\|\nabla e^{-tL^*}E_\lambda\varphi\|_n &\leqslant Ct^{-3/4}\|E_\lambda\varphi\|_{2n/3} \\ &\leqslant Ct^{-3/4}\|A^{(n-3)/4}E_\lambda\varphi\|_2 \leqslant Ct^{-3/4}\lambda^{(n-3)/4}\|\varphi\|_2.\end{aligned}$$

Combining these gives the desired result.

Proof of Theorem 4.1. We here prove only (i) and (ii), employing Lemma 4.2 (i), since the proof of (iii) is the same as in [5] if we employ Lemma 4.2 (ii). The arguments below are essentially the same as those developed in [4], [5]. We multiply the equation

$$\frac{du}{dt} + Lu + P(\bar{u}\cdot\nabla)u = 0$$

by u and integrate by parts to get

$$\frac{1}{2}\cdot\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_2^2 + \langle u\cdot\nabla w, u\rangle = 0.$$

But, since (see [28], [29])

$$\left\|\frac{u}{|x|}\right\|_2 \leqslant \frac{2}{n-2}\|\nabla u\|_2$$

for $u\in W_0^{1,2}(\Omega)$, $n\geqslant 3$, we have

$$|\langle u\cdot\nabla w,u\rangle|=|\langle w,u\cdot\nabla u\rangle|\leqslant\|w\|\cdot\||x|^{-1}u\|_2\|\nabla u\|_2\leqslant\frac{2}{n-2}\|w\|\cdot\|\nabla u\|_2^2.$$

It follows that

$$\langle Lu,u\rangle=\langle u,L^*u\rangle=\|\nabla u\|_2^2+\langle u\cdot\nabla w,u\rangle\geqslant\left(1-\frac{2}{n-2}\|w\|\right)\|\nabla u\|_2^2.$$

Thus, if $\|w\|<\frac{1}{2}(n-2)$, then both L and L^* are regularly accretive in L^2_σ; $N(L)=N(L^*)=0$ in L^2_σ; so $R(L)$ is dense in L^2_σ; and therefore

$$\lim_{t\to\infty}\|e^{-tL}a\|_2=0\quad\text{for all } a\in L^2_\sigma.$$

Furthermore, we get

$$\frac{d}{dt}\|u\|_2^2+2C_0\|\nabla u\|_2^2\leqslant 0 \tag{4.2}$$

for some $C_0>0$, so that

$$\|u(t)\|_2\leqslant\|a_k\|_2\leqslant\|a\|_2\quad\text{and}\quad 2C_0\int_0^\infty\|\nabla u\|_2^2\,ds\leqslant\|a_k\|_2^2\leqslant\|a\|_2^2. \tag{4.3}$$

Here, we apply the estimate

$$\|\nabla u\|_2^2=\|A^{1/2}u\|_2^2\geqslant\int_\varrho^\infty\lambda\,d\|E_\lambda u\|_2^2\geqslant\varrho(\|u\|_2^2-\|E_\varrho u\|_2^2)$$

for any fixed $\varrho>0$, to obtain from (4.2)

$$2\|u\|_2\frac{d}{dt}\|u\|_2+2C_0\varrho\|u\|_2^2\leqslant 2C_0\varrho\|E_\varrho u\|_2^2.$$

Since $\|E_\varrho u\|_2\leqslant\|u\|_2$, we finally obtain

$$\frac{d}{dt}\|u\|_2+C_0\varrho\|u\|_2\leqslant C_0\varrho\|E_\varrho u\|_2. \tag{4.4}$$

On the other hand, integral equation (4.1) gives the estimate

$$\|E_\varrho u\|_2\leqslant\|e^{-tL}a\|_2+\int_0^t\|E_\varrho e^{-(t-\tau)L}P(\bar{u}\cdot\nabla)u\|_2\,d\tau.$$

Applying Lemma 4.2 (i) then yields

$$\begin{aligned}\|E_\varrho u\|_2&\leqslant\|e^{-tL}a\|_2+C\varrho^{n/4-1/2-\delta}\int_0^t(t-\tau)^{-1/2}\|u\|_2^{1-2\delta}\|\nabla u\|_2^{1+2\delta}\,d\tau\\&\leqslant\|e^{-tL}a\|_2+C\varrho^{n/4-1/2-\delta}F_1(t)^{1/2-\delta}F_2(t)^{1/2+\delta},\end{aligned} \tag{4.5}$$

where

$$F_1(t)=\int_0^t (t-\tau)^{-1/2}\|u\|_2^2\,d\tau,\quad F_2(t)=\int_0^t (t-\tau)^{-1/2}\|\nabla u\|_2^2\,d\tau.$$

We thus obtain from (4.4) and (4.5)

$$\frac{d}{dt}\|u\|_2+C_0\varrho\|u\|_2\leqslant C_0\varrho(\|e^{-tL}a\|_2+C\varrho^{n/4-1/2-\delta}F_1^{1/2-\delta}F_2^{1/2+\delta}). \tag{4.6}$$

Now, set $\varrho=m(C_0t)^{-1}$ with a sufficiently large integer $m>0$ and then multiply both sides of (4.6) by t^m to obtain

$$\frac{d}{dt}(t^m\|u\|_2)\leqslant mt^{m-1}(\|e^{-tL}a\|_2+Ct^{1/2+\delta-n/4}F_1^{1/2-\delta}F_2^{1/2+\delta})$$

and therefore,

$$\begin{aligned}\|u(t)\|_2\leqslant{}& t^{-m}\int_0^t m\tau^{m-1}\|e^{-\tau L}a\|_2\,d\tau\\ &+Ct^{1/2+\delta-n/4}\left(t^{-1}\int_0^t F_1\,d\tau\right)^{1/2-\delta}\left(t^{-1}\int_0^t F_2\,d\tau\right)^{1/2+\delta}.\end{aligned}\tag{4.7}$$

Since $\|\nabla u\|_2^2\in L^1(\mathbf{R}_+)$ by (4.3), we see that

$$t^{-1}\int_0^t F_2\,d\tau\leqslant Ct^{-1/2},$$

so we get from (4.7)

$$\|u(t)\|_2\leqslant t^{-m}\int_0^t m\tau^{m-1}\|e^{-\tau L}a\|_2\,d\tau+Ct^{1/4+\delta/2-n/4}\left(t^{-1}\int_0^t F_1\,d\tau\right)^{1/2-\delta}. \tag{4.8}$$

Now, $\|u\|_2\in L^\infty(\mathbf{R}_+)$ by (4.3); so it follows from (4.8) that

$$\|u(t)\|_2\leqslant t^{-m}\int_0^t m\tau^{m-1}\|e^{-\tau L}a\|_2\,d\tau+Ct^{1/2-n/4}\to 0,$$

since $\|e^{-tL}a\|_2\to 0$. This shows assertion (i) of Theorem 1.1. Furthermore, (4.8) yields

$$\|u(t)\|_2\leqslant C(t^{-\alpha}+t^{1/2-n/4})$$

provided $\|e^{-tL}a\|_2=O(t^{-\alpha})$, and this shows assertion (ii) for $\alpha\leqslant\frac14 n-\frac12$. If $\alpha>\frac14 n-\frac12$, we have $\|u\|_2^2=O(t^{-1/2})$; so $F_1\in L^\infty(\mathbf{R}_+)$, and therefore

$$t^{-1}\int_0^t F_1\,d\tau\leqslant C.$$

It thus follows from (4.8) that

$$\|u(t)\|_2 \leqslant C(t^{-\alpha}+t^{1/4+\delta/2-n/4}) \leqslant C(t^{-\alpha}+t^{3/8-n/4}).$$

This shows the result for $\alpha \leqslant \frac{1}{4}n-\frac{3}{8}$. If $\alpha > \frac{1}{4}n-\frac{3}{8}$, we have $\|u\|_2^2=O(t^{-3/4})$ so that

$$\left(t^{-1}\int_0^t F_1\, d\tau\right)^{1/2-\delta} \leqslant Ct^{-1/8+\delta/4} \leqslant Ct^{-1/16},$$

and therefore (4.8) gives

$$\|u(t)\|_2 \leqslant C(t^{-\alpha}+t^{3/16+\delta/2-n/4}) \leqslant C(t^{-\alpha}+t^{5/16-n/4}),$$

which shows the result for $\alpha \leqslant \frac{1}{4}n-\frac{5}{16}$. If $\alpha > \frac{1}{4}n-\frac{5}{16}$, we have $\|u\|_2^2=O(t^{-7/8})$ so that

$$\left(t^{-1}\int_0^t F_1\, d\tau\right)^{1/2-\delta} \leqslant Ct^{-3/16+3\delta/8} \leqslant Ct^{-3/32},$$

and therefore (4.8) gives

$$\|u(t)\|_2 \leqslant C(t^{-\alpha}+t^{5/32+\delta/2-n/4}) \leqslant C(t^{-\alpha}+t^{9/32-n/4}),$$

which shows the result for $\alpha \leqslant \frac{1}{4}n-\frac{9}{32}$.

Repeating these processes, we arrive at the situation where

$$\begin{aligned} &\|u(t)\|_2^2 = O(t^{-2\alpha}) \quad \left(\alpha \leqslant \tfrac{1}{4}n-(2^{l-1}+1)/2^{l+1}\right), \\ &\|u(t)\|_2^2 \leqslant C(t^{-2\alpha}+t^{-1+1/2^l}) \quad \left(\alpha > \tfrac{1}{4}n-(2^{l-1}+1)/2^{l+1}\right), \end{aligned} \tag{4.9}$$

for an arbitrarily given integer $l>0$. Since $\frac{1}{2}n-(2^{l-1}+1)/2^l \geqslant 1-2^{-l}$, we have $\|u\|_2^2= O(t^{-1+1/2^l})$ in the latter case of (4.9), which implies

$$\left(t^{-1}\int_0^t F_1\, d\tau\right)^{1/2-\delta} \leqslant Ct^{(1/2^l-1/2)(1/2-\delta)},$$

and so (4.8) gives

$$\|u(t)\|_2 \leqslant C(t^{-\alpha}+t^{\delta-n/4+\mu}),$$

where $\mu=\left(\frac{1}{2}-\delta\right)/2^l$. Since we may take l so that $\frac{1}{4}n-\delta-\mu \geqslant \frac{3}{4}-\delta-\mu > \frac{1}{2}$, we can set

$$\tfrac{1}{4}n-\delta-\mu \geqslant \tfrac{1}{2}+\varkappa$$

with $\varkappa>0$. Thus, we obtain

$$\|u(t)\|_2^2 \leqslant C(t^{-2\alpha}+t^{-1-2\varkappa}). \tag{4.10}$$

Suppose now that $n\geqslant 4$. Then, in view of (4.9) and (4.10), we may assume that $2\alpha> n/2-(2^{l-1}+1)/2^l\geqslant 3/2-1/2^l$ for some large l. So, (4.10) implies that $\|u\|_2^2\in L^1(\mathbf{R}_+)$. Hence,

$$\left(t^{-1}\int_0^t F_1\,d\tau\right)^{1/2-\delta}\leqslant Ct^{-1/4+\delta/2},$$

which, together with (4.8), yields

$$\|u(t)\|_2\leqslant C(t^{-\alpha}+t^{\delta-n/4}) \tag{4.11}$$

and the proof is complete. Consider next the case $n=3$. Then, if $2\alpha>1-1/2^l$, we obtain $\|u\|_2^2=O(t^{-1+1/2^l})$ from (4.9). So, the argument above implies $\|u\|_2=O(t^{-\alpha})$ in case $\alpha<\frac{1}{2}$. When $\alpha>\frac{1}{2}$, the same argument as above shows $\|u\|_2^2=O(t^{-1+1/2^l})$ for all l; so we get (4.10) for some $\varkappa>0$, which in turn implies $\|u\|_2^2=O(t^{-1-\eta})$ for some $\eta>0$ since $2\alpha>1$. Hence, $\|u\|_2^2\in L^1(\mathbf{R}_+)$ and we arrive at the desired result (4.11). There remains to discuss the case $\alpha=\frac{1}{2}$. By (4.3), (4.9) and (4.10) we may assume

$$\|u(t)\|_2^2\leqslant C((t+1)^{-1}+(t+1)^{-1-2\varkappa}) \tag{4.12}$$

for some $\varkappa>0$. It follows from (4.12) that

$$\left(t^{-1}\int_0^t F_1\,d\tau\right)^{1/2-\delta}\leqslant C([t^{-1/2}\log(t+1)]^{1/2-\delta}+t^{-1/4+\delta/2})$$

which, together with (4.8), implies that

$$\|u(t)\|_2\leqslant C(t^{-1/2}+t^{-1+\delta/2}(\log t)^{1/2-\delta}+t^{\delta-3/4})=O(t^{-1/2}).$$

This completes the proof of Theorem 4.1.

