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Uniformization of Kähler manifolds with vanishing Bochner tensor

by

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Dedicated to Professor Frank Raymond for his sixtieth birthday

Introduction

In 1948, S. Bochner introduced a curvature tensor on Hermitian manifolds [1]. He defined it as an analogue to the Weyl conformal curvature tensor. When, on a Riemannian manifold M^n, the Weyl conformal curvature tensor $(n>3)$ or the Schouten–Weyl tensor $(n=3)$ vanishes, then M^n is said to be a conformally flat manifold. In this case, M^n can be uniformized over the n-sphere S^n with respect to the group of conformal transformations $\mathrm{Conf}(S^n)$. It is natural in Geometry to determine the class of compact Kähler manifolds for which the Bochner curvature tensor vanishes. The Bochner curvature tensor B on a complex manifold with a Kähler metric is defined as follows:

$$B_{\alpha\bar\beta\varrho\bar\sigma} = R_{\alpha\bar\beta\varrho\bar\sigma} - \frac{1}{n+2}(R_{\alpha\bar\beta}g_{\varrho\bar\sigma} + R_{\varrho\bar\beta}g_{\alpha\bar\sigma} + g_{\alpha\bar\beta}R_{\varrho\bar\sigma} + g_{\varrho\bar\beta}R_{\alpha\bar\sigma}) + \frac{R}{(n+1)(n+2)}(g_{\alpha\bar\beta}g_{\varrho\bar\sigma} + g_{\varrho\bar\beta}g_{\alpha\bar\sigma}).$$

Here, $R_{\alpha\bar\beta\varrho\bar\sigma}$ is the curvature tensor, and $R_{\varrho\bar\sigma} = R_{\alpha}{}^{\alpha}{}_{\varrho\bar\sigma}$ and $R = g^{\varrho\bar\sigma}R_{\varrho\bar\sigma}$ are the Ricci tensor and the scalar curvature respectively.

The purpose of this note is to show that when the Bochner curvature tensor B vanishes with respect to a Kähler metric g, the Kähler manifold M^{2n} can be uniformized over the Kähler manifold $Y^n_{\mathbf{C}}$ with a canonical Kähler metric with respect to the transitive group $\mathcal{G}$ consisting of transformations preserving the geometric structure of $Y^n_{\mathbf{C}}$.

Recall that a uniformization of M^{2n} is a maximal collection of charts $\{(\phi_\alpha, U_\alpha)\}_{\alpha\in\Lambda}$ satisfying the following:

(*) $M=\bigcup_{\alpha\in\Lambda} U_\alpha$, $\phi_\alpha: U_\alpha \to \phi_\alpha(U_\alpha)$ $(\subset Y^n_{\mathbf{C}})$ is a homeomorphism,

(**) if $U_\alpha \cap U_\beta \neq \varnothing$, then the local change of coordinates $g_{\alpha\beta}=\phi_\beta\circ\phi_\alpha^{-1}: \phi_\alpha(U_\alpha\cap U_\beta)\to \phi_\beta(U_\alpha\cap U_\beta)$ extends to an element of $\mathcal{G}$.

(See [10] for example.)

We shall consider a local contactization, which in our case is a procedure for obtaining a *CR*-manifold with a characteristic *CR* vector field from a Kähler manifold. Then, the Bochner tensor can be interpreted geometrically as the obstruction for the *CR*-manifold to be locally modelled on the sphere S^{2n+1}, where S^{2n+1} is the boundary of complex hyperbolic space. (Compare that the vanishing of the Weyl tensor implies that a manifold is locally modelled on S^n, where S^n is the boundary of a real hyperbolic space.) The proof uses an idea which goes back to Webster [14]. Webster observed that the Chern–Moser curvature tensor of the *CR*-manifold coincides with the Bochner tensor when $n>1$. Let $R_{,\alpha\beta}$ $(\alpha,\beta=1,\dots,n)$ be the covariant differentiation for the scalar curvature R of M^{2n}. Define a tensor $\hat{R}$ to be $\hat{R}_{\alpha\beta}=R_{,\alpha\beta}$.

