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Classification of amenable subfactors of type II

by

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Contents

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0. Introduction

A central problem arising in the theory of subfactors, after Jones initiated it in the early 80's, is the classification of subfactors $N\subset M$ of finite index of the hyperfinite (or approximately finite dimensional) factors M. Besides its intrinsic interest, the classification of subfactors provides a natural approach for the problem of classification of actions by groups of automorphisms on (single) von Neumann algebras as well. Further motivations towards solving this general problem comes from low dimensional quantum field theories ([L1], see also [FRS], [Fr]), as the main problems in the theory of superselection sectors ([DHR]) can be formulated in terms of the classification of endomorphisms of finite statistics ($=$ index$^{1/2}$) of the hyperfinite factors. The physically relevant invariant for a subfactor, that can be "observed" in practice, is the lattice of its higher relative commutants $\{M_i'\cap M_j\}_{i,j}$ in the Jones tower $N\subset M\subset M_1 \dots$ ([Po5], [GHJ], [J3]). We will call this invariant the *standard invariant* of $N\subset M$ and denote it $\mathcal{G}_{N,M}$. The inclusions between these higher relative commutants are described by Jones' *principal* (or *standard*) *graph* $\Gamma_{N,M}$ ([J3]). An abstract characterisation of the invariant $\mathcal{G}_{N,M}$ is given in [Oc2] as the *paragroup* of all irreducible correspondences (bimodules, see [C3]) generated under Connes' composition (or fusion) rule by $N\subset M$, and $\Gamma_{N,M}$ as its "fusion rule matrix" (a Cayley-type graph).

We prove in this paper that for a large class of subfactors, that we call strongly amenable, this invariant is a complete invariant. All the examples of subfactors coming out this far from quantum field theories and polynomial invariants for knots ([J3], [We1], [We2]) are strongly amenable. Also, the classical problems of classifying actions of amenable discrete groups and compact Lie groups on hyperfinite factors were given equivalent formulations in terms of classification of certain subfactors that are strongly amenable ([Po6], [PoWa]). Our result is in some sense the most general that can be obtained, as we show that strongly amenable subfactors give the largest class of subfactors which can be reconstructed (generated) in a direct way from their standard invariants.

We now state in more details the main results of the paper.

A main concept that we introduce here for studying subfactors is that of representation of inclusions. This provides both the proper set-up for defining a conceptually suitable notion of amenability and the tools for proving the classification results. While for a single von Neumann algebra M a representation is simply an embedding $M\subset\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, for an inclusion of type II_1 factors $N\subset M$ we define a representation as an embedding of $N\subset M$ into an inclusion of von Neumann algebras $\mathcal{N}\overset{\mathcal{E}}{\subset}\mathcal{M}$, with $N\subset\mathcal{N}$, $M\subset\mathcal{M}$ and with $\mathcal{E}:\mathcal{M}\to\mathcal{N}$ a conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ that restricted to M agrees with the trace preserving expectation of M onto N. Typically, both $\mathcal{N}$ and $\mathcal{M}$ are direct sums of algebras of the form $\mathcal{B}(\mathcal{H})$. In particular, the suitably defined standard representation $\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$ of $N \subset M$ is of this type and the inclusion (or multiplicity) matrix of $\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$ coincides with $\Gamma_{N,M}$, a fact that justifies our terminology. The inclusion $N \subset M$ is called *amenable* if whenever represented smoothly in some $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ there exists an M-hypertrace φ on $\mathcal{M}$ (in the sense of [C3]) satisfying $\varphi = \varphi \circ \mathcal{E}$. This is equivalent to the existence of a norm one projection of $\mathcal{N} \subset \mathcal{M}$ onto $N \subset M$. $N \subset M$ is *strongly amenable* if in addition the graph $\Gamma_{N,M}$ is ergodic. Our first result gives alternative numerical characterizations of these notions.

THEOREM 1. *Let $N \subset M$ be an extremal inclusion* ([PiPo1], [PiPo2]) *of finite index of factors of type* II_1. *Then $N \subset M$ is strongly amenable if and only if M is the hyperfinite type* II_1 *factor,* $\|\Gamma_{N,M}\|^2 = [M:N]$ *and $\Gamma_{N,M}$ is ergodic. Also, if $\Gamma_{N,M}$ is ergodic, then $N \subset M$ is strongly amenable if and only if $N \subset M$ has a hypertrace in its standard representation. In particular, if M is the hyperfinite type* II_1 *factor and if $N \subset M$ has finite depth, i.e., if $\Gamma_{N,M}$ is finite, or if $[M:N] \leqslant 4$ then $N \subset M$ is strongly amenable.*

For the subfactors associated to actions of finitely generated discrete groups G in [Po6] the graph $\Gamma_{N,M}$ coincides with the Cayley graph of the group G and the above condition $\|\Gamma_{N,M}\|^2 = [M:N]$ becomes Kesten's characterisation of amenability for G. We will in fact introduce the notion of (strong) amenability for a standard invariant (or paragroup) $\mathcal{G}_{N,M}$, one equivalent characterisation of which is "$\|\Gamma_{N,M}\|^2 = [M:N]$ (and $\Gamma_{N,M}$ ergodic)". The above theorem thus states that $N \subset M$ is strongly amenable if and only if M is amenable and $\mathcal{G}_{N,M}$ is strongly amenable.

To state the next theorem we denote by $N^{\mathrm{st}} \subset M^{\mathrm{st}}$ the canonical model coming from the invariant $\mathcal{G}_{N,M}$, i.e., from the higher relative commutants.

THEOREM 2. *$N \subset M$ is isomorphic to $N^{\mathrm{st}} \subset M^{\mathrm{st}}$ if and only if it is strongly amenable. In particular the isomorphism class of such an inclusion is completely determined by its standard invariant $\mathcal{G}_{N,M}$.*

Since we will prove that the strong amenability of the standard invariant (or of the paragroup) is equivalent to the entropic condition "$H(M^{\mathrm{st}} | N^{\mathrm{st}}) = H(M|N)$", the above theorem states that a subfactor of the hyperfinite type II_1 factor is isomorphic to its canonical model if and only if it has the same entropy as its model.

Theorem 2 can be regarded as the analogue to the case of inclusions of factors of Connes' classification result for (single) amenable factors of type II ([C2]). To prove Theorem 2 we will first use the amenability condition to prove an appropriate Følner-type condition, in the spirit of [C2], by "decodifying" the hypertraces of $N \subset M$ into almost left invariant finite dimensional vector spaces that are right modules over the higher relative commutants. These finite dimensional vector spaces have no multiplicative structure, but by a "local quantization" method, similar to the one we used in ([Po3]), one converts these vector spaces into genuine finite dimensional algebras of higher relative commutants, locally approximating the inclusion. Thus, through this technique, the amenability condition is used to recover the structure of the space of representation into the initial algebras, by means of the norm one projection (or hypertraces). Thus, in the single algebra case $M\subset\mathcal{B}(\mathcal{H})=\mathcal{M}$, the projection of $\mathcal{B}(\mathcal{H})$ onto M takes the matricial structure of $\mathcal{B}(\mathcal{H})$ back into M, while in the case of representations of $N\subset M$ into $\mathcal{N}=\bigoplus\mathcal{B}(\mathcal{K}_l)\subset\bigoplus\mathcal{B}(\mathcal{H}_k)=\mathcal{M}$ the projection of $\mathcal{N}\subset\mathcal{M}$ onto $N\subset M$ gives rise to inclusions of finite dimensional commuting squares of algebras inside $N\subset M$, having the same inclusion matrix as $\mathcal{N}\subset\mathcal{M}$. In particular, in the case $\mathcal{N}\subset\mathcal{M}$ is the standard representation of $N\subset M$ these finite dimensional inclusions come from $\mathcal{G}_{N,M}$.

Before commenting on the use of the above theorems for genuine subfactor problems, let us note that they give alternative proofs to some of the classical results in single von Neumann algebras and can be used to derive some new ones as well. For instance, Theorem 2 applied to $N=M\subset M$ with M an amenable type II_1 factor, implies Connes' fundamental theorem on the uniqueness of the amenable type II_1 (and thus type II_∞) factor. Taking M to be the hyperfinite type II_1 factor, $p\in M$ a projection,

$$\alpha: pMp\simeq(1-p)M(1-p)$$

an isomorphism and $N=\{x\oplus\alpha(x)\,|\,x\in pMp\}$ one obtains Connes' theorem on the classification, up to outer conjugacy, of single automorphisms of the hyperfinite type II_1 and II_∞ factors. A similar construction for actions of discrete groups with 0 entropy ([A], [KaiVe]), such as groups with subexponential growth, gives the uniqueness, up to cocycle conjugacy, of outer actions of such groups on the hyperfinite type II factors, more generally the classification of G-kernels on such factors, i.e., [J1] and part of the general such result in [Oc1]. Also, Theorem 2 is used in [PoWa] to obtain the classification of minimal actions of compact Lie groups.

While Theorem 2 gives a complete classification of strongly amenable inclusions of type II, in terms of their standard invariants, for the actual listing of all such subfactors one still has to solve the graph theoretical problem of enumerating all such invariants, say for a given index. In this direction, for type II_1 subfactors, the best set-up until now is the formalism of Ocneanu ([Oc2], [Ka], [IzKa]), who used it to give a full list of paragroups of index <4. The physically motivated point of view in [L1] of investigating the tensor category generated by an endomorhism of range N regarded as a Connes correspondence (bimodule), proved to be useful as well (cf. [Iz]). In some situations ad hoc arguments work out ([GHJ], [Po6]). Altogether the following exhaustive list of possible invariants coming from subfactors of index $\leqslant 4$ can be obtained:

For type II_1 factors and index less than 4, the graph $\Gamma_{N,M}$ is of one of the forms A_n, $n\geqslant 1$, D_{2n}, $n\geqslant 2$, E_6, E_8 and there is a unique possible paragroup (standard invariant) for each of the graphs A_n (cf. [J2]), a unique one for each D_{2n} and two for each E_6, E_8 (cf. [Oc2], see also [Ka], [Iz], [SuVa]). If $[M:N]=4$ then $\Gamma_{N,M}$ is of the form $A_{2n-1}^{(1)}$ (with n possible paragroups), $n\geqslant 1$, or $A_\infty^{(1)}$ (with one paragroup), or $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}=E_9$ (one for each), or $D_n^{(1)}$ ($n-2$ for each), $n\geqslant 4$, or A_∞, D_∞ (with one paragroup each). (All this, by using [GHJ], [Oc2], [Po6], [IzKa].)

From Theorem 2 we thus have:

COROLLARY 1. *The above is the complete list of subfactors of index* $\leqslant 4$ *of the hyperfinite type* II_1 *factors. In particular, the Jones subfactor of index* $4\cos^2\pi/(n+1)$, $n\leqslant\infty$, *is the unique subfactor with graph* $\Gamma=A_n$ *of the hyperfinite type* II_1 *factor, i.e., with higher relative commutants generated by the Jones projections.*

The Jones subfactors come from positive Markov traces on representations of Hecke algebras (or Braid groups). Their construction, in turn, is strongly related to the polynomial invariant for knots ([J3]). The general construction along these lines, which uses the general Markov traces on Hecke algebras ([Ocetal]), is obtained in [We1], [We2]. The associated subfactors have index larger than 4 but were shown to have finite depth at roots of unity ([We2]) and strongly amenable in general ([Wa2], [PoWa]). By Theorem 1 it follows that one can recognise them by merely "observing" their invariants.

COROLLARY 2. *The subfactors associated to positive Markov traces on representation of Hecke algebras of type A, B, C, D of* [We1], [We2] *are completely determined by their standard invariants.*

Finally, let us mention that a number of results have been recently obtained in [H] on the first possible subfactors with finite depth beyond 4 and on the accumulation point of their indices. The previously known subfactor with finite depth and smallest index but larger than 4 was Jones' subfactor of index $3+\sqrt{3}$ (cf. [GHJ], [Ok]).

Like in the single von Neumann algebra case, by using the decomposition methods of [C1], [CT], the classification of inclusions of hyperfinite type III_λ factors, $0<\lambda\leqslant 1$, reduces (cf. [Lo]) to firstly classify the associated inclusion of type II, which is solved by Theorem 2, and secondly to classify trace scaling actions on it. This second problem, of classifying properly outer actions on strongly amenable subfactors of type II, is solved in a parallelly circulated paper ([Po9]) thus showing that strongly amenable subfactors of type III_λ, $0<\lambda<1$, are completely classified by their appropriately defined standard invariant as well. A similar result holds true for subfactors of type III_1 (paper in preparation).

Part of the results in this paper have been announced by the author in a number of lectures during 1989–1991 and in a C. R. Acad. Sci. Paris note in 1990 ([Po7]). A preliminary form of the paper has been circulated since the summer of 1991 and was the subject of a one year course at UCLA in 1990–1991. The final version of the paper

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1. Basics of the theory of subfactors

1.1. General inclusions

The generic notation for finite von Neumann algebras, i.e., von Neumann algebras which have a faithful normal tracial state, will be N, M, and τ will usually denote the trace. The Hilbert norm given by τ on M is denoted $\|x\|_2=\tau(x^*x)^{1/2}$, $x\in M$, and the completion of M in this norm is denoted by $L^2(M,\tau)$. When regarded as vectors in $L^2(M,\tau)$ ($\supset M$) the elements $x\in M$ are denoted by $\hat{x}$. Note that M acts on $L^2(M,\tau)$ by left and right multiplication. We identify M, as an algebra, with its action on $L^2(M,\tau)$ by left multiplication. If $J_M: L^2(M,\tau)\to L^2(M,\tau)$ is defined by $J_M(\hat{x})=\hat{x}^*$, then JxJ is the operator of right multiplication by x^* and we have $M^{\mathrm{op}}\overset{\mathrm{def}}{=}JMJ=M'$, the commutant of M in $\mathcal{B}(L^2(M,\tau))$. J_M is called the *canonical conjugation* on $L^2(M,\tau)$. Note that the elements of $L^2(M,\tau)$ can be identified with the square summable operators affiliated with M. Similarly, the predual of M, M_* can be identified with $L^1(M,\tau)$, the summable operators affiliated with M.

1.1.1. *Conditional expectations.* A major feature of the von Neumann subalgebras of a finite algebra is the existence of nice expectations onto them. If $N\subset M$ is a von Neumann subalgebra of M then we denote by E_N the unique τ-preserving conditional expectation of M onto N. $e_N: L^2(M,\tau)\to L^2(N,\tau)$ denotes the orthogonal projection onto $L^2(N,\tau)$, the closure of N in the norm $\|\cdot\|_2$, in $L^2(M,\tau)$. Then $e_N(\hat{x})=\widehat{E_N(x)}$ and e_N, E_N are related by the following important algebraic relations:

(a) $e_N x e_N=E_N(x)e_N$, $x\in M$.

(b) $N=\{x\in M\,|\,[x,e_N]=0\}$.

(c) $b\in N, be_N=0\Rightarrow b=0$.

(d) $[J_M, e_N]=0$.

One still denotes by E_N the extension of the conditional expectation of M onto N to an N–N bimodule projection from the summable operators of $L^1(M,\tau)$ ($=M_*$) to $L^1(N,\tau)$ ($=N_*$), i.e., when regarded as elements in M_*, $E_N(\xi)(x)=\xi(E_N(x))$.

1.1.2. *Extensions by subalgebras.* The von Neumann algebra generated by M and e_N in $\mathcal{B}(L^2(M,\tau))$ is denoted by $\langle M, N\rangle$ or $\langle M, e_N\rangle$ and it is called the *extension of M by N*. The construction of $\langle M, N\rangle$ is typically quantum theoretical, since even if N, M are abelian, $\langle M, N\rangle$ is not abelian, unless $N=M$ and it is abelian.

We have the following important properties:

(a) $\langle M,N\rangle = \mathrm{vN}(M,e_N) = \overline{\mathrm{sp}}^w\{xe_Ny \mid x,y\in M\} = J_M N' J_M$.

(b) $e_N\langle M,N\rangle e_N = Ne_N$.

(c) There exists a unique faithful normal trace, $\mathrm{Tr}_{\langle M,N\rangle}$, on $\langle M,N\rangle$ such that $\mathrm{Tr}_{\langle M,N\rangle}(xe_N)=\tau(x)$, $x\in M$.

(d) $N'\cap M \ni x \mapsto J_M x^* J_M \in M'\cap\langle M,N\rangle$ is an antiisomorphism which takes the center of N, $\mathcal{Z}(N)$, onto the center of $\langle M,N\rangle$. If $z\in\mathcal{Z}(N)$ then $z'=J_M z^* J_M$ is the unique element of $\mathcal{Z}(\langle M,N\rangle)$ such that $ze_N=z'e_N$. Since $\mathcal{Z}(N), \mathcal{Z}(\langle M,N\rangle)$ are commutative, $z\mapsto z'$ is an isomorphism, called the *canonical isomorphism*.

The construction of $\langle M,N\rangle$ with the trace $\mathrm{Tr}_{\langle M,N\rangle}$ is called the *basic construction* ([J2], [Ch], [Sk]). We write it $N\subset M\subset^{e_N}\langle M,N\rangle$. This proves to be a very powerful tool of investigation in the theory of inclusions of algebras. Note that properties (a), (b), (c) describe $\langle M,N\rangle$ abstractly, in the sense that, if $\mathcal{B}$ is a von Neumann algebra containing M and a projection e such that $\mathcal{B}=\overline{\mathrm{sp}}\, MeM$ and with a trace Tr such that $[e,N]=0$, $exe=E_N(x)e$, $\mathrm{Tr}(xe)=\tau(x)$, $x\in M$, then one has $\mathcal{B}=\langle M,e_N\rangle$, with e corresponding to e_N.

1.1.3. *Orthonormal bases.* We have the following general facts:

(a) There always exists a family $\{\xi_i\}_{i\in I}\subset L^2(M,\tau)$ such that

(i) $E_N(\xi_i^*\xi_j)=\delta_{ij}f_j$, with f_j projections in N.

Such a family is called an *orthonormal basis* of M over N. The orthonormal basis is unique, in the sense that if $\{\eta_j\}_{j\in J}$ is another basis of M over N with $E_N(\eta_j^*\eta_j)=g_j\in\mathcal{P}(N)$ then $v\overset{\mathrm{def}}{=}(E_N(\xi_i^*\eta_j))_{i\in J, j\in J}$ is a unitary element from $\bigoplus_{i\in I} f_iL^2(N,\tau)$ onto $\bigoplus_{j\in J} g_jL^2(N,\tau)$ satisfying $\eta=\xi v$, where $\eta=\{\eta_j\}$, $\xi=\{\xi_i\}$, i.e., $\eta_j=\sum_{i\in I}\xi_iE_N(\xi_i^*\eta_j)$, $j\in J$. Also, $\sum_i Ct_N(f_i)=\sum_j Ct_N(g_j)$, where Ct_N is the central trace on N and $\sum_i \xi_i\xi_i^*=\sum_j \eta_j\eta_j^*$.

(ii) $\overline{\sum_i \xi_i N}=L^2(M,\tau)$ and $\xi=\sum_j \xi_jE_N(\xi_j^*\xi)$, $\forall\xi\in L^2(M,\tau)$.

(b) If N, M are factors then the family $\{\xi_i\}_{i\in I}$ can be chosen such that in addition one has:

(i) If $\langle M,N\rangle$ is an infinite (still semifinite) factor then

$$E_N(\xi_i^*\xi_i)=1, \quad \text{for all } i\in I.$$

(ii) If $\langle M,N\rangle$ is a finite factor then I is finite and one has $E_N(\xi_i^*\xi_i)=1$ for all $i\in I$ but possibly one. Moreover in this case all ξ_i follow automatically bounded operators, thus $\xi_i\in M$.

The proof of (a) is trivial by a maximality argument. To prove part (i) of (b), let $\{\xi_i\}_{i\in I_0}$ be a maximal family such that $E_N(\xi_i^*\xi_j)=\delta_{ij}$, $i,j\in I_0$. If $\overline{\sum_{i\in I_0}\xi_iN}\neq L^2(M,\tau)$ then let $\{\xi_j\}_{j\in I_1}$ be a family in $L^2(M,\tau)$ such that $\overline{\sum_{i\in I_0}\xi_iN}+\overline{\sum_{j\in I_1}\xi_jN}=L^2(M,\tau)$

and such that $E_N(\xi_i^*\xi_j)=\delta_{ij}f_j$, if $j\in I_1$, $i\in I_0\cup I_1$. If $\sum_{j\in I_1}\tau(f_j)\geqslant 1$ then there exists some partial isometries $v_j\in N$ such that $v_jv_j^*\leqslant f_j$ and $\sum_j v_jv_j^*=1$. Let $\xi=\sum_j \xi_jv_j$. Then $E_N(\xi^*\xi)=1$ and $E_N(\xi^*\xi_i)=0$ for all $i\in I_0$, contradicting the maximality of $\{\xi_i\}_{i\in I_0}$. If $\sum\tau(f_j)<1$ then we can take the above v_j so that $v_jv_j^*=f_j$ and thus get $\xi=\sum\xi_jv_j$ so that $\overline{\sum_i \xi_iN+\xi N}=L^2(M,\tau)$. Thus I_0 is infinite (otherwise $\langle M,N\rangle$ follows finite). So, there exists a countable infinite set $I_2\subset I_0$. Let $f=E_N(\xi^*\xi)$ and label the set of indices I_2 by $\{1,2,\dots\}$. For $i\in I_2$ put $\eta_i'=\xi_i(1-f)$, $\eta_i''=\xi_if$. Let $\xi_1'=\xi+\eta_1'$ and $\xi_n'=\eta_{n+1}'+\eta_n''$, $n\geqslant 2$. Then clearly $\{\xi_i\}_{i\in I_0\setminus I_2}\cup\{\xi_n'\}_{n\in I_2}$ satisfies the requirements.

By [PiPo1], part (ii) of (b) is now clear.

Let us finally note that orthonormal bases can be used to give a very intuitive description of $\langle M,N\rangle$ as some amplification of N. To this end, note that if $\xi\in L^2(M,\tau)$ is regarded as a (possibly unbounded) operator affiliated with M, then $E_N(\xi^*\xi)$ is a projection if and only if $e_N\xi^*\xi e_N$ is a projection and if and only if ξe_N is a partial isometry. Note that this partial isometry must then be in $\langle M,N\rangle$.

(c) If $\{\xi_i\}_i\subset L^2(M,\tau)$ is an orthonormal basis of M over N like in (a) then any element $\tilde{x}$ in $\langle M,N\rangle$ can be uniquely written as $\tilde{x}=\sum_{i,j}\xi_ix_{ij}e_N\xi_j^*$, where $x_{ij}\in f_iNf_j$ and where in fact x_{ij} is the unique element x of N such that $xe_N=e_N\xi_i^*\tilde{x}\xi_je_N$. Moreover, if an infinite sum $\sum_{i,j}\xi_ix_{ij}e_N\xi_j^*$, with $x_{ij}\in N$, defines a bounded operator on $L^2(M,\tau)$, then that operator is in $\langle M,N\rangle$. In particular, $\sum_i\xi_ie_N\xi_i^*=1_{\langle M,N\rangle}$.

(d) The canonical isomorphism $z\mapsto z'$ of 1.1.2(d), from $\mathcal{Z}(N)$ onto $\mathcal{Z}(\langle M,N\rangle)$ coincides with the application $z\mapsto\sum_i\xi_ie_Nz\xi_i^*$, i.e., $\sum\xi_ie_Nz\xi_i^*$ is the unique element z' in $\mathcal{Z}(\langle M,N\rangle)$ such that $z'e_N=ze_N$.

1.1.4. *Markov traces on the extension algebra.* If M has a finite orthonormal basis over N, $\{\xi_i\}_{i\in I}$, $\operatorname{card} I<\infty$, then $\langle M,N\rangle$ is clearly a finite von Neumann algebra and $\mathrm{Tr}_{\langle M,N\rangle}$ is a finite trace on $\langle M,N\rangle$. In fact we have the following:

(a) $\langle M,N\rangle$ is finite if and only if $\sum_i Ct_N(f_i)$ is a densely defined (but possibly unbounded) operator affiliated with $\mathcal{Z}(N)$, where Ct_N is as in 1.1.3 the central trace on N and $f_i=E_N(\xi_i^*\xi_i)$, $\{\xi_i\}_{i\in N}$ being an orthonormal basis of M over N.

This is trivial if one regards $\xi_ie_N\xi_i^*\in\langle M,N\rangle$ as the cyclic projection $[\xi_iN]$ of $L^2(M,\tau)$ onto $\overline{\xi_iN}$.

(b) The following conditions are equivalent and if they are satisfied then $\mathrm{Tr}_{\langle M,N\rangle}$ is finite:

(i) If $\{\xi_i\}_{i\in I}$ is an orthonormal basis of M over N then $\sum_i Ct_N(f_i)$ is a bounded operator in $\mathcal{Z}(N)$, where $f_i=E_N(\xi_i^*\xi_i)$, $i\in I$.

(ii) M has a finite orthonormal basis over N.

(iii) $\langle M,N\rangle=\operatorname{sp} Me_NM$.

Indeed, (iii) $\Rightarrow$ (ii) by the Gram–Schmidt algorithm and the rest is trivial by (a) and by 1.1.3.

(c) If $\mathrm{Tr}_{\langle M,N\rangle}$ is finite and we denote by $\lambda=\mathrm{Tr}_{\langle M,N\rangle}(1_{\langle M,N\rangle})^{-1}$ and by $\tau_1=\tau_{\langle M,N\rangle}\overset{\mathrm{def}}{=}\lambda\mathrm{Tr}_{\langle M,N\rangle}$, its normalization, then the following conditions are equivalent:

(i) $\tau_1|_M=\tau$.

(i′) There exists an extension $\tilde{\tau}$ of the trace τ to $\langle M,N\rangle$ such that $\tilde{\tau}(xe_N)=\lambda'\tau(x)$, $\forall x\in M$, some $\lambda'\in\mathbf{C}$.

(ii) If $E_1=E_M^{\langle M,N\rangle}$ is the τ_1-preserving conditional expectation of $\langle M,N\rangle$ onto M then $E_1(e_N)=\alpha 1$ is a scalar multiple of the identity.

(ii′) There exists a conditional expectation $\widetilde{E}$ of $\langle M,N\rangle$ onto M such that $\widetilde{E}(e_N)=\alpha' 1$, $\alpha'\in\mathbf{C}$.

(iii) There exists an orthonormal basis $\{\xi_i\}$ of M over N with $\sum_i\xi_i\xi_i^*=\beta 1$, $\beta\in\mathbf{R}_+$.

Moreover, if the above conditions are satisfied then $\lambda=\lambda'=\alpha=\alpha'=\beta^{-1}$ and given any orthonormal basis $\{\eta_j\}$ we have $\sum_j\eta_j\eta_j^*=\lambda^{-1}1$. And then τ_1 is the unique trace on $\langle M,N\rangle$ satisfying (i′) and E_1 is the unique expectation satisfying (ii′).

Indeed, by the definition of τ_1, (i) $\Rightarrow$ (i′) with $\tilde{\tau}=\tau_1$ and $\lambda'=\lambda$. Clearly (i′) $\Rightarrow$ (ii′), by taking $\widetilde{E}$ to be the unique $\tilde{\tau}$-preserving expectation and $\alpha'=\lambda'$. Since for any orthonormal basis $\{\xi_i\}$ we have $\sum\xi_ie_N\xi_i^*=1$, by (ii′) we get (iii) with $\beta=\alpha'^{-1}$. If (iii) is satisfied and $x\in M$ then $x=x\sum_i\xi_ie_N\xi_i^*$ so that $\tau_1(x)=\sum\tau_1(x\xi_ie_N\xi_i')=\lambda\sum\tau(x\xi_i\xi_i^*)=\lambda\beta\tau(x)$, so that $\lambda\beta=1$ and $\tau_1|_M=\tau$. Thus (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii). If $E_1(e_N)=\alpha 1$ then $\tau_1(e_Nx)=\lambda\tau_1(x)=\lambda\tau(x)$, so that $\lambda=\alpha$ and (ii) $\Rightarrow$ (i).

An inclusion $N\subset M$ with a trace τ for which the above equivalent conditions (c) are satisfied is called a *λ-Markov inclusion* and $\tau_{\langle M,N\rangle}=\tau_1$ a *λ-Markov trace.*

Note that by [J2] a finite dimensional inclusion $N\subset M$ with irreducible inclusion matrix T is Markov if and only if the trace on M is given by the unique Perron–Frobenius eigenvector of TT^t.

Note that if $(N\subset M,\tau)$ is a λ-Markov inclusion and $\{\xi_i\}_i$ is an orthonormal basis of M over N then $\{\lambda^{-1/2}\xi_ie_N\}$ is an orthonormal basis of $M_1=\langle M,N\rangle$ over M and $(M\subset M_1,\tau_1)$ is a Markov inclusion. One can thus obtain by iteration a whole tower of inclusions $N\subset M\subset^{e_1}M_1\subset^{e_2}M_2\subset\ldots$ and a trace τ_∞ on $\bigcup M_n$, where $e_{n+1}=e^{M_n}_{M_{n-1}}$, $M_{n+1}=\mathrm{sp}\, M_ne_{n+1}M_n=\langle M_n,M_{n-1}\rangle$, $\tau(e_{n+1}x)=\lambda\tau(x)$, $x\in M_n$ and in fact $e_ne_{n+1}e_n=\lambda e_n$. It is called the Jones tower associated to the Markov inclusion $N\subset M$.

1.1.5. *Commuting squares.* We now define the suitable notion of morphism between the objects that we study here, namely between inclusions of algebras. So, let $N\subset M$ and $Q\subset P$ be inclusions of finite von Neumann algebras such that $P\subset M$, $Q\subset N$. Then

$$\begin{matrix} N & \subset & M \\ \cup & & \cup \\ Q & \subset & P \end{matrix}$$

is called a *commuting square* if $E_N^M E_P^M = E_P^M E_N^M = E_Q^M$. Note that for this relation to hold true it is sufficient to have $E_N^M(P) \subset Q$ or $E_P^M(N) \subset Q$. This concept was initially introduced in [Po3] to study the various mutual position and normalization properties of subalgebras of a type II_1 factor. It later proved to be extremely important for the theory of subfactors as well ([PiPo1], [We1], [HSc]). The key observation that makes commuting squares appear naturally whenever we consider subalgebras of a von Neumann algebra is the following:

Example ([Po3]). If $B \subset N \subset M$ are finite von Neumann algebras then

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ B' \cap N & \subset & B' \cap M \end{array}$$

is a commuting square.

The type of "morphisms" that we are interested in are the commuting squares

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ Q & \subset & P \end{array}$$

in which M is somehow minimally generated by N and P. For instance, if $Q=P$ then we would like to only consider embeddings of $Q \subset P = Q$ into some $N \subset M$ for which $N=M$. We thus define a commuting square as above to be *nondegenerate* if P generates M as a right N module. We actually have:

PROPOSITION. *The following conditions are equivalent*:
(i) $\mathrm{supp}\{x e_N^M y \mid x, y \in P\} = \bigvee \{u e_N^M u^* \mid u \in \mathcal{U}(P)\} = 1$.
(ii) $\bigvee \{v e_P^M v^* \mid v \in \mathcal{U}(N)\} = 1$.
(iii) *Any orthonormal basis of P over Q is an orthonormal basis of M over N.*
(iv) *Any orthonormal basis of N over Q is an orthonormal basis of M over P.*
(v) $\overline{\mathrm{sp}\, PN} = M$ (*i.e., the commuting square is nondegenerate*).
(vi) $\overline{\mathrm{sp}\, NP} = M$.

Proof. If $u \in P$ then $u e_N^M u^*$ is the orthogonal projection onto $\overline{uN} \subset L^2(M, \tau)$. Taking $\xi = u - E_N^M(u) = u - E_Q^P(u)$ it follows that $\xi \in P$ and that $u e_N^M u^* \vee e_N^M = [\hat{1}N] \oplus [\xi N]$. By replacing ξ with $\xi E_Q^P(\xi^* \xi)^{-1/2}$ it follows that we may also assume $E_N^M(\xi^* \xi) = E_Q^P(\xi^* \xi) = f$, where f is a projection in Q. More generally, if $\{\xi_i\}_{i \in I} \in L^2(P, \tau)$ is an orthonormal basis of P over Q and $\mathcal{H}_0 = \overline{\sum_i \xi_i N}$ and if $\mathcal{H}_0 \neq L^2(M, \tau)$ then it follows that there exists $u \in P$ such that $\overline{uN} \not\subset \overline{\sum \xi_i N}$. Let $\xi = u - \sum_i E_N^M(\xi_i^* u) = u - \sum_i E_Q^P(\xi_i^* u) \in L^2(P, \tau)$. Then $\xi \neq 0$ and $\xi \in L^2(P, \tau)$, $\xi \perp \sum \xi_i N$, in particular $\xi \perp \sum \xi_i Q$, a contradiction. This proves

(i) $\Rightarrow$ (iii). Clearly (iii) $\Rightarrow$ (i), because (iii) means the existence of $\{\xi_i\} \subset L^2(P,\tau)$ such that $\xi_i e_N^M \xi_i^*$ are mutually orthogonal projections with $\sum \xi_i e_N^M \xi_i^* = 1$.

Let $\{\eta_j\}_{j\in J} \subset L^2(N,\tau)$ be an orthonormal basis of N over Q and $\{\xi_i\}_{i\in I}$ an orthonormal basis of P over Q which is assumed to be also an orthonormal basis of M over N. By the commuting square relation η_j are orthonormal with respect to P as well. Assume $\eta \in L^2(M,\tau)$ is so that $\eta \perp \sum_j \eta_j P$, so that $\eta \perp \sum_{i,j} \eta_j \xi_i Q$. Since $\overline{Q \sum \xi_i Q} = \overline{\sum \xi_i Q} = L^2(P,\tau)$ and $\overline{\sum \eta_j Q} = L^2(N,\tau)$, it follows that $\eta \perp NP$. Thus $\eta^* \perp P^* N^* = PN$ and so $\eta^* \perp \sum \xi_i N$. Thus $\eta = 0$. Thus (iv) follows from (iii). Clearly (iv) is equivalent to (ii) the same way (i) $\Leftrightarrow$ (iii).

Clearly (iii) $\Rightarrow$ (v) and (iv) $\Rightarrow$ (vi). Conversely, if say (v) holds true and if $\{\xi_j\}_j \subset L^2(P,\tau)$ is a maximal orthonormal system for M over N so that $\overline{\sum_j \xi_j N} \neq L^2(M,\tau) = \overline{\text{sp}\, PN}$ (the last closure is taken this time in $L^2(M,\tau)$) then there exists $x \in P$ such that $\hat{x} \notin \overline{\sum \xi_j N}$. But then the projection of $\hat{x}$ onto the Hilbert space $\overline{\sum \xi_j N}$, $P_{\sum \xi_j N}(\hat{x})$, lies in $L^2(P,\tau)$ (since $P_{\sum \xi_j N}(\hat{x}) = \sum \xi_j E_N(\xi_j^* x) = \sum \xi_j E_Q(\xi_j^* x)$), and thus $\hat{x} - \hat{P}_{\sum \xi_j N}(\hat{x}) \neq 0$ is in $L^2(P,\tau)$ and it is orthogonal to $\sum \xi_j N$, thus contradicting the maximality of $\{\xi_j\}_j$. $\square$

COROLLARY. *Let*

$$\begin{array}{ccccc} R & \subset & P & \subset & M \\ \cup & & \cup & & \cup \\ S & \subset & Q & \subset & N \end{array}$$

be commuting squares. Then the two small commuting squares are nondegenerate if and only if the big commuting square is nondegenerate.

Proof. Assume the big commuting square is nondegenerate. Thus $M = \text{sp}\, NR = \text{sp}\, NP$ and $P = E_P(M) = \text{sp}\, QR$, so that the two small commuting squares follow nondegenerate. Conversely, if $P = \text{sp}\, QR$, $M = \text{sp}\, NP$ then $M = \text{sp}\, NQR = \text{sp}\, NR$. $\square$

1.1.6. *Basic construction for commuting squares.* A main feature of the nondegenerate commuting squares is their well behavior to extensions:

PROPOSITION. *Let*

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

be a nondegenerate commuting square. Let $P \subset M \subset^{e_P^M} \langle M, P\rangle = \langle M, e_P^M\rangle$ *and* $Q \subset N \subset^{e_Q^N} \langle N, Q\rangle = \langle N, e_Q^N\rangle$ *be the basic constructions for the horizontal inclusions. If* $\langle N, e_P^M\rangle$ *denotes the von Neumann algebra generated in* $\mathcal{B}(L^2(M,\tau))$ *by* N *and* e_P^M *then we have:*

(1) *If* $\{\xi_i\}$ *is an orthonormal basis of* N *over* Q *then*

$$\langle N, e_P^M\rangle = \left\{ \sum_{i,j} \xi_i x_{ij} e_P^M \xi_j^* \,\middle|\, x_{ij} \in f_i Q f_j \right\}.$$

(2) *There exists a unique isomorphism* θ *of* $\langle N, e_P^M\rangle$ *onto* $\langle N, e_Q^N\rangle = \langle N, Q\rangle$ *satisfying* $\theta(x)=x$, $x\in N$, $\theta(e_P^M)=e_Q^N$. θ *can be defined by* $\theta(\sum \xi_i x_{ij} e_P^M \xi_j^*)=\sum \xi_i x_{ij} e_Q^N \xi_j^*$ *and it is trace preserving, i.e.,* $\mathrm{Tr}_{\langle N,Q\rangle}\circ\theta = \mathrm{Tr}_{\langle M,P\rangle}|_{\langle N, e_P^M\rangle}$.

(3) *The application* $E_{\langle N,Q\rangle}^{\langle M,P\rangle}: \langle M,P\rangle = \langle M, e_P^M\rangle \to \langle N, e_P^M\rangle \simeq \langle N, Q\rangle$ *with*

$$E_{\langle N,Q\rangle}^{\langle M,P\rangle}\left(\sum_{i,j} \xi_i x_{ij} e_P^M \xi_j^*\right) = \sum \xi_i E_Q^P(x_{ij}) e_P^M \xi_j^*,$$

is a normal faithful conditional expectation, and it is trace preserving on $\langle M, e_P^M\rangle$, *i.e.,* $\mathrm{Tr}\circ E_{\langle N,Q\rangle}^{\langle M,P\rangle} = \mathrm{Tr}$.

(4) $E_{\langle N,Q\rangle}^{\langle M,P\rangle}$ *satisfies the commuting square relation* $E_{\langle N,Q\rangle}^{\langle M,P\rangle}|M = E_N^M$.

(5) *If* $z\in Q'\cap P$ *then* $\sum \xi_i e_P^M z \xi_i^*$ *is the unique element* $z'\in\langle N,Q\rangle'\cap\langle M,P\rangle$ *such that* $z' e_P^M = z e_P^M$ *and* $z\mapsto z'$ *is an isomorphism of* $Q'\cap P$ *onto* $\langle N,Q\rangle'\cap\langle M,P\rangle$, *which coincides with the canonical isomorphism of* $Z(P)$ *onto* $Z(\langle M,P\rangle)$, *resp. of* $Z(Q)$ *onto* $Z(\langle N,Q\rangle)$ *when restricted to* $Z(P)$ *resp.* $Z(Q)$.

Proof. (1), (2), (3), (5) follow trivially from 1.1.2, 1.1.5. To prove (4) note that for $x\in M$ we have $x = x\sum \xi_j e_P^M \xi_j^* = \sum \xi_i E_P^M(\xi_i^* x \xi_j) e_P^M \xi_j^*$ so that:

$$\begin{aligned}
E_{\langle N,Q\rangle}^{\langle M,P\rangle}(x) &= \sum \xi_i E_Q^M(\xi_i^* x \xi_j) e_P^M \xi_j^* \\
&= \sum \xi_i E_Q^M(E_N^M(\xi_i^* x \xi_j)) e_P^M \xi_j^* \\
&= \sum \xi_i E_Q^M(\xi_i^* E_N^M(x) \xi_j) e_P^M \xi_j^* \\
&= \sum \xi_i E_Q^N(\xi_i^* E_N^M(x) \xi_j) e_P^M \xi_j^* \\
&= \sum_j \left(\sum_i \xi_i E_Q^N(\xi_i^* E_N^M(x) \xi_j)\right) e_P^M \xi_j^* \\
&= \sum_j E_N^M(x) \xi_j e_P^M \xi_j^* = E_N^M(x).
\end{aligned}$$

□

The construction of the inclusions

$$\begin{array}{ccccc}
P & \subset & M & \subset & \langle M,P\rangle \\
\cup & & \cup & & \cup \\
Q & \subset & N & \subset & \langle N,Q\rangle
\end{array}$$

with the conditional expectation $E_{\langle N,Q\rangle}^{\langle M,P\rangle}$ is called the *basic construction* for the commuting square

$$\begin{array}{ccc}
P & \subset & M \\
\cup & & \cup \\
Q & \subset & N
\end{array} \tag{1}$$

and the commuting square

$$\begin{array}{ccc} M & \subset & \langle M,P\rangle \\ E_N^M \cup & & \cup\, E_{\langle N,Q\rangle}^{\langle M,P\rangle} \\ N & \subset & \langle N,Q\rangle \end{array} \tag{2}$$

is called the *extension of the commuting square* (1).

Note that $\langle N,Q\rangle\subset\langle M,P\rangle$ is an amplification of $Q\subset P$ and that the whole construction can be regarded as a Takesaki type duality for inclusions.

1.1.7. *The probabilistic index of an expectation* ([PiPo1]). We will adopt the point of view in [PiPo1] for defining the index of an inclusion of arbitrary von Neumann algebras $\mathcal{N}\subset^{\mathcal{E}}\mathcal{M}$, with a normal conditional expectation $\mathcal{E}$ of $\mathcal{M}$ onto $\mathcal{N}$. The index will in fact be attached to the expectation $\mathcal{E}$. We thus put $\operatorname{Ind}\mathcal{E}=(\max\{\lambda\geqslant 0\,|\,\mathcal{E}(x)\geqslant\lambda x\,|\,x\in\mathcal{M}_+\})^{-1}$ and call it the *index* of $\mathcal{E}$.

If $\mathcal{N}'\cap\mathcal{M}=\mathbf{C}$ (so that in particular $\mathcal{N},\mathcal{M}$ are factors) then there exists a unique normal conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ (cf. [C1]) so we may as well denote

$$[\mathcal{M}:\mathcal{N}]\overset{\text{def}}{=}\operatorname{Ind}\mathcal{E}.$$

If $\operatorname{Ind}\mathcal{E}<\infty$ and we denote $\mathcal{B}=\mathcal{N}'\cap\mathcal{N}=\mathcal{Z}(N)$, $\mathcal{A}=\mathcal{N}'\cap\mathcal{M}$, then $\mathcal{E}(\mathcal{A})=\mathcal{B}$ and from $\mathcal{E}x\geqslant\lambda x$, $x\in\mathcal{A}_+$, $\lambda=(\operatorname{Ind}\mathcal{E})^{-1}>0$ and the abelianness of $\mathcal{B}$ one sees that $\mathcal{A}$ must be of the form $\bigoplus_{k=1}^n\mathcal{A}_k$ with $\mathcal{A}_k$ homogeneous of type I_{n_k} with $\sup n_k$ finite (cf. e.g. [PoWa]). By [T2] we also deduce that if $\dim\mathcal{N}'\cap\mathcal{M}<\infty$ then any other normal faithful expectation of $\mathcal{M}$ onto $\mathcal{N}$ will also have finite index. We then say that $\mathcal{N}\subset\mathcal{M}$ has finite index in the sense of [PiPo1]. By using the weak compactness of the set of expectations with finite index, it has been shown in [Hi] that if $\mathcal{N}\subset\mathcal{M}$ has finite index then there exists a normal expectation $\mathcal{E}_{\min}$ so that $\operatorname{Ind}\mathcal{E}_{\min}=\inf\{\operatorname{Ind}\mathcal{E}\,|\,\mathcal{E}:\mathcal{M}\to\mathcal{N}\}$. In the case either M or N is a factor von Neumann algebra there exists a unique such normal expectation, called the expectation of minimal index.