5. Fractional powers of the Stokes operator in Lorentz spaces

In this section we study the Stokes operator A in the Lorentz spaces over an exterior domain Ω. The result will be applied in §6 to show the existence of a global-in-time solution u of problem (1.5), satisfying

$$u(t)\in L^n_w,\quad \nabla u(t)\in L^{n/2}_w\quad \text{for all } t\geqslant 0.$$

Notice that the stationary solutions w obtained in §2 all satisfy these properties.

We first recall the definition of the Lorentz spaces (see [2], [45]). Let $1<r<\infty$ and $1\leqslant q\leqslant\infty$. A measurable function f defined on a domain $D\subset\mathbf{R}^n$ is said to belong to $L^{(r,q)}(D)$ if and only if

$$\int_0^\infty (t^{1/r}f^*(t))^q\,\frac{dt}{t}<+\infty\quad(q<\infty),\quad \sup_{t>0} t^{1/r}f^*(t)<+\infty\quad(q=\infty),\tag{5.1}$$

where f^* is the nonincreasing rearrangement of f. It is well known that the quantity

$$\|f\|^*_{r,q}=\begin{cases}\left(\dfrac{q}{r}\displaystyle\int_0^\infty (t^{1/r}f^*(t))^q\,\frac{dt}{t}\right)^{1/q} & (q<\infty),\\ \sup_{t>0} t^{1/r}f^*(t) & (q=\infty)\end{cases}$$

defines a quasinorm on the vector space $L^{(r,q)}(D)$. Notice also that $L^{(r,r)}=L^r$ and $L^{(r,\infty)}=L^r_w$ in our previous notation. As shown in §2, L^r_w is a Banach space with respect to a norm which is equivalent to the quasinorm $\|\cdot\|^*_{r,\infty}$. The same is true of general $L^{(r,q)}$; indeed, the interpolation theory of Banach spaces [2], [45] gives the following result.

THEOREM 5.1. *Let* $1<r<\infty$ *and* $1\leqslant q\leqslant\infty$.

(i) *We have*

$$L^{(r,q)}(D)=(L^1(D),L^\infty(D))_{\theta,q},\quad 1/r=1-\theta,$$

where $(\cdot,\cdot)_{\theta,q}$ *stands for the real interpolation spaces constructed via the* K*-method.*

(ii) *If* $1<r_0<r_1<\infty$, $1\leqslant q\leqslant\infty$ *and* $0<\theta<1$, *then*

$$(L^{r_0}(D),L^{r_1}(D))_{\theta,q}=L^{(r,q)}(D),\quad 1/r=(1-\theta)/r_0+\theta/r_1.$$

(iii) *If* $1<r<\infty$ *and* $1\leqslant q<\infty$, *then*

$$(L^{(r,q)}(D))^*=L^{(r',q')}(D),\quad 1/r'=1-1/r,\ 1/q'=1-1/q.$$

For the proof of (i) and (iii) we refer to [2], [45]. (ii) follows from the reiteration theorem in the interpolation theory. In this section we denote the norm of the space $L^{(r,q)}$ by $\|\cdot\|_{r,q}$.

We next consider the Helmholtz decomposition of the space $L^{(r,q)}(\Omega)^n$ over a smooth exterior domain $\Omega\subset\mathbf{R}^n$, $n\geqslant2$. Let $P=P_r$ be the projector associated with the Helmholtz decomposition of $L^r(\Omega)^n$. Then, P defines a bounded projector on each of $L^{(r,q)}(\Omega)^n$, $1<r<\infty$, $1\leqslant q\leqslant\infty$.

THEOREM 5.2. *Let* $L^{(r,q)}_\sigma=R(P)$ *and* $G^{(r,q)}=N(P)$. *Then*

$$L^{(r,q)}(\Omega)^n=L^{(r,q)}_\sigma\oplus G^{(r,q)}\tag{5.2}$$

and

$$L_\sigma^{(r,q)}=\{u\in L^{(r,q)}(\Omega)^n:\nabla\cdot u=0, u\cdot\nu|_S=0\},$$
$$G^{(r,q)}=\{\nabla p\in L^{(r,q)}(\Omega)^n: p\in L_{\mathrm{loc}}^{(r,q)}(\bar\Omega)\}. \tag{5.3}$$

Furthermore, if $1\leqslant q<\infty$, *then*

$$(L_\sigma^{(r,q)})^*=L_\sigma^{(r',q')},\quad (G^{(r,q)})^*=G^{(r',q')}. \tag{5.4}$$

Proof. Since P is a bounded projector, (5.2) is obvious. Relation (5.4) follows from Theorem 5.1 (iii) by interpolating the relation: $P_r^*=P_{r'}$ in L_σ^r. It is also easy from the interpolation theory to see that the spaces on the left-hand sides of (5.3) are included in the right-hand sides. So, it suffices to show that the intersection of the spaces on the right-hand side consists only of 0. But, this is easily obtained from the following lemma, so the proof is complete.

LEMMA 5.3. *Let p be a distribution on an exterior domain Ω of $\mathbf{R}^n$, $n\geqslant 2$, with smooth boundary S. Suppose $\nabla p\in L^{(r,\infty)}(\Omega)^n$ for some $1<r<\infty$. If*

$$\Delta p=0 \ \ in\ \Omega,\quad \frac{\partial p}{\partial\nu}\Big|_S=0,$$

then $\nabla p=0$.

Proof. By assumption ∇p is harmonic in Ω. Taking $x\in\Omega$ with $B(x,\frac12|x|)\subset\Omega$, we apply the mean value theorem for harmonic functions to get

$$\nabla p(x)=\left|B(x,\tfrac12|x|)\right|^{-1}\int_B\nabla p(y)\,dy,\quad B=B(x,\tfrac12|x|).$$

Then

$$\begin{aligned}|\nabla p(x)|&\leqslant\left|B(x,\tfrac12|x|)\right|^{-1}\int_B|\nabla p|\,dy\\ &\leqslant\left|B(x,\tfrac12|x|)\right|^{-1}\|\nabla p\|_{r,\infty}\left|B(x,\tfrac12|x|)\right|^{1-1/r}\\ &=\|\nabla p\|_{r,\infty}\left|B(x,\tfrac12|x|)\right|^{-1/r}\to 0,\end{aligned}$$

as $|x|\to\infty$. Hence, the expansion theorem for harmonic functions yields

$$|\nabla p(x)|=\begin{cases}O(|x|^{2-n}) & (n\geqslant 3),\\ O(|x|^{-1}) & (n=2).\end{cases}$$

Thus, $|p(x)|$ is bounded in Ω if $n\geqslant 4$ and $|p(x)|\leqslant C_1+C_2\log|x|$ for large $|x|$ in case $n=2,3$. Applying again the expansion theorem yields

$$p(x)=\begin{cases}p_0+p_1|x|^{2-n}+O(|x|^{1-n}) & (n\geqslant 3),\\ p_0+p_1\log|x|+O(|x|^{-1}) & (n=2),\end{cases}$$

with some constants p_0 and p_1. But, since the assumption implies

$$\int_{|x|=r} \frac{\partial p}{\partial r}\, dS = 0$$

for large r, it follows from the expansion theorem that $p_1=0$. We thus obtain

$$p(x) = p_0 + O(|x|^{1-n}) \quad \text{so that} \quad |\nabla p(x)| = O(|x|^{-n})$$

as $|x|\to\infty$. Let B_r be the open ball of radius r centered at the origin. Since p has zero flux through each point of S, we can integrate by parts to see that the function $q=p-p_0$ satisfies, for large $r>0$,

$$\int_{B_r\cap\Omega} |\nabla q|^2\, dx = \int_{|x|=r} q\frac{\partial q}{\partial r}\, dS \leqslant Cr^{-n}.$$

Letting $r\to\infty$ yields $\|\nabla q\|_2=0$ and so $\nabla q=\nabla p=0$. The proof is complete.

THEOREM 5.4. *Let* $0<r_0<r_1<\infty$, $1\leqslant q\leqslant\infty$, $0<\theta<1$ *and*

$$1/r = (1-\theta)/r_0 + \theta/r_1.$$

Then we have

$$L_\sigma^{(r,q)} = (L_\sigma^{r_0}, L_\sigma^{r_1})_{\theta,q}, \quad G^{(r,q)} = (G^{r_0}, G^{r_1})_{\theta,q}.$$

Proof. The result is easily verified since P is a bounded projector (see [45, pp. 22–23]).

THEOREM 5.5. *Let* $1<r<\infty$ *and* $1\leqslant q<\infty$. *Then the set* $C_{0,\sigma}^\infty(\Omega)$ *of smooth solenoidal vector fields with compact support in* Ω *is dense in* $L_\sigma^{(r,q)}$.

Proof. By Theorem 5.4 and general theory of interpolation spaces, $L_\sigma^{r_0}\cap L_\sigma^{r_1}$ is dense in $L_\sigma^{(r,q)}$. It thus suffices to show that $C_{0,\sigma}^\infty(\Omega)$ is dense in the space $L_\sigma^{r_0}\cap L_\sigma^{r_1}$. Let

$$f\in (L_\sigma^{r_0}\cap L_\sigma^{r_1})^* = L_\sigma^{r_0'} + L_\sigma^{r_1'}$$

so that we can decompose (non-uniquely) $f=f_0+f_1$, $f_j\in L_\sigma^{r_j'}$, $j=0,1$, and assume that f annihilates $C_{0,\sigma}^\infty(\Omega)$. By [38, Theorem 17′], we have $f=\nabla p$ for some distribution p on Ω. But, then

$$\Delta p = 0 \ \text{ in } \Omega, \quad \left.\frac{\partial p}{\partial \nu}\right|_S = 0.$$

Since $\Delta(\nabla p)=0$ with $\nabla p\in L^{r_0'}+L^{r_1'}$, it follows from the mean value property of harmonic functions that $|\nabla p(x)|=O(|x|^{2-n})$ if $n\geqslant 3$, and $|\nabla p(x)|=O(|x|^{-1})$ if $n=2$. By the same reasoning as in the proof of Lemma 5.3, we get $\nabla p=0$. This proves the result.

We next study the Helmholtz decomposition in more detail when $1<r<n$. We begin by extending the Sobolev-type inequality as given in §§2 and 3 to Lorentz spaces.

LEMMA 5.6. *Let* $n\geqslant 2$, $1<r<n$, $1\leqslant q\leqslant\infty$ *and* $1/r^*=1/r-1/n$.

(i) *If* $f\in L^{(p,\infty)}(\mathbf{R}^n)$ *for some* $p<\infty$ *and if* $\nabla f\in L^{(r,q)}(\mathbf{R}^n)^n$, *then* $f\in L^{(r^*,q)}(\mathbf{R}^n)$ *and the estimate*

$$\|f\|_{r^*,q,\mathbf{R}^n}\leqslant C\|\nabla f\|_{r,q,\mathbf{R}^n}$$

holds with $C>0$ *independent of* f.

(ii) *If* $\nabla f\in L^{(r,q)}(\mathbf{R}^n)^n$ *for some distribution* f, *then there is a function*

$$g\in L^{(r^*,q)}(\mathbf{R}^n)\quad \text{so that}\quad \nabla g=\nabla f.$$

Proof. (i) We may assume f is smooth in $\mathbf{R}^n$. Indeed, $C_0^\infty(\mathbf{R}^n)$ is dense in $L^{(r,q)}(\mathbf{R}^n)$ when $q<\infty$; and when $q=\infty$, we take $f_t=e^{t\Delta}f$ $(t>0)$, the convolution of f by the heat kernel, which belongs to $C^\infty(\mathbf{R}^n)$ and satisfies

$$\lim_{t\to 0}\|f_t\|_{r,\infty}=\|f\|_{r,\infty},\quad \lim_{t\to 0}\|\nabla f_t\|_{r,\infty}=\|\nabla f\|_{r,\infty}.$$

For any fixed x we set

$$\psi_N(y)=\psi((y-x)/N)$$

where $\psi\in C_0^\infty(\mathbf{R}^n)$, $0\leqslant\psi\leqslant 1$, $\psi(y)=1$ if $|y|\leqslant 1$, and $\psi(y)=0$ if $|y|\geqslant 2$. Using

$$f(x)=\psi_N(x)f(x)=-\int_0^\infty\frac{d}{dt}(\psi_N f)(x+t\omega)\,dt,\quad |\omega|=1,$$

we get

$$|f(x)|\leqslant C\int|x-y|^{1-n}|\nabla(\psi_N f)(y)|\,dy.$$

The right-hand side is estimated as

$$\begin{aligned}&\leqslant C\int|x-y|^{1-n}(|\psi_N\nabla f|+|f\nabla\psi_N|)(y)\,dy\\&\leqslant C\int|x-y|^{1-n}|\nabla f(y)|\,dy+CN^{-n}\int_{N\leqslant|x-y|\leqslant 2N}|f(y)|\,dy.\end{aligned}$$

Since $p<\infty$, the last term is estimated as

$$\leqslant CN^{-n}\int_{B(x,2N)}|f|\,dy\leqslant CN^{-n/p}\|f\|_{p,\infty}\to 0$$

as $N\to\infty$, so we get

$$|f(x)|\leqslant C\int|x-y|^{1-n}|\nabla f(y)|\,dy.$$

Assertion (i) follows from the boundedness of the Riesz potential $|x|^{1-n}$ from $L^{(r,q)}(\mathbf{R}^n)$ to $L^{(r^*,q)}(\mathbf{R}^n)$.