We shall prove the following:

THEOREM A. *Let M be a Kähler manifold of dimension $2n$. Suppose that the Bochner tensor B (with respect to the Kähler metric) vanishes when $n>1$ or that the tensor $\hat{R}=0$ when $n=1$. Then M is uniformized over $Y^n_{\mathbf{C}}$ with respect to $\mathcal{G}$. Here $Y^n_{\mathbf{C}}$ is a connected simply connected Kähler manifold with a canonical Kähler metric, equipped with a transitive group of biholomorphic transformations $\mathcal{G}$. More precisely, $(\mathcal{G}, Y^n_{\mathbf{C}})$ is one of the following geometries*:

(1) *a projective geometry* $(PU(n+1), \mathbf{CP}^n)$,

(2) *a similarity geometry* $(\mathbf{C}^n \rtimes (U(n)\times \mathbf{R}^+), \mathbf{C}^n)$,

(3) *a hyperbolic geometry* $(PU(n,1), \mathbf{H}^n_{\mathbf{C}})$,

(4) *a projective-hyperbolic geometry*

$$(PU(m,1)\times PU(n-m+1), \mathbf{H}^m_{\mathbf{C}}\times \mathbf{CP}^{n-m}), \quad m=1,2,\dots,n-1.$$

Remark 1. The above geometries (1), (2) and (3) are subgeometries of the projective geometry $(\mathrm{PGL}_{n+1}(\mathbf{C}), \mathbf{CP}^n)$. Except for (2), the group $\mathcal{G}$ is a transitive group of $Y^n_{\mathbf{C}}$ preserving the canonical Kähler structure. The Euclidean group $E_{\mathbf{C}}(n)=\mathbf{C}^n \rtimes U(n)$ preserves the Kähler structure of $\mathbf{C}^n$ as well.

In the compact case, we derive the following corollary from the theorem.

COROLLARY B. *Let M be a $2n$-dimensional compact Kähler manifold. Suppose that the Bochner tensor B vanishes when $n>1$ or that the tensor $\hat{R}=0$ when $n=1$. Then M is holomorphically isometric to*:

(1) *the complex projective space* $\mathbf{CP}^n$,

(2) *a complex Euclidean space form* $T^n_{\mathbf{C}}/F$, $F\subset U(n)$,
(3) *a complex hyperbolic space form* $\mathbf{H}^n_{\mathbf{C}}/\Gamma$, $\Gamma\subset \mathrm{PU}(n,1)$,
(4) *the fiber space* $\mathbf{H}^m_{\mathbf{C}}\times\mathbf{CP}^{n-m}/\Gamma$ *where*

$$\Gamma\subset PU(m,1)\times PU(n-m+1),\quad m=1,2,...,n-1.$$

Here F is a finite group and Γ is a discrete cocompact subgroup, both acting properly discontinuously.

Remark 2. Similar results have been obtained (cf. [2], [3], [16], [17]) under some assumptions on Ricci tensors, scalar curvatures on compact Kähler manifolds, or on certain relations between Chern classes.

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1. Contactization

Let M be a $2n$-dimensional Kähler manifold with fundamental 2-form Ω. For each $x\in M$, choose an open subset U in M homeomorphic to a ball. Put $\Omega_U=\Omega|_U$, the restriction of Ω to U. Since Ω_U is exact, there exists a 1-form ω_U on U such that $d\omega_U=\Omega_U$. In the product space $M(U)=\mathbf{R}\times U$, we define a 1-form

$$\omega=dt+p^*\omega_U$$

where t is the coordinate on $\mathbf{R}$ and $p: M(U)\to U$ is the projection. Let $\xi=d/dt$ be a vector field induced by the $\mathbf{R}$-action. Since $\omega(\xi)=1$ and $d\omega=p^*\Omega_U$ for the Kähler form Ω_U, it follows that $\omega\wedge(d\omega)^n\neq 0$. Thus ω is a contact form. It is easy to check that:

LEMMA 1.1. (1) *ω is a contact form on $M(U)$.*
(2) *ξ is a characteristic vector field, i.e., $\omega(\xi)=1$ and $d\omega(\xi,V)=0$ for all $V\in TM(U)$.*
(3) *$\mathbf{R}$ acts as contact transformations of $M(U)$.*

1.2. Let J be a complex structure on M. When $TM\otimes\mathbf{C}=T^{1,0}+T^{0,1}$ is the canonical splitting, it implies that $[T^{1,0},T^{1,0}]\subset T^{1,0}$. If we put $\mathrm{Null}\,\omega=\{X\in TM(U)\,|\,\omega(X)=0\}$, then it is a codimension 1 subbundle of the tangent bundle $TM(U)$. Since $p_*:\mathrm{Null}\,\omega\to TM$

is an isomorphism at each point, we define an almost complex structure $\tilde{J}$ on $\mathrm{Null}\,\omega$ to be the pullback of J by p_*,

$$\tilde{J}(X)=p_*^{-1}\circ J\circ p_*(X) \quad \text{for } X\in \mathrm{Null}\,\omega_{(t,x)}.$$

If $\mathrm{Null}\,\omega\otimes\mathbf{C}=\widetilde{T}^{1,0}+\widetilde{T}^{0,1}$ is a splitting for $\tilde{J}$, then $[\widetilde{T}^{1,0},\widetilde{T}^{1,0}]\subset\widetilde{T}^{1,0}$ because $p_*(\widetilde{T}^{1,0})=T^{1,0}$ and $p_*([X,Y])=[p_*(X),p_*(Y)]$. Therefore $\tilde{J}$ is a complex structure on $\mathrm{Null}\,\omega$. By definition, the pair $(\omega,\tilde{J})$ is a pseudo-Hermitian structure on $M(U)$. (See [4], [8], [13], [14].)

Let $\{\phi_t\}_{|t|<\infty}$ be the one-parameter group $\mathbf{R}$ of contact transformations. Obviously $\tilde{J}\circ(\phi_t)_*=(\phi_t)_*\circ\tilde{J}$ and so ϕ_t is a pseudo-Hermitian diffeomorphism. In this case, ξ is called a characteristic *CR* vector field and $(\omega,\tilde{J},\xi)$ is said to be a standard pseudo-Hermitian structure (cf. [8], [13]).

Now $(\mathrm{Null}\,\omega,\tilde{J})$ is a *CR* structure on $M(U)$ for which $M(U)$ is the trivial principal bundle $\mathbf{R}\to M(U)\xrightarrow{p}U$ over the Kähler manifold U where ξ generates $\mathbf{R}$. At this stage, Webster finds the relationship between the *CR* curvature tensor on the total space and the curvature tensor of the base space, which is crucial to our argument.

THEOREM 1.3 ([14], [13], [4]). *The Bochner tensor B of U coincides with the fourth order Chern–Moser tensor of the strictly pseudoconvex CR-manifold $M(U)$. B vanishes identically when $n=1$.*

Moreover, Chern and Moser defined the second-order curvature tensor Q on a *CR*-manifold, which satisfies $Q=(n+1)^{-1}(n+2)^{-1}\widehat{R}$, i.e., $Q_{\alpha\beta}=(n+1)^{-1}(n+2)^{-1}R_{\ ,\alpha\beta}$ in this case.

In general, the fourth order Chern–Moser curvature tensor of a *CR*-manifold M^{2n+1} vanishes when $n>1$ (the tensor Q vanishes when $n=1$) if and only if M^{2n+1} can be uniformized over the $(2n+1)$-dimensional sphere S^{2n+1} with respect to the group of *CR* transformations $\mathrm{Aut}_{CR}(S^{2n+1})$. The group $\mathrm{Aut}_{CR}(S^{2n+1})$ is isomorphic to the group of biholomorphic transformations $PU(n+1,1)$ of complex hyperbolic space $\mathbf{H}^{n+1}_{\mathbf{C}}$. The sphere S^{2n+1} is viewed as the boundary of $\mathbf{H}^{n+1}_{\mathbf{C}}$. In this case, M^{2n+1} is said to be a spherical *CR*-manifold. Let $\mathrm{Aut}_{CR}(M^{2n+1})$ denote the group of all *CR* transformations of M^{2n+1} onto itself. The usual monodromy argument shows that there exists a homomorphism $\varrho\colon\mathrm{Aut}_{CR}(\widetilde{M})\to\mathrm{Aut}_{CR}(S^{2n+1})$ such that $\pi_1(M)\subset\mathrm{Aut}_{CR}(\widetilde{M})$ and an immersion $\mathrm{dev}\colon\widetilde{M}\to S^{2n+1}$ which preserves the *CR* structure, where $\widetilde{M}$ is the universal covering space of M. Note that:

1.4. Given a spherical *CR* structure, the developing pair (ϱ,dev) is uniquely determined up to conjugacy by an element of $PU(n+1,1)$.

2. Fibration of spherical *CR*-manifolds

Let $(\mathrm{Null}\,\omega, \tilde{J})$ be a *CR* structure on $M(U)$ for an open subset U of M. Note that $\mathbf{R}\subset \mathrm{Aut}_{CR}(M(U))$.

Suppose that the Bochner tensor B vanishes on M^{2n} ($\widehat{R}=0$ when $n=1$) with respect to a Kähler metric. Then, it follows from Theorem 1.3 that $M(U)$ is a spherical *CR*-manifold. Since $M(U)$ is simply connected, we have a developing pair

$$(\varrho, \mathrm{dev})\colon (\mathbf{R}, M(U)) \to (PU(n+1,1), S^{2n+1}).$$

We can assume that $\mathrm{dev}\colon 0\times U \to \mathrm{dev}(0\times U)$ is a homeomorphism for each U.

Notation 2.1. Let G be the closure of the holonomy $\varrho(\mathbf{R})$ in $PU(n+1,1)$.

LEMMA 2.2. *Let* $(\varrho_i, \mathrm{dev}_i)\colon (\mathbf{R}, M(U_i))\to(PU(n+1,1), S^{2n+1})$ *be developing pairs for* $i=1,2$. *If* $U_1\cap U_2\neq\varnothing$, *then they define the same CR structure, i.e., there is an element* $h\in PU(n+1,1)$ *such that* $\mathrm{dev}_2=h\circ\mathrm{dev}_1$ *on* $M(U_1\cap U_2)$. *Moreover,* $G_2=h\cdot G_1\cdot h^{-1}$ *where* $G_i=\overline{\varrho_i(\mathbf{R})}$.

Proof. Choose an open subset homeomorphic to a ball in each component of $U_1\cap U_2$ and let W be a finite union of such open subsets. Let $(\omega_i, \tilde{J}_i)$ be pseudo-Hermitian structures on $M(U_i)$ for $i=1,2$. Since $d(\omega_1-\omega_2)=0$ on $M(U_1\cap U_2)$, the 1-form $\omega_1-\omega_2$ represents a cocycle in $H^1(M(W);\mathbf{R})$ which is zero, so there is a smooth map $\chi\colon M(W)\to\mathbf{R}$ with $\omega_1-\omega_2=d\chi$. Define a diffeomorphism $f\colon M(W)\to M(W)$ as $f(t,w)=(t+\chi(w),w)$. Then it is easy to see that $f_*(d/dt)=d/dt$ and $f^*\omega_2=\omega_1$ on $M(W)$. Since $p\circ f=p$ for the projection $p\colon M(W)\to W$, it follows from the definition that $\tilde{J}_2\circ f_*=f_*\circ\tilde{J}_1$ on $\mathrm{Null}\,\omega_1$. Therefore f is a pseudo-Hermitian diffeomorphism and so $(\mathrm{Null}\,\omega_i, \tilde{J}_i)$ define the same *CR* structure on $M(W)$. Since $(\varrho_i, \mathrm{dev}_i)$ is a developing pair for $(\mathrm{Null}\,\omega_i, \tilde{J}_i)$ $(i=1,2)$, restricted to $M(W)$, 1.4 implies that for some element $h\in PU(n+1,1)$,

$$\mathrm{dev}_2 = h\circ\mathrm{dev}_1 \quad \text{on } M(W), \tag{2.3}$$

$$\varrho_2(t) = h\cdot\varrho_1(t)\cdot h^{-1} \quad \text{for } t\in\mathbf{R}. \tag{2.4}$$