It is an easy exercise to show that if $\mathcal{N}\subset\mathcal{M}$ are atomic algebras with finite inclusion (multiplicity) matrix Γ then $\mathcal{E}_{\min}$ is the unique conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ preserving the trace with the weights given by a Perron–Frobenius eigenvector of $\Gamma\Gamma^t$.

Note also that (by 2.1 in [PiPo1]) if $(N\subset M,\tau)$ is a λ-Markov inclusion like in 1.1.5 then $M\subset\langle M,N\rangle$ is also Markov and $\operatorname{Ind}E_M^{\langle M,N\rangle}=\lambda^{-1}$.

1.2. Subfactors of finite index

If $N\subset M$ are type II_1 factors then $\langle M,N\rangle$ is either of type II_1 or type II_∞. If $\dim_N\mathcal{K}$ denotes the Murray and von Neumann coupling constant of N when acting on the Hilbert space $\mathcal{K}$, then the *Jones index* of an inclusion of type II_1 factors $N\subset M$ is by definition $\dim_N L^2(M,\tau)$, and it is denoted by $[M:N]$. We actually have:

1.2.1. (a) $[M:N]=\dim_N\mathcal{K}/\dim_M\mathcal{K}$, for any representation of M on a Hilbert space $\mathcal{K}$ with $\dim_M\mathcal{K}<\infty$.

(b) If $\dim_M\mathcal{K}<\infty$ then $[M:N]<\infty$ if and only if $\langle M,N\rangle$ (or N') is a type II_1 factor.

(c) If $P\subset N\subset M$ then $[M:P]=[M:N][N:P]$.

(d) If $[M:N]<\infty$ then $N\subset M$ is an $[M:N]^{-1}$-Markov inclusion.

1.2.2. *The probabilistic characterization of the index* ([PiPo1]). $[M:N]=\operatorname{Ind}E_N$, with $\operatorname{Ind}E_N$ defined in 1.1.6, i.e., $[M:N]^{-1}=\max\{c\geqslant 0\,|\,E_N(x)\geqslant cx, x\in M_+\}$, and with the convention $0^{-1}=\infty$. Moreover, if $[M:N]<\infty$, then there exist projections $e_0\in M$ such that $E_N(e_0)=[M:N]^{-1}\cdot 1$. Such projections on M are called *Jones projections* and they have a remarkable significance.

1.2.3. *The downward basic construction* ([J2], [PiPo1]). Any Jones projection $e_0\in M$ satisfies the following: if $N_1=N\cap\{e_0\}'$ then M is the basic construction for $N_1\subset N$, i.e.,

(i) $e_0xe_0=E_{N_1}(x)e_0$, $x\in M$,

(ii) $M=\mathrm{sp}\{xe_0y\,|\,x,y\in N\}$,

(iii) there exists a trace preserving $*$-isomorphism of $\langle N,N_1\rangle$ onto M carrying N as a subalgebra of $\langle N,N_1\rangle$ identically onto N as a subalgebra of M and e_{N_1} into e_0.

Moreover, e_0 is unique up to conjugacy by a unitary element in N (cf. [PiPo1]), i.e., if $e_0^1\in M$ is another Jones projection, i.e., so that $E_N(e_0^1)=[M:N]^{-1}\cdot 1$, then $e_0^1=ue_0u^*$, for some $u\in\mathcal{U}(N)$.

The construction of a subfactor $N_1\subset N$ with a projection $e_0\in M$ as above is called a *downward basic construction* for $N\subset M$. Unlike the usual basic construction, which is unique and canonical, we see that the downward construction is unique only up to conjugacy by unitaries in N.

1.2.4. *Relative commutants* ([J2]). If $N\subset M$ has finite index then $N'\cap M$ is finite dimensional and in fact we have:

(i) $[M:N]=[pMp:N_p]/\tau(p)\tau'(p)$, for any projection $p\in N'\cap M$, τ' being the unique normalized trace on N'.

(ii) $[M:N]=\sum_i[p_iMp_i:Np_i]/\tau(p_i)$, for any partition of the unity $\{p_i\}$ with projections in $N'\cap M$.

1.2.5. *Extremal inclusions* ([PiPo1]). The inclusion $N\subset M$ is called *extremal* if $E_{N'\cap M}(e_0)\in\mathbf{C}1$ for one (and thus all!) Jones projection $e_0\in M$. We have:

(i) ([PiPo1]) $N\subset M$ is extremal if and only if the antiisomorphism $N'\cap M\ni x\mapsto J_Mx^*J_M\in M'\cap\langle M,N\rangle$ is trace preserving, equivalently if $\tau(p)=\tau'(p)$, $\forall p\in N'\cap M$, τ' being the normalized trace on $\langle M,N\rangle=JN'J$.

(ii) ([PiPo1]) $N\subset M$ is extremal if and only if

$$[pMp:Np]=[M:N]\tau(p)^2,\quad \forall p\in\mathcal{P}(N'\cap M).$$

(iii) ([PiPo2]) $N\subset M$ is extremal if and only if $M\subset\langle M,N\rangle$ is extremal.

(iv) $N\subset P$ and $P\subset M$ are extremal if and only if $N\subset M$ is extremal.

(v) $N\subset M$ is extremal if and only if E_N coincides with the expectation of minimal index (1.1.7).

Property (iv) was recently proved in [L3] for arbitrary factors; we give here our independent proof in the type II_1 case for the sake of completeness. If $N\subset M$ is extremal then $H(M|N)=\ln[M:N]=\ln[M:P]+\ln[P:N]$ and $H(P|N)\leqslant\ln[P:N]$, $H(M|P)\leqslant\ln[M:P]$, $H(M|N)\leqslant H(M|P)+H(P|N)$ forces $H(M|P)=\ln[M:P]$, $H(P|N)=\ln[P:N]$. Conversely, assume $N\subset P$, $P\subset M$ are extremal. Let $e\in P$ be a Jones projection for the inclusion $N\subset P$, i.e., such that $E_N(e)=[P:N]^{-1}1$. Let also $f\in eMe$ be a Jones projection for the inclusion $ePe\subset eMe$, i.e., $f\leqslant e$, $E_{ePe}(f)=E_P(f)=[eMe:ePe]e=[M:P]^{-1}e$. We then have

$$E_N(f)=E_NE_P(f)=[M:P]^{-1}E_N(e)=[M:P]^{-1}[P:N]^{-1}1.$$

Let $N_1=\{e\}'\cap N$, then $N_1e=ePe$. Since $P\subset M$ satisfies the extremality condition, it follows that $ePe\subset eMe$ is still extremal so that $E_{(ePe)'\cap eMe}(f)=[M:P]^{-1}e$. It follows that there exist unitary elements $u_1,\dots,u_n\in ePe$ such that $\|(1/n)\sum u_ifu_i^*-[M:P]^{-1}e\|_2<\varepsilon_1$. Since $ePe=N_1e$ we can find unitaries $v_i\in N_1$ such that $v_ie=u_i$. Thus $v_if=u_if$ and we get

$$\Big\|\frac{1}{n}\sum_i v_ifv_i^*-[M:P]^{-1}e\Big\|_2<\varepsilon_1.$$

Since $N\subset P$ is also extremal, there exist unitaries $w_1,\dots,w_m\in N$ such that

$$\Big\|\frac{1}{m}\sum_j w_jew_j^*-[P:N]^{-1}1\Big\|_2<\varepsilon_2.$$

Altogether we get:

$$\Big\|\frac{1}{nm}\sum_{i,j} w_jv_ifv_i^*w_j^*-[M:N]^{-1}1\Big\|_2<\varepsilon_1+[M:P]^{-1}\varepsilon_2$$

letting $\varepsilon_{1,2}\to0$ it follows that $N\subset M$ is extremal. □

1.3. The standard invariant

Throughout this subsection $N\subset M$ will be an inclusion of type II_1 factors with finite index.

1.3.1. *The tower of factors* ([J2]). Denote by $M_1=\langle M,N\rangle$, $e_1=e_N$, the projection and the II_1 factor obtained from the basic construction for $N\subset M$. Since $N\subset M$ is an $[M:N]^{-1}$-Markov inclusion and M_1 is a type II_1 factor, we get $[M_1:M]=[M:N]<\infty$, and one can repeat this construction to get $M_2=\langle M_1,M\rangle$, $e_2=e_M$. More generally, by iterating the construction, one gets an increasing sequence of type II_1 factors, called the Jones *tower of factors*, $N\subset M\subset M_1\subset\ldots$, with projections $e_1,e_2,\ldots$, so that $M_{i-1}\subset M_i\subset^{e_{i+1}}M_{i+1}$ is the basic construction for $M_{i-1}\subset M_i$, for each $i\geqslant 1$ (where $M_{-1}=N$, $M_0=M$). Thus, $\{M_i,e_i\}_{i\geqslant -1}$ satisfy:

(i) $[M_i:M_{i-1}]=[M:N]=\tau(e_i)^{-1}$, $i\geqslant 1$.

(ii) $e_{i+1}xe_{i+1}=E_{M_{i-1}}(x)e_{i+1}$, $x\in M_i$. In particular $e_{i+1}e_ie_{i+1}=[M:N]^{-1}e_{i+1}$. Also, $e_ie_{i+1}e_i=[M:N]^{-1}e_i$. Moreover $M_{i+1}=\mathrm{sp}\,M_ie_{i+1}M_i$.

(iii) $e_{i+1}\in M_{i-1}'\cap M_{i+1}$. In particular $[e_i,e_j]=0$, for $|i-j|\geqslant 2$, and $e_l,e_{l+1},\ldots,e_k\in M_{l-2}'\cap M_k$, $1\leqslant l\leqslant k$.

(iv) $\tau(xe_{i+1})=[M:N]^{-1}\tau(x)$, $x\in M_i$.

1.3.2. *The tunnel of factors* ([PiPo1], [GHJ], [Po5]). Similarly, by iterating the downward basic construction, one obtains a decreasing sequence of type II_1 factors, $M\supset N\supset N_1\supset N_2\supset\ldots$, called *a tunnel of factors*, with corresponding projections $e_0,e_{-1},e_{-2},\ldots$. They satisfy the conditions:

(i′) $[N_i:N_{i+1}]=[M:N]=\tau(e_{-i})^{-1}$, $i\geqslant 0$.

(ii′) $e_{-i}xe_{-i}=E_{N_{i+1}}(x)e_{-i}$, $x\in N_i$, $i\geqslant 0$. Moreover $N_i=\mathrm{sp}\,N_{i+1}e_{-i-1}N_{i+1}$, $i\geqslant -1$.

(iii′) $e_{-i}\in N_{i+1}'\cap N_{i-1}$, $e_0,e_{-1},\ldots,e_{-k}\in N_{k+1}'\cap M$.

(iv′) $\tau(xe_{-i})=[M:N]^{-1}\tau(x)$, $x\in N_i$.

As mentioned in 1.2.4, each step of this construction is unique up to conjugacy by a unitary element in the last chosen subfactor. If we iterate this construction say up to N_k, we call it *a choice of the tunnel up to k*, and denote it $M\supset^{e_0}N\supset^{e_1}N_1\supset\ldots\supset^{e_{-k+1}}N_{k-1}\supset N_k$. Note that this means that the projection e_{-k} $(\in N_{k-1})$, which would uniquely determine one more subfactor, $N_{k+1}=\{e_{-k}\}'\cap N_k$, has not been chosen.

Note that by the product rule 1.2.1 (c) ([J2]) for indices of consecutive inclusions one has $[M_k:M_l]=[M:N]^{k-l}<\infty$ and similarly for the inclusions in the tunnel. Moreover by [PiPo2], any of the inclusions $M_i\subset M_{i+k}\subset M_{i+2k}$, or $N_{i+2k}\subset N_{i+k}\subset N_i$, $i\geqslant -1$, $k\geqslant 1$, is a basic construction, with the appropriate Jones projection obtained from the e_j's (resp. e_{-j}'s), by taking a certain scalar multiple of the word of maximal length in these e_j's.

Also, note that, if $M\supset^{e_0}N\supset^{e_{-1}}\ldots\supset^{e_{k+1}}N_{k-1}\supset N_k$ is a choice of the tunnel up to

k then there exists a unique representation of the tower up to $k+1$ on $L^2(M,\tau)$, $N\subset M\subset^{e_1}M_1\subset\ldots\subset^{e_{k+1}}M_{k+1}$, such that $J_M N_i' J_M = M_{i+1}$, $J_M e_{-i+1} J_M = e_{i+1}$, $1\leqslant i\leqslant k$ (cf. 3.1 in [Po5]).

1.3.3. *Duality* ([PiPo1]). If $\{m_r\}_r$ is an orthonormal basis of M_{k+1} over M_k and

$$M_k \subset M_{k+1} \overset{e_{k+1}}{\subset} M_{k+2} \subset \ldots \subset M_i \overset{e_{i+1}}{\subset} M_{i+1} \overset{e_{i+2}}{\subset} M_{i+2}$$

is part of the tower, then $\{\lambda^{k-i} m_i e_{k+2} e_{k+3} \ldots e_{i+2}\}_i$ is an orthonormal basis of M_{i+2} over M_{i+1} and if $\alpha=[M:N]$ then

$$\sigma((x_{rs})) = \sum_{r,s} m_r e_{k+2} \ldots e_{i+2} x_{rs} e_{i+2} \ldots e_{k+2} m_s^*$$

defines an isomorphism from the α-amplification of M_i onto M_{i+2} carrying the α-amplification of the whole sequence of inclusions $M_k\subset M_{k+1}\subset\ldots\subset M_i$, i.e., $M_k^\alpha\subset\ldots\subset M_i^\alpha$, onto the sequence of inclusions $M_{k+2}\subset\ldots\subset M_{i+2}$, i.e., $\sigma(M_l^\alpha)=M_{l+2}$, $i\leqslant l\leqslant k$.

1.3.4. *The higher relative commutants.* Since $[M_k:M_l]<\infty$ and $[N_l:N_k]<\infty$, the relative commutants $M_l'\cap M_k$ and $N_l'\cap N_k$, $k>l\geqslant 1$, are all finite dimensional algebras. They satisfy the commuting square condition, namely:

(i) $E_{N_j'\cap N_k} E_{N_i} = E_{N_j'\cap N_i}$, $i\geqslant k$.

(ii) $E_{M_j'\cap M_k} E_{M_i} = E_{M_j'\cap M_i}$, $k\geqslant i$.

Since the Jones tower of factors is canonical, the sequence of finite dimensional algebras of higher relative commutants $\{M_i'\cap M_j\}_{j>i\geqslant 0}$ is canonically associated to $N\subset M$.

The tunnel is not canonical, yet the τ-preserving isomorphism class of the corresponding sequence of higher relative commutants $\{N_k'\cap N_l\}_{k>l\geqslant -1}$ does not depend on the choice of the tunnel, but just on the initial inclusion $N\subset M$ ([Po5]).

Moreover, by the last remark in 1.2.1 (see 3.1 in [Po5]), it follows that there exists a canonical antiisomorphism between (the isomorphism classes of) the two sequences $\{N_k'\cap N_l\}_{k\geqslant l\geqslant -1}$ and $\{M_i'\cap M_j\}_{j\geqslant i\geqslant 0}$, carrying $N_k'\cap N_l$ onto $M_{l+1}'\cap M_{k+1}$ and e_{-i} onto e_{i+2}. This antiisomorphism is trace preserving if and only if $N\subset M$ is extremal ([PiPo2], [Po5]).

1.3.5. *The standard, or principal graph and matrix of* $N\subset M$ ([J3], [GHJ]). The sequence of consecutive inclusions of higher relative commutants of M, $\mathbf{C}=M'\cap M\subset M'\cap M_1\subset\ldots$, is completely determined by just one pointed matrix over $\mathbf{Z}_+$, called the *standard matrix* of $N\subset M$ and denoted $\Gamma_{N,M}=(a_{kl})_{k\in K, l\in L}$, $k_0\in K$, a fact that was first noted by Jones in early 1983. Alternatively, $\Gamma_{N,M}$ can also be regarded as a bipartite (pointed) graph, with the points of the sets K as *even vertices* and the set L as *odd vertices* and having a_{kl} edges from k to l. This graph is called the *standard* (or the *principal*) *graph* of $N \subset M$ and it is still denoted by $\Gamma_{N,M}$ with the vertex $k_0 \in K$ sometimes denoted by $*$. For the proof of the next properties of $\Gamma_{N,M}$ see [Po5].

The matrix Γ is always irreducible, equivalently, as a graph, Γ is connected. Following [Oc1], [GHJ], if Γ is finite we say that $N \subset M$ has *finite depth.*

Let $K_0 = \{k_0\}$, $L_i = \{l \in L \mid a_{kl} \neq 0, \text{for some } k \in K_{i-1}\}$ and $K_i = \{k \in K \mid a_{kl} \neq 0, \text{for some } l \in L_i\}$, $i \geqslant 1$, so that $K = \bigcup_i K_i$, $L = \bigcup_i L_i$.

Then the sets of irreducible components of $M' \cap M_{2i}$ and $N_{2i-1} \cap M$ (resp. $M' \cap M_{2i+1}$ and $N'_{2i} \cap M$) are identified with K_i (resp. L_i) and the embeddings $K_i \subset K_{i+1}$ by the correspondence between the centers of $M' \cap M_{2i}$ and $M' \cap M_{2i+2}$ given by

$$\mathcal{Z}(M' \cap M_{2i}) \ni z \mapsto \text{unique } z_1 \in \mathcal{Z}(M' \cap M_{2i+2}), \quad \text{with } e_{2i+2} z_1 = e_{2i+2} z$$

and similarly for $L_i \subset L_{i+1}$.

With these conventions, the embedding matrix for $M' \cap M_{2i} \subset M' \cap M_{2i+1}$ is $\Gamma|_{K_i}$ and for $M' \cap M_{2i+1} \subset M' \cap M_{2i+2}$ it is $\Gamma^t|_{L_i}$.

The corresponding inclusions of higher relative commutants in the tunnel are given by Γ for the identification of $N'_{i-1} \cap M$ with $M' \cap M_i$ given by the canonical antiisomorphism.

Note that by duality (1.3.3) we have $\Gamma_{N,M} = \Gamma_{N_{2i}, N_{2i-1}} = \Gamma_{M_{2i-1}, M_{2i}}$, $i \geqslant 1$, but that in general $\Gamma_{N,M}$ may not be equal to $\Gamma_{N_1,N}$ (see e.g. [Oc2], [Ka1]). The next result relates the norm of $\Gamma_{N,M}$ with the size of the higher relative commutants.

PROPOSITION.

$$\begin{aligned} \|\Gamma_{N,M}\|^2 &= \lim_{n\to\infty} (\dim M' \cap M_n)^{1/n} \\ &= \lim_{n\to\infty} (\dim N' \cap M_n)^{1/n} = \|\Gamma_{M,M_1}\|^2 \leqslant [M:N]. \end{aligned}$$

Proof. Let $T = \Gamma_{N,M} \Gamma^t_{N,M}$. If δ_{k_0} denotes the vector $\delta_{k_0} = (\delta_{kk_0})_{k \in K}$ then, by the definition of $\Gamma_{N,M}$, $\dim(M' \cap M_{2n}) = \|T^n \delta_{k_0}\|_2^2$, the norm being in $l^2(K)$. We thus get $\limsup_{n\to\infty} (\dim(M' \cap M_{2n}))^{1/2n} \leqslant \lim_n \|T^n\|^{1/n} = \|T\|$. Conversely, let $\varepsilon > 0$ and let n_0 be large enough such that $K_0 = \{k \in K \mid k \text{ is connected with } k_0 \text{ after at most } n_0 \text{ steps on } T = \Gamma_{N,M}\Gamma^t_{N,M}\}$ satisfies $\|T_{K_0}\| \geqslant \|T\| - \varepsilon$, where $T_{K_0} = \delta_{K_0} T|_{K_0}$ is the restriction (on both sides) of T to K_0. Since $T^n \delta_{k_0} \geqslant (T_{K_0})^n \delta_{k_0}$ and since by the Perron–Frobenius theorem we have $\lim_{n\to\infty} \|(T_{K_0})^n \delta_{k_0}\|_2^{1/n} = \|T_{K_0}\|$, we get $\liminf_n \|T^n \delta_{k_0}\|_2^{1/n} \geqslant \|T_{K_0}\| \geqslant \|T\| - \varepsilon$. Since ε is arbitrary, $\liminf_n \|T^n \delta_{k_0}\|_2^{1/n} \geqslant \|T\|$. We thus get $\|T\| \geqslant \limsup_n \|T^n \delta_{k_0}\|_2^{1/n} \geqslant \liminf_n \|T^n \delta_{k_0}\|_2^{1/n} \geqslant \|T\|$, so that $\|T\| = \lim_{n\to\infty} (\dim M' \cap M_{2n})^{1/2n}$. Also, we have

$$\begin{aligned} \|T\|^2 &= \lim_n (\dim(M' \cap M_{2n}))^{1/n+1} \leqslant \liminf_n (\dim(M' \cap M_{2n+1}))^{1/n+1} \\ &\leqslant \limsup_n (\dim(M' \cap M_{2n+1}))^{1/n+1} \leqslant \lim_n (\dim(M' \cap M_{2n+2}))^{1/n+1} = \|T\|^2. \end{aligned}$$

This shows that $\lim_n(\dim(M'\cap M_n))^{1/n}=\|T\|=\|\Gamma_{N,M}\|^2$. The inequalities

$$\begin{aligned}\|T\|^2&=\lim_n(\dim(M'\cap M_{2n}))^{1/n+1}\\&\leqslant\liminf_n(\dim(N'\cap M_{2n}))^{1/n+1}\\&\leqslant\limsup_n(\dim(N'\cap M_{2n}))^{1/n+1}\\&\leqslant\limsup_n(\dim(N_1'\cap M_{2n}))^{1/n+1}\\&\leqslant\limsup_n(\dim(M'\cap M_{2n+2}))^{1/n+1}=\|T\|^2\end{aligned}$$

show that $\lim_n(\dim(N'\cap M_n))^{1/n}=\|T\|$ as well. Since $\dim M'\cap M_n\leqslant[M_n:M]=[M:N]^n$ (see e.g. [J1]), by the first part we get $\|T\|\leqslant[M:N]$. □

COROLLARY. *Assume $N\subset M$, $Q\subset P$ are inclusions of type* II_1 *factors with the same index and such that* $\dim(M'\cap M_{2i})\leqslant\dim(P'\cap P_{2i})$, $\forall i$. *Then* $\|\Gamma_{N,M}\|\leqslant\|\Gamma_{Q,P}\|$.

Proof. Trivial by the proposition. □

Finally note that the standard graph $\Gamma_{N_i,M}$ of $N_i\subset M$ is equal to $\Gamma_{N,M}\Gamma^t_{N,M}\dots$, the product taking alternatively $\Gamma_{N,M},(\Gamma_{N,M})^t$, $i+1$ times in total. So for i odd $\Gamma_{N_i,M}$ is indexed by K on both sides and for i even it is by K to the left and L to the right. As a pointed graph, $\Gamma_{N_i,M}$ has the same $k_0\in K$ (or $*$) as $\Gamma_{N,M}$.

While Γ completely determines the algebras $M'\cap M_i$, $N_j'\cap M$ and their consecutive inclusions, up to isomorphism, the trace will be determined by:

1.3.6. *The standard eigenvector.* There exist unique vectors $\vec{s}=(s_k)_{k\in K}$, $\vec{t}=(t_l)_{l\in L}$ such that the trace of the minimal projections of the kth summand of $M'\cap M_{2i}$ (respectively the lth summand of $M'\cap M_{2i+1}$) is given by $[M:N]^{-i}s_k$ (resp. $[M:N]^{-i}t_l$).

Similarly, for $N_j'\cap M$, the traces are described by the vectors $\vec{s}'=(s_k')_{k\in K}$, $\vec{t}'=(t_l')_{l\in L}$.

One has the relations:

(i) $\Gamma\vec{t}=\vec{s}$, $\Gamma\vec{t}'=\vec{s}'$;

(ii) $\Gamma^t\vec{s}=[M:N]\vec{t}$, $\Gamma^t\vec{s}'=[M:N]\vec{s}'$;

so that, for $\Gamma\Gamma^t$, $\Gamma^t\Gamma$ they are actually eigenvectors:

(iii) $\Gamma^t\Gamma\vec{t}=[M:N]\vec{t}$, $\Gamma^t\Gamma\vec{t}'=[M:N]\vec{t}'$;

(iv) $\Gamma\Gamma^t\vec{s}=[M:N]\vec{s}$, $\Gamma\Gamma^t\vec{s}'=[M:N]\vec{s}'$.

The vectors $\vec{s}$, $\vec{s}'$, $\vec{t}$, $\vec{t}'$ are called the *standard eigenvectors* (*or weights*) of Γ.

The fact that $\vec{s}$ may be different from $\vec{s}'$ corresponds to the case when the anti-isomorphism from $N_i'\cap M$ to $M'\cap M_{i+1}$ is not trace preserving, i.e., when $N\subset M$ is not extremal.

When $N\subset M$ is extremal we have, $\vec{s}=\vec{s}'$, $\vec{t}=\vec{t}'$ and $(s_k)_{k\in K}$ has a remarkable significance: if $k\in K_i$ and p is a minimal projection in the kth summand of $M'\cap M_{2i}$ (or $N'_{2i-1}\cap M$) then $[pM_{2i}p:Mp]=s_k^2=[pMp:N_{2i-1}p]$.

We will now show that even if $N\subset M$ is not extremal the local indices will still give an eigenvector for $\Gamma_{N,M}\Gamma^t_{N,M}$, corresponding to a remarkable eigenvalue.

PROPOSITION. (i) *If $k\in K$ then for any i for which $k\in K_i$ and any minimal projection p (resp. p') in the kth summand of $M'\cap M_{2i}$ (resp. $N'_{2i-1}\cap M$) we have*:

$$[pM_{2i}p:Mp]=[p'Mp':N_{2i-1}p']=s_k s'_k.$$

(i′) *If $l\in L$ then for any j for which $l\in L_j$ and any minimal projection q (resp. q') in the lth summand of $M'\cap M_{2i+1}$ (resp. $N'_{2i}\cap M$) we have*

$$[qM_{2i+1}q:Mq]=[q'Mq':N_{2i}q']=[M:N]t_l t'_l.$$

(ii) *If $v_k=(s_k s'_k)^{1/2}$, $u_l=([M:N]t_l t'_l)^{1/2}$ and $\alpha=(\sum a_{k_0,l}u_l)^2$ then we have*

$$\Gamma\vec{u}=\alpha^{1/2}\vec{v},\quad \Gamma^t\vec{v}=\alpha^{1/2}\vec{u},$$
$$\Gamma^t\Gamma\vec{u}=\alpha\vec{u},\quad \Gamma\Gamma^t\vec{v}=\alpha\vec{v}.$$

(iii) *α coincides with the minimal index of $N\subset M$, i.e., $\alpha=\operatorname{Ind}E_{\min}^{N,M}\leqslant[M:N]$ and $\alpha=[M:N]$ if and only if $N\subset M$ is extremal.*

(iv) $\|\Gamma_{N,M}\|^2\leqslant\alpha$.

Proof. (i) and (i′) follow trivially from Jones' formula 1.2.5 (i), since $\tau'(p)=\tau(p')$, $\tau'(q)=\tau(q')$.

Direct computation shows that $E_{\min}^{M_1,M}$ is the unique expectation of M_1 onto M such that $E_{\min}^{M_1,M}(q)$ are proportional to $[qM_1q:Mq]^{1/2}$, for $q\in M'\cap M_1$ and that $(\operatorname{Ind}E_{\min}^{M_1,M})^{1/2}$ is the factor of proportionality (see [Hi]). Thus, since $\sum_q E_{\min}(q)=1$, one has

$$\begin{aligned}(\operatorname{Ind}E_{\min}^{M_1,M})^{1/2}&=\sum_q(\operatorname{Ind}E_{\min}^{M_1,M})^{1/2}E_{\min}^{M_1,M}(q)\\&=\sum_q[qM_1q:Mq]^{1/2}=\sum_l a_{k_0l}u_l=\alpha^{1/2},\end{aligned}$$

where the sums are taken over a partition of 1 with minimal projections of $M'\cap M_1$. The same computation for $N\subset M$ shows that $\operatorname{Ind}E_{\min}^{M,N}=\alpha$ as well (since $[qM_1q:Mq]=[q'Mq':Nq']=u_l^2$). By [L3] $E_{\min}^{M_k,M}=E_{\min}^{M_1,M}\dots E_{\min}^{M_k,M_{k-1}}$ and $E_{\min}^{M_k,M}|_{M'\cap M_k}$ defines a trace τ_m on $\bigcup_k(M'\cap M_k)$ with $\tau_m(q_l^{2i+1})=\alpha^{-1}\tau_m(q_l^{2i-1})$, $\tau_m(p_k^{2i+2})=\alpha^{-1}\tau_m(p_k^{2i})$, for $k\in K_i$, $l\in L_i$ and $q_l^{2i\pm1}, p_k^{2i+2}, p_k^{2i}$ minimal projections in the corresponding direct summands of $M'\cap M_{2i\pm1}$, $M'\cap M_{2i}$, $M'\cap M_{2i+2}$. It follows by (i) and (i′) that $\tau_m(p_k^{2i})=\alpha^{-i}v_k$, $\tau_m(q_l^{2i+1})=\alpha^{-i-1/2}u_l$ and we get (ii) by the relations $\sum_l a_{kl}\tau_m(q_l^{2i+1})=\tau_m(p_k^{2i})$, $\sum_k a_{kl}\tau_m(p_k^{2i+2})=\tau_m(q_l^{2i+1})$.

To show (iv), note that

$$\begin{aligned}\|\Gamma_{N,M}\|^2 &= \lim_n(\langle(\Gamma_{N,M}\Gamma_{N,M}^t)^n\delta_{k_0},\delta_{k_0}\rangle)^{1/n}\\ &\leqslant \lim_n(\langle(\Gamma\Gamma^t)^n\delta_{k_0},\vec{v}\rangle)^{1/n}=\lim_n(\langle\delta_{k_0},(\Gamma\Gamma^t)^n\vec{v}\rangle)^{1/n}\\ &= \lim_n(\langle\delta_{k_0},\alpha^n\vec{v}\rangle)^{1/n}=\alpha.\end{aligned}$$ □

We call the vectors $\vec{v}=(v_k)_{k\in K}$, $\vec{u}=(u_l)_{l\in L}$ in the above proposition the *standard vectors of even* (resp. *odd*) *local indices* in the Jones tower.

Remark. Note that the proofs of 1.3.5 and of (iii) in the above proposition show that the following holds true: if A is an arbitrary irreducible symmetric matrix operator over $\mathbf{N}$ with nonnegative entries and $\delta\in l^2(\mathbf{N})$ is a nonnegative vector $\neq 0$, then $\lim_n\|A^n\delta\|_2^{1/n}=\|A\|$ and $\|A\|\leqslant\inf\{\alpha>0\mid \text{there exists } \vec{v}=(v_k)_{k\in\mathbf{N}}, v_k\geqslant 0 \text{ not all zero, such that } A\vec{v}=\alpha\vec{v}\}$.

Note that in the finite depth case $\|\Gamma_{N,M}\|^2=[M:N]$ ([J3]), $N\subset M$ follows automatically extremal and the vector $\vec{s}=\vec{s}\,'$ coincides with the unique Perron–Frobenius eigenvector of $\Gamma\Gamma^t$ normalized so that $s_{k_0}=1$ (see e.g. [Po5]). In particular, $\vec{s}$ is uniquely determined once $\Gamma_{N,M}$ is known. The extremality property is in fact a consequence of the relation $\|\Gamma_{N,M}\|^2=[M:N]$. Indeed we have:

Corollary. (i) *If $\|\Gamma_{N,M}\|^2=[M:N]$ then $N\subset M$ is extremal.*

(ii) *If $N\subset M$ is not extremal, then*:

$$\limsup_k s_k=\limsup_k s'_k=\limsup_l t_k=\limsup_l t'_l=\infty.$$

Proof. (i) If $N\subset M$ is not extremal then $\alpha=\operatorname{Ind}E_{\min}^{N,M}<[M:N]$. But $\alpha\geqslant\|\Gamma_{N,M}\|^2$.

(ii) If $\{s_k\}$ is bounded then there exists $c'>0$ such that $c'\vec{s}\leqslant\vec{v}$ (since $\vec{v}$ is bounded from below by $v_{k_0}=1$). Let $c=\sup\{c'>0\mid c'\vec{s}\leqslant\vec{v}\}$. Then $\vec{v}-c\vec{s}\geqslant 0$ and we have

$$0\leqslant\Gamma\Gamma^t(\vec{v}-c\vec{s})=\alpha\vec{v}-[M:N]c\vec{s}=\alpha(\vec{v}-([M:N]/\alpha)c\vec{s}).$$

Since $[M:N]/\alpha>1$ when $N\subset M$ is not extremal, this contradicts the choice of c. Similarly, we get that $\vec{s},\vec{t},\vec{t}\,'$ are unbounded. □

Further remarks. (1) Since $v_k=(s_k s'_k)^{1/2}$, $u_l=([M:N]t_l t'_l)^{1/2}$ are local indices, by Jones' theorem $v_k, u_l<2$ implies they are of the form $2\cos\pi/n$. This observation can be used to exclude a number of matrices from being standard matrices of inclusions.

(2) The result in ([PiPo1, 1.7]) translates into the following property of $(\Gamma_{N,M},\vec{s})$: if $N'\cap M=\mathbf{C}$ (equivalently $L_0=\{l_0\}$) then $\operatorname{card}\{k\in K_1\,|\,s_k=1\}=\operatorname{card}\{k\in K_1\,|\,a_{kl}\neq 0$ only if $l=l_0\}=\operatorname{card}\mathcal{N}_{M_1}(M)/\mathcal{U}(M)=\operatorname{card}G$, where $\mathcal{N}(M)''=M\rtimes G$. In particular, if n denotes this number then $[M:N]/n<4$ implies $[M:N]/n=4\cos^2\pi/n$, for some $n\geqslant 3$. Also, 1.9 in [PiPo1] translates into the following: if for some k in K_1 we have $s_k<n+1$ then $a_{k,l_0}\leqslant n$. In particular, if $s_k<2$ then $a_{k,l_0}=1$.

(3) If Γ is an infinite irreducible matrix with an eigenvector $\xi=(\xi_k)_{k\in K}$, $\xi_k>0$, $\lim\xi_k=0$, $\Gamma\Gamma^t\xi=[M:N]\xi$, then Γ cannot be the standard matrix of $N\subset M$, with $N\subset M$ extremal. Indeed, because otherwise $\Gamma\Gamma^t s=[M:N]s$ and $s_k\geqslant s_{k_0}$, $\forall k$, so that there exists $c>0$ such that $s-c\xi\geqslant 0$ and $s_k-c\xi_k=0$ for at least one k, but only finitely many. Since $\Gamma\Gamma^t$ is irreducible, $(\Gamma\Gamma^t)^n(s-c\xi)$ has only positive entries for large n and is a multiple of $s-c\xi$, a contradiction.

Note that the same argument shows that if $\Gamma\Gamma^t$ admits an eigenvector ξ with $\limsup_k\xi_k<\xi_{k_0}$ then (Γ,k_0) cannot be a standard matrix for a subfactor of index $\|\Gamma\|^2$.

1.3.7. *The standard invariant* (*or the paragroup*) *of* $N\subset M$. The standard graph $(\Gamma_{N,M},*)$ with its standard weight $\vec{s}$ do not completely determine the "tableau" of all higher relative commutants and of all their inclusions $\{M'_i\cap M_j\}_{j\geqslant i\geqslant 0}$ (or $\{N'_i\cap N_j\}_{i\geqslant j\geqslant -1}$) with their corresponding traces, but just the part $\{M'\cap M_i\}_i$ (resp. $\{N'_i\cap M\}_i$).

Since by the last part of 1.3.4 (see also [PiPo2]) the isomorphism 1.3.3 (see [PiPo1], [PiPo2]) from the $[M:N]$-amplification of M_i to M_{i+2}, implements an isomorphism from $M'_j\cap M_i$ onto $M'_{j+2}\cap M_{i+2}$, it follows that in fact all the information is contained in the sequence of consecutive inclusions $\{M'_1\cap M_i\subset M'\cap M_i\}_{i\geqslant 1}$.

We will call the trace preserving isomorphism class of the sequence of canonical commuting squares

$$\begin{array}{ccccccccc}
\mathbf{C}=M'\cap M & \subset & M'\cap M_1 & \subset & M'\cap M_2 & \subset \dots \subset & M'\cap M_i & \subset \\
 & & \cup & & \cup & & \cup & \\
 & & \mathbf{C}=M'_1\cap M_1 & \subset & M'_1\cap M_2 & \subset \dots \subset & M'_1\cap M_i & \subset
\end{array}$$

the *opposite standard invariant* of $N\subset M$ and denote it $\mathcal{G}^{\mathrm{op}}_{N,M}$.

We will call the trace preserving isomorphism class of the sequence of canonical commuting sequences

$$\begin{array}{ccccccccc}
\mathbf{C}=M'\cap M & \subset & N'\cap M & \subset & N'_1\cap M & \subset \dots \subset & N'_i\cap M & \subset \\
 & & \cup & & \cup & & \cup & \\
 & & \mathbf{C}=N'\cap N & \subset & N'_1\cap N & \subset \dots \subset & N'_i\cap N & \subset
\end{array}$$

the *standard invariant* of $N\subset M$ and denote it $\mathcal{G}_{N,M}$. If $N^0\subset M^0$ is another inclusion, then an isomorphism between $\mathcal{G}_{N,M}$ and $\mathcal{G}_{N^0,M^0}$ means a trace preserving isomorphism of $\bigcup(N_i'\cap M)$ onto $\bigcup(N_i^{0\prime}\cap M^0)$ carrying $N_j'\cap M$ onto $N_j^{0\prime}\cap M^0$ and $N_j'\cap N$ onto $N_j^{0\prime}\cap N^0$, $\forall j$. Similarly for antiisomorphisms. Note that in the case $N\subset M$ is extremal, $\mathcal{G}_{N,M}$ is antiisomorphic to $\mathcal{G}_{N,M}^{\mathrm{op}}$ and also antiisomorphic to $\mathcal{G}_{N^{\mathrm{op}},M^{\mathrm{op}}}$ ([Po5]).

In the case $N\subset M$ has finite depth, $\mathcal{G}_{N,M}$, $\mathcal{G}_{N,M}^{\mathrm{op}}$ are uniquely determined by just one commuting square:

$$\begin{array}{ccc} M'\cap M_{i_0} & \subset & M'\cap M_{i_0+1} \\ \cup & & \cup \\ M_1'\cap M_{i_0} & \subset & M_1'\cap M_{i_0+1} \end{array}$$

with i_0 large enough (e.g. $i_0\geqslant \operatorname{card} K$). Note that if $i_0\geqslant \operatorname{card} K$ then such a commuting square is nondegenerate and $[M:N]^{-1}$-Markov, the rest of the higher relative commutants in $\mathcal{G}_{N,M}$ resulting from the basic construction. A finite dimensional nondegenerate Markov commuting square which is isomorphic to such a commuting square of higher relative commutants is called a *standard commuting square*.

In general, the commuting square condition and the standard matrix describing the inclusions, are not however sufficient to characterize $\mathcal{G}_{N,M}$. In the finite depth case an axiomatization for $\mathcal{G}_{N,M}$ was given in [Oc2] who calls it in this case the paragroup of $N\subset M$ and uses Connes' correspondences (or bimodule) multiplicative structure ([C6], [Po6]) to describe it.

Ocneanu's idea is to regard $\mathcal{G}_{N,M}$ as the (tensor) category with 2 objects $\{N,M\}$ and with the morphisms given by all Q–P correspondences $\mathcal{H}$, with $Q,P\in\{N,M\}$, that are contained in a correspondence generated (multiplicatively) by $L^2(M)$,

$$\mathcal{H}\subset L^2(M)\underset{P_1}{\otimes}L^2(M)\underset{P_2}{\otimes}L^2(M)\ldots,\quad P_i\in\{N,M\}$$

(only $P_i=N$ are of course sufficient). Also, each $\mathcal{H}$ is regarded with a weight assigned to it, which is given by $(\dim_{Q,P}\mathcal{H})^{1/2}$, the dimension of $\mathcal{H}$ as defined in [Po6]. The principal part of the fusion rule matrix of this category coincides with the standard graph $\Gamma_{N,M}$ and the above weights with the standard vectors of $\Gamma_{N,M}$.

Following [Oc2], when regarded in this equivalent way $\mathcal{G}_{N,M}$ will be called the *paragroup* of $N\subset M$.

Note that the invariant $\mathcal{G}_{N,M}$ cannot distinguish one from another the amplifications of $N\subset M$, more precisely, $\mathcal{G}_{N,M}=\mathcal{G}_{N^\alpha,M^\alpha}$, $\mathcal{G}_{N,M}^{\mathrm{op}}=\mathcal{G}_{N^\alpha,M^\alpha}^{\mathrm{op}}$, for any $\alpha>0$, where $N^\alpha\subset M^\alpha$ is defined as $N\otimes M_{n\times n}(\mathbf{C})\subset M\otimes M_{n\times n}(\mathbf{C})$ for $\alpha=n$ and as a reduced of such an inclusion by a projection of trace α/n if $n-1<\alpha<n$, $n\geqslant 1$. This fact is trivial by the definitions.

1.4. Core and model inclusions

The standard invariant of an inclusion gives rise to a natural model inclusion of hyperfinite type II_1 algebras that we will now describe.

1.4.1. *Definitions.* Let $M_\infty=\overline{\bigcup_k M_k}$, the closure being taken in the weak topology given by the unique trace on $\bigcup_k M_k$. M_∞ is thus a type II_1 factor, called the *enveloping algebra* of $N\subset M$.

Then the (isomorphism class of the) inclusion of algebras $M_1'\cap M_\infty\subset M'\cap M_\infty$ is called the *opposite model* of $N\subset M$. The algebras $M_1'\cap M_\infty$, $M'\cap M_\infty$ are not factors in general. Since $\{e_i\}_{i\geqslant 3}\subset M_1'\cap M_\infty$, they are always of type II_1 and they are clearly approximately finite dimensional (i.e., hyperfinite). The opposite model is uniquely determined by $\mathcal{G}^{\mathrm{op}}_{N,M}$.

If $\{N_k\}_{k\geqslant 1}$ is a choice of a tunnel, then the inclusion

$$S=\overline{\bigcup_k (N_k'\cap N)}\subset \overline{\bigcup_k (N_k'\cap M)}=R$$

is called the *core* associated to $\{N_k\}_k$. By 1.3.5, the isomorphism class of $S\subset R$ depends only on $N\subset M$. It is called the *standard part* (*or model*) of $N\subset M$ and it is denoted $N^{\mathrm{st}}\subset M^{\mathrm{st}}$. $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ need not be factors but they are hyperfinite of type II_1. The standard model with its trace is uniquely determined by $\mathcal{G}_{N,M}$.

In case $N\subset M$ is extremal, the standard model is antiisomorphic to the opposite model (cf. 1.3.5).