(ii) By assertion (i) the space

$$Y^{(r,q)}=\{f\in L^{(r^*,q)}(\mathbf{R}^n):\nabla f\in L^{(r,q)}(\mathbf{R}^n)^n\}$$

is a Banach space with norm $\|\nabla f\|_{r,q,\mathbf{R}^n}$. To prove (ii), it suffices to show that the map

$$\nabla: Y^{(r,q)}\to G^{(r,q)}(\mathbf{R}^n)$$

is an isomorphism. To this end we first prove that

$$\nabla:\widehat{H}_0^{1,r}(\mathbf{R}^n)\to G^r(\mathbf{R}^n)\quad (1<r<n) \tag{5.5}$$

is an isomorphism, where

$$\widehat{H}_0^{1,r}(\mathbf{R}^n)=\{f\in L^{r^*}(\mathbf{R}^n):\nabla f\in L^r(\mathbf{R}^n)^n\}$$

is a Banach space with norm $\|\nabla f\|_{r,\mathbf{R}^n}$ (see [17], [19]). To show (5.5), observe that (i) shows that $R(\nabla)$ is closed in $G^r(\mathbf{R}^n)$. It thus suffices to show that $R(\nabla)$ is dense in $G^r(\mathbf{R}^n)$. Suppose $f=\nabla p\in G^{r'}(\mathbf{R}^n)=G^r(\mathbf{R}^n)^*$ annihilates $R(\nabla)$. Then $\Delta p=0$ in $\mathbf{R}^n$, so $\Delta(\nabla p)=0$ in $\mathbf{R}^n$, and therefore $f=\nabla p=0$ since $\nabla p\in L^{r'}(\mathbf{R}^n)$. This proves (5.5).

Now, let $1<r_j<n$ $(j=0,1)$, $r_0\neq r_1$, $1\leqslant q\leqslant\infty$, $0<\theta<1$ and $1/r=(1-\theta)/r_0+\theta/r_1$. Then, (5.5) implies that

$$\nabla:(\widehat{H}_0^{1,r_0}(\mathbf{R}^n),\widehat{H}_0^{1,r_1}(\mathbf{R}^n))_{\theta,q}\to G^{(r,q)}(\mathbf{R}^n)$$

is an isomorphism. But, we easily see that

$$(\widehat{H}_0^{1,r_0}(\mathbf{R}^n),\widehat{H}_0^{1,r_1}(\mathbf{R}^n))_{\theta,q}\subset\{f\in L^{(r^*,q)}(\mathbf{R}^n):\nabla f\in L^{(r,q)}(\mathbf{R}^n)^n\}=Y^{(r,q)},$$

and so ∇ is surjective from $Y^{(r,q)}$ to $G^{(r,q)}(\mathbf{R}^n)$. Since it is obviously injective, we get the desired result (ii). The proof is complete.

The following is a refinement of the Gagliardo–Nirenberg inequality [15].

LEMMA 5.7. *Let $n<r<\infty$, $1\leqslant q\leqslant\infty$, and suppose that $f\in L^{(r,q)}(\mathbf{R}^n)$ with $\nabla f\in L^{(r,q)}(\mathbf{R}^n)^n$. Then $f\in L^\infty(\mathbf{R}^n)$ and we have the estimate*

$$\|f\|_{\infty,\mathbf{R}^n}\leqslant C\|\nabla f\|_{r,q,\mathbf{R}^n}^{n/r}\|f\|_{r,q,\mathbf{R}^n}^{1-n/r}.$$

Proof. Since $L^{(r,q)}\subset L^{(r,\infty)}$, we need only to consider the case $q=\infty$. This time we use

$$g(x)=-\int_0^\infty\frac{d}{dt}[e^{-t}(\psi_N g)(x+t\omega)]\,dt,\quad |\omega|=1,$$

to obtain, as in the proof of Lemma 5.6 (i),

$$\begin{aligned}|g(x)| &\leqslant C\int |x-y|^{1-n}e^{-|x-y|}(|g|+|\nabla g|)(y)\,dy \\ &\leqslant C\int_{|x-y|<1}|x-y|^{1-n}(|g|+|\nabla g|)(y)\,dy + C\int_{|x-y|\geqslant 1}e^{-|x-y|}(|g|+|\nabla g|)(y)\,dy \\ &\equiv C(I_1+I_2).\end{aligned}$$

Using the definition of the norm $\|g\|_{r,w}$ as given in §2, we evaluate the above two integrals in terms of the measure $\mu=(|g|+|\nabla g|)\,dy$, to obtain

$$\begin{aligned}I_1+I_2 &= \int_1^\infty \mu[\{y:|x-y|^{1-n}>t\}]\,dt + \int_0^{1/e}\mu[\{y:e^{-|x-y|}>t\}]\,dt \\ &= \int_1^\infty \mu[B(x,t^{-1/(n-1)})]\,dt + \int_0^{1/e}\mu[B(x,\log t^{-1})]\,dt \\ &\leqslant C_n(\|g\|_{r,\infty}+\|\nabla g\|_{r,\infty})\left(\int_1^\infty t^{-n'/r'}\,dt + \int_0^{1/e}(\log t^{-1})^{n/r'}\,dt\right) \\ &= C_{r,n}(\|g\|_{r,\infty}+\|\nabla g\|_{r,\infty}),\end{aligned}$$

since $n'/r'>1$. We thus have

$$\|g\|_{\infty,\mathbf{R}^n} \leqslant C(\|g\|_{r,\infty,\mathbf{R}^n}+\|\nabla g\|_{r,\infty,\mathbf{R}^n}).$$

We then insert $g(x)=f_\lambda(x)=f(x/\lambda)$, $\lambda>0$, to obtain

$$\|f\|_{\infty,\mathbf{R}^n} \leqslant C(\lambda^{n/r}\|f\|_{r,\infty,\mathbf{R}^n}+\lambda^{-1+n/r}\|\nabla f\|_{r,\infty,\mathbf{R}^n}),$$

with $C>0$ independent of $\lambda>0$. Taking the minimum with respect to $\lambda>0$ gives the desired result.

LEMMA 5.8. *Let $n\geqslant 2$ and let D be a smooth bounded domain in $\mathbf{R}^n$. For $1<r<n$ and $1\leqslant q\leqslant\infty$ we have the following*:

(i) *If $\nabla f\in L^{(r,q)}(D)^n$ for a distribution f on D, then $f\in L^{(r^*,q)}(D)$ and we have*

$$\left\|f-|D|^{-1}\int_D f\,dx\right\|_{r^*,q,D} \leqslant C\|\nabla f\|_{r,q,D}, \tag{5.6}$$

with C independent of f.

(ii) *If $f\in L^{(r,q)}(D)$ and $\nabla f\in L^{(r,q)}(D)^n$, then $f\in L^{(r^*,q)}(D)$ and the estimate*

$$\|f\|_{r^*,q,D} \leqslant C(\|f\|_{r,q,D}+\|\nabla f\|_{r,q,D})$$

holds with $C>0$ independent of f.

Proof. (i) The result is known to be valid with $L^{(r,q)}(D)$ replaced by $L^r(D)$. This implies in particular that the linear operator

$$T: G^r(D) \ni \nabla f \mapsto f - |D|^{-1} \int_D f\,dx \in L^{r^*}(D)$$

is well defined and bounded from $G^r(D)$ to $L^{r^*}(D)$. By interpolation, the same operator T is bounded from $G^{(r,q)}(D)$ to $L^{(r^*,q)}(D)$. This proves (i).

(ii) From (5.6) we have

$$\begin{aligned} \|f\|_{r^*,q,D} &\leqslant C\|\nabla f\|_{r,q,D} + \left\| |D|^{-1} \int_D f\,dx \right\|_{r^*,q,D} \\ &\leqslant C\|\nabla f\|_{r,q,D} + |D|^{-1} \left| \int_D f\,dx \right| \cdot \|1\|_{r^*,q,D}. \end{aligned}$$

Since the constant function 1 belongs to $L^{(p,s)}(D)$ for all $1<p<\infty$ and $1 \leqslant s \leqslant \infty$ because D is bounded, the last integral is estimated as

$$\left| \int_D f\,dx \right| \leqslant \|f\|_{r,q,D} \|1\|_{r',q',D} = C\|f\|_{r,q,D}.$$

This proves (ii).

We are now in a position to establish

THEOREM 5.9. *Let Ω be a smooth exterior domain in $\mathbf{R}^n$, $n \geqslant 2$.*

(i) *If $f \in L^{(r^*,q)}(\Omega)$ and $\nabla f \in L^{(r,q)}(\Omega)^n$ for some $1<r<n$ and $1 \leqslant q \leqslant \infty$, then*

$$\|f\|_{r^*,q} \leqslant C\|\nabla f\|_{r,q}, \tag{5.7}$$

with $C>0$ independent of f.

(ii) *Let $1<r<n$ and $1 \leqslant q \leqslant \infty$. If $\nabla f \in L^{(r,q)}(\Omega)^n$ for a distribution f on Ω, there exists a function $g \in L^{(r^*,q)}(\Omega)$ such that $\nabla g = \nabla f$.*

Proof. (i) Suppose the contrary; then there exists a sequence f_j such that

$$\|f_j\|_{r^*,q} \equiv 1, \quad \|\nabla f_j\|_{r,q} \to 0 \tag{5.8}$$

as $j \to \infty$. Since $L^{(r^*,q)} \subset L^{(r^*,\infty)}$ with continuous injection, we may assume that f_j converges weakly* in $L^{(r^*,\infty)}$ to a function f. So, $\nabla f = 0$, and therefore $f=0$ since $f \in L^{(r^*,\infty)}(\Omega)$ and $r^* < \infty$. On the other hand, we see from (5.8) that for any smooth bounded subdomain $D \subset \Omega$, f_j are bounded in $L^s(D)$ for any $s<r^*$ and $\nabla f_j \to 0$ in $L^p(D)^n$ for any $p<r$. So, the Rellich–Kondrachov compactness theorem applies to see that $f_j \to 0$

in $L^p(D)$. Since we can take p, s and s_0 so that $p<r<s<s_0<r^*$ and since f_j is bounded in $L^{s_0}(D)$, it follows that f_j converges also in $L^s(D)$. But, since $(L^s \cap L^p)(D) \subset L^{(r,q)}(D)$ with continuous injection, and since $f_j \to 0$ in $(L^s \cap L^p)(D)$, we conclude that $f_j \to 0$ in $L^{(r,q)}(D)$. Therefore, by Lemma 5.8 (ii),

$$f_j \to 0 \quad \text{in } L^{(r^*,q)}(D). \tag{5.9}$$

On the other hand, let $\varphi \in C^\infty(\mathbf{R}^n)$ be such that $0 \leqslant \varphi \leqslant 1$, $\varphi=1$ for large $|x|$ and $\varphi=0$ in a neighborhood of the complement of Ω. Then Lemma 5.6 implies

$$\begin{aligned} \|(f_j-f_k)\varphi\|_{r^*,q,\mathbf{R}^n} &\leqslant C\|\nabla((f_j-f_k)\varphi)\|_{r,q,\mathbf{R}^n} \\ &\leqslant C(\|\nabla(f_j-f_k)\|_{r,q}+\|(f_j-f_k)\nabla\varphi\|_{r,q}) \\ &\leqslant C(\|\nabla(f_j-f_k)\|_{r,q}+\|f_j-f_k\|_{r,q,D}) \to 0, \end{aligned}$$

where D is a neighborhood in Ω of the compact set $\operatorname{supp} \nabla\varphi$. This, together with (5.9), implies that $f_j \to 0$ in $L^{(r^*,q)}(\Omega)$, which contradicts the assumption $\|f_j\|_{r^*,q} \equiv 1$. Thus, we get estimate (5.7).