Recall that $S^{2n+1}=PU(n+1,1)/PU(n+1,1)_\infty$ is analytic where $PU(n+1,1)_\infty$ is the stabilizer at the point $\{\infty\}$, and the local change

$$\mathrm{dev}_2\circ\mathrm{dev}_1^{-1}\colon \mathrm{dev}_1(0\times U_1\cap U_2)\to\mathrm{dev}_2(0\times U_1\cap U_2)$$

is a smooth map on the domains of S^{2n+1}, so it is a restriction of an analytic map. By (2.3), $\mathrm{dev}_2\circ\mathrm{dev}_1^{-1}=h$ on $\mathrm{dev}_1(0\times W)$; it follows that $\mathrm{dev}_2=h\circ\mathrm{dev}_1$ on $0\times U_1\cap U_2$. Since dev_i is equivariant with respect to ϱ_i and by (2.4), $\mathrm{dev}_2=h\circ\mathrm{dev}_1$ on $M(U_1\cap U_2)$. □

22−945202 *Acta Mathematica* 172. Imprimé le 28 juin 1994

Let $(\varrho, \mathrm{dev})\colon (\mathbf{R}, M(U)) \to (PU(n+1,1), S^{2n+1})$ and G be as above.

If $\mathcal{F}$ is the fixed point set in the sphere with respect to the group G, then we have $\mathrm{dev}^{-1}(\mathcal{F}) = \varnothing$, because $\mathbf{R}$ acts freely on $M(U)$ and dev is an immersion. In particular it follows that

$$\mathrm{dev}(M(U)) \subset S^{2n+1} - \mathcal{F} \quad \text{and} \quad G \subset \mathrm{Aut}_{CR}(S^{2n+1} - \mathcal{F}). \tag{2.5}$$

We examine the fixed point set of G in S^{2n+1} using the result of [9]. Recall that if G is noncompact, then G has a fixed point $\{\infty\}$ in S^{2n+1}. If G is compact, then either G has no fixed points on S^{2n+1} (in this case, up to conjugacy G has a unique fixed point at the origin of hyperbolic space $\mathbf{H}_{\mathbf{C}}^{n+1}$ so that $G \subset U(n+1)$, the maximal compact subgroup of $PU(n+1,1)$), or G has the fixed point set S^{2m-1} up to conjugacy for each $m = 1, 2, \ldots, n$. Here S^{2m-1} is the boundary of the totally geodesic subspace $\mathbf{H}_{\mathbf{C}}^{m}$ of $\mathbf{H}_{\mathbf{C}}^{n+1}$. These are all the possible cases of fixed point sets for G. Given a CR structure, the developing pair is unique up to conjugacy from 1.4, so we fix those fixed point sets. If the developing pairs (ϱ, dev) define the same CR structure, then the corresponding groups G are conjugate so that the fixed point sets $\mathcal{F}$ are isomorphic. In particular, each fixed point set as above is mutually distinct. From (2.5) now follows

PROPOSITION 2.6. *For the spherical CR structure* $(\mathrm{Null}\,\omega, \tilde{J}, \xi)$ *on* $M(U)$, *either the developing pair* (ϱ, dev) *satisfies that* $\mathrm{dev}(M(U)) \subset S^{2n+1}$ *for which* $G \subset U(n+1)$, *or it determines a refinement* $(\mathrm{Aut}_{CR}(X), X)$ *uniquely, where* X *is one of the domains* $S^{2n+1} - \{\infty\}$, $S^{2n+1} - S^{2m-1}$ $(m = 1, 2, \ldots, n)$. *None of these are CR equivalent.*

2.7. *Model space* $(\mathcal{G}, Y_{\mathbf{C}})$. We consider fibrations for X of Proposition 2.6. First, for the sphere S^{2n+1} with a canonical metric, put $S^1 = \mathcal{Z}U(n+1)$, the center of $U(n+1)$. We have an equivariant fibration:

$$(S^1, S^1) \to (U(n+1), S^{2n+1}) \xrightarrow{\nu} (PU(n+1), \mathbf{CP}^n). \tag{2.8}$$

The projective space $\mathbf{CP}^n$ carries the Fubini-Study metric.