Note that if $N\subset M$ has finite depth then all the algebras N^{st}, M^{st}, $M_1'\cap M_\infty$, $M'\cap M_\infty$ follow factors and $[M:N]=\|\Gamma_{N,M}\|^2$ (cf. [Po5], [J3]). Also, $N\subset M$ is then necessarily extremal (cf. [Po5] or 1.3.6 above). In fact even $N^{\mathrm{st}}\subset M^{\mathrm{st}}$, $M_1'\cap M_\infty\subset M'\cap M_\infty$ follow extremal. Indeed, since these two inclusions are antiisomorphic (by the extremality of $N\subset M$), we only need to prove the extremality of $N^{\mathrm{st}}\subset M^{\mathrm{st}}$. But the traces of the minimal central projections of $N_{2n-1}'\cap M$ are given by $\vec{c}_n=\vec{s}\cdot(\lambda\Gamma_{N,M}\Gamma_{N,M}^t)^n\delta_{k_0}$. By the Perron–Frobenius theorem, as n tends to ∞, $\vec{c}_n$ has the entries tending to a multiple of $\vec{s}\cdot\vec{s}=(s_k^2)_{k\in K}$. Similarly for $N_{2n+1}'\cap N$. Using Example 2.2.2 in [PiPo3] it follows by 2.7 in [PiPo3] that

$$\lim_{n\to\infty} E_{(N_{2n+1}'\cap N)'\cap(N_{2n+1}'\cap M)}(e_0)=\lambda 1,$$

showing that $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ is extremal.

1.4.2. *Ergodicity of the core and of* $\Gamma_{N,M}$. The factorality of M^{st} (or $M'\cap M_\infty$) is strongly related to the ergodic properties of the weighted graph $(\Gamma_{N,M},\vec{s})$.

PROPOSITION. *The following conditions are equivalent*:

(i) M^{st} *is a factor.*

(ii) *Up to a scalar multiple,* $\vec{s}$ *is the unique* $\vec{s}$*-bounded eigenvector of* $\Gamma_{N,M}\Gamma_{N,M}^t$ *corresponding to the eigenvalue* $[M:N]$.

(iii) *If* $P=\lambda s\Gamma_{N,M}\Gamma_{N,M}^t s^{-1}$, *s being the diagonal operator given by* $\vec{s}$ *and* $\lambda=[M:N]^{-1}$, *then* $\lim_n \|P^n\delta_{k_1}-P^n\delta_{k_0}\|_1=0$, $\forall k_1\in K$.

Proof. (ii) and (iii) are known to be equivalent from the general theory of Markov chains (cf. e.g. [Or] and [HaG]). If $p_1,p_2\in N'_{2i}\cap M$ are minimal projections in the k_1 resp. k_2 summands of $N'_{2i_0-1}\cap M$ then the traces of $(p_jp_k^n)_{k\in K_n}$ corresponding to the summand k ($\in K_n$), are given by the vector $\lambda^n s(\Gamma\Gamma^t)^{n-i_0}\delta_{k_j}$. Thus (iii) is equivalent to the fact that the traces of $(p_jp_k^n)_{k\in K_n}$ tend to be proportional to the numbers s_{k_j} (which are proportional to the traces of p_j). This is the same as saying that

$$\|Ct_{N'_{2n-1}\cap M}(p_j)-\tau(p_j)1\|_1\to 0,$$

which is the same as saying that $N'_{2n-1}\cap M$ tends to a factor. Thus (iii) $\Leftrightarrow$ (i). $\square$

Definition. If $N\subset M$ satisfies the equivalent conditions of the preceding proposition then we say that $N\subset M$ has *ergodic core*. Also, in case $(\Gamma_{N,M},\vec{s})$ satisfies condition (iii) we say that $\Gamma_{N,M}$ is *ergodic*. Note that in the case $N\subset M$ is extremal these conditions are also equivalent to the factoriality of $M'\cap M_\infty$ (by trace preserving antiisomorphism). This fact is actually true in general, but the proof is longer and will not be given here.

Typically, subfactors with small index have ergodic $\Gamma_{N,M}$: we already pointed this out in 1.3.6 for $\Gamma_{N,M}$ finite, thus for $[M:N]<4$. From the previous considerations we can deduce here some further ergodical properties.

COROLLARY. (i) *If* $N\subset M$ *is extremal then* $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ *is weakly irreducible, i.e.,* $\mathcal{Z}(N^{\mathrm{st}})\cap\mathcal{Z}(M^{\mathrm{st}})=\mathbf{C}$. *Thus* $\mathcal{Z}(N^{\mathrm{st}})$, $\mathcal{Z}(M^{\mathrm{st}})$ *are either both atomic or both diffuse.*

(ii) *If* $\Gamma_{N,M}$ *coincides with* A_∞ *outside a finite set* (*of vertices*) *then* $\Gamma_{N,M}$ *is ergodic.*

(iii) *If* $[M:N]\leqslant 2+\sqrt{5}$ *then* $\Gamma_{N,M}$ *is ergodic.*

Proof. (i) If $\mathcal{Z}(N^{\mathrm{st}})\cap\mathcal{Z}(M^{\mathrm{st}})\neq\mathbf{C}$ then by antiisomorphism we get $\mathcal{Z}(M_1'\cap M_\infty)\cap\mathcal{Z}(M'\cap M_\infty)\neq\mathbf{C}$, thus $\mathcal{Z}((M_1'\cap M_\infty)'\cap M_\infty)\cap\mathcal{Z}((M'\cap M_\infty)'\cap M_\infty)\neq\mathbf{C}$. Thus, there exists $p\notin\mathbf{C}1$, $[p,M]=0=[p,M_1]$ and $[p,e_i]=0$, $\forall i\geqslant 2$. But $\mathrm{vN}(M_1,e_2,e_3,\ldots)=M_\infty$ and $\mathcal{Z}(M_\infty)=\mathbf{C}1$, a contradiction.

(ii) Follows from the previous proposition and [HaG].

(iii) By [GHJ] all matrices $\Gamma_{N,M}$ of norm $\leqslant 2+\sqrt{5}$ satisfy (ii), are finite, or are of the form $A_\infty^{(1)}=A_{\infty,\infty}$ or $T_{2,\infty,\infty}$ ([GHJ]). But $\Gamma_{N,M}=A_\infty^{(1)}$ implies $N'\cap M\neq\mathbf{C}$ so that if $[M:N]\leqslant 2+\sqrt{5}$ then $N'\cap M=\mathbf{C}^2$ and $N\subset M$ is necessarily locally trivial, thus ergodic.

Since $\|T_{2,\infty,\infty}\|^2=2+\sqrt{5}$, if $[M:N]\leqslant 2+\sqrt{5}$ and $\Gamma_{N,M}=T_{2,\infty,\infty}$ then $[M:N]=2+\sqrt{5}$ so that $T_{2,\infty,\infty}(T_{2,\infty,\infty})^t\vec{s}=(2+\sqrt{5})\vec{s}$. But an easy computation shows that there is a unique $\vec{s}$ satisfying this. (Note that in fact this unique $\vec{s}$ is square summable so by Further remarks (3) at 1.3.6, $T_{2,\infty,\infty}$ cannot in fact be the graph of a subfactor of index $2+\sqrt{5}$.) □

1.4.3. *Basic construction for cores and models.* If $S\subset R$ is a core corresponding to some tunnel $M\supset^{e_0}N\supset^{e_{-1}}N_1\supset^{e_{-2}}N_2\supset\dots$ then the Jones projection $e_1\in M_1$ implements the trace preserving conditional expectation of R onto S and $\mathrm{vN}(R,e_1)=\mathrm{sp}\,Re_1R$, so that $\mathrm{sp}\,Re_1R=\langle R,e_1\rangle=\langle R,e_s\rangle=\langle R,S\rangle$. Moreover $\langle R,e_1\rangle=\overline{\bigcup_k(N_k'\cap M_1)}$ and $R\subset\langle R,e_1\rangle$ is the core of $M\subset M_1$ corresponding to the tunnel $M_1\supset M\supset N\supset N_1\supset\dots$. Furthermore, if $S_1=\{e_0\}'\cap N$ then $S_1=\overline{\bigcup(N_k'\cap N_1)}$ and $S_1\subset S$ is the core of $N_1\subset N$ associated to the tunnel $N\supset N_1\supset\dots$.

All this follows trivially once we observe that if $\{m_j\}_j$ is an orthonormal basis of $\mathrm{vN}\{e_0,e_{-1},\dots\}$ $(\subset R)$ over $\mathrm{vN}\{e_{-1},e_{-2},\dots\}$ $(\subset S)$ then it is also an orthonormal basis of R over S and of M over N. Thus

$$\begin{aligned}\overline{\bigcup_k(N_k'\cap M_1)}\subset\sum_j m_je_1E_M\big(\overline{\bigcup_k(N_k'\cap M_1)}\big)&=\sum_j m_je_1\big(\overline{\bigcup_k(N_k'\cap M)}\big)\\ &=\sum_j m_je_1R\subset\mathrm{sp}\,Re_1R.\end{aligned}$$

Also, since $\sum m_jm_j^*=[M:N]$, $(N^{\mathrm{st}}\subset M^{\mathrm{st}})=(S\subset R)$ are $[M:N]^{-1}$-Markov inclusions.

As for the opposite model $M_1'\cap M_\infty\subset M'\cap M_\infty$, if $N\subset M$ is extremal, then $M_1'\cap M_\infty\subset M'\cap M_\infty\subset N'\cap M_\infty$ is antiisomorphic to $S\subset R\subset\langle R,e_1\rangle$ with the antiisomorphism carrying e_1 into e_1, and we have $N'\cap M_\infty=\langle M'\cap M_\infty,e_1\rangle=\langle M'\cap M_\infty,M_1'\cap M_\infty\rangle$ in the sense of 1.1.2. But when $N\subset M$ is not extremal e_1 does not implement the trace preserving expectation of $M'\cap M_\infty$ onto $M_1'\cap M_\infty$.

1.4.4. *Stability of model inclusions.* If $\alpha>0$ and $N^\alpha\subset M^\alpha$ is the α-amplification of $N\subset M$ then we have $((N^\alpha)^{\mathrm{st}}\subset(M^\alpha)^{\mathrm{st}})\simeq(N^{\mathrm{st}}\subset M^{\mathrm{st}})$ and $(M_1^{\alpha\prime}\cap M_\infty^\alpha\subset M^{\alpha\prime}\cap M_\infty^\alpha)\simeq(M_1'\cap M_\infty\subset M'\cap M_\infty)$. In fact we already pointed out the existence of such isomorphisms for the corresponding standard invariants (or paragroups) in 1.3.7.

In particular it follows that $(M_{2i-1}^{\mathrm{st}}\subset M_{2i}^{\mathrm{st}})\simeq(N^{\mathrm{st}}\subset M^{\mathrm{st}})$ and that $(M_{2i+1}'\cap M_\infty\subset M_{2i}'\cap M_\infty)\simeq(M_1'\cap M_\infty\subset M'\cap M_\infty)$, if $i\in\mathbf{Z}$, where $M_j=N_{-j-1}$ for $j<0$. Indeed, this is a consequence of 1.3.3 and the above.

In case $N^{\mathrm{st}},M^{\mathrm{st}}$ are factors it is easy to see that the inclusion $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ is stable (cf. e.g. [Bi1], [Po8]). One can in fact easily obtain the same result even if $N^{\mathrm{st}},M^{\mathrm{st}}$ are not factors:

PROPOSITION. *If $N \neq M$ and P is a copy of the hyperfinite type* II_1 *factor, or of an $n\times n$ matrix algebra then $(N^{\mathrm{st}}\subset M^{\mathrm{st}})\simeq(N^{\mathrm{st}}\otimes P\subset M^{\mathrm{st}}\otimes P)$ and $(M_1'\cap M_\infty\subset M'\cap M_\infty)\simeq((M_1'\cap M_\infty)\otimes P\subset(M'\cap M_\infty)\otimes P)$. Moreover, if $M\supset^{e_0}N\supset^{e_{-1}}N_1\supset\ldots$ is a choice of the tunnel, $S=\overline{\bigcup_k(N_k'\cap M)}\subset\overline{\bigcup_k(N_k'\cap M)}=R$ is the corresponding core and $P^0\subset S$ is a hyperfinite factor (finite or infinite dimensional) so that $R^0=(P^0)'\cap R$, $S^0=(P^0)'\cap S$ satisfy $R^0\overline{\otimes}P^0\simeq R^0\vee P^0=R$, $S^0\overline{\otimes}P^0\simeq S^0\vee P^0=S$, then $S^0\subset R^0$ is also a core, i.e., there exists a choice of the tunnel $M\supset N\supset N_1^0\supset N_2^0\supset\ldots$ so that*

$$R^0=\overline{\bigcup_k((N_k^0)'\cap M)},\quad S^0=\overline{\bigcup_k((N_k^0)'\cap N)}.$$

Proof. Both inclusions $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ and $M_1'\cap M_\infty\subset M'\cap M_\infty$ have the property that given any finite number of elements $x_1,\ldots,x_n$ in M^{st} (resp. $M'\cap M_\infty$) there exist unitary elements $u,v\in N^{\mathrm{st}}$ (resp. $M_1'\cap M_\infty$) such that $uv=-vu$ and

$$\|[u,x_i]\|_2<\varepsilon,\quad \|[v,x_i]\|_2<\varepsilon,\quad 1\leqslant i\leqslant n.$$

But then, following McDuff's original argument ([McD]), exactly like in the factorial case proved in [Bi1], by using the central trace instead of the scalar trace in all the estimates, one gets a sequence of commuting 2×2 matrix subalgebras $\{P_j\}_j$ in N^{st} (resp. in $M_1'\cap M_\infty$) such that $P\overset{\mathrm{def}}{=}\bigvee_k P_k=\overline{\bigotimes}_k P_k$ satisfies the requirements, proving the first part of the statement.

To prove the last part, let $\sigma:(S\overline{\otimes}P\subset R\overline{\otimes}P)\simeq(S\subset R)$ be an isomorphism and denote $e_{-k}^0=\sigma(e_{-k}\otimes 1)$, $S_k^0=\sigma(S_k\otimes P)$, where $S_0=S$, $S_k=\overline{\bigcup_i(N_i'\cap N_k)}=R\cap N_k$.

Note first that $S_k=\{e_{-k+1}\}'\cap S_{k-1}$. Indeed, $S_k=N_k\cap S_{k-1}=(\{e_{-k+1}\}'\cap N_{k-1})\cap S_{k-1}=\{e_{-k+1}\}'\cap S_{k-1}$. Then denote $N_k^0=\{e_{-k+1}^0\}'\cap N_{k-1}^0$, $k\geqslant 1$, where $N_0^0=N_0=N$. Since $e_{-k+1}^0\in S_k^0\subset N_k^0$ and since $E_{N_{k-1}^0}(e_{-k+1}^0)=E_{S_{k-1}^0}(e_{-k+1}^0)=\lambda 1$ (by the commuting square relations), it follows recursively that $M\supset^{e_0^0}N\supset^{e_{-1}^0}N_1^0\supset\ldots$ is a choice of the tunnel. Also, we have that $(N_k^0)'\cap M\subset(S_k^0)'\cap R$, by the definitions.

Since e_0 is in $\mathrm{vN}\{e_0,e-1,\ldots\}$, and e_0^0 in $\mathrm{vN}\{e_0^0,e_{-1}^0,\ldots\}$, which are subfactors of R, both e_0, e_0^0 have scalar central trace in R so that they are conjugate in R, say by a unitary v_0, i.e. $e_0^0=v_0e_0v_0$. Then, like in [PiPo1], $u_0=\lambda^{-1}E_S(v_0e_0)\in S$ is a unitary element satisfying $u_0e_0u_0^*=e_0^0$, $u_0S_1u_0^*=S_1^0$. One obtains in this way recursively some unitary elements $u_i\in S_i^0$ such that $\mathrm{Ad}\,u_iu_{i-1}\ldots u_0(e_j)=e_j^0$, $0\leqslant j\leqslant i$, $\mathrm{Ad}\,u_i\ldots u_0(N_j'\cap M)=N_j^{0\prime}\cap M$, $\mathrm{Ad}\,u_i\ldots u_0(S_j'\cap R)=S_j^{0\prime}\cap R$.

It follows that if we denote

$$\sigma_0:\overline{\bigcup_j(S_j'\cap R)}\to\overline{\bigcup_j((S_j^0)'\cap R)},\quad \sigma_0(x)=\lim_i\mathrm{Ad}\,u_i\ldots u_0(x),$$

then σ_0 is a trace preserving onto isomorphism and $\sigma_0(\overline{\bigcup_j(N_j'\cap M)})=\overline{\bigcup_j((N_j^0)'\cap M)}$. Since

$$\overline{\bigcup_j(S_j'\cap R)}=R,\quad \overline{\bigcup_j(N_j'\cap M)}=R,$$

we have

$$\overline{\bigcup_j((S_j^0)'\cap R)}=\sigma(R\otimes 1)\quad\text{and}\quad \overline{\bigcup_j((N_j^0)'\cap M)}\subset\overline{\bigcup_j((S_j^0)'\cap R)}=\sigma(R\otimes 1).$$

Thus

$$\overline{\bigcup_j((N_j^0)'\cap M)}=\sigma_0(\overline{\bigcup_j(N_j'\cap M)})=\sigma_0(R)=\sigma_0(\overline{\bigcup_j(S_j'\cap R)})=\overline{\bigcup_j((S_j^0)'\cap R)}=\sigma(R\otimes 1).\ \square$$

Note that the proof of the above proposition actually shows the following:

COROLLARY. *If $Q\subset P$ is isomorphic to $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ and if it is represented in $N\subset M$ (i.e., as a commuting square), then there exists a choice of a tunnel $M\supset N\supset N_1\supset\ldots$ such that the corresponding core coincides with $Q\subset P$.*

Remark. Note that if N^{st}, M^{st} are factors then $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ has the so called *generating property* ([Po5]), meaning that there exists a choice of the tunnel $M^{\mathrm{st}}\supset N^{\mathrm{st}}\supset N_1^{\mathrm{st}}\supset\ldots$ such that $(N_k^{\mathrm{st}})'\cap M^{\mathrm{st}}\nearrow M^{\mathrm{st}}$, in particular $(N^{\mathrm{st}}\subset M^{\mathrm{st}})^{\mathrm{st}}=N^{\mathrm{st}}\subset M^{\mathrm{st}}$ (i.e., the standard part of $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ coincides with that of $N\subset M$). Indeed, this is trivial since if $M\supset N\supset\ldots$ is a choice of a tunnel for $N\subset M$ then $N_k'\cap M\subset(N_k^{\mathrm{st}})'\cap M^{\mathrm{st}}$ (with the appropriate identifications). The same is true for $M_1'\cap M_\infty\subset M'\cap M_\infty$.

2. Representation theory for subfactors

Let $N\subset M$ be an inclusion of type II_1 factors with finite index. Unlike the single von Neumann algebras M for which the representations are simply normal embeddings of the given algebra M in other von Neumann algebras, e.g. in $\mathcal{B}(H)$ for some Hilbert space $\mathcal{H}$, in the case of inclusions of finite index $N\subset M$ one also has to take into account the expectation E_N. Since the morphisms between inclusions are the commuting squares, it is natural to define a representation as an embedding of $N\subset M$ into an inclusion of arbitrary von Neumann algebras $\mathcal{N}\subset\mathcal{M}$ with a conditional expectation $\mathcal{E}:\mathcal{M}\to\mathcal{N}$ which is compatible with E_N when restricted to M. A nondegeneracy condition similar to 1.1.5 has to be imposed as well. Equivalently we only want to consider representations of $N\subset M$ into inclusions $\mathcal{N}\subset\mathcal{M}$ with the same [PiPo1] index as $N\subset M$. Finally, for a finer theory, one needs to have some compatibility between the higher relative commutants of $\mathcal{N}\subset\mathcal{M}$ and $N\subset M$.

2.1. Definitions and motivations

2.1.1. *Definition.* Let $\mathcal{N}\subset\mathcal{M}$ be arbitrary von Neumann algebras with $\mathcal{E}:\mathcal{M}\to\mathcal{N}$ a normal faithful conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$. Assume $N\subset\mathcal{N}$, $M\subset\mathcal{M}$ and that $\mathcal{E}|_M=E_N$. We then say that $N\subset M$ is *embedded* or is *represented* in $(\mathcal{N}\subset\mathcal{M})$, or that

$$\begin{array}{ccc} \mathcal{N} & \overset{\mathcal{E}}{\subset} & \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

is a *commuting square.* The embedding (or representation, or commuting square) is *nondegenerate* if $\overline{\operatorname{span} M\mathcal{N}}=\mathcal{M}$.

This condition is quite natural to impose. As we will later see, it is equivalent to the fact that the [PiPo1] index of $\mathcal{N}\subset^{\mathcal{E}}\mathcal{M}$ is equal to that of $N\subset M$. In other words, we only consider representations within the category of inclusions with the same index as the given one.

Representation theories are considered in order to get more insight into the structure of the represented objects. In our case, for the structure and classification problems for inclusions of subfactors $N\subset M$, we are interested in getting finite dimensional subalgebras $Q\subset P$ in $N\subset M$ that make a commuting square with $N\subset M$ and that approximate some given set of finitely many elements in M. The next example shows why such inner structure is reflected into our representation theory.

2.1.2. PROPOSITION. *Let $Q\subset P$ be subalgebras in $N\subset M$ satisfying the commuting square condition $E_PE_N=E_NE_P=E_Q$. If this commuting square is nondegenerate* (1.1.4) *and if*

$$\begin{array}{ccc} \mathcal{N} & \overset{\mathcal{E}}{\subset} & \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

is its extension (1.1.4), *where $\mathcal{M}=\langle M,e_P^M\rangle$, $\mathcal{N}=\langle N,e_P^M\rangle\simeq\langle N,e_Q^N\rangle$ and $\mathcal{E}=E_{\langle N,Q\rangle}^{\langle M,P\rangle}$ is the amplification of E_Q^P, then it gives a nondegenerate representation of $N\subset M$ and* $\operatorname{Ind}\mathcal{E}=$ $[M:N]$*. Moreover, if $Q\subset P$ are finite dimensional and T is its inclusion matrix, which we assume irreducible, then the nondegeneracy of the commuting square is equivalent with the condition $\|T\|^2=[M:N]$. Also, in this case, $\mathcal{N}\subset\mathcal{M}$ are atomic von Neumann algebras with inclusion matrix T (via the identification of the centers of Q and $\mathcal{N}$, respectively P and $\mathcal{M}$).*

Proof. If $\{m_j\}$ is an orthonormal basis of P over Q which is also a basis of M over N then $[m_j,e_P^M]=0$ and $\sum m_j(\operatorname{sp} Ne_P^MN)=\operatorname{sp} Me_P^MN=\operatorname{sp}\sum Mm_je_P^MN=\operatorname{sp} Me_P^MM$, showing by density that $\sum m_j\mathcal{N}=\mathcal{M}$. If $Q\subset P$ are finite dimensional then clearly $\mathcal{M}=\langle M,P\rangle$

and $\mathcal{N}=\langle N,Q\rangle$ are atomic and since $e_P^M\mathcal{M}e_P^M=Pe_P^M$, $e_P^M\mathcal{N}e_P^M=Qe_P^M$, the inclusion matrix of $\mathcal{N}\subset\mathcal{M}$ is the above T. If $\operatorname{sp}Pe_P^MP\ni 1$ then by [J2] we have $\|T\|^2=\tau(e_P^M)^{-1}=[M:N]$. The converse holds true by [PiPo3].

The equality $\operatorname{Ind}\mathcal{E}=[M:N]$ will be proved later in this section for arbitrary nondegenerate representations. A different argument works here as well:

By density and linearity, like in 2.1 of [PiPo1], we only need to show that

$$E_{\langle N,Q\rangle}^{\langle M,P\rangle}\left(\sum_{i,j}\xi_i x_i x_j^* e_P^M \xi_j^*\right)\geqslant\lambda\sum_{i,j}\xi_i x_i x_j^* e_P^M\xi_j^*,$$

where $\lambda=[M:N]^{-1}$, $x_i\in P$, only finitely many nonzero. By 1.1.5 and the commuting square condition, this amounts to $(E_N(x_ix_j^*))\geqslant\lambda((x_ix_j^*))$. But if the number of nonzero x_i's is say n, then by [PiPo1],

$$\operatorname{Ind}E_N\otimes i_n=[M\otimes M_{n\times n}(\mathbf{C}):N\otimes M_{n\times n}(\mathbf{C})]=[M:N]=\operatorname{Ind}E_N,$$

thus the above inequality holds true, showing that $\operatorname{Ind}E_{\langle N,Q\rangle}^{\langle M,P\rangle}\leqslant[M:N]$. But if $e_0\in M$ is a Jones projection, then $E_{\langle N,Q\rangle}^{\langle M,P\rangle}(e_0)=E_N^M(e_0)=\lambda 1$, thus we have the equality. □

2.1.3. PROPOSITION. *An embedding of $(N\subset M)$ in $(\mathcal{N}\subset\mathcal{M})$ is nondegenerate if and only if any orthonormal basis $\{m_j\}_j$ of M over N is an orthonormal basis of $\mathcal{M}$ over $\mathcal{N}$, i.e., $T=\sum_j m_j\mathcal{E}(m_j^*T)$, $T\in\mathcal{M}$.*

Proof. Assume $\overline{\operatorname{sp}M\mathcal{N}}=\mathcal{M}$ and let $\{m_j\}_j$ be an orthonormal basis of M over N. If $T=yX$, with $y\in M, X\in\mathcal{N}$, then $y=\sum_j m_jE_N(m_j^*y)$ so that $yX=\sum m_jE_N(m_j^*y)X=\sum m_j\mathcal{E}(m_j^*yX)=\sum m_j\mathcal{E}(m_j^*T)$.

Taking linear combinations and weak closures it follows that $T=\sum m_j\mathcal{E}(m_j^*T)$ for all $T\in\overline{\operatorname{sp}M\mathcal{N}}=\mathcal{M}$.

The converse is trivial. □

2.2. Basic construction for representations

2.2.1. LEMMA. *Let $\mathcal{N}\subset\mathcal{M}$ be arbitrary von Neumann algebras with a normal faithful conditional expectation $\mathcal{E}$ and assume $\mathcal{M}$ has a finite orthonormal basis over $\mathcal{N}$, i.e., a finite set $\{m_j\}_j\subset\mathcal{M}$ such that $T=\sum m_j\mathcal{E}(m_j^*T)$, $T\in\mathcal{M}$.*

Also assume $\sum m_jm_j^*=\lambda^{-1}1$, for some scalar $\lambda>0$. Let $\mathcal{M}$ be represented normally and faithfully on a Hilbert space, $\mathcal{M}\subset\mathcal{B}(H)$, such that for some projection $e_1\in\mathcal{B}(H)$ we have (*cf.* [T2]):

$$[e_1,\mathcal{N}]=0$$
$$e_1Te_1=\mathcal{E}(T)e_1,\quad T\in\mathcal{M}$$
$$\overline{\operatorname{sp}\mathcal{M}e_1\mathcal{H}}=\mathcal{H}.$$

Then e_1 *and* $\mathcal{M}_1 \overset{\text{def}}{=} \mathrm{vN}(\mathcal{M}, e_1) = (\mathcal{M} \cup \{e_1\})''$ *satisfy*:

(a) $\sum m_j e_1 m_j^* = 1$.

(b) $\mathcal{M}_1 = \{\sum m_i T_{ij} e_1 m_j^* \mid T_{ij} \in f_i \mathcal{N} f_j\}$ *and if* $T \in \mathcal{M} \subset \mathcal{M}_1$ *then*

$$T = \sum m_i \mathcal{E}(m_i^* T m_j) e_1 m_j^*.$$

(c) $\mathcal{E}_1(\sum_{i,j} m_i T_{ij} e_1 m_j^*) = \lambda \sum_{i,j} m_i T_{ij} m_j^*$ *defines a normal faithful conditional expectation of* $\mathcal{M}_1$ *onto* $\mathcal{M}$. *This conditional expectation satisfies* $\mathcal{E}_1(x e_1 y) = \lambda xy$, *for* $x, y \in \mathcal{M}$, *so that, in particular, it does not depend on the choice of* $\{m_j\}_j$.

(d) $\{\lambda^{-1} m_j e_1\}_j$ *is an orthonormal basis of* $\mathcal{M}_1$ *over* $\mathcal{M}$, *with respect to* $\mathcal{E}_1$, *i.e.*, $T_1 = \lambda^{-2} \sum m_j e_1 \mathcal{E}_1(e_1 m_j^* T_1)$, *for all* $T_1 \in \mathcal{M}_1$, *and it satisfies* $\sum_j (\lambda^{-1} m_j e_1)(e_1 m_j^* \lambda^{-1}) = \lambda^{-1} 1$.

(e) *The probabilistic index* [PiPo1] *of* $\mathcal{E}_1$ *is equal to* λ^{-1}, *i.e.*, $\mathcal{E}_1(T) \geqslant \lambda T$, $T \in \mathcal{M}_{1+}$ *and* λ *is the best constant for which the inequality holds, in fact* $\mathcal{E}_1(e_1) = \lambda 1$.

Moreover, if $\mathcal{B}$ *is any von Neumann algebra containing* $\mathcal{M}$ *and a projection* e_1' *such that* $[e_1', \mathcal{N}] = 0$, $e_1' T e_1' = \mathcal{E}(T) e_1'$, $\bigvee\{u e_1' u^* \mid u \in \mathcal{U}(M)\} = 1$ *then* $\mathrm{vN}(\mathcal{M}, e_1')$ *is isomorphic to* $\mathcal{M}_1$, *by letting* $x \mapsto x$, $x \in \mathcal{M}$, $e_1 \mapsto e_1'$.

Proof. (a) follows from $\overline{\mathrm{sp}\, \mathcal{M} e_1 \mathcal{H}} = \mathcal{H}$. The first part of (b) is trivial. Then, if $T \in \mathcal{M}$ we have $T = T \sum m_j e_1 m_j^* = \sum_{i,j} m_i \mathcal{E}(m_i^* T m_j) e_1 m_j^*$. Clearly $\mathcal{E}_1(\mathcal{M}_1) \subset \mathcal{M}$ and if $T = \sum m_i \mathcal{E}(m_i^* T m_j) e_1 m_j^* \in \mathcal{M}$ then $\mathcal{E}_1(T) = \lambda \sum m_i \mathcal{E}(m_i^* T m_j) m_j^* = \lambda \sum T m_j m_j^* = T$.

(d) is trivial and the proof of (e) is identical to the proof of the similar statement for finite algebras in [PiPo1]. □

2.2.2. *Definition.* If $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ satisfies the hypothesis of 2.2.1 then $\mathcal{M} \subset^{\mathcal{E}_1} \mathcal{M}_1$ as constructed in 2.2.1 is called the *extension of* $\mathcal{N} \subset \mathcal{M}$. One also denotes $\mathcal{M}_1 = \langle \mathcal{M}, e_1 \rangle$. This construction is called the *basic construction for* $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$. Since the new inclusion also satisfies 2.2.1, one obtains recursively a whole *tower of inclusions* $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots$ with $\mathcal{M}_{i+1} = \langle \mathcal{M}_i, e_{i+1} \rangle$ and $\mathcal{E}_{i+1} : \mathcal{M}_{i+1} \to \mathcal{M}_i$, $\mathcal{E}_{i+1}(T_1 e_{i+1} T_2) = \lambda T_1 T_2$, $T_{1,2} \in \mathcal{M}_i$.

2.2.3. LEMMA. *Let*

$$\begin{array}{ccc} \mathcal{N} & \overset{\mathcal{E}}{\subset} & \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

be a nondegenerate embedding. Then $\mathcal{N} \subset \mathcal{M}$ *satisfies the hypothesis of* 2.2.1 *and if* $\mathcal{M} \subset^{\mathcal{E}_1} \mathcal{M}_1 = \langle \mathcal{M}, e_1 \rangle$ *denotes the extension of* $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ *then* $\langle M, e_1 \rangle \overset{\text{def}}{=} \mathrm{vN}(M, e_1)$ *is isomorphic to* $M_1 = \langle M, e_N \rangle$ *via the map* $x \to x$, $x \in M$, $e_1 \to e_N$, *and we have the nondegenerate embedding*

$$\begin{array}{ccc} \mathcal{M} & \overset{\mathcal{E}_1}{\subset} & \mathcal{M}_1 \\ \cup & & \cup \\ M & \subset & M_1 \end{array}$$

e.g. $\overline{\mathrm{sp}\, M_1 \mathcal{M}} = \mathcal{M}_1$, $\mathcal{E}|_{M_1} = E_M^{M_1}$. *More generally, by iterating the construction one gets a tower of nondegenerate embeddings*:

$$\begin{array}{ccccccc} \mathcal{N} & \overset{\mathcal{E}}{\subset} & \mathcal{M} & \overset{\mathcal{E}_1}{\subset} & \mathcal{M}_1 & \overset{\mathcal{E}_2}{\subset} & \ldots \\ \cup & & \cup & & \cup & & \\ N & \subset & M & \subset & M_1 & \subset & \ldots \end{array}$$

Proof. Trivial by 2.2.1. □

2.2.4. *Definition.* The nondegenerate embedding (or representation, or commuting square)

$$\begin{array}{ccc} \mathcal{M} & \overset{\mathcal{E}_1}{\subset} & \mathcal{M}_1 \\ \cup & & \cup \\ M & \subset & M_1 \end{array}$$

is called the *extension of*

$$\begin{array}{ccc} \mathcal{M} & \subset & \mathcal{M} \\ \cup & & \cup \\ N & \subset & M. \end{array}$$

Its construction is called the *basic construction.*

The next result shows that any representation comes from a basic construction, in other words that we can make the *downward basic construction* for representations, thus getting from the initial $\mathcal{N} \subset \mathcal{M}$ both a *tower* and a *tunnel of representations.*

PROPOSITION. *Let $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ be a nondegenerate representation of $N \subset M$ and $e_0 \in M$ be a Jones projection and put $\mathcal{N}_1 \overset{\mathrm{def}}{=} \{e_0\}' \cap \mathcal{N}$, $\mathcal{E}_{-1}: \mathcal{N} \to \mathcal{N}$ by $\mathcal{E}_{-1}(x) \overset{\mathrm{def}}{=} [M:N]\mathcal{E}(e_0 x e_0)$, for $x \in \mathcal{N}$. Then $\mathcal{E}_{-1}$ is a conditional expectation of $\mathcal{N}$ onto $\mathcal{N}_1$, $\mathcal{N}_1 \subset^{\mathcal{E}_{-1}} \mathcal{N}$ is a nondegenerate representation of $\{e_0\}' \cap N = N_1 \subset N$ and its extension is $\mathcal{N} \subset \mathcal{M}$, in particular $e_0 x e_0 = \mathcal{E}_{-1}(x) e_0$, $x \in \mathcal{N}$.*

Proof. Let $\{m_j\}$ be an orthonormal basis of N over N_1, so that $\{\lambda^{-1/2} m_j e_0\}$ is a basis of M over N (and thus of $\mathcal{M}$ over $\mathcal{N}$). We want to show that $\sum m_j \mathcal{N}_1 = \mathcal{N}$, with $\mathcal{N}_1 = \{e_0\}' \cap \mathcal{N}$. Assume there exists $y \in e_0 \mathcal{N} e_0$ with $\mathcal{E}_{-1}(y) = 0$. We then get $\mathcal{E}(e_0 m_j^* y) e_1 = e_1 e_0 m_j^* y e_1 = E_{N_1}^N(m_j) \mathcal{E}_{-1}(y) e_1 = 0$, so that $y = 0$, or otherwise $\sum m_j e_0 \mathcal{N} \neq \mathcal{M}$.

Let us prove that $[\mathcal{E}_{-1}(x), e_0] = 0$, $x \in \mathcal{N}$. We have this if and only if

$$[e_2 e_1 e_0 x e_0 e_1 e_2, e_0 e_2] = 0,$$

because for $y \in \mathcal{M}$, $y \mapsto y e_2$ is an isomorphism and because $e_2 e_1 e_0 x e_0 e_1 e_2 = \lambda^2 \mathcal{E}_{-1}(x) e_2$. But

$$\begin{aligned} e_0(e_2 e_1 e_0 x e_0 e_1 e_2) &= \lambda(e_2 e_0 x e_0 e_1 e_2) = \lambda(e_0 x e_0 e_2 e_1 e_2) \\ &= \lambda^2(e_0 x e_0 e_2) = \lambda(e_2 e_1 e_0 x e_0 e_2) \\ &= (e_2 e_1 e_0 x e_0 e_1 e_2) e_0. \end{aligned}$$

Since for $x \in \mathcal{N}_1$, clearly $\mathcal{E}_{-1}(x)=x$, we get by Tomiyama's theorem that $\mathcal{E}_{-1}$ is a conditional expectation of $\mathcal{N}$ onto $\mathcal{N}_1$ and that $e_0 x e_0 = \mathcal{E}_{-1}(x) e_0$, $x \in \mathcal{N}$, because

$$\mathcal{E}(e_0 x e_0 - \mathcal{E}_{-1}(x) e_0) = 0$$

and the first part of the proof then shows that $e_0 x e_0 - \mathcal{E}_{-1}(x) e_0 = 0$. Finally, we have $\sum m_j \mathcal{N}_1 = \mathcal{N}$ if and only if $\sum m_j \mathcal{N}_1 e_0 \supseteq \mathcal{N} e_0$. But

$$\sum m_j \mathcal{N}_1 e_0 = \sum m_j e_0 \mathcal{N} e_0 = \Big(\sum m_j e_0 \mathcal{N}\Big) e_0 = \mathcal{M} e_0 \supset \mathcal{N} e_0. \qquad \square$$

COROLLARY. *If $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is a representation of $N \subset M$ then* $\operatorname{Ind} \mathcal{E} = \operatorname{Ind} E_N = [M:N]$.

Proof. By 2.2.3 the extension of a representation has index $[M:N]$, so the above proposition applies. $\square$

2.3. Smooth representations

2.3.1. *Definition.* The nondegenerate embedding (or representation, or commuting square)

$$\begin{array}{ccc} \mathcal{N} & \overset{\mathcal{E}}{\subset} & \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

is *smooth* if $\mathcal{N}' \cap \mathcal{M}_k \supset N' \cap M_k$, for all k, equivalently, if and only if $\mathcal{N}' \cap M_k = N' \cap M_k$, for all k. Note that in this case we have $\mathcal{M}_i' \cap \mathcal{M}_k \supset M_i' \cap M_k$, for all $i \leqslant k$, and in fact $\mathcal{M}_i' \cap M_k = M_i' \cap M_k$.

2.3.2. PROPOSITION. *If*

$$\begin{array}{ccc} \mathcal{N} & \overset{\mathcal{E}}{\subset} & \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

is a smooth embedding then

$$\begin{array}{ccc} \mathcal{M} & \overset{\mathcal{E}_1}{\subset} & \mathcal{M}_1 \\ \cup & & \cup \\ M & \subset & M_1 \end{array}$$

is also a smooth embedding.

Proof. If $x \in M' \cap M_k$ then $x \in N' \cap M_k \subset \mathcal{N}' \cap \mathcal{M}_k$ and $[x, m_j] = 0$, for all j, where $\{m_j\}_j$ is an orthonormal basis of M over N. Thus $x = \lambda \sum m_j x m_j^*$ and we have for

$T \in \mathcal{M}$,

$$\begin{aligned} Tx &= \sum_j Tm_j x m_j^* = \sum_{i,j} m_i \mathcal{E}(m_i^* T m_j) x m_j^* \\ &= \sum_{i,j} m_i x \mathcal{E}(m_i^* T m_j) m_j^* \\ &= \sum_i m_i x \sum_j \mathcal{E}(m_i^* T m_j) m_j^* = \sum_i m_i x m_i^* T = xT. \end{aligned} \qquad \square$$

2.3.3. *Example.* (a) Let

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ Q & \subset & P \end{array}$$

be a nondegenerate commuting square and assume $N' \cap M_k \subset Q' \cap P_k$, $\forall k$. Then the representation

$$\begin{array}{ccc} \mathcal{N} = \langle N, Q \rangle & \subset & \langle M, P \rangle = \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

is smooth. Indeed, if $x \in N' \cap M_k$ then $x \in P_k$ by hypothesis so that x commutes with $e_{P_k}^{M_k} = e_P^M$. Thus $x \in (N \cup \{e_P^M\})' \cap M_k = \mathcal{N}' \cap M_k$.

The above hypothesis $N' \cap M_k \subset Q' \cap P_k$, $\forall k$, is fulfilled if for instance one takes $Q \subset P$ to be a core associated to some tunnel for $N \subset M$. This class of smooth representations will play an important role in the sequel.

(b) Let P be a type II_1 factor and G a discrete group with finitely many generators $g_1, \ldots, g_n \in G$. Let $\sigma: G \to \operatorname{Aut} P / \operatorname{Int} P$ be a faithful G-kernel on P ([J1]). Let $\theta_0 = \mathrm{id}$, $\theta_i \in \operatorname{Aut} P$, $\varepsilon \theta_i = \sigma(g_i)$, $1 \leqslant i \leqslant n$. Let $M^\theta \overset{\text{def}}{=} M_{n+1}(\mathbf{C}) \otimes P$ and

$$N^\theta \overset{\text{def}}{=} \left\{ \sum_{i=0}^n \theta_i(x) \otimes e_{ii} \,\middle|\, x \in P \right\} \subset M^\theta,$$

where $\{e_{ij}\}_{i,j}$ is a matrix unit of $M_{n+1}(\mathbf{C})$. For more on this example of subfactors see 5.1.5.

Let $\mathcal{P}$ be a von Neumann algebra that contains P and assume there exists a G-kernel (not necessarily faithful) $\tilde{\sigma}$ on $\mathcal{P}$ with some automorphisms $\tilde{\theta}_i$ representing $\tilde{\sigma}(g_i)$ such that $\tilde{\theta}_i|_P = \theta_i$. Define $\mathcal{M}^{\tilde{\theta}} = M_{n+1}(\mathbf{C}) \otimes \mathcal{P}$ and $\mathcal{N}^{\tilde{\theta}} = \{\sum_{i=0}^n \tilde{\theta}_i(X) \otimes e_{ii} \,|\, X \in \mathcal{P}\}$. Then $\mathcal{N}^{\tilde{\theta}} \subset \mathcal{M}^{\tilde{\theta}}$ is a smooth representation of $N^\theta \subset M^\theta$. To see this, note that the Jones tower and the higher relative commutants for the inclusion $\mathcal{N}^{\tilde{\theta}} \subset \mathcal{M}^{\tilde{\theta}}$ are constructed in the same way as for $N^\theta \subset M^\theta$ (see e.g. 5.1.5 (4)). Conversely, if $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is a smooth representation for $N^\theta \subset M^\theta$ then $\mathcal{N}' \cap \mathcal{M} \supset N^{\theta\prime} \cap M^\theta$, in particular $e_{ii} \in \mathcal{N}' \cap \mathcal{M}$. Thus, if $\mathcal{P} \simeq \mathcal{N} e_{00}$, then $\tilde{\theta}_i \overset{\text{def}}{=} (n+1)\mathcal{E}(e_{0i} \cdot e_{i0})$ is easily seen to define an automorphism of $\mathcal{P} \supset P$ with $\tilde{\theta}_i|_P = \theta_i$ and that in fact $(\mathcal{N} \subset \mathcal{M}) \simeq (\mathcal{N}^{\tilde{\theta}} \subset \mathcal{M}^{\tilde{\theta}})$.

Another important class of smooth representations is given by the embeddings satisfying the following stronger condition.