(ii) Consider the Banach space

$$X_r = \{f \in L^{r^*}(\Omega) : \nabla f \in L^r(\Omega)^n\} \quad \text{with norm } \|\nabla f\|_r.$$

We already know (see [34]) that when $1<r<n$,

$$\nabla : X_r \to G^r \quad \text{is an isomorphism.}$$

Hence, by interpolation

$$\nabla : (X_{r_0}, X_{r_1})_{\theta,q} \to G^{(r,q)} \quad \text{is an isomorphism.}$$

But, we easily see that

$$(X_{r_0}, X_{r_1})_{\theta,q} \subset \{f \in L^{(r^*,q)}(\Omega) : \nabla f \in L^{(r,q)}(\Omega)^n\} \equiv Y^{(r,q)},$$

and so ∇ maps $Y^{(r,q)}$ onto $G^{(r,q)}$. Since it is also injective, it follows that

$$(X_{r_0}, X_{r_1})_{\theta,q} = Y^{(r,q)}$$

and ∇ defines an isomorphism between $Y^{(r,q)}$ and $G^{(r,q)}$. The proof is complete.

The proof of Theorem 5.9 (ii) shows in particular the following

COROLLARY 5.10. *Let $n \geqslant 2$, $1<r<n$, $1 \leqslant q \leqslant \infty$ and $1/r^* = 1/r - 1/n$. Then, on a smooth exterior domain Ω we have*

$$G^{(r,q)} = \{\nabla p \in L^{(r,q)}(\Omega)^n : p \in L^{(r^*,q)}(\Omega)\}.$$

We now examine the fractional power $A^{1/2}$ of the Stokes operator A in the Lorentz spaces over an exterior domain. Let $n \geqslant 3$ and $1<r<\infty$. We know by [8] that the linear operators $\nabla^j (A+I)^{-1}$, $j=0,1,2$, are bounded from L^r_σ to $L^r(\Omega)$. Interpolating between the indices r_0 and r_1 with $r_0 \neq r_1$, we see that the same operators are bounded from $L^{(r,q)}_\sigma$ to $L^{(r,q)}(\Omega)$, $1 \leqslant q \leqslant \infty$; hence we obtain the estimate

$$\sum_{j=0}^{2} \|\nabla^j u\|_{r,q} \leqslant C(\|Au\|_{r,q} + \|u\|_{r,q}). \tag{5.10}$$

Thus, A defines a closed linear operator in $L^{(r,q)}_\sigma$ with domain

$$D_{(r,q)}(A) = \{u \in L^{(r,q)}_\sigma : \nabla^j u \in L^{(r,q)}(\Omega),\ j=1,2,\ u|_S = 0\}.$$

Notice that $D_{(r,q)}(A)$ is dense in $L^{(r,q)}_\sigma$ provided $q<\infty$; indeed, in this case $C^\infty_{0,\sigma}(\Omega)$ is dense.

Suppose next that $u \in D_{(r,q)}(A)$ and $Au=0$. Applying (5.10) and Theorem 5.9 repeatedly, we see that $u \in L^{(p,q)}(\Omega)$ and $\nabla u \in L^{(p,q)}(\Omega)$ for some $p>n$, so by Lemma 5.7, $u \in L^\infty(\Omega)$. (Note that $u|_S=0$.) Hence,

$$u \in (L^\infty \cap L^{(p,q)})(\Omega) \subset (L^\infty \cap L^{(p,\infty)})(\Omega) \subset L^s(\Omega)$$

for all s with $p<s<\infty$, and therefore

$$\int_{\Omega \cap \{|x| \leqslant R\}} |u|^2 |x|^{-n}\, dx = o(\log R) \quad \text{as } R \to \infty.$$

The uniqueness theorem of Chang and Finn [10, Theorem 6] then implies $u=0$. This shows that A is injective in $L^{(r,q)}_\sigma$. The parabolic resolvent estimate [8]

$$\|(\lambda + A)^{-1} u\|_r \leqslant C\|u\|_r / |\lambda| \quad (1<r<\infty)$$

is also extended via interpolation to the space $L^{(r,q)}_\sigma$, and this implies that the semigroup $\{e^{-tA}\}_{t \geqslant 0}$ in the space $L^{(r,q)}_\sigma$ is bounded and analytic, and so the fractional powers of A are well defined. However, notice that this semigroup is not strongly continuous at $t=0$ if $q=\infty$, since in this case $D_{(r,\infty)}(A)$ is not dense in $L^{(r,\infty)}_\sigma$.

Consider next the space

$$D^1_r = \text{the completion of } D(A_r) \text{ in the norm } \|Au\|_r \ \left(1<r<\tfrac{1}{2}n\right).$$

As shown in [4], [8],

$$D^1_r = \{u\in L^{r^{**}}_\sigma : \nabla u \in L^{r^*}(\Omega)^{n^2},\ \nabla^2 u\in L^r(\Omega)^{n^3},\ u|_S=0\},$$

with $1/r^{**}=1/r-2/n$ and $1/r^*=1/r-1/n$; and the estimate

$$C^{-1}\|\nabla^2 u\|_r \leqslant \|Au\|_r \leqslant C\|\nabla^2 u\|_r, \quad u\in D^1_r,$$

holds with $C>0$ independent of $u\in D^1_r$. The first inequality above implies in particular that

$$\|\nabla^2(\lambda+A)^{-1}u\|_r \leqslant C\|u\|_r \quad \left(1<r<\tfrac{1}{2}n\right),$$

for $u\in L^r_\sigma$, with $C>0$ independent of $\lambda>0$; so we see by interpolation that if we set

$$D^1_{(r,q)} = (D^1_{r_0}, D^1_{r_1})_{\theta,q},$$

for $r_0\neq r_1$, $1\leqslant q\leqslant\infty$, $0<\theta<1$ and $1/r=(1-\theta)/r_0+\theta/r_1$, then

$$C^{-1}\|\nabla^2 u\|_{r,q} \leqslant \|Au\|_{r,q} \leqslant C\|\nabla^2 u\|_{r,q} \quad \left(1<r<\tfrac{1}{2}n\right),$$

for $u\in D^1_{(r,q)}$. Combining these with Theorem 5.9 gives

THEOREM 5.11. *Let $n\geqslant 3$, $1<r<\frac{1}{2}n$ and $1\leqslant q\leqslant\infty$. Then*

$$D^1_{(r,q)} = \{u\in L^{(r^{**},q)}_\sigma : \nabla u\in L^{(r^*,q)}(\Omega)^{n^2},\ \nabla^2 u \in L^{(r,q)}(\Omega)^{n^3},\ u|_S=0\},$$

where $1/r^{**}=1/r-2/n$ and $1/r^*=1/r-1/n$. Furthermore, A maps $D^1_{(r,q)}$ injectively onto $L^{(r,q)}_\sigma$, with estimate

$$C^{-1}\|\nabla^2 u\|_{r,q} \leqslant \|Au\|_{r,q} \leqslant C\|\nabla^2 u\|_{r,q}.$$

The second assertion of Theorem 5.11 follows via interpolation from the fact that if $1<r<\frac{1}{2}n$, then A maps D^1_r injectively onto L^r_σ.

Let $n\geqslant 3$, $1<r<n$ and

$$D^{1/2}_r = \text{the completion of } D(A^{1/2}_r) \text{ in the norm } \|A^{1/2}u\|_r.$$

We next characterize

$$D^{1/2}_{(r,q)} = (D^{1/2}_{r_0}, D^{1/2}_{r_1})_{\theta,q}.$$

To do so, we introduce

$$\widehat{H}^{1,r}_{0,\sigma}(\Omega) = \text{the completion of } C^\infty_{0,\sigma}(\Omega) \text{ in the } L^r\text{-Dirichlet norm } \|\nabla u\|_r.$$

As shown in [4] we have

$$\widehat{H}^{1,r}_{0,\sigma}(\Omega) = \{u\in L^{r^*}_\sigma : \nabla u\in L^r(\Omega)^{n^2},\ u|_S=0\}.$$

We also recall the following result (see [4, Theorem 4.4]):

THEOREM 5.12. *Let $n \geqslant 3$ and $1<r<n$. Then*

$$D_r^{1/2} = \widehat{H}_{0,\sigma}^{1,r}(\Omega)$$

and we have

$$C^{-1}\|\nabla u\|_r \leqslant \|A^{1/2}u\|_r \leqslant C\|\nabla u\|_r \tag{5.11}$$

for all $u \in D_r^{1/2}$ with $C>0$ independent of u.

Starting from Theorem 5.12, we now prove

THEOREM 5.13. *If $n \geqslant 3$, $1<r<n$, $1 \leqslant q \leqslant \infty$ and $1/r^* = 1/r - 1/n$, then*

$$D_{(r,q)}^{1/2} = \{u \in L_\sigma^{(r^*,q)} : \nabla u \in L^{(r,q)}(\Omega)^{n^2},\ u|_S = 0\},$$

and the estimate

$$C^{-1}\|\nabla u\|_{r,q} \leqslant \|A^{1/2}u\|_{r,q} \leqslant C\|\nabla u\|_{r,q} \tag{5.12}$$

holds for $u \in D_{(r,q)}^{1/2}$ with $C>0$ independent of u.

Proof. Since $R(A_r^{1/2})$ is dense in L_σ^r, Theorem 5.12 implies that $A^{1/2}$ is an isomorphism between $D_r^{1/2}$ and L_σ^r. By interpolation we see that

$$A^{1/2} : (D_{r_0}^{1/2}, D_{r_1}^{1/2})_{\theta,q} = D_{(r,q)}^{1/2} \to L_\sigma^{(r,q)} \quad \text{is an isomorphism.} \tag{5.13}$$

Furthermore, interpolating between $\nabla : D_r^{1/2} \to L^r(\Omega)$ yields

$$(D_{r_0}^{1/2}, D_{r_1}^{1/2})_{\theta,q} \subset \{u \in L_\sigma^{(r^*,q)} : \nabla u \in L^{(r,q)}(\Omega)^{n^2},\ u|_S = 0\} \equiv Y^{(r,q)}$$

with continuous injection, and so

$$\|\nabla u\|_{r,q} \leqslant C\|A^{1/2}u\|_{r,q}, \quad u \in D_{(r,q)}^{1/2}. \tag{5.14}$$

Consider next the operator Z as introduced in [4, §4]. We know by [4] that if $1<r<n$, then

$$Z : \widehat{H}_{0,\sigma}^{1,r}(\mathbf{R}^n) \to D_r^{1/2}$$

is bounded and, with E_0 denoting the zero-extension of functions,

$$ZE_0 u = u \quad \text{for } u \in D_r^{1/2}.$$

Thus,

$$Z : (\widehat{H}_{0,\sigma}^{1,r_0}(\mathbf{R}^n), \widehat{H}_{0,\sigma}^{1,r_1}(\mathbf{R}^n))_{\theta,q} \to (D_{r_0}^{1/2}, D_{r_1}^{1/2})_{\theta,q} = D_{(r,q)}^{1/2}$$

is bounded and, by definition of Z given in [4, §4],

$$ZE_0u = u \quad \text{for } u \in Y^{(r,q)}. \tag{5.15}$$

But, as shown below, we have

$$(\widehat{H}^{1,r_0}_{0,\sigma}(\mathbf{R}^n), \widehat{H}^{1,r_1}_{0,\sigma}(\mathbf{R}^n))_{\theta,q} = \{u \in L^{(r^*,q)}_\sigma(\mathbf{R}^n) : \nabla u \in L^{(r,q)}(\mathbf{R}^n)^{n^2}\}. \tag{5.16}$$

It follows from (5.15)–(5.16) that $ZE_0 : Y^{(r,q)} \to D^{1/2}_{(r,q)}$ is bounded, so $Y^{(r,q)} \subset D^{1/2}_{(r,q)}$, and we have

$$\|A^{1/2}u\|_{r,q} = \|A^{1/2}ZE_0u\|_{r,q} \leqslant C\|\nabla E_0u\|_{r,q} = C\|\nabla u\|_{r,q}, \quad u \in Y^{(r,q)}. \tag{5.17}$$

The result follows from (5.14) and (5.17).

