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The distortion problem

by

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1. Introduction

An infinite dimensional Banach space X is *distortable* if there exists an equivalent norm $|\cdot|$ on X and $\lambda>1$ such that for all infinite dimensional subspaces Y of X,

$$\sup\{|y|/|z| : y,z\in S(Y;\|\cdot\|)\}>\lambda, \tag{1.1}$$

where $S(Y;\|\cdot\|)$ is the unit sphere of Y. R. C. James [11] proved that l_1 and c_0 are not distortable. In this paper we prove that l_2 is distortable. In fact we shall prove that l_2 is *arbitrarily distortable* (for every $\lambda>1$ there exists an equivalent norm on l_2 satisfying (1.1)).

The distortion problem is related to stability problems for a wider class of functions than the class of equivalent norms. A function $f: S(X)\to\mathbf{R}$ is *oscillation stable* on X if for all subspaces Y of X and for all $\varepsilon>0$ there exists a subspace Z of Y with

$$\sup\{|f(y)-f(z)| : y,z\in S(Z)\}<\varepsilon. \tag{1.2}$$

(By *subspace* we shall mean a closed infinite dimensional linear subspace unless otherwise specified.) It was proved by V. Milman (see e.g., [28, p. 6] or [26], [27] that every Lipschitz (or even uniformly continuous) function $f: S(X)\to\mathbf{R}$ is *finitely* oscillation stable (a subspace Z of arbitrary finite dimension can be found satisfying (1.2)). V. Milman also proved in his fundamental papers [26], [27] that if all Lipschitz functions on every unit sphere of every Banach space were oscillation stable, then every X would isomorphically contain c_0 or l_p for some $1\leqslant p<\infty$. Of course Tsirelson's famous example [38] dashed such hopes and caused Milman's paper to be overlooked. However Milman's work contains the result that if X does not contain c_0 or l_p $(1\leqslant p<\infty)$ then some subspace of X

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admits a distorted norm. Thus the general distortion problem (does a given X contain a distortable subspace?) reduces to the case $X=l_p$ $(1<p<\infty)$.

For a given space X, every Lipschitz function $f\colon S(X)\to\mathbf{R}$ is oscillation stable if and only if every uniformly continuous $g\colon S(X)\to\mathbf{R}$ is oscillation stable. Indeed if such a g were not oscillation stable then there exist a subspace Y of X and reals $a<b$ such that

$$C=\{y\in S(Y):g(y)<a\} \quad\text{and}\quad D=\{y\in S(Y):g(y)>b\}$$

are both asymptotic for Y (C is *asymptotic* for Y if $C_\varepsilon\cap S(Z)\neq\varnothing$ for all subspaces Z of Y and all $\varepsilon>0$ where $C_\varepsilon=\{x:d(C,x)<\varepsilon\}$). Since g is uniformly continuous, $d(C,D)\equiv \inf\{\|c-d\|:c\in C,\ d\in D\}>0$ and so $f(x)\equiv d(C,x)$ is a Lipschitz function on $S(X)$ that does not stabilize in Y.

If C and D are asymptotic sets for a uniformly convex space X with $d(C,D)>0$ then X contains a distortable subspace. For example, the norm $|\cdot|$ on X whose unit ball is the closed convex hull of $(A\cup -A\cup\delta\,\mathrm{Ba}\,X)$ is a distortion of a subspace for sufficiently small δ and any choice $A\in\{C,D\}$. If $X=c_0$ or l_p $(1\leqslant p<\infty)$, then by the minimality of X one obtains that every uniformly continuous $f\colon S(X)\to\mathbf{R}$ is oscillation stable if and only if $S(X)$ does not contain two asymptotic sets a positive distance apart. If $X=l_p$ $(1<p<\infty)$ then this is, in turn, equivalent to X is not distortable.

T. Gowers [8] proved that every uniformly continuous function $f\colon S(c_0)\to\mathbf{R}$ is oscillation stable. Every uniformly continuous $f\colon S(l_1)\to\mathbf{R}$ is oscillation stable if and only if l_2 (equivalently l_p, $1<p<\infty$) is not distortable. This is seen by considering the Mazur map [25] $M\colon S(l_1)\to S(l_2)$ given by $M(x_i)_{i=1}^\infty=\big((\operatorname{sign}x_i)\sqrt{|x_i|}\,\big)_{i=1}^\infty$. M is a uniform homeomorphism between the two unit spheres (see e.g., [32, Lemma 1]). Moreover, since M preserves subspaces spanned by block bases of the respective unit vector bases of l_1 and l_2, C is an asymptotic set for l_1 if and only if $M(C)$ is an asymptotic set for l_2.

Gowers theorem combined with our main result and that of Milman's yields

THEOREM 1.1. *Let X be an infinite dimensional Banach space. Then every Lipschitz function $f\colon S(X)\to\mathbf{R}$ is oscillation stable if and only if X is c_0-saturated.*

(X is *c_0-saturated* if every subspace of X contains an isomorph of c_0.)

In §2 we consider a generalization of the Mazur map. The Mazur map satisfies for $h=(h_i)\in S(l_1)^+$ with h finitely supported, $M(h)=x$ where $x\in S(l_2)^+$ maximizes $E(h,y)\equiv \sum_i h_i\log y_i$ over $S(l_2)^+$. Furthermore in this case $h=x^*\circ x$ where x^* is the unique support functional of x and $\circ$ denotes pointwise multiplication of the sequences x and x^*. These facts are well known. We give a proof in Proposition 2.5.

The generalization is given as follows. Let X have a 1-*unconditional* normalized basis (e_i). This just means that $\|\,|x|\,\|=\|x\|$ for all $x=\sum a_ie_i\in X$ where $|x|=\sum |a_i|e_i$.

We regard X as a discrete lattice. c_{00} denotes the linear space of finitely supported sequences on $\mathbf{N}$. Thus $X\cap c_{00}=\{x\in X : \operatorname{supp} x \text{ is finite}\}$ where $\operatorname{supp}(\sum a_i e_i)=\{i : a_i\neq 0\}$. For $B\subseteq\mathbf{N}$ and $x=\sum x_i e_i\in X$ we set $Bx=\sum_{i\in B} x_i e_i$. We often write $x=(x_i)$. l_1 is a particular instance of such an X and we use the same notational conventions for l_1.

The generalization F_X of the Mazur map is defined in terms of an auxilliary map, the *entropy* function $E: (l_1\cap c_{00})\times X\to[-\infty,\infty)$ given by $E(h,x)\equiv E(|h|,|x|)\equiv\sum_i |h_i|\log|x_i|$ where $h=(h_i)\in l_1\cap c_{00}$ and $x=(x_i)\in X$ under the convention $0\log 0\equiv 0$. Fix $h\in l_1\cap c_{00}$ and $B=\operatorname{supp} h$. Then there exists a unique $x=(x_i)\in S(X)$ satisfying

(i) $E(h,x)\geqslant E(h,y)$ for all $y\in S(X)$,

(ii) $\operatorname{supp} h=\operatorname{supp} x=B$,

(iii) $\operatorname{sign} x_i=\operatorname{sign} h_i$ for $i\in B$.

This unique x we denote by $F_X(h)$ and we set

$$E_X(h)=E(h,F_X(h))=\max\{E(h,y) : y\in S(X)\}.$$

Indeed the function $E(h,\cdot): \{x\in S(X)^+ : \operatorname{supp} x\subseteq B\}\to[-\infty,0]$ is continuous taking real values on those x's with $\operatorname{supp} x=B$ and taking the value $-\infty$ otherwise. Thus there exists $x\in S(X)^+$ satisfying (ii) and $E(h,x)\geqslant E(h,y)$ if $y\in S(X)^+$, $\operatorname{supp} y\subseteq B$. Since (e_i) is 1-unconditional and $E(h,y)=E(h,By)$ for all $y\in X$, we obtain (i). (iii) is then achieved by changing the signs of x_i as needed. The uniqueness of x follows from the strict concavity of the log function. If $\operatorname{supp} x=\operatorname{supp} y=B$ and $x\neq y$ then $E\big(h,\frac{1}{2}(|x|+|y|)\big)>\frac{1}{2}E(h,|x|)+\frac{1}{2}E(h,|y|)$.

We discovered the map E in a paper of Gillespie [7] and we thank L. Weis for bringing that paper to our attention. A similar map is considered in [37]. As noted there other authors have also worked with this map in various contexts ([20], [21], [13], [30], [36], [14]). The central objective of some of these earlier papers was to show that elements of $S(l_1)$ could be written as $x^*\circ x$ with $\|x^*\|=\|x\|=1$. Our additional focal point is the map F_X itself. For certain X, F_X is uniformly continuous. In general F_X is not uniformly continuous, but retains enough structure (Proposition 2.3) to be extremely useful in §3. In addition it is known (e.g., [37, Lemma 39.3]) that whenever $x=F_X(h)$ there exists $x^*\in S(X^*)$ with $x^*\circ x=h$.

We prove (Theorem 2.1) that if X has an unconditional basis and if X does not contain l_∞^n uniformly in n, then there exists a uniform homeomorphism $F: S(l_1)\to S(X)$. We prove this by reducing the problem, this follows easily from the work of [6] and [23], to the case where X has a 1-unconditional basis and is q-concave with constant 1 for some $q<\infty$. X is *q-concave* with constant $M_q(X)$ if

$$\left(\sum_{i=1}^n \|x^i\|^q\right)^{1/q}\leqslant M_q(X)\left\|\left(\sum_{i=1}^n |x^i|^q\right)^{1/q}\right\| \tag{1.3}$$

whenever $(x^i)_{i=1}^n \subseteq X$. The vector on the right side of (1.3) is computed coordinatewise with respect to (e_j). In this particular case the uniform homeomorphism F is the map F_X described above (see the remark before Proposition 2.9).