2.3.4. *Definition.* A representation of $N \subset M$ into $\mathcal{N} \subset \mathcal{M}$ is *exact* if $(M' \cap \mathcal{N}) \vee M = \mathcal{M}$. Note that if this condition is satisfied and $P = M' \cap \mathcal{N}$, then $\mathcal{M}_k = P \vee M_k$, $k \geqslant -1$. This is because $[M' \cap \mathcal{N}, e_k] = 0$, $k \geqslant 1$, so that

$$\mathcal{M}_k = \langle \mathcal{M}, e_1, e_2, ..., e_k \rangle = P \vee (\langle M, e_1, e_2, ..., e_k \rangle).$$

But this implies $\mathcal{N}' \cap \mathcal{M}_k = N' \cap M_k$, $\forall k$, so that the exact representations are smooth.

2.3.5. *Example.* Let $N \subset M$ be an inclusion of finite depth and $M \supset N \supset ... \supset N_k$ be a choice of the tunnel up to k that reaches the depth. Thus $Q = N_k' \cap N \subset N_k' \cap M = P$ contains an orthonormal basis of $N \subset M$ (cf. §1.3) so that by 2.1.2

$$\begin{array}{ccc} \mathcal{N} = \langle N, Q \rangle & \subset & \langle M, P \rangle = \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

is a nondegenerate representation. More than that, this representation is in fact exact and thus smooth. More generally, we have the following:

PROPOSITION. *Let $N \subset M$ be an inclusion of finite index and assume there exists a subalgebra $B \subset N$ such that $Q = B' \cap N \subset B' \cap M = P$ contains an orthonormal basis of $N \subset M$ (equivalently, $(Q \subset P)$ is nondegenerate in $(N \subset M)$). Then the representation*

$$\begin{array}{ccc} \mathcal{N} = \langle N, Q \rangle & \subset & \langle M, P \rangle = \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

is exact and in fact $M' \cap \mathcal{N} = J\tilde{B}J$, $\mathcal{N} = N \vee J\tilde{B}J$, $\mathcal{M} = M \vee J\tilde{B}J$, where $\tilde{B} = P' \cap N \supset B$.

Moreover, if P is an atomic algebra (e.g. finite dimensional), then this representation is atomic.

Proof. Note first that e_N^M implements the canonical conditional expectation $\mathcal{E} = E_{\langle N, Q \rangle}^{\langle M, P \rangle}$ of $\mathcal{M} = \langle M, P \rangle$ onto $\mathcal{N} = \langle N, Q \rangle$. Indeed, since $(Q \subset P)$ is nondegenerate in $(N \subset M)$, any orthonormal basis $\{\xi_j\}_j$ of N over Q is an orthonormal basis of M over P. Thus, the elements of $\mathcal{M}$ are of the form $x = \sum_{i,j} \xi_i p_{ij} e_P^M \xi_j^*$, with $p_{ij} \in P$. But $[e_N^M, \xi_i] = 0$, $[e_N^M, e_P^M] = 0$ so that $e_N^M x e_N^M = \sum_{i,j} \xi_i E_Q^P(p_{ij}) e_P^M \xi_j^* e_N^M = \mathcal{E}(x) e_N^M$.

Thus $\{e_N^M\}' \cap \mathcal{M} = \mathcal{N}$ and we have

$$\begin{aligned} M' \cap \mathcal{N} &= M' \cap \{e_N^M\}' \cap \mathcal{M} = JMJ \cap \{e_N^M\}' \cap JP'J \\ &= J((\{e_N^M\}' \cap M) \cap P')J = J(N \cap P')J = J\tilde{B}J. \end{aligned}$$

Also, since $\tilde{B}' \cap M = B' \cap M = P$, we have $(M \vee J\tilde{B}J)' = J(\tilde{B}' \cap M)'J = JP'J$, so that $M \vee J\tilde{B}J = \mathcal{M}$. □

2.4. The standard representations

Roughly speaking, the standard representation of $N \subset M$ is defined as the "smallest" representation of $N \subset M$ which, as a representation of M, contains the standard representation of M as a direct summand. We will now give a constructive and rigorous definition of it, and along the line provide a constructive way of getting all possible exact representations of $N \subset M$.

Let $N \subset M$ be an inclusion of type II_1 factors with finite index and let $\lambda = [M:N]^{-1}$. Let P be another von Neumann algebra (in fact even a C^*-algebra would do). Typically, P will be M^{op}, the opposite algebra of M. Then the closure of $N \otimes P$ in $M \otimes_{\max} P$ coincides with $N \otimes_{\max} P$ (indeed, since given any cyclic representation π_φ of $N \otimes P$, the representation $\pi_{\varphi \circ E_N \otimes \mathrm{id}}$ extends it to all $M \otimes_{\max} P$). We will simply denote this inclusion by $N \otimes P \subset M \otimes P$. We also have an expectation $E = E_N \otimes \mathrm{id}$ from $M \otimes P$ onto $N \otimes P$.

Let further $(N \otimes P)^{**} \subset^{E^{**}} (M \otimes P)^{**}$ be the bidual of this inclusion. It will still satisfy $E^{**}(x) \geqslant \lambda x$, $x \in (M \otimes P)^{**}_+$. Also $x = \sum_j m_j E^{**}(m_j^* x)$, i.e., $\{m_j\}$ is an orthonormal basis of $(M \otimes P)^{**}$ over $(N \otimes P)^{**}$. Indeed, this is easily checked for x in the dense $*$-subalgebra $M \otimes P$, by writing $M \otimes P = (\mathrm{sp}\, N e_0 N) \otimes P$ (resp. $M \otimes P = (\sum m_j N) \otimes P$), $e_0 \in M$ being a Jones projection.

If p_N is the maximal projection in $N' \cap (N \otimes P)^{**}$ such that N_{p_N} is a normal embedding of N in $(N \otimes P)^{**}$ then p_N also commutes with P so that $p_N \in \mathcal{Z}((N \otimes P)^{**})$. Similarly, the analogue projection p_M for $(M \otimes P)^{**}$ belongs to $\mathcal{Z}((M \otimes P)^{**})$. Clearly $p_M \leqslant p_N$. Also, if $(x_i)_i \subset N$, $\|x_i\| \leqslant 1$, and $(x_i)_i$ converges weakly to x in N, then $\{E_N(m_k^* x_i m_l)\}_i$ converges in N to $E_N(m_k^* x m_l)$, for each k, l. Thus $x_i \sum m_k p_N m_k^* = \sum_{l,k} m_l E_N(m_l^* x_i m_k) p_N m_K^*$ converges weakly to $\sum_{l,k} m_l E_N(m_l^* x m_k) p_N m_k^* = x$. This shows that $\sum m_k p_N m_k^*$ is supported by p_N, thus m_k commute with p_N so that $p_N \in M' \cap (M \otimes P)^{**}$, showing that $p_N \in \mathcal{Z}((M \otimes P)^{**})$. Next, if $(x_i)_i$ is in M and $x_i \to x \in M$ then $x_i p_N = \sum_k m_k (E_N(m_k^* x_i) p_N) \to \sum_k m_k E_N(m_k^* x) p_N$. Altogether this shows that $p_N \leqslant p_M$ as well, so $p_N = p_M = p_{\mathrm{nor}}$.

Now take q_N, q_M to be the atomic parts of $(N \otimes P)^{**}$, respectively $(M \otimes P)^{**}$. By [PoWa], $q_N = q_M$. Indeed, if $q \in (N \otimes P)^{**}$ is an atom and $(p_i)_{i \in I}$ is a partition of the unity with projections in $q(M \otimes P)^{**} q$ then $E^{**}(p_i) = \alpha_i q \geqslant \lambda p_i$. Thus $\operatorname{card} I \leqslant \lambda^{-1}$, showing that $(N \otimes P)' \cap q(M \otimes P)^{**} q$ is finite dimensional, and thus $q(M \otimes P)^{**} q$ is atomic. Thus $q_M \geqslant q_N$. Also, since $E^{**}(q_M)^{-1} E^{**}(x) q_M$ is a normal faithful conditional expectation of $(M \otimes P)^{**} q_M$ onto $(N \otimes P)^{**} q_M$, we have that $(N \otimes P)^{**} q_M$ is atomic so that $q_M \leqslant q_N$ as well, thus $q_M = q_N = q_{\mathrm{at}}$.

Since we are particularly interested in studying inclusions of separable type II_1 factors, in fact of separable hyperfinite factors, all that really matters is when each simple summand of $(M \otimes P)^{**}$ in which M is normally embedded is separable (e.g. when

P is a separable C^*-algebra or when P is a separable von Neumann algebra and we only take simple summands on which P is also normally embedded).

Note that by construction, $N \subset M$ is represented in $(N \otimes P)^{**}_{p_{\text{nor}}} \subset (M \otimes P)^{*}_{p_{\text{nor}}}$ and since $\{m_j\}_j$ is an orthonormal basis of $(M \otimes P)^{**}_{p_{\text{nor}}}$ over $(N \otimes P)^{*}_{p_{\text{nor}}}$, the representation is nondegenerate. This is the case also if we cut further with a projection q in $\mathcal{Z}((N \otimes P)^{**}) \cap \mathcal{Z}((M \otimes P)^{**})$ which is under p_{nor}. Furthermore, if $N \subset M \subset^{e_1} M_1 \subset \ldots$ is the Jones tower for $N \subset M$, then we have an associated tower of representations:

$$\begin{array}{ccccccc} (N \otimes P)^{**}_q & \overset{E^{**}}{\subset} & (M \otimes P)^{**}_q & \overset{E_1^{**}}{\subset} & (M_1 \otimes P)^{**}_q & \subset & \ldots \\ \cup & & \cup & & \cup & & \\ N & \subset & M & \subset & M_1 & \subset & \ldots \end{array}$$

where $E_1^{**} = (E_M^{M_1} \otimes \mathrm{id})^{**}$.

Note that by construction we clearly have $M' q \cap (N \otimes P)^{**} q \supset (1 \otimes P)$, so that all such representations are exact and thus smooth.

Note also that $\mathcal{Z}((N \otimes P)^{**}) \cap \mathcal{Z}((M \otimes P)^{**})_{p_{\text{nor}} q_{\text{at}}}$ is an atomic abelian von Neumann algebra and that every atom in $\mathcal{Z}((M \otimes P)^{**} p_{\text{nor}})$ is majorized by an atom (i.e., minimal projection) in this common center.

2.4.1. *Notation.* We denote by $\mathcal{N}^u \subset^{\mathcal{E}} \mathcal{M}^u$ the inclusion of atomic algebras

$$\bigoplus (N \otimes P)^{**}_{p_{\text{nor}} q_{\text{at}}} \overset{E^{**}}{\subset} \bigoplus (M \otimes P)^{**}_{p_{\text{nor}} q_{\text{at}}},$$

the sums being taken over all separable C^*-algebras P (so that each atom in $\mathcal{M}^u$ is a $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ separable). We denote the corresponding smooth representation of $N \subset M$ by

$$\begin{array}{ccc} \mathcal{N}^u & \overset{\mathcal{E}}{\subset} & \mathcal{M}^u \\ \cup & & \cup \\ N & \subset & M. \end{array}$$

The next result describes this representation by an intrinsic universality property.

2.4.2. PROPOSITION. *Let $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ be an inclusion of atomic von Neumann algebras and assume*

$$\begin{array}{ccc} \mathcal{N} & \overset{\mathcal{E}}{\subset} & \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

is a nondegenerate representation of $N \subset M$. Then this representation is equivalent to a direct summand of the representation $\mathcal{N}^u \subset \mathcal{M}^u$ if and only if it is exact.

Proof. Let $\mathcal{P} = M' \cap \mathcal{N}$. If $\mathcal{N} \subset \mathcal{M}$ is a direct summand of $\mathcal{N}^u \subset \mathcal{M}^u$, i.e., $(\mathcal{N} \subset \mathcal{M}) = (\mathcal{N}^u_q \subset \mathcal{M}^u_q)$ for some $q \in \mathcal{Z}(\mathcal{N}^u) \cap \mathcal{Z}(\mathcal{M}^u)$, then $\mathcal{P} \supset P$ so that $M \vee \mathcal{P} \supset \mathrm{vN}(M, P) = \mathcal{M}$.

Conversely, if $\mathcal{P}\vee M=\mathcal{M}$ then we can simply take $P=\mathcal{P}$, or if P is required to be a direct sum of separable C^*-algebras, then we can take dense parts of $\mathcal{P}$. $\square$

2.4.3. *Definitions.* $\mathcal{N}^u\subset\mathcal{M}^u$ is called the *universal* (exact) *atomic representation* of $N\subset M$. If $q\in\mathcal{Z}(\mathcal{N}^u)\cap\mathcal{Z}(\mathcal{M}^u)$ is the minimal projection majorizing the atom q in $\mathcal{Z}(\mathcal{M}^u)$ corresponding to the direct summand $\mathcal{B}(L^2(M))\subset\mathcal{M}^u$, then $\mathcal{N}^u_q\subset\mathcal{M}^u_q$ is called the *standard representation* of $N\subset M$ and it is denoted $\mathcal{N}^{\mathrm{st}}\subset\mathcal{M}^{\mathrm{st}}$. If $\mathcal{N}\subset^{\mathcal{E}}\mathcal{M}$ is a representation of $N\subset M$ so that $\mathcal{Z}(\mathcal{N})\cap\mathcal{Z}(\mathcal{M})=\mathbf{C}1$, then the represenation is called *weakly irreducible.*

We see from all the above considerations that $\mathcal{N}^{\mathrm{st}}\subset\mathcal{M}^{\mathrm{st}}$ is exact (thus smooth) and weakly irreducible.

Let K^u (resp. L^u) be a labeling of the minimal projections in $\mathcal{Z}(\mathcal{M}^u)$ (resp. $\mathcal{Z}(\mathcal{N}^u)$). Let $A^u_{N,M}=(a_{kl})_{k\in K^u,l\in L^u}$ be the multiplicity matrix of the inclusion $\mathcal{N}^u\subset\mathcal{M}^u$. By the inequality (see the considerations above), if $\mathcal{B}(\mathcal{K}_l)$, resp. $\mathcal{B}(\mathcal{H}_k)$, are the simple direct summands of $\mathcal{N}^u$ resp. $\mathcal{M}^u$ corresponding to the labels $l\in L^u, k\in K^u$, then

$$\dim(\mathcal{B}(\mathcal{K}_l)'\cap\mathcal{B}(\mathcal{H}_k))\leqslant\lambda^{-1},$$

showing that $a_{kl}\leqslant\lambda^{-1}$. Thus A^u is a matrix over $\mathbf{Z}_+$.

The matrix $A^u_{N,M}$ is called the *universal* (exact) *matrix* of the inclusion $N\subset M$ and can alternatively be regarded as a bipartite graph, called the *universal graph* of $N\subset M$.

Note that for each minimal projection q in $\mathcal{Z}(\mathcal{N}^u)\cap\mathcal{Z}(\mathcal{M}^u)$ the inclusion matrix of $\mathcal{N}^u q\subset\mathcal{M}^u q$ corresponds to an irreducible direct summand of $A^u_{N,M}$ and vice versa, each irreducible part of A^u corresponds to such a q.

To interpret all this more thoroughly in terms of Connes' correspondences, let us take in the above considerations P to be separable type II_1 factors and let $r_M\in\mathcal{Z}(\mathcal{M}^u)$, $r_N\in\mathcal{Z}(\mathcal{N}^u)$ be the maximal central projections on which $1\otimes P$ is embedded normally in $\mathcal{M}^u$, resp. $\mathcal{N}^u$. Clearly $r_M=r_N=r$. Note that any minimal central projection in $\mathcal{Z}(\mathcal{M}^u)r$ (resp. $\mathcal{Z}(\mathcal{N}^u)r$) corresponds to an irreducible M–P (resp. N–P) Connes correspondence, as defined in [Po4]. Thus, if we still denote by K^u, resp. L^u, the sets of simple summands (or minimal central projections) of $\mathcal{M}^u r$, resp. $\mathcal{N}^u r$, then the matrix A^u (or the graph Γ^u) is just a generalized function, or correspondence, in the classical sense of Hurwitz, between the sets K^u, L^u. In turn K^u (resp. L^u) can be regarded as the set of all classes of irreducible M–P (resp. N–P) correspondences, in the operator algebra sense of Connes.

Let K^{uf}, resp. L^{uf}, be the subsets of K^u, resp. L^u, corresponding to all classes of irreducible M–P (resp. N–P) correspondences $\mathcal{H}_k$ (resp. $\mathcal{K}_l$) with finite index, i.e.,

$$\dim_{M,P}\mathcal{H}_k \overset{\mathrm{def}}{=} \dim_M\mathcal{H}_k \dim_P\mathcal{H}_k<\infty,$$
$$\dim_{N,P}\mathcal{K}_l \overset{\mathrm{def}}{=} \dim_P\mathcal{K}_l \dim_N\mathcal{K}_l<\infty.$$

If $s_M \in \mathcal{Z}(\mathcal{M}^u)$ is the central projection corresponding to K^{uf} and $s_N \in \mathcal{Z}(\mathcal{N}^u)$ to L^{uf} then clearly $s_M \leqslant s_N$ (since $[M:N]<\infty$ in the Jones sense). Let us show that in fact $s_M = s_N$ (see also [PoWa]). Let $p \in (N' \cap P')q_k$, where q_k is minimal in $\mathcal{Z}(\mathcal{M}^u r)$, with $\mathcal{M}^u q_k = \mathcal{B}(\mathcal{H}_k)$, be so that $\dim_N p\mathcal{H}_k \cdot \dim_P p\mathcal{H}_k < \infty$. Then $\dim_P \mathcal{H}_k < \infty$ and $Q = P' \cap \mathcal{B}(\mathcal{H}_k)$ is a type II_1 factor with the properties $N \subset M \hookrightarrow Q$, $p \in N' \cap Q$, $[pQp:Np]<\infty$, $M' \cap Q = \mathbf{C}$. Assume $M' \cap \mathcal{B}(\mathcal{H}_k)$ is of type II_∞. Then $N' \cap \mathcal{B}(\mathcal{H}_k)$ is also of type II_∞ and Q' $(=P)$ is of type II_1 with $(Q')' \cap (M' \cap \mathcal{B}(\mathcal{H}_k)) = \mathbf{C}$ and $p \in (Q')' \cap (N' \cap \mathcal{B}(\mathcal{H}_k))$. But $p(N' \cap \mathcal{B}(\mathcal{H}_k))p$ has finite index over $Q'p$ so that p is a finite projection in $N' \cap \mathcal{B}(\mathcal{H}_k)$. Since we have $(Q')' \cap (M' \cap \mathcal{B}(\mathcal{H}_k)) = \mathbf{C}$, the expected value of p on $M' \cap \mathcal{B}(\mathcal{H}_k)$ (with respect to the trace preserving expectation of $N' \cap \mathcal{B}(\mathcal{H}_k)$ onto $M' \cap \mathcal{B}(\mathcal{H}_k)$) is a nonzero scalar. This contradicts the finiteness of p. Thus, $M' \cap \mathcal{B}(\mathcal{H}_k)$ is finite so that $\dim_M \mathcal{H}_k < \infty$. Altogether this shows that $s_N \leqslant s_M$ as well so that $s_N = s_M$.

2.4.4. *Notation.* We identify K^{uf} (resp. L^{uf}) with the set of isomorphism classes of all irreducible M–P (resp. N–P) correspondences of finite index $\mathcal{H}_k$ (resp. $\mathcal{K}_l$). We put $\mathcal{M}^{uf} = \bigoplus_{k \in K^{uf}} \mathcal{B}(\mathcal{H}_k)$ and $\mathcal{N}^{uf} = \bigoplus_{l \in L^{uf}} \mathcal{B}(\mathcal{K}_l)$. By the above proof we have $\mathcal{N}^{uf} \subset \mathcal{M}^{uf}$ is a unital inclusion and its multiplicity matrix is $A^{uf}_{N,M}$, the restriction of $A^u_{N,M}$ to K^{uf} (or L^{uf}). Note that k_0 $(=*)$ is contained in K^{uf}, so that $\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$ is a subrepresentation of $\mathcal{N}^{uf} \subset \mathcal{M}^{uf}$.

From this moment on, it is useful to adopt Connes' philosophy of regarding isomorphism classes of (irreducible) correspondences alternatively either as (irreducible) bimodules as above, or as isomorphism classes of (irreducible) embeddings (see e.g. [Po4]).

We denote by i_k (resp. j_l) the inclusion of M in $Q = (P^{\mathrm{op}})' \cap \mathcal{B}(\mathcal{H}_k)$ (resp. of N in $(P^{\mathrm{op}})' \cap \mathcal{B}(\mathcal{K}_l)$), which have finite index by hypothesis (since $k \in K^{uf}$, $l \in L^{uf}$). We denote by $|i_k|$ (resp. $|j_l|$) the index of M in $Q = (P^{\mathrm{op}})' \cap \mathcal{B}(\mathcal{K}_l)$) equivalently $|i_k| = \dim_{M,P} \mathcal{H}_k$, $|j_l| = \dim_{N,P} \mathcal{K}_l$. Since $\mathcal{B}(\mathcal{K}_l)$ has multiplicities a_{kl} in $\mathcal{B}(\mathcal{H}_k)$, it follows that the irreducible correspondence j_l appears with multiplicity a_{kl} in i_k. Also, by Jones' local index formula [J2] (see 1.2.5) we have $[Q:N] = \sum_l a_{kl} |j_l| / \tau_Q(p_l)$ with p_l a minimal projection in $N' \cap Q$ giving the irreducible inclusion j_l (i.e., the bimodule $\mathcal{K}_l$). Also, by the product formula for indices, $[Q:N] = [Q:M][M:N] = [M:N]|i_k|$.

Now, if we assume $N \subset M$ is extremal (see 1.2.6) then by 1.2.6 $N \subset Q$ is also extremal (since $M \subset Q$ is) so that by [PiPo1], $|j_l| / \tau_Q(p_l)^2 = [Q:N]$. Thus $[Q:N]^{1/2} = |j_l|^{1/2} / \tau_Q(p_l)$ and we get that $[M:N]^{1/2} |i_k|^{1/2} = [Q:N]^{1/2} = (\sum a_{kl} |j_l| / \tau(p_l)) / [Q:N]^{1/2} = \sum_l a_{kl} |j_l|^{1/2}$.

Before stating the result that summarizes the main properties of this construction let us observe that the M–M (resp. N–M) correspondences labeled by K^{st} (resp. L^{st}) are simply the irreducible inclusions of M (resp. N) in the Jones tower of higher relative commutants. To see this recall that the irreducible inclusions of M in M_{2n} are labeled by K_n (resp. those of N in M_{2n} by L_n), by their identification with $M' \cap M_{2n}$, via $J_{M_n} \cdot J_{M_n}$, with the natural identification of K_n (resp. L_n) as a subset of K_{n+1} (resp. L_{n+1}) (cf. [Po5]). The corresponding bimodules are $L^2(p_k^n M_{2n} p_{k_0}^n)$ (resp. $L^2(q_l^n M_{2n} p_{k_0}^n)$), with $k \in K_n$ and p_k^n a minimal projection in $M' \cap M_{2n}$, in its kth summand (resp. $l \in L_n$ and q_l^n minimal in the lth summand of $N' \cap M_{2n}$). Note that, as correspondences (i.e., as either inclusions or bimodules), $\mathcal{H}_k = L^2(p_k^n M_{2n} p_{k_0}^n)$ coincides with $L^2(p_k^m M_{2m} p_{k_0}^m)$, $m \geqslant n$ (and $\mathcal{K}_l = L^2(q_l^n M_{2n} p_{k_0}^n)$ with $L^2(q_l^m M_{2m} p_{k_0}^m)$, $m \geqslant n$). Also, with all these identifications, L_n coincides with the set of irreducible N–M correspondences that are subcorrespondences of an irreducible M–M correspondence labeled by K_n and K_{n+1} is the set of irreducible M–M correspondences in which some N–M correspondence in L_n appears as a subcorrespondence. Altogether we get:

2.4.5. THEOREM. (i) $\|A_{N,M}^{uf}\|^2 \leqslant \operatorname{Ind} E_{\min}^{M,N} \leqslant [M:N]$.

(ii) *If* $\vec{v} = (v_k)_{k \in K^{uf}}$ *and* $\vec{u} = (u_l)_{l \in L^{uf}}$, *with* $v_k = |i_k|^{1/2}$ *and* $u_l = |j_l|^{1/2}$, *then* $A\vec{u} = \alpha^{1/2}\vec{v}$, *where* $\alpha = \operatorname{Ind} E_{\min}^{M,N}$.

(iii) *If we identify the set of simple summands of irreducible* M_1–P *correspondences with* L^u, *via the identification of* M_1 *with the* $[M:N]$*-amplification of* N, *i.e., of* $e_1 \mathcal{M}_1^u e_1$ *with* $\mathcal{N}^u$, *then* $A_{M,M_1}^u = (A_{N,M}^u)^t$ *and* $A^t\vec{v} = \alpha^{1/2}\vec{u}$, *with* α *as before.*

(iv) *If* $N \subset M \subset Q$ *then* $A_{N,M}^u A_{M,Q}^u = A_{N,Q}^u$. *Also* $AA^t\vec{v} = \alpha\vec{v}$ *and* $A^tA\vec{u} = \alpha\vec{u}$.

(v) *There is a natural identification of* $A_{M,N}^{\mathrm{st}} = (a_{kl})_{k \in K^{\mathrm{st}}}$, $l \in L^{\mathrm{st}}$ *with the transpose of the standard matrix of* $N \subset M$, $\Gamma_{N,M}^t$.

Proof. (iii) is clear once we observe that $\mathcal{N}^u \subset \mathcal{M}^u \subset^{e_1} \mathcal{M}_1^u$ is the (algebraic) basic construction (as defined in [Po5]) of $\mathcal{N}^u \subset^{\mathcal{E}} \mathcal{M}^u$. The rest is clear, by the above considerations, with (i) following from (iv) and 1.3.6 . □

2.4.6. *Remarks.* (a) The above construction will be of important use in some future work. We just point out here that it provides a proper set up for a short proof of a recent result of D. Bisch ([Bi3]), showing that if $N \subset M \subset P$ are subfactors of finite index and $N \subset P$ has finite depth then both $N \subset M$ and $M \subset P$ have finite depth. Indeed, by the above theorem, $\Gamma_{N,P}$ is just the connected component of $\Gamma_{N,M}^u \Gamma_{M,P}^u$ which contains the vertex $*$ of $\Gamma_{M,P}$. If this is finite then the connected component of $\Gamma_{M,P}$ containing $*$ is finite as well, thus $M \subset P$ has finite depth. Also, we have the inclusions $P \subset \langle P, M\rangle \subset \langle P, N\rangle$ with the big inclusion having finite depth, as being the basic construction of $N \subset P$. Thus, from the above we get that $\langle P, M\rangle \subset \langle P, N\rangle$ has finite depth. But this inclusion is an amplification of $N \subset M$ (e.g. by [Ch]) so that $N \subset M$ has finite depth as well.

(b) Finally, let us point out that if $N \subset M$ has finite depth, if k is so that $N_{2k-1} \subset \ldots \subset N \subset M$ reaches the depth, and if $Q = N_{2k-1}' \cap N \subset N_{2k-1}' \cap M = P$, then the representation of $N \subset M$ into $\mathcal{N} \subset \mathcal{M}$ of Example 2.3.5 is equivalent to a reduced of the representation $\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$, more precisely, there exists $p \in M' \cap \mathcal{N}^{\mathrm{st}}$, with $\tau'(p) = [M:N]^{-k}$ such that the representation of $N\subset M$ into $p\mathcal{N}^{\mathrm{st}}p\subset p\mathcal{M}^{\mathrm{st}}p$ is equivalent to its representation into $\mathcal{N}\subset\mathcal{M}$.

3. Amenability for inclusions of type II_1 factors

3.1. Definitions and motivations

The concept of amenability was first introduced in the theory of groups, by a functional analytical characterisation: A discrete group G is called amenable if it has a (left) invariant mean, i.e., a state $\varphi\in(l^\infty(G))^*$ such that $\varphi({}_gf)=\varphi(f)$, $\forall g\in G$, $\forall f\in l^\infty(G)$, where ${}_gf(h)=f(g^{-1}h)$, $h\in G$. In the quantum theory of von Neumann algebras, the analogue of an invariant mean (and thus of amenability) is Connes' concept of a hypertrace. If M is a von Neumann factor and $M\subset\mathcal{M}$ then an *M-hypertrace* on the von Neumann algebra $\mathcal{M}$ is a state φ on $\mathcal{M}$ such that $\varphi(xT)=\varphi(Tx)$, $\forall x\in M$, $\forall T\in\mathcal{M}$. Thus, a single von Neumann algebra M is *amenable* if given any representation $M\subset\mathcal{M}$, $\mathcal{M}$ has an M-hypertrace.

In this section we will introduce the concept of amenability and strong amenability for inclusions of (type II_1) factors of finite index. Like in the group and single algebra cases, this means defining the proper concept of invariant means (or hypertraces) for $N\subset M$, which in this case, besides the usual hypertrace properties, will also have to be compatible with certain E_N-related expectations.

3.1.1. *Definition.* Let $N\subset M$ be an inclusion of finite type II_1 factors with finite index. $N\subset M$ is an *amenable inclusion* if given any smooth representation (or embedding) of $N\subset M$

$$\begin{array}{ccc} \mathcal{N} & \overset{\mathcal{E}}{\subset} & \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

(i.e., such that $\mathcal{E}|_M=E_N^M$, $\overline{\mathrm{sp}\,M\mathcal{N}}=\mathcal{M}$, and $\mathcal{N}'\cap M_k=N'\cap M_k$), there exists an M-hypertrace φ on $\mathcal{M}$ satisfying $\varphi\circ\mathcal{E}=\varphi$. Such a state on $\mathcal{M}$ is called an *$(N\subset M)$-hypertrace* on $\mathcal{N}\subset\mathcal{M}$. $N\subset M$ is a *strongly amenable* inclusion if $N\subset M$ is amenable and has ergodic core (i.e., has ergodic $\Gamma_{N,M}$).

3.1.2. *Definition.* Let

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

be a nondegenerate commuting square of finite von Neumann subalgebras of the finite von Neumann algebra M, i.e., $E_N^M|_P=E_Q^P$ and $\mathrm{sp}\,PN=M$ (cf. 1.1.4). Let $\mathcal{E}=E_{\langle N,Q\rangle}^{\langle M,P\rangle}$ be the canonical conditional expectation of $\langle M,P\rangle=\langle M,e_N^M\rangle$ onto $\langle N,Q\rangle=\langle N,e_Q^N\rangle\simeq\langle N,e_P^M\rangle\subset\langle M,e_P^M\rangle$. Then $N\subset M$ is *amenable relative to* $Q\subset P$ if there exists an $(N\subset M)$-hypertrace on $\langle N,Q\rangle\subset\langle M,P\rangle$.

Note that if we take $N=M$ and $P=Q$ then the amenability of $(N\subset M)$ relative to $(Q\subset P)$ reduces to the usual amenability of M relative to P, as defined in single von Neumann algebra theory ([Po4]). In particular, if $P=Q=\mathbf{C}$ (and $M=N$) then this reduces to the amenability of M, i.e., existence of hypertraces for M in its standard representation on $L^2(M,\tau)$ (since $\langle M,\mathbf{C}\rangle=\mathcal{B}(L^2(M,\tau))$).

Note that in case $N=M$, the core is reduced to the scalars $\mathbf{C}\subset\mathbf{C}$ and the amenability of $N\subset M$ reduces to the amenability of M. The same with strong amenability.

3.1.3. *Examples.* (a) Let $N\subset M$ be a subfactor of finite index of the amenable (thus hyperfinite [C3]) type II_1 factor M. Assume there exists an increasing sequence of finite dimensional inclusions $Q_n\subset P_n$ in $N\subset M$, making nondegenerate commuting squares and such that $\overline{\bigcup_{n\geqslant 1}P_n}=M$ (so that $\overline{\bigcup_{n\geqslant 1}Q_n}=N$ too). Let $\mathcal{N}=\bigoplus_n\langle N,Q_n\rangle\subset\bigoplus_n\langle M,P_n\rangle=\mathcal{M}$ and note that $N\subset M$ is naturally represented into $\mathcal{N}\subset\mathcal{M}$ and that this representation is atomic if $Q_n\subset P_n$ are atomic (e.g. finite dimensional). Then $\phi(X)\overset{\text{def}}{=}\lim_{n\to\omega}\mathrm{Tr}_{\langle M,P_n\rangle}(Xe_{P_n}^M)$, $X\in\mathcal{M}$, for ω a free ultrafilter on $\mathbf{N}$, is an $(N\subset M)$-hypertrace on $\mathcal{N}\subset\mathcal{M}$. Indeed, if $x\in P_n$ and $Y\in\mathcal{M}$, then $[e_{P_m}^M,x]=0$, for $m\geqslant n$, thus $\mathrm{Tr}_{\langle M,P_m\rangle}(xYe_{P_m}^M)=\mathrm{Tr}_{\langle M,P_m\rangle}(Yxe_{P_m}^M)$, so that $\phi(xY)=\phi(Yx)$, for $x\in\bigcup_n P_n$. Also $\phi(x)=\tau(x)$, $x\in M$, so that by [C6] $\phi(xY)=\phi(Yx)$, $\forall x\in M$, $Y\in\mathcal{M}$.

(b) If $N^\theta\subset M^\theta$ is as in 2.3.3 (b) (see also 5.1.5) for some faithful G-kernel on a type II_1 factor P, then this inclusion is amenable if and only if the factor P is an amenable algebra (i.e. hyperfinite by [C3]) and G is an amenable group. Indeed, by 2.3.3 (b) any smooth representation of $N^\theta\subset M^\theta$ is given by some inclusion $P\subset\mathcal{P}$ and an extension of the G-kernel σ on P to a G-kernel $\tilde\sigma$ on $\mathcal{P}$. If P is amenable and G as well then there exists a P-hypertrace ϕ on $\mathcal{P}$ such that $\phi(\tilde\sigma_g(X))=\phi(X)$, $X\in\mathcal{P}$, $g\in G$. But then ϕ extends trivially to an $(N^\theta\subset M^\theta)$-hypertrace on $\mathcal{N}^{\tilde\theta}\subset\mathcal{M}^{\tilde\theta}$. The converse will be proved later (in 5.1.5).

(c) If $Q\subset P$ makes a nondegenerate commuting square in $N\subset M$ and if there is a partition of the unity in Q with central projections $(q_i)_{i\in I}$ such that $\mathrm{Ind}(q_iQ\subset q_iNq_i)<\infty$, $\forall i\in I$, then $N\subset M$ is amenable relative to $Q\subset P$. Indeed, if the set I is finite, then $\mathrm{Tr}_{\langle M,P\rangle}$ is finite, so that its normalization gives the desired hypertrace. If I is infinite, one takes a Banach limit of such hypertraces.

3.2. Basic properties

We first prove some simple properties of the notions we just introduced.

3.2.1. PROPOSITION. *If $N \subset M$ is an amenable inclusion then:*

(1) *N and M are amenable.*

(2) *$N \subset M$ is amenable relative to any inclusion $Q \subset P$, where*

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ Q & \subset & P \end{array}$$

is a nondegenerate commuting square.

Proof. (2) is trivial by the definitions and (1) follows from the fact that, for a chosen core R, $\langle M, R\rangle$ is amenable, since it is the commutant of the hyperfinite algebra $J_M R J_M$. Thus there exists a conditional expectation of $\mathcal{B}(L^2(M,\tau))$ onto $\langle M, R\rangle$ which composed with the $(N \subset M)$-hypertrace on $\langle M, R\rangle$ gives an M-hypertrace on $\mathcal{B}(L^2(M,\tau))$. □

The next result shows that the existence of $(N \subset M)$-hypertraces is equivalent to the existence of norm 1 projections onto $(N \subset M)$.

3.2.2. PROPOSITION. *Let $N \subset M$ be an inclusion of finite von Neumann algebras and $\mathcal{N} \subset \mathcal{M}$ an inclusion of arbitrary von Neumann algebras with an expectation $\mathcal{E}: \mathcal{M} \to \mathcal{N}$. Assume $N \subset \mathcal{N}$, $M \subset \mathcal{M}$ and $\mathcal{E}|_M = E_N^M$. There exists an M-hypertrace φ on $\mathcal{M}$ such that $\varphi \circ \mathcal{E} = \varphi$ if and only if there exists a conditional expectation $\Phi: \mathcal{M} \to M$ such that $E_N^M \Phi = \Phi \mathcal{E}$.*

Proof. If such a Φ exists then $\varphi = \tau \circ \Phi$ is clearly an $(N \subset M)$-hypertrace on $\mathcal{M}$. Conversely if φ is an $(N \subset M)$-hypertrace on $\mathcal{M}$ then

$$\mathcal{M} \ni T \longmapsto \varphi(\cdot T) \in M^*$$

gives a positive linear application Φ from $\mathcal{M}$ into M^* which takes 1 into the trace τ. By the positivity of Φ and by the faithfulness of τ it follows that if $0 \leqslant T \leqslant 1$ then $0 \leqslant \Phi(T) \leqslant \tau$, so that $\Phi(T)$ is a normal functional in M_* and in fact $\Phi(T)$ corresponds to the Radon–Nikodým derivative of τ, $\Phi(T) = \tau(\cdot t)$, for some $t \in M$, $0 \leqslant t \leqslant 1$. By identifying M with its image in M_* via Radon–Nikodým derivatives of τ, it follows that Φ can be regarded as an application from $\mathcal{M}$ onto M. Φ is clearly an M–M bimodule map and leaves M fixed. Thus it is a conditional expectation and we have

$$\begin{aligned} \Phi(\mathcal{E}(T)) &= \varphi(\cdot \mathcal{E}(T)) = \varphi(\mathcal{E}(\cdot \mathcal{E}(T))) \\ &= \varphi(\mathcal{E}(\cdot)\mathcal{E}(T)) = \varphi(\mathcal{E}(\cdot)T) = \varphi(E_N^M(\cdot)T) = E_N^M(\Phi(T)) \end{aligned}$$ □

3.2.3. PROPOSITION (Hereditarity). *Let*

$$\begin{array}{ccccc} P_0 & \subset & P & \subset & M \\ \cup & & \cup & & \cup \\ Q_0 & \subset & Q & \subset & N \end{array}$$

be nondegenerate commuting squares.

(i) *If $q\in Q$ is a projection with $q, 1-q$ of central support 1 in Q then $qNq\subset qMq$ is amenable relative to $qQq\subset qPq$ if and only if $N\subset M$ is amenable relative to $Q\subset P$.*

(ii) *Assume that there exists an orthonormal basis $(\eta_i)_i$ of N over Q which is either finite or so that $E_Q(\eta_i^*\eta_i)\in Q_0$, $\forall i$. Then $N\subset M$ is amenable relative to $Q_0\subset P_0$ if and only if $N\subset M$ is amenable relative to $Q\subset P$ and $Q\subset P$ is amenable relative to $Q_0\subset P_0$.*

(iii) *If $p\in N$ is a nonzero projection then $N\subset M$ is amenable if and only if $pNp\subset pMp$ is amenable. Also, if P^0 is an $n\times n$ matrix algebra then $N\subset M$ is amenable if and only if $N\otimes P^0\subset M\otimes P^0$ is amenable.*

(iv) *$N\subset M$ is amenable if and only if $M\subset M_1$ is amenable.*

Proof. (i) If $N\subset M$ is amenable relative to $Q\subset P$ and φ is the corresponding $(N\subset M)$-hypertrace on $\langle M, e_P^M\rangle$ then we have $\langle qMq, qPq\rangle \simeq \langle qMq, e_P^M q\rangle \subset \langle M, P\rangle$, so we may just define φ_q on $\langle qMq, qPq\rangle$ by $\varphi_q(x)=\tau(q)^{-1}\varphi(x)$, $x\in\langle qMq, e_P^M q\rangle$ which will clearly be a $(qNq\subset qMq)$-hypertrace.

Conversely, assume $qNq\subset qMq$ is amenable relative to $qQq\subset qPq$. We claim that, since the central support of q in Q is one, there exists an orthonormal basis of N over $Q, (\eta_i)_i$, such that $[\eta_i, q]=0$. Indeed, just take $(\eta_i')_i$ to be an orthonormal basis of qNq over qQq and $(\eta_i'')_i$ to be an orthonormal basis of $(1-q)N(1-q)$ over $(1-q)Q(1-q)$ and define $\eta_i=\eta_i'+\eta_i''$. Thus

$$\begin{aligned} q\langle M, P\rangle q &= q\Big\{\sum_{i,j}\eta_i p_{ij} e_P^M \eta_j^* \,\Big|\, p_{ij}\in P\Big\}q \\ &= \Big\{\sum_{i,j}\eta_i' q p_{ij} q (e_P^M q)\eta_j'^* \,\Big|\, qp_{ij}q\in qPq\Big\} \\ &= \langle qMq, e_P^M q\rangle = \langle qMq, qPq\rangle. \end{aligned}$$

Thus, if Φ_q denotes a conditional expectation of $\langle qMq, qPq\rangle$ onto qMq satisfying the commuting square condition 3.2.2, then Φ_q can be regarded as defined on $q\langle M, P\rangle q$. Since the support of q in Q is 1, there exists an amplification of Φ_q to a conditional expectation Φ of $\langle M, P\rangle$ onto M, satisfying $\Phi(vqxqw^*)=v\Phi_q(qxq)w^*$, for all $x\in\langle M, P\rangle$, $v, w\in Q$, $w^*w, v^*v\leqslant q$ (see for example [St]). Then Φ clearly satisfies 3.2.3 and thus $N\subset M$ is amenable relative to $Q\subset P$.

(ii) If $N\subset M$ is amenable relative to $Q_0\subset P_0$ and if φ_0 is an $(N\subset M)$-hypertrace on $\langle M,P_0\rangle$ then $\varphi=\varphi_0|_{\langle M,P\rangle}$ is clearly an $(N\subset M)$-hypertrace on $\langle M,P\rangle$. If $(\eta_i)_i$ is finite then $\varphi_0(e^M_{P_0})\neq 0$ and so the normalization of $\varphi_0|_{\langle P,e^M_{P_0}\rangle}$ will be a $(Q\subset P)$-hypertrace on $(\langle Q,e^M_{P_0}\rangle\subset\langle P,e^M_{P_0}\rangle)\simeq(\langle Q,e^P_{P_0}\rangle\subset\langle P,e^P_{P_0}\rangle)$.

Next assume that $(\eta_i)_i$ is such that $E_Q(\eta_i^*\eta_i)\in Q_0$, for all i. Since $e^M_P\langle M,P_0\rangle e^M_P=\langle P,P_0\rangle e^M_P$, $e^M_P\langle N,Q_0\rangle e^M_P=\langle Q,Q_0\rangle e^M_P$, by denoting

$$\begin{aligned}\widetilde{P}&=\Big\{\sum\eta_i p_{ij}e^M_P\eta_j^*\,\Big|\,p_{ij}\in P\Big\}=\langle M,P\rangle,\\ \widetilde{P}_0&=\Big\{\sum\eta_i p^0_{ij}e^M_P\eta_j^*\,\Big|\,p^0_{ij}\in P_0\Big\},\\ \widetilde{Q}&=\Big\{\sum\eta_i q_{ij}e^M_P\eta_j^*\,\Big|\,q_{ij}\in Q\Big\}=\langle N,Q\rangle,\\ \widetilde{Q}_0&=\Big\{\sum\eta_i q^0_{ij}e^M_P\eta_j^*\,\Big|\,q^0_{ij}\in Q_0\Big\},\end{aligned}$$

it follows that

$$\begin{array}{ccccc}\widetilde{P}_0&\subset&\widetilde{P}&\subset&\langle M,P_0\rangle\\ \cup&&\cup&&\cup\\ \widetilde{Q}_0&\subset&\widetilde{Q}&\subset&\langle N,Q_0\rangle\end{array}$$

is an amplification of

$$\begin{array}{ccccc}P_0\simeq P_0e^M_P&\subset&Pe^M_P\simeq P&\subset&\langle P,P_0\rangle\simeq e^M_P\langle M,P_0\rangle e^M_P\\ \cup&&\cup&&\cup\\ Q_0\simeq Q_0e^M_P&\subset&Qe^M_P\simeq Q&\subset&\langle Q,Q_0\rangle\simeq e^M_P\langle M,Q_0\rangle e^M_P.\end{array}$$

Then the $(N\subset M)$-hypertrace φ_0 on $\langle M,P_0\rangle$ is clearly a $(\widetilde{Q}\subset\widetilde{P})$-hypertrace, so $\widetilde{Q}\subset\widetilde{P}$ is amenable relative to $\widetilde{Q}_0\subset\widetilde{P}_0$. By (i) it follows that $Q\subset P$ is amenable relative to $Q_0\subset P_0$.

