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Helmholtz operators on harmonic manifolds

by

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Introduction

Singularities of solutions of differential equations have always attracted attention. Frobenius, Hadamard, and Painlevé, to name a few authors, developed tools for their investigation. A typical problem is the occurence of a logarithmic singularity, but also, where a logarithmic term is expected, its absence. The present paper studies this problem for a particular class of differential equations which naturally arises from the analysis on manifolds.

Let (M,g) be an analytic pseudo-Riemannian manifold of dimension $n>2$. A linear analytic differential operator L on M is called *Laplace-like* if it has the same principal part as the Laplace–Beltrami operator Δ with respect to g. Note that L is elliptic if g is properly Riemannian, hyperbolic if g is Lorentzian, and ultrahyperbolic otherwise. Let $r(x,y)$ denote the distance between two points x and y in M (the definition is given below—for definite metrics it is the geodesic distance). Hadamard's *elementary solution* is a solution $u=u(x,y)$ of $L_x u=0$ which qualitatively behaves like $r(x,y)^{2-n}$ as the running point x approaches the fixed (but arbitrary) point of origin y. It was shown by Hadamard (see [8], [7]) that there exists an elementary solution of the form

$$u=Ur^{2-n}+\mathcal{U}\log r$$

where $U=U(x,y)$ and $\mathcal{U}=\mathcal{U}(x,y)$ are defined and analytic in a neighborhood of the diagonal of $M\times M$, and $U(x,x)\equiv 1$ on the diagonal. If n is odd we always have $\mathcal{U}\equiv 0$, but if n is even the logarithmic term is generally present. The exceptional behaviour in which we are interested here is precisely the absence of this term (that is, $\mathcal{U}\equiv 0$). In this case the elementary solution u is called *logarithm-free*.

In addition to being mathematically interesting, the search for logarithm-free elementary solutions also has some physical relevance. In the even-dimensional Lorentzian (hyperbolic) case, the absence of the logarithmic term for the operator L is equivalent to the property that its formal adjoint L^* obeys Huygens' principle locally, that is, solutions to the equation $L^*u=0$ propagate sharply, just as waves in even-dimensional Minkowski space-time are known to do (see [7]). Furthermore, in the properly Riemannian (elliptic) case the equation $Lu=0$ can sometimes be interpreted as the field equation for scalar gravitation; in this case the elementary solution is the gravitational potential of a central point mass, and a log-term is expected to act as a potential barrier for a point particle moving to infinity. In Newton's theory we have $L=\Delta$; certain deviations from this are discussed under the name of the 'fifth force' (see [15], [19]). In particular, the *Helmholtz equation* $(\Delta+\lambda)u=0$, where λ is a constant, has been taken as a candidate for a gravitational field equation.

It is a long-standing unsolved problem to generally characterize the Laplace-like differential operators which admit a logarithm-free elementary solution. Tractable subproblems emerge by restriction of the class of differential operators and the class of spaces under consideration. Here we restrict to the *Helmholtz operators*

$$L=L_\lambda=\Delta+\lambda, \quad \lambda=\text{const}\in\mathbf{C},$$

on *harmonic* manifolds. We call λ an *exceptional Helmholtz number* for (M,g) if L_λ admits a logarithm-free elementary solution.

As an example, consider the even-dimensional Euclidean space $\mathbf{R}^n=\mathbf{R}^{2m+2}$ with the flat Riemannian metric. Then $r(x,y)=|x-y|$. In this case, if $\lambda=-a^2\neq 0$, then L_λ has the elementary solution $u=cr^{-m}K_m(ar)$, where $c\neq 0$ and K_m is a modified Bessel function of the second kind ([25, pp. 47–48]). Since the latter exhibits a logarithmic part, λ is not exceptional. If $\lambda=0$ then $L_0=\Delta$ has the elementary solution $u=r^{-2m}$. Hence 0 is the only exceptional Helmholtz number for the flat $\mathbf{R}^n$.

In the first part of this paper we associate to any even-dimensional harmonic manifold a polynomial h of degree $m=\frac{1}{2}(n-2)$, which we call the *Hadamard polynomial*. The main result in this part is the observation that λ is exceptional if and only if it is a root of the Hadamard polynomial (Theorem 3.1). Hence there are at least one, and at most m exceptional Helmholtz numbers for (M,g). The Hadamard polynomial gets its name from a close relationship to the Hadamard coefficients of Δ, but it can also be described independently by a simple algebraic recursion formula (Theorem 3.2). Finally, we also establish (in Theorem 3.3) the existence of a solution $u=u(x,y)$ to the $(2m)$th order equation $h(-\Delta)u=0$ of the form

$$u=r^{-2}+Z, \tag{1}$$

where Z is defined and analytic in a neighborhood of the diagonal. It follows that if λ is a root of h then $(\Delta+\lambda)^{-1}h(-\Delta)u$ is a logarithm-free elementary solution to L_λ (up to a constant multiple).

In the second part we explicitly determine the exceptional Helmholtz numbers for the (even-dimensional) isotropic spaces, a family of homogeneous harmonic spaces that includes the two-point homogeneous spaces (which are properly Riemannian) and the spaces of constant curvature (not necessarily properly Riemannian). In particular we thus obtain the complete answer to the problem of determining the exceptional Helmholtz numbers for the closed simply connected harmonic manifolds, as well as for the Lorentzian harmonic manifolds, since it is known that a manifold of the first type is two-point homogeneous (see [27]), and that a manifold of the second type has constant curvature (see [21, p. 68]). (There are however open properly Riemannian harmonic manifolds which are not isotropic, see [4].) The exceptional Helmholtz numbers are found by explicit determination of a function u as in (1), which is annihilated by an mth order polynomial in $-\Delta$. Consequently this polynomial is a constant multiple of h, and its roots $\lambda_1, \ldots, \lambda_m$ are the exceptional Helmholtz numbers (some of which may occur twice). The fact that the product of the operators $\Delta+\lambda_k$ $(k=1, \ldots, m)$ has a logarithm-free elementary solution which becomes singular like $r(x,y)^{-2}$ as $x \to y$, is a curved analog of the fact that in the flat even-dimensional $\mathbf{R}^n$ the iterated Laplace operator Δ^m has the logarithm-free elementary solution $u=r^{-2}$ (whereas Δ^k for $k>m$ requires logarithms, see [25, p. 47]). The proof is strongly related to Helgason's analysis of integral (Radon) transforms for the two-point homogeneous spaces ([9], [10] or [12, Chapter I, §4]).

For the even-dimensional Lorentzian hyperbolic spaces we have thus obtained all the values λ for which the *Klein–Gordon equation* $(\Delta+\lambda)u=0$ obeys Huygens' principle locally; there are precisely m different such values $\lambda_1, \ldots, \lambda_m$ (these can be shown to be exceptional as well by an argument of conformal invariance, as was also observed independently by Ørsted (unpublished)). In addition we obtain that the product of the operators $\Delta+\lambda_k$ $(k=1, \ldots, m)$ obeys Huygens' principle, even globally: It has a fundamental solution supported on the boundary of the light cone. We derive this result from Helgason's inversion formula for the orbital integrals for these spaces ([12, Chapter I, Theorem 6.17]). It is a curved analog of the fact that in the flat even-dimensional Lorentzian $\mathbf{R}^n$ the iterated d'Alembertian Δ^m has such a fundamental solution (see [14, §7.1]).

