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Contact geometry and CR structures on S^3

by

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I. Introduction

1.

One of the most beautiful theorems in one complex variable is the Riemann mapping theorem: Any simply connected open set which is a proper subset of $\mathbf{C}^1$ is biholomorphically equivalent to the unit disc. Moreover, the biholomorphism can be determined either from knowledge of the Green's function for the Laplacian of the domain (vis-à-vis the electric potential for a charged plate), or from the complete metric of constant negative curvature (via the exponential map). This theorem is a beautiful example of the intimate relationship between the complex analysis, function theory, and the geometry of invariant metrics.

One of the quests in several complex variables is to determine how this theorem generalizes. In a ground-breaking paper [L1], Lempert established fundamental results concerning the Kobayashi metric for strongly convex domains in $\mathbf{C}^n$ which again intimately connected the complex analytic properties of a domain with canonical maps from the unit ball to the domain via the exponential map for the Kobayashi metric and the plurisubharmonic Green's function. In [BD1], these results were used to describe and parameterize the moduli space of pointed strongly convex domains up to biholomorphic equivalence. (The results mentioned here will be elaborated upon later in the introduction.)

One new feature which arises in several complex variables is that much of the analysis for a domain can be reduced to analysis on the boundary of the domain. More precisely, the complex structure from $\mathbf{C}^n$ restricts to the boundary of a strongly convex domain to define a CR structure on the boundary. (Once again, definitions and more complete descriptions of these ideas will be provided later in the introduction.) Two strongly

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convex domains in $\mathbf{C}^n$ are biholomorphically equivalent if and only if their boundaries are CR equivalent. On the other hand, the CR structure can be described as an intrinsic structure on the boundary. This immediately raises the imbeddability question: Which (strongly pseudoconvex) CR structures on a $(2n+1)$-dimensonal manifold M can be realized as the boundaries of strongly convex domains in $\mathbf{C}^{n+1}$. It is well known that if $n \geqslant 2$, then they can all be realized as the boundaries of some open complex manifolds, while for $n=1$, that is not the case [N].

Returning to the question of biholomorphic equivalence of domains, the fact that it can be reduced to a question of the CR equivalence of their boundaries indicates that there should be appropriate analogues of the Riemann mapping theorem, the plurisubharmonic Green's function, and the Kobayashi metric which rely completely upon the intrinsic geometry of the boundary. Moreover, if these analogues are 'correct', then they should shed light upon the imbeddability question.

One of the purposes of this paper is to indicate a generalization of the Riemann mapping theorem to the space of abstract CR manifolds which are small perturbations of the standard CR structure on the unit sphere in $\mathbf{C}^2$. The main technique will be to study the interplay between contact geometry and CR geometry, and to use this interplay to obtain a normal form for the CR structure on the manifold. The analysis will effectively intertwine several different objects—the complex analytic structure of the domain with the CR structure on the boundary, the Kobayashi metric on the domain with a canonical foliation of the boundary by circles, the plurisubharmonic Green's function for the domain with a normalized choice for a contact form on the boundary, a Riemann mapping theorem with the structure of a complex line bundle over $\mathbf{P}^1$, and the moduli space for convex domains with a normal form for CR structures on the boundary.

All of the results contained in this paper generalize to higher dimensions. Most of them can be pushed much farther than small perturbations of the standard CR structure for the sphere. However, the purpose of this paper is to set down as clearly as possible the approach to the problem, and to indicate how this approach intertwines such varied objects as described in the previous paragraph. To achieve this purpose, we have for the most part narrowed our focus to small perturbations of the standard CR structure on S^3. (The three dimensional case has the added interest of addressing the question of global obstructions to the imbeddability of CR structures.) However, we have tried to introduce as many of the crucial ideas as possible. In a forthcoming paper, we will indicate the modifications necessary to extend this approach to higher dimensions.

Organization of the paper. This paper will contain a rather lengthy introduction. The intent of this introduction is to introduce rather carefully all of the major concepts and structures required throughout the paper, and to indicate how this solution to the equivalence problem for the space of CR structures effectively ties together such varied objects as described above. It is also our hope to make the relevant sections readable for those who are expert in one area without requiring knowledge in the remaining areas.