One way to attack the distortion problem is to find a distortable space X with a 1-unconditional basis and having say $M_2(X)=1$ and possessing a describable pair of separated asymptotic sets. Then use the map F_X to pull these sets back to a separated pair (easy) of asymptotic sets (not easy) in $S(l_1)$. Our original proof that l_2 is distortable was a variation of this idea using $X=T_2^*$, the dual of convexified Tsirelson space. However much more is possible as was shown to us by B. Maurey. Maurey's elegant argument is given in §3 (Theorem 3.4). We thank him for permitting us to include it in this paper.

In §3 we use the map F_X for $X=S^*$, the dual space of the arbitrarily distortable space constructed in [34] (see also [35]). As shown in [10] and implicitly in [34], [35] this space contains a sequence of nearly biorthogonal sets: $A_k \subseteq S(S)$, $A_k^* \subseteq \mathrm{Ba}(S^*)$ with A_k asymptotic in S for all k. By "nearly biorthogonal" we mean that for some sequence $\varepsilon_i \downarrow 0$, $|x_k^*(x_j)| < \varepsilon_{\min(k,j)}$ if $k \neq j$, $x_k^* \in A_k^*$, $x_j \in A_j$, and A_k^* $(1-\varepsilon_k)$-norms A_k. The latter means that for all $x_k \in A_k$ there exists $x_k^* \in A_k^*$ with $x_k^*(x_k) > 1-\varepsilon_k$. The particular description of these sets is used along with the mapping F_{S^*} to show that the sets

$$C_k \equiv \{x \in l_2 : |x| = (|x_k^* \circ x_k| / \|x_k^* \circ x_k\|_1)^{1/2} \text{ for some } x_k^* \in A_k^*,\ x_k \in A_k \text{ with } \|x_k^* \circ x_k\|_1 \geqslant 1-\varepsilon_k\}$$

are nearly biorthogonal in l_2 (easy) and that C_k is asymptotic in l_2. By $x^* \circ x$ we mean again the element of l_1 given by the operation of pointwise multiplication. Thus if $x^* = \sum a_i e_i^*$ and $x = \sum b_i e_i$, $x^* \circ x = (a_i b_i)_{i=1}^\infty$. $\|\cdot\|_1$ is the l_1-norm.

The sets C_k easily lead to an arbitrary distortion of l_2. In fact using an argument of [10] one can prove the following (see also Theorem 3.1).

THEOREM 1.2. *For all $1<p<\infty$, $\varepsilon>0$ and $n \in \mathbf{N}$ there exists an equivalent norm $|\cdot|$ on l_p such that for any block basis (y_i) of the unit vector basis of l_p there exists a finite block basis $(z_i)_{i=1}^n$ of (y_i) which is $(1+\varepsilon)$-equivalent to the first n terms of the summing basis, $(s_i)_{i=1}^n$.*

The summing basis norm is

$$\left\| \sum_{i=1}^n a_i s_i \right\| = \sup \left\{ \left| \sum_{i=1}^l a_i \right| : l \leqslant n \right\}.$$

Thus for all $\lambda > 1$ there exists an equivalent norm $|\cdot|$ on l_p such that no basic sequence in l_p is λ-unconditional in the $|\cdot|$ norm. The sets C_k, in addition to being nearly biorthogonal, are unconditional and spreading (defined in §3 just before the statement of Theorem 3.4) and seem likely to prove useful elsewhere.

T. Gowers [9] proved the conditional theorem that if every equivalent norm on l_2 admits an almost symmetric subspace, then l_2 is not distortable. Theorem 1.2 shows that one cannot even obtain an almost 1-unconditional subspace in general.

The paper by Lindenstrauss and Pełczyński [17] also contains some nice results on distortion. They consider a restricted form of distortion in which the subspace Y of (1.1) is isomorphic to X.

Our notation is standard Banach space terminology as may be found in the books [18] and [19]. In §2 we use a number of results in [6] although we cite the corresponding statements in [19].

Thanks are due to numerous people, especially B. Maurey and N. Tomczak-Jaegermann. As we noted, Maurey gave us the elegant argument of §3. The idea of exploiting the ramifications of being able to write elements of $S(l_2)$ as $\sqrt{x^*\circ x}$ with x in the sphere of a Tsirelson-type space X and $x^*\in S(X^*)$ in attacking the distortion problem is due to Tomczak-Jaegermann.

2. Uniform homeomorphisms between unit spheres

The main result of this section is

THEOREM 2.1. *Let X be a Banach space with an unconditional basis. Then $S(X)$ and $S(l_1)$ are uniformly homeomorphic if and only if X does not contain l_∞^n uniformly in n.*

A *uniform homeomorphism* between two metric spaces is an invertible map such that both the map and its inverse are uniformly continuous. Many results are known concerning uniform homeomorphisms between Banach spaces (see [1] for a nice survey of these results). Our focus however is on the unit spheres of Banach spaces. The prototype of such maps is the Mazur map discussed in the introduction.

Before proceeding we set some notation. Unless stated otherwise X shall be a Banach space with a normalized 1-unconditional basis (e_i). We regard X as a discrete lattice. $x=(x_i)\in X$ means that $x=\sum x_ie_i$, $|x|=(|x_i|)$, and $\mathrm{Ba}(X)^+=\{x\in \mathrm{Ba}(X): x=|x|\}$. $\mathrm{Ba}(X)$ is the closed unit ball of X. For $1\leqslant p<\infty$, X is *p-convex* with p-convexity constant $M^p(X)$ if for all $(x^i)_{i=1}^n\subseteq X$,

$$\left\|\left(\sum_{i=1}^n |x^i|^p\right)^{1/p}\right\| \leqslant M^p(X)\left(\sum_{i=1}^n \|x^i\|^p\right)^{1/p},$$

where $M^p(X)$ is the smallest constant satisfying the inequality. The *p-convexification* of X is the Banach space given by

$$X^{(p)} = \left\{(x_i) : \|(x_i)\|_{(p)} \equiv \left\|\sum_i |x_i|^p e_i\right\|^{1/p} < \infty\right\}.$$

The unit vector basis of $X^{(p)}$, which we still denote by (e_i), is a 1-unconditional basis for $X^{(p)}$ and $M^p(X^{(p)})=1$. These facts may be found in [19, §1.d].

Let $F_X: l_1 \cap c_{00} \to S(X)$ be as defined in the introduction. As we shall see in Proposition 2.5, F_X generalizes the Mazur map. If $X=l_p$ $(1<p<\infty)$ and $h \in S(l_1)^+ \cap c_{00}$ then $F_X(h)=(h_i^{1/p})$. Even in this nice setting however we cannot use our definitions directly on infinitely supported elements. Indeed one can find $h \in S(l_1)$ with $E_{l_2}(h)=-\infty$. The map F_{l_2} is uniformly continuous on $S(l_1)\cap c_{00}$, though, and thus extends to a map on $S(l_1)$. E_X is not uniformly continuous on $S(l_1)\cap c_{00}$ but has some positive features as the next proposition reveals. Some of our arguments could be shortened by referring to the papers [20], [21], [13], [37] and [7] but we choose to present complete proofs.

First we define a function $\psi(\varepsilon)$ that appears in Proposition 2.3. Note that there exists a function $\eta: (0,1) \to (0,1)$ so that

$$\log \frac{1}{2}\left(\sqrt{a}+\frac{1}{\sqrt{a}}\right) > \eta(\varepsilon) \quad \text{if } |a-1| > \varepsilon \text{ with } a > 0. \tag{2.1}$$

Indeed, let $g(a)=\log \frac{1}{2}(a+1/a)$ for $a>0$. g is continuous on $(0,\infty)$, strictly decreasing on $(0,1)$ and strictly increasing on $(1,\infty)$. The minimum value of g is $g(1)=0$. Thus there exists $\eta: (0,1) \to (0,1)$ so that $|a-1|>\varepsilon$ implies $g(\sqrt{a})>\eta(\varepsilon)$. □

Definition 2.2. $\psi(\varepsilon)=\varepsilon\eta(\varepsilon)$ for $\varepsilon \in (0,1)$.