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1. Harmonic manifolds

Let us review here some well-known properties of harmonic manifolds. For details, see [2], [21]. For a smooth pseudo-Riemannian manifold (M,g) the distance $r(x,y)\geqslant 0$ between two points x and y can be defined as follows: Firstly one defines *Synge's function* $\sigma(x,y)$ by

$$\sigma(x,y):=\tfrac{1}{2}g(y)(\mathrm{Exp}_y^{-1}x,\mathrm{Exp}_y^{-1}x)\in\mathbf{R}$$

where Exp_y denotes the exponential map from T_yM to M. It is defined and smooth in a neighborhood of the diagonal of $M\times M$ and takes the value 0 on the diagonal. Secondly

$$r(x,y):=\sqrt{2|\sigma(x,y)|}.$$

If M is properly Riemannian then $r(x,y)$ equals the geodesic distance between x and y, for x in a normal neighborhood of y.

It follows from the definition of σ that it satisfies the differential equation

$$\langle \mathrm{grad}\,\sigma, d\sigma\rangle=2\sigma. \tag{2}$$

Here and in the following $\langle\cdot,\cdot\rangle$ denotes the natural pairing between vector fields and 1-forms, and the differentiations take place with respect to x.

From σ some other useful quantities are derived. The *normal volume function* $\varrho(x,y)$ is defined in local coordinates $(x_1,...,x_n)$ on M by

$$\varrho(x,y):=\sqrt{\bar g(x)\bar g(y)}\,|\det\partial_{x_i}\partial_{y_j}\sigma(x,y)|^{-1},$$

for (x,y) in a neighborhood of the diagonal of $M\times M$. Here

$$\bar g=|\det g_{ij}|,\quad g_{ij}=g(\partial_{x_i},\partial_{x_j}),\quad \partial_{x_i}=\frac{\partial}{\partial x_i}.$$

In fact ϱ is independent of the choice of local coordinates, it is the Radon–Nikodym derivative of the Riemannian volume element on M with respect to the push-forward by Exp_y of the volume element on T_yM. On the diagonal we have $\varrho(y,y)=1$. Let $D(x,y)=\varrho(x,y)^{-1}$ be the *van Vleck determinant*, and

$$\mu(x,y):=\tfrac{1}{2}(\Delta_x\sigma(x,y)-n).$$

It is easily seen that $\mu(y,y)=0$.

A function u on a neighborhood of the diagonal in $M\times M$ is called *radial* if it factors through σ, that is $u(x,y)=f(\sigma(x,y))$. When applied to the first variable of a smooth radial function $u(x,y)=f(\sigma(x,y))$, the Laplace operator reduces by means of (2) to the expression

$$\Delta_x u = 2\sigma f''(\sigma)+(2\mu+n)f'(\sigma). \tag{3}$$

A smooth manifold (M,g) is called *harmonic* if the Laplace operator Δ_x maps smooth radial functions to radial functions again. It follows from (3) that M is harmonic if and only if μ (or equivalently, $\Delta_x\sigma$) is radial. Another equivalent criterion reads that ϱ (or D) is radial (see [21, p. 35]). In this case a radial elementary solution to the Laplace equation $\Delta u=0$ is represented by the integral

$$u=\int \sigma^{-m-1}D(\sigma)\,d\sigma, \tag{4}$$

and we have the expression

$$\mu=\sigma\frac{d}{d\sigma}\log\varrho=\tfrac{1}{2}r\frac{d}{dr}\log\varrho \tag{5}$$

(see [21, pp. 39–40]).

Every harmonic manifold is Einstein. As a consequence a properly Riemannian harmonic manifold is analytic (see [3, Theorem 5.26]). Likewise, a Lorentzian harmonic manifold has constant curvature, and such a manifold is known to be analytic (see [12, Theorem I.6.1]). Hence an assumption of analyticity (to be imposed later) will only (possibly) be a proper restriction in the non-Riemannian, non-Lorentzian case.

2. Hadamard's approach

Let (M,g) be a smooth pseudo-Riemannian manifold. To every Laplace-like operator L on M there is associated a sequence of smooth functions $H_k=H_k(x,y)$ on a neighborhood of the diagonal in $M\times M$, called the *Hadamard coefficients*, defined by a certain differential-recursion system (see [7, p. 155]). Assume for simplicity that $L=\Delta+W$, where the so-called Weitzenböck term W is a smooth function on M. Then the definition of the Hadamard coefficients reads

$$\begin{aligned}\langle \operatorname{grad}\sigma, dH_0\rangle+\mu H_0&=0, \quad H_0(y,y)=1,\\ \langle \operatorname{grad}\sigma, dH_k\rangle+(\mu+k)H_k&=LH_{k-1}, \quad k\geqslant 1.\end{aligned} \tag{6}$$

(As always the differentiations take place with respect to x.) As shown in loc. cit. the H_k are uniquely determined by this system.

Assume that $n=\dim M$ is even, and let $m=\frac{1}{2}(n-2)$. By definition an (Hadamard) elementary solution $u=u(x,y)$ to L is a solution of $Lu=0$ of the form

$$u=U\sigma^{-m}+\mathcal{U}\log|\sigma|, \tag{7}$$

where U and $\mathcal{U}$ are defined and smooth on a neighborhood of the diagonal in $M\times M$, and $U\equiv 1$ on the diagonal (thus u is only defined outside the set where $\sigma=0$). Notice that in general the functions U and $\mathcal{U}$ in (7) are not uniquely determined by the elementary solution u alone. We say that u is *logarithm-free* if it allows an expression of the form (7) in which $\mathcal{U}=0$.

PROPOSITION 2.1 (see [8], [20], [7, p. 227], [22]). *Let $n=2m+2\geqslant 4$ be even and let u be an elementary solution of the form (7) to the Laplace-like operator $L=\Delta+W$. Then:*

(i) *We have $L\mathcal{U}=o(|\sigma|^k)$ for all $k=0,1,2,\ldots$.*

(ii) *There exists, for each $p=0,\ldots,m$, a smooth function R_p on a neighborhood of the diagonal such that*

$$U=\sum_{k=0}^{p-1}2^{-k}\binom{m-1}{k}^{-1}H_k\frac{\sigma^k}{k!}+R_p\sigma^p. \tag{8}$$

(iii) *There exists, for each $p=m+1,m+2,\ldots$, a smooth function R_p on a neighborhood of the diagonal such that*

$$-\mathcal{U}=\frac{1}{(m-1)!}\sum_{k=0}^{p-m-1}(-1)^k2^{-k-m}H_{k+m}\frac{\sigma^k}{k!}+R_p\sigma^{p-m}. \tag{9}$$

(iv) *u is logarithm-free if and only if the m-th Hadamard coefficient H_m admits σ as a factor (that is $\sigma^{-1}H_m$ is smooth on a neighborhhood of the diagonal).*

Proof. Since it is difficult to point out a specific reference we sketch the proof briefly. It follows from the equation $Lu=0$ that $\sigma^m\log|\sigma|L\mathcal{U}$ is smooth over the diagonal. This implies (i). Using a similar procedure as in (6) one can define a sequence of smooth functions $R_0,R_1,\ldots,R_m$ recursively by $R_0=U$ and

$$2(m-k+1)[\langle\operatorname{grad}\sigma,dR_k\rangle+(\mu+k+1)R_k]=LR_{k-1}+\sigma^{m-k+1}L(\log|\sigma|\mathcal{U}) \tag{10}$$

for $1\leqslant k\leqslant m$ (the expression on the right involving $\log|\sigma|$ is smooth because of (i)). It is easily seen that the sequence

$$2^k\binom{m-1}{k}k!\,(R_k-R_{k+1}\sigma),\quad k=0,1,\ldots,m-1,$$

obeys exactly the same recursion system (6) as does H_k. The uniqueness of the latter implies that these sequences are identical, and (8) follows by induction.