The remainder of the paper will proceed as follows. Chapter II will contain the necessary preliminary material, setting up the basic notation, introducing the various operators and recalling the basic facts from the Hodge theory on S^3. Chapter III will then introduce a linear structure on the space of diffeomorphisms of S^3, and describe an 'integrability condition'—the condition that the diffeomorphism corresponds to a contact diffeomorphism; this integrability condition is a nonlinear PDE which the vector field parameterizing the diffeomorphism must satisfy. Chapter IV will study the solution space to this PDE, and show that if an anisotropic Sobolev space structure is placed on the full group of diffeomorphisms, then the solution space forms a Hilbert submanifold; that is, the space of contact diffeomorphisms which are sufficiently near the identity admits an anisotropic Sobolev space structure—one which considers L^2 estimates only on those derivatives in directions which are tangent to the contact distribution. Chapter V considers the action of the contact diffeomorphisms on the CR structure, and shows that the contact diffeomorphism group can be used to place the CR structure in various normal forms. These results basically follow from writing down the action at a linearized level, obtaining the normal form at the linearized level, and concluding that the nonlinear results holds in a neighbourhood by the implicit function theorem for Banach spaces. Chapter VI contains the basic imbedding results, and discusses the geometry of the situation. In this chapter, we discuss a more general situation, and try to indicate that the basic ingredient which is necessary for the analysis in this paper is a strongly pseudoconvex CR manifold for which the underlying contact structure admits a S^1 action.

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2. Outline of the results

A *contact structure* on S^3 is a codimension one subbundle of the real tangent space satisfying a nondegeneracy condition best described as follows. Let η be a 1-form dual to this distribution. Then the hyperplane distribution is a *contact distribution* if $\eta \wedge d\eta$ is a non-vanishing volume form. Thus, the hyperplane distribution is a contact distribution if it is as far from being integrable as possible. The form η is said to be a *contact form*. Notice that η is only defined up to multiple by a nonvanishing function.

A *CR structure* on S^3 is an 1-dimensional subbundle $H_{(1,0)}$ of the complexified tangent space such that $H_{(1,0)} \oplus \overline{H}_{(1,0)}$ is a subbundle of complex codimension one; in this case, the intersection of this subbundle with the real tangent space to S^3 is a real codimension one subbundle. We set $H_{(0,1)} := \overline{H}_{(1,0)}$. Let T be a real globally defined transverse vector field to this distribution. Then the CR structure is said to be *nondegenerate* if for any nonzero $Z \in H_{(1,0)}$, the bracket $[Z, \overline{Z}] = -i\lambda T \pmod{Z, \overline{Z}}$ for some nonvanishing function λ, and it is said to be *strongly pseudoconvex* if this function λ is positive.

It follows immediately from the definitions that a CR structure defines in a natural way a hyperplane distribution of the real tangent space, and that this hyperplane distribution is a contact distribution (fully nonintegrable) precisely when the CR structure is nondegenerate. Indeed, let η be a real 1-form dual to the hyperplane distribution $H_{(1,0)} \oplus H_{(0,1)}$; then the nondegeneracy implies that $\eta \wedge d\eta$ is nonvanishing. In this sense, a strongly pseudoconvex CR structure can be thought of as a contact structure together with a smoothly varying complex (or conformal) structure on the hyperplane sections.

This is the approach in this paper. Consider a CR structure to be described via a two step procedure. First, define a hyperplane distribution on M—that is, the codimension one subbundle of the complexified tangent bundle which consists of the holomorphic tangent space and its complex conjugate; specifying this distribution is equivalent to specifying a nonvanishing real one form η which is dual to it. (Recall that the strong pseudoconvexity of the CR structure guarantees that the hyperplane distribution is fully non-integrable; that is, it is a contact distribution. In terms of the dual one form η, this is the condition that $\eta \wedge d\eta \neq 0$.) Second, on each hyperplane in the distribution, specify the splitting into the holomorphic and the conjugate holomorphic directions.