It remains to prove (5.16). As shown in the proof of Lemma 5.6 (ii),

$$(\widehat{H}^{1,r_0}_0(\mathbf{R}^n), \widehat{H}^{1,r_1}_0(\mathbf{R}^n))_{\theta,q} = \{f \in L^{(r^*,q)}(\mathbf{R}^n) : \nabla f \in L^{(r,q)}(\mathbf{R}^n)^n\}. \tag{5.18}$$

Let $\widetilde{P}$ be the bounded projector associated with the Helmholtz decomposition of $L^r(\mathbf{R})^n$. Since

$$(\widetilde{P}u)_j = \sum_{k=1}^{n}(\delta_{jk} + R_jR_k)u_k, \quad j = 1, \dots, n,$$

where R_j are the Riesz transforms [43], we see that

$$\widehat{H}^{1,r}_{0,\sigma}(\mathbf{R}^n) = \widetilde{P}\widehat{H}^{1,r}_0(\mathbf{R}^n)^n.$$

Since $\widetilde{P}$ defines a bounded projector on $L^{(r,q)}(\mathbf{R}^n)^n$, it follows from (5.18) that

$$\begin{aligned}(\widehat{H}^{1,r_0}_{0,\sigma}(\mathbf{R}^n), \widehat{H}^{1,r_1}_{0,\sigma}(\mathbf{R}^n))_{\theta,q} &= \widetilde{P}\{u \in L^{(r^*,q)}(\mathbf{R}^n)^n : \nabla u \in L^{(r,q)}(\mathbf{R}^n)^{n^2}\} \\ &= \{u \in L^{(r^*,q)}_\sigma(\mathbf{R}^n) : \nabla u \in L^{(r,q)}(\mathbf{R}^n)^{n^2}\}.\end{aligned}$$

This proves (5.16).

6. Stability in L^n_w

We now discuss the existence and asymptotic behavior of strong solutions of perturbation equation (1.5), assuming that the initial perturbations a are small in the Banach space

$$L^n_{\sigma,w} = \{u \in L^n_w(\Omega)^n : \nabla \cdot u = 0,\ u \cdot \nu|_S = 0\}. \tag{6.1}$$

Notice that in (6.1) the trace $u\cdot\nu|_S$ makes sense, since $L^r_w(\Omega)\subset L^q_{\rm loc}(\bar\Omega)$ with continuous injection, whenever $1\leqslant q<r<\infty$.

As is now well known, the space L^n_σ is the basic space in which to find strong solutions for the Navier–Stokes system, i.e., equation (1.5) with $w=0$. In other words, it has so far been possible to get a global-in-time strong solution of the Navier–Stokes system only when a is sufficiently small in L^n_σ. As shown by Kozono and Ogawa [25], the same is true for perturbation equation (1.5) if

$$w\in L^n\cap L^\infty,\quad \nabla w\in L^{n/2}\cap L^\infty, \tag{6.2}$$

and if $\|w\|_n+\|\nabla w\|_{n/2}$ is small enough. We establish in this section an $L^n_{\sigma,w}$-version of the global existence result of [25]. To be more precise, we shall show that if

$$|w|\leqslant C/|x|,\quad |\nabla w|\leqslant C/|x|^2, \tag{6.3}$$

and if $\|w\|+\|\nabla w\|$ is small enough, then equation (1.5) admits a unique strong solution in $L^n_{\sigma,w}$ defined for all $t\geqslant 0$ provided that a is small in $L^n_{\sigma,w}$. Since $L^n_\sigma\subset L^n_{\sigma,w}$, this includes the global existence result of [25] as a special case. We further remark that condition (6.2) is much stronger than our condition (6.3) when $n=3$. In fact, when $n=3$, we have shown in Theorem 2.4 that conditions (6.2) and $F\in L^{3/2}$ together imply the vanishing of the total net force:

$$\int_S \nu\cdot(T[w,p]-w^*\otimes w^*+F)\,dS=0,$$

which would not always be valid for our stationary flows.

As in [25], we systematically use in this section the L^p-L^q estimates for the semigroups $\{e^{-tL}\}_{t\geqslant 0}$ and $\{e^{-tL^*}\}_{t\geqslant 0}$ as established in §3, so the size of the derivatives ∇w in $L^{n/2}_w\cap L^\infty$ plays an important role. Kozono and Ogawa [25] deduced their version of L^p-L^q estimates for the semigroups $\{e^{-tL}\}_{t\geqslant 0}$ and $\{e^{-tL^*}\}_{t\geqslant 0}$, and applied it to discussing the solvability in L^n_σ of equation (1.5). Since our version of the L^p-L^q estimates improves theirs, most of our results in this section are deduced in essentially the same way as in [25]. However, we state our proofs in detail, since most readers would not be familiar with the use of Lorentz spaces in the study of nonlinear differential equations and since our results include an L^∞ estimate, which is not discussed in [25].

We now introduce our class of strong solutions of problem (1.5), or equivalently, that of the evolution equation

$$\frac{du}{dt}+Lu+P(u\cdot\nabla)u=0\ \ (t>0),\quad u(0)=a. \tag{6.4}$$

Definition 6.1. Let w satisfy assumption (6.3). Given $a\in L^n_{\sigma,w}$, a measurable function u defined on $\Omega\times(0,T)$ is called a *strong solution* of (6.4) on $(0,T)$ if

(1) $u\in C_w([0,T);L^n_{\sigma,w})\cap C^1((0,T);L^n_{\sigma,w})$;

(2) $Lu\in C((0,T);L^n_{\sigma,w})$;

and the function u satisfies (6.4). Here C_w stands for the weak* continuity. (Recall that $L^r_w=L^{(r,\infty)}=(L^{(r',1)})^*$.)

Our first results are the following; the second results (L^∞ estimate) will be stated in Theorems 6.8 and 6.9 in the final paragraph.

THEOREM 6.2. *There exists a (small) number $\lambda=\lambda(n)>0$ so that if*

$$\|a\|_{n,w}\leqslant\lambda \quad \text{and} \quad \|w\|+\|\nabla w\|\leqslant\lambda, \tag{6.5}$$

then there is a unique strong solution u defined for all $t\geqslant 0$ satisfying

$$u\in \mathrm{BC}_w([0,\infty);L^n_{\sigma,w})\cap\mathrm{BC}((0,\infty);L^n_{\sigma,w}),\quad t^{1/4}u(\,\cdot\,)\in\mathrm{BC}((0,\infty);L^{2n}_\sigma),$$

where BC *stands for the space of bounded continuous functions. Moreover, for each $n<r<\infty$ there is an $\eta=\eta(n,r)>0$ so that if*

$$\|w\|+\|\nabla w\|\leqslant\eta, \tag{6.6}$$

then the solution u obtained above satisfies

$$\|u(t)\|_q\leqslant Ct^{-(1-n/q)/2} \quad \text{for } n<q\leqslant r \tag{6.7}$$

with some $C=C(n,r,q)>0$.

THEOREM 6.3. (i) *Let $1<r<n$ and $a\in L^r_{\sigma,w}\cap L^n_{\sigma,w}$. Then, there is a positive number*

$$\lambda'=\lambda'(n,r)\leqslant\lambda$$

so that if

$$\|a\|_{n,w}\leqslant\lambda' \quad \text{and} \quad \|w\|+\|\nabla w\|\leqslant\lambda', \tag{6.8}$$

then the solution u given in Theorem 6.2 satisfies

$$u\in\mathrm{BC}_w([0,\infty);L^r_{\sigma,w})\cap\mathrm{BC}((0,\infty);L^r_{\sigma,w}). \tag{6.9}$$

Moreover, under the assumption (6.8) we have

$$t^{1/2}\nabla u(\,\cdot\,)\in\mathrm{BC}((0,\infty);L^r_w). \tag{6.10}$$

(ii) *Let* $1<p<n$, $a\in L^p_{\sigma,w}\cap L^n_{\sigma,w}$ *and assume* (6.8) *with* $r=p$. *Then for each* $p<r<\infty$ *there is a positive* $\eta'=\eta'(n,p,r)\leqslant\eta$ *so that if*

$$\|w\|+\|\nabla w\|\leqslant\eta', \tag{6.11}$$

the function u *satisfies, as* $t\to\infty$,

$$\|u(t)\|_q=O(t^{-(n/p-n/q)/2}) \quad \text{for } p<q\leqslant r. \tag{6.12}$$

Moreover, suppose $p<r<n$ *and* (6.8) *holds. Then under the assumption* (6.11), *the function* u *satisfies, as* $t\to\infty$,

$$\|\nabla u(t)\|_q=O(t^{-(n/p-n/q)/2-1/2}) \quad \text{for } p<q\leqslant r<n. \tag{6.13}$$

(iii) *Let* $1<p<n$. *For each* $\varepsilon>0$ *there is a number* $\mu=\mu(p,\varepsilon)>0$ *so that if*

$$\|w\|+\|\nabla w\|\leqslant\mu,$$

then

$$\|\nabla u(t)\|_n=O(t^{\varepsilon-n/2p}) \quad \text{as } t\to\infty. \tag{6.14}$$

THEOREM 6.4. (i) *Under the assumption of Theorem* 6.2, *the solution* u *satisfies*

$$\lim_{t\to\infty}\|u(t)\|_{n,w}=0$$

provided a *is in the* L^n_w*-closure of* $C^\infty_{0,\sigma}(\Omega)$.

(ii) *Let* $a\in L^n_{\sigma,w}$, $\nabla a\in L^{n/2}_w$ *and* $a|_S=0$. *Then, under the assumption of Theorem* 6.2, *we have*

$$\nabla u\in \mathrm{BC}_w([0,\infty);L^{n/2}_w)\cap \mathrm{BC}((0,\infty);L^{n/2}_w).$$

We shall deduce Theorems 6.2, 6.3 and 6.4, solving the integral equation

$$u(t)=e^{-tL}a-\int_0^t e^{-(t-\tau)L}P(u\cdot\nabla)u(\tau)\,d\tau \tag{6.15}$$

in the class of *mild solutions* in the following sense:

Definition 6.5. Given $a\in L^n_{\sigma,w}$ and $n<r<\infty$, a measurable function u on $\Omega\times(0,T)$ is called a mild solution of (1.5) in the class $S_r(0,T)$ if it satisfies

(1) $u\in\mathrm{BC}_w([0,T);L^n_{\sigma,w})\cap\mathrm{BC}((0,T);L^n_{\sigma,w})$; $t^{(1-n/r)/2}u(\cdot)\in\mathrm{BC}([0,T);L^r_\sigma)$; and

(2) $\langle u(t),\phi\rangle=\langle e^{-tL}a,\phi\rangle+\int_0^t\langle u\otimes u(\tau),\nabla e^{-(t-\tau)L^*}\phi\rangle\,d\tau$,

for all $\phi\in C^\infty_{0,\sigma}(\Omega)$ and $0<t<T$.

Our definition of mild solution is essentially due to [25]. To establish an existence and uniqueness theorem for equation (6.15) in the class of mild solutions, we interpolate between estimates (3.34) and (3.35) to get similar estimates in Lorentz spaces.

PROPOSITION 6.6. *Let* $n \geqslant 3$, $1<p<\infty$ *and* $1 \leqslant q \leqslant \infty$, *and let* $\|\cdot\|_{p,q}$ *denote the norm of the space* $L^{(p,q)}(\Omega)$.

(i) *For each* r *with* $p \leqslant r < \infty$ *there is a number* $\eta(p,q,r)>0$ *so that if*

$$\|w\|+\|\nabla w\| \leqslant \eta(p,q,r),$$

then we have

$$\|e^{-tL}a\|_{r,q},\ \|e^{-tL^*}a\|_{r,q} \leqslant Mt^{-(n/p-n/r)/2}\|a\|_{p,q}. \tag{6.16}$$

Furthermore, when $p \leqslant r < n$, *there is a number* $\eta'(p,q,r)>0$ *so that if*

$$\|w\|+\|\nabla w\| \leqslant \eta'(p,q,r),$$

then

$$\|\nabla e^{-tL}a\|_{r,q},\ \|\nabla e^{-tL^*}a\|_{r,q} \leqslant Mt^{-1/2-(n/p-n/r)/2}\|a\|_{p,q}. \tag{6.17}$$

(ii) *For each* r *with* $p<r<\infty$ *there is a number* $\lambda(p,r)>0$ *so that if*

$$\|w\|+\|\nabla w\| \leqslant \lambda(p,r),$$

then we have

$$\|e^{-tL}a\|_{r},\ \|e^{-tL^*}a\|_{r} \leqslant M_{p,r}t^{-(n/p-n/r)/2}\|a\|_{p,w}. \tag{6.18}$$

Furthermore, when $p<r<n$, *there is a number* $\lambda'(p,r)>0$ *so that if*

$$\|w\|+\|\nabla w\| \leqslant \lambda'(p,r),$$

then

$$\|\nabla e^{-tL}a\|_{r},\ \|\nabla e^{-tL^*}a\|_{r} \leqslant M_{p,r}t^{-1/2-(n/p-n/r)/2}\|a\|_{p,w}. \tag{6.19}$$

Proof. (i) follows by interpolating estimates (3.34) and (3.35) between the spaces $L^{(p,q)}_\sigma$. Assertion (ii) is easily obtained from estimates (6.16) and (6.17) with $q=\infty$, since

$$L^{p_0}_w \cap L^{p_1}_w \subset L^p \quad \text{and} \quad \|f\|_p \leqslant C(p_0,p_1,\theta)\|f\|^{1-\theta}_{p_0,w}\|f\|^{\theta}_{p_1,w}$$

provided that

$$p_0 \neq p_1, \quad 0<\theta<1 \quad \text{and} \quad 1/p=(1-\theta)/p_0+\theta/p_1.$$

The proof is complete.