The recursion in (10) breaks down when $k=m+1$. Instead we define R_{m+1} as follows:

$$2[\langle\operatorname{grad}\sigma,dR_{m+1}\rangle+(\mu+m+2)R_{m+1}]=LR_m+\log|\sigma|L\mathcal{U}. \tag{11}$$

It is then easily seen that the smooth quantity $-2^m(m-1)!(\mathcal{U}+R_{m+1}\sigma)$ satisfies the equation for H_m in (6), and we obtain (9) for $p=m+1$. In order to obtain (9) for all p we define $R_{m+2}, R_{m+3}, \dots$ by

$$2(m-k+1)[\langle \operatorname{grad}\sigma, dR_k\rangle+(\mu+k+1)R_k]=LR_{k-1}+\sigma^{m-k+1}L\mathcal{U}$$

for $k\geqslant m+2$. By similar argumentation as above we get that

$$(-1)^{k-m}2^k(k-m)!\,(m-1)!\,(R_k-R_{k+1}\sigma)=H_k$$

for $k>m$, and (9) follows.

Finally (iv) is obtained as follows. If u is logarithm-free then we may assume $\mathcal{U}=0$, and it follows immediately from (iii) that H_m admits σ as a factor. Conversely, assuming that H_m admits σ as a factor, it follows from (iii) that also $\mathcal{U}$ admits σ as a factor (for any expression (7) for u). We claim that then $\mathcal{U}$ also admits σ^{k+1} as a factor for all $k\geqslant 1$. In order to establish this claim by induction assume that $\mathcal{U}$ admits σ^k as a factor, and define a smooth function S by

$$2k[\langle \operatorname{grad}\sigma, dS\rangle+(\mu+m+k+2)S]=L(\sigma^{-k}\mathcal{U})-\sigma^{-k}L\mathcal{U}$$

(the expression on the right is smooth because of the induction hypothesis and (i)). It is then easily seen that the smooth quantity $T=\sigma^{-k}\mathcal{U}+S\sigma$ satisfies

$$\langle \operatorname{grad}\sigma, dT\rangle+(\mu+m+k)T=0,$$

from which it follows that $T=0$. Hence $\mathcal{U}$ admits σ^{k+1} as a factor, and the induction is complete. The property of $\mathcal{U}$ we have obtained implies that $\sigma^m\log|\sigma|\,\mathcal{U}$ is smooth, and then the expression

$$u=(U+\sigma^m\log|\sigma|\,\mathcal{U})\sigma^{-m}+0\log|\sigma|$$

shows that u is logarithm-free. □

Notice that Proposition 2.1 does not address the problem of existence of elementary solutions. It was proved by Hadamard that if (M,g) and L are analytic then a real analytic elementary solution u exists. Hence in this case the condition in (iv) that H_m admits σ as a factor is necessary and sufficient for L to admit a logarithm-free elementary solution. This is a generalization of Hadamard's famous criterion for Huygens' principle. It follows from (iv) that the necessity holds in the general smooth case as well. Moreover if L is elliptic or hyperbolic (or equivalently, (M,g) is properly Riemannian or Lorentzian) the sufficiency can be generalized to the smooth case (see [22], [20, §90]), but in the ultrahyperbolic smooth case the existence of elementary solutions seems to be an open problem.

Let (M,g) be smooth harmonic and assume that the Weitzenböck term is constant, $W=\lambda\in\mathbf{C}$. We claim that it follows that the Hadamard coefficients are radial, that is,

$$H_k(x,y)=F_k(\sigma(x,y)), \quad k=0,1,\dots . \tag{12}$$

Under the assumption of (12) the recursion system (6) reduces by means of (2) to the following system of ordinary differential equations:

$$\begin{gathered}2\sigma F_0'+\mu F_0=0, \quad F_0(0)=1,\\ 2\sigma F_k'+(\mu+k)F_k=L_\lambda F_{k-1}, \quad k\geqslant 1.\end{gathered} \tag{13}$$

Both existence and uniqueness of a smooth solution system F_k to this recursion system is easily seen (see for example [1, pp. 208–209]). Together with the unicity of the Hadamard coefficients this justifies (12) (see also [23, Theorem 3.1]).

Assume now in addition that (M,g) is analytic (as noted in §1 this is only a possible restriction in the ultrahyperbolic case). We claim that there exists a *radial* elementary solution. Indeed Hadamard's existence proof shows that (9) with $p\to\infty$ gives a converging expansion for a real analytic function $\mathcal{U}$ satisfying $L\mathcal{U}=0$. Hence this function is radial. We now define R_{m+1} by (9) with $p=m+1$; this function is also radial. By the lemma below it follows that (11), interpreted as an equation for an unknown quantity R_m, admits a radial solution. Inserting this R_m in (8) with $p=m$ we obtain a radial function U, which inserted together with $\mathcal{U}$ into (7) yields a radial elementary solution u.

LEMMA 2.1. *Let (M,g) be an analytic harmonic manifold, and let f be a real analytic radial function. Then the inhomogeneous equation $L_\lambda z=f$ admits a real analytic radial solution z in a neighborhood of $\sigma=0$.*

Proof. The proof is elementary. Since z is required to be radial the equation is an ordinary differential equation by means of (3). This equation is of Fuchsian type and can be solved by standard power series methods. □

3. The Hadamard polynomial

Let (M,g) be a smooth pseudo-Riemannian manifold of even dimension $n=2m+2\geqslant 4$, and let $H_k(x,y,\lambda)$ denote the kth Hadamard coefficient to the Helmholtz operator L_λ.

PROPOSITION 3.1. *The Hadamard coefficients $H_k(x,y,\lambda)$ to L_λ are determined by the Hadamard coefficients $H_k(x,y,0)$ to $L_0=\Delta$ as follows:*

$$H_k(x,y,\lambda)=\sum_{l=0}^{k} H_{k-l}(x,y,0)\frac{\lambda^l}{l!}. \tag{14}$$

Proof. Both sides of (14) obey the recursion system (6). □

In particular, we infer that $H_k(x,y,\lambda)$ is a polynomial of degree k in λ. We now assume that (M,g) is harmonic. Then it is a consequence of the radiality of the Hadamard coefficients that the polynomials $H_k(x,x,\lambda)=F_k(0,\lambda)$ do not depend on x. We put $h(\lambda)=H_m(x,x,\lambda)$, and call it the *Hadamard polynomial.*

THEOREM 3.1. *Let (M,g) be an even-dimensional harmonic manifold, and $\lambda\in\mathbf{C}$ a constant.*

(i) *If λ is an exceptional Helmholtz number then it is a root of the Hadamard polynomial h.*

(ii) *Assume in addition that (M,g) is analytic. Then the roots of the Hadamard polynomial h are exceptional Helmholtz numbers.*

Proof. An immediate consequence of the results in the previous section (alternatively the existence in (ii) follows from Theorem 3.3 below). □

The evaluation of the Hadamard polynomial h from its definition is quite complicated, because it involves the successive determination of the Hadamard coefficients $H_1,...,H_m$ by means of the differential recursive system (6). We shall now derive an algebraic recursion formula for it, which is independent of the Hadamard theory.

We define a sequence of polynomials $h_k(\lambda)$, $k=0,1,...,m$, recursively by $h_0=1$ and

$$kh_k(\lambda)+\sum_{l=0}^{k-1}2^{k-l}\binom{m-l}{k-l}\mu_{k-l}h_l(\lambda)=\lambda h_{k-1}(\lambda),\quad k\geqslant 1, \tag{15}$$

where μ_k denotes the kth derivative of $\mu=\mu(\sigma)$ at $\sigma=0$. Clearly h_k has degree k, for all k.