The main technique in the paper is to use contact geometry and the analysis associated to the $\bar{\partial}_b$ operator to obtain a normal form for the pair consisting of a contact structure and a conformal structure on the contact distribution. This is achieved as follows:

Step I. A well-known result from contact geometry [G] states that any two nearby contact structures are equivalent (via a diffeomorphism which may change the contact form.) Since we are interested in the space of CR structures up to equivalence, we may as well fix once and for all the underlying contact structure. This relatively simple normalization has the property that it immediately simplifies much of the remaining analysis, and repeatedly does so at several different stages.

(a) This normalization immediately reduces the remaining action of the diffeomorphism group to action of the group of *contact diffeomorphisms* (those which fix the contact structure). This is not only a much smaller group, it is also much better behaved.

(b) Fixing the contact structure, and restricting attention to small deformations of the standard CR structure which have the same underlying contact structure allows for a particularly simple representation of the space of CR structures in terms of deformation tensors. Indeed, let e be a local section of the standard holomorphic tangent space. Then a local section of the deformed holomorphic tangent space can be taken to be of the form $\hat{e}=e-\bar{\phi}(e)$, where $\phi\in\mathrm{Hom}(H_{(0,1)},H_{(1,0)})$. (Notice that we have used the conjugate of the deformation tensor in the defining equation in order to agree with standard deformation theory—in which case the deformation tensor is considered to be a vector valued $(0,1)$ form.)

(c) Specifying a contact form fixes a splitting of the tangent space necessary to make the $\bar{\partial}_b$ operator well defined. Since all of the analysis will be done using the initial structure and its associated $\bar{\partial}_b$ operator, the analysis always uses the same contact form and the same splitting.

(d) The anisotropic Folland–Stein spaces which are adapted to the $\bar{\partial}_b$ analysis are fully commensurate with the underlying contact structure, and the contact diffeomorphisms preserve these Folland–Stein spaces.

Step II. In order to understand the action of the space of contact diffeomorphisms on the space of CR structures, we first introduce natural Banach space structures on the various spaces of objects. The space of CR structures already has a natural linear structure when it is represented as the space of deformation tensors. The space of contact diffeomorphisms can be given a natural linear structure in various ways; however, since we will eventually be using $\bar{\partial}_b$ analysis in order to normalize the CR structure, we will require a Banach space structure on the space of contact diffeomorphisms which uses the weighted (or anisotropic) Sobolev spaces referred to as Folland–Stein spaces (coming from the context of $\bar{\partial}_b$ geometry [FS]). Notice that these spaces are also 'natural' in the context of contact geometry, since they are precisely the spaces which are preserved under contact diffeomorphisms.

Step III. We show that the space of contact diffeomorphisms can be parameterized by a single real valued function p on S^3. Using this parameterization, the linearization at the origin of the action of the contact diffeomorphisms on the CR structures defined by the deformation tensors is given by

$$\phi\mapsto\phi+\bar{\partial}_b{}^{\#}\bar{\partial}_b p \tag{2.1}$$

where $^{\#}$ is the inverse to the operator $\iota\colon H_{(1,0)} \to H^{(0,1)}$ defined by $Z \mapsto (Z \lrcorner d\eta)$. (Here $H^{(0,1)}$ is defined as in equation (6.1) by the splitting

$$T^*_{\mathbf{C}}(S^3) = \mathbf{C}\cdot\eta \oplus H^{(1,0)}(S^3) \oplus H^{(0,1)}(S^3),$$

where $d\eta$ is contained in the wedge of the last two factors.) By using an inverse mapping theorem in Banach spaces, we show that the CR structure can be normalized to lie in a complementary subspace to the image of the operator $\bar{\partial}_b{}^{\#}\bar{\partial}_b$ applied to a real valued function.