Suppose that G has a fixed point $\{\infty\}$ in S^{2n+1}. The complement $S^{2n+1} - \{\infty\}$ is CR equivalent to the Heisenberg nilpotent Lie group $\mathcal{N}$ such that $G \subset \mathrm{Aut}_{CR}(\mathcal{N})$. Here $\mathcal{N}$ has the central extension $1 \to \mathcal{R} \to \mathcal{N} \to \mathbf{C}^n \to 1$ and $\mathrm{Aut}_{CR}(\mathcal{N}) = \mathcal{N} \rtimes (U(n) \times \mathbf{R}^+)$ (cf. [6], [12]). As $\mathcal{R}$ is a normal subgroup of $\mathrm{Aut}_{CR}(\mathcal{N})$, we have an equivariant principal bundle:

$$(\mathcal{R}, \mathcal{R}) \to (\mathcal{N} \rtimes (U(n) \times \mathbf{R}^+), \mathcal{N}) \xrightarrow{\nu} (\mathbf{C}^n \rtimes (U(n) \times \mathbf{R}^+), \mathbf{C}^n). \tag{2.9}$$

Suppose that G has the fixed point set S^{2m-1} for each $m = 1, 2, \ldots, n$. The complement $S^{2n+1} - S^{2m-1}$ is CR equivalent to the quotient of the product of a Lorentz hyperbolic space form and a sphere, $P(V_{-1}^{2m+1} \times S^{2(n-m)+1})$ and, $\mathrm{Aut}_{CR}(S^{2n+1} - S^{2m-1}) =$

$P(U(m,1)\times U(n-m+1))$ (cf. [8], [9], [7]). As $\mathrm{Aut}_{CR}(S^{2n+1}-S^{2m-1})$ has the center $S^1=P(\mathcal{Z}U(m,1)\times\mathcal{Z}U(n-m+1))$, we have an equivariant principal bundle:

$$\begin{aligned}(S^1,S^1)&\longrightarrow(\mathrm{Aut}_{CR}(S^{2n+1}-S^{2m-1}),S^{2n+1}-S^{2m-1})\\&\xrightarrow{\nu}(PU(m,1)\times PU(n-m+1),\mathbf{H}^m_{\mathbf{C}}\times\mathbf{CP}^{n-m}).\end{aligned}\tag{2.10}$$

In particular when $m=n$, $S^{2n+1}-S^{2n-1}=P(V^{2n+1}_{-1}\times S^1)=V^{2n+1}_{-1}$ and

$$(S^1,S^1)\to(U(n+1,1),V^{2n+1}_{-1})\xrightarrow{\nu}(PU(n+1,1),\mathbf{H}^n_{\mathbf{C}})$$

is a principal bundle over the complex hyperbolic space.

3. Construction of uniformization

For simplicity, let $(T,T)\to(H,X)\xrightarrow{\nu}(\mathcal{G},Y_{\mathbf{C}})$ be the principal bundle representing one of (2.8), (2.9), (2.10), where $T=\mathcal{R}$ or S^1. We call $(\mathcal{G},Y_{\mathbf{C}})$ the model space for our uniformization. Let M^{2n} be a Kähler manifold with vanishing Bochner tensor ($\widehat{R}=0$ when $n=1$). It follows from Proposition 2.6 that for an open subset U of a point x, there exists a developing pair $(\varrho,\mathrm{dev})\colon(\mathbf{R},M(U))\to(\mathrm{Aut}_{CR}(X),X)$. Note that $\mathrm{Aut}_{CR}(X)=H$ when $X=\mathcal{N}$, $S^{2n+1}-S^{2m-1}$ and if $X=S^{2n+1}$, H is the maximal compact subgroup in $\mathrm{Aut}_{CR}(X)=PU(n+1,1)$. Since H acts transitively on X, we can assume that $\mathrm{dev}(0\times U)$ is transverse to the fiber T. Then we can define a map $\varphi\colon U\to Y_{\mathbf{C}}$ by setting