PROPOSITION 2.3. *Let X have a 1-unconditional basis.*

(A) *Let $h \in S(l_1)^+ \cap c_{00}$, let $\varepsilon>0$ and $v \in \mathrm{Ba}(X)^+$ be such that $E(h,v) \geqslant E_X(h)-\psi(\varepsilon)$. Then if $u=F_X(h)$ there exists $A \subseteq \operatorname{supp} h$ satisfying $\|Ah\|>1-\varepsilon$ and $(1-\varepsilon)Au \leqslant Av \leqslant (1+\varepsilon)Au$ (the latter inequalities being pointwise in the lattice sense).*

(B) *Let $h_1, h_2 \in S(l_1)^+ \cap c_{00}$ with $\|h_1-h_2\| \leqslant 1$. Let $x_i=F_X(h_i)$ for $i=1,2$. Then*

$$\left\|\tfrac{1}{2}(x_1+x_2)\right\| \geqslant 1-\sqrt{\|h_1-h_2\|}.$$

Proof. (A) Let $u=(u_i)$ and $v=(v_i)$ be as in the statement of (A). We may assume that $\operatorname{supp} u = \operatorname{supp} v = B \equiv \operatorname{supp} h$. $E(h,v) \geqslant E_X(h)-\psi(\varepsilon)$ yields

$$\psi(\varepsilon) \geqslant \sum_{i \in B} h_i(\log u_i - \log v_i). \tag{2.2}$$

Since $\frac{1}{2}(u+v)\in \mathrm{Ba}(X)^+$ and $u=F_X(h)$ we obtain from (2.2)

$$\begin{aligned}\psi(\varepsilon) &\geqslant \sum_{i\in B} h_i\left[\log \tfrac{1}{2}(u_i+v_i)-\log v_i\right]\\ &=\sum_{i\in B} h_i\left[\tfrac{1}{2}\log u_i+\tfrac{1}{2}\log v_i+\log\tfrac{1}{2}(u_i+v_i)-\log\sqrt{u_iv_i}-\log v_i\right]\\ &=\frac{1}{2}\sum_{i\in B} h_i(\log u_i-\log v_i)+\sum_{i\in B} h_i\log\frac{1}{2}\left(\sqrt{\frac{v_i}{u_i}}+\sqrt{\frac{u_i}{v_i}}\right).\end{aligned}$$

The first term in the last expression is nonnegative so

$$\psi(\varepsilon)\geqslant \sum_{i\in B} h_i\log\frac{1}{2}\left(\sqrt{\frac{v_i}{u_i}}+\sqrt{\frac{u_i}{v_i}}\right). \tag{2.3}$$

Now $|v_i/u_i-1|\leqslant\varepsilon$ if and only if $(1-\varepsilon)u_i\leqslant v_i\leqslant(1+\varepsilon)u_i$. Let $I=\{i\in B:|v_i/u_i-1|>\varepsilon\}$. For $i\in I$,

$$\log\frac{1}{2}\left(\sqrt{\frac{u_i}{v_i}}+\sqrt{\frac{v_i}{u_i}}\right)>\eta(\varepsilon)\quad\text{(by (2.1))}. \tag{2.4}$$

Let $J=\{i\in B:\log\frac{1}{2}\left(\sqrt{u_i/v_i}+\sqrt{v_i/u_i}\right)>\eta(\varepsilon)\}$. Thus $I\subseteq J$ by (2.4) and from (2.3),

$$\sum_{i\in I} h_i\leqslant\sum_{i\in J} h_i\leqslant\frac{1}{\eta(\varepsilon)}\sum_{i\in J} h_i\log\frac{1}{2}\left(\sqrt{\frac{u_i}{v_i}}+\sqrt{\frac{v_i}{u_i}}\right)\leqslant\frac{\psi(\varepsilon)}{\eta(\varepsilon)}=\varepsilon.$$

Thus (A) follows with $A=B\setminus I$.

(B) Let $\|\frac{1}{2}(x_1+x_2)\|\equiv 1-2\varepsilon$. Set $\tilde{x}_1=x_1+\varepsilon x_2$ and $\tilde{x}_2=x_2+\varepsilon x_1$. Thus $\operatorname{supp}\tilde{x}_1=\operatorname{supp}\tilde{x}_2=\operatorname{supp}h_1\cup\operatorname{supp}h_2$ and $\|\frac{1}{2}(\tilde{x}_1+\tilde{x}_2)\|\leqslant 1-\varepsilon$. We may assume $\varepsilon>0$. For $j\in\operatorname{supp}\tilde{x}_1$, $|\log\tilde{x}_{1,j}-\log\tilde{x}_{2,j}|\leqslant|\log\varepsilon|$ where $\tilde{x}_i=(\tilde{x}_{i,j})$ for $i=1,2$.

From this and $\tilde{x}_1\geqslant x_1$ we obtain

$$\begin{aligned}E(h_1,\tilde{x}_1)\geqslant E(h_1,x_1)\geqslant E\left(h_1,\frac{\tilde{x}_1+\tilde{x}_2}{2(1-\varepsilon)}\right)\\ =E\left(h_1,\tfrac{1}{2}(\tilde{x}_1+\tilde{x}_2)\right)+|\log(1-\varepsilon)|\\ \geqslant\tfrac{1}{2}E(h_1,\tilde{x}_1)+\tfrac{1}{2}E(h_1,\tilde{x}_2)+|\log(1-\varepsilon)|.\end{aligned}$$

Thus

$$|\log(1-\varepsilon)|\leqslant\tfrac{1}{2}(E(h_1,\tilde{x}_1)-E(h_1,\tilde{x}_2)).$$

Similarly,

$$|\log(1-\varepsilon)|\leqslant\tfrac{1}{2}(E(h_2,\tilde{x}_2)-E(h_2,\tilde{x}_1)).$$

Averaging the two inequalities yields

$$\begin{aligned}\varepsilon &\leqslant |\log(1-\varepsilon)| \leqslant \tfrac{1}{4}(E(h_1,\tilde{x}_1)-E(h_1,\tilde{x}_2)-E(h_2,\tilde{x}_1)+E(h_2,\tilde{x}_2))\\ &= \tfrac{1}{4}\sum_{j\in B}(h_{1,j}-h_{2,j})(\log\tilde{x}_{1,j}-\log\tilde{x}_{2,j})\\ &\leqslant \tfrac{1}{4}\|h_1-h_2\|\cdot|\log\varepsilon| \leqslant \tfrac{1}{4}\|h_1-h_2\|\varepsilon^{-1}.\end{aligned}$$

Thus $\varepsilon\leqslant\frac{1}{2}\|h_1-h_2\|^{1/2}$. Hence $\|\frac{1}{2}(x_1+x_2)\|=1-2\varepsilon\geqslant 1-\|h_1-h_2\|^{1/2}$. □

PROPOSITION 2.4. *Let X be a uniformly convex Banach space with a 1-unconditional basis. The map $F_X\colon S(l_1)\cap c_{00}\to S(X)$ is uniformly continuous. Moreover the modulus of continuity of F_X depends solely on the modulus of uniform convexity of X.*

Proof. The uniform continuity of F_X on $S(l_1)^+\cap c_{00}$ follows immediately from Proposition 2.3 (B).

Precisely, there is a function $g(\varepsilon)$, depending solely upon the modulus of uniform convexity of X, which is continuous at 0 with $g(0)=0$ and satisfies

$$\|F_X(h_1)-F_X(h_2)\|\leqslant g(\|h_1-h_2\|)$$

for $h_1,h_2\in S(l_1)^+\cap c_{00}$. A consequence of this is that if $h\in S(l_1)^+\cap c_{00}$, $x=F_X(h)$ and $I\subseteq\mathbf{N}$ is such that $\|Ih\|<\varepsilon$ then $\|Ix\|<g(2\varepsilon)$. Indeed if $J=\mathbf{N}\setminus I$,

$$\left\|h-\frac{Jh}{\|Jh\|}\right\| = \|Ih\|+\left\|Jh-\frac{Jh}{\|Jh\|}\right\| < 2\varepsilon.$$

Thus since $Ix=I(F_X(h)-F_X(Jh/\|Jh\|))$,

$$\|Ix\|\leqslant\left\|F_X(h)-F_X\left(\frac{Jh}{\|Jh\|}\right)\right\|<g(2\varepsilon).$$

For the general case let $h_1,h_2\in S(l_1)\cap c_{00}$ with $\|h_1-h_2\|=\varepsilon$. Let $F_X(|h_i|)=|x_i|$ for $i=1,2$. Then $x_i\equiv\operatorname{sign}h_i\circ|x_i|$, $\circ$ denoting pointwise multiplication, satisfies $x_i=F_X(h_i)$ for $i=1,2$. Also $\||h_1|-|h_2|\|\leqslant\|h_1-h_2\|$. Thus if $I=\{j:\operatorname{sign}x_{1,j}\neq\operatorname{sign}x_{2,j}\}$,

$$\begin{aligned}\|x_1-x_2\| &\leqslant \||x_1|-|x_2|\|+\left\|\sum_{j\in I}(|x_{1,j}|+|x_{2,j}|)e_j\right\|\\ &\leqslant g(\||h_1|-|h_2|\|)+\|I|x_1|\|+\|I|x_2|\|\\ &\leqslant g(\varepsilon)+g(2\varepsilon)+g(2\varepsilon).\end{aligned}$$ □

Here is a fact we promised earlier.

PROPOSITION 2.5. *Let $X=l_p$, $1<p<\infty$. Then F_X is the Mazur map, i.e., if $h\in S(l_1)^+\cap c_{00}$ then $F_X(h)=(h_i^{1/p})$.*

Proof. Let $h\in S(l_1)^+\cap c_{00}$, $B=\operatorname{supp} h$ and $F_X(h)=x$. Then $\operatorname{supp} x=B$ and the vector $(x_i)_{i\in B}$ maximizes the function $\mathbf{R}_+^B\ni(y_i)\mapsto\sum_{i\in B}h_i\log y_i$ under the restriction $\sum_{i\in B}y_i^p=1$. By the method of Lagrange multipliers this implies that there is a number $c\neq 0$ so that $h_i/x_i=cpx_i^{p-1}$ for $i\in B$. Thus $x_i=(cp)^{-1/p}h_i^{1/p}$. Since $\|x\|_p=1$,

$$c=p^{-1} \quad\text{and}\quad x_i=h_i^{1/p} \ \text{for } i\in B. \qquad \square$$

If X is uniformly convex, by Proposition 2.4 the map F_X extends uniquely to a uniformly continuous map, which we still denote by F_X, from $S(l_1)\to S(X)$.