Conversely, if $Q\subset P$ is amenable relative to $Q_0\subset P_0$ and $N\subset M$ is amenable relative to $Q\subset P$ then, again by (i), $\widetilde{Q}\subset\widetilde{P}$ is amenable relative to $\widetilde{Q}_0\subset\widetilde{P}_0$, so there exist conditional expectations $\psi_1\colon\langle M,P_0\rangle\to\widetilde{P}=\langle M,P\rangle$ and $\psi_2\colon\langle M,P\rangle\to\mathcal{M}$ so that $\Phi=\psi_2\circ\psi_1$ satisfies the appropriate commuting square.

(iii) If $N\subset M$ is amenable and

$$\begin{array}{ccc}\mathcal{N}&\subset&\mathcal{M}\\ \cup&&\cup\\ N\otimes P^0&\subset&M\otimes P^0\end{array}$$

is a smooth embedding for $N\otimes P^0\subset M\otimes P^0$ then it will be also a smooth embedding for $N\subset M$ so there exists an $(N\subset M)$-hypertrace φ on $\mathcal{M}$. But then averaging φ over $\mathcal{U}(P^0)$ gives a hypertrace for $N\otimes P^0\subset M\otimes P^0$. Conversely if $N\otimes P^0\subset M\otimes P^0$ is amenable and

$(N\subset M)$ is smoothly embedded in $\mathcal{N}\subset\mathcal{M}$ then $N\otimes P^0\subset M\otimes P^0$ is smoothly embedded in $\mathcal{N}\otimes P^0\subset\mathcal{M}\otimes P^0$ and any $(N\otimes P^0\subset M\otimes P^0)$-hypertrace on $\mathcal{M}\otimes P^0$ gives an $(N\subset M)$-hypertrace when restricted to $\mathcal{M}$.

If $p\in N$ is a projection and $pNp\subset pMp$ is amenable then let $v_i\in N$ be partial isometries such that $v_i^*v_i\leqslant p$, $\sum v_iv_i^*=1$ and let $(N\subset M)$ be smoothly embedded into $\mathcal{N}\subset\mathcal{M}$. In particular $pNp\subset pMp$ is smoothly embedded into $p\mathcal{N}p\subset p\mathcal{M}p$. Let φ_0 be a $(pNp\subset pMp)$-hypertrace on $p\mathcal{M}p$ and define $\varphi(x)=\tau(p)\sum_i\varphi_0(v_i^*xv_i)$ on $\mathcal{M}$. Then φ is trivially an $(N\subset M)$-hypertrace on $\mathcal{M}$. Thus $N\subset M$ follows amenable. The converse implication follows by first amplifying with some P^0 and by using the above first part.

(iv) By (iii) and by using that $N\subset M$ is the reduced of the inclusion $M_1\subset M_2$ ([PiPo1]) it follows that it is sufficient to prove one implication.

Assume $M\subset M_1$ is amenable. Let $(N\subset M)$ be smoothly embedded into $\mathcal{N}\subset\mathcal{M}$ and let $(M\subset M_1)$ be smoothly embedded in $\mathcal{M}\subset\mathcal{M}_1$ by the usual extension (see §1.4). Then there exists an $(M\subset M_1)$-hypertrace φ_1 on $\mathcal{M}_1$. Define φ on $\mathcal{M}$ by $\varphi=\varphi_1|_{\mathcal{M}}$. Note that for $T\in\mathcal{M}$ we have

$$\begin{aligned}\varphi(T)&=\varphi_1(T)=\lambda^{-1}\varphi_1(\mathcal{E}_1(Te_1))=\lambda^{-1}\varphi_1(Te_1)=\lambda^{-1}\varphi_1(e_1Te_1)\\&=\lambda^{-1}\varphi_1(\mathcal{E}(T)e_1)=\lambda^{-1}\varphi_1(\mathcal{E}_1(\mathcal{E}(T)e_1))=\varphi_1(\mathcal{E}(T))=\varphi(\mathcal{E}(T)).\end{aligned}$$

Also, if $x\in M$ then

$$\begin{aligned}\varphi(xT)&=\lambda^{-1}\varphi_1(e_1xT)=\lambda^{-1}\varphi_1(Te_1x)=\lambda^{-1}\varphi_1(\mathcal{E}_1(Te_1x))\\&=\varphi_1(Tx)=\varphi(Tx).\end{aligned}$$

□

3.2.4. Proposition. *If $N\subset M$ is amenable relative to one of its cores (respectively has ergodic core) then:*

(i) *$N_k\subset M$ is amenable relative to one of its cores (respectively has ergodic core) for any k and any choice of the tunnel up to k.*

(ii) *$pNp\subset pMp$ is amenable relative to one of its cores (respectively has ergodic core), for any projection $p\in N$.*

Proof. (i) By conjugating if necessary by a unitary element in N, we may assume $N\subset M$ is amenable relative to a core $S\subset R$ so that the tunnel $\{N_k^0\}_{k\geqslant1}$ with $\overline{\bigcup((N_l^0)'\cap M)}=R$ coincides with N_i up to k, i.e., $N_l^0=N_l$ for $l\leqslant k$. Let $\Phi\colon\langle M,e_R^M\rangle=\langle M,R\rangle\to M$ be the conditional expectation of 3.2.2, satisfying $\Phi(T)\subset N$ for $T\in\langle N,e_R^M\rangle$. But $\Phi(e_{-i})=e_{-i}$, $1\leqslant i\leqslant k-1$, e_{-i} being the Jones projections of the tunnel. Since $\Phi(\langle N_k,e_R^M\rangle)\subset\Phi(\langle N,e_R^M\rangle)\subset N$ and since $[\langle N_k,e_R^M\rangle,e_{-i}]=0$, $1\leqslant i\leqslant k-1$, it follows that $[\Phi(\langle N_k,e_R^M\rangle),e_{-i}]=0$, so that $\Phi(\langle N_k,e_R^M\rangle)\subset N_k$.

(ii) By 3.2.3, if $N\subset M$ is amenable relative to the core $S\subset R$ and if P is a finite subfactor of $S\subset N$ such that $p\in P$ and such that $S=S^0\otimes P$, $R=R^0\otimes P$, with $S^0\subset R^0$ still a core inclusion (cf. §1.4), then $pNp\subset pMp$ is amenable relative to $S^0p\subset R^0p$. □

4. Approximation of amenable inclusions by higher relative commutants

The structure theorem for amenable groups is Følner's characterisation as those groups that locally behave like finite permutations (approximately). The operator algebra analogue of the Følner condition for amenable factors was discovered by Connes ([C3]). The structure theorem for amenable factors is a much more precise result though, stating that a factor is amenable if and only if it is hyperfinite, i.e., approximable in the strong operator topology by its finite dimensional subalgebras. This is Connes' fundamental theorem ([C3]), which in fact, in its original proof, does not use much the Følner condition. The later proof of this result in [Po10], though, uses in a crucial way the Følner condition of [C3] to obtain the finite dimensional approximation in a rather direct way. In this section we will first prove a Følner-type characterisation of the amenability for inclusions, in the same spirit as Connes' single algebra case. Then we will use the same techniques as in [Po10] to show that amenable inclusions can be approximated by the finite dimensional algebras of higher relative commutants. This finite dimensional approximation of amenable inclusions by higher relative commutants, resulting into a complete classification of such inclusions by their standard invariant, is the main technical result of this paper. It can be regarded as the analogue of Connes' theorem to the case of inclusions of algebras.

4.1. Statement of results

We now state the main theorems of this section:

4.1.1. THEOREM. *Let $N\subset M$, $N\neq M$ be type II_1 factors with finite index. Consider the following conditions*:

(i) *$N\subset M$ is amenable.*

(ii) *$N\subset M$ is amenable relative to any of its cores.*

(iii) *$N\subset M$ is amenable relative to one of its cores.*

(iv) *$N\subset M$ satisfies the Følner condition in $\langle N,e_R^M\rangle\subset\langle M,e_R^M\rangle$, where $S\subset R$ is one of the cores of $N\subset M$, i.e., $\forall\varepsilon>0$ and all unitary elements $u_1,\dots,u_m\in M$, there exists a finite projection $p\in\langle N,e_R^M\rangle$ such that $\|u_ipu_i^*-p\|_{2,\mathrm{Tr}}<\varepsilon\|p\|_{2,\mathrm{Tr}}$.*

(v) *$N\subset M$ can be locally approximated by higher relative commutants, i.e., $\forall\varepsilon>0$, $\forall x_1,\dots,x_n\in M$, $\exists m$ and a continuation of the tunnel up to m, $M\supset N\supset N_1\supset\dots\supset N_m$, with* *a projection* $s \in N'_m \cap N$ *such that*

$$\|[s, x_i]\|_2 < \varepsilon \|s\|_2,$$
$$\|E_{sN'_m \cap Ms}(sx_i s) - sx_i s\|_2 < \varepsilon \|s\|_2.$$

(vi) $\forall \varepsilon > 0$, $\forall x_1, \ldots, x_n \in M$, $\exists m$ *and a continuation of the tunnel,* $M \supset N \supset N_1 \supset \ldots \supset N_m \supset \ldots$, *with a projection* $s_0 \in N'_m \cap N$, *a projection* $f_0 \in N_m \cap R$ *of scalar central trace in* $\mathcal{Z}(N_m \cap R)$ *and a central projection* $z_0 \in \mathcal{Z}(R \cap N)$, *where* $R = \overline{\bigcup (N'_k \cap M)}$, *such that* $s = s_0 f_0$ *satisfies*:

$$\|[s, x_i]\|_2 < \varepsilon \|s\|_2,$$
$$\|E_{sN'_m \cap Ms}(sx_i s) - sx_i s\|_2 < \varepsilon \|s\|_2,$$
$$\|s_0 - z_0\|_2 < \varepsilon \|s_0\|_2.$$

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi).

4.1.2. THEOREM. *Let* $N \subset M$, $N \neq M$, *be type* II_1 *factors with finite index. Then the following conditions are equivalent*:

(i) $N \subset M$ *is strongly amenable.*

(ii) $N \subset M$ *is amenable relative to one of its cores and it has ergodic core.*

(iii) $N \subset M$ *can be globally approximated by higher relative commutants, i.e.,* $\forall \varepsilon > 0$, $\forall x_1, \ldots, x_n \in M$, *there exist an* m *and a continuation of the tunnel up to* m, $M \supset N \supset \ldots \supset N_m$, *such that*

$$\|E_{N'_m \cap M}(x_i) - x_i\|_2 < \varepsilon, \quad \forall i.$$

If in addition M *is separable then they are also equivalent to*:

(iv) $N \subset M$ *has the generating property, i.e., there exists a tunnel* $M \supset N \supset N_1 \supset \ldots$ *such that* $\overline{\bigcup (N'_k \cap M)} = M$.

(v) $N \subset M$ *is isomorphic to its standard part* $N^{\mathrm{st}} \subset M^{\mathrm{st}}$ *(as defined in* 1.3.7).

(vi) (a) M *is amenable*;

(b) M *satisfies the bicommutant condition in* M_∞, *i.e.,* $(M' \cap M_\infty)' \cap M_\infty = M$.

Our purpose in this section is to prove these theorems. More precisely we will prove (iii) $\Leftrightarrow$ (iv) of 4.1.1 in §4.2, (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (iv) of 4.1.1 in §4.3, then (ii) $\Rightarrow$ (iv) of 4.1.2 in §4.4 and (iv) $\Rightarrow$ (iii) $\Rightarrow$ (vi) $\Rightarrow$ (i) of 4.1.2 in §4.5. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) of 4.1.1 and (i) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (v) of 4.1.2 are of course trivial. We mention that the implication (iii) $\Rightarrow$ (i) of 4.1.1, which would make all the conditions in 4.1.1 equivalent is also true, but it will be proved elsewhere.

4.2. A Følner type condition: proof of (iii) $\Leftrightarrow$ (iv) in Theorem 4.1.1

From the $(N\subset M)$-hypertraces we will obtain now vector subspaces of M which are finite dimensional relative to the core, make appropriate commuting squares with $N\subset M$ and are almost invariant to a given finite set of elements in M. This will be our Følner type condition for inclusions, in the spirit of Connes' single algebra case ([C3]).

4.2.1. THEOREM. *$N\subset M$ is amenable relative to $Q\subset P$ if and only if given any finite set of unitary elements in M, $u_1, ..., u_n$, and any $\varepsilon>0$, there exists a finite projection p in $\langle N, e_R^M\rangle$ such that*

$$\|u_i p u_i^* - p\|_{2,\mathrm{Tr}} < \varepsilon \|p\|_{2,\mathrm{Tr}}.$$

Proof. Let φ_0 denote an $(N\subset M)$-hypertrace on $M_1=\langle M,P\rangle$. Let also $E=E_{\langle N,Q\rangle}^{\langle M,P\rangle}$: $M_1=\langle M,P\rangle\to\langle N,Q\rangle$ be the canonical conditional expectation satisfying $E(e_R^M)=e_R^M$, $E(x)=E_Q^P(x)$, for $x\in P$.

Step 1 (Day's trick). Let

$$\mathcal{L}=\{(\psi(E(\cdot))-\psi(\cdot), \psi(u_1^*\cdot u_1)-\psi(\cdot), ..., \psi(u_n^*\cdot u_n)-\psi(\cdot)) \mid \psi \text{ a state in } M_{1*}\}.$$

Then $\mathcal{L}$ is a bounded convex set in $(M_{1*})^{n+1}$. Since the states $\psi\in M_{1*}$ are $\sigma(M_1^*, M_1)$ dense in M_1^* it follows that the $\sigma((M_1^*)^{n+1}, (M_1)^{n+1})$ closure $\bar{\mathcal{L}}^w$ of $\mathcal{L}$ contains any $(n+1)$-tuple $(\varphi(E(\cdot))-\varphi(\cdot), \varphi(u_1^*\cdot u_1)-\varphi(\cdot), ..., \varphi(u_n^*\cdot u_n)-\varphi(\cdot))$ with φ a state in M_1^*. In particular $\bar{\mathcal{L}}^w$ contains $(\varphi_0(E(\cdot))-\varphi_0(\cdot), \varphi_0(u_1^*\cdot u_1)-\varphi_0(\cdot), ..., \varphi_0(u_n^*\cdot u_n)-\varphi_0(\cdot))=(0,...,0)$, φ_0 being the $(N\subset M)$-hypertrace. But since $(0,...0)$ is in $(M_{1*})^{n+1}$ and the dual of $(M_{1*})^{n+1}$ is $(M_1)^{n+1}$ it follows that the $\sigma((M_{1*})^{n+1}, (M_1)^{n+1})$ closure of $\mathcal{L}$ in $(M_{1*})^{n+1}$ is equal to the norm closure of $\mathcal{L}$ and thus that $(0,...,0)$ is norm adherent to $\mathcal{L}$. It follows that given any $\delta>0$ there exists some element b in the dense subspace $(M_1\cap L^1(M_1,\mathrm{Tr}))_+$ of $L^1(M_1,\mathrm{Tr})_+$ such that $\mathrm{Tr}(b)=1$ and such that

$$\|\mathrm{Tr}(E(\cdot)b)-Tr(\cdot b)\| < \tfrac{1}{3}\delta,$$
$$\|\mathrm{Tr}((u_k^*\cdot u_k)b)-\mathrm{Tr}(\cdot b)\| < \tfrac{1}{3}\delta, \quad 1\leqslant k\leqslant n.$$

Thus we get:

$$\|\mathrm{Tr}(\cdot E(b))-\mathrm{Tr}(\cdot b)\| < \tfrac{1}{3}\delta,$$
$$\|\mathrm{Tr}(\cdot u_k b u_k^*)-\mathrm{Tr}(\cdot b)\| < \tfrac{1}{3}\delta, \quad 1\leqslant k\leqslant n,$$

so that

$$\|E(b)-b\|_{1,\mathrm{Tr}} < \tfrac{1}{3}\delta,$$
$$\|u_k b u_k^*-b\|_{1,\mathrm{Tr}} < \tfrac{1}{3}\delta, \quad 1\leqslant k\leqslant n.$$

By replacing b with $E(b)$ in the second set of inequalities, we get an element $b_0=E(b)\in N_1=\langle N, e_R^M\rangle$, $b_0\geqslant 0$, $\mathrm{Tr}(b_0)=1$, satisfying

$$\|u_k b_0 u_k^* - b_0\|_{1,\mathrm{Tr}} < \delta, \quad 1\leqslant k\leqslant n.$$

Step 2 (Powers–Størmer's inequality). Let $a=b_0^{1/2}\in\langle N, e_R^M\rangle$. By the Powers–Størmer inequality and Step 1 we have:

$$\|u_k a u_k^* - a\|_{2,\mathrm{Tr}}^2 \leqslant \|u_k b_0 u_k^* - b_0\|_{1,\mathrm{Tr}} \leqslant \delta = \delta\|a\|_{2,\mathrm{Tr}}^2.$$

If we denote $\|(x_1,...,x_n)\|_{2,\sim}=(\sum_k \|x_k\|_{2,\mathrm{Tr}}^2)^{1/2}$, for $x_1,...,x_n\in M_1$, and if we put $\tilde{a}=(a,...,a)$, $\tilde{u}=(u_1,...,u_n)\in(M_1)^n$, then we get

$$\|\tilde{u}\tilde{a}\tilde{u}^* - \tilde{a}\|_{2,\sim}^2 < \delta\|a\|_{2,\sim}^2.$$

Step 3 (Connes' trick). By 1.1 in [C1] there exists a probability measure space (X,μ) with two functions $h,k\in L^\infty(X,\mu)$ such that given any Borel functions f,g on $\mathbf{R}_+$, continuous at 0 and vanishing at 0, we have

$$\|f(\tilde{u}\tilde{a}\tilde{u}^*)-g(\tilde{a})\|_{2,\sim}^2 = \|f(h)-g(k)\|_2^2.$$

Step 4 (Connes–Namioka's trick). We then have

$$\begin{aligned}
&\int_{s>0} \|E_{[s^{1/2},\infty)}(\tilde{u}\tilde{a}\tilde{u}^*)-E_{[s^{1/2},\infty)}(\tilde{a})\|_{2,\sim}^2\, ds \\
&\qquad = \int_{s>0} \|E_{[s^{1/2},\infty)}(h)-E_{[s^{1/2},\infty)}(k)\|_2^2\, ds \\
&\qquad = \int_{s>0} \|E_{[s,\infty)}(h^2)-E_{[s,\infty)}(k^2)\|_2^2\, ds \\
&\qquad = \int_{s>0} \left(\int_X |\chi_{[s,\infty)}(h^2(x))-\chi_{[s,\infty)}(k^2(x))|^2\, d\mu(x)\right) ds \\
&\qquad = \int_X \left(\int_{s>0} |\chi_{[s,\infty)}(h^2(x))-\chi_{[s,\infty)}(k^2(x))|\, ds\right) d\mu(x) \\
&\qquad = \int_X |h^2(x)-k^2(x)|\, d\mu(x) = \|h^2-k^2\|_1 \\
&\qquad \leqslant \|h-k\|_2\|h+k\|_2 \leqslant 2\|\tilde{u}\tilde{a}\tilde{u}^*-\tilde{a}\|_{2,\sim}\|\tilde{a}\|_{2,\sim} \\
&\qquad \leqslant 2\delta^{1/2}\|\tilde{a}\|_{2,\sim}^2 = 2\delta^{1/2}\int_{s>0}\|E_{[s^{1/2},\infty)}(\tilde{a})\|_2^2\, ds.
\end{aligned}$$

Thus, there exists some $s>0$ such that $\tilde{p}=E_{[s^{1/2},\infty)}(\tilde{a})$ and $\tilde{u}\tilde{p}\tilde{u}=\tilde{u}E_{[s^{1/2},\infty)}(\tilde{a})\tilde{u}=E_{[s^{1/2},\infty)}(\tilde{u}\tilde{a}\tilde{u}^*)$ satisfy the inequality

$$\|\tilde{u}\tilde{p}\tilde{u}^*-\tilde{p}\|_{2,\sim} < 2\delta^{1/2}\|\tilde{p}\|_{2,\sim}.$$

But

$$E_{[s^{1/2},\infty)}(a,...,a)=(E_{[s^{1/2},\infty)}(a),...,E_{[s^{1/2},\infty)}(a))$$

so that if we define $p=E_{[s^{1/2},\infty)}(a)$ then p is a finite projection in $\langle N,e_P^M\rangle$ and

$$\|u_kpu_k^*-p\|_{2,\mathrm{Tr}}<2\delta^{1/2}n\|p\|_{2,\mathrm{Tr}},\quad 1\leqslant k\leqslant n.$$

Thus, if we take δ so that $2\delta^{1/2}n\leqslant\varepsilon$ then we are done.

Conversely now, let $(\mathcal{U}_i)_{i\in I}$ be the family of all finite sets of unitaries in M. For each i let p_i be a finite projection in $\langle N,e_P^M\rangle$ such that

$$\|up_iu^*-p_i\|_{2,\mathrm{Tr}}<\frac{1}{|i|}\|p\|_{2,\mathrm{Tr}},\quad u\in\mathcal{U}_i.$$

Let ω be a free ultrafilter on I and put $\varphi_0(T)=\lim_\omega\mathrm{Tr}(Tp_i)/\mathrm{Tr}(p_i)$. Then φ_0 is clearly a $(N\subset M)$-hypertrace. □

We can now prove (iii)$\Leftrightarrow$(iv) of Theorem 2.1.4, which for convenience we restate here:

4.2.2. THEOREM. *$N\subset M$ is amenable relative to one of its cores if and only if given any $\varepsilon>0$ and any finite set of unitary elements $u_1,...,u_n$ in M there exists a core $S\subset R$ for $N\subset M$ and a finite projection p in $\langle N,e_R^M\rangle$ such that*

$$\|u_ipu_i^*-p\|_{2,\mathrm{Tr}}<\varepsilon\|p\|_{2,\mathrm{Tr}},\quad 1\leqslant i\leqslant m.$$

Moreover, the core $S\subset R$ and the projection p can be chosen such that the value of the generalized trace of $\langle N,e_R^M\rangle$ on the projection p is an integer multiple of a central projection. Thus one can write $p=\sum p_i$ with p_i projections such that $p_i\sim e_R^Mz$ for all i and some common central projection $z\in\mathcal{Z}(R)$.

Proof. The first part is just a particular case of the previous theorem.

By the first part of the theorem there exists a finite projection p_1 in $\langle N,e_R^M\rangle$ such that

$$\|u_ip_1u_i^*-p_1\|_{2,\mathrm{Tr}}<\frac{1}{4m}\varepsilon\|p_1\|_{2,\mathrm{Tr}}.$$

By the stability of the core it follows that given any $n\times n$ matrix subalgebra P in S there exists another choice of the tunnel $\{N_k^0\}_{k\geqslant1}$ such that $S^0=\overline{\bigcup((N_k^0)'\cap N)}$, $R^0=\overline{\bigcup((N_k^0)'\cap M)}$ satisfies $R=R^0\vee P=R^0\otimes P$, $S=S^0\vee P=S^0\otimes P$. Note that $\mathcal{Z}(\langle N,e_R^M\rangle)=\mathcal{Z}(\langle N,e_{R^0}^M\rangle)$ and that if $C\,\mathrm{Tr}_{\langle N,e_R^M\rangle}$, $C\,\mathrm{Tr}_{\langle N,e_{R^0}^M\rangle}$ denote the generalized central traces on $\langle N,e_R^M\rangle$ respectively $\langle N,e_{R^0}^M\rangle$ then

$$C\,\mathrm{Tr}_{\langle N,e_{R^0}^M\rangle}(p_1)=n^2C\,\mathrm{Tr}_{\langle N,e_R^M\rangle}(p_1).$$

More generally

$$C\,\mathrm{Tr}_{\langle N, e^M_{R^0}\rangle}(x) = n^2 C\,\mathrm{Tr}_{\langle N, e^M_R\rangle}(x), \quad \forall x \in \langle N, e^M_R\rangle \subset \langle N, e^M_{R^0}\rangle.$$

Let $z_1 = C\,\mathrm{Tr}_{\langle N, e^M_{R^0}\rangle}(p_1)$. Given any k_0 there exists n such that if z_0 is the spectral projection of z_1 corresponding to $[k_0+1, \infty)$ then $\|p_1 - p_1 z_0\|_{2,\mathrm{Tr}} < (1/8m)\varepsilon \|p z_0\|_{2,\mathrm{Tr}}$. Then we still have:

$$\|u_i(p_1 z_0)u_i^* - p_1 z_0\|_{2,\mathrm{Tr}} < \frac{1}{2m}\varepsilon \|p_1 z_0\|_{2,\mathrm{Tr}}.$$

Write z_0 as a sum of central projections $z_0 = z^1 + z^2 + \ldots + z^l$, $z^i \in \mathcal{Z}(\langle N, e^M_{R^0}\rangle)$ such that $z_0 z_1 - \sum c_i z^i \leqslant \delta$ for some scalars $c_i \geqslant k_0 + 1$ (by spectral decomposition of $z_0 z_1$). It follows that we have

$$\begin{aligned}\sum_i \|(u_j(p_1 z_0)u_j^* - p_1 z_0)z^i\|^2_{2,\mathrm{Tr}} &= \|u_j(p_1 z_0)u_j^* - p_1 z_0\|^2_{2,\mathrm{Tr}} \\ &< \left(\frac{1}{2m}\varepsilon\right)^2 \|p_1 z_0\|^2_{2,\mathrm{Tr}} = \left(\frac{1}{2m}\varepsilon\right)^2 \sum_i \|(p_1 z_0)z^i\|^2_{2,\mathrm{Tr}}.\end{aligned}$$

It follows that for some i we have

$$\|u_j(p_1 z^i)u_j^* - p_1 z^i\|_{2,\mathrm{Tr}} < \tfrac{1}{2}\varepsilon \|p_1 z^i\|_{2,\mathrm{Tr}}, \quad \forall j.$$

Moreover $C\,\mathrm{Tr}(p_1 z^i) = C\,\mathrm{Tr}(p_1) z^i = z_1 z^i$ and $0 \leqslant z_1 z^i - c_i z^i \leqslant \delta$, $c_i \geqslant k_0 + 1$. Now since $\langle M, R^0\rangle$ is of type II it follows that there exists a finite projection $p' \in \langle N, e^M_{R^0}\rangle$ such that $C\,\mathrm{Tr}(p') = n_0 z^i$ where n_0 is the integer part of c_i.

Since $n_0 z^i \leqslant c_i z^i \leqslant z_1 z^i = C\,\mathrm{Tr}(p_1 z^i)$ it follows that p' is majorized by $p_1 z^i$. Let $p \leqslant p_1 z^i$ be so that $p' \sim p$. We thus get $\|p_1 z^i - p\|^2_{2,\mathrm{Tr}} \leqslant (1+\delta)\,\mathrm{Tr}(z^i e^M_{R^0})$ so that we have the estimate

$$\begin{aligned}\|u_j p u_j^* - p\|_{2,\mathrm{Tr}} &\leqslant 2((1+\delta)\,\mathrm{Tr}(z^i e^M_{R^0}))^{1/2} + \tfrac{1}{2}\varepsilon \|p_1 z^i\|_{2,\mathrm{Tr}} \\ &\leqslant 2((1+\delta)\,\mathrm{Tr}(z^i e^M_{R^0}))^{1/2} + \varepsilon((1+\delta)\,\mathrm{Tr}(z^i e^M_{R^0}))^{1/2} + \tfrac{1}{2}\varepsilon \|p\|_{2,\mathrm{Tr}}, \quad \forall j.\end{aligned}$$

Since $\|p\|^2_{2,\mathrm{Tr}} = n_0\,\mathrm{Tr}(z^i e^M_{R^0})$, it follows that if n_0 (and thus k_0) and δ are chosen such that $2((1+\delta)/n_0)^{1/2} + \varepsilon((1+\delta)/n_0)^{1/2} < \frac{1}{2}\varepsilon$ then we get

$$\|u_j p u_j^* - p\|_{2,\mathrm{Tr}} < \varepsilon \|p\|_{2,\mathrm{Tr}}$$

and $C\,\mathrm{Tr}(p) = n_0 z^i$, where n_0 is an integer and z^i is a central projection of $\langle N, e^M_{R^0}\rangle$. Then $z = J_M z^i J_M$ and p satisfy the requirements in the last part of the theorem. $\square$

4.2.3. COROLLARY. *Let $N \subset M$ be an amenable inclusion. Given any finite set of unitary elements $\mathcal{U}_0$ in M and any $\varepsilon > 0$, there exists a continuation of the tunnel up to some m, $M \supset N \supset N_1 \supset \ldots \supset N_m$, and a finite set of elements $\{x_i\}_{1 \leqslant i \leqslant n}$ in N such that*:

(1) $E_{N'_m \cap M}(x_i^* x_j) = \delta_{ij} f$ *with f a central projection of $N'_m \cap N$ satisfying* $\|f - z\|_2 < \varepsilon \|f\|_2$ *for some central projection $z \in \mathcal{Z}(S)$ where $S \subset R$ is the core associated to some continuation of the tunnel $\{N_k\}_{1 \leqslant k \leqslant m}$.*

(2) $2n\tau(f) - 2\sum_{i,j} \|E_{N'_m \cap M}(x_i v x_j^*)\|_2^2 < \varepsilon n \tau(f)$ *for all $v \in \mathcal{U}_0$.*

Moreover, if in addition $N \subset M$ has ergodic core then the tunnel $\{N_k\}_{1 \leqslant k \leqslant m}$ and $\{x_i\}_{1 \leqslant i \leqslant n} \subset N$ can be chosen so that:

(1′) $E_{N'_m \cap M}(x_i^* x_j) = \delta_{ij}$.

(2′) $2n - \sum_{i,j} \|E_{N'_m \cap M}(x_i^* v x_j)\|_2^2 < \varepsilon n$, $v \in \mathcal{U}_0$.

Proof. Let $S \subset R$ be the core and $p \in \langle N, e_R^M \rangle$ be the finite projection given by 4.2.2, such that

$$\|upu^* - p\|_{2,\mathrm{Tr}} < \varepsilon \|p\|_{2,\mathrm{Tr}}, \quad u \in \mathcal{U}_0,$$

and such that $p = \sum p_j$ for some projections p_j with $p_j \sim e_R^M z$ for some projection $z \in \mathcal{Z}(S)$ and all j.

In particular p_i are all cyclic projections so there exist $\xi_1, \ldots, \xi_n \in L^2(N, \tau)$ such that $p_i = [\xi_i R]$ (the orthogonal projection onto $\overline{\xi_i R}$). By replacing if necessary ξ_i by $\xi_i(\xi_i^* \xi_i)^{-1/2}$ we may assume $E_R^M(\xi_i^* \xi_i) = z$ for all i. By the mutual orthogonality of p_i we also have $E_R^M(\xi_i^* \xi_j) = 0$ if $i \neq j$. Moreover $p_i = [\xi_i R] = \xi_i e_R^M \xi_i^*$ so that, $p = \sum \xi_i e_R^M \xi_i^*$ and $E_R^M(\xi_i^* \xi_j) = \delta_{ij} z$, equivalently $e_R^M \xi_i^* \xi_j e_R^M = \delta_{ij} z e_R^M$.

Let $\{N_k\}_{k \geqslant 1}$ be the choice of the tunnel such that $R = \overline{\bigcup_k (N'_k \cap M)}$, $S = \overline{\bigcup_k (N'_k \cap N)} = R \cap N$. Since $N'_k \cap M \subset R$, $E_{N'_k \cap M}(\xi_i^* \xi_j) = 0$, for all $k, i \neq j$. Moreover by the definition of Tr we have for all $v \in \mathcal{U}_0$:

$$\begin{aligned}
&\left\| v\left(\sum \xi_i e_R^M \xi_i^*\right) v^* - \sum \xi_i e_R^M \xi_i^* \right\|_{2,\mathrm{Tr}}^2 \\
&\qquad = 2\sum_i \mathrm{Tr}(\xi_i e_R^M \xi_i^*) - 2\sum_{i,j} \mathrm{Tr}(\xi_i e_R^M \xi_i^* v \xi_j e_R^M \xi_j^* v^*) \\
&\qquad = 2\sum_i \mathrm{Tr}(e_R^M \xi_i^* \xi_i e_R^M) - 2\sum_{i,j} \mathrm{Tr}(e_R^M (\xi_j^* v^* \xi_i) e_R^M (\xi_i^* v \xi_j) e_R^M) \\
&\qquad = 2\sum_i \mathrm{Tr}(E_R^M(\xi_i^* \xi_i) e_R^M) - \sum_{i,j} \mathrm{Tr}(E_R^M(\xi_j^* v^* \xi_i) E_R^M(\xi_i^* v \xi_j) e_R^M) \\
&\qquad = 2n\tau(z) - 2\sum_{i,j} \|E_R^M(\xi_i^* v \xi_j)\|_2^2.
\end{aligned}$$

Thus we get

$$2n\tau(z) - 2\sum_{i,j} \|E_R^M(\xi_i^* v \xi_j)\|_2^2 < \varepsilon n \tau(z).$$

Since $N'_m \cap M \uparrow R$ and $\|E^M_{N'_m \cap R}(x) - E^M_R(x)\|_2 \to 0$, $x \in M$, for m large enough we obtain a projection $f' \in \mathcal{Z}(N'_m \cap N)$ such that $\|z - f'\|_2 < \delta \|f'\|_2$ and such that we still have for all $v \in \mathcal{U}_0$:

$$2n\tau(f') - 2\sum_{i,j} \|E^M_{N'_m \cap M}(\xi_i^* v \xi_j)\|_2^2 < \varepsilon n \tau(f').$$

Now each $\xi_i \in L^2(N,\tau)$ can be approximated arbitrarily well by some $x_i^1 = x_i^1 f' \in N$, $\|\xi_i - x_i^1\|_2 < \delta_1^1$, $1 \leqslant i \leqslant n$, with δ_1^1 chosen small independently of all the choices before, e.g. $N'_m \cap M$, etc. Taking $x_1 = x_1^1 f' E_{N'_m \cap M}(x_1^{1*} x_1^1)^{-1/2} f' \in N$ and $f_1 = E_{N'_m \cap M}(x_1^* x_1) \in N'_m \cap N$, we will still have $\|\xi_1 - x_1\|_2 < \delta_1$, $\|f' - f_1\|_2 < \delta_1 \|f'\|$ with δ_1 depending on δ_1^1 and $\delta_1 \to 0$ as $\delta_1^1 \to 0$. By the Gram–Schmidt orthogonalization method one can then find $x_2^{11} \in N$ such that $E_{N'_m \cap M}(x_2^{11*} x_1) = 0$ and $\|\xi_2 - x_2^{11}\| < \delta_2^1$, with δ_2^1 depending on δ_1, δ_1^1, but so that $\delta_2^1 \to 0$ as $\delta_1^1 \to 0$. Again, since $N'_m \cap M$ is finite dimensional, by taking $x_2 = x_2^{11} f' E_{N'_m \cap M}(x_2^{11\prime} x_2^{11})^{-1/2} f' \in N$ we still have $\|\xi_1 - x_2\|_2 < \delta_2$, with $\delta_2 \to 0$ as $\delta^1 \to 0$, but also $E_{N'_m \cap M}(x_2^* x_1) = 0$, $E_{N'_m \cap M}(x_2^* x_2) = f_2 \in N'_m \cap N$. Recursively, we obtain this way some elements $x_1, \ldots, x_n \in N$ such that

$$E_{N'_m \cap M}(x_i^* x_j) = \delta_{ij} f_j,$$

with $f_n \leqslant f_{n-1} \leqslant \ldots \leqslant f_1 \leqslant f'$ in $N'_m \cap N$, $\|f_n - f'\|_2 < \delta_n \|f'\|$, $\|\xi_i - x_i\|_2 < \delta_i$, $1 \leqslant i \leqslant n$, with $\delta_i \to 0$ as $\delta_1^1 \to 0$.

But since $N'_m \cap M$ is finite dimensional, the $\|\cdot\|_2$ topology on $N'_m \cap M$ coincides with the weak operator topology and since $x_i \to \xi_i$ in the norm $\|\cdot\|_2$ implies $E_{N'_m \cap M}(x_i^* v x_j) \to E_{N'_m \cap M}(\xi_i^* v \xi_j)$ in the weak operator topology (by the Cauchy–Schwartz inequality), it follows that $\|E_{N'_m \cap M}(x_i^* v x_j)\|_2 \to \|E_{N'_m \cap M}(\xi_i^* v \xi_j)\|_2^2$ for all $v \in \mathcal{U}_0$, so that for δ_1^1 small enough we will still have, for $f = f_n$ and x_i replaced by $x_i f = x_i f_n$, the estimates:

$$2n\tau(f) - 2\sum_{i,j} \|E_{N'_m \cap M}(x_i^* v x_j)\|_2^2 < \varepsilon n \tau(f),$$
$$E_{N'_m \cap M}(x_i^* x_j) = \delta_{ij} f,$$
$$\|z - f\|_2 < \varepsilon \|z\|_2.$$

In the ergodic core case by the previous theorem it follows that we can choose all the $\xi_i \in L^2(N,\tau)$ such that $z = E^M_R(\xi_i^* \xi_i) = 1$, i.e.,

$$E_R(\xi_i^* \xi_j) = \delta_{ij}, \quad 1 \leqslant i,j \leqslant n,$$

and then the first part of the proof gives the m and the $x_i \in N$ such that

$$E_{N'_m \cap M}(x_i^* x_j) = \delta_{ij},$$
$$2n - 2\sum_{i,j} \|E_{N'_m \cap M}(x_i^* v x_j)\|_2^2 < \varepsilon n. \qquad \square$$

4.3. Local approximation: proof of (iv) ⇔ (v) ⇔ (vi) of Theorem 4.1.1

We will use now the Rohlin type theorem in A.1 to translate the Følner conditions in §4.2 into local approximation by higher relative commutants. Here the term "local" means under a projection (typically of very small trace) which is almost invariant to the given finite set of elements. This way we will prove the equivalence of the conditions (iv), (v), (vi) of Theorem 4.1.1. For convenience we restate them here.

4.3.1. THEOREM. *Let $N \subset M$ be an inclusion with $N \neq M$. Assume $N \subset M$ satisfies the Følner condition relative to one of its cores. If $Y \subset M$ is a finite set of elements and $\varepsilon > 0$, then there exist a continuation of the tunnel up to some m, $M \supset N \supset N_1 \supset \ldots \supset N_m$, and a projection $s \in N'_m \cap N$ such that*

$$\|[y, s]\|_2 < \varepsilon \|s\|_2, \quad y \in Y,$$
$$\|E_{s(N'_m \cap M)s}(sys) - sys\|_2 < \varepsilon \|s\|_2, \quad y \in Y.$$

Also, there exist an m, a continuation of the tunnel $M \supset N \supset \ldots \supset N_m \supset \ldots$ and projections $f_0 \in N'_m \cap N$, $s_0 \in N_m \cap R$, $z_0 \in \mathcal{Z}(S)$, where $R = \overline{\bigcup (N'_k \cap M)}$, $S = R \cap N$, such that if $s = s_0 f_0$ then we have

$$\|[y, s]\|_2 < \varepsilon \|s\|_2,$$
$$\|E_{sN'_m \cap Ms}(sys) - sys\|_2 < \varepsilon \|s\|, \quad y \in Y,$$
$$\|f_0 - z_0\|_2 < \varepsilon \|f_0\|_2.$$

If in addition $N \subset M$ has ergodic core then we may take $f_0 = z_0 = 1$ (so that $s = s_0 \in N_m \cap R$).

Conversely, if either of these local approximation properties holds true then $N \subset M$ satisfies the Følner condition relative to any of its cores.