$$\varphi(u)=\nu\circ\mathrm{dev}(0,u)\quad\text{for } u\in U.\tag{3.1}$$

Obviously φ is a homeomorphism onto its image. In general, the choice of $Y_{\mathbf{C}}$ is uniquely determined by the developing map dev. If (V,ψ) is another pair with $x\in V$, then there is a developing map dev′ for $M(V)$. Then by Lemma 2.2 and Proposition 2.6, dev′ maps into the same space X as that of dev. Thus ψ maps V into the same space $Y_{\mathbf{C}}$ as φ. For each point of M choose such a pair. Then we obtain a collection of charts $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in\Lambda}$ of M. By the above remark, the collection of charts uniquely determines the model space $Y_{\mathbf{C}}$.

Suppose that $U_\alpha\cap U_\beta\neq\varnothing$. Let $\mathrm{dev}_\alpha,\mathrm{dev}_\beta$ be the developing maps corresponding to $\varphi_\alpha,\varphi_\beta$. When $X=\mathcal{N}$, $S^{2n+1}-S^{2m-1}$ $(m=1,2,\dots,n)$, Lemma 2.2 and Proposition 2.6 imply that both dev_α and dev_β define the same $(\mathrm{Aut}_{CR}(X),X)$-structure for which $\mathrm{dev}_\beta=h\circ\mathrm{dev}_\alpha$. Thus h lies in $H=\mathrm{Aut}_{CR}(X)$. On the other hand, as T $(=\mathcal{R},S^1)$ is normal in H, the map ν takes h into an element $\hat{h}$ in $\mathcal{G}$. It is easy to see that $\varphi_\beta\circ\varphi_\alpha^{-1}=\hat{h}$ on $U_\alpha\cap U_\beta$. Hence, when $(\mathcal{G},Y_{\mathbf{C}})=(\mathbf{C}^n\rtimes(U(n)\times\mathbf{R}^+),\mathbf{C}^n)$, or

$$(PU(m,1)\times PU(n-m+1),\mathbf{H}^m_{\mathbf{C}}\times\mathbf{CP}^{n-m}),\quad m=1,2,\dots,n,$$

the collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ gives a uniformization of M.

Let $X=S^{2n+1}$. Consider the developing maps $\mathrm{dev}_\alpha: M(U_\alpha)\to S^{2n+1}$ and dev_β: $M(U_\beta)\to S^{2n+1}$. The holonomy homomorphisms $\varrho_\alpha, \varrho_\beta$ map $\mathbf{R}$ into $U(n+1)$ by Proposition 2.6. Let G_α, G_β be the closures as before. Then G_α and G_β stabilize a unique point at the origin in complex hyperbolic space $\mathbf{H}_{\mathbf{C}}^{n+1}$. On the other hand, $G_\beta=h\cdot G_\alpha\cdot h^{-1}$ by Lemma 2.2, and the element h also fixes the origin by uniqueness. Hence $h\in U(n+1)$ which induces an element $\hat{h}\in PU(n+1)$. Similarly as above, we have $\varphi_\beta\circ\varphi_\alpha^{-1}=\hat{h}$ on $U_\alpha\cap U_\beta$. Therefore M is uniformizable over $\mathbf{CP}^n$ with respect to $PU(n+1)$.

This proves Theorem A.