Normal forms for the CR structure. In order to normalize the CR structure, we use the fact that the underlying CR structure admits a natural S^1 action. This is the circular action induced by the standard imbedding of S^3 as the unit sphere in $\mathbf{C}^2$, and it is generated by the vector field dual to the standard contact form on S^3. This S^1 action on S^3 induces an action on the function spaces and the full tensor algebra of S^3. We use this action to decompose the tensor algebra according to its Fourier components, and express the normal forms in terms of the vanishing of certain of the Fourier components.

(Before continuing, we should briefly mention two interpretations of these Fourier coefficients. Complex analytically, any function—or tensor—can be restricted to the boundary of any complex line which passes through the origin. This is the boundary of a unit disc, and the Fourier components restricted to the boundary of this disc are the standard Fourier components; in particular, any data with no negative coefficients on the boundary of this disc admits a holomorphic extension to the entire disc. Geometrically, the sphere can be interpreted as the unit sphere bundle of the tautological line bundle over $\mathbf{P}^1$; the fibres of this bundle correspond to the boundaries of the complex discs referred to above. In this case, functions restrict to any fibre as a function on a unit circle in a complex line, and the Fourier decomposition again agrees with the one dimensional version. Data with no negative Fourier components admits an extension to the entire unit disc bundle over $\mathbf{P}^1$—a complex manifold—in such a way that it is holomorphic in the fibre directions.)

Since the contact diffeomorphism is parameterized by a real valued function, it is completely determined by either its negative Fourier coefficients or its positive Fourier coefficients. (Notice that we are being a little sloppy here in regards to the zeroth—or S^1 invariant—coefficient; we have to treat this with special care in the paper.) Furthermore, the $\bar{\partial}_b{}^{\#}\bar{\partial}_b$ operator respects the Fourier decomposition. Thus we can normalize either the negative or the positive Fourier components of the CR structure. If we attempt to normalize the negative coefficients to be zero, we find that there is an infinite dimensional obstruction; this obstruction corresponds to CR structures which do not bound convex domains. If these bad negative coefficients vanish, then it is easy to conclude that the deformation tensor extends holomorphically to define a complex manifold of which the CR manifold is the boundary. On the other hand, if we normalize the positive coefficients, we find that there is no obstruction, and that we can always make the deformation tensor have only negative Fourier coefficients. In this case, the CR structure extends holomorphically to the exterior of the unit circles (or, in terms of the dual bundle, holomorphically to the interior) to define a complex manifold for which the CR structure is the pseudoconcave boundary.

3. Background results—complex analysis

The Kobayashi metric. The *infinitesimal Kobayashi metric* at a point $p \in D$ assigns a length to each tangent vector $v \in T_pD$ as follows:

$$\|v\| := (\sup\{\lambda \mid f\colon \triangle \to D \text{ is holomorphic; } f(0)=p,\ f'(0)=\lambda v\})^{-1}$$

where $\triangle$ is the unit disc in $\mathbf{C}^1$. The *indicatrix* for the Kobayashi metric at the point $p \in D$ is the sublevel set in T_pD of the infinitesimal metric corresponding to all vectors of Kobayashi length less than one. This is a circular domain in the tangent space at p.

In [L1], Lempert showed that for a strongly convex domain D, the infinitesimal Kobayashi metric defines a *Finsler* metric on D (that is, it restricts to the tangent space T_pD at any point $p \in D$ as a norm), and that the appropriately renormalized exponential map at any point $p \in D$ is a homeomorphism from the indicatrix B_p onto the domain D, and a diffeomorphism away from the origin. (Recall that the exponential map for a metric is a map from the tangent space to the domain which takes straight lines through the origin to geodesics—distance minimizing curves. The appropriate normalization and invariant description of this map was due to Patrizio [P].) Furthermore, the restriction of this map to any complex line through the origin is holomorphic, and an isometry relative to the Kobayashi metric on the indicatrix (thought of as a circular domain inside the tangent space T_pD with its natural complex structure) and the Kobayashi metric on the domain D. This map is called the *circular representation*,

$$\Psi_p\colon B_p \to D$$

between the indicatrix and the domain. This result is a natural generalization of the Riemann mapping theorem to the class of strongly convex domains in $\mathbf{C}^n$.