Proof. By writing each y as a linear combination of unitary elements we see that we may assume $y \in Y$ are all unitary elements. By 4.2.3 there exist some $m_0 \geqslant 1$, a choice of the tunnel $M \supset N \supset N_1^0 \supset \ldots \supset N_{m_0}^0$, elements $x_1, \ldots, x_n \in N$ and projections $f \in \mathcal{Z}((N_{m_0}^0)' \cap N)$, $z \in \mathcal{Z}(R^0 \cap N)$, where $R^0 = \overline{\bigcup ((N_k^0)' \cap M)}$, such that

$$\|f - z\|_2 < \varepsilon \|f\|_2,$$
$$E_{(N_{m_0}^0)' \cap M}(x_i^* x_j) = \delta_{ij} f, \quad 1 \leqslant i, j \leqslant n,$$
$$2n\tau(f) - 2 \sum_{i,j} \|E_{(N_{m_0}^0)' \cap M}(x_i^* y x_j)\|_2^2 < \tfrac{1}{2} \varepsilon n \tau(f).$$

By A.1, given any δ (independently, at this point, on all elements chosen before, in particular on $\|x_i\|$) there exists a projection $q\in N^0_{m_0}$ such that

$$\|qx_i^*yx_jq - E_{(N^0_{m_0})'\cap M}(x_i^*yx_j)q\|_2 < \delta\|q\|_2, \quad y\in Y,$$
$$\|qx_i^*x_jq-\delta_{ij}fq\|_2<\delta\|q\|_2.$$

By A.2 there exist some partial isometries $v_i\in N$ (the ambient algebra of the x_i's) such that

$$v_i^*v_j=\delta_{ij}fq,$$
$$\|v_i^*yv_j - E_{(N^0_{m_0})'\cap M}(x_i^*yx_j)q\|_2 < f(\delta)\|q\|_2,$$
$$\|x_iq-v_i\|_2<f(\delta)\|q\|_2,$$

for all $y\in Y$, where $f(\delta)\to 0$ when $\delta\to 0$. Let m be large enough such that $(N^0_m)'\cap N^0_{m_0}$ contains an $n\times n$ matrix algebra P^0 with matrix unit $\{e^0_{ij}\}_{1\leqslant i,j\leqslant n}$ and $\|1_{P_0}-1_M\|_2 < \delta\|q\|_2$. Since $N^0_m\cap R^0$ is of type II_1, there exists a projection $s^0\in N^0_m\cap R^0$ such that $\tau(s^0)\tau(e^0_{ii})=\tau(q)$. Denote $s^0e^0_{11}=q^0$ and $s^0e^0_{j1}=v^0_j$, $1\leqslant j\leqslant n$. Since $N^0_{m_0}$ is a factor there exists a partial isometry $w\in N^0_{m_0}$ such that $ww^*=q$, $w^*w=q^0$. Let $v=\sum_i v_iwv_i^{0*}$ and let $V\in N$ be a unitary element that extends the partial isometry v, i.e., $v=Vv^*v$. Denote by $N_k=VN^0_kV^*$ and $R=\overline{\bigcup(N_k'\cap M)}$. Denote $f_0=VfV^*$, $z_0=VzV^*$, $s_0=Vs^0V^*$. Note that

$$f_0\in\mathcal{Z}(N'_{m_0}\cap N),$$
$$v_i=Vv_i^0V^*\in Vs^0(R^0\cap N^0_{m_0})s^0V^*=s_0R\cap N_{m_0}s_0,$$
$$s_0=Vs^0V^*\in R\cap N_m,$$
$$q=Vq^0V^*\in R\cap N_{m_0} \quad\text{and}\quad z_0\in\mathcal{Z}(R\cap N).$$

Moreover, if we let $s=s_0f_0$, we have:

$$\begin{aligned}
\|[s,y]\|_2^2 &= \|ysy^*-s\|_2^2 = 2\tau(s)-2\tau(ysy^*s)\\
&\leqslant 2n\tau(f)\tau(q)-2\sum_{i,j}\tau(yv_iv_i^*y^*v_jv_j^*)\\
&= 2n\tau(f)\tau(q)-2\sum_{i,j}\|v_j^*yv_i\|_2^2\\
&\leqslant 2n\tau(f)\tau(q)-2\sum_{i,j}\|E_{(N^0_m)'\cap M}(x_i^*yx_j)\|_2^2+2n^2f(\delta)^2\|q\|_2^2\\
&< \tfrac{1}{2}\varepsilon(n\tau(f))\tau(q)+2n^2f(\delta)^2(n\tau(f))^{-1}(n\tau(f))\tau(q)\\
&< \varepsilon\tau(s)=\varepsilon\|s\|_2^2
\end{aligned}$$

if δ is chosen small enough to have $2n^2f(\delta)^2(n\tau(f))^{-1}<\frac{1}{2}\varepsilon$. Also we get:

$$\begin{aligned}
&\|E_{s_0f_0N'_m\cap Mf_0s_0}(sys)-sys\|_2^2=\|E_{Vs^0f(N_m^0)'\cap Mfs^0V^*}(sys)-sys\|_2^2\\
&\quad=\|E_{s^0f(N_m^0)'\cap Mfs^0}(v^*yv)-v^*yv\|_2^2\\
&\quad=\sum_{i,j}\|E_{s^0f(N_m^0)'\cap Mfs^0}(v_i^0w^*v_i^*yv_jwv_j^0)-v_i^{0*}w^*v_i^*yv_jwv_j^0\|_2^2\\
&\quad\leqslant\sum_{i,j}\|E_{s^0f(N_m^0)'\cap Mfs^0}(w^*v_i^*yv_jw)-w^*v_i^*yv_jw\|_2^2+2n^2\|1_{P_0}-1_M\|_2^2\\
&\quad\leqslant\sum_{i,j}\|E_{s^0f(N_m^0)'\cap Mfs^0}(w^*E_{(N_{m_0}^0)'\cap M}(x_i^*yx_j)qw)-w^*v_i^*yv_jw\|_2^2+n^2f(\delta)\|q\|_2^2\\
&\quad=\sum_{i,j}\|E_{s^0f(N_m^0)'\cap Mfs^0}(E_{(N_{m_0}^0)'\cap M}(x_i^*yx_j))q^0-w^*v_i^*yv_jw\|_2^2+nf(\delta)\|s\|_2^2\\
&\quad=\sum_{i,j}\|E_{(N_{m_0}^0)'\cap M}(x_i^*yx_j)q-v_i^*yv_j\|_2^2+nf(\delta)\|s\|_2^2\leqslant 2nf(\delta)\|s\|_2^2.
\end{aligned}$$

Finally, we get:

$$\|z_0-f_0\|_2=\|VzV^*-VfV^*\|_2=\|z-f\|_2<\varepsilon\|f\|_2=\varepsilon\|VfV^*\|_2=\varepsilon\|f_0\|.$$

This proves the last part of the statement. The first part and the case when R is a factor follow now trivially.

Conversely, let $S\subset R$ be a core for $N\subset M$. Let $u_1,\dots,u_n\in M$ be some unitary elements and $\varepsilon>0$. By local approximation there exist a choice of the tunnel $M\supset N\supset\dots\supset N_m$ and a projection $s\in N'_m\cap N$ such that

$$\|[u_i,s]\|_2<\varepsilon\|s\|,$$
$$\|E_{sN'_m\cap Ms}(su_is)-su_is\|_2<\varepsilon\|s\|_2.$$

Let $v\in N$ be a unitary element such that $vRv^*\supset N'_m\cap M$, $vSv^*\supset N'_m\cap N$. Define

$$p\in\langle M,R\rangle=\langle M,e_R\rangle$$

by $p=sve_Rv^*=ve_Rv^*s$. Then we have

$$\begin{aligned}
\|u_ipu_i^*-p\|_{2,\mathrm{Tr}}^2&=2\tau(s)-2\|E_{svRv^*s}(su_is)\|_2^2\\
&\leqslant 3\varepsilon\tau(s)+2\|s-u_isu_i^*\|_2^2\leqslant 5\varepsilon\tau(s)=5\varepsilon\|p\|_{2,\mathrm{Tr}}.
\end{aligned}$$ □

4.4. Global approximation: proof of (ii) $\Rightarrow$ (iv) in Theorem 4.1.2

We will now use a maximality argument to obtain from the local approximation in the previous §4.3, the actual (global) approximation by higher relative commutants. Under the assumption that $N \subset M$ is amenable relative to a core and has ergodic core, we will prove first that:

(*) Given any $\varepsilon > 0$ and any finite set of elements Y in M and any choice of the tunnel up to some k_0, $M \supset N \supset N_1 \supset \ldots \supset N_{k_0}$, there exist $m > k_0$ and a continuation of the tunnel up to m, $N_{k_0-1} \supset N_{k_0} \supset \ldots \supset N_m$ such that $\|E_{N'_m \cap M}(y) - y\|_2 < \varepsilon$, $y \in Y$.

Let $\mathcal{T}$ be the set of all the finite continuations of the tunnel

$$T = (M \supset N \supset N_1 \supset \ldots \supset N_{k_0} \supset \ldots \supset N_p).$$

For such a tunnel T denote $l(T) = p$ and $N(T) = N_p$. Let $\mathcal{S} = \{(T, s) \mid T \in \mathcal{T}, s \in \mathcal{P}(N(T)), s \neq 0\}$. Let $\mathcal{J}$ be the set of all the families $(S_i)_{i \in I}$ of couples $S_i = (T_i, s_i)$ in $\mathcal{S}$, with $(s_i)_{i \in I}$ mutually orthogonal and such that if $s = \sum s_i$ then

$$\left\| \sum_{i \in I} E_{s_i (N(T_i)' \cap M) s_i}(s_i y s_i) - (y - (1-s) y (1-s)) \right\|_2^2 < \varepsilon \|s\|_2^2, \quad \forall y \in Y.$$

The set $\mathcal{J}$ with the order given by inclusion is clearly inductively ordered. Let $(S_j^0)_{j \in J_0}$ be a maximal element of $\mathcal{J}$ and let $S_j^0 = (T_j^0, s_j^0)$. Assume $s^0 = \sum s_j^0 \neq 1$. Let $f = 1 - s^0 \in N_{k_0}$. By §3.2, $f N_{k_0} f \subset f M f$ is also amenable relative to a core and has ergodic core. By applying §4.3 to $f M f \supset f N_{k_0} f$ it follows that there exist a continuation of the tunnel up to some $m_0 > k_0$, $M \supset N \supset N_1 \supset \ldots \supset N_{k_0} \supset \ldots \supset N_{m_0}$, with $f \in N_{m_0}$, and a projection $s_0 \in N_{m_0}$, $0 \neq s_0 \leqslant f$, such that

$$\|E_{s_0 (N'_{m_0} \cap M) s_0}(s_0 y s_0) - (f y f - (f - s_0) y (f - s_0))\|_2^2 < \varepsilon \|s_0\|_2^2, \quad y \in Y.$$

Denote $T_0 = (M \supset N \supset \ldots \supset M_{m_0})$, $S_0 = (T_0, s_0)$ and $J = J_0 \cup \{0\}$. Then $(S_j)_{j \in J}$, with $s_j = s_j^0$ for $j \in J_0$, is an element of $\mathcal{J}$. Indeed, for $s = s^0 + s_0$ we have:

$$\begin{aligned}
&\left\| \sum_{i \in J} E_{s_i (N(T_i)' \cap M) s_i}(s_i y s_i) - (y - (1-s) y (1-s)) \right\|_2^2 \\
&\quad = \left\| \sum_{i \in J_0} E_{s_i^0 (N(T_i^0)' \cap M) s_i^0}(s_i^0 y s_i^0) - (y - (1-s^0) y (1-s^0)) \right\|_2^2 \\
&\qquad + \|E_{s_0 (N(T_0)' \cap M) s_0}(s_0 y s_0) - (f y f - (f - s_0) y (f - s_0))\|_2^2 \\
&\quad < \varepsilon \|s^0\|_2^2 + \varepsilon \|s_0\|_2^2 = \varepsilon \|s\|_2^2.
\end{aligned}$$

This contradicts the maximality of $(S_j^0)_{j\in J_0}$ thus showing that $\sum s_j^0=1$. Since $\tau(s_j^0)\neq 0$ for all j (because $s_j^0\neq 0$) it follows that J_0 is countable and that we can find a large finite subset $s_1^0, ..., s_k^0$ among the s_j^0, $j\in J_0$, such that $\tau(\sum_{i=1}^k s_i^0)\geqslant 1-\delta$. Let $m=\max_{1\leqslant j\leqslant k} l(T_j^0)$. We claim that there exists a continuation of the tunnel up to m, $M\supset N\supset N_1\supset ...\supset N_{k_0}\supset ...\supset N_m$, such that $s_i^0\in N_m$, $1\leqslant i\leqslant k$, and such that $s_i^0 N_j s_i^0 = s_i^0 N_j(T_i^0)s_i^0$, $j\leqslant l(T_i^0)$, $1\leqslant i\leqslant k$.

To see this let $M\supset N\supset ...\supset N_{k_0}\supset N_{k_0+1}^1\supset ...\supset N_m^1$ be any continuation of the tunnel up to m and let $s_i^1\in N_m^1$ be mutually orthogonal projections such that $\tau(s_i^1)=\tau(s_i^0)$. Let U be a unitary element in N_{k_0} such that $Us_i^1U^*=s_i^0$ and such that $Us_i^1N_k^1s_i^1U^*=s_i^0N_k(T_i^0)s_i^0$ (this is possible because by [PiPo1] any two choices of the tunnel $s_i^0Ms_i^0\supset ...\supset s_i^0N_ks_i^0$ are conjugate). Then $N_k=UN_k^1U^*$ satisfies the requirements.

Since s_i^0, $1\leqslant i\leqslant k$, are mutually orthogonal projections and since the core is of type II_1, there exists a continuation of the tunnel $N_m\supset N_{m+1}\supset ...\supset N_k$ such that $N_k'\cap N_m$ δ^1-contains all the s_i^0. It follows that the algebra $B_0=\sum s_i^0(N_m'\cap M)s_i^0+\mathbf{C}(1-\sum s_i^0)$ is almost contained in $N_k'\cap M$ so that if δ^1 is small enough we get

$$\begin{aligned}\|E_{N_k'\cap M}(y)-y\|_2 &\leqslant \|E_{B_0}(y)-y\|_2+\varepsilon\\ &\leqslant \left\|\sum_{i=1}^n E_{s_i(N_m'\cap M)s_i^0}(s_i^0ys_i^0)-\left(y-\left(1-\sum s_i^0\right)y\left(1-\sum s_i^0\right)\right)\right\|_2\\ &\quad+\left\|1-\sum s_i^0\right\|_2\|y\|+\varepsilon\leqslant 3\varepsilon.\end{aligned}$$

This ends the proof of (*).

Let then $\{x_n\}_n$ be a $\|\cdot\|_2$-dense sequence in M. We construct recursively some integers $k_1<k_2<...$ and some choices of the tunnel $M\supset N\supset ...\supset N_{k_1}\supset ...\supset N_{k_2}\supset ...$ such that

$$\|E_{N_{k_i}'\cap M}(x_j)-x_j\|_2<2^{-i},\quad 1\leqslant j\leqslant i.$$

Suppose we made the choice up to some i. By (*) it follows that there exists $m>k_i$ and a continuation of the tunnel $M\supset N\supset ...\supset N_{k_i}\supset ...\supset N_m$ such that

$$\|E_{N_m'\cap M}(x_j)-x_j\|_2<2^{-i-1},\quad 1\leqslant j\leqslant i+1.$$

Taking $k_{i+1}=m$ we are done.

But then $R=\overline{\bigcup(N_k'\cap M)}$ satisfies

$$E_R(x_j)=\lim_k E_{N_k'\cap M}(x_j)=x_j$$

so that $M=R$ by density. □

Let us finally mention some global properties of the amenable inclusions that follow from 4.1.1 and from a similar maximality argument like the one used above. Note that part (2) of the next theorem contains Connes' fundamental theorem. Note also that in case $N\subset M$ comes from an action of a group G like in [Po6] then condition (3) states that the Cayley matrix of the group G has maximal spectral radius, i.e., Kesten's characterization of amenability. This condition will be investigated in more detail in Section 5 where it will be shown that conversely, if $\Gamma_{N,M}$ is ergodic, $\|\Gamma_{N,M}\|^2=[M:N]$ and M is amenable then $N\subset M$ is strongly amenable. Also, in a forthcoming paper we will show that $\|\Gamma_{N,M}\|^2=[M:N]$ and M amenable implies $N\subset M$ amenable.

4.4.1. THEOREM. *If $N\subset M$ is an amenable inclusion then:*

(1) *If $N\neq M$, $N\subset M$ can be globally approximated by finite dimensional commuting squares which come locally from higher relative commutants, i.e., $\forall\varepsilon>0$, $\forall x_1,\dots x_n\in M$, $\exists Q\subset P$ finite dimensinal subalgebras satisfying the commuting square condition, $E_PE_N=E_Q$, such that $\|E_P(x_j)-x_j\|_2<\varepsilon$, $1\leqslant j\leqslant n$, and there exist a finite set I_0 and for each $i\in I_0$ a continuation of the tunnel up to some k_i, $M\supset N\supset N_1^i\supset\dots\supset N_{k_i}^i$, and some projections $s_i\in(N_{k_i}^i)'\cap N$ such that $\sum s_i=1$ and*

$$\sum s_i((N_{k_i}^i)'\cap N)s_i=Q,\quad \sum s_i((N_{k_i}^i)'\cap M)s_i=P.$$

(2) *If N, M are separable, then both N and M are isomorphic to the hyperfinite type* II_1 *factor and $(N\subset M)\simeq(N\otimes P\subset M\otimes P)$ where P is a copy of the hyperfinite type* II_1 *factor.*

(3) *If $N\subset M$ is extremal then $\|\Gamma_{N,M}\|^2=[M:N]$.*

Proof. (1) The proof is identical to the one for (ii) $\Rightarrow$ (iv) of Theorem 4.1.2 completed at the beginning of this subsection, by using 2.1.7 for the hereditarity of amenability to reduced algebras.

(2) If $N\subset M$ is amenable then, by §3.2, N and M are amenable, so that $N\subset M_{2\times2}(N)$ and $M\subset M_{2\times2}(M)$ are obviously amenable and have ergodic cores. By (i) $\Rightarrow$ (iv) of 4.1.2, N, M follow isomorphic to the same hyperfinite type II_1 factor. Moreover, if $x_1,\dots,x_n\in M$ and $\varepsilon>0$ and if $N_{k_i}^i, s_i$ are as in part (1) above, then let $u_i,v_i\in N_{k_i}^i$ be unitary elements such that $u_iv_i=-v_iu_i$ and define $u=\sum u_is_i$, $v=\sum v_is_i$. It follows that u,v are unitary elements in N, $uv=-vu$ and

$$\|[u,x_i]\|_2<2\varepsilon+\|[u,E_P(x_i)]\|_2=2\varepsilon.$$

By [Bi2], [Po8], it follows that $(N\subset M)\simeq(N\otimes R\subset M\otimes R)$.

(3) By [PiPo2] we see that (1) implies $\|\Gamma_{N,M}\|^2=[M:N]$. □

4.5. Bicommutant condition and existence of hypertraces: end of the proof of 4.1.2

We now prove the remaining implications in Theorem 4.1.2.

4.5.1. *Proof of* (iv) $\Rightarrow$ (iii) $\Rightarrow$ (vi) *of* 4.1.2. The first implication is trivial. To prove the second, let $M \supset^{e_0^0} N \supset^{e_{-1}^0} N_1^0 \supset \ldots \supset N_k^0$ be a choice of the tunnel up to some k let $\varepsilon > 0$ and let $\{u_j^i\}_{1 \leqslant j \leqslant n} \subset N_i$ be unitary elements such that

$$\left\| \frac{1}{n} \sum_j u_j^i e_{-i}^0 u_j^{i*} - E_{(N_i^0)' \cap M}(e_{-i}^0) \right\|_2 < \varepsilon.$$

By (iii) there exist m and a choice of the tunnel up to m, $M \supset^{e_0} N \supset N_1 \supset \ldots \supset N_m$, such that $u_j^i, e_{-i}^0 \in_\varepsilon N_m' \cap M$, $0 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$. Fix some k_0. Since $\{e_{-i}^0\}_{i \geqslant 0}$, $\{e_{-i}\}_{i \geqslant 0}$ generate factors (by Jones' theorem [J2]), there exists k large enough such that the central trace of $e_0^0, \ldots, e_{-k_0}^0$ in $\mathrm{Alg}\{1, e_0^0, \ldots e_{-k}^0\}$, respectively of $e_0, \ldots, e_{-k_0}$ in $\mathrm{Alg}\{1, e_0, \ldots, e_{-k}\} \subset N_m' \cap M$, is as close to $\lambda 1$ as we please. It follows that there exists a unitary $w_0 \in N_m' \cap M$ such that $\|w_0 e_0 w_0^* - e_0^0\|_2 \leqslant f(\varepsilon)$. Taking $v_0' = \lambda^{-1} E_N(w_0 e_0) = \lambda^{-1} E_{N_m' \cap N}(w_0 e_0)$ it follows by [PiPo1] that $v_0' e_0 v_0'^* = w_0 e_0 w_0^*$ so that $\lambda v_0' v_0'^* = E_N(v_0' e_0 v_0'^*)$ is close to $E_N(e_0^0) = \lambda 1$. Thus $v_0' \in N_m' \cap N$ is close to a unitary. By perturbing v_0' if necessary we get a unitary $v_0 \in N_m' \cap N$ so that

$$\|v_0^* e_0^0 v_0 - e_0\|_2 \leqslant f_0(\varepsilon)$$

where $f_0(\varepsilon) \to 0$ when $\varepsilon \to 0$. But then $\{v_0^* e_0^0 v_0\}' \cap N = v_0^* N_1^0 v_0$ is close to $\{e_0\}' \cap N = N_1$ (in the distance defined in [Ch]) so that $v_0^* e_{-1}^0 v_1 \in N$ expects close to $\lambda 1$ on N_1. Like before, it follows that there exists a unitary $v_1 \in N_m' \cap N_1$ such that

$$\|v_1^* v_0^* e_{-1}^0 v_0 v_1^* - e_{-1}\|_2 \leqslant f_1(\varepsilon)$$

with $f_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. Recursively, we get unitary elements $v_i \in N_m' \cap N_i$ such that

$$\|v_i^* v_{i-1}^* \ldots v_0^* e_{-i}^0 v_0 v_1 \ldots v_i - e_{-i}\|_2 < f_i(\varepsilon),$$
$$\|v_i^* v_{i-1}^* \ldots v_0^* u_j^i v_0 \ldots v_{i-1} v_i - E_{N_m' \cap N_i}(v_i^* \ldots v_0^* u_j^i v_0 \ldots v_i)\|_2 < f_i(\varepsilon)$$

for $0 \leqslant i \leqslant k_0$, $1 \leqslant j \leqslant n$, where $f_i(\varepsilon) \to 0$ as $\varepsilon \to 0$.

It follows that

$$\|E_{(N_m' \cap N_i)' \cap N_{i-1}}(e_{-i}) - E_{N_i' \cap N_{i-1}}(e_{-i})\|_2 < f(\varepsilon)$$

for $0 \leqslant i \leqslant k_0$, where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Thus, letting $\varepsilon\to 0$, we get that

$$E_{(R\cap N_i)'\cap(R\cap N_{i-1})}(e_{-i})=E_{N_i'\cap N_{i-1}}(e_{-i}).$$

Now, if $N\subset M$ is extremal it follows that $E_{(R\cap N_i)'\cap(R\cap N_{i-1})}(e_{-i})=\lambda 1$, so that by the antiisomorphism in [Po5], $E_{(M_i'\cap M_\infty)'\cap M_\infty}(e_{i+1})=\lambda 1$ for all $i\geqslant 0$. It follows that for $x\in M_{i+1}=\operatorname{sp} M_i e_{i+1} M_i$ we have $E_{M_i}(x)=E_{(M_i'\cap M_\infty)'\cap M_\infty}(x)$. Applying this n times it follows that for $x\in M_{i+n}$ we have

$$\begin{aligned}E_{M_i}(x)&=E_{(M_i'\cap M_\infty)'\cap M_\infty}E_{(M_{i+1}'\cap M_\infty)'\cap M_\infty}\cdots E_{(M_{i+n-1}'\cap M_\infty)'\cap M_\infty}(x)\\&=E_{(M_i'\cap M_\infty)'\cap M_\infty}(x).\end{aligned}$$

Thus, we get

$$E_M(x)=E_{(M'\cap M_\infty)'\cap M_\infty}(x),\quad x\in\bigcup M_i,$$

and by weak continuity for all $x\in M_\infty$, so that $M=(M'\cap M_\infty)'\cap M_\infty$.

For general $N\subset M$ (not necessarily extremal) one first takes the 2-step inclusion $N_1\subset M$ as $N\subset M$ then for this $N\subset M$ one replaces the projections e_{-i}, e^0_{-i} by the canonical projections $e'_{-i}\in N'_{i+1}\cap N_{i-1}$, $(e^0_{-i})'\in(N^0_{i+1})'\cap N^0_{i-1}$ defined in [PiPo1] which still implement conditional expectations (not necessarily trace preserving though) but satisfy $E_{N_i'\cap N_{i-1}}(e'_{-i})=\lambda 1$, $E_{(N_i^0)'\cap N^0_{i-1}}((e^0_{-i})')=\lambda 1$ (instead of $E_{N_i}(e_{-i})=\lambda 1$, $E_{N_i^0}(e^0_{-i})=\lambda$). Arguing like above we get that $E_{(N'_m\cap N_i)'\cap N_{i-1}}(e'_{-i})$ is close to $\lambda 1$. By [Po12], there exists a trace preserving antiisomorphism of $N'_m\cap N_{i-1}$ onto $M'_i\cap M_{m+1}$ carrying $N'_m\cap N_i$ onto $M'_{i+1}\cap M_{m+1}$ and e'_{-i} onto e_{i+2}. We thus obtain that

$$E_{(M_i'\cap M_\infty)'\cap M_\infty}(e_{i+1})=\lambda 1,\quad i\geqslant 0,$$

like before. The same argument then shows that $M=(M'\cap M_\infty)'\cap M_\infty$. □

The argument already used above twice is due to Skau (unpublished, see also [GHJ]). Since it will be of later use as well, we display it here, in a more general form:

LEMMA. *Let $N\subset M\subset^{e_1} M_1\subset^{e_2}\ldots$ be the Jones tower of factors and let $B\subset M'\cap M_\infty$ be a von Neumann subalgebra such that $B_k=B\cap(M_k'\cap M_\infty)$ satisfy the commuting square relations*

$$E_B E_{M_k'\cap M_\infty}=E_{B_k},$$

and so that

$$E_{B_k'\cap M_\infty}(e_{k+1})=\lambda 1,\quad k\geqslant 0.$$

Then $B'\cap M_\infty = M$.

Proof. If $x=\sum_i y_i^1 e_{j+1} y_i^2 \in M_{j+1}$, where $y_i^j \in M_j$, then

$$\begin{aligned} E_{B'\cap M_\infty}(x) &= E_{B'\cap M_\infty} E_{B'_j\cap M_\infty}(x) \\ &= E_{B'\cap M_\infty}\Big(\sum_i y_i^1 E_{B'_j\cap M_\infty}(e_{j+1}) y_i^2\Big) \\ &= E_{B'\cap M_\infty}\Big(\lambda \sum_i y_i^1 y_i^2\Big) \in E_{B'\cap M_\infty}(M_j). \end{aligned}$$

Thus $E_{B'\cap M_\infty}(M_{j+1}) \subset E_{B'\cap M_\infty}(M_j)$, $j\geqslant 0$, so that

$$E_{B'\cap M_\infty}\big(\bigcup_{j\geqslant 1} M_j\big) \subset E_{B'\cap M_\infty}(M) = M. \qquad \square$$

4.5.2. *Proof of* (v) $\Rightarrow$ (i) *of* 4.1.2. Let

$$\begin{array}{ccc} \mathcal{N} & \overset{\mathcal{E}}{\subset} & \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

be a smooth embedding of $N\subset M$. Let $\mathcal{N}\subset^{\mathcal{E}}\mathcal{M}\subset^{\mathcal{E}_1}\mathcal{M}_1 = \langle \mathcal{M}, e_1\rangle \subset \ldots$ be the associated tower. Let ψ be a faithful normal state on $\mathcal{N}$ extending the trace of N and still denote by ψ the state $\psi(T)=\psi(\mathcal{E}(\ldots\mathcal{E}_{n-2}(\mathcal{E}_{n-1}(\mathcal{E}_n(T)\ldots))))$, for $T\in\mathcal{M}_n$, and thus for $T\in\bigcup_n \mathcal{M}_n$. Let $(\pi_\psi, \mathcal{H}_\psi, \xi_\psi)$ be the GNS construction for ψ and $\bigcup_n \mathcal{M}_n$, and denote $\mathcal{M}_\infty = \overline{\pi_\psi(\bigcup_n \mathcal{M}_n)}$. Note that if p is the orthogonal projection of $\mathcal{H}_\psi = \overline{\pi_\psi(\bigcup_n \mathcal{M}_n)\xi_\psi}$ onto $\overline{\pi_\psi(\bigcup_n M_n)\xi_\psi}$, then $[p, \pi_\psi(\bigcup_n M_n)]=0$ and in fact $\overline{\pi_\psi(\bigcup_n M_n)}p \simeq M_\infty$. Since M is amenable M_{2k} are all amenable (as amplifications of M) so that M_∞ is also amenable. Thus, there exists a conditional expectation Φ_∞ of $\mathcal{M}_\infty$ onto $M_\infty \subset \mathcal{M}_\infty$. Since $M'\cap M_k \subset \mathcal{M}'\cap\mathcal{M}_k$, the embedding being smooth, it follows that $[\mathcal{M}, M'\cap M_\infty]=0$ so that $[\Phi_\infty(\mathcal{M}), M'\cap M_\infty]=0$. Thus $\Phi_\infty(\mathcal{M}) \subset (M'\cap M_\infty)'\cap M_\infty = M$. Also $[\Phi_\infty(\mathcal{N}), N'\cap M_\infty]=0$ so that we have $[\Phi_\infty(\mathcal{N}), e_1]=0$. Thus $\Phi_\infty(\mathcal{N})\subset N$. But then $\tau\circ\Phi_\infty|_{\mathcal{M}}$ is an $(N\subset M)$-hypertrace (see §3.2). $\square$

5. Classification by standard invariants

We can now restate more explicitly the classification result following from the approximation by higher relative commutants in the previous section. This will be applied to classify various classes of subfactors that check the amenability conditions. Also, we will give combinatorial characterizations of the amenality that are easy to check and, along the line, will introduce the concept of strongly amenable paragroup. Throughout this section, M will be a separable type II_1 factor, $N\subset M$ a subfactor of finite index.

5.1. The main result

5.1.1. THEOREM. *$N\subset M$ is strongly amenable if and only if $N\subset M$ is isomorphic to its standard model $N^{\mathrm{st}}\subset M^{\mathrm{st}}$. If in addition $N\subset M$ is extremal then it is strongly amenable if and only if it is antiisomorphic to its opposite model $M_1'\cap M_\infty\subset M'\cap M_\infty$. In particular strongly amenable subfactors $N\subset M$ are completely classified by their standard invariants (or paragroups) $\mathcal{G}_{N,M}$.*

Proof. If $N\subset M$ is strongly amenable then for some choice of the tunnel $\{N_k\}$, $M=\overline{\bigcup_k(N_k'\cap M)}$ so that $N=\overline{\bigcup_k(N_k'\cap N)}$ and thus $(N\subset M)\simeq(N^{\mathrm{st}}\subset M^{\mathrm{st}})$. Conversely, if $(N\subset M)\simeq(N^{\mathrm{st}}\subset M^{\mathrm{st}})$ then $N\subset M$ has the generating property, since the inclusion $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ can be recaptured from $\mathcal{G}_{N,M}$ (cf. Remark 1.4.4). The rest is trivial. □

For inclusions of type II_∞ factors $N^\infty\subset M^\infty$ of finite [PiPo] index we will adopt the simple minded point of view of defining the (strong) amenability by requiring the (strong) amenability of the type II_1 inclusion $(N\subset M)=(pN^\infty p\subset pM^\infty p)$ obtained by reducing with finite projections in N^∞. By Section 3 this does not depend on the projection p and, by 1.3.7, neither $\mathcal{G}_{N,M}$ does depend on p. So we can define the standard invariant of $N^\infty\subset M^\infty$ as $\mathcal{G}_{N^\infty,M^\infty}\overset{\mathrm{def}}{=}\mathcal{G}_{N,M}$ (for all this see also [Po9]). We thus get from the above theorem:

COROLLARY. *An inclusion of type II_∞ factors $N^\infty\subset M^\infty$ is strongly amenable if and only if it is isomorphic to $(N^{\mathrm{st}}\subset M^{\mathrm{st}})\otimes\mathcal{B}(\mathcal{H})$. In particular, such inclusions are completely classified by their standard invariant.*

5.1.2. *Finite depth and index $\leqslant 4$.* As we will now point out for low indices and finite depth subfactors the strong amenability is automatic. Stronger results in this direction will be proved in §5.3.

COROLLARY. *Let M be the hyperfinite type II_1 factor and $N\subset M$ a subfactor with finite index.*

(i) *If $[M:N]\leqslant 4$ then $N\subset M$ is extremal and strongly amenable.*

(ii) *If $N\subset M$ has finite depth, i.e., $\Gamma_{N,M}$ is finite, then $N\subset M$ is extremal and strongly amenable.*

Thus, in all these cases $(N\subset M)\simeq(N^{\mathrm{st}}\subset M^{\mathrm{st}})\overset{\mathrm{op}}{\simeq}(M_1'\cap M_\infty\subset M'\cap M_\infty)$ and $N\subset M$ is completely classified by its standard invariant (or paragroup) $\mathcal{G}_{N,M}$.

Proof. If $N\subset M$ has finite depth then 1.4.1 shows that $M_1'\cap M_\infty\subset M'\cap M_\infty$ is an extremal inclusion of factors. But then $M\subset M_1$ also has finite depth (since

$$E_M(\mathcal{Z}(N_k'\cap M_1))\subset\mathcal{Z}(N_k'\cap M)$$

and by the [PiPo1] inequality), so that $M_1'\cap M_\infty\subset M_2'\cap M_\infty$ is also extremal. By trace preserving isomorphism we then have $E_{(M_k'\cap M_\infty)'}(e_{k+1})=[M:N]^{-1}1$ for all k and Skau's lemma in 4.5.1 shows that the bicommutant condition holds, thus $N\subset M$ is strongly amenable.

If $[M:N]\leqslant 4$ and $P_k=\mathrm{vN}(e_k,e_{k+1},\ldots)$ with e_k the Jones projections in the tower, then by [J2] $P_k'\cap P_{k-1}=\mathbf{C}$ so that $E_{P_k'\cap P_{k-1}}(e_{k-1})=[M:N]^{-1}1$ and again we find that $(M'\cap M_\infty)'\cap M_\infty=M$ by the lemma in 4.5.1. So $N\subset M$ is strongly amenable in this case as well. □

We mention that the finite depth case of the above result has been previously proved in [Po5] (see also [Oc2]), by using a completely different approach. The proof in [Po5] however, while quite simple and very elementary, cannot be adapted to work beyond the finite depth case.

5.1.3. *Subfactors coming from representations of Braid groups.* An important class of subfactors that are hyperfinite and check the finite depth condition are the Jones subfactors and their generalisation, the so-called Wenzl subfactors ([We1], [We2]), coming from certain unitary representations π of the Braid group on infinitely many generators $g_0,g_1,\ldots$ that admit positive Markov traces. These are representations that factor through representations of Hecke algebras of type A,B,C,D (at roots of unity) and they have been shown by Wenzl to produce finite depth subfactors by using the same method used in [J2], i.e. by taking $N=\pi(\mathrm{Alg}\{g_1,\ldots\})\subset\pi(\mathrm{Alg}\{g_0,g_1,\ldots\})=M$. In the simplest case of type A and certain additional relations, they coincide with the Jones' subfactors [J2]. 5.1.1 shows that they can be recognized by merely observing their invariants.

COROLLARY. *The subfactors of* [We1], [We2] *are uniquely determined by their standard invariants* (*paragroups*).

5.1.4. *Subfactors coming from actions of compact groups.* Already in [J2] it has been pointed out that the Jones subfactor of index 4 and standard graph A_∞, which on one hand can be constructed like in 5.1.3 when the value of the parameter (roots of unity) tends to 1, can also be obtained as an inclusion of fixed point algebras of a product type action of SU(2). This type of construction of subfactors was further exploited in [GHJ] to produce more examples of subfactors of index 4. In an independent work, Wassermann generalized this to arbitrary minimal actions of compact groups as follows ([Wa2]): let P be a copy of the hyperfinite type II_1 factor and $\sigma: G\to\mathrm{Aut}\,P$ a faithful minimal action of G on P (i.e., so that the fixed point algebra P^G is irreducible in P, $(P^G)'\cap P=\mathbf{C}$). Let $\pi: G\to\mathrm{End}\,V$ be a unitary representation of π on the finite dimensional Hilbert space V. Then $N=P^G\subset(P\otimes\mathrm{End}\,V)^G=M$ is an inclusion of type II_1 factors of index $(\dim V)^2$ which is irreducible if and only if π is irreducible and whose standard graph $\Gamma_{N,M}$ equals the multiplicity matrix of π (as the Jones tower is just $P^G \subset (P \otimes \mathrm{End}\, V)^G \subset (P \otimes \mathrm{End}\, V \otimes \mathrm{End}\, \overline{V})^G \subset \ldots$, and the higher relative commutants $N' \cap M_k = (\mathrm{End}\, V \otimes \mathrm{End}\, \overline{V} \ldots)^G = (\pi \otimes \bar{\pi} \otimes \ldots)(G)' \cap \mathrm{End}(V \otimes \overline{V} \ldots))$.

These subfactors were checked to be strongly amenable in [PoWa], by using the invariance principle ([Wa2]) to show that they have the same higher relative commutants as their cores (standard parts). Note that in the case G is a semisimple compact Lie group the subfactors have infinite depth.

COROLLARY. *The subfactors* $P^G \subset (P \otimes \mathrm{End}\, V)^G$ *of* [Wa1] *are uniquely determined by their paragroups.*

We note that such subfactors were further investigated in [PoWa], where it is shown that if one takes appropriate (large) finite dimensional representations π of G then the isomorphism class of such a subfactor determines uniquely the class of the action σ. As the corresponding paragroups do not depend on σ, the uniqueness of the minimal actions of G on P is obtained ([PoWa]). More precisely, if one takes $\mathrm{End}\, V$ to contain the trivial representation of G and a finite set of irreducible representations that generate (via tensor product) all other finite dimensional irreducible representations of G, taken with distinct multiplicities, then two outer minimal actions σ_1, σ_2 of G on P are conjugate if and only if their associated subfactors are isomorphic. Also, the subfactors are strongly amenable and their standard invariants only depend on G not on σ ([PoWa]).

5.1.5. *Subfactors coming from actions of discrete groups.* Let P be a von Neumann factor and $\theta_0 = \mathrm{id}, \theta_1, \theta_2, \ldots$ some $m+1$ (not necessarily distinct) automorphisms of P, where m may be finite or infinite. Denote by $P^0 = \mathcal{B}(l^2(m+1))$, where $l^2(m+1)$ is the $(m+1)$-dimensional Hilbert space (so that $P^0 = M_{(m+1)\times(m+1)}(\mathbf{C})$ if m is finite) and by $\{e^0_{ij}\}_{i,j \geqslant 0}$ its canonical matrix unit. Let $M^\theta = P \otimes P^0$ and $N^\theta = \{\sum_{i \geqslant 0} \theta_i(x) e^0_{ii} \mid x \in P\}$. We call $N^\theta \subset M^\theta$ *the inclusion associated to* $(\theta_i)_{i \geqslant 0}$. It is trivial to note that $N^\theta \subset M^\theta$ has finite [PiPo1] index (1.1.7) if and only if m is finite, in which case $\mathrm{Ind}\, E_{\min} = 1/(m+1)$. More than providing examples of subfactors (of finite index), such inclusions have the important feature of translating problems on classifications of actions by automorphisms into problems on classification of subfactors. Indeed, the isomorphism class of $N^\theta \subset M^\theta$ "encodes" the outer conjugacy class of θ due to the following facts, reminiscent of Connes' 2×2 matrix trick:

Facts. (1) $\{e^0_{ii}\}_i$ are minimal projections in $(N^\theta)' \cap M^\theta$ and $e^0_{ii} \sim e^0_{jj}$ in $(N^\theta)' \cap M^\theta$ if and only if there exists a unitary element $v_{ij} \in P \simeq P \otimes 1$ such that $\theta_i = \mathrm{Ad}\, v_{ij} \theta_j$. In this case $v_{ij} e^0_{ij} \in (N^\theta)' \cap M^\theta$. This is trivial, by the definition of $N^\theta \subset M^\theta$.

(2) If $\theta' = (\theta'_0 = \mathrm{id}, \theta'_1, \ldots)$ is another $(m+1)$-tuples of automorphisms of P then θ' is conjugate to θ (i.e., there exists $\sigma \in \mathrm{Aut}\, P$ such that $\sigma \theta'_i \sigma^{-1} = \sigma_i$, $\forall i$) if and only if there

exists $\tilde{\sigma}:(N^{\theta}\subset M^{\theta})\simeq(N^{\theta'}\subset M^{\theta'})$ and $\tilde{\sigma}(e^0_{ij})=e^0_{ij}$, $\forall i,j$. Also, θ' is outer conjugate to θ (i.e., there exists $\sigma\in\operatorname{Aut}P$ such that $\sigma\theta'_i\sigma^{-1}=\theta_i$ mod $\operatorname{Int}P$, $\forall i$) if and only if there exists $\tilde{\sigma}:(N^{\theta'}\subset M^{\theta'})\simeq(N^{\theta}\subset M^{\theta})$ and $\tilde{\sigma}(e^0_{ii})=e^0_{ii}$, $\forall i$. Again, this is a simple consequence of the definition.

(3) If $(\sigma_0=\mathrm{id},\sigma_1,\ldots)$ is a set of $k+1$ distinct automorphisms, $k\leqslant\infty$, $(\sigma'_0=\mathrm{id},\sigma'_1,\ldots)$ $k+1$ other automorphisms and $\{n_i\}_{i\geqslant 0}$ a set of $k+1$ distinct multiplicities then let $\theta=(\theta_0,\theta_1,\ldots,\theta_m)$, $\theta'=(\theta'_0,\theta'_1,\ldots,\theta'_m)$ be the two $(m+1)$-tuples obtained by repeating each σ_i (resp. σ'_i) n_i times, where $m+1=\sum n_i$. Then $(N^{\theta'}\subset M^{\theta'})=(N^{\theta}\subset M^{\theta})$ if and only if there exists $\alpha\in\operatorname{Aut}P$ such that $\alpha\sigma'_i\alpha^{-1}=\sigma_i$ mod $\operatorname{Int}P$, $\forall i$. This is trivial by (1) and (2).

(4) Let m be finite and $\theta=(\theta_0,\theta_1,\ldots,\theta_m)\subset\operatorname{Aut}P$. For an automorphism $\sigma\in\operatorname{Aut}P$ we convene to still denote by σ the automorphisms $\sigma\otimes 1$ on $M^{\theta}=P\otimes P^0$, $M^{\theta}_1=P\otimes P^0\otimes P^1$, ..., where $P^i\simeq M_{(m+1)\times(m+1)}(\mathbf{C})$, $\forall i$, and with matrix unit $\{e^i_{lk}\}_{l,k\geqslant 0}\subset P^i$. Define the embeddings $M^{\theta}_j\subset M^{\theta}_{j+1}$ by

$$M^{\theta}_{2j}\ni x\mapsto\sum_{i=0}^{m}\theta_i^{-1}(x)e^{2j+1}_{ii}\in M^{\theta}_{2j+1},$$
$$M^{\theta}_{2j-1}\ni x\mapsto\sum_{i}\theta_i(x)e^{2j}_{ii}\in M^{\theta}_{2j},$$

and put $e_{j+1}=(1/(m+1))\sum_{i,i'}e^j_{ii'}e^{j+1}_{ii'}\in M^{\theta}_{j+1}$. Then

$$N^{\theta}\subset M^{\theta}\overset{e_1}{\hookrightarrow}M^{\theta}_1\overset{e_2}{\hookrightarrow}M^{\theta}_2\ldots$$

is the Jones tower of factors for $N^{\theta}\subset M^{\theta}$ with the expectation $E_{\min}$ (for which $E_{\min}(e^0_{ii})=1/(m+1)$, $\forall i$). In general, if the expectation E from M^{θ} onto N^{θ} is given by $E(e^0_{ii})=t_i$, then $E_{2j}(e^{2j}_{ii})=t_i$, $E_{2j+1}(e^{2j+1}_{ii})=t'_i=t_i^{-1}/\sum_k t_k^{-1}$ and the Jones projections are given by:

$$e_{2j+1}=\sum_{i,i'}\sqrt{t_it_{i'}}\,e^{2j}_{ii'}e^{2j+1}_{ii'},$$
$$e_{2j+2}=\sum_{i,i'}\sqrt{t'_it'_{i'}}\,e^{2j+1}_{ii'}e^{2j+2}_{ii'},$$

and $\operatorname{Ind}E=\sum_i t_i^{-1}=\sum_i t_i'^{-1}=(t_jt'_j)^{-1}$, $\forall j$. All this follows by using the abstract characterization of the basic construction and Jones projections in [PiPo2] and 2.2.2, and by direct computation. Note that if $p\in N^{\theta}$ then the tower for $pN^{\theta}p\subset pM^{\theta}p$ is obtained by reducing the tower for $N^{\theta}\subset M^{\theta}$.

(5) Let G be a discrete group and $\sigma\colon G\to\operatorname{Aut}P/\operatorname{Int}P$ a free G-kernel, i.e., an injective group morphism. Assume G is finitely generated, say by $g_0=e,g_1,\ldots,g_k$, with $g_i\neq g_j$ if $i\neq j$. Let $n_0,n_1,\ldots,n_k\geqslant 1$ be some fixed multiplicities and let $\theta=(\theta_0,\theta_1,\ldots,\theta_m)$ with $m=$

$\sum n_i$ be such that $\theta_0=\theta_1=\ldots=\theta_{n_0}=\mathrm{id}$, $\theta_{n_0+1}=\ldots=\theta_{n_0+n_1}$ a lifting in $\mathrm{Aut}\,P$ of $\sigma(g_1)$, ..., $\theta_{n_0+\ldots+n_{k-1}+1}=\ldots=\theta_m$ a lifting of $\sigma(g_k)$ in $\mathrm{Aut}\,P$ (thus, each lifting of $\sigma(g_i)$ is repeated n_i times).