PROPOSITION 2.6. *Let X be a uniformly convex uniformly smooth Banach space with a 1-unconditional basis. Then $F_X\colon S(l_1)\to S(X)$ is invertible and $(F_X)^{-1}$ is uniformly continuous, with modulus of continuity depending only on the modulus of uniform smoothness of X. For $x\in S(X)$, $F_X^{-1}(x)=\operatorname{sign}(x)\circ x^*\circ x=|x^*|\circ x$ where x^* is the unique support functional of x.*

Proof. For $x\in S(X)$ there exists a unique element $x^*\in S(X^*)$ such that $x^*(x)=1$. The biorthogonal functionals (e_i^*) are a 1-unconditional basis for X^* and thus we can express $x^*=\sum x_i^*e_i^*$ and write $x^*=(x_i^*)$. The element $x^*\circ x\in S(l_1)^+$ and $\operatorname{sign}x^*=\operatorname{sign}x$. Let $G(x)=|x^*|\circ x$. G is uniformly continuous. Indeed the map $S(X)\ni x\mapsto x^*$, the supporting functional, is uniformly continuous since X is uniformly smooth. The modulus of continuity of this map depends solely on the modulus of uniform smoothness of X (see e.g., [4, p. 36]). Let $G(x_i)=h_i=|x_i^*|\circ x_i$ for $i=1,2$. Then

$$\begin{aligned}\|h_1-h_2\|&=\||x_1^*|\circ x_1-|x_2^*|\circ x_2\|\leqslant\||x_1^*|\circ(x_1-x_2)\|+\|(|x_1^*|-|x_2^*|)\circ x_2\|\\&\leqslant\|x_1^*\|\cdot\|x_1-x_2\|+\||x_1^*|-|x_2^*|\|\cdot\|x_2\|\leqslant\|x_1-x_2\|+\|x_1^*-x_2^*\|\end{aligned}$$

which proves that G is uniformly continuous.

It remains only to show that $G=F_X^{-1}$. Since $G(x)=\operatorname{sign}x\circ G(|x|)$ we need only show that $G(F(h))=h$ for $h\in S(l_1)^+\cap c_{00}$ and $F(G(x))=x$ for $x\in S(X)^+\cap c_{00}$.

If $h\in S(l_1^+)\cap c_{00}$ and $x=F_X(h)$ then, as in the proof of Proposition 2.5, the method of Lagrange multipliers yields that $\vec{\nabla}E(h,x)=(h_i/x_i)_{i\in\operatorname{supp}h}$ equals a multiple of $(x_i^*)_{i\in\operatorname{supp}h}$ where x^* is the support functional of x. This multiple must be 1 and $h_i=x_i^*\circ x_i$ or $G(F(h))=h$.

That $F(G(x))=x$ follows once we observe that if $h=x^*\circ x=y^*\circ y$, all norm 1 elements, then $x=y$. Assume for simplicity $\operatorname{supp}h=\{1,2,\dots,n\}$. Define $f(z)=\|z\|-E(h,z)$ for $z\in U$, a convex open subset of the positive cone $\operatorname{Ba}(\langle e_i\rangle_{i=1}^n)^+$ which contains both x and

y and is bounded away from the boundary of the cone. $f(z)$ is strictly convex so $\vec{\nabla} f(z)=\vec{0}$ for at most one point. But $\vec{\nabla} f(z)=\vec{0}$ if and only if $h=z^*\circ z$. $\square$

COROLLARY 2.7 [37, Lemma 39.3]. *Let X have a 1-unconditional basis and let $h\in S(l_1^+)\cap c_{00}$ with $x\in F_X(h)$. Then there exists $x^*\in S(X^*)$ with $x^*\circ x=h$.*

Proof. We may restrict our attention to $X=\langle e_i\rangle_{i\in\operatorname{supp} h}$. The result follows if X is smooth from the proof of Proposition 2.6. Let $\|\cdot\|_n$ be a sequence of smooth norms on X with $\|\cdot\|_n\to\|\cdot\|$ and such that $x/\|x\|_n\in F_{X_n}(h)$. Then use a compactness argument. $\square$

Before proving Theorem 2.1 we need one more proposition. Recall that $X^{(p)}$ is the p-convexification of X. The map G_p below is another generalization of the Mazur map.

PROPOSITION 2.8. *Let $1<p<\infty$ and let X be a Banach space with a 1-unconditional basis. The map $G_p\colon S(X^{(p)})\to S(X)$ given by $G_p(x)=\operatorname{sign}(x)\circ|x|^p=((\operatorname{sign} x_i)|x_i|^p)$ for $x=(x_i)$ is a uniform homeomorphism. Moreover the modulus of continuity of G_p and G_p^{-1} are functions solely of p.*

Proof. As usual (e_i) denotes the normalized 1-unconditional basis of both X and $X^{(p)}$. Let $x,y\in S(X^{(p)})$ with $\delta\equiv\|x-y\|_{(p)}$. We shall show that

$$2^{1-p}\delta^p\leqslant\|G_p(x)-G_p(y)\|\leqslant\delta^p+\delta^{p/2}+2\big(1-(1-\sqrt{\delta})^p\big)$$

which will complete the proof.

Let $x=\sum x_ie_i$ and $y=\sum y_ie_i$.

$$\begin{aligned}\|G_p(x)-G_p(y)\|&=\Big\|\sum_{i=1}^{\infty}(\operatorname{sign}(x_i)|x_i|^p-\operatorname{sign}(y_i)|y_i|^p)e_i\Big\|\\&=\Big\|\sum_{i\in I_+}(|x_i|^p-|y_i|^p)e_i+\sum_{i\in I_-}(|x_i|^p+|y_i|^p)e_i\Big\|\end{aligned}$$

where

$$I_+=\{i:\operatorname{sign}(x_i)=\operatorname{sign}(y_i)\}\quad\text{and}\quad I_-=\{i:\operatorname{sign}(x_i)\neq\operatorname{sign}(y_i)\}.$$

We denote the two terms in the last norm expression as d_+ and d_-, respectively.

Since $a^p-b^p\geqslant(a-b)^p$ and $a^p+b^p\geqslant2^{1-p}(a+b)^p$ for $a\geqslant b\geqslant0$ we deduce from the 1-unconditionality of (e_i) that

$$\begin{aligned}\|d_++d_-\|&\geqslant\Big\|\sum_{i\in I_+}\big||x_i|-|y_i|\big|^pe_i+2^{1-p}\sum_{i\in I_-}(|x_i|+|y_i|)^pe_i\Big\|\\&\geqslant2^{1-p}\Big\|\sum|x_i-y_i|^pe_i\Big\|=2^{1-p}\|x-y\|_{(p)}^p.\end{aligned}$$

To prove the upper estimate we begin by noting that

$$\|d_-\| \leqslant \Big\| \sum_{i\in I_-} |x_i - y_i|^p e_i \Big\| \leqslant \|x-y\|_{(p)}^p = \delta^p.$$

Set $q=1-\sqrt{\delta}$ and $c=(1-q)^{-p}=\delta^{-p/2}$. For $a,b\geqslant 0$ with $0\leqslant b\leqslant qa$ we have

$$c(a-b)^p-(a^p-b^p)\geqslant c(1-q)^p a^p - a^p = a^p(c(1-q)^p-1)=0. \tag{2.5}$$

Let $I'_+=\{i\in I_+ : |y_i|<q|x_i| \text{ or } |x_i|<q|y_i|\}$ and $I''_+=I_+\setminus I'_+$. Write $d_+=d'_+ + d''_+$ where $d'_+=\sum_{i\in I'_+}(|x_i|^p-|y_i|^p)e_i$ and $d''_+=d_+-d'_+$. Thus (2.5) yields that

$$\|d'_+\| \leqslant c \Big\| \sum_{i\in I'_+} \big||x_i|-|y_i|\big|^p e_i \Big\| \leqslant \delta^{-p/2}\|x-y\|_{(p)}^p = \delta^{p/2}.$$

Furthermore,

$$\|d''_+\| \leqslant (1-q^p) \Big\| \sum_{i\in I''_+} (|x_i|^p+|y_i|^p)e_i \Big\| \leqslant 2(1-q^p) \leqslant 2\big(1-(1-\sqrt{\delta})^p\big). \qquad \square$$

Proof of Theorem 2.1. It follows quickly from work of Enflo that if X contains l_∞^n uniformly in n then $S(X)$ is not uniformly homeomorphic to a subset of $S(l_1)$. Indeed Enflo [5] proved that a certain family of finite subsets of $\mathrm{Ba}(l_\infty^n)$, $n\in\mathbf{N}$, cannot be uniformly embedded into $\mathrm{Ba}(l_2)$ and hence neither into $\mathrm{Ba}(l_1)$. But $B(l_\infty^n)$ embeds isometrically into $S(l_\infty^{n+1})$ and hence these finite subsets embed uniformly into $S(X)$.

For the converse assume that X does not contain l_∞^n uniformly in n. We may suppose that X has a 1-unconditional basis (e_i). Indeed if (e_i) is a normalized basis for X, $|x|\equiv\|\sum|x_i|e_i\|$ is an equivalent 1-unconditional norm. Furthermore the map $x\mapsto x/\|x\|$ is easily seen to be a uniform homeomorphism between $S(X,|\cdot|)$ and $S(X,\|\cdot\|)$.