THEOREM 3.2. *The Hadamard polynomial $h=H_m(x,x,\cdot)$ is identical with the polynomial h_m.*

Proof. Let λ be fixed, and let $U_m(x,y)$ denote the summation term in the expression (8) for U (with $p=m$), that is,

$$U_m(x,y)=\sum_{k=0}^{m-1}2^{-k}\binom{m-1}{k}^{-1}H_k(x,y,\lambda)\frac{\sigma^k}{k!} \tag{16}$$

where $\sigma=\sigma(x,y)$ (notice however that we are not assuming the existence of U). Then $U_m(x,y)$ is smooth and radial. Let u_k for $k=0,\dots,m-1$ be the coefficients in the Taylor decomposition

$$U_m=\sum_{k=0}^{m-1}2^{-k}\binom{m-1}{k}^{-1}u_k\frac{\sigma^k}{k!}+z(\sigma)\sigma^m \tag{17}$$

with z smooth. We claim that

$$u_k=h_k(\lambda) \tag{18}$$

for $k=0,\dots,m-1$. In order to prove this claim we apply L_λ to $U_m\sigma^{-m}$. It follows easily from (3), (13) and (16) that

$$L_\lambda(U_m\sigma^{-m})=2^{1-m}\frac{1}{(m-1)!}(L_\lambda H_{m-1})\sigma^{-1}. \tag{19}$$

On the other hand it follows from (3) and (17) that

$$L_\lambda(U_m\sigma^{-m})=\sum_{k=0}^{m-1}2^{-k}\binom{m-1}{k}^{-1}u_k\frac{1}{k!}[-2(m-k)(\mu(\sigma)+k)\sigma^{k-m-1}+\lambda\sigma^{k-m}]+L_\lambda z.$$

Inserting the Taylor series for $\mu(\sigma)$ at $\sigma=0$ (with a suitable remainder term; recall that $\mu(0)=0$) and reorganizing the terms we get an expression

$$L_\lambda(U_m\sigma^{-m})=\sum_{k=0}^{m-1}2^{1-k}\frac{(m-k)!}{(m-1)!}\left[-ku_k-\sum_{l=0}^{k-1}\binom{m-l}{k-l}2^{k-l}\mu_{k-l}u_l+\lambda u_{k-1}\right]\sigma^{k-m-1}$$
$$+2^{1-m}\frac{1}{(m-1)!}\left[-\sum_{l=0}^{m-1}2^{m-l}\mu_{m-l}u_l+\lambda u_{m-1}\right]\sigma^{-1}+\zeta$$

where ζ is smooth. Now a comparison with (19) yields

$$-ku_k-\sum_{l=0}^{k-1}\binom{m-l}{k-l}2^{k-l}\mu_{k-l}u_l+\lambda u_{k-1}=0 \tag{20}$$

for $k=0,\dots,m-1$, and

$$L_\lambda H_{m-1}=-\sum_{l=0}^{m-1}2^{m-l}\mu_{m-l}u_l+\lambda u_{m-1}+2^{m-1}(m-1)!\,\sigma\zeta. \tag{21}$$

The claimed identity (18) is an immediate consequence of (20) and the definition (15) of h_k.

Furthermore, inserting the identity (18) just obtained into (21) and using the definition of h_m we get $L_\lambda H_{m-1}=mh_m(\lambda)+2^{m-1}(m-1)!\,\sigma\zeta$, that is, $L_\lambda H_{m-1}(x,x)=mh_m(\lambda)$. But from (13) it follows that $L_\lambda H_{m-1}$ equals mH_m on the diagonal. Hence $H_m(x,x)=h_m(\lambda)$, and the theorem is proved. □

COROLLARY 3.1. *Let (M,g) be an even-dimensional analytic harmonic manifold, and let $h_0,\dots,h_m=h$ be the associated polynomials as above. There exists, for each $\lambda\in\mathbf{C}$, a radial elementary solution u_λ to L_λ of the form*

$$u_\lambda(\sigma)=\sum_{k=0}^{m-1}2^{-k}\binom{m-1}{k}^{-1}h_k(\lambda)\frac{\sigma^{k-m}}{k!}+Z_\lambda(\sigma)-2^{-m}\frac{1}{(m-1)!}h(\lambda)\mathcal{V}_\lambda(\sigma)\log\sigma,$$

where the radial functions Z_λ and $\mathcal{V}_\lambda$ are analytic in a neighborhood of $\sigma=0$, and $L\mathcal{V}_\lambda=0$, $\mathcal{V}_\lambda(0)=1$.

Proof. The corollary follows immediately from Proposition 2.1 and the proof of Theorem 3.2. □

Notice that if M is simply harmonic, which means that $\mu=0$ (for example if M is the flat $\mathbf{R}^n$), then it follows from (15) that $h_k(\lambda)=\lambda^k/k!$, and hence exactly $\lambda=0$ is exceptional (for $\mathbf{R}^n$ this was already seen in the introduction).

The absolute terms $h_k(0)$ of the polynomials $h_k(\lambda)$ can be expressed in terms of the derivatives of the van Vleck determinant $D(\sigma)$:

PROPOSITION 3.2. *There holds $h_k(0)=2^k\binom{m}{k}D^{(k)}(0)$ for $k=0,\dots,m$.*

Proof. The van Vleck determinant is determined by

$$\sigma D'+\mu D=0,\quad D(0)=1.$$

Here we differentiate k times and set $\sigma=0$, then

$$kD^{(k)}(0)+\sum_{l=0}^{k-1}\binom{k}{l}\mu_{k-l}D^{(l)}(0)=0,$$

from which the relation to h_k immediately follows. □

As a consequence 0 is an exceptional Helmholtz number if and only if $D^{(m)}(0)=0$ (this could also be seen from the integral formula (4)).

We have seen above (in the analytic case) that each of the operators $L_\lambda=\Delta+\lambda$, where λ is a root of h, admits a logarithm-free elementary solution. The following result shows that a similar statement holds for the product of these operators (roots counted with multiplicities). Up to a constant multiple, this product is the operator $h(-\Delta)$ of degree $2m$.

THEOREM 3.3. *Let (M,g) be an even-dimensional analytic harmonic manifold, and let h be the associated Hadamard polynomial. There exists a real analytic function Z on a neighborhood of $\sigma=0$ such that the radial function u given by*

$$u(\sigma)=\sigma^{-1}+Z(\sigma)$$

solves the equation $h(-\Delta)u=0$.

Proof. For $k=0,1,\ldots$ let r_k denote the real analytic function given by

$$\mu(\sigma)=\sum_{j=1}^{k}\mu_j\frac{\sigma^j}{j!}+r_k(\sigma)\frac{\sigma^{k+1}}{k!}$$

in the Taylor expansion of $\mu(\sigma)$ at $\sigma=0$. Note that $kr_{k-1}=\mu_k+\sigma r_k$ for $k\geqslant 1$. The proof of the theorem is based on the identity

$$kh_k(-\Delta)[\sigma^{k-m-1}]=\sum_{l=0}^{k-1}2^{k-l}\binom{m-l}{k-l}h_l(-\Delta)[\sigma^{k-m}r_{k-l}],\quad k=0,\ldots,m, \tag{22}$$

which can be seen by induction on k as follows. For $k=0$ both sides of (22) vanish. For the induction step one uses that

$$kh_k(-\Delta)=-h_{k-1}(-\Delta)\Delta-\sum_{l=0}^{k-1}2^{k-l}\binom{m-l}{k-l}\mu_{k-l}h_l(-\Delta) \tag{23}$$

by (15), together with the identity (from (3))

$$\Delta\sigma^{k-m-1}=2(k-m-1)(k-1+\mu)\sigma^{k-m-2}=2(k-m-1)(k-1+\sigma r_0)\sigma^{k-m-2};$$

application of (23) to σ^{k-m-1} easily leads to (22), when $(k-1)h_{k-1}(-\Delta)\sigma^{k-m-2}$ is determined by means of the induction hypothesis.