Proof of Theorem 6.2. Consider now the iteration scheme:

$$\begin{aligned} u_0(t) &= e^{-tL}a, \\ u_{j+1}(t) &= u_0(t)+v_j(t), \quad j \geqslant 0, \\ v_j(t) &= -\int_0^t e^{-(t-\tau)L}P(u_j\cdot\nabla)u_j(\tau)\,d\tau. \end{aligned} \tag{6.20}$$

Assuming $a \in L^n_{\sigma,w}$ to be sufficiently small, we show that the functions u_j are well defined and converge in an appropriate sense to a mild solution. The mild solution obtained below is actually a strong solution; but, its proof is standard and so omitted in this paper. The following argument is essentially the same as in [18], which deals with the case of the Cauchy problem for the Navier–Stokes system in $\mathbf{R}^2$. Let

$$K_j = \max\big(\sup_{t>0} t^{1/4}\|u_j(t)\|_{2n}, \sup_{t>0} \|u_j(t)\|_{n,w}\big), \quad j=0,1,2,\dots .$$

Proposition 6.6 and Lemma 2.1 (i) together imply that

$$\begin{aligned}|\langle u\otimes u, \nabla e^{-(t-\tau)L^*}\varphi\rangle| &\leqslant \|u\otimes u\|_{2n/3,w}\|\nabla e^{-(t-\tau)L^*}\varphi\|_{(2n/3)',1}\\ &\leqslant CM'\|u\|_{2n}\|u\|_{n,w}(t-\tau)^{-3/4}\|\varphi\|_{n',1}\end{aligned}$$

and

$$|\langle u\otimes u, \nabla e^{-(t-\tau)L^*}\varphi\rangle| \leqslant \|u\otimes u\|_n\|\nabla e^{-(t-\tau)L^*}\varphi\|_{n'} \leqslant M''\|u\|_{2n}^2(t-\tau)^{-3/4}\|\varphi\|_{(2n)'}$$

for $\varphi \in C^\infty_{0,\sigma}(\Omega)$. We thus have

$$\|e^{-(t-\tau)L}P(u\cdot\nabla u)\|_{n,w} \leqslant CM'(t-\tau)^{-3/4}\|u\|_{2n}\|u\|_{n,w}$$

and

$$\|e^{-(t-\tau)L}P(u\cdot\nabla u)\|_{2n} \leqslant M''(t-\tau)^{-3/4}\|u\|_{2n}^2.$$

This implies that

$$K_0 \leqslant M_1\|a\|_{n,w}, \quad K_{j+1} \leqslant M_1\|a\|_{n,w} + M_2\beta K_j^2, \quad j \geqslant 0, \tag{6.21}$$

where

$$\begin{gathered} M_1 = \max(M_{n,2n}, M_n), \quad M_2 = \max(CM', M''), \\ \beta = \max\big(B(\tfrac14,\tfrac12), B(\tfrac14,\tfrac34)\big), \end{gathered}$$

and $B(\cdot,\cdot)$ is the beta function. Thus, an elementary calculation shows that if

$$4M_1M_2\beta\|a\|_{n,w} < 1,$$

then for each $j>0$,

$$K_j \leqslant k \equiv \frac{1-\sqrt{1-4M_1M_2\beta\|a\|_{n,w}}}{2M_2\beta} < \frac{1}{2M_2\beta}, \tag{6.22}$$

so that

$$\|u_j(t)\|_{n,w} \leqslant k, \quad \|u_j(t)\|_{2n} \leqslant kt^{-1/4}. \tag{6.23}$$

In the same way, the relation

$$w_j(t) \equiv u_{j+1}(t) - u_j(t) = -\int_0^t e^{-(t-\tau)L}(P(w_{j-1}\cdot\nabla)u_j + P(u_{j-1}\cdot\nabla)w_{j-1})(\tau)\,d\tau$$

implies that if we fix $T>0$, then

$$\begin{aligned} \|w_j(t)\|_{n,w} &\leqslant 2M_2k\beta \sup_{0<t\leqslant T} \|w_{j-1}(t)\|_{n,w}, \\ t^{1/4}\|w_j(t)\|_{2n} &\leqslant 2M_2k\beta \sup_{0<t\leqslant T} t^{1/4}\|w_{j-1}(t)\|_{2n}, \end{aligned} \tag{6.24}$$

for $t\in(0,T]$. Since $2M_2k\beta<1$ by (6.22), it follows from (6.24) that there exists a function u satisfying

$$\|u(t)\|_{n,w} \leqslant k, \quad \|u(t)\|_{2n} \leqslant kt^{-1/4}, \tag{6.23$'$}$$

such that $t^{1/4}u_j(\cdot)$ and $u_j(\cdot)$ converge uniformly for bounded $t>0$ to $t^{1/4}u(\cdot)$ and $u(\cdot)$, respectively. It then easily follows that

$$\langle u(t),\phi\rangle = \langle e^{-tL}a,\phi\rangle + \int_0^t \langle u\otimes u(\tau), \nabla e^{-(t-\tau)L^*}\phi\rangle\,d\tau \tag{6.25}$$

for all $\phi\in C_{0,\sigma}^\infty$. Estimate (6.23′) ensures the absolute convergence in $L^n_{\sigma,w}\cap L^{2n}_\sigma$ of the integral on the right-hand side of (6.15); so it is continuous in $t\geqslant 0$ with values in $L^n_{\sigma,w}$, and continuous in $t>0$ with values in L^{2n}_σ. The linear term $e^{-tL}a$ is continuous for $t>0$ with values in $L^n_{\sigma,w}\cap L^{2n}_\sigma$, and weakly* continuous at $t=0$ with values in $L^n_{\sigma,w}$. These observations, together with (6.23′), imply that

$$u\in \mathrm{BC}_w([0,\infty);L^n_{\sigma,w})\cap \mathrm{BC}((0,\infty);L^n_{\sigma,w}), \quad t^{1/4}u(\cdot)\in \mathrm{BC}((0,\infty);L^{2n}_\sigma).$$

Thus, u is a mild solution of class S_{2n}. The proof of uniqueness is standard, so omitted.

We next prove (6.7). Since $u\in S_{2n}$, we obtain

$$\|u(t)\|_q \leqslant C\|u(t)\|^\theta_{2n,w}\|u(t)\|^{1-\theta}_{n,w} \leqslant C\|u(t)\|^\theta_{2n}\|u(t)\|^{1-\theta}_{n,w} \leqslant Ct^{-(1-n/q)/2}$$

for $n<q<2n$, where $1/q=(1-\theta)/n+\theta/2n$. This, together with (6.23′), shows (6.7) with $r=2n$. When $q>2n$, we apply this with $q=2n$ to (6.25), to get by Proposition 6.6,

$$\begin{aligned} \|u(t)\|_q &\leqslant C_q\|a\|_{n,w}t^{-(1-n/q)/2} + M_{n,q}k^2\int_0^t (t-\tau)^{-1+n/2q}\tau^{-1/2}\,d\tau \\ &= (C_q\|a\|_{n,w} + M'_{n,q}k^2)t^{-(1-n/q)/2} \equiv K_q t^{-(1-n/q)/2}. \end{aligned}$$

This completes the proof of (6.7).

Proof of Theorem 6.3. We first prove (6.10). Note that if $1<r<n$, then $1/r+1/q<1$ for sufficiently large q. Thus, Proposition 6.6 and (6.7) together imply that if $\|\nabla u_j(t)\|_{r,w}\leqslant C_j t^{-1/2}$, then

$$\begin{aligned}\|\nabla u_{j+1}(t)\|_{r,w} &\leqslant C_0 t^{-1/2}\|a\|_{r,w}+M\int_0^t (t-\tau)^{-1/2-n/2q}\|u_j\|_q\|\nabla u_j\|_{r,w}(\tau)\,d\tau \\ &\leqslant C_0 t^{-1/2}\|a\|_{r,w}+MK_qC_j\int_0^t (t-\tau)^{-1/2-n/2q}\tau^{-1+n/2q}\,d\tau \\ &= C_0 t^{-1/2}\|a\|_{r,w}+MK_qC_jB(1/2-n/2q,n/2q)t^{-1/2},\end{aligned}$$

where K_q is the constant in the estimate $\|u_j(t)\|_q\leqslant K_q t^{-(1-n/q)/2}$. Hence,

$$C_{j+1}\leqslant C_0\|a\|_{r,w}+MK_qB(1/2-n/2q,n/2q)C_j.$$

Since we may assume K_q sufficiently small as shown in the proof of (6.7), we obtain (6.10).

We can now prove (6.9), using (6.7) and (6.10). Let $1<r<n$ and choose $q>1$ so that $1/r+1/q<1$. Since $L^r_{\sigma,w}=(L^{(r',1)}_\sigma)^*$ by Theorem 5.2, the estimate

$$\begin{aligned}|\langle u\cdot\nabla u(\tau),e^{-(t-\tau)L^*}\phi\rangle| &\leqslant C\|u\|_{q,w}(\tau)\|\nabla u\|_{r,w}(\tau)\|e^{-(t-\tau)L^*}\phi\|_{qr'/(q-r'),1} \\ &\leqslant C\tau^{-(1-n/q)/2}\tau^{-1/2}(t-\tau)^{-n/2q}\|\phi\|_{r',1} \\ &= C\tau^{-1+n/2q}(t-\tau)^{-n/2q}\|\phi\|_{r',1}\end{aligned}$$

implies that

$$\begin{aligned}\|u(t)\|_{r,w} &\leqslant M_r\|a\|_{r,w}+C\int_0^t (t-\tau)^{-n/2q}\tau^{-1+n/2q}\,d\tau \\ &= M_r\|a\|_{r,w}+CB(1-n/2q,n/2q).\end{aligned}$$

This completes the proof of (6.9).

We next prove (6.12) and (6.13). When $q=p=r$, these results are just (6.9) and (6.10), respectively. Let $1<p<r$, and suppose that

$$0<1/p-1/r<1/n. \tag{6.26}$$

We take a large s so that $1/s+1/p-1/r<1/n$ and apply (6.10) with the exponent p and (6.7) with the exponent s to get by Proposition 6.6,

$$\begin{aligned}\|u(t)\|_r &\leqslant Mt^{-(n/p-n/r)/2}\|a\|_{p,w}+M\int_0^t (t-\tau)^{-(n/s+n/p-n/r)/2}\tau^{-1+n/2s}\,d\tau \\ &= Mt^{-(n/p-n/r)/2}\|a\|_{p,w}+M't^{-(n/p-n/r)/2},\end{aligned}$$

since $\frac{1}{2}(n/s+n/p-n/r)<1$. This proves (6.12) for $p<q=r$ under the assumption (6.26). The result in case $p<q<r$ is deduced via the interpolation inequality:

$$\|u\|_q \leqslant C\|u\|_r^\theta \|u\|_{p,w}^{1-\theta}, \quad 1/q=(1-\theta)/p+\theta/r.$$

The proof of (6.12) is thus complete under the assumption (6.26). In the same way, we see that if $r<n$ and if (6.26) holds, Proposition 6.6 yields

$$\begin{aligned}\|\nabla u(t)\|_r &\leqslant Mt^{-1/2-(n/p-n/r)/2}\|a\|_{p,w}+M\int_0^t (t-\tau)^{-1/2-(n/s+n/p-n/r)/2}\tau^{-1+n/2s}\,d\tau \\ &= Mt^{-1/2-(n/p-n/r)/2},\end{aligned}$$

since $\frac{1}{2}+\frac{1}{2}(n/s+n/p-n/r)<1$. This shows (6.13) for $p<q=r<n$ under the assumption (6.26); and the case $p<q<r<n$ is deduced via interpolation.