By a theorem of Maurey and Pisier [23], X has cotype q' for some $q'<\infty$. This implies that X is q-concave for all $q>q'$ ([19, p. 88]). Fix $q>q'$. There exists an equivalent norm on X for which (e_i) is still 1-unconditional and for which $M_q(X)=1$ ([19, p. 54]). The 2-convexification of X in this norm, $X^{(2)}$, satisfies $M_{2q}(X^{(2)})=1=M^2(X^{(2)})$ ([19, p. 54]). In particular $X^{(2)}$ is uniformly convex and uniformly smooth ([19, p. 80]) and so $F_{X^{(2)}}: S(l_1)\to S(X^{(2)})$ is a uniform homeomorphism by Proposition 2.6. Thus $G_2\circ F_{X^{(2)}}: S(l_1)\to S(X)$ is a uniform homeomorphism by Proposition 2.8. $\square$

Remark. If X has a 1-unconditional basis and $M_q(X)=1$ for some $q<\infty$, the map $G_2\circ F_{X^{(2)}}=F_X$. Furthermore the modulus of continuity of F_X and F_X^{-1} are functions solely of q.

The uniform homeomorphism theorem extends to unit balls by the following simple proposition.

PROPOSITION 2.9. *Let X and Y be Banach spaces and let $F: S(X) \to S(Y)$ be a uniform homeomorphism. For $x \in \mathrm{Ba}(X)$ let $\bar{F}(x) = \|x\| F(x/\|x\|)$ if $x \neq 0$ and $\bar{F}(0) = 0$. Then $\tilde{F}$ is a uniform homeomorphism between* $\mathrm{Ba}(X)$ *and* $\mathrm{Ba}(Y)$.

Proof. Clearly $\bar{F}$ is a bijection. Since $\bar{F}^{-1}(y) = \|y\| F^{-1}(y/\|y\|)$ for $y \neq 0$, it suffices to show that $\bar{F}$ is uniformly continuous. Let f be the modulus of continuity of F, i.e., $\|F(x_1) - F(x_2)\| \leqslant f(\|x_1 - x_2\|)$.

Let $x_1, x_2 \in \mathrm{Ba}(X)$ with $\|x_1 - x_2\| = \delta$, $\lambda_1 = \|x_1\|$, $\lambda_2 = \|x_2\|$ and $\lambda_1 \geqslant \lambda_2$.

$$\begin{aligned}\|\bar{F}(x_1) - \bar{F}(x_2)\| &= \left\| \lambda_1 F\left(\frac{x_1}{\lambda_1}\right) - \lambda_2 F\left(\frac{x_2}{\lambda_2}\right) \right\| \\ &\leqslant (\lambda_1 - \lambda_2) + \lambda_2 \left\| F\left(\frac{x_1}{\lambda_1}\right) - F\left(\frac{x_2}{\lambda_2}\right) \right\|.\end{aligned}$$

If $\lambda_2 < \delta^{1/4}$ this is less than $\delta + 2\delta^{1/4}$. Otherwise

$$\begin{aligned}\left\| \frac{x_1}{\lambda_1} - \frac{x_2}{\lambda_2} \right\| &= \frac{1}{\lambda_1 \lambda_2} \|\lambda_2 x_1 - \lambda_1 x_2\| \\ &\leqslant \frac{1}{\lambda_1 \lambda_2} [\lambda_1 \|x_1 - x_2\| + \lambda_1 - \lambda_2] \leqslant \frac{\delta}{\lambda_2} + \frac{\delta}{\lambda_1 \lambda_2} \\ &\leqslant \frac{2\delta}{\lambda_1 \lambda_2} \leqslant \frac{2\delta}{\sqrt{\delta}} = 2\sqrt{\delta}.\end{aligned}$$

Thus

$$\|\bar{F}(x_1) - \bar{F}(x_2)\| \leqslant \max\left(\delta + f(2\sqrt{\delta}), \delta + 2\delta^{1/4}\right). \qquad \square$$

Remark. It is not possible, in general, to replace "uniformly homeomorphic" by "Lipschitz equivalent" in Theorem 2.1. Indeed if $S(X)$ and $S(Y)$ are Lipschitz equivalent, then an argument much like that of Proposition 2.9, yields that X and Y are Lipschitz equivalent which need not be true (see [1]).

There exist separable infinite dimensional Banach spaces X not containing l_∞^n's uniformly such that $\mathrm{Ba}(X)$ does not embed uniformly into l_2. For example the James' nonoctohedral space [12] has this property. Indeed, Y. Raynaud [31] proved that if X is not reflexive and $\mathrm{Ba}(X)$ embeds uniformly into l_2, then X admits an l_1-spreading model.

Fouad Chaatit [2] has extended Theorem 2.1. He showed one can replace the hypothesis that X has an unconditional basis with the more general assumption that X is a separable infinite dimensional Banach lattice. N. J. Kalton [15] and M. Daher [3] have subsequently discovered proofs of this result using complex interpolation theory.

3. l_2 is arbitrarily distortable

Let X be a Banach space with a basis (e_i). A *block subspace* of X is any subspace spanned by a block basis of (e_i). X is *sequentially arbitrarily distortable* if there exist a sequence of equivalent norms $\|\cdot\|_i$ on X and $\varepsilon_i \downarrow 0$ such that:

$\|\cdot\|_i \leqslant \|\cdot\|$ for all i and for all subspaces Y of X, and for all $i_0 \in \mathbf{N}$ there exists $y \in S(Y, \|\cdot\|_{i_0})$ with $\|y\|_i \leqslant \varepsilon_{\min(i,i_0)}$ for $i \neq i_0$.

We note that if X contains an asymptotic biorthogonal system with vanishing constant (see [10]), then X is sequentially arbitrarily distortable.

If X is sequentially arbitrarily distortable then X is arbitrarily distortable. Indeed fix $i>1$ and let Y be a subspace of X. Choose $x \in Y$ with $\|x\|_i=1$ and $\|x\|_1 \leqslant \varepsilon_1$. Let $\|\cdot\|_1 \leqslant \|\cdot\| \leqslant C_1 \|\cdot\|_1$ and $\tilde{x}=x/\|x\|$. Then $\|\tilde{x}\|_i = 1/\|x\| \geqslant 1/C_1\varepsilon_1$. Choose $y \in Y$ with $\|y\|_{i+1}=1$ and $\|y\|_i \leqslant \varepsilon_i$. Then for $\tilde{y}=y/\|y\|$, $\|\tilde{y}\|_i \leqslant \varepsilon_i/\|y\| \leqslant \varepsilon_i$. Thus $\|\tilde{x}\|_i/\|\tilde{y}\|_i \geqslant 1/C_1\varepsilon_1\varepsilon_i$. Furthermore we have

THEOREM 3.1. *Let X be a sequentially arbitrarily distortable Banach space with a basis (e_i). For all $n \in \mathbf{N}$ and $\varepsilon>0$ there exists an equivalent norm $|\cdot|$ on X with the following property. Let $(y_i)_{i=1}^n$ be a normalized monotone basis for an n-dimensional Banach space. Then every block basis of (e_i) admits a further finite block basis $(x_i)_{i=1}^n$ which is $(1+\varepsilon)$-equivalent to $(y_i)_{i=1}^n$.*

The space S of [34] was shown in [10] to be sequentially arbitrarily distortable. The argument used to prove Theorem 3.1 is a slight variation of an argument which appears in [10] which, in turn, has its origins in [24].

Proof of Theorem 3.1. Choose for $n \in \mathbf{N}$ and $\varepsilon>0$, $(B_i)_{i=1}^{k(n)}$ a finite sequence of n-dimensional Banach spaces, each having a normalized monotone basis, such that every normalized monotone basis of length n is $(1+\varepsilon)$-equivalent to the basis of some B_i^n. Let $(w_i)_{i=1}^\infty$ be a normalized monotone basis for $W \equiv (\sum_{n,i} B_i^n)_{l_2}$ such that the monotone basis of each B_i^n is 1-equivalent to $(w_i)_{i \in A_i^n}$ for some segment $A_i^n \subseteq \mathbf{N}$. Let (w_i^*) be the biorthogonal functionals of (w_i).

It suffices to prove that for all $n \in \mathbf{N}$ there exists an equivalent norm $|\cdot|$ on X such that every block basis of (e_i) admits a further block basis $(x_i)_{i=1}^n$ which is $(1+8/n)$-equivalent to $(w_i)_{i=1}^n$.

Let $n \in \mathbf{N}$, $\varepsilon_i \downarrow 0$ and let $\|\cdot\|_i$ be a sequence of equivalent norms on X satisfying the definition of sequentially arbitrarily distortable. Let $\varepsilon>0$ with $n^5\varepsilon<1$. We may assume that $\max_i \varepsilon_i < \frac{1}{4}\varepsilon$.

Let $X_i=(X, \|\cdot\|_i)$. Let $(z_i^*)_{i=2}^\infty$ be an enumeration of all elements of the linear span of (e_i^*) which have rational coordinates. Set

$$\Gamma=\Big\{z^*=\sum_{i=1}^n b_i\sum_{j=(i-1)n+1}^{in} z^*_{k_j} : k_1<\ldots<k_{n^2},\ (z^*_{k_i})_{i=1}^{n^2} \text{ is a finite block basis of } (e_i^*) \text{ with } z^*_{k_1}\in 3\,\mathrm{Ba}(X_1^*),$$
$$z^*_{k_{i+1}}\in 3\,\mathrm{Ba}(X^*_{k_i}) \text{ for } 1\leqslant i\leqslant n^2-1 \text{ and } \sum_{i=1}^n b_i w_i^*\in \mathrm{Ba}(W^*)\Big\}.$$

Define $|\cdot|$ on X by

$$|x|=\sup\{|z^*(x)|:z^*\in\Gamma\}.$$

Then $3\|x\|_1\leqslant|x|\leqslant 6n^2\|x\|$ for all $x\in X$ and so $|\cdot|$ is an equivalent norm on X.