For $k=m$ the right side of (22) is real analytic, hence $h(-\Delta)\sigma^{-1}$ is real analytic. By successive use of Lemma 2.1 there is a real analytic radial solution Z to the inhomogeneous equation $h(-\Delta)Z+h(-\Delta)\sigma^{-1}=0$, on a neighborhood of the diagonal, and the theorem follows. □

In the following sections we shall present an explicit expression for the u in Theorem 3.3 in the case of the isotropic spaces (see Theorem 5.1).

The following can be said on multiple roots. Let the *multiplicity* of an exceptional Helmholtz number denote the largest number $p\leqslant m$ for which the pth power L_λ^p of L_λ admits a logarithm-free elementary solution, that is a solution $u=u(x,y)$ to $L_\lambda^p u=0$ of the form

$$u=U(\sigma)\sigma^{p-1-m}$$

with U smooth.

THEOREM 3.4. *Let (M, g) be an even-dimensional analytic harmonic manifold, and let h be the associated Hadamard polynomial. The following assertions are equivalent:*

(i) *λ is an exceptional Helmholtz number of multiplicity at least p.*

(ii) *$H_k(x, x, \lambda)=0$ for $k=m-p+1, \ldots, m$.*

(iii) *λ is a p-fold root of the Hadamard polynomial h.*

In particular it follows that counted with multiplicities there are exactly m exceptional Helmholtz numbers.

Proof. The equivalence of (i) and (ii) is shown in the forthcoming paper [24]. Item (iii) means by definition that

$$\frac{\partial^k}{\partial \lambda^k} H_m(x, x, \lambda) = 0$$

for all $k \leqslant p-1$. By Proposition 3.1 this is equivalent to item (ii). □

Notice the formal resemblance of the problem of the exceptional Helmholtz numbers and the eigenvalue problem for the Laplace operator. In fact there is a relation between the two problems if M is closed:

THEOREM 3.5. *If two closed properly Riemannian harmonic manifolds of even dimension have the same spectrum then they have the same exceptional Helmholtz numbers with the same multiplicities.*

Proof. It is well known from spectral geometry (see for example [1, p. 215]) that if two closed manifolds are isospectral then they exhibit the same sequence

$$\int_M H_k(x, x, 0)\, d\mathrm{vol}(x) \quad \text{for } k = 0, 1, \ldots .$$

The integrands are constants for harmonic manifolds, thus the sequence reduces to

$$H_k(x, x, 0)\, \mathrm{Vol}\, M \quad \text{for } k = 0, 1, \ldots .$$

By Proposition 3.1 the subsequence for $k \leqslant m$ contains exactly the same information as the polynomial $H_m(x, x, \lambda)$. Now the result follows from Theorem 3.1. □

Notice that the converse of Theorem 3.5 is false: The sphere S^n and the projective space $\mathcal{P}^n(\mathbf{R})$ are locally isomorphic, hence they have the same exceptional Helmholtz numbers (which will be determined below), but they have different spectra.

4. The isotropic spaces

Particular examples of harmonic manifolds are the pseudo-Riemannian *isotropic spaces*. Recall (see [29, p. 377]) that a pseudo-Riemannian manifold (M, g) is called isotropic if it has the following property: Given a point $y_0 \in M$ the group of isometries of (M, g) leaving y_0 fixed acts transitively on the level sets of the quadratic form $Q(\xi) = g(y_0)(\xi, \xi)$ on $T_{y_0}M \setminus \{0\}$. By [29, Lemma 11.6.6] the group $G = I(M)$ of isometries of (M, g) then acts transitively on M.

In order to show that an isotropic manifold is harmonic, notice first that σ is invariant for the action of G on $M \times M$ defined by $a \cdot (x, y) = (a \cdot x, a \cdot y)$, $a \in G$, hence G acts on each level set of σ in $M \times M$. Now the isotropy implies that this action is transitive off the diagonal: If $\sigma(x_1, y_1) = \sigma(x_2, y_2)$ we choose $a \in G$ such that $y_1 = a \cdot y_2$. Then we have

$$\sigma(a^{-1} \cdot x_1, y_2) = \sigma(x_1, y_1) = \sigma(x_2, y_2),$$

hence there exists an element $h \in G$ such that

$$h \cdot y_2 = y_2 \quad \text{and} \quad a^{-1} \cdot x_1 = h \cdot x_2,$$

by the isotropy, and then

$$(x_1, y_1) = ah \cdot (x_2, y_2) \in G(x_2, y_2)$$

as claimed. It follows that a smooth function on $M \times M$ is radial if and only if it is G-invariant, and since the Laplace–Beltrami operator is G-invariant, it maps smooth radial functions to smooth radial functions. Hence M is harmonic.

The isotropic spaces have been classified (see [29, p. 390]). The properly Riemannian isotropic spaces are precisely the *two-point homogenous spaces* (that is, Riemannian manifolds with the property that for any two pairs $x_1, y_1 \in M$, $x_2, y_2 \in M$ satisfying $r(x_1, y_1) = r(x_2, y_2)$ there exists an isometry of M taking x_1 to x_2 and simultaneously y_1 to y_2, see [11, p. 535]). According to the classification, a two-point homogeneous space is (up to local isometry) either a Euclidean space $\mathbf{R}^N$ or a (compact or non-compact) Riemannian symmetric space of rank one (see [11, p. 535]), that is one of the projective, compact spaces $\mathcal{P}^N(\mathbf{A})$ or one of the hyperbolic, non-compact spaces $\mathcal{H}^N(\mathbf{A})$. Here $\mathbf{A}$ is one of the algebras $\mathbf{R}, \mathbf{C}, \mathbf{H}$ or $\mathbf{O}$ of real numbers, complex numbers, quaternions or octonions, respectively, and N is the dimension of the space over this algebra. The real dimension is then given by $n = N\nu$ with $\nu = \dim_{\mathbf{R}} \mathbf{A} = 1, 2, 4, 8$.

More generally the isotropic spaces are (modulo local isometry) the flat spaces $\mathbf{R}^{p,q} = \mathbf{R}^{p+q}$ with the metric of signature (p, q) and the projective hyperbolic spaces

$$\mathcal{H}_p^N(\mathbf{A}) = U(p+1, q; \mathbf{A}) / U(1; \mathbf{A}) \times U(p, q; \mathbf{A})$$

of signature $(\nu p, \nu q)$, where $\mathbf{A}$ is one of the four algebras above, $0 \leqslant p \leqslant N$, $q = N - p$, and where $U(p, q; \mathbf{A})$ denotes the group of (p, q)-pseudoorthogonal matrices over $\mathbf{A}$. Then we have $\mathcal{P}^N(\mathbf{A})=\mathcal{H}_N^N(\mathbf{A})$ and $\mathcal{H}^N(\mathbf{A})=\mathcal{H}_0^N(\mathbf{A})$. For $\mathbf{A}=\mathbf{O}$ only $N=2$ is allowed, $G=U(p+1,q;\mathbf{O})$ is to be interpreted as an exceptional Lie group of type F_4, and $H=U(1;\mathbf{O})\times U(p,q;\mathbf{O})$ as a subgroup of type $SO(9)$ in the definite cases or $SO(1,8)$ in the indefinite case.

In particular, the Lorentzian isotropic spaces are exactly (modulo local isometry) $\mathbf{R}^{1,q}$, $\mathcal{H}_{N-1}^N(\mathbf{R})$ and $\mathcal{H}_1^N(\mathbf{R})$.