Consider next the case

$$1/n \leqslant 1/p-1/r < 2/n \tag{6.27}$$

and choose l with $p<l<r$ so that

$$0<1/p-1/l<1/n, \quad 0<1/l-1/r<1/n.$$

Writing the integral equation (6.15) in the form

$$u(t)=e^{-tL/2}u\left(\tfrac{1}{2}t\right)-\int_{t/2}^t e^{-(t-\tau)L}P(u\cdot\nabla)u(\tau)\,d\tau, \tag{6.28}$$

and bearing in mind the estimate

$$\|\nabla u(t)\|_l \leqslant Ct^{-1/2-(n/p-n/l)/2},$$

we take $s>1$ so that $1/s+1/l-1/r<1/n$ and apply (6.7) with $q=s$, to get

$$\begin{aligned}\|u(t)\|_r &\leqslant Ct^{-(n/l-n/r)/2}\left\|u\left(\tfrac{1}{2}t\right)\right\|_l \\ &\qquad +C\int_{t/2}^t (t-\tau)^{-(n/s+n/l-n/r)/2}\tau^{-1/2-(n/p-n/l)/2}\tau^{-(1-n/s)/2}\,d\tau \\ &\leqslant Ct^{-(n/p-n/r)/2}+C\int_{t/2}^t (t-\tau)^{-(n/s+n/l-n/r)/2}\tau^{-1-(n/p-n/s-n/l)/2}\,d\tau \\ &\leqslant Ct^{-(n/p-n/r)/2}\end{aligned}$$

for $t>0$, since s can be taken so that $1/p>1/l+1/s$. This shows (6.12) with $p<q=r$ and the case $p<q<r$ is deduced via interpolation. Thus, we have proved (6.12) under the

assumption (6.27). Similarly, since $\frac{1}{2}+\frac{1}{2}(n/s+n/l-n/r)<1$, assuming $r<n$ we obtain

$$\begin{aligned}\|\nabla u(t)\|_r &\leqslant Ct^{-1/2-(n/p-n/r)/2}\\ &\quad +C\int_{t/2}^{t}(t-\tau)^{-1/2-(n/s+n/l-n/r)/2}\tau^{-1-(n/p-n/s-n/l)/2}\,d\tau\\ &\leqslant Ct^{-1/2-(n/p-n/r)/2}.\end{aligned}$$

This shows (6.13) for $p<q=r<n$ under the assumption (6.27); and the case $p<q<r<n$ is deduced via interpolation.

When

$$j/n\leqslant 1/p-1/r<(j+1)/n,\quad j\geqslant 2,$$

we choose l so that $p<l<r$ and

$$0<1/p-1/l<j/n,\quad 0<1/l-1/r<1/n,$$

and repeat the above processes to get (6.12) and (6.13) in all cases.

To prove (6.14), we rewrite (6.28) in the form

$$u(t)=e^{-tA/2}u\left(\tfrac{1}{2}t\right)-\int_{t/2}^{t}e^{-(t-\tau)A}(Bu+P(u\cdot\nabla u))(\tau)\,d\tau \tag{6.28'}$$

in terms of the Stokes operator A and apply the estimates of Iwashita [23]:

$$\begin{aligned}\|e^{-tA}a\|_r &\leqslant Ct^{-(n/q-n/r)/2}\|a\|_q \quad (1<q\leqslant r<\infty),\\ \|\nabla e^{-tA}a\|_r &\leqslant Ct^{-1/2-(n/q-n/r)/2}\|a\|_q \quad (1<q\leqslant r\leqslant n).\end{aligned} \tag{6.29}$$

The function

$$v_1(t)=e^{-tA/2}u\left(\tfrac{1}{2}t\right)-\int_{t/2}^{t}e^{-(t-\tau)A}P(u\cdot\nabla u)(\tau)\,d\tau$$

is estimated as in the proof of (6.13) and we obtain

$$\|\nabla v_1(t)\|_n=O(t^{-n/2p})\quad \text{as } t\to\infty.$$

The remaining term

$$v_2(t)=-\int_{t/2}^{t}e^{-(t-\tau)A}Bu(\tau)\,d\tau$$

is estimated as follows. Given a small $\delta>0$, we apply (6.29) to get

$$\begin{aligned}\|\nabla v_2(t)\|_n &\leqslant C\int_{t/2}^{t}(t-\tau)^{-1+\delta}(\|w\cdot\nabla u\|_{n/2(1-\delta)}+\|u\cdot\nabla w\|_{n/2(1-\delta)})\,d\tau\\ &\leqslant C\int_{t/2}^{t}(t-\tau)^{-1+\delta}(\|w\|_{n/(1-3\delta)}\|\nabla u\|_{n/(1+\delta)}+\|\nabla w\|_{n/(2-3\delta)}\|u\|_{n/\delta})(\tau)\,d\tau\\ &\leqslant C\|\nabla w\|_{n/(2-3\delta)}\int_{t/2}^{t}(t-\tau)^{-1+\delta}\|\nabla u\|_{n/(1+\delta)}(\tau)\,d\tau\\ &\leqslant C\|\nabla w\|\int_{t/2}^{t}(t-\tau)^{-1+\delta}\|\nabla u\|_{n/(1+\delta)}(\tau)\,d\tau.\end{aligned}$$

Since $\|\nabla u\|_{n/(1+\delta)} \leqslant C\tau^{-n/2p+\delta/2}$ provided w is small depending on $\delta>0$ and p, it follows that

$$\|\nabla v_2(t)\|_n \leqslant C\int_{t/2}^t (t-\tau)^{-1+\delta}\tau^{-n/2p+\delta/2}\,d\tau = O(t^{-n/2p+3\delta/2}),$$

which completes the proof of (6.14). The proof of Theorem 6.3 is complete.

Proof of Theorem 6.4. Let u_1 and u_2 be the solutions with the initial data a_1 and a_2, respectively. Then $v=u_1-u_2$ satisfies

$$v(t) = e^{-tL}(a_1-a_2) - \int_0^t e^{-(t-\tau)L}P((v\cdot\nabla)u_1+(u_2\cdot\nabla v))(\tau)\,d\tau,$$

so that

$$\begin{aligned}\|v(t)\|_{n,w} &\leqslant M_1\|a_1-a_2\|_{n,w} + M_2\int_0^t (t-\tau)^{-3/4}(\|u_1\|_{2n}+\|u_2\|_{2n})\|v\|_{n,w}(\tau)\,d\tau \\ &\leqslant M_1\|a_1-a_2\|_{n,w} + 2M_2k\beta\sup_{\tau>0}\|v(\tau)\|_{n,w}.\end{aligned}$$

This shows that if we assume $4M_1M_2\beta\|a_j\|_{n,w}<1$ for $j=1,2$, so that (6.22) holds, then

$$\sup_{\tau>0}\|u_1(\tau)-u_2(\tau)\|_{n,w} \leqslant C\|a_1-a_2\|_{n,w},$$

which shows that the map $a\mapsto u$ is continuous from a neighborhood of 0 in $L^n_{\sigma,w}$ to $\mathrm{BC}_w([0,\infty); L^n_{\sigma,w})$. On the other hand, (6.12) shows that if $a\in L^r_{\sigma,w}\cap L^n_{\sigma,w}$ for some $1<r<n$, then $\|u(t)\|_{n,w}\to 0$ as $t\to\infty$. Hence, Theorem 6.4 (i) follows through approximation of the initial value a.

We finally prove Theorem 6.4 (ii). To this end we need

LEMMA 6.7. *Let $n\geqslant 3$ and $1<r<n$. Under the assumptions of Theorem* 3.10 (i) *or* (ii), *we have the estimates*

$$\|\nabla e^{-tL}a\|_{r,w},\ \|\nabla e^{-tL^*}a\|_{r,w} \leqslant M_r\|\nabla a\|_{r,w} \tag{6.30}$$

for all $t\geqslant 0$ and $a\in D^{1/2}_{r,w}\equiv D^{1/2}_{(r,\infty)}$.

Assuming Lemma 6.7 for a moment, we prove Theorem 6.4 (ii). Let $a\in L^n_{\sigma,w}$, $\nabla a\in L^{n/2}_w$ and $a|_S=0$ so that $a\in D^{1/2}_{n/2,w}$ by Theorem 5.13. Consider the iteration scheme (6.20). Lemma 6.7 and (6.16) yield

$$\|u_0(t)\|_{n,w} + \|\nabla u_0(t)\|_{n/2,w} \leqslant M(\|a\|_{n,w}+\|\nabla a\|_{n/2,w}). \tag{6.31}$$

Let

$$K_j = \sup_{t>0}(\|u_j(t)\|_{n,w}+\|\nabla u_j(t)\|_{n/2,w}). \tag{6.32}$$

The proof of Theorem 6.2 (i) shows that

$$\|u_{j+1}(t)\|_{n,w} \leqslant \|u_0(t)\|_{n,w} + C_n \int_0^t (t-\tau)^{-3/4} \|u_j\|_{2n} \|u_j\|_{n,w}(\tau)\, d\tau. \tag{6.33}$$

Furthermore, (6.17) and the weak Hölder inequality together imply

$$\begin{aligned} \|\nabla u_{j+1}(t)\|_{n/2,w} &\leqslant \|\nabla u_0(t)\|_{n/2,w} + C_n \int_0^t (t-\tau)^{-3/4} \|u \cdot \nabla u\|_{2n/5,w}(\tau)\, d\tau \\ &\leqslant \|\nabla u_0(t)\|_{n/2,w} + C_n \int_0^t (t-\tau)^{-3/4} \|u_j\|_{2n} \|\nabla u_j\|_{n/2,w}(\tau)\, d\tau. \end{aligned} \tag{6.34}$$

Since $\|u_j(t)\|_{2n} \leqslant k t^{-1/4}$ by (6.23), it follows from (6.32)–(6.34) that

$$K_{j+1} \leqslant K_0 + C_n k B\left(\tfrac{1}{4}, \tfrac{3}{4}\right) K_j.$$

Since K_0 is finite by (6.31), assuming k sufficiently small, we get a uniform bound for K_j. Similarly, we can show the convergence of u_j and ∇u_j by estimating $w_j = u_{j+1} - u_j$. This proves Theorem 6.4 (ii).

Proof of Lemma 6.7. We consider only the case of operator L. The case of L^* is discussed similarly. Suppose first $1 < r < \frac{1}{2}n$ and consider the Neumann series expansion

$$(\lambda + L)^{-1} u = (\lambda + A)^{-1} \sum_{j=0}^{\infty} (-B(\lambda + A)^{-1})^j u = \sum_{j=0}^{\infty} (-(\lambda + A)^{-1} B)^j (\lambda + A)^{-1} u. \tag{6.35}$$

Since $1 < r < \frac{1}{2}n$, we get $\|\nabla^2(\lambda + A)^{-1}\| \leqslant C \|A(\lambda + A)^{-1}\| \leqslant C$; and so

$$\|\nabla^2(\lambda + A)^{-1} B v\|_r \leqslant C \|Bv\|_r \leqslant C_r(\|w\| + \|\nabla w\|) \|\nabla^2 v\|_r.$$

Hence, (6.35) gives

$$\|\nabla^2(\lambda + L)^{-1} u\|_r \leqslant \left(\sum_{j=0}^{\infty} [C_r(\|w\| + \|\nabla w\|)]^j \right) \|\nabla^2(\lambda + A)^{-1} u\|_r.$$

But, since $1 < r < \frac{1}{2}n$, we have

$$\begin{aligned} \|\nabla^2(\lambda + A)^{-1} u\|_r &\leqslant C \|A(\lambda + A)^{-1} u\|_r = C \|(\lambda + A)^{-1} A u\|_r \\ &\leqslant C \|Au\|_r / |\lambda| \leqslant C \|\nabla^2 u\|_r / |\lambda|. \end{aligned}$$

Thus, assuming $\|w\| + \|\nabla w\|$ to be small, we get from (6.35)

$$\|\nabla^2(\lambda + L)^{-1} u\|_r \leqslant C \|\nabla^2 u\|_r / |\lambda| \tag{6.36}$$

for $u \in D^1_r$. Now, according to [4, Proposition 4.3 (ii)], the space $D^{1/2}_r$ equals the complex interpolation space $[D^1_r, L^r_\sigma]_{1/2}$. Thus, interpolating between (3.15) and (6.36) shows that if $1<r<\frac{1}{2}n$, then we have

$$\|\nabla(\lambda+L)^{-1}u\|_r \leqslant C\|\nabla u\|_r/|\lambda|, \quad u \in D^{1/2}_r. \tag{6.37}$$

Suppose next that $n'<r<n$. We again appeal to the Neumann series expansion (6.35) and get (3.23):

$$\|\nabla(\lambda+L)^{-1}u\|_r \leqslant \left(\sum_{j=0}^{\infty}(C\|w\|)^j\right)\|\nabla(\lambda+A)^{-1}u\|_r.$$

Since $r<n$, we have

$$\begin{aligned}\|\nabla(\lambda+A)^{-1}u\|_r &\leqslant C\|A^{1/2}(\lambda+A)^{-1}u\|_r = C\|(\lambda+A)^{-1}A^{1/2}u\|_r \\ &\leqslant C\|A^{1/2}u\|_r/|\lambda| \leqslant C\|\nabla u\|_r/|\lambda|\end{aligned}$$

for $u \in D^{1/2}_r$. Since $\|w\|$ is small by assumption, this shows (6.37) for $n'<r<n$. Now let $1<r<n$ and choose $1<r_0<\frac{1}{2}n$ and $n'<r_1<n$ with $r_0<r<r_1$. Then, by [4, Proposition 4.3 (iii)] and [4, Theorem 4.4 (iii)], we have

$$[D^{1/2}_{r_0}, D^{1/2}_{r_1}]_\theta = D^{1/2}_r, \quad 1/r=(1-\theta)/r_0+\theta/r_1. \tag{6.38}$$

Hence, it follows via interpolation that (6.37) holds for all $1<r<n$.