Let Z be any block subspace of X. Since X being distortable cannot contain l_1 [11], we may assume by [33] that Z is spanned by a normalized weakly null block basis of (e_i), denoted (z_i). Using the argument that a subsequence of (z_i) is nearly monotone for any given norm $|\cdot|_i$ and a diagonal argument we may suppose that for all i, $\|P_A\|_i<2.5$ whenever $A\subseteq\mathbf{N}$ is a *segment* of $\mathbf{N}$ with $i\leqslant\min A$. (Here P_A is the projection $P_A(\sum a_iz_i)=\sum_{i\in A}a_iz_i$.)

From our hypotheses we can then choose block bases $(\bar{x}_i)_{i=1}^{n^2}$ of (z_i), and $(z^*_{k_i})_{i=1}^{n^2}$ of (e_i^*) satisfying $k_1<k_2<\ldots<k_{n^2}$ and

(i) $z^*_{k_1}\in 3\,\mathrm{Ba}(X_1^*)$ and $z^*_{k_{i+1}}\in 3\,\mathrm{Ba}(X^*_{k_i})$ for $1\leqslant i<n^2$,

(ii) $z^*_{k_i}(\bar{x}_j)=\delta_{ij}$ for $1\leqslant i,\ j\leqslant n^2$,

(iii) $\|\bar{x}_i\|_j<\frac{1}{3}\varepsilon$ if $j\neq k_{i-1}$ and $\|\bar{x}_i\|_{k_{i-1}}\leqslant 1$.

Let $x_i=(1/n)\sum_{j=(i-1)n+1}^{in}\bar{x}_j$ for $1\leqslant i\leqslant n$, and let $\|\sum_1^n a_iw_i\|=1=\sum_1^n a_ib_i$ where $\|\sum_1^n b_iw_i^*\|=1$. Let

$$z^*=\sum_{i=1}^n b_i\sum_{j=(i-1)n+1}^{in} z^*_{k_j}$$

and note that $z^*\in\Gamma$. Thus

$$\Big|\sum_1^n a_ix_i\Big|\geqslant z^*\Big(\sum_1^n a_ix_i\Big)=\sum_1^n a_ib_i=1.$$

For the reverse inequality, let $\bar{z}^*=\sum_{i=1}^n c_i\sum_{j=(i-1)n+1}^{in} z^*_{m_j}\in\Gamma$ with $z^*_{m_1}\in 3\,\mathrm{Ba}(X_1^*)$, $z^*_{m_{i+1}}\in 3\,\mathrm{Ba}(X^*_{m_i})$ for $i<n^2$ and $\|\sum_1^n c_iw_i^*\|\leqslant 1$. Let j_0 be the smallest integer such that $m_{j_0}\neq k_{j_0}$. We first deduce from the definition of Γ and the choice of $(\bar{x}_i)$ that $|z^*_{m_i}(\bar{x}_j)|<\varepsilon$ and $|z^*_{m_j}(\bar{x}_i)|<\varepsilon$ if $i<j_0$, $j\leqslant n^2$ and $i\neq j$. Secondly we claim that

$$\{m_{j_0},m_{j_0+1},\ldots,m_{n^2}\}\cap\{k_{j_0},k_{j_0+1},\ldots,k_{n^2}\}=\varnothing.$$

Indeed, if not, let $j\geqslant j_0$ be the smallest integer such that $m_j=k_i$ for some $i\geqslant j_0$. If $j=j_0$ then $i>j_0$. But then (letting $k_0\equiv 1$) $z^*_{m_j}\in 3\,\mathrm{Ba}(X^*_{k_{j_0-1}})$ and $\|\bar{x}_i\|_{k_{j_0-1}}<\frac{1}{3}\varepsilon$ which contradicts $z^*_{k_i}(\bar{x}_i)=1$. If $j>j_0$ then $z^*_{m_j}\in 3\,\mathrm{Ba}(X^*_{m_{j-1}})$ and $\|\bar{x}_i\|_{m_{j-1}}<\frac{1}{3}\varepsilon$ since $m_{j-1}\neq k_{i-1}$, yielding again a contradiction to $z^*_{k_i}(\bar{x}_i)=1$.

It follows that $|z^*_{m_{j_0}}(\bar{x}_i)|<\varepsilon$ if $i\neq j_0$ and $|z^*_{m_j}(\bar{x}_i)|<\varepsilon$ if $j>j_0$ and $i\leqslant n^2$. Let $j_0=i_0n+s_0$ with $0\leqslant i_0<n$, $1\leqslant s_0\leqslant n$. Then

$$\begin{aligned}\left|\bar{z}^*\left(\sum_{i=1}^n a_ix_i\right)\right| &= \left|\left(\sum_{i=1}^n c_i\sum_{j=(i-1)n+1}^{in} z^*_{m_j}\right)\left(\sum_{i=1}^n a_i\frac{1}{n}\sum_{j=(i-1)n+1}^{in}\bar{x}_j\right)\right|\\ &\leqslant \left|\sum_{i=1}^{i_0} c_ia_i+\frac{s_0-1}{n}c_{i_0+1}a_{i_0+1}\right|+3\left|\frac{c_{i_0+1}a_{i_0+1}}{n}\right|+n^4\varepsilon\max_i|a_ic_i|\\ &\leqslant \left\|\sum_1^n a_iw_i\right\|\left[1+\frac{6}{n}+\frac{2}{n}\right].\end{aligned}$$

We used that from monotonicity the first term in the next to last inequality does not exceed

$$\max\left(\left|\sum_{i=1}^{i_0}c_ia_i\right|,\left|\sum_{i=1}^{i_0+1}c_ia_i\right|\right)\leqslant\left\|\sum_1^n a_iw_i\right\|$$

and $|c_ia_i|\leqslant 2$ for all i. □

Remark. The proof of Theorem 3.1 requires only the following condition. For all $\varepsilon>0$ there exists a sequence of equivalent norms $\|\cdot\|_i\leqslant\|\cdot\|$ on X such that for all subspaces Z of X and all $i_0\in\mathbf{N}$ there exists $y\in S(Z,\|\cdot\|_{i_0})$ with $\|y\|_i<\varepsilon$ if $i\neq i_0$. Theorem 1.2 is a special case of Theorem 3.1.

Theorem 1.2 yields that a sequentially arbitrarily distortable Banach space can be renormed to not contain an almost bimonotone basic sequence. Since $\|s_1-2s_2\|=1$, the best constant that can be achieved for the norm of the tail projections of a basic sequence is 2.

Other curious norms can be put on sequentially arbitrarily distortable spaces X. For example let $(w_i)_{i=1}^n$ be a normalized 1-unconditional 1-subsymmetric finite basic sequence and let $\varepsilon>0$. One can find a norm on X such that every block basis contains a further block basis (z_i) with $(z_{k_i})_{i=1}^n\overset{1+\varepsilon}{\sim}(w_i)_{i=1}^n$ whenever $k_1<...<k_n$. This is accomplished by taking (using the terminology of the proof of Theorem 3.1)

$$\begin{aligned}\Gamma=\Bigg\{z^*=\sum_{i=1}^n b_i\sum_{j=(k_i-1)n+1}^{k_in} z^*_{m_j} &: (z^*_{m_j})_{j=1}^\infty \text{ is a block basis of } (e_i^*)\\ &\text{with } z^*_{m_1}\in 3\,\mathrm{Ba}(X_1^*),\ z^*_{m_{j+1}}\in 3\,\mathrm{Ba}(X^*_{m_j}) \text{ for } j\in\mathbf{N},\\ &k_1<k_2<...<k_n \text{ and } \left\|\sum_1^n b_iw_i^*\right\|\leqslant 1\Bigg\}.\end{aligned}$$

THEOREM 3.2. *For $1<p<\infty$, l_p is sequentially arbitrarily distortable.*

In order to prove Theorem 3.2 we will make use of the Banach space S introduced in [34].

The space S has a 1-unconditional 1-subsymmetric normalized basis (e_i) whose norm satisfies the following implicit equation

$$\|x\| = \max\left\{\|x\|_{c_0}, \sup_{\substack{l\geqslant 2 \\ E_1<E_2<\ldots<E_l}} \frac{1}{\phi(l)}\sum_{i=1}^{l}\|E_i x\|\right\}$$

where $\phi(l)=\log_2(1+l)$.

The fact that S is arbitrarily distortable [34] and complementably minimal [35] hinges heavily on two types of vectors which live in all block subspaces: l_1^n+ averages and averages of rapidly increasing $l_1^{n_i}+$ averages or RIS vectors. Precisely, following the terminology of [10], we call $x\in S$ an l_1^n+ *average with constant* C if $\|x\|=1$ and $x=\sum_{i=1}^{n} x_i$ for some block basis $(x_i)_{i=1}^n$ of (e_i) where $\|x_i\|\leqslant Cn^{-1}$ for all i.