The isotropic spaces can geometrically be realized as follows when $\mathbf{A}=\mathbf{R},\mathbf{C},\mathbf{H}$. For $x,y\in\mathbf{A}^{N+1}$ let

$$[x,y]=\bar{y}_0x_0+\ldots+\bar{y}_px_p-\bar{y}_{p+1}x_{p+1}-\ldots-\bar{y}_Nx_N,$$

then

$$\mathcal{H}_p^N(\mathbf{A})=\{x\in\mathbf{A}^{N+1}\mid[x,x]=1\}/U(1;\mathbf{A}),$$

where the action of $U(1;\mathbf{A})$ on $\mathbf{A}^{N+1}$ is given by right multiplication (for $\mathbf{A}=\mathbf{O}$ this realization breaks down, see [28] for a substitute). We prefer to use the pseudo-Riemannian structure on $\mathcal{H}_p^N(\mathbf{A})$ induced by the form $-[\cdot,\cdot]$, of signature $(\nu q,\nu p)$, because this is related to the Killing form of G by a positive multiple. In particular, this normalization of g implies that it is negative definite on the compact properly Riemannian spaces and positive definite on the non-compact properly Riemannian spaces. The real spaces $\mathcal{H}_p^N(\mathbf{R})$ have constant sectional curvature -1 for all N,p. The origin is the point $o=(1,0,\ldots,0)$.

Using the invariance of σ it is easily seen that

$$\sigma(x,y)=\begin{cases}\ \ \frac{1}{2}r^2 & \text{if } |[x,y]|=\cosh r\geqslant 1,\\ -\frac{1}{2}r^2 & \text{if } |[x,y]|=\cos r\leqslant 1,\end{cases}$$

for x in a neighborhood of y. Arguing as in [12, p. 165 and p. 206] one then obtains that

$$\varrho(x,y)=\begin{cases}\left(\dfrac{\sinh r}{r}\right)^{n-1}\cosh^{\nu-1}r & \text{if } \sigma(x,y)\geqslant 0,\\[2ex] \left(\dfrac{\sin r}{r}\right)^{n-1}\cos^{\nu-1}r & \text{if } \sigma(x,y)\leqslant 0.\end{cases}\tag{24}$$

This formula remains true in the case $\mathbf{A}=\mathbf{O}$ since the derivation is only based on calculations with root vectors. In this case we use the pseudo-Riemannian structure given by the Killing form, suitably normalized. In the flat case we have $\varrho(x,y)=1$.

Notice that ϱ is independent of the signature of the metric, but that the pseudo-Riemannian structure that we use is negative definite for the compact two-point homogeneous spaces ($q=0$), so that in this case only $\sigma\leqslant 0$ occurs, and positive definite for the non-compact two-point homogeneous spaces ($p=0$), so that only $\sigma\geqslant 0$ occurs. In all other cases both signs occur.

By insertion of (24) into (4) one obtains an explicit elementary solution of $\Delta u=0$ for the spaces under consideration.

From (5) and (24) we readily obtain

$$2\mu=\begin{cases}(n-1)(r\coth r-1)+(\nu-1)r\tanh r & \text{if } \sigma=\ \tfrac{1}{2}r^2>0,\\ (n-1)(r\cot r-1)-(\nu-1)r\tan r & \text{if } \sigma=-\tfrac{1}{2}r^2<0.\end{cases} \tag{25}$$

5. The exceptional Helmholtz numbers for the isotropic spaces

Let (M,g) be one of the isotropic spaces $\mathcal{H}_p^N(\mathbf{A})$ of even real dimension $n=\nu N=2m+2\geqslant 4$, equipped with the pseudo-Riemannian metric as above. Define the numbers $\lambda_1,\dots,\lambda_m$ by

$$\lambda_k=2k(2m+\nu-2k),\quad k=1,\dots,m, \tag{26}$$

and let P be the polynomial

$$P(\lambda)=(\lambda+\lambda_1)\cdot\ldots\cdot(\lambda+\lambda_m). \tag{27}$$

THEOREM 5.1. *Let*

$$u=u(x,y)=\begin{cases}\sin^{-2}r & \textit{if } \sigma(x,y)=-\tfrac{1}{2}r^2<0,\\ -\sinh^{-2}r & \textit{if } \sigma(x,y)=\ \tfrac{1}{2}r^2>0,\end{cases}$$

for $\sigma(x,y)\neq 0$. Then

$$P(\Delta)u=0 \tag{28}$$

for $\sigma\neq 0$. Moreover, let S be a subset of $\{1,\dots,m\}$, with l elements, $l<m$. Then there exists a smooth radial function $U(\sigma)$ on a neighborhood of $\sigma=0$ such that

$$\prod_{k\in S}(\Delta+\lambda_k)u=U(\sigma)\sigma^{-l-1} \tag{29}$$

and $U(0)\neq 0$.

Remark. Notice that we have the simple expression $u(x,y)=(1-|[x,y]|^2)^{-1}$ except in the case $\mathbf{A}=\mathbf{O}$ where it does not make sense.

Proof. It is easily seen from (3) and (25) that the radial part of Δ is given by

$$\begin{cases}\dfrac{d^2}{dr^2}+[(n-1)\coth r+(\nu-1)\tanh r]\dfrac{d}{dr} & \text{if } \sigma>0,\\ -\dfrac{d^2}{dr^2}-[(n-1)\cot r-(\nu-1)\tan r]\dfrac{d}{dr} & \text{if } \sigma<0.\end{cases} \tag{30}$$

Let u_k be the radial function $u_k(\sigma)=u(\sigma)^k$ for $k=1,2,\ldots$, where u is as above, then $u_1=u$, and a straightforward computation shows that

$$\Delta u_k = 4k(m-k)u_{k+1}-\lambda_k u_k. \tag{31}$$

Now (28) is immediate. Moreover it follows from (31) that

$$\prod_{k\in S}(\Delta+\lambda_k)u_1$$

is a linear combination of $u_1,\ldots,u_{l+1}$, and that the coefficient to u_{l+1} is given by $c_l=\prod_{k=1}^{l}4k(m-k)$. This shows (29). □

COROLLARY 5.1. *The exceptional Helmholtz numbers for* (M,g) *are the numbers given by* (26).

Hence $\lambda\mapsto P(-\lambda)$ is identical with the Hadamard polynomial, up to a constant factor (which can be determined from Proposition 3.2 and (24)).

Proof. It follows from (28) that (29) is a logarithm-free elementary solution for the operator $\prod_{k\notin S}(\Delta+\lambda_k)$, up to a constant multiple. This shows that the numbers λ_k are exceptional, with at least the multiplicity by which they appear in this list. Now Theorem 3.4 ensures that there are no other exceptional Helmholtz numbers. □

Remark. It is easily seen that (30) with the substitution

$$z=\begin{cases} \sin^2 r & \text{if } \sigma=-\tfrac{1}{2}r^2\leqslant 0,\\ -\sinh^2 r & \text{if } \sigma=\ \tfrac{1}{2}r^2\geqslant 0,\end{cases}$$

becomes the operator

$$4z(z-1)\frac{d^2u}{dz^2}+2(\nu z+n(z-1))\frac{du}{dz}.$$

Hence $(\Delta+\lambda)u=0$ becomes a hypergeometric equation, so that for each λ an elementary solution can be given explicitly in terms of a hypergeometric function. The expansion at $z=0$ can then be deduced from [5, p. 63, (18)]. In this fashion an alternative proof of Corollary 5.1 can be obtained, independently of the results in §3. See also [6, pp. 382–384].