Thus we may regard the subfactors associated to an $(m+1)$-tuple of automorphisms of P as associated to a G-kernel. By (3), if the multiplicities are properly chosen then $N^\theta \subset M^\theta$ determines the conjugacy class of the G-kernel σ.

(6) $\{e^1_{i_1i_1}e^2_{i_2i_2}\ldots e^j_{i_ji_j}\}_{0\leqslant i,i_2,\ldots,i_j\leqslant m}$ are minimal projections in $M'\cap M_j$ and we have $e^1_{i_1i_1}e^2_{i_2i_2}\ldots e^j_{i_ji_j}\sim e^1_{i'_1i'_1}\ldots e^j_{i'_ji'_j}$ in $M'\cap M_j$ if and only if $\theta^{-1}_{i_1}\theta_{i_2}\ldots=\mathrm{Ad}\,u\theta^{-1}_{i'_1}\theta_{i'_2}\ldots$, for some $u\in P$ (product of k alternative terms θ_i^{-1},θ_i). In this case $ue^1_{i_1i'_1}e^2_{i_2i'_2}\ldots e^j_{i_ji'_j}\in M'\cap M_j$ and such partial isometries generate $M'\cap M_j$.

(7) If G is the group generated by $\theta_0,\theta_1,\ldots,\theta_m$ in $\mathrm{Aut}\,P/\mathrm{Int}\,P$, then $\Gamma_{N^\theta,M^\theta}=(a_{hg})_{h,g\in G}$, where $*=e$ and $a_{hg}=$ the number of times $g\in G$ can be obtained as $\theta_i^{-1}h$, $0\leqslant i\leqslant m$. Also $\Gamma_{M,M_1}=(a'_{hg})_{h,g\in G}$, where $*=e$ and $a'_{hg}=$ the number of times $g\in G$ can be obtained as $\theta_i h$, $0\leqslant i\leqslant m$. If $E=E_{\min}$ then $E|_{\bigcup_k(M'\cap M_k)}$ is a trace with all the minimal projections in $M'\cap M_k$ having equal trace. In general $E_\infty|_{\bigcup_k(M'\cap M_k)}$ is a trace, where $E_\infty=EE_1E_2\ldots$, if and only if there exists a group morphism $\beta\colon G\to\mathbf{R}^*_+$ such that $s_g=\beta(g)$. Indeed, the first part is clear by (9). Also, if $\vec{s}$ is given like this and $g=\theta^{-1}_{i_1}\theta_{i_2}\ldots=\theta^{-1}_{i'_1}\theta_{i'_2}\ldots=g'$ mod $\mathrm{Int}\,P$, (equivalently $e=e^1_{i_1i_1}\ldots e^k_{i_ki_k}\sim e^1_{i'_1i'_1}\ldots e^k_{i'_ki'_k}=e'$), then $\beta(g)=\beta(g')$ so that $E_\infty(e)=E_\infty(e')$. The converse follows by showing recursively on generators that (s_g) defines a morphism.

PROPOSITION. (i) *Let P be a type* II *factor and G a discrete group with finitely many generators $g_1,\ldots,g_k$. Let σ,σ' be two free cocycle actions of G on P with* $\mathrm{Mod}\,\sigma=\mathrm{Mod}\,\sigma'$. *Let $\theta=(\mathrm{id},\theta_1,\ldots,\theta_m)$, $\theta'=(\mathrm{id},\theta'_1,\ldots,\theta'_m)$ with $\theta_i\in\{\sigma(g_j)\}_j$, $\theta'_i\in\{\sigma'(g_j)\}_j$ appearing with the same multiplicity $n_i\geqslant 1$. Then $\mathcal{G}_{N^\theta,M^\theta}\simeq\mathcal{G}_{N^{\theta'},M^{\theta'}}$.*

(ii) *More generally, if σ,σ' are free G-kernels of* $\mathrm{Aut}\,P/\mathrm{Int}\,P$ *and* $(\mathrm{Ob}\,\sigma,\mathrm{Mod}\,\sigma)=(\mathrm{Ob}\,\sigma',\mathrm{Mod}\,\sigma')$ *(see* [C3], [J1], [Oc1], [Su]*) then $\mathcal{G}_{N^\theta,M^\theta}=\mathcal{G}_{N^{\theta'},M^{\theta'}}$.*

Proof. If σ,σ' are genuine actions then (i) is trivial. Then the rest of the statement follows by direct computation. □

We can deduce now the classification of actions of [J1] and, for most amenable groups (such as groups with subexponential growth), of [Oc1] as well:

COROLLARY. *Let P be a type* II *factor and G a discrete finitely generated group. Let $g_0=e,g_1,\ldots,g_k\in G$ be a fixed set of generators and $n_0,n_1,\ldots,n_k\geqslant 1$ some multiplicities. Let $\sigma\colon G\to\mathrm{Aut}\,P/\mathrm{Int}\,P$ be a free G-kernel and $N^\theta\subset M^\theta$ the inclusion associated to σ, $(g_i)_i$ and $(n_i)_i$ as in* (5) *before. Then we have:*

(i) *$N^\theta\subset M^\theta$ is amenable if and only if P is amenable and G is amenable.*

(ii) *$N^\theta \subset M^\theta$ is strongly amenable if and only if P is amenable and G has* 0 *entropy with respect to the measure μ on it given by $(n_i)_i$ and by the morphism $\beta: G \to R_+^*$, $\beta(g) = \operatorname{Mod} \sigma(g)$* ([C1]). *This condition is satisfied for any choice of $(g_i), (n_j)$ and any β, in case G has subexponential growth.*

(iii) *If P is hyperfinite of type* II, *G satisfies* (ii) (*e.g. if it has subexponential growth*) *and the multiplicities $(n_i)_i$ are all distinct, then two free G-kernels σ, σ' are outer conjugate if and only if $(N^\theta \subset M^\theta) \simeq (N^{\theta'} \subset M^{\theta'})$ if and only if $(\operatorname{Ob}\sigma, \operatorname{Mod}\sigma) = (\operatorname{Ob}\sigma', \operatorname{Mod}\sigma')$. In particular, any two properly outer actions of G (with the same* Mod *function in the type II_∞ case) are outer conjugate and any* 2*-cocycle vanishes.*

Proof. (i) follows by 4.4.1, by Kesten's characterization of amenability of G and by 3.1.3 (b).

(ii) is clear by §1.4 and 4.1.2 (or 5.1.1).

(iii) is then a simple consequence of the previous proposition and (3). □

5.2. A list of subfactors of index $\leqslant 4$

A list of all possible matrices Γ over $\mathbf{Z}_+$ of norm $\leqslant 2$ has been obtained in [GHJ]: for index <4 these are the Coxeter graphs A_n, D_n, E_6, E_7, E_8 and for index $=4$ the Coxeter-type graphs $A_n^{(1)}, 2 \leqslant n \leqslant \infty$, $D_n^{(1)}, n \geqslant 4$, A_∞, D_∞, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)} = E_9$. It is already implicit in [J2] that for each $n \leqslant \infty$ there exists a unique possible standard invariant with standard graph A_n, given by the so-called Jones subfactor of index $4\cos^2 \pi/(n+1)$. An example of subfactor (thus paragroup) with graph E_6 was obtained in [BN] and with graphs $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $D_n^{(1)}$ in [GHJ].

The complete list of all possible paragroups with index <4 was then obtained by Ocneanu ([Oc2], see also [Ka1] for details of the proof), showing that there exist no paragroups with graph D_{2n+1} or E_7 (for simpler proofs see [Iz], [SuVa]) and that for each $n \geqslant 2$ there exists a unique one with graph D_{2n} while for each of E_6 and E_8 there are actually two.

For each $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ there is a unique paragroup ([Ka1]). For $A_{2n-1}^{(1)}$, $n \geqslant 1$, there are n, as they are in bijection with the number of elements in $H^3(\mathbf{Z}/n\mathbf{Z}, \mathbf{T})$, i.e., of the n-roots of unity, and there is only one for $A_\infty^{(1)}$ (all this by 5.1.5 and [C2], [C6]). Any other graph of square norm equal to 4 must be of the form $D_n^{(1)}$. It was shown in [Po6] that such paragroups are in one to one correspondence with the elements of the subgroup of $H^3(D_m, \mathbf{T})$, $2 \leqslant m \leqslant \infty$, that vanish on the two generators, where D_m is the dihedral group with generators α, β, $\alpha^2 = \beta^2 = 1$ and with the period of $\alpha\beta$ being m. Thus, these subfactors are classified by some well understood classical objects as well. The relation between these paragroups and their corresponding graphs has been obtained in [IzKa] where it is shown that for finite n there are $n-2$ paragroups corresponding to $D_n^{(1)}$ and they correspond to the elements in $H^3(D_{n-2}, \mathbf{T})$ considered above and that there is a unique paragroup with graph D_∞, corresponding to the fact that the subgroup in $H^3(D_\infty, \mathbf{T})$ is trivial.

By Theorem 5.1.1 we can now deduce that the above list coincides in fact with the list of all subfactors of index $\leqslant 4$.

5.2.1. COROLLARY. *The following is a listing of all subfactors of index* $\leqslant 4$ *of the hyperfinite type* II_1 *factor*:

standard graph	*number of subfactors*
$A_n,\ n\geqslant 2$	1
$D_{2n},\ n\geqslant 2$	1
E_6	2
E_8	2
$E_6^{(1)}$	1
$E_7^{(1)}$	1
$E_8^{(1)}$	1
$A_{2n-1}^{(1)},\ n\geqslant 1$	n
$D_n^{(1)},\ n\geqslant 4$	$n-2$
$A_\infty^{(1)}=A_{\infty,\infty}$	1
A_∞	1
D_∞	1

5.2.2. *Detecting Jones' irreducible subfactors.* From the above we can now formulate more explicitly some simple criteria for detecting the Jones subfactors.

COROLLARY. *Let* $N\subset M$ *be hyperfinite type* II_1 *factors with* $[M:N]\leqslant 4$.

(i) *If* $N'\cap M_k=\mathrm{Alg}\{1,e_1,\dots,e_k\}$, $\forall k$, *then* $N\subset M$ *is isomorphic to the Jones subfactor* $R^s\subset R$ *of index* $s=[M:N]$ (*i.e.*, $R=\mathrm{vN}\{f_i\}_{i\geqslant 0}$, $R^s=\mathrm{vN}\{f_j\}_{j\geqslant 1}$, *where* $\{f_i\}_{i\geqslant 0}$ *are the Jones projections of trace* $\tau(f_i)=[M:N]^{-1}=s^{-1}$).

(ii) *If* $[M:N]=4\cos^2\pi/(2n+1)$ *then* $N\subset M$ *is isomorphic to the Jones subfactor of index* $4\cos^2\pi/(2n+1)$.

(iii) *If* $[M:N]=4\cos^2\pi/(2n+2)$ *and* $N'\cap M_{n-1}=\mathrm{Alg}\{1,e_1,\dots,e_{n-1}\}$ *then* $N\subset M$ *is isomorphic to the Jones subfactor of index* $4\cos^2\pi/(2n+2)$.

(iv) *If* $[M:N]=4$ *and* $N'\cap M_5=\mathrm{sp}\{1,e_1,\dots,e_5\}$ *then* $N\subset M$ *is isomorphic to the Jones subfactor of index* 4.

Proof. (i), (ii) are reformulations of part of 5.2.1 and (iii), (iv) follow because of the shape of graphs of that norm. □

5.2.3. *Connes' uniqueness of trace scaling automorphisms and detection of Jones' locally trivial subfactors.* If $s>4$, the Jones subfactors $R^s\subset R$ of [J2] were proved to be locally trivial in [PiPo1], i.e., of the form $R^s=\{x\oplus\sigma(x)\,|\,x\in pRp\}$, R being the hyperfinite type II_1 factor and $\sigma: pRp\simeq(1-p)R(1-p)$ a surjective isomorphism, where $p\in\mathcal{P}(R)$, $\tau(p)\tau(1-p)=s^{-1}$. It has also been shown that if $N_\sigma\subset M$ is locally trivial as above, then $(N_\sigma^{\mathrm{st}}\subset M^{\mathrm{st}})\simeq(R^s\subset R)$ and an explicit representation of all higher relative commutants (= standard invariant) was obtained, showing that $\mathcal{G}_{N_\sigma,M}=\mathcal{G}_{R^s,R}$ and $\Gamma_{N_\sigma,M}=\Gamma_{R^s,R}=A_{-\infty,\infty}$. From that representation we also see that $E_{(R'\cap R_\infty)'\cap R_\infty}(e_1)\in\mathbf{C}$. Thus also $E_{(M'\cap M_\infty)'\cap M_\infty}(e_1)\in\mathbf{C}$ and $E_{(N_\sigma'\cap M_\infty)'\cap M_\infty}(e_1)\in\mathbf{C}$. Thus $(M'\cap M_\infty)'\cap M_\infty=M$, by Skau's lemma, and $N\subset M$ follows strongly amenable. By 5.1.1 one thus get both the uniqueness of Jones' subfactors of index >4 (by their graphs) and the uniqueness of the locally trivial subfactors. Since isomorphism of locally trivial subfactors $N_{\sigma_1}\subset M$, $N_{\sigma_2}\subset M$ amounts to outer conjugacy of σ_1,σ_2, (like in 5.1.5) which in turn are easily seen to come from $t/(1-t)$-scaling auto-morphisms $\tilde\sigma_1,\tilde\sigma_2$ of $R\overline{\otimes}\mathcal{B}(l^2(\mathbf{N}))$, where $t(1-t)=s^{-1}$, one also obtains Connes' uniqueness of trace scaling automorphisms up to outer conjugacy ([C3]).

It is easy to see that to detect such subfactors little information is needed:

COROLLARY. (i) *If* $[M:N]=s>4$, *then* $N\subset M$ *is a locally trivial subfactor with* $N'\cap M=\mathbf{C}^2$ *if and only if* $\Gamma_{N,M}=A_{-\infty,\infty}$ *and if and only if* $\mathrm{Ind}(E_{\min})=4$.

(ii) $N\subset M$ *is locally trivial if and only if* $H(M|N)=-t\ln t-(1-t)\ln(1-t)$ *and if and only if* $N'\cap M=\mathbf{C}p+\mathbf{C}(1-p)$, *where* $t(1-t)=\tau(p)\tau(1-p)=[M:N]^{-1}$.

(iii) *Any locally trivial subfactor of index* $s>4$ *and* $\dim N'\cap M=2$ *of the hyperfinite factor is isomorphic to the Jones subfactor of index* s.

Proof. (ii), (iii) are trivial by the above discussion and [PiPo1]. Then $\Gamma_{N,M}=A_{-\infty,\infty}$ implies $N'\cap M=\mathbf{C}p+\mathbf{C}(1-p)$. If $[pMp:Np]\neq1$ and $p_1=J_MpJ_M\in M'\cap M_1$, then $Npp_1\subset pMpp_1\subset pp_1M_1pp_1$ is a basic construction, so that $(Npp_1)'\cap pp_1M_1pp_1\neq\mathbf{C}$ (since the index $\neq1$). But this means that $N'\cap M_1$ has more than $\mathrm{Alg}\{p,p_1,1\}$ on its diagonal, contradicting $\Gamma_{N,M}=A_{-\infty,\infty}$. □

5.3. Strong amenability for standard invariants and paragroups

The (strong) amenability condition introduced in this paper is conceptually and technically best suited but it has the disadvantage of being difficult to check. In practical situations one does not have enough aprioric knowledge of the representation theory for $N\subset M$ to be able to decide on the existence of $N\subset M$ hypertraces. Also the amenability relative to a core assumes the knowledge of the higher relative commutants (via the core), which anyway are needed afterwards to actually distinguish the subfactor by its invariant $\mathcal{G}_{N,M}$. So, one needs characterizations of the strong amenability of an inclusion $N\subset M$, in terms of the behavior of its higher relative commutants. We will do this in the present paragraph. As it turns out, the amenability of $N\subset M$ splits into 2 complementary notions: the amenability (thus hyperfiniteness) of M and a growth property of its standard invariant (the paragroup), analoguous to the strong amenability for groups as defined in [Po6] and that we will now introduce in several equivalent forms.

5.3.1. THEOREM. *Let $N\subset M$ be an extremal inclusion of type* II_1 *factors with finite index. The following conditions are equivalent:*

(i) $\dim N^{\mathrm{st}\prime}\cap M^{\mathrm{st}}=\dim N'\cap M$.

(ii) $N^{\mathrm{st}}, M^{\mathrm{st}}$ *are factors and* $\mathcal{G}_{N^{\mathrm{st}},M^{\mathrm{st}}}$ *is isomorphic to* $\mathcal{G}_{N,M}$.

(iii) *If $M\supset N\supset N_1\supset\ldots$ is an arbitrary choice of the tunnel, $S\subset R$ is the associated core and $S_k=S\cap N_k$ then $S_k'\cap S_j=N_k'\cap N_j$, $k\geqslant j\geqslant -1$, where $N_{-1}=M$, $S_{-1}=R$.*

(iv) $(M'\cap M_\infty)'\cap M_\infty=M$ *and* $(N'\cap M_\infty)'\cap M_\infty=N$.

(v) $(M'\cap M_\infty)'\cap M_\infty=M$.

(vi) $\|\Gamma_{N,M}\|^2=[M:N]$ *and* $\Gamma_{N,M}$ *is ergodic.*

(vii) $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ *is an extremal inclusion, i.e.,* $E_{N^{\mathrm{st}\prime}\cap M^{\mathrm{st}}}(e_0)\in\mathbf{C}1, e_0$ *being the Jones projection.*

(viii) $H(M|N)=H(M^{\mathrm{st}}|N^{\mathrm{st}})=\lim_k H(N_k'\cap M|N_k'\cap N)$. *In this case one also has* $H(M|N)=\lim_k H(M'\cap M_{k+1}|M'\cap M_k)$.

Proof. (i)$\Rightarrow$(iv). Assume that (i) holds true. Since $S'\cap R\supset N'\cap M$ it follows that $S'\cap R=N'\cap M$. By expecting on N and using the commuting square relation we get

$$E_{S'\cap S}=E_SE_{S'\cap R}=E_SE_{N'\cap M}=E_SE_NE_{N'\cap M}=E_SE_{N'\cap N}=E_{\mathbf{C}}.$$

Thus $S'\cap S=\mathbf{C}$ and S is a factor. Also, by [PiPo1] e_0 cannot commute with the nontrivial projections in $N'\cap M$, which shows that none of the central projections of $N'\cap M=S'\cap R$ can be central projections of R. Thus R is also a factor. Moreover, $[R:S]=[M:N]$, since the probabilistic index [PiPo1] of S in R is $\lambda=[M:N]^{-1}$. But if $p\in S'\cap R=N'\cap M$ is a minimal projection then by the formula of the trace preserving conditional expectation of pMp onto Np, $E_{N_p}(pxp)=\tau(p)^{-1}E_N(pxp)p$, it follows that the probabilistic index of $Sp\subset pRp$ is majorized by that of $Np\subset pMp$, so that $[pMp:Np]\geqslant[pRp:Sp]$. By Jones' formula we thus have:

$$[R:S]=\sum[pRp:Sp]/\tau(p)\leqslant\sum[pMp:Np]/\tau(p)=[M:N]=[R:S].$$

Thus $[pRp:Sp]=[pMp:Np]$ for each $p\in N'\cap M=S'\cap R$. By the formula of $E_{M'\cap M_1}(e_1)$ in [PiPo1] it follows that $E_{M'\cap M_1}(e_1)=E_{R'\cap R_1}(e_1)$.

Thus $R \subset R_1$ is extremal. But then, by [PiPo2], $S \subset R$ is also extremal. By trace preserving isomorphism (since $N \subset M \subset M_1$ are extremal) it follows that $M_1' \cap M_\infty \subset M' \cap M_\infty \subset N' \cap M_\infty$ are extremal inclusions of factors. Thus all $M_k' \cap M_\infty \subset M_{k-1}' \cap M_\infty$ are extremal and by Skau's lemma 4.5.1 we get (iv).

(iv) $\Rightarrow$ (ii). If $(M' \cap M_\infty)' \cap M_\infty = M$ and $(N' \cap M_\infty)' \cap M_\infty = N$ then in particular $M' \cap M_\infty$ and $N' \cap M_\infty$ are factors so by 1.3.9 $M_k' \cap M_\infty$ are all factors. By antiisomorphism, $N^{\rm st}, M^{\rm st}$ are factors. We have $(M_k' \cap M_\infty)' \cap (M_j' \cap M_\infty) = M_k \cap (M_j' \cap M_\infty) = M_j' \cap M_k$, so by antiisomorphism we get (ii).

(iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) follows since we always have, by definitions, $S_k' \cap S_j \supset N_k' \cap N_j$, $k \geqslant j \geqslant -1$.

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (i) $(M' \cap M_\infty)' \cap M_\infty = M$ implies that $M' \cap M_\infty$ is a factor and that

$$(M' \cap M_\infty)' \cap (N' \cap M_\infty) = N' \cap M$$

so that, by antiisomorphism,

$$\dim(N^{\rm st}{}' \cap M^{\rm st}) = \dim(M^{\rm st}{}' \cap M_1^{\rm st}) = \dim((M' \cap M_\infty)' \cap (N' \cap M_\infty)) = \dim N' \cap M.$$

(ii) $\Rightarrow$ (vi) is trivial, since by (ii) we get that $N^{\rm st} \subset M^{\rm st}$ is extremal and $\Gamma_{N,M} = \Gamma_{N^{\rm st},M^{\rm st}}$, and since $\|\Gamma_{N^{\rm st},M^{\rm st}}\|^2 = [M:N]$ (for instance by [PiPo3], or by 4.4.1).

(vi) $\Rightarrow$ (iv) Since $\Gamma_{N_1,M} = \Gamma_{N,M} \Gamma_{N,M}^t$ (1.3.5), if $\|\Gamma_{N,M}\|^2 = [M:N]$ then $\|\Gamma_{N_1,M}\|^2 = [M:N_1]$. Also, since $\Gamma_{N,M}$ is ergodic $\Gamma_{N_1,M}$ is ergodic (as $M^{\rm st}$ is the same for $N_1 \subset M$ and $N \subset M$). By Corollary 1.3.5, since $N_1^{\rm st} \subset M^{\rm st}$ has larger higher relative commutants than $N_1 \subset M$, we get

$$\|\Gamma_{N_1,M}\|^2 \leqslant \|\Gamma_{N_1^{\rm st},M^{\rm st}}\|^2 \leqslant [M^{\rm st} : N_1^{\rm st}] = [M : N_1] = \|\Gamma_{N_1,M}\|^2,$$

thus $\|\Gamma_{N_1^{\rm st},M^{\rm st}}\|^2 = [M^{\rm st} : N_1^{\rm st}]$. By Corollary 1.3.6, $N_1^{\rm st} \subset M^{\rm st}$ is extremal, so that all $N_{2i+1}^{\rm st} \subset N_{2i+1}^{\rm st}$ are extremal by duality. By antiisomorphism, $M_{2i+2}' \cap M_\infty \subset M_{2i}' \cap M_\infty$ are all extremal, so that by Skau's lemma, $(M' \cap M_\infty)' \cap M_\infty = M$.

(iii) $\Rightarrow$ (vii) $\Rightarrow$ (viii) are trivial (by [PiPo1]). If $N^{\rm st} \subset M^{\rm st}$ would be factors, by [PiPo1], [PiPo2] we would get from (viii) that $N^{\rm st} \subset M^{\rm st}$, $M^{\rm st} \subset M_1^{\rm st}$ are extremal and Skau's lemma applies. If we do not assume this (apriorically), then let $\{e_k^i\}_{k \in K_i'}$ be the the minimal central projections of $N_{2i}' \cap N$ and $\{f_l^i\}_{l \in L_i'}$ the minimal central projections of $N_{2i}' \cap M$ (note that the inclusion matrix of $N_{2i}' \cap N \subset N_{2i}' \cap M$ is given by $(a_{kl})_{k \in K', l \in L_i'}$ where

$(a_{kl})_{k\in K',l\in L'}=\Gamma_{N_1,N})$. By [PiPo1], [PiPo3],

$$\begin{aligned}H(N'_{2i}\cap M\mid N'_{2i}\cap N)&=\sum_{k,l}\tau(e^i_kf^i_l)\ln(a^2_{kl}\tau(e^i_k)\tau(f^i_l)/\tau(e^i_kf^i_l)^2)\\&=\sum_l\tau(f^i_l)\Big(\sum_k\tau(e^i_kf^i_l)/\tau(f^i_l)\Big)\ln(a^2_{kl}\tau(e^i_k)\tau(f^i_l)/\tau(e^i_kf^i_l)^2)\\&\leqslant\sum_l\tau(f^i_l)\ln\Big(\sum_k a^2_{kl}\tau(e^i_k)/\tau(e^i_kf^i_l)\Big)\\&\leqslant\ln\max_l\Big(\sum_k a^2_{kl}\tau(e^i_k)/\tau(e^i_kf^i_l)\Big)\\&=\ln\big(\operatorname{Ind}E^{N'_{2i}\cap M}_{N'_{2i}\cap N}\big)\leqslant\ln[M:N].\end{aligned}$$

where the above inequalities follow by first using that ln is convex and then that it is increasing. By hypothesis (viii) the first sum tends to $\ln[M:N]$. It follows that if $\varepsilon>0$, $\exists i_0$ such that if $i\geqslant i_0$ then $S_i=\{l\in L'_i\mid[M:N]-\sum a^2_{kl}\tau(e^i_k)/\tau(e^i_kf^i_l)<\varepsilon\}$ satisfies $\sum_{l\in S_i}\tau(f^i_l)\geqslant 1-\frac{1}{2}\varepsilon$. Also, for i large enough we may assume $S_i\subset L'_{i-1}$ (since by [Po5], $\sum_{l\in L'_{i-1}}f^i_l=\operatorname{supp}(N'_{2i}\cap N)e_0(N'_{2i}\cap N)$, e_0 being the Jones projection). There exists an orthonormal basis $\{m^i_j\}_j$ of $N'_{2i}\cap M$ over $N'_{2i}\cap N_1$ such that

$$\Big\|\sum_j E_{N'_{2i}\cap N_1}(m^i_jm^{i*}_j)-(n+\tau(f))1\Big\|_2\to 0\quad\text{as } i\to\infty,$$

where $f\in\mathcal{P}(N_1)$. By [Po5], $f^i_l=\sum_j m^i_jf^{i-1}_le_0m^{i*}_j$, $\forall l\in L'_i$. Thus if

$$S'_i=\{l\in S_i\mid|\tau(f^i_l)-\tau(f^{i-1}_l)|\leqslant\varepsilon\tau(f^{i-1}_l)\}$$

then for $l\in S'_i$ and $k\in K'_i$ with $a_{kl}\neq 0$ we have

$$\begin{aligned}\tau(f^i_l)/\tau(e^i_kf^i_l)&=(\dim(N'_{2i}\cap M)f^i_l/\dim(N'_{2i}\cap N)e^i_k)^{1/2}\\&\leqslant(\dim(N'_{2i}\cap M)f^i_l/\dim(N'_{2i}\cap N_1)f^{i-1}_l)^{1/2}\leqslant\lambda^{-1}(1+\varepsilon),\end{aligned}$$

by using that the trace of a minimal projection under f^i_l is λ times the trace of a minimal projection under f^{i-1}_l.

From the above inequalities involving the convexity of ln we thus get that there exists $S''_i\subset S'_i$ such that for each $l\in S''_i$, $a^2_{kl}\tau(e^i_k)\tau(f^i_l)/\tau(e^i_kf^i_l)^2$ are close to λ^{-1}, $\forall k$ with $a_{kl}\neq 0$, and $\sum_{s\in S''_i}\tau(f^i_l)$ close to 1. Let $\partial S''_i=\{l\in S''_i\mid\exists l'\in L'_i\setminus S''_i$ such that $a_{kl}a_{kl'}\neq 0$ for some $k\in K'_i\}=\{l\subset S''_i\mid\exists l'\in L'_i\setminus S''_i$ with $\tau(f^i_{l'}f^{i-1}_l)\neq 0\}$. Since by the [PiPo1] inequality $\tau(f^i_{l'}f^{i-1}_l)\geqslant\lambda^2\tau(f^{i-1}_l)$ and for each l' the number of l's with $\tau(f^i_{l'}f^{i-1}_l)\neq 0$ is bounded by λ^{-1}, we get that $\sum\{\tau(f^{i-1}_l)\mid l\in\partial S''_i\}$ is small for large enough i. Finally, note that since the central support of e_0 in $M^{\mathrm{st}}=\overline{\bigcup_i(N'_{2i}\cap M)}$ is 1, for i large the central support of e_0 in $N'_{2i}\cap M$ is close to 1, so that we may assume $|\tau(e_0f_l^i)-\lambda\tau(f_l^i)|<\varepsilon\tau(f_l^i)$, $\forall l\in S_i''\setminus\partial S_i''$. Altogether, if we denote $T_i=S_i''\setminus\partial S_i''$, we get that for large i,

(a) $|a_{kl}^2\tau(e_k^i)\tau(f_l^i)/\tau(e_k^if_l^i)^2-\lambda^{-1}|<\varepsilon$, $\forall l\in\Gamma^t\Gamma(T_i)$, $\forall k$ with $a_{kl}\neq 0$;

(b) $|\tau(e_0f_l^i)-\lambda\tau(f_l^i)|<\varepsilon\tau(f_l^i)$, $\forall l\in T_i$;

(c) $|\tau(f_l^i)-\tau(f_l^{i-1})|<\varepsilon\tau(f_l^i)$, $\forall l\in T_i$;

(d) $\sum_{l\in T_i}\tau(f_l^i)\geqslant 1-\varepsilon$.

In the inclusion $N'_{2i}\cap N_1\subset N'_{2i}\cap N\subset N'_{2i}\cap M$ the Jones projection e_0 is represented so that there exist projections $q_{lk}\in A_{lk}$ such that $E_{B_{lk}}(q_{lk})=1/a_{kl}1_{A_{lk}}=E_{B'_{lk}\cap A_{lk}}(q_{lk})$ and $e_k^if_l^if_l^{i-1}e_0e_k^if_l^if_l^{i-1}=\lambda a_{kl}s_k/t_lq_{kl}e_k^if_l^if_l^{i-1}$, where

$$A_{lk}=(N'_{2i}\cap N_1e_k^if_l^if_l^{i-1})'\cap e_k^if_l^{i-1}N'_{2i}\cap Me_k^if_l^{i-1}f_l^i$$

and

$$B_{lk}=(N'_{2i}\cap N_1e_k^if_l^if_l^{i-1})'\cap f_l^{i-1}N'_{2i}\cap Nf_l^{i-1}e_k^if_l^i.$$

It follows that if $m_l^i=\dim N'_{2i}\cap Mf_l^i$ and $n_k^i=\dim N'_{2i}\cap Ne_k^i$ then we have

$$\begin{aligned}E_{(N'_{2i}\cap N)'\cap(N'_{2i}\cap M)}(e_0f_l^i)&=\sum_k(m_l^{i-1}/n_k^i)(\lambda s_k/t_l)f_l^ie_k^i\\&=\sum_k(a_{kl}m_l^{i-1}\lambda s_k)^2/a_{kl}^2(n_k^is_k)(m_l^{i-1}t_l)f_l^ie_k^i\\&=\sum_k\tau(f_l^{i-1}e_k^i)^2/a_{kl}^2\tau(f_l^{i-1})\tau(e_k^i)f_l^ie_k^i.\end{aligned}\tag{$*$}$$

Similarly we get

$$E_{(N'_{2i}\cap M)'\cap(N'_{2i}\cap M_1)}(e_1)=\sum_k\tau(f_l^ie_k^i)^2/a_{kl}^2\tau(f_l^i)\tau(e_k^i)f_l^ie_k^{i-1}.\tag{$**$}$$

(a) above then shows that $(**)$ is close to $\lambda 1$. Since $\tau(f_l^i)=\lambda m_l^i/m_l^{i-1}\tau(f_l^{i-1})$, by (c), we also get that $(*)$ is close to $\lambda 1$. Altogether we obtain that $E_{N'^{\mathrm{st}}\cap M^{\mathrm{st}}}(e_0)=\lambda 1, E_{M'^{\mathrm{st}}\cap M_1^{\mathrm{st}}}(e_1)=\lambda 1$. By antiisomorphism and duality

$$E_{(M'_j\cap M_\infty)'\cap M_\infty}(e_{j+1})=\lambda 1,\quad \forall j\geqslant 0,$$

and Skau's lemma applies to give $(M'\cap M_\infty)'\cap M_\infty=M$, i.e., (viii) $\Rightarrow$ (v). $\square$

A similar statement holds true for nonextremal inclusions as well.

5.3.2. THEOREM. *If $N\subset M$ is not extremal (so that $\alpha=\operatorname{Ind} E_{\min}<[M:N]$) then the following are equivalent*:

(i) $\dim N^{\mathrm{st}\prime}\cap M^{\mathrm{st}}=\dim N'\cap M$.

(ii) $N^{\mathrm{st}}, M^{\mathrm{st}}$ *are factors and* $\mathcal{G}_{N^{\mathrm{st}},M^{\mathrm{st}}}=\mathcal{G}_{N,M}$.

(iii) *If $S\subset R$ is the core for the tunnel $\{N_k\}$ then $S'_k\cap S_j=N'_k\cap N_j$, $\forall k,j\geqslant -1$.*

(iv) $(M'\cap M_\infty)'\cap M_\infty=M$, $(N'\cap M_\infty)'\cap M_\infty=N$.

(v) $(M'\cap M_\infty)'\cap M_\infty=M$.

(vi) $E_{N^{\mathrm{st}\prime}\cap M^{\mathrm{st}}}(e_0)=E_{N'\cap M}(e_0)$ *and* $E_{M^{\mathrm{st}\prime}\cap M_1^{\mathrm{st}}}(e_0)=E_{M'\cap M_1}(e_1)$.

Moreover, the condition

(vii) $\|\Gamma_{N,M}\|^2=\operatorname{Ind} E_{\min}^{N,M}$ *and* $\Gamma_{N,M}$ *ergodic*,

implies any of the above conditions (i)–(vi).

Proof. The proof of (i) $\Rightarrow$ (iv) in 5.3.1, which does not depend on $N\subset M$ being extremal, shows that if $S\subset R$ is a core and if one assumes (i) then S,R are factors, $S'\cap R=N'\cap M$, the local indices for $S\subset R$, $N\subset M$ coincide and $E_{R'\cap R_1}(e_1)=E_{M'\cap M_1}(e_1)$. But by Jones' formula 1.2.5, the local indices for $R\subset R_1$ will also coincide and by the same argument as for e_1 we get $E_{R'_1\cap R_2}(e_2)=E_{M'_1\cap M_1}(e_2)$. This shows (i) $\Rightarrow$ (vi). By the stability of the core $E_{M_{j-1}^{\mathrm{st}\prime}\cap M_j^{\mathrm{st}}}(e_j)=E_{M'_{j-1}\cap M_j}(e_j)$, $\forall j$. If e'_{-j} are the projections in [PiPo1, 2.2] for which $E_{N'_j\cap N_{j-1}}(e'_{-j})\in\mathbf{C}$, constructed by modifying e_{-j} (but still, $e'_{-j}\in N'_{j+1}\cap N_{n-1}$, $e'_{-j}\vee e_{-j}\in N'_{j+1}\cap N_{j-1}$), then it follows that $E_{N_j^{\mathrm{st}\prime}\cap N_{j-1}^{\mathrm{st}}}(e'_{-j})\in\mathbf{C}$. By [Po12], there exists a trace preserving (noncanonical!) isomorphism of $N'_{2i-1}\cap M$ onto $M'\cap M_{2i}$ carrying $N'_{2i-1}\cap N_1$ onto $M'_2\cap M_{2i}$ and $e'_{-1}e'_0e'_{-2}e'_{-1}$ onto $e_3e_2e_4e_3$ (which is the Jones projection for $M_2\subset M_4$. Thus: $E_{(M'_2\cap M_\infty)'\cap M_\infty}(e_3e_2e_4e_3)\in\mathbf{C}$ and by duality $E_{(M'_{2i}\cap M_\infty)'\cap M_\infty}(f_i)\in\mathbf{C}$, where f_i are the Jones projections for $M_{2i}\subset M_{2i+2}$. By Skau's lemma $(M'\cap M_\infty)'\cap M_\infty=M$. Thus (vi) $\Rightarrow$ (v). But

$$(N'\cap M_\infty)'\cap M_\infty=(\{e_1\}\cup(M'\cap M_\infty))'\cap M_\infty=\{e_1\}'\cap M=N,$$

so that (v) $\Leftrightarrow$ (iv).

Suppose (iv) holds true. Then the same argument as in (vi) $\Rightarrow$ (v) above shows that $E_{S'_i\cap R}(e'_{-i})\in\mathbf{C}$. If $R_\infty=\overline{\bigcup_k(N'_k\cap M_\infty)}=\overline{\bigcup_{k,j}(N'_k\cap M_j)}$, then Skau's lemma applied to $M'\cap M_\infty\subset N'\cap M_\infty$ shows that $S'_k\cap R_\infty=N'_k\cap M_\infty$. Thus, by projecting on S_l, $S'_k\cap S_l=N'_k\cap N_l$, and we get (iv) $\Rightarrow$ (iii). Next, (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i) are trivial.

To prove (vii) $\Rightarrow$ (v), note that $\Gamma_{N_1,M}=\Gamma_{N,M}\Gamma^t_{N,M}$ and thus $\|\Gamma_{N_1,M}\|^2=\|\Gamma_{N,M}\|^4=(\operatorname{Ind} E_{\min}^{M,N})^2=\operatorname{Ind} E_{\min}^{N,N_1}\operatorname{Ind} E_{\min}^{M,N}=\operatorname{Ind} E_{\min}^{M,N_1}$. By 1.3.5 we have

$$\|\Gamma_{N_1,M}\|^2\leqslant\|\Gamma_{N_1^{\mathrm{st}},M^{\mathrm{st}}}\|^2\leqslant\operatorname{Ind} E_{\min}^{M^{\mathrm{st}},N_1^{\mathrm{st}}}\leqslant\operatorname{Ind} E_{\min}^{M,N_1}=\|\Gamma_{N_1,M}\|^2.$$

Thus $\operatorname{Ind} E_{\min}^{M^{\mathrm{st}},N_1^{\mathrm{st}}}=\operatorname{Ind} E_{\min}^{M,N_1}$ which by the formula of the Jones projection for $N_1\subset M$ in [PiPo1, 4.4] shows that $E_{M^{\mathrm{st}\prime}\cap M_2^{\mathrm{st}}}(f_1)=E_{M'\cap M_2}(f_1)$, where $f_1=[M:N]e_1e_2e_0e_1$ is the Jones projection for $M \subset M_2$. By the previously proved equivalence of (vii) and (v), applied to $N_1 \subset M$, we get that $(M' \cap M_\infty)' \cap M_\infty = M$. □

We mention that condition (vii) in 5.3.2 is in fact equivalent to (i)–(vi), but the proof of the converse (i)–(vi) $\Rightarrow$ (vii) will be detailed elsewhere.

5.3.3. *Definition.* The standard invariant (or paragroup) $\mathcal{G}_{N,M}$ of an extremal (resp. nonextremal) inclusion $N \subset M$ is *strongly amenable* if any of the equivalent conditions (i)–(vii) of 5.3.1 (resp. (i)–(vi) of 5.3.2) is satisfied.

Note that if we define $\mathcal{G}_{N,M}$ to be *ergodic* if both $N^{\rm st}, M^{\rm st}$ are factors (i.e., $\Gamma_{N_1,N}$, $\Gamma_{N,M}$ ergodic) and $\mathcal{G}^{\rm st}_{N,M} \overset{\rm def}{=} \mathcal{G}_{N^{\rm st},M^{\rm st}}$, $\Gamma^{\rm st}_{N,M} \overset{\rm def}{=} \Gamma_{N^{\rm st},M^{\rm st}}$ (for ergodic $\mathcal{G}_{N,M}$) then conditions (i)–(iii) (in 5.3.1, 5.3.2) can be reformulated, for ergodic $\mathcal{G} = \mathcal{G}_{N,M}$, as follows:

5.3.4. *$\mathcal{G}$ is strongly amenable if and only if $\mathcal{G} = \mathcal{G}^{\rm st}$ if and only if $\Gamma = \Gamma^{\rm st}$ and in fact if and only if $\Gamma^{\rm st}$ has the same number of edges starting at $*$ as Γ does.*

The entropic condition (viii) in 5.3.1 can be regarded as a Shanon–McMillan–Breimann type condition, analogous to the entropic characterization of strongly amenable groups of Avez and Kaimonivici–Vershik ([A], [KV], [Po6]). Our formulation is in terms of conditional entropies, more suitable in the noncommutative operator algebra setting. If one considers the random walk on the graph $\Gamma_{N,M} \circ \Gamma^t_{N,M} \circ \Gamma_{N,M} \circ \ldots$ (the composition means simply glueing the bypartite graphs Γ, Γ^t at vertices with the same label) starting at $*$ and with probabilities determined by the weights $\vec{s}, \vec{t}$, then $H(M' \cap M_{k+1} | M' \cap M_k)$ gives the conditional entropy from step k to step $k+1$ of the random walk. As this entropy is always majorized by $H(M_{k+1} | M_k) = H(M|N)$ ([PiPo1]), one can interpret condition (viii) in 5.3.1 as follows:

5.3.5 (A Shanon–McMillan–Breimann type condition). *$\mathcal{G}$ is strongly amenable if and only if it has maximal entropy, $H(\mathcal{G})$ ($\overset{\rm def}{=} \lim H(N_k' \cap M | N_k' \cap N)) = H(M|N)$, i.e., if and only if the associated random walk tends to have maximal entropy, at infinity.*

Finally, condition (vi) in 5.3.1 coincides in the case of subfactors associated with actions of finitely generated discrete groups with a characterization of amenability by Kesten, showing that such a group G is amenable if and only if its Cayley graph has maximal norm ([Ke]). So, for extremal $N \subset M$ with $N^{\rm st}, M^{\rm st}$ factors we have:

5.3.6 (A Kesten type condition). *$\mathcal{G}_{N,M}$ strongly amenable* $\Leftrightarrow$ $\|\Gamma_{N,M}\|^2 = [M:N]$ $\Leftrightarrow$ $\|\Gamma^{\rm st}_{N,M}\|^2 = [M:N]$.

At this point we should note that while in general the two conditions $\Gamma_{N,M}$ ergodic and $\|\Gamma_{N,M}\|^2 = [M:N]$ are complementary, it is shown in [Po6] that for the locally trivial subfactors $N^\sigma \subset M^\sigma$ coming from actions of groups the first condition implies the second! The next result clarifies this relation.

5.3.7. COROLLARY. *Let $N \subset M$ be an extremal inclusion and assume the weight vector $\vec{s}$ is bounded (equivalently, the set of all indices at irreducible inclusions in the Jones tower is bounded). Then $\mathcal{G}_{N,M}$ is strongly amenable if and only if $\Gamma_{N,M}$ is ergodic if and only if $\Gamma_{N_1,N}$ is ergodic.*

Proof. By Corollary 1.3.6, if $N_1^{\mathrm{st}} \subset M^{\mathrm{st}}$ is not extremal and $\vec{r}$ denotes its standard weights, giving the traces of the minimal projections in $M^{\mathrm{st}\prime} \cap M_j^{\mathrm{st}}$, then $\sup_{k'} r_{k'} = \infty$. But since $M' \cap M_j \subset M^{\mathrm{st}\prime} \cap M_j^{\mathrm{st}}$, any s_k is a sum of some $r_{k'}$, so $\sup s_k = \infty$, a contradiction. By 5.3.1 we are done. □

Along these lines, in order to introduce some nice sufficient conditions for paragroups to be strongly amenable we now consider one more concept.