Let $M_\phi(x)=\phi^{-1}(36x^2)$ for $x\in\mathbf{R}$. A block basis $(x_i)_{i=1}^N$ is an *RIS of length N with constant* $C\equiv 1+\varepsilon<2$ if each x_k is an $l_1^{n_k}+$ average with constant C,

$$n_1\geqslant 2CM_\phi(N/\varepsilon)/2\varepsilon\ln 2$$

and

$$\tfrac{1}{2}\varepsilon\phi(n_k)^{1/2}\geqslant|\operatorname{supp}(x_{k-1})|\quad\text{for } k=2,\ldots,N.$$

The vector $x=(\sum_{i=1}^N x_i)/\|\sum_{i=1}^N x_i\|$ is called an *RIS vector of length N and constant C* and we say that the RIS sequence $(x_i)_{i=1}^N$ *generates* x.

LEMMA 3.3 [10]. *Let $\varepsilon_i\downarrow 0$. There exist integers $p_k\uparrow\infty$ and reals $\delta_k\downarrow 0$ with*

$$(1+2\delta_k)^{-1}>1-\varepsilon_k$$

so that if

$$A_k=\{x\in S: x \text{ is an RIS vector of length } p_k \text{ with constant } 1+\delta_k\}$$

and

$$A_k^*=\left\{x^*\in S^*: x^*=\frac{1}{\phi(p_k)}\sum_{1}^{p_k}x_i^* \text{ where } (x_i^*)_1^{p_k} \text{ is a block sequence in } \mathrm{Ba}(S^*)\right\}$$

then:

(a) $|x_k^*(x_l)|<\varepsilon_{\min(k,l)}$ *if $k\neq l$, $x_k^*\in A_k^*$ and $x_l\in A_l$.*

(b) *For all $k\in\mathbf{N}$ and $x\in A_k$ there exists $x^*\in A_k^*$ with $x^*(x)>1-\varepsilon_k$. This follows from the fact if x is generated by $(x_i)_{i=1}^{p_k}$, then $\|\sum_1^{p_k}x_i\|\leqslant(1+2\delta_k)p_k/\phi(p_k)$.*

Moreover A_k is asymptotic in S for all $k\in\mathbf{N}$.

Using the sets A_k and A_k^* we can define the following subsets of l_1

$$B_k = \{x_k^* \circ x_k / |x_k^*|(|x_k|) : x_k^* \in A_k^*,\ x_k \in A_k \text{ and } |x_k^*|(|x_k|) = \|x_k^* \circ x_k\|_1 \geqslant 1 - \varepsilon_k\}.$$

A set of sequences B is *unconditional* if $x=(x_i)\in B$ implies that $(\pm x_i)\in B$ for all choices of signs and B is *spreading* if $x=(x_i)\in B$ implies $\sum_i x_i e_{n_i}\in B$ for all increasing sequences (n_i). Note that $A_k^*\subseteq \mathrm{Ba}(S^*)$ and the sets A_k and A_k^* are unconditional and spreading. Thus the sets $B_k\subseteq S(l_1)$ are also spreading and unconditional.

THEOREM 3.4. *The sets $B_k\subseteq S(l_1)$, $k\in\mathbf{N}$, are unconditional, spreading and asymptotic.*

We postpone the proof of Theorem 3.4.

Proof of Theorem 3.2. We first give the argument for $p=2$. Let $C_k=\{v\in S(l_2): |v|^2\in B_k\}$. C_k is just the image of B_k in $S(l_2)$ under the Mazur map. Since the Mazur map preserves block subspaces and is a uniform homeomorphism, C_k is asymptotic in l_2 for all k. Moreover the C_k's are nearly biorthogonal. Indeed if $v_k\in C_k$, $v_l\in C_l$ with $k\neq l$ let $|v_k|^2=(x_k^*\circ x_k)/|x_k^*|(|x_k|)$ and $|v_l|^2=(x_l^*\circ x_l)/|x_l^*|(|x_l|)$ be as in the definition of B_k and B_l. Then letting $\lambda=(1-\varepsilon_1)^{-1}$

$$\begin{aligned}\langle |v_k|,|v_l|\rangle &\leqslant \lambda \sum_j |x_k^*(j)x_k(j)x_l^*(j)x_l(j)|^{1/2}\\ &\leqslant \lambda\Big(\sum_j |x_k^*(j)x_l(j)|\Big)^{1/2}\Big(\sum_j |x_l^*(j)x_k(j)|\Big)^{1/2} \quad \text{(by Cauchy–Schwarz)}\\ &= \lambda \langle |x_k^*|,|x_l|\rangle^{1/2}\langle |x_l^*|,|x_k|\rangle^{1/2} \leqslant \lambda\varepsilon_{\min(k,l)} \quad \text{(by Lemma 3.3).}\end{aligned}$$

Define $\|x\|_k=\sup\{|\langle x,v\rangle| : v\in C_k\cup\varepsilon_k\,\mathrm{Ba}(l_2)\}$.

If $p\neq 2$ we use a similar argument. Let $C_k=\{v\in S(l_p): |v|^p\in B_k\}$ and $D_k=\{v\in S(l_q): |v|^q\in B_k\}$ where $1/p+1/q=1$. Define $\|\cdot\|_k$ on l_p by

$$\|x\|_k = \sup\{|\langle x,v\rangle| : v\in D_k\cup\varepsilon_k\,\mathrm{Ba}(l_q)\}.$$

Again, via the Mazur map, C_k is asymptotic in l_p.

Let $v_k\in C_k$ and $v_l\in D_l$ with $k\neq l$. Let $|v_k|^p=(x_k^*\circ x_k)/|x_k^*|(|x_k|)$ and $|v_l|^q=(x_l^*\circ x_l)/|x_l^*|(|x_l|)$ be as in the definition of B_k and B_l. Assume $p>2$. Then

$$\begin{aligned}|\langle |v_k|,|v_l|\rangle| &\leqslant \lambda\sum_j |x_k^*(j)x_k(j)|^{1/p}|x_l^*(j)x_l(j)|^{1/q}\\ &= \lambda\sum_j |x_k^*(j)x_k(j)x_l^*(j)x_l(j)|^{1/p}|x_l^*(j)x_l(j)|^{1/q-1/p}.\end{aligned}$$

Using Hölder's inequality with exponents $\frac{1}{2}p$ and $p/(p-2)$ and the fact that $1/q-1/p=(p-2)/p$ we obtain that the last expression is

$$\leqslant \lambda\Big(\sum_j |x_k^*(j)x_k(j)x_l^*(j)x_l(j)|^{1/2}\Big)^{2/p}\Big(\sum_j |x_l^*(j)x_l(j)|\Big)^{(p-2)/p} \leqslant \lambda\varepsilon_{\min(k,l)}^{2/p}$$

from the first part of the proof. The same estimates prevail if $p<2$. □

Remark. The proof yields that for $1<p<\infty$, $1/p+1/q=1$ there exist sequences $C_k\subseteq S(l_p)$ and $D_k\subseteq S(l_q)$ of nearly biorthogonal asymptotic unconditional spreading sets.

It remains only to prove Theorem 3.4 which entails only showing that each B_k is asymptotic. This will follow from the following

LEMMA 3.5. *Let Y be a block subspace of l_1 and let $\varepsilon>0$, $m\in\mathbf{N}$. There exists a vector $u\in S$ which is an l_1^m+ average with constant $1+\varepsilon$ and $u^*\in\mathrm{Ba}(S^*)$ with $d(u^*\circ u, S(Y))<\varepsilon$.*

Indeed assume that the lemma is proved and let $k\in\mathbf{N}$ and $\varepsilon>0$ with

$$(1+\varepsilon)^{-1}(1+2\delta_k)^{-1}>1-\varepsilon_k.$$

From the lemma we can find finite block sequences $(u_i)_{i=1}^{p_k}\subseteq S(S)$ and $(u_i^*)_{i=1}^{p_k}\subseteq\mathrm{Ba}(S^*)$ along with a normalized block sequence $(y_i)_{i=1}^{p_k}\subseteq S(Y)$ and $1\leqslant\lambda_i<1+\varepsilon$ for $i\leqslant p_k$ such that

(1) $u=(\sum_{i=1}^{p_k}u_i)/\|\sum_{i=1}^{p_k}u_i\|$ is an RIS vector of length p_k and constant $(1+\delta_k)$ generated by the RIS $(u_i)_{i=1}^{p_k}$,

(2) $\|u_i^*\circ u_i-y_i\|_1<\varepsilon$ for $i\leqslant p_k$,

(3) $u_i^*\circ u_j=0$ if $i\neq j$ and $\|\lambda_i u_i^*\circ u_i\|_1=1$ for $i\leqslant p_k$.

Let $u^*=(1/(1+\varepsilon)\phi(p_k))\sum_1^{p_k}\lambda_i u_i^*$. Then $u^*\in A_k^*$ and from Lemma 3.3 (b)

$$\|u^*\circ u\|_1=\frac{1}{(1+\varepsilon)\phi(p_k)}\cdot\frac{p_k}{\|\sum_1^{p_k}u_i\|}\geqslant\frac{1}{(1+\varepsilon)(1+2\delta_k)}>1-\varepsilon_k.$$

Thus $(u^*\circ u)/\|u^*\circ u\|_1\in B_k$. Now $(u^*\circ u)/\|u^*\circ u\|_1=(1/p_k)\sum_1^{p_k}\lambda_i u_i^*\circ u_i$ and so using (2)

$$\Big\|\frac{u^*\circ u}{\|u^*\circ u\|_1}-\frac{1}{p_k}\sum_1^{p_k}y_i\Big\|\leqslant\frac{1}{p_k}\sum_1^{p_k}\|\lambda_i u_i^*\circ u_i-y_i\|_1<2\varepsilon.$$

This proves that B_k is asymptotic in l_1.