6. The fundamental solutions

In the previous section we explicitly determined Hadamard's elementary solutions $u=u(x,y)$ for the equations $(\Delta+\lambda_k)u=0$ on the isotropic spaces. From these the fundamental solutions, that is, distributions u on M satisfying $(\Delta+\lambda_k)u=\delta_o$ where δ_o is the

Dirac measure centered at the origin, can be derived. Here we do this in the properly Riemannian case. We also point out the relations to Helgason's results mentioned in the introduction.

Let (M,g) be a properly Riemannian isotropic space, that is, a two-point homogeneous space, of even dimension $n=2m+2\geqslant 4$. Assume that M is $\mathcal{P}^N(\mathbf{A})$ or $\mathcal{H}^N(\mathbf{A})$ with $n=N\nu$, and let g be normalized as earlier. In particular this means that it is negative definite in the compact cases and positive definite otherwise. Let $y=o\in M$ be the origin, and δ_o the Dirac measure centered at y. Let u be defined on $M\setminus\{y\}$ as in Theorem 5.1, that is,

$$u(x)=\begin{cases} \sin^{-2} r(x,y) & \text{if } M \text{ is compact,} \\ -\sinh^{-2} r(x,y) & \text{if } M \text{ is non-compact,} \end{cases}$$

and let $\lambda_1,\ldots,\lambda_m$ and the polynomial P be defined as in the previous section.

THEOREM 6.1. *The function u on $M\setminus\{y\}$ is locally integrable on M with respect to the Riemannian measure. This function, considered as a distribution on M, satisfies $P(\Delta)u=c\delta_o$ for some non-zero constant c. Thus $(1/c)u$ is a fundamental solution for $P(\Delta)$.*

Proof. For a radial function $f(x)=F(r(x,y))$ on M its integral with respect to the Riemannian measure is given by

$$\int_M f(x)\,dx=\int_0^\infty F(r)A(r)\,dr \tag{32}$$

in the non-compact cases, where $A(r)=\Omega_n\varrho(r)r^{n-1}$ with Ω_n the area of the unit sphere S^{n-1}. In the compact cases we have the same formula, except that the upper limit of the integral is replaced by $\frac{1}{2}\pi$. It follows that u is locally integrable.

The function $u_k(\sigma)=u(\sigma)^k$ is locally integrable when $k\leqslant m$, hence it can be considered as a distribution on M. We claim that for $k<m$ the equation (31) holds on M in the distributional sense, whereas for $k=m$ it has to be replaced by

$$\Delta u_m=c_m\delta_o-\lambda_m u_m \tag{33}$$

for some constant $c_m\neq 0$. To show these claims it suffices to apply the distributions to radial test functions, because the distributions themselves are radial. Then the assertions follow by means of (32) and (24), using the following formula of partial integration:

$$\int_a^b((\Delta\phi)\psi-\phi(\Delta\psi))(r)A(r)\,dr=\pm[(\phi'\psi-\phi\psi')(r)A(r)]_{r=a}^b$$

with $\pm=\operatorname{sign}\sigma$. Clearly the theorem is implied by (31) and (33). □

For the case $\mathbf{A}=\mathbf{R}$ the statement of Theorem 6.1 can also be derived quite easily from Helgason's inversion formula for the $2m$-dimensional Radon transform on M, [12, Chapter I, Theorems 4.5, 4.7] (see for example loc. cit., (28)). For the case $\mathbf{A}=\mathbf{C}$ it similarly follows from loc. cit., Theorems 4.11, 4.17. The proof given above for the general case is essentially identical with Helgason's proofs for these two cases. In the remaining cases, $\mathbf{A}=\mathbf{H}, \mathbf{O}$, Theorem 6.1 cannot be derived directly from the corresponding results of Helgason.

The fundamental solution for each individual operator $\Delta+\lambda_k$ is immediately found from Theorem 6.1, one only has to apply the remaining factors of $P(\Delta)$ to $(1/c)u$.

Notice that the equation $P(\Delta)u=c\delta_o$ is not valid for the real sphere S^n, the double cover of $\mathcal{P}^n(\mathbf{R})$. Here one has to replace the upper limit in the integral (32) by π instead of $\frac{1}{2}\pi$, with the result that δ_o should be replaced by $\delta_o+\delta_a$ where δ_a is the Dirac measure centered at the antipodal point of o.

Let us now turn to the Lorentzian isotropic spaces. In this case the fundamental solutions for the exceptional Helmholtz operators $\Delta+\lambda_k$ can be found from Helgason's inversion formula, [12, Chapter I, Theorem 6.17], for the orbital integrals on M.

Let M be one of the Lorentzian spaces G/H where (G,H) is one of the symmetric pairs $(SO_0(n,1), SO_0(n-1,1))$ or $(SO_0(2,n-1), SO_0(1,n-1))$, $n=2m+2\geqslant 4$. Then M is a two-fold cover of $\mathcal{H}^n_{n-1}(\mathbf{R})$ or $\mathcal{H}^n_1(\mathbf{R})$, respectively. We geometrically realize M as the hyperbolic space

$$M=\{x\in\mathbf{R}^{n+1}\mid [x,x]=x_1^2+x_2^2+\varepsilon(x_3^2+...+x_n^2)-x_{n+1}^2=1\}$$

with $\varepsilon=1,-1$ for the two cases, and we use the metric of signature $(1,n-1)$, respectively $(n-1,1)$, given by the form $-[x,x]$. The origin is $o=(1,0,...,0)$. Let $\Gamma\subset M$ be given by

$$\Gamma=\{x\in M\mid \sigma(x,o)=0\}=\{x\in M\mid x_1=1\},$$

and let $\Gamma_+\subset\Gamma$ consist of those points $x\in\Gamma$ for which $x_{n+1}>0$ if $\varepsilon=1$ and $x_2>0$ if $\varepsilon=-1$. It is easily seen that H acts transitively on Γ_+. The closure of Γ_+ is called the boundary of the forward light cone, or sometimes just the light cone.

LEMMA 6.1. *There is a unique (up to constant factors) positive H-invariant measure α_0 on Γ_+.*

Proof. Consider the point $\gamma_+=(1,1,0,...,0,1)$ in Γ_+. We will determine its stabilizer in H.

Let L be the stabilizer in H of the point $(1,0,...,0,1)$, respectively $(1,1,0,...,0)$, then $L\simeq SO(n-1)$ and L is a maximal compact subgroup of H. Let $A\subset H\subset G$ be the one-parameter subgroup of $(n+1)\times(n+1)$-matrices of the form

$$a_s=\begin{pmatrix}1&&&&&\\&\cosh s&&&&\sinh s\\&&1&&&\\&&&\ddots&&\\&&&&1&\\&\sinh s&&&&\cosh s\end{pmatrix},\quad s\in\mathbf{R}$$

(with zeroes outside the indicated entries), then $a_s\cdot\gamma_+=(1,e^s,0,\dots,0,e^s)$. It is easily seen that there is an Iwasawa decomposition $H=LAN$ of H such that N stabilizes γ_+. The stabilizer in L of γ_+ is the subgroup $M_0\simeq SO(n-2)$ of matrices of the form

$$\begin{pmatrix}1&&&\\&1&&\\&&v&\\&&&1\end{pmatrix}$$

with $v\in SO(n-2)$. The stabilizer in H of γ_+ is now determined as M_0N.