3.2. *Proof of Corollary* B. Let M be a $2n$-dimensional compact Kähler manifold. Suppose that the Bochner tensor B vanishes when $n>1$ or that the tensor $\widehat{R}=0$ when $n=1$. Then we have a uniformization from Theorem A, and so the monodromy argument implies that there exists a developing pair $(\varrho, \mathrm{dev})\colon (\pi_1(M), \widetilde{M})\to(\mathcal{G}, Y_{\mathbf{C}}^n)$ (cf. [10]). Note that $\mathcal{G}$ acts as isometries of $Y_{\mathbf{C}}^n$ with respect to the Kähler metric except in case (2). Then it is easy to see that dev is a covering map onto $Y_{\mathbf{C}}^n$ because M is compact. As $Y_{\mathbf{C}}^n$ is simply connected, dev is a homeomorphism. It follows that $M\approx\mathbf{CP}^n$, $\mathbf{H}_{\mathbf{C}}^n/\Gamma$ or $\mathbf{H}_{\mathbf{C}}^m\times\mathbf{CP}^{n-m}/\Gamma$ according to whether (1), (3) or (4) holds.

Consider case (2). If we note that $\mathcal{G}=\mathbf{C}^n\rtimes(U(n)\times\mathbf{R}^+)\subset\mathrm{Sim}(\mathbf{R}^{2n})$, which is the group of similarity transformations, then M is a similarity manifold. It follows from the result of [5] (cf. also [11], [12]) that either M is a (complex) Euclidean space form $\mathbf{C}^n/\Gamma$ where $\Gamma\subset E_{\mathbf{C}}(n)=\mathbf{C}^n\rtimes U(n)$, or some finite covering is a Hopf manifold $(\mathbf{C}^n-\{0\})/\mathbf{Z}^+=S^{2n-1}\times S^1$ where $\mathbf{Z}^+\subset U(n)\times\mathbf{R}^+$. However since $S^{2n-1}\times S^1$ is not Kähler when $n>1$, the latter case does not occur.

Finally consider the case when $n=1$. Suppose M is a closed 2-manifold. In this case, M is always Kähler and some multiple of the fundamental 2-form $\lambda\cdot\Omega$ is integral, i.e., $[\lambda\cdot\Omega]\in H^2(M;\mathbf{Z})$. Then it represents a principal circle bundle $1\to S^1\to V\xrightarrow{p} M$. Using a connection form on V, we have a contact form ω such that $d\omega=p^*\Omega$ (cf. [8]). If $\widehat{R}=0$, then V is a spherical CR 3-manifold by the fact that $Q=\frac{1}{6}\widehat{R}$ (cf. Theorem 1.3). Moreover S^1 acts as CR transformations of V. Note that the Euler class of this bundle is nonzero. Then the classification of [9] shows that V is CR equivalent to the principal circle bundle over a complex curve of constant holomorphic sectional curvature. On the other hand, the Hopf torus $S^1\times S^1$ is not a flat one. Again this case does not occur. This completes the proof of Corollary B.

Problems. (1) Let (ω, J) be a strictly pseudoconvex pseudo-Hermitian structure on M^{2n-1}. There exists a symplectization $W=M\times\mathbf{R}$ where $\Omega=d(t\omega)$. Since ω is a contact form on M, (W,Ω) is a symplectic manifold. Let ξ be a characteristic vector field on

M for ω. Putting $\bar{J}(\xi)=-d/dt$ and $\bar{J}(d/dt)=\xi$, the complex structure J extends to an almost complex structure $\bar{J}$ on W. (Note that if ξ is a characteristic CR vector field (cf. 1.2), then $\bar{J}$ is a complex structure on W.) Thus we have the Bochner tensor B on an almost Kähler manifold W. Then, is there any relationship between the fourth order Chern–Moser tensor $S_{\alpha\bar{\beta}\varrho\bar{\sigma}}$ (cf. [4], [14]) on M^{2n-1} and the Bochner tensor $B_{\alpha\bar{\beta}\varrho\bar{\sigma}}$ on W^{2n}?

(2) As the Weyl conformal tensor is an invariant of the conformal class of Riemannian manifolds, define an equivalence class of Kähler manifolds on which the Bochner tensor B will be an invariant (cf. [15]).

(3) Construct an invariant such as Chern–Simons type (secondary characteristic class) for which the invariant is stationary in the space of equivalence classes of (2) if and only if the Bochner tensor B vanishes.

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