In order to prove Lemma 3.5 we first need a sublemma. We denote the maps $E_{S^*}(h)$ and $F_{S^*}(h)$ by $E_*(h)$ and $F_*(h)$, respectively.

SUBLEMMA 3.6. *Let m, K be integers and let $0<\tau<1$ be such that $\log\phi(m^K)<\tau K$. Let $(h_i)_{i=1}^{m^K}$ be a normalized block sequence in l_1^+. Then there exist in l_1^+ a normalized block basis $(b_i)_{i=1}^m$ of $(h_i)_1^{m^K}$ such that*

$$\sum_{j=1}^m E_*(b_j)-E_*\Big(\sum_{j=1}^m b_j\Big)\leqslant \tau m. \tag{3.1}$$

Proof. For each $i\leqslant m^K$, let $v_i=F_*(h_i)$. Now $(1/\phi(m^K))\sum_1^{m^K} v_i\in \mathrm{Ba}(S^*)$ and so

$$\begin{aligned}E_*\Big(\sum_1^{m^K} h_i\Big) &\geqslant E\Big(\sum_1^{m^K} h_i, \frac{1}{\phi(m^K)}\sum_1^{m^K} v_i\Big)\\ &=\sum_1^{m^K} E(h_i,v_i)-m^K\log\phi(m^K)\\ &=\sum_1^{m^K} E_*(h_i)-m^K\log\phi(m^K).\end{aligned}\tag{3.2}$$

Let $\sum_{i=1}^{m^K} h_i=\sum_{j=1}^m d_j^1$ where $(d_j^1)_{j=1}^m$ is a block basis of (h_i), each d_j^1 consisting of the sum of m^{K-1} of the h_i's. Break each d_j^1 into m successive pieces, each containing m^{K-2} of the h_i's to obtain $d_j^1=\sum_{l=1}^m d_{j,l}^2$ and continue to define $d_{\alpha,j}^l$ for $l\leqslant k$ and $\alpha\in\{1,\dots,m\}^{l-1}$ in this fashion. Consider the telescoping sum

$$\begin{aligned}\sum_{i=1}^{m^K} E_*(h_i)-E_*\Big(\sum_{i=1}^{m^K} h_i\Big) &=\sum_{j=1}^m E_*(d_j^1)-E_*\Big(\sum_{j=1}^m d_j^1\Big)\\ &\quad+\sum_{j=1}^m\Big[\sum_{l=1}^m E_*(d_{j,l}^2)-E_*\Big(\sum_{l=1}^m d_{j,l}^2\Big)\Big]+\dots .\end{aligned}$$

For $1\leqslant s\leqslant K$, the sth level of this decomposition is the sum of m^{s-1} nonnegative terms of the form (for $\alpha\in\{1,\dots,m\}^{s-1}$)

$$\sum_{l=1}^m E_*(d_{\alpha,l}^s)-E_*\Big(\sum_{l=1}^m d_{\alpha,l}^s\Big). \tag{3.3}$$

If each of these terms is greater than τm^{K-s+1} then the sum of all terms on the sth level is greater than τm^K and so the sum over all K levels yields

$$\sum_{i=1}^{m^K} E_*(h_i)-E_*\Big(\sum_1^{m^K} h_i\Big)>K\tau m^K$$

which contradicts (3.2).

Thus the number (3.3) does not exceed the value τm^{K-s+1} for some s and multi-index α. Let $b_l = d^s_{\alpha,l}/\|d^s_{\alpha,l}\|$. Using $E_*(ah)=aE_*(h)$ for $a>0$ and $\|d^s_{\alpha,l}\|=m^{K-s}$ we obtain

$$\sum_{l=1}^m E_*(b_l) - E_*\left(\sum_{l=1}^m b_l\right) \leqslant \frac{\tau m^{K-s+1}}{m^{K-s}} = \tau m. \qquad \square$$

Proof of Lemma 3.5. Let $\varepsilon>0$, $m\in\mathbf{N}$ and let Y be a block subspace of l_1 with block basis (h_i). By unconditionality in S it suffices to consider only the case where $(h_i)\subseteq S(l_1)^+$. Let $0<\tau<\psi(\varepsilon)/m$ (see Definition 2.2) and choose $K\in\mathbf{N}$ such that $\tau K> \log(\phi(m^K))$. By Sublemma 3.6 choose a block basis $(b_i)_1^m$ of $(h_i)_{i=1}^{m^K}$, $(b_i)_1^m\subseteq S(l_1^+)$ with

$$\sum_1^m E_*(b_j) - E_*\left(\sum_{j=1}^m b_j\right) < \tau m. \tag{3.4}$$

Choose $x^* = F_*(\sum_{j=1}^m b_j)$ and write $x^*=\sum_{j=1}^m x_j^*$ with $\operatorname{supp} x_j^* = \operatorname{supp} b_j$. For $j\leqslant m$ let $w_j^* = F_*(b_j)$. As we noted in §2, for each j there exists $w_j\in S(S)^+$ with $b_j = w_j^*\circ w_j$ and $\operatorname{supp} w_j = \operatorname{supp} b_j$. By (3.4) we have

$$\sum_{j=1}^m E(b_j,w_j^*) - E\left(\sum_{j=1}^m b_j, x^*\right) = \sum_{j=1}^m [E(b_j,w_j^*)-E(b_j,x_j^*)] < \tau m < \psi(\varepsilon).$$

Since each term in the middle expression is nonnegative we obtain

$$E(b_j,x_j^*) > E(b_j,w_j^*) - \psi(\varepsilon) \quad \text{for } j\leqslant m.$$

By Proposition 2.3 (A) there exists sets $H_j\subseteq \operatorname{supp} b_j$ such that $\|H_jb_j\|_1>1-\varepsilon$ and $(1-\varepsilon)H_jw_j^*\leqslant H_jx_j^*\leqslant(1+\varepsilon)H_jw_j^*$ pointwise for all $1\leqslant j\leqslant m$.

$H_jb_j = H_jw_j^*\circ w_j$ and $\|H_jx_j^*-H_jw_j^*\|\leqslant\varepsilon$ so $\|H_jb_j - H_jx_j^*\circ w_j\|_1\leqslant\varepsilon$. Thus

$$\|b_j - H_jx_j^*\circ w_j\|_1 \leqslant 2\varepsilon \quad \text{for } 1\leqslant j\leqslant m. \tag{3.5}$$

From this we first note that $H_jx_j^*(w_j)\geqslant 1-2\varepsilon$ and so for a_i's nonnegative,

$$\left\|\sum_1^m a_jw_j\right\| \geqslant x^*\left(\sum_1^m a_jw_j\right) \geqslant \sum_{j=1}^m a_jH_jx_j^*(w_j) \geqslant \left(\sum_{j=1}^m a_j\right)(1-2\varepsilon).$$

By unconditionality $(w_j)_{j=1}^m$ is an l_1^m sequence with constant $(1-2\varepsilon)^{-1}$.

Secondly, set

$$\bar{w} = \frac{1}{m}\sum_{j=1}^m w_j \quad \text{and} \quad w = \frac{1}{\|\sum_{j=1}^m w_j\|}\sum_{j=1}^m w_j.$$

w is an l_1^m average with constant $(1-2\varepsilon)^{-1}$. Furthermore

$$\Big\|\frac{1}{m}\sum_{j=1}^m b_j-\Big(\bigcup_{j=1}^m H_j\Big)x^*\circ w\Big\|_1 \leqslant \Big\|\frac{1}{m}\sum_{j=1}^m b_j-\Big(\bigcup_{j=1}^n H_j\Big)x^*\circ \bar{w}\Big\|_1+\|w-\bar{w}\|$$
$$\leqslant \frac{1}{m}\sum_{j=1}^m \|b_j-H_jx^*\circ w_j\|_1+\|w-\bar{w}\|.$$

The first term is $<2\varepsilon$ by (3.5). Since $\|\sum_{j=1}^m w_j\|\geqslant m(1-2\varepsilon)$, $\|w-\bar{w}\|\leqslant 2\varepsilon/(1-2\varepsilon)$. Thus

$$d\Big(\Big(\bigcup_{j=1}^m H_j\Big)x^*\circ w, S(Y)\Big)<2\varepsilon+\frac{2\varepsilon}{1-2\varepsilon}$$

which proves Lemma 3.5. □

Remark 3.7. Our proof of Theorem 3.2 actually shows that l_p admits an asymptotic biorthogonal system with vanishing constant (see [10]). B. Maurey [22] has recently extended the results above. He has proven that if X has an unconditional basis and does not contain l_1^n uniformly, then X contains an arbitrarily distortable subspace. B. Maurey and the second named author have independently shown that one can construct the sets B_k to be symmetric ($(x_i)\in B_k\Rightarrow(x_{\pi(i)})\in B_k$ if π is a permutation of $\mathbf{N}$).

N. Tomczak-Jaegermann and V. Milman [29] have proven that if X has bounded distortion, then X contains an "asymptotic l_p or c_0". X has bounded distortion if for some $\lambda<\infty$, no subspace of X is λ-distortable. A space with a basis (e_i) is an asymptotic l_p if for some $C<\infty$ for all n whenever

$$e_n<x_1<\ldots<x_n,\quad \|x_i\|=1\quad(i=1,\ldots,n),$$

then $(x_i)_1^n$ is C-equivalent to the unit vector basis of l_p^n.

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