Since both H and M_0N are unimodular the lemma follows from [12, Chapter I, Theorem 1.9]. More explicitly, it follows from loc. cit., Proposition 5.1, that we can take

$$\int f\,d\alpha_0=\int_{-\infty}^{\infty}\int_L f(la_s\cdot\gamma_+)\,dl\,e^{(n-2)s}\,ds, \tag{34}$$

for $f\in C(\Gamma_+)$, where dh, dl are Haar measures on H and L, respectively, and ds denotes the Lebesgue measure on $\mathbf{R}$. □

THEOREM 6.2. *Let P be the polynomial defined by* (27), *and let α_0 be the above measure on Γ_+, considered as a distribution on M. Then $P(\Delta)\alpha_0=c\delta_o$ for some non-zero constant c. Thus $(1/c)\alpha_0$ is a fundamental solution for $P(\Delta)$.*

As a consequence $P(\Delta)$, as well as all its factors, obeys *Huygens' principle*: the fundamental solution is supported on the closure of Γ_+.

In the proof of Theorem 6.2 we shall need the following lemma. For $r>0$ let $x_r^{\pm1}\in M$ be given by $x_r^1=(\cosh r,0,\dots,0,\sinh r)$ and $x_r^{-1}=(\cos r,\sin r,0,\dots,0)$ then these points are both located at distance r from o. Let $\mathbf{S}_r^{\pm1}=H\cdot x_r^{\pm1}\subset M$ denote the H-orbits through x_r^1 and x_r^{-1}, then these sets are two of the four components of the set of points at distance r from o. It is easily seen in the two cases $\varepsilon=\pm1$ that $\mathbf{S}_r^\varepsilon$ approaches Γ_+ as $r\to0$. The isotropy subgroup of H at x_r^ε is the maximal compact subgroup $L=SO(n-1)$, in the two cases, and hence there is a unique (up to constant multiplication) H-invariant positive measure α_r on $\mathbf{S}_r^\varepsilon$ given by

$$\int_{\mathbf{S}_r^\varepsilon} f(x)\,d\alpha_r(x)=\int_H f(h\cdot x_r^\varepsilon)\,dh. \tag{35}$$

LEMMA 6.2. *Let the measures α_r be determined by* (35) *with a fixed Haar measure dh on H. Then*

$$\lim_{r\to 0} r^{n-2}\int_{S_r^\varepsilon} f(x)\,d\alpha_r(x) = c\int_{\Gamma_+} f(x)\,d\alpha_0(x) \tag{36}$$

for $f\in C_c(M)$, with a suitable positive constant c.

Proof. It follows from the Cartan decomposition of H and [12, Chapter I, Theorem 5.8] that (35) equals

$$c\int_0^\infty\int_L f(la_s\cdot x_r^\varepsilon)\,dl\,\sinh^{n-2}s\,ds,$$

which is

$$c\int_0^\infty\int_L f(l\cdot(\cosh r, \sinh s\sinh r, 0, \dots, 0, \cosh s\sinh r))\,dl\,\sinh^{n-2}s\,ds$$

if $\varepsilon=1$, and

$$c\int_0^\infty\int_L f(l\cdot(\cos r, \cosh s\sin r, 0, \dots, 0, \sinh s\sin r))\,dl\,\sinh^{n-2}s\,ds$$

if $\varepsilon=-1$. After the substitutions $e^t=\cosh s\sinh r$, respectively $e^t=\cosh s\sin r$, dominated convergence shows that both these integrals, when multiplied by r^{n-2}, converge to c times (34). □

Remark. The following alternative proof of Lemma 6.2 was suggested by Professor Helgason: By [12, p. 216, (31)] and its proof the limit in (36) exists and gives a positive H-invariant measure α supported on the closure $\Gamma_+\cup\{o\}$ of Γ_+. Its restriction to Γ_+ must equal $c\alpha_0$. Writing $f=f\chi+f(1-\chi)$ where χ is the characteristic function of the ball of radius η around o, we have $\alpha(f(1-\chi))=c\alpha_0(f(1-\chi))$. The proof of [12, p. 216, (31)] with $K=\eta$ shows that $\alpha(f\chi)$ is bounded by $C\eta$ for some constant C. Since $\alpha_0(f\chi)$ is also bounded by $C\eta$ we obtain $\alpha(f)=c\alpha_0(f)$ in the limit $\eta\to 0$.

Proof of Theorem 6.2. Let $f\in\mathcal{D}(M)$ be a test function. We must show that $\alpha_0(P(\Delta)f)=cf(o)$. By [12, Chapter I, Theorem 6.17] we have

$$f(o)=(4\pi)^{-m}\frac{1}{(m-1)!}\lim_{r\to 0} r^{n-2}P(\Delta_r)F(r) \tag{37}$$

where Δ_r is the radial part of Δ and

$$F(r)=\int_H f(h\cdot x_r^\varepsilon)\,dh$$

is the *orbital integral* of f. By loc. cit., Theorem 6.13 and Corollary 6.16, (37) is equivalent to

$$f(o)=(4\pi)^{-m}\frac{1}{(m-1)!}\lim_{r\to 0} r^{n-2}\int_H (P(\Delta)f)(h\cdot x_r^\varepsilon)\,dh$$

(see also loc. cit., (34)). The theorem now follows, by Lemma 6.2 above. □

Remark. For the spaces $\mathcal{H}_p^n(\mathbf{R})$ with p odd and n even, a generalization of Theorem 6.2 can be derived from [16, Theorem 2.37]. Again $P(\Delta)$ admits a singularly supported fundamental solution.

7. Further remarks

For the spaces $\mathcal{H}_p^n(\mathbf{R})$ of constant curvature there is a parametrization of the numbers $\lambda_1, \dots, \lambda_m$ which is more convenient than (26):

LEMMA 7.1. *For $\nu=1$ the exceptional Helmholtz numbers are also given by*

$$\lambda=m(m+1)-l(l+1),\quad l=0,1,\dots,m-1.$$

In particular, all multiplicities equal one.

Proof. The transformation from (26) reads

$$l=\begin{cases} m-2k & \text{if } 2k\leqslant m,\\ 2k-m-1 & \text{if } 2k>m.\end{cases}$$ □

Notice that for $\nu>1$ it follows from (26) that some of the exceptional Helmholtz numbers may have multiplicity 2 (but not higher). More precisely, the values of (26), for $k=\frac{1}{2}\nu+j-1$ with $1\leqslant j\leqslant \frac{1}{4}(n-\nu)$, occur twice.

Returning to the real case considered in Lemma 7.1, we obtain for $l=0$ the exceptional Helmholtz number $\lambda=m(m+1)$ which can also be written as $-(n-2)R/4(n-1)$, where R is the scalar curvature (which is related to the sectional curvature $K=-1$ by $R=n(n-1)K$). In this case the Helmholtz operator

$$\Delta+\lambda=\Delta-\frac{(n-2)R}{4(n-1)} \tag{38}$$

coincides with the *conformal Laplacian.* It has been known for long by an argument of conformal invariance, that (38) admits a logarithm-free elementary solution. More generally, the same conformal transformations relate the exceptional operators $\Delta+\lambda_k$ to a set of Huygens-type operators constructed by Stellmacher (see [26]). Since the property of having a logarithm-free elementary solution is conformally invariant (see [18], [17] or [13, §6]), the statement that the λ_k found here (for $\nu=1$) are exceptional is thus equivalent with the well known fact that Stellmacher's operators obey Huygens' principle (see [22, Proposition 3.4] for details).

For low n, we get the following exceptional Helmholtz operators for $\mathcal{H}_p^n(\mathbf{R})$ (any p):

$$\begin{aligned} n=4: &\quad \Delta-\tfrac{1}{6}R, \\ n=6: &\quad \Delta-\tfrac{1}{5}R, \quad \Delta-\tfrac{2}{15}R, \\ n=8: &\quad \Delta-\tfrac{3}{14}R, \quad \Delta-\tfrac{5}{28}R, \quad \Delta-\tfrac{3}{28}R. \end{aligned}$$

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