5.3.8. *Definition.* Let $N \subset M$ be extremal. $\mathcal{G}_{N,M}$ has *subexponential growth* if $\lim_n (\sum_{k \in K_n} v_k^2)^{1/n} = 1$, where $K_n \subset K$ has the usual significance of the set of even vertices of $\Gamma_{N,M}$ that can be reached after n steps, starting from $*$, and $\vec{v} = (v_k)_{k \in K}$ is the standard vector of even local indices in the Jones tower (see 1.3.6). Recall that $\Gamma\Gamma^t \vec{v} = \alpha \vec{v}$, where $\alpha = \operatorname{Ind} E_{\min}^{M,N}$ and that $\alpha = [M:N]$, $\vec{v} = \vec{s}$, when $N \subset M$ is extremal.

The next result is analogous to the well known similar statement for groups, where however the ergodicity condition is redundant (see also 5.4.6).

COROLLARY. *If $\Gamma_{N,M}$ is ergodic and $\mathcal{G}_{N,M}$ has subexponential growth then $\mathcal{G}_{N,M}$ is strongly amenable.*

Proof. Let $\vec{v}_{K_n} = (v_k)_{k \in K_n}$. Then $\lim (\sum_{k \in K_n} v_k^2)^{1/n} = 1$ implies that

$$\liminf_n \frac{\|v_{K_n}\|_2}{\|v_{K_{n-1}}\|_2} = 1.$$

But if $B = \Gamma\Gamma^t$ then $(B\vec{v}_{K_n})_{K_{n-1}} = \alpha \vec{v}_{K_{n-1}}$ so that $\alpha \|\vec{v}_{K_{n-1}}\|_2 \leqslant \|B\vec{v}_{K_n}\|_2$ showing that $\|B\| = \alpha$, where $\alpha = \operatorname{Ind} E_{\min}^{M,N}$. □

5.4. Further characterizations of amenability for inclusions

With Theorems 5.3.1 and 5.3.2 we can now divide the strong amenability of an inclusion into two separate properties: the amenability of M on the one hand and the strong amenability of $\mathcal{G}_{N,M}$ on the other. Also, we can now deduce that the existence of hypertraces for the standard representation of $N \subset M$ is sufficient to ensure its amenability.

5.4.1. THEOREM. *$N \subset M$ is strongly amenable if and only if M is the hyperfinite type II_1 factor and $\mathcal{G}_{N,M}$ is strongly amenable. If $N \subset M$ is extremal then $N \subset M$ is* *strongly amenable if and only if there exists an $(N \subset M)$-hypertrace on $\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$ (the standard representation of $N \subset M$) and $\Gamma_{N,M}$ is ergodic.*

Proof. The first part is a consequence of 4.1.1 and 5.3.1. If $N \subset M$ is extremal and $\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$ has a $(N \subset M)$-hypertrace, then by 4.4.1 the inclusion graph of $\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$, $\Gamma^t_{N,M}$, has norm $\|\Gamma^t_{N,M}\|^2 = [M:N]$. By 5.3.1, this and the ergodicity of $\Gamma_{N,M}$ imply the bicommutant condition which by Section 4 implies the strong amenability of $N \subset M$. □

In a future paper we will prove a result similar to 5.3.1 for characterizing the amenability of $\mathcal{G}_{N,M}$ only (i.e., without assuming ergodicity properties on $\Gamma_{N,M}$ and the core), in which one of the equivalent conditions is $\|\Gamma_{N,M}\|^2 = [M:N]$. Also, we will prove that $N \subset M$ is amenable if and only if M is amenable and $\mathcal{G}_{N,M}$ is amenable (i.e., $\|\Gamma_{N,M}\|^2 = [M:N]$). We point out that in fact there exist no examples up to now of irreducible subfactors $N \subset M$ for which $\|\Gamma_{N,M}\|^2 = [M:N]$ but $\Gamma_{N,M}$ is not ergodic. In fact, the only examples of amenable subfactors which are not strongly amenable are the locally trivial subfactors in [Po6] associated to actions of finitely generated amenable, but not strongly amenable groups. Examples of such groups are given in [KaiVe]. Let us mention here the following:

5.4.2. *Problem.* Is the amenability equivalent to strong amenability for irreducible subfactors? Such examples must have infinite depth, i.e., $\Gamma_{N,M}$ infinite. One should point out that, up to now, there are no other examples of irreducible strongly amenable subfactors with infinite depth other than Wassermann's subfactors ([Wa2]) (see 5.1.3) or tensor products of such subfactors with finite depth ones. In particular, there are no examples of irreducible amenable subfactors of infinite depth with index $4 < [M:N] < 8$. The next problems seem of even more interest:

5.4.3. *Problem.* If $\mathcal{J}^a = \{[M:N] \mid N \subset M$ irreducible, amenable$\}$ then is $\mathcal{J}^a$ a closed set? Is it countable? Can it contain an interval? Is the set of isomorphism classes of strongly amenable subfactors, or at least the one of subfactors with finite depth of the hyperfinite factor, countable? Note that by the main result of this paper (4.1.1 or 5.1.1), this amounts to evaluate the number of distinct strongly amenable, or merely finite depth, paragroups. To evaluate cardinality one needs to show that only countably many paragroups may exist with the same standard graph. (It has been pointed out to us by A. Ocneanu that this problem is not solved even in the finite depth case.) Note that in fact any information on the set $\mathcal{J}^a$ would be interesting to know. Haagerup seems to have recently found a candidate for the first limit point α_1 of $\mathcal{J}^a$ with $\alpha_1 > \alpha_0 = 4$ and proved that $\alpha_1 \notin \mathcal{J}^a$! There are no examples of limit points "from above" in $\mathcal{J}^a$. A more approachable problem along these lines seems to be to show that if there exists a strongly amenable subfactor with infinite graph Γ then there exist finite depth subfactors with graphs Γ_n such that $\|\Gamma_n\|<\|\Gamma\|$ and $\|\Gamma\|=\lim\|\Gamma_n\|$ (or even $\Gamma=\lim\Gamma_n$, asymptotically). In some situations (that should be understood!) this may even be possible by taking Γ_n to be the restrictions of Γ to K_n. Equivalently, one can formulate this question by asking whether indices of infinite depth strongly amenable subfactors can be isolated points in $\mathcal{J}^a$. Note that conversely, if $(\Gamma_n,\vec{v}_n)$ are weighted graphs of finite depth subfactors that tend (in an appropriate sense) to a weighted graph $(\Gamma,\vec{v})$ and if $\|\Gamma\|=\lim_n\|\Gamma_n\|$ and if $(\Gamma,\vec{v})$ is ergodic, then Γ is the graph of a (strongly) amenable subfactor. Indeed, if $\widetilde{N}=\prod_\omega N_n\subset\prod_\omega M_n=\widetilde{M}$, then $\Gamma_{\widetilde{N},\widetilde{M}}=\lim_\omega\Gamma_n=\Gamma$ so that if Γ is ergodic then $N=(\widetilde{N})^{\mathrm{st}}\subset(\widetilde{M})^{\mathrm{st}}=M$ has $\Gamma_{N,M}=\Gamma$.

Let us mention that for subfactors of small index and ergodic graphs, nonamenability automatically entails the trivality of the higher relative commutants:

5.4.4. Corollary. *If* $4<[M:N]<\left(1+\sqrt{2}\right)^2=3+2\sqrt{2}$, $\mathcal{G}_{N,M}$ *is ergodic (i.e.,* $\Gamma_{N,M}$, $\Gamma_{N_1,N}$ *ergodic) but not strongly amenable (i.e., for extremal* $N\subset M$, $\|\Gamma_{N,M}\|^2<[M:N]$*), then* $\Gamma_{N,M}=A_\infty$*, i.e., the higher relative commutants are generated by the Jones projections only. Also, if* $N\subset M$ *has infinite depth,* $4<[M:N]<2+\sqrt{5}$ *and* $N'\cap M=C$*, then* $\Gamma_{N,M}=A_\infty$.

Proof. If $4<[M:N]<\left(1+\sqrt{2}\right)^2$ then either $N\subset M$ is locally trivial with $N'\cap M=\mathbf{C}^2$ or $N'\cap M=\mathbf{C}$ (cf. [PiPo1]). Since locally trivial subfactors are strongly amenable, we must have that $N'\cap M=\mathbf{C}$. Since $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ are factors of index $[M^{\mathrm{st}}:N^{\mathrm{st}}]=[M:N]<\left(1+\sqrt{2}\right)^2$, again, either $N^{\mathrm{st}\prime}\cap M^{\mathrm{st}}=\mathbf{C}$ or $N^{\mathrm{st}\prime}\cap M^{\mathrm{st}}=\mathbf{C}^2$ and $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ is locally trivial. In the first case, by 5.1.1 $N\subset M$ would be strongly amenable. The second case means that $N'\cap M_k=\mathrm{Alg}\{1,e_1,\dots,e_k\}$, meaning that $\Gamma_{N,M}=A_\infty$.

If $[M:N]\leqslant 2+\sqrt{5}$ then by Corollary 1.4.2, $\mathcal{G}_{N,M}$ is ergodic (i.e., $\Gamma_{N,M},\Gamma_{N_1,N}$ are ergodic) so that if $\|\Gamma\|<[M:N]$ then the first part applies to get $\Gamma_{N,M}=A_\infty$. If $\|\Gamma\|=[M:N]$ then by 1.3.6, Further Remarks 3, we get a contradiction. $\square$

We mention that Haagerup announced a result along this line, showing that in fact any irreducible subfactor of index $4<[M:N]<4.3$ has graph A_∞.

As for the ergodicity of $\Gamma_{N,M}$ versus Γ_{M,M_1}, we note:

5.4.5. Corollary. *If either* $[M:N]\leqslant 5$ *or* $\|\Gamma_{N,M}\|^2=[M:N]$*, then* $\Gamma_{N,M}$ *is ergodic if and only if* Γ_{M,M_1} *is ergodic.*

Proof. If $\|\Gamma_{N,M}\|^2=[M:N]$, then $\|\Gamma_{M,M_1}\|^2=\|\Gamma_{N,M}\|^2=[M:N]=[M_1:M]$ so that by 5.1.1, $\Gamma_{N,M}$ is ergodic $\Leftrightarrow$ $\mathcal{G}_{N,M}$ is strongly amenable $\Leftrightarrow$ $\mathcal{G}_{M,M_1}$ is strongly amenable $\Leftrightarrow$ Γ_{M,M_1} is ergodic.

By considering $M\subset M_1$ instead of $N\subset M$, it is sufficient to prove $\Gamma_{N_1,N}$ $(=\Gamma_{M,M_1})$ ergodic $\Rightarrow$ $\Gamma_{N,M}$ ergodic. By §1.4, $Q=N^{\mathrm{st}}\subset M^{\mathrm{st}}=P$ is a λ-Markov inclusion of [PiPo1] index equal to $\lambda^{-1}=[M:N]$. If Q is a factor then by [PiPo1], $\tau(p)\geqslant\lambda$, for $p\in\mathcal{Z}(P)$. We need the following:

LEMMA. *If $Q\subset P=\bigoplus_{i=1}^{n}P_i$ is an inclusion with Q, P_i factors of type* II_1 *then there exists a projection $e_0\in P$ of central trace $\lambda=(\operatorname{Ind}E_Q^P)^{-1}$ such that $E_Q^P(e_0)=\lambda$ if and only if $\sum_j[P_j:Q]=[P_i:Q]/\tau(p_i)=\lambda^{-1}$, $\forall i$, where $p_i=1_{P_i}\in P$, if and only if $Q\subset P$ is λ-Markov. In addition, $E_{Q'\cap P}(e_0)=\lambda 1$ if and only if $Q\subset P_i$ is extremal, $\forall i$.*

Proof. If $f\in P$ then $E_Q(fp_i)\leqslant E_Q(f)$. Since $E_Q^P(x_ip_i)=E_Q^{P_i}(x_i)\tau(p_i)$ it follows that $(\operatorname{Ind}E_Q^P)^{-1}=\min[P_i:Q]^{-1}\tau(p_i)$ and the first two conditions follow equivalent. An orthonormal basis $\{m_k\}$ for P over Q can be obtained as $\{\tau(p_i)^{-1/2}m_j^ip_i\}_{i,j}$, where $\{m_j^i\}_j$ is a basis of P_i over Q. Then $\sum_k m_km_k^*=\sum_{i,j}\tau(p_i)^{-1}m_j^im_j^{i*}p_i=\sum_i[P_i:Q]/\tau(p_i)p_i$ and the second equivalence follows. The last part is trivial by the representation of $e_0\in P$ with $\operatorname{Ctr}(e_0)=\lambda$ and $E_Q(e_0)=(\operatorname{Ind}E_Q^P)^{-1}$: let $f_i\in Q$ be projections with $\sum f_i=1$ and $\tau(f_i)=\tau(p_i)$. Let $e_i^0\in f_iP_if_i$ be Jones projections for $f_iQf_ip_i\subset f_iP_if_i$. Then $e_0=\sum e_i^0$ satisfies $E_Q^P(e_0)=\lambda 1$ and $E_{Q'\cap P}(e_0)=\lambda 1$ if and only if $E_{f_iQf_ip_i'\cap f_iP_if_i}(e_1^0)=\lambda/\tau(p_i)p_if_i=[P_i:Q]^{-1}p_if_i$ if and only if $f_iQf_ip_i\subset f_iP_if_i$ (thus $Q\subset P_i$) are all extremal. $\square$

End of the proof of 5.4.5. If $M^{\mathrm{st}}=P=\sum P_i$ with P_i factors and $[P_i:Q]\leqslant 4$, $\forall i$, then $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ follows extremal by the previous lemma and then 5.1.1 implies that $\mathcal{G}_{N,M}$ is strongly amenable, thus M^{st} is actually a factor. So, in order for M^{st} not to be a factor when N^{st} is, it is necessary that $M^{\mathrm{st}}=\bigoplus_{i=1}^{n}P_i$ with $n\geqslant 2$ and at least one P_i, say P_1, has index $[P_1:Q]>4$. But then $[M^{\mathrm{st}}:N^{\mathrm{st}}]=[P_i:Q]/\tau(p_i)$, $\forall i$, and summing up $[M^{\mathrm{st}}:N^{\mathrm{st}}]=\sum_i[P_i:Q]/\sum_i\tau(p_i)=\sum_i[P_i:Q]>4+1=5$. $\square$

Note that the above proof shows that if N^{st} is a factor but M^{st} is not and if $[M:N]<6$ then $(N^{\mathrm{st}}\subset M^{\mathrm{st}})=(Q\subset P_1\oplus P_2)$ with $Qp_2=P_2$ and with $Qp_1\subset P_1$ a locally trivial Jones subfactor. Recently Haagerup constructed a subfactor $N\subset M$ of index $8\cos^2\pi/5=3+\sqrt{5}$ and ergodic $\Gamma_{N,M}$ but with $\Gamma_{N_1,N}$ nonergodic, by taking appropriate inclusions $N\subset P\subset M$ with $[P:N]=2$, $[M:P]=4\cos^2\pi/5$. From the above comments, since $3+\sqrt{5}<6$, we see that $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ must be of the form described above.

5.4.6. *Problem.* The known examples of subfactors $N\subset M$ of infinite depth and subexponential growth $\mathcal{G}_{N,M}$ (5.3.8) are the subfactors $N^\sigma\subset M^\sigma$ coming from actions σ of discrete groups G with subexponential growth (5.1.5) and Wassermann's subfactors (5.1.4). In all such cases the ergodicity of $\Gamma_{N,M}$ follows automatically. Does it follow, in general, that $\mathcal{G}_{N,M}$ with subexponential growth implies $\mathcal{G}_{N,M}$ strongly amenable? Does this follow at least when one also assumes $\sup_k s_k<\infty$? We should mention that there are no known examples of subfactors with bounded vector $\vec{s}$ which have infinite depth and are strongly amenable (thus, with ergodic $\Gamma_{N,M}$ by 5.3.7) other than those of the form $(N^\sigma \subset M^\sigma)\otimes$finite depth, in particular there are no examples of irreducible such subfactors.

5.4.7. *Problem.* As we will show in another paper, the standard invariant $\mathcal{G}_{N,M}$ determines completely $N\subset M$ even if one only assumes $N\subset M$ amenable (thus, $\Gamma_{N,M}$ not necessarily ergodic). It seems possible that the class of amenable inclusions is as much as the standard invariants can uniquely determine. It would be important to prove (or disprove) that given any non-amenable subfactor $N\subset M$ of the hyperfinite type II_1 factor (for extremal $N\subset M$ this means $\|\Gamma_{N,M}\|<[M:N]$) there exists another subfactor $P\subset M$ so that $\mathcal{G}_{N,M}\simeq\mathcal{G}_{P,M}$ but $(N\subset M)\not\simeq(P\subset M)$.

Appendix

A.1. A local quantization principle

The type of result that we will discuss now proves to be very useful in exploiting the noncommutative ergodic phenomena specific to the theory of type II_1 factors (see [Po1], [Po2], [Po4], [Po5], [Po10]). Such results allow obtaining from an almost invariant finite dimensional vector space $\mathcal{H}_0$, an (almost) invariant matrix algebra by taking $\mathcal{H}_0 q \mathcal{H}_0^*$, with q an "infinitesimal" projection satisfying $q\mathcal{H}_0^*\mathcal{H}_0 q\approx \mathbf{C}q$. The proof of this result is essentially contained in [Po1], [Po10] but we will give it here anyway, for the sake of completeness.

A.1.1. LEMMA. *Let $B\subset B_1$ be finite von Neumann algebras with a normal finite faithful trace τ. Let $X\subset B_1$ be a finite set of elements such that $E_{B\vee(B'\cap B_1)}(x)=0$, for all $x\in X$. Given any $\delta>0$, there exists a partition of the unity $\{p_i\}_i$ with projections in B such that*

$$\Big\|\sum_i p_i x p_i\Big\|_2<\delta,\quad \forall x\in X.$$

Proof. Step 1. We first prove that if $x\in B_1$ is so that $E_{B\vee B'\cap B_1}(x)=0$ then there exists $v\in\mathcal{U}(B)$ such that $\|vxv^*-x\|_2\geqslant\|x\|_2$. Let K be the weak closure of the set of Dixmier averagings of x, i.e., $K=\overline{\mathrm{co}}^w\{vxv^*\mid v\in\mathcal{U}(B)\}$, which is a w-compact convex subset in B_1. Since K is weakly compact, by the inferior semicontinuity of the norm $\|\cdot\|_2$ it follows that there exists an element $y_0\in K$ such that

$$\|y_0\|_2=\inf\{\|y\|_2\mid y\in K\}.$$

Since $\|\cdot\|_2$ is a Hilbert norm and K is convex, it follows that y_0 is the unique element in K with this property. But $vKv^*\in K$ for all $v\in\mathcal{U}(B)$, in particular $vy_0v^*\in K$. Since

$\|vy_0v^*\|_2=\|y_0\|_2$, by the uniqueness of y_0 it follows that $vy_0v^*=y_0$ so that $[v,y_0]=0$, for all $v\in\mathcal{U}(B)$. Thus $y_0\in B'\cap B_1$. But $E_{B\vee(B'\cap B_1)}(x)=0$ implies $E_{B\vee(B'\cap B_1)}(vxv^*)=0$, for $v\in B$, so that by weak limits $E_{B\vee(B'\cap B_1)}(y_0)=0$. This is in contradiction with $y_0\in B'\cap B_1$, unless $y_0=0$.

This proves that $0\in K$.

Suppose now that $\|vxv^*-x\|_2\leqslant\|x\|_2$ for all $v\in\mathcal{U}(B)$. Thus

$$\|vxv^*-x\|_2^2=\|vxv^*\|_2^2+\|x\|_2^2-2\operatorname{Re}\tau(x^*vxv^*)\leqslant\|x\|_2^2$$

so that $2\operatorname{Re}\tau(x^*vxv^*)\geqslant\|x\|_2^2$, for all v, and thus, by taking convex combinations and weak limits, $2\operatorname{Re}(x^*y)\geqslant\|x\|_2^2$ for all $y\in K$. In particular, for $y_0=0$, we get $0\geqslant\|x\|_2^2$ a contradiction, unless $x=0$.

Step 2. We now prove that if $x\in B_1$, $x\neq0$, is so that $E_{B\vee(B'\cap B_1)}(x)=0$ and if $\{p_i^0\}_i\subset\mathcal{P}(B)$ is a given finite partition of the unity with projections in B, then there exists a finite partition of the unity $(p_j^1)_j\subset\mathcal{P}(B)$, refining (p_i^0) such that

$$\left\|\sum p_j^1xp_j^1\right\|_2<\tfrac{3}{4}\|x\|_2.$$

To do this, we apply Step 1 to $x_1=\sum p_i^0xp_i^0$ (instead of x), $\sum p_i^0Bp_i^0$ (instead of B), by taking into account that $(\sum p_i^0Bp_i^0)'\cap B_1=\sum p_i^0(B'\cap B_1)p_i^0$ so that

$$\begin{aligned}E_{(\sum p_i^0Bp_i^0)\vee(\sum p_i^0Bp_i^0)'\cap B_1}(x_1)&=E_{\sum p_i^0(B\vee B'\cap B_1)p_i^0}(x)\\&=E_{\sum p_i^0(B\vee B'\cap B_1)p_i^0}E_{B\vee B'\cap B_1}(x)=0.\end{aligned}$$

We thus get a unitary element $v_1\in\mathcal{U}(\sum p_i^0Bp_i^0)$ such that $\|v_1x_1v_1^*-x_1\|_2>\|x_1\|_2$. Let e_i be some spectral projections of v_1 so that $\sum e_i=1$ and so that for suitable scalars λ_i, $|\lambda_i|=1$, we have $\|\sum\lambda_ie_i-v_1\|$ small enough to ensure that

$$\left\|\left(\sum\lambda_ie_i\right)x_1\left(\sum\bar\lambda_je_j\right)-x_1\right\|_2\geqslant\|x_1\|_2.$$

Using the mutual orthogonality, with respect to the scalar product given by the trace, of the elements $\{e_ix_1e_j\}_{i,j}$ and the inequality $2\geqslant|\lambda_i\bar\lambda_j-1|$, we get:

$$\begin{aligned}4\|x_1\|_2^2-4\left\|\sum_ie_ix_1e_i\right\|_2^2&=4\left\|\sum_{i\neq j}e_ix_1e_j\right\|_2^2\\&\geqslant\left\|\sum_{i\neq j}(\lambda_i\bar\lambda_j-1)e_ix_1e_j\right\|_2^2\\&=\left\|\left(\sum_i\lambda_ie_i\right)x_1\left(\sum_j\bar\lambda_je_j\right)-x_1\right\|_2^2\geqslant\|x_1\|_2^2.\end{aligned}$$

From the first and last terms of the inequality we get:

$$\left\|\sum e_i x_1 e_i\right\|_2^2 \leqslant \tfrac{3}{4}\|x_1\|_2^2.$$

Taking (p_j^1) to be the projections $(e_l p_k^0)_{l,k}$ we get

$$\left\|\sum p_j^1 x p_j^1\right\|_2^2 = \left\|\sum p_j^1 x_1 p_j^1\right\|_2^2 \leqslant \tfrac{3}{4}\|x_1\|_2^2 \leqslant \tfrac{3}{4}\|x\|_2^2.$$

Step 3. By applying recursively Step 2 k times for each $x\in X$, with k so that $(\frac{3}{4})^k<\delta^2$, we get the required partition. □

A.1.2. THEOREM. *Let $B\subset M$ be type* II_1 *factors. Given any $\varepsilon>0$ and any finite set of elements $Y\subset M$, there exists a projection $q\in B$ such that $\|qyq-E_{B'\cap M}(y)q\|_2<\varepsilon\|q\|_2$, for all $y\in Y$.*

Proof. For each $y\in Y$ let $y'=E_{B\vee(B'\cap M)}(y)$ and $y''=y-y'$.

Let $\varepsilon>6\operatorname{card}Y\delta_0>0$. For each y' as above there exists some finite number of elements $b\in B$, $b'\in B'\cap M$ such that

$$\left\|y'-\sum bb'\right\|_2<\delta_0. \tag{0}$$

Denote by S the finite set of all such elements b, corresponding to all y' coming from $y\in Y$.

We claim that there exists a separable subfactor $B_0\subset B$ such that $S\subset B_0$. Indeed, by Dixmier's theorem, given any countable set $T\subset B$, there exists a countable set of unitaries $\mathcal{U}(T)$ such that $\tau(y)1\in\overline{\mathrm{co}}^w\{vyv^*\mid v\in\mathcal{U}(T)\}$, for all $y\in T$. Thus, by defining recursively $T_0=S\cup S^*$, $T_{i+1}=\mathcal{U}(T_i)\cup\mathcal{U}(T_i)^*$ and by taking $B_0=\mathrm{vN}(\bigcup_i T_i)$, it follows that B_0 has the Dixmier property and thus has a unique trace. Thus B_0 is a factor and by construction $S\subset B_0$ and B_0 is separable (having a faithful trace and being countably generated).

But if B_0 is a separable type II_1 factor then we claim that there exists a hyperfinite subfactor $R\subset B_0$ such that $R'\cap B_0=\mathbf{C}$ (cf. [Po4]). Indeed, if $\{x_n\}_n$ is a countable subset of B_0, dense in B_0 in the norm $\|\cdot\|_2$, then one constructs recursively mutually commuting matrix subalgebras $N_i\simeq M_{k_i\times k_i}(\mathbf{C})$, $i\geqslant 1$ in B_0 such that, if $M_n=N_1\vee\ldots\vee N_n$, then

$$\|E_{M_n'\cap B_0}(x_i)-\tau(x_i)1\|_2<2^{-n},\quad 1\leqslant i\leqslant n.$$

For suppose we made the construction up to n. Let $A\subset M_n'\cap B_0$ be a maximal abelian subalgebra, so that $A'\cap(M_n'\cap B_0)=A$. By taking an "approximation" of A by finite dimensional subalgebras, it follows that there exists an abelian finite dimensional subalgebra $A_0\subset A$ such that

$$\|E_{A_0'\cap(M_n'\cap B_0)}(E_{M_n'\cap B_0}(x_i))-E_{A_0}(E_{M_n'\cap B_0}(x_i))\|_2<2^{-n-1}$$

for all $1 \leqslant i \leqslant n+1$, and so that all the minimal projections of A_0 have the same trace. Let then N_{n+1} be a matrix subalgebra of $M_n' \cap B_0$ having A_0 as a diagonal. Then we have for $M_{n+1} = M_n \vee N_{n+1}$,

$$\begin{aligned}\|E_{M_{n+1}' \cap B_0}(x_i) - \tau(x_i)1\|_2 &= \|E_{N_{n+1}' \cap B_0}(E_{A_0' \cap M_n' \cap B_0}(E_{M_n' \cap B_0}(x_i))) - \tau(x_i)1\|_2 \\ &\leqslant \|E_{N_{n+1}' \cap B_0}(E_{A_0}(E_{M_n' \cap B_0}(x_i))) - \tau(x_i)1\|_2 + 2^{-n-1} = 2^{-n-1}.\end{aligned}$$

Now, by taking $R = \overline{\bigcup M_n}^w$ it follows that $E_{R' \cap B_0}(x_i) = \tau(x_i)1$, for all i, so that, by density $E_{R' \cap B_0}(x) = \tau(x)$, for all $x \in B_0$.

Let now R be represented as an infinite tensor product $R = \overline{\bigotimes}_{g \in S_\infty}(M_{2\times 2}(\mathbf{C}))_g$ with the countable index set S_∞ being the infinite symmetric group (the finite permutations of $\{1, 2, \ldots\}$). It follows that S_∞ acts on R by shifting the indices, i.e., for $g \in S_\infty$, $\bigotimes x_h \in R$, we put $\sigma(g)(\bigotimes_h x_h) = \bigotimes_h x_{gh}$. This action is easily seen to be ergodic, i.e., if $\sigma(g)(x) = x$ for all $g \in S_\infty$ then $x \in \mathbf{C}1$.

Let $R_n = R^{S_n}$ be the fixed point algebra of R under the restriction of this action σ to $S_n \subset S_\infty$. Clearly $\bigcap_n R_n = \mathbf{C}1$ (by the ergodicity of σ) and $R_n' \cap R = \mathbf{C}$ for each n (since the action of S_n by σ is properly outer). From $R_n \downarrow \mathbf{C}1$ it follows that

$$\lim_n \|E_{R_n}(x) - \tau(x)1\|_2 = 0, \quad \forall x \in M.$$

Thus, for n_0 large enough,

$$\|E_{R_{n_0}}(b) - \tau(b)1\|_2 < \delta, \quad \forall b \in S. \tag{1}$$

Since $R_{n_0}' \cap R = \mathbf{C}$, by A.1.1 there exists a finite dimensional abelian subalgebra $A_1 \subset R_{n_0}$ such that

$$\|E_{A_1' \cap R}(E_R(b) - E_{R_{n_0}}(b))\|_2 < \delta, \quad \forall b \in S. \tag{2}$$

Moreover, since $S \subset B_0$ and $R' \cap B_0 = \mathbf{C}$, by A.1.1 there exists a finite dimensional abelian subalgebra $A_2 \subset R$, with $A_2 \supset A_1$, such that

$$\|E_{A_2' \cap B_0}(b - E_R(b))\|_2 < \delta, \quad \forall b \in S. \tag{3}$$

Putting together (1), (2), (3) we get:

$$\begin{aligned}\|E_{A_2' \cap B}(b) - \tau(b)1\|_2 &= \|E_{A_2' \cap B_0}(b) - \tau(b)1\|_2 \\ &\leqslant \delta + \|E_{A_2' \cap B_0}(E_R(b)) - \tau(b)1\|_2 \\ &= \delta + \|E_{A_2' \cap R}(E_R(b) - E_{R_{n_0}}(b)) + E_{A_2' \cap R}(E_{R_{n_0}}(b) - \tau(b)1)\|_2 \\ &\leqslant 2\delta + \|E_{A_1' \cap R}(E_R(b) - E_{R_{n_0}}(b))\|_2 \leqslant 3\delta.\end{aligned}$$

Now, going back to (0), it follows that if δ is chosen from the beginning very small (e.g. $\delta<\delta_0(4\sum\|b'\|+1)^{-1}$ will do), then we get:

$$\begin{aligned}\|E_{A_2'\cap M}(y')-E_{B'\cap M}(y')\|_2 &\leqslant 2\delta_0+\left\|E_{A_2'\cap M}\left(\sum bb'\right)-E_{B'\cap M}\left(\sum bb'\right)\right\|_2\\ &\leqslant 2\delta_0+\sum\|b'\|\|E_{A_2'\cap M}(b)-\tau(b)1\|_2\\ &\leqslant 2\delta_0+3\delta\sum\|b'\|<\varepsilon/2\operatorname{card}Y.\end{aligned}$$

Finally, by applying A.1.1 to the inclusion $B\subset M$ and to the finite set of elements y'' (which are orthogonal to $B\vee(B'\cap M)$ by construction) we get a refinement $A_3\subset B$ of A_2, i.e., A_3 abelian, finite dimensional, such that

$$\|E_{A_3'\cap M}(y'')\|_2<\varepsilon/2\operatorname{card}Y,\quad \forall y''.$$

Thus we get

$$\|E_{A_3'\cap M}(y)-E_{B'\cap M}(y)\|_2<\varepsilon/\operatorname{card}Y$$

for all $y\in Y$. If $\{q_i\}$ are the minimal projections of A_3 then this yields:

$$\begin{aligned}\sum_i\|q_i(y-E_{B'\cap M}(y))q_i\|_2^2 &=\left\|\sum_i q_i(y-E_{B'\cap M}(y))q_i\right\|_2^2\\ &=\|E_{A_3'\cap M}(y)-E_{B'\cap M}(y)\|_2^2<(\varepsilon/\operatorname{card}Y)^2\\ &=\sum_i(\varepsilon/\operatorname{card}Y)^2\|q_i\|_2^2.\end{aligned}$$

So, there exists some $q=q_i$ for which

$$\|qyq-E_{B'\cap M}(y)q\|_2<\varepsilon\|q\|_2,\quad \forall y\in Y. \qquad \square$$

The importance of A.1.2 will become clear through the following consequence, that can be regarded as a local quantization principle, as it gives the possibility of obtaining local algebras from vector spaces:

A.1.3. COROLLARY. *Let $B\subset M$ be type* II_1 *factors and assume $B'\cap M$ is finite dimensional. Let $\mathcal{H}_0\subset M$ be a finite dimensional vector space such that $\mathcal{H}_0$ is a $B'\cap M$ right module, i.e., $\mathcal{H}_0B'\cap M=\mathcal{H}_0$, and such that it is in fact a free $B'\cap M$ module, i.e., $\mathcal{H}_0\simeq(B'\cap M)^n$ as modules, for some n. Given any $\varepsilon>0$ there exists a projection $q\in B$ such that $\mathcal{H}_0q\mathcal{H}_0^*$ is $\varepsilon\tau(q)$-close to a finite dimensional algebra B_0 of the form $M_{n\times n}(B'\cap M)$.*

Proof. Let $x_i\in\mathcal{H}_0$ be so that $\mathcal{H}_0=\sum_{i=1}^n x_iB'\cap M$. Then, by the Gram–Schmidt process, replacing x_1 by $x_1E_{B'\cap M}(x_1^*x_1)^{-1/2}$, we may assume $E_{B'\cap M}(x_1^*x_1)=1$ and more generally that $E_{B'\cap M}(x_i^* x_j)=\delta_{ij}$. Apply A.1.2 to $\{x_i^* x_j\}_{i,j}$ and to $\delta>0$ to get a projection $q\in B$ such that $\|qx_i^* x_j q-\delta_{ij}q\|_2<\delta\|q\|_2$. By using the perturbation result proved next, one can then find partial isometries $(v_i)_i\subset M$ such that

$$v_i^* v_j = \delta_{ij} q,$$
$$\|x_i q - v_i\|_2 < f(\delta)\|q\|_2$$

where $f(\delta)\to 0$ when $\delta\to 0$. By taking $B_0=\sum_{i,j} v_i B'\cap M v_j^*$ and δ very small, the statement follows. □

Let us finally point out a more general similar result, that can be obtained in the nonfactorial case:

A.1.4. THEOREM. *Let $B\subset M$ be type II_1 von Neumann algebras (not necessarily factors). Let $x_1, ..., x_n \in M$ and $\varepsilon>0$. There exists a projection $q\in B$ such that*

$$\|qx_i q - E_{B'\cap M}(x)q\|_2 < \varepsilon\|q\|, \quad 1\leqslant i\leqslant n.$$

Proof. The only change that has to be made in the proof of Theorem A.1.2 to get this nonfactorial version is the proof of the fact that given a finite set of elements Y in B and $\delta>0$ there exists a partition of the unity $\{p_i\}$ in B such that $\|\sum p_i y p_i - \tau(y)\|_2<\delta$. The first step in proving this was the construction of a hyperfinite subfactor $R\subset B$ in any separable type II_1 factor B such that $R'\cap B_0=\mathbf{C}$. The second step consisted in constructing a decreasing sequence of type II_1 subfactors $R_n\subset R$ with $R_n\downarrow\mathbf{C}$, $R_n'\cap R=\mathbf{C}$. This second step does not depend on B, M or B_0 being factors. The first step has to be changed into the existence of a hyperfinite factor R with $R'\cap B_0=\mathcal{Z}(B_0)$. The existence of such an R is proved in [Po11]. The rest of the proof is identical. □

A.2. A perturbation result

We will prove now a perturbation result needed when applying Theorem A.1.2 (cf. A.1.3). Namely, that if $\{x_{ij}\}$ is a finite set of elements that almost satisfies the axioms of a matrix unit then $\{x_{ij}\}$ is close to a matrix unit. Results of this type appeared first in the work of Murray and von Neumann, in the context of von Neumann algebras and in the work of Glimm, in the context of C^*-algebras. It has become since then a standard technique in operator algebra.

A.2.1. LEMMA. *Let $\varepsilon>0$ and $(y_i)_{1\leqslant i\leqslant n}$ a finite set of elements in the type II_1 factor M, with $n\geqslant 1$ and $\varepsilon\leqslant 1/n$. Assume*

$$\|y_i^* y_j - \delta_{ij} q\|_2 < \varepsilon\|q\|_2$$

for all $1 \leqslant i, j \leqslant n$ *and some projection* q.

Then there exist partial isometries $(v_i)_{i \leqslant i \leqslant n}$ *such that if we denote by*

$$\alpha_0 = 3\max\{1, \|y_1\|, \dots, \|y_n\|\}$$

and we put $\beta_0 = \alpha_0^4(\alpha_0^2+1)^{n-2}$, *then*:

$$v_i^* v_j = \delta_{ij} q,$$
$$\|v_i - y_i\|_2 \leqslant \beta_0 \varepsilon \|q\|_2 \quad \textit{for all } 1 \leqslant i \leqslant n.$$

Proof. Assume first $n=1$ and put $\varepsilon_1 = \varepsilon$. By taking $y_1 q$ instead of y_1 we may assume that $s(y_1^* y_1) \leqslant q$. If $1 > \alpha > 0$ then the spectral projection of $y_1^* y_1$ corresponding to the interval $(1-\alpha, 1+\alpha)$, $p = E_{(1-\alpha, 1+\alpha)}(y_1^* y_1)$ will satisfy

$$(1-\alpha)^2 \tau(q-p) \leqslant \|y_1^* y_1 - q\|_2^2 \leqslant \varepsilon^2 \|q\|_2^2 = \varepsilon^2 \tau(q).$$

Let $v_1^1 = y_1 (y_1^* y_1)^{-1/2} E_{(1-\alpha, 1+\alpha)}(y_1^* y_1)$. Then we have $v_1^{1*} v_1^1 = p$ and the estimates

$$\begin{aligned} \|v_1^1 - y_1\|_2^2 &= \tau(p) + \tau(y_1^* y_1) - 2\tau(|y_1| p) \\ &\leqslant \tau(p) + \tau(y_1^* y_1) - 2\tau(p)(1-\alpha)^{1/2} \end{aligned}$$

so that by the Cauchy-Schwartz inequality we get

$$\begin{aligned} &\leqslant \tau(p) + \tau(q) + \tau((y_1^* y_1 - q) q) - 2\tau(p)(1-\alpha)^{1/2} \\ &\leqslant \tau(p) + \tau(q) + \varepsilon^2 \tau(q) - 2\tau(p)(1-\alpha)^{1/2} \\ &\leqslant 2\tau(p)(1-(1-\alpha)^{1/2}) + \varepsilon^2(1+(1-\alpha)^{-2})\tau(q) \\ &\leqslant \beta \tau(q) \end{aligned}$$

where $\beta = 2(1-(1-\alpha)^{1/2}) + \varepsilon^2(1+(1-\alpha)^{-2})$. So, by taking $\alpha = \varepsilon^2$, we get $\beta \leqslant 2\varepsilon^2 + 2\varepsilon^2 = 4\varepsilon^2$. Thus, if we take v_1 to be any partial isometry which extends v_1^1 and $y_1(y_1^* y_1)^{-1/2}$ to all q, then

$$\begin{aligned} \|v_1 - y_1\|_2^2 &= \|v_1^1 - y_1 p\|_2^2 + \|(v_1 - y_1)(q-p)\|_2^2 \\ &\leqslant 5\varepsilon^2 \tau(q) + \tau(q-p) + \tau(y_1^* y_1 (q-p)) \\ &\leqslant (7\varepsilon^2 + 2\varepsilon^2)\tau(q) = 9\varepsilon^2 \tau(q). \end{aligned}$$

This proves the statement for $n=1$.

Assume we proved that there exist partial isometries $v_1, v_2, \dots, v_k$ such that

$$v_i^* v_j = \delta_{ij} q, \quad 1 \leqslant i, j \leqslant k,$$
$$\|v_i - y_i\|_2 \leqslant \varepsilon_i \|q\|_2,$$

for some $\varepsilon_i>0$, $1\leqslant i\leqslant k$, where $\varepsilon_1=3\varepsilon$. Let $y^1_{k+1}=(1-\sum_{i=1}^k v_iv_i^*)y_{k+1}$. Note that we have:

$$\begin{aligned}\|y^{1*}_{k+1}y^1_{k+1}-q\|_2 &\leqslant \varepsilon\|q\|_2+\sum_{i=1}^k \|y^*_{k+1}v_iv_i^*y_{k+1}\|_2\\ &\leqslant \varepsilon\|q\|_2+\|y_{k+1}\|\sum_i \|y^*_{k+1}v_i\|_2\\ &\leqslant \varepsilon\|q\|_2+\|y_{k+1}\|\Big(\sum_i \|y^*_{k+1}(v_i-y_i)\|_2+\sum_i\|y^*_{k+1}y_i\|_2\Big)\\ &\leqslant \varepsilon\|q\|_2+\|y_{k+1}\|^2\|q\|_2\sum_{i=1}^k\varepsilon_i+k\varepsilon=\varepsilon^1_{k+1}\|q\|_2.\end{aligned}$$

By the proof of step $n=1$ it follows that we can find first v^1_{k+1} so that $v^{1*}_{k+1}v^1_{k+1}\leqslant q$ and $v^{1*}_{k+1}v_i=0$ and

$$\|v^1_{k+1}-y^1_{k+1}\|_2\leqslant 2\varepsilon^1_{k+1}\|q\|_2.$$

Since $\tau(1-\sum_{i=1}^k v_iv_i^*)\geqslant(n-k)\tau(q)$ (because $\tau(q)<1/n$) it follows that we can extend v^1_{k+1} to a partial isometry v_{k+1} such that $v^*_{k+1}v_{k+1}=q$ and $v_{k+1}v^*_{k+1}\leqslant 1-\sum_{i=1}^k v_iv_i^*$ and such that $v^*_{k+1}y^1_{k+1}\geqslant 0$. By the step $n=1$ we have

$$\|v_{k+1}-y^1_{k+1}\|_2\leqslant 3\varepsilon^1_{k+1}\|q\|_2.$$

Thus we get

$$\begin{aligned}\|v_{k+1}-y_{k+1}\|_2 &\leqslant 3\varepsilon^1_{k+1}\|q\|_2+\sum_i\|v_i^*y_{k+1}\|_2\\ &\leqslant 3\varepsilon^1_{k+1}\|q\|_2+k\varepsilon\|q\|_2+\|y_{k+1}\|\sum_{i=1}^k\varepsilon_i\|q\|_2\\ &=\Big(4(k+1)\varepsilon+(3\|y_{k+1}\|^2+\|y_{k+1}\|)\sum_{i=1}^k\varepsilon_i\Big)\|q\|_2\\ &\leqslant(3\max\{\|y_{k+1}\|,1\})^2\sum_{i=1}^k\varepsilon_i\|q\|_2=\varepsilon_{k+1}\|q\|_2.\end{aligned}$$

By induction, it follows that if we put $\alpha_0=3\max\{1,\|y_1\|,\ldots,\|y_n\|\}$ then $\varepsilon_1\leqslant\alpha_0^2\varepsilon$, $\varepsilon_2\leqslant\alpha_0^2\varepsilon_1\leqslant\alpha_0^4\varepsilon$, ..., $\varepsilon_{k+1}\leqslant\alpha_0^2\sum_{i=1}^k\varepsilon_i\leqslant\alpha_0^4(\alpha_0^2+1)^{k-1}\varepsilon$ and the statement follows. □

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