Now, (6.37) implies

$$\|\nabla e^{-tL}a\|_r \leqslant M_r\|\nabla a\|_r, \quad a \in D^{1/2}_r \ (1<r<n). \tag{6.39}$$

But, since

$$D^{1/2}_{(r,\infty)} = (D^{1/2}_{r_0}, D^{1/2}_{r_1})_{\theta,\infty}, \quad 1/r=(1-\theta)/r_0+\theta/r_1,$$

estimate (6.30) follows by applying interpolation to (6.39). This proves Lemma 6.7.

Remarks. (i) In this section we have discussed only the existence of a (unique) global-in-time mild solution in the sense of Definition 6.5. But, the properties of the obtained mild solution as described above ensure that they are in fact strong solutions in the sense of Definition 6.1. Since the proof is standard, the details are omitted.

(ii) The method of this section applies also to the proof of the existence of a (unique) local-in-time strong solution if we take the initial value a from the usual L^p spaces instead of the weak L^p spaces. This case is discussed in detail in [25] (but, under assumption (6.2)), except for the fact that if $a \in L^n_\sigma$ and $\nabla a \in L^{n/2}$, then $u(t) \in L^n_\sigma$ and $\nabla u(t) \in L^{n/2}$

for all $t>0$ in the existence interval of the strong solution u. This latter result is proved in almost the same way as Theorem 6.4 (ii), by applying Theorem 4.4 of [4].

(iii) For large initial data a in $L^n_{\sigma,w}$, we do not know if there exists a corresponding local-in-time strong solution. As noticed in [18], the main difficulty arises from the fact that the semigroups $\{e^{-tA}\}_{t\geqslant 0}$, $\{e^{-tL}\}_{t\geqslant 0}$ and $\{e^{-tL^*}\}_{t\geqslant 0}$ are not strongly continuous at $t=0$ in the weak L^p spaces, while they are all strongly continuous in the usual L^p spaces.

We conclude this paper with deducing decay rates of L^∞-norm of strong solutions.

THEOREM 6.8. *Let $a\in L^n_{\sigma,w}\cap L^p_{\sigma,w}$ for some $1<p<n$ and let u be the corresponding strong solution given in Theorem 6.2.*

(i) *For each $\varepsilon>0$ there is a number $\mu=\mu(p,\varepsilon)$ so that if*

$$\|w\|+\|\nabla w\|\leqslant \mu,$$

then $u(t)\in L^\infty$ for large $t>0$ and

$$\|u(t)\|_\infty=o(t^{\varepsilon-n/2p}) \quad \text{as } t\to\infty.$$

(ii) *Let $\nabla w\in L^q_w\cap L^\infty$ for some $n'\leqslant q<\frac{1}{2}n$ in case $n\geqslant 4$, and let $\nabla w\in L^q\cap L^\infty$ for some $1<q<\frac{3}{2}$ in case $n=3$. There exists a number $\mu=\mu(p,n)>0$ so that if*

$$\|w\|+\|\nabla w\|_{q,w}+\|\nabla w\|_\infty\leqslant \mu \quad (n\geqslant 4),$$

or if

$$\|w\|+\|\nabla w\|_q+\|\nabla w\|_\infty\leqslant \mu \quad (n=3),$$

then $u(t)\in L^\infty$ and $\nabla u(t)\in L^n$ for large $t>0$ and

$$\|u(t)\|_\infty=O(t^{-n/2p}),\ \ \|\nabla u(t)\|_n=O(t^{-n/2p}) \quad \text{as } t\to\infty.$$

Here, the number $\mu(p,n)>0$ is taken so that the semigroup $\{e^{-tL}\}_{t\geqslant 0}$ satisfies estimates of Theorem 3.13 with $s=p$ and $s=\frac{2}{3}n$.

Remark. When $w=0$, Kozono, Ogawa and Sohr [26] deduce the decay result

$$\|u(t)\|_\infty=O(t^{-n/2p}(\log t)^{1-1/n}) \quad \text{as } t\to\infty, \tag{6.40}$$

for their strong solutions u corresponding to $a\in L^n_\sigma\cap L^p_\sigma$ for some $1<p\leqslant 2$, via a variant of Trudinger's inequality and estimates (6.29) of Iwashita [23]. Contrary to [26], we establish Theorem 6.8, applying Proposition 3.11 and Theorem 3.13, as well as Theorem 6.3 (ii). Note that Theorem 6.8 (ii) improves (6.40).

On the other hand, when $n=3$ and $w\neq 0$, Heywood [22] proved that if $a\in L^3_\sigma\cap L^2_\sigma$, then

$$\|u(t)\|_\infty = O(t^{-1/4}) \quad \text{as } t\to\infty. \tag{6.41}$$

Theorem 6.8 (i) is stronger than (6.41) and is valid also in higher space dimensions, while (6.41) holds also in the case $w^\infty\neq 0$.

Proof of Theorem 6.8. (i) The function u satisfies

$$u(t)=e^{-tA/2}u\left(\tfrac{1}{2}t\right)-\int_{t/2}^{t} e^{-(t-\tau)A}(Bu+P(u\cdot\nabla u))(\tau)\,d\tau. \tag{6.42}$$

Applying Proposition 3.11, we see as in the proof of (6.14) that

$$\|e^{-tA}Bu\|_\infty \leqslant Ct^{-1+\delta}(\|w\|+\|\nabla w\|)\|\nabla u\|_{n/(1+\delta)}$$

with $C>0$ depending on $\delta>0$, and

$$\|e^{-tA}P(u\cdot\nabla u)\|_\infty \leqslant Ct^{-1+\delta}\|u\|_{n/(1-3\delta)}\|\nabla u\|_{n/(1+\delta)}.$$

These estimates, together with (6.12) and (6.13), yield

$$\begin{aligned}\|u(t)\|_\infty &\leqslant Ct^{-n/2p}\left\|u\left(\tfrac{1}{2}t\right)\right\|_p + C\int_{t/2}^{t}(t-\tau)^{-1+\delta}(\tau^{-n/2p+\delta/2}+\tau^{1/2-n/p-\delta})\,d\tau\\ &\leqslant O(t^{-n/2p})+O(t^{-n/2p+3\delta/2})+O(t^{1/2-n/p})=O(t^{-n/2p+3\delta/2}),\end{aligned}$$

since $p<n$. Choosing $\delta>0$ sufficiently small, we get (i).

(ii) We write

$$u(t)=e^{-tL/2}u\left(\tfrac{1}{2}t\right)-\int_{t/2}^{t} e^{-(t-\tau)L}P(u\cdot\nabla u)(\tau)\,d\tau \tag{6.43}$$

and apply (6.12), (6.13) and Theorem 3.13 with $s=p$ and $s=\frac{2}{3}n$, to obtain

$$\begin{aligned}\|u(t)\|_\infty &\leqslant Ct^{-n/2p}\left\|u\left(\tfrac{1}{2}t\right)\right\|_p + C\int_{t/2}^{t}(t-\tau)^{-3/4}\|u\cdot\nabla u\|_{2n/3}(\tau)\,d\tau\\ &\leqslant Ct^{-n/2p}+C\int_{t/2}^{t}(t-\tau)^{-3/4}(\|u\|_\infty\|\nabla u\|_{2n/3})(\tau)\,d\tau\\ &\leqslant Ct^{-n/2p}+C\int_{t/2}^{t}(t-\tau)^{-3/4}\tau^{1/4-n/2p}\|u\|_\infty(\tau)\,d\tau\end{aligned}$$

and

$$\|\nabla u(t)\|_n \leqslant Ct^{-n/2p}+C\int_{t/2}^{t}(t-\tau)^{-3/4}(\|u\|_{2n}\|\nabla u\|_n)(\tau)\,d\tau$$
$$\leqslant Ct^{-n/2p}+C\int_{t/2}^{t}(t-\tau)^{-3/4}\tau^{1/4-n/2p}\|\nabla u\|_n(\tau)\,d\tau.$$

Thus, if we set $V(t)=\sup_{T<\tau<t}\tau^{n/2p}(\|u\|_\infty+\|\nabla u\|_n)(\tau)$ for a fixed $T>0$, we easily obtain

$$V(t)\leqslant C_1+C_2t^{1/2-n/2p}V(t)\leqslant C_1+C_2T^{1/2-n/p}V(t)$$

since $1/2-n/2p<0$. Taking $T>0$ sufficiently large, we obtain

$$V(t)\leqslant C_1+\tfrac{1}{2}V(t)\quad \text{for } t\geqslant T.$$

Hence we get $V(t)\leqslant C$ for large t, which proves (ii).

Finally, we prove a refined version of Theorem 6.8.

THEOREM 6.9. *Let w satisfy the assumption of Theorem* 6.8 (ii) *with the same number $\mu=\mu(p,n)>0$ as given there. Let u be the strong solution given in Theorem* 6.2 *with initial value $a\in L^p_\sigma\cap L^n_\sigma$. If $1<p\leqslant n'$ or $p=2$, then*

$$\|u(t)\|_\infty=o(t^{-n/2p})\quad \textit{as } t\to\infty.$$

Proof. In view of the integral representation (6.43) and the calculations that follow, it suffices to show that $u(t)\in L^p_\sigma$ for all $t\geqslant 0$ and

$$\|u(t)\|_p\to 0\quad \text{as } t\to\infty. \tag{6.44}$$

For $p=2$ this follows from the result of §4, since in the present situation u is the only one weak solution corresponding to a. So, we need only to discuss the cases where $1<p\leqslant n'$.

The argument below is due to [4, §5]. The assumption implies $a\in L^2_\sigma$, and so our strong solution u is in the class of weak solutions. Thus, we have

$$\langle u(t),\varphi\rangle=\langle u(s),e^{-(t-s)L^*}\varphi\rangle-\int_s^t\langle u\cdot\nabla v(\tau),e^{-(t-\tau)L^*}\varphi\rangle\,d\tau \tag{6.45}$$

for all $\varphi\in C^\infty_{0,\sigma}(\Omega)$ and $0\leqslant s\leqslant t$. The boundedness of the semigroup $\{e^{-tL^*}\}_{t\geqslant 0}$ in $L^{p'}_\sigma$ implies

$$|\langle u\cdot\nabla u,e^{-(t-\tau)L^*}\varphi\rangle|\leqslant C\|\varphi\|_{p'}\|u\|_{2p'/(p'-2)}\|\nabla u\|_2$$
$$\leqslant\|\varphi\|_{p'}\|u\|_2^{1-n/p'}\|u\|_{2n/(n-2)}^{n/p'}\|\nabla u\|_2$$
$$\leqslant C\|\varphi\|_{p'}\|u\|_2^{1-n/p'}\|\nabla u\|_2^{1+n/p'}.$$

Hence, (6.45) gives

$$\|u(t)\|_p \leqslant C\left(\|e^{-(t-s)L}u(s)\|_p + \int_s^t \|u\|_2^{1-n/p'}\|\nabla u\|_2^{1+n/p'}d\tau\right). \tag{6.46}$$

By assumption we have $1-n/p' \geqslant 0$ and $1+n/p' \leqslant 2$; so the integral in (6.46) with $s=0$ is finite, since u is a weak solution. Furthermore, (6.12) and (6.13) together imply that

$$\|u\|_2^{1-n/p'}\|\nabla u\|_2^{1+n/p'} \leqslant C\tau^{-1/2-n/2p} = C\tau^{-1-\alpha}, \quad \alpha = n/2p - 1/2 > 0,$$

for large $\tau > 0$. It follows from (6.46) that

$$\|u(t)\|_p \leqslant C\left(\|e^{-(t-s)L}u(s)\|_p + \int_s^\infty \|u\|_2^{1-n/p'}\|\nabla u\|_2^{1+n/p'}d\tau\right) \tag{6.47}$$

and the integral on the right-hand side is finite for any fixed $s \geqslant 0$. This shows in particular that $u(t) \in L^p_\sigma$ for all $t \geqslant 0$. Applying Theorem 3.10 (v) to (6.47) gives

$$\limsup_{t\to\infty} \|u(t)\|_p \leqslant C\int_s^\infty \|u\|_2^{1-n/p'}\|\nabla u\|_2^{1+n/p'}d\tau \to 0$$

by letting $s \to \infty$. This shows (6.44) and the proof of Theorem 6.9 is complete.

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