The plurisubharmonic Green's function. One possible generalization of the harmonic Green's function from one complex variable to several variables is known as the *plurisub-*

harmonic Green's function. It is defined as the function u_p which satisfies the homogeneous Monge–Ampère equation

$$\begin{cases} u \text{ is plurisubharmonic} & \text{in } D, \\ (\partial\bar{\partial}u)^n = 0 & \text{in } D\backslash\{p\}, \\ u = 0 & \text{on } \partial D, \\ u(z) = \log|z-p| + O(1) & \text{as } z \to p. \end{cases}$$

In the same paper [L1], Lempert showed that if $\delta_p: D \to \mathbf{R}$ denotes the Kobayashi distance from the point p and τ_p denotes the real valued function on D defined by the formula

$$\tau_p(q) := \tanh^2(\delta_p(q)), \tag{3.1}$$

then the function $\log(\tau_p)$ is smooth away from p, and satisfies the homogeneous Monge–Ampère equation with logarithmic singularity at p. This indicates that the plurisubharmonic Green's function with logarithmic singularity at p is naturally determined by the Kobayashi distance from p.

Conversely, the behaviour for the Kobayashi metric centred at p (and consequently, the Riemann map) can be completely determined by the plurisubharmonic Green's function u_p. First, it is clear that the Kobayashi distance from p is determined from u_p by using the relation (3.1); more is true, though. Since $\partial\bar{\partial}u$ is a closed two form of constant rank $n-1$, the two dimensional distribution on the tangent space which is annihilated by this form is integrable, and the integral submanifolds of this distribution are complex curves which correspond to the geodesics for the Kobayashi metric. Since there is a canonical Poincaré metric determined on each of these curves, the Riemann map centred at p is again completely determined by the function u_p.

Finally, we should note that the Riemann map pulls back the Green's function from the domain D to the Green's function for the circular domain B_p; in the case of the circular domains, the Green's function is the same as the logarithm of the Kobayashi norm on T_pD.

The moduli space. Since the Kobayashi metric is a biholomorphic invariant of the domain, the circular representation is a biholomorphic invariant of the pair (D,p) and can be used to construct moduli for the domain. A pair of pointed domains (D,p) and (D',p') are said to be *equivalent* if there is a biholomorphism $f: D \to D'$ with $f(p)=p'$. Because the Kobayashi metric is a biholomorphic invariant, the linear equivalence $B_p \cong B'_p$ follows and there is a commutative diagram

$$\begin{array}{ccc} B_p & \xrightarrow{\Psi_p} & D \\ {\scriptstyle df_p}\downarrow & & \downarrow{\scriptstyle f} \\ B'_p & \xrightarrow{\Psi'_p} & D. \end{array} \tag{3.2}$$

Thus, the pointed domains are equivalent if and only if the equivalence factors through a linear equivalence of their circular representations.

The above observations lead to a natural construction of a moduli space for pointed domains up to biholomorphic equivalence. First, we use the circular representation to pull back the complex structure from the domain D, and represent it as a deformation of the complex structure on the circular domain B_p; we refer to the circular domain with this deformed complex structure as the *circular model*. Then, two pointed domains will be biholomorphically equivalent if and only if their circular models are linearly equivalent. The description of the moduli space is thus reduced to describing the moduli space of circular models. The power in this approach lies in the fact that the space of circular models admits a very elegant description, and it can be effectively parameterized. (See [BD1] for details.)

Restriction to the boundary. If S^3 is differentiably imbedded as the boundary of a strongly convex set in $\mathbf{C}^2$, then the complex structure from $\mathbf{C}^2$ restricts to the image of S^3 to define a one complex dimensional subbundle of the complexified tangent bundle to the image—a CR structure. Furthermore, since the image is the boundary of a strongly convex set (strongly pseudoconvex would be sufficient), the CR structure thus defined is strongly pseudoconvex.

When studying such questions as the equivalence of bounded convex domains in $\mathbf{C}^2$, it is sufficient to restrict one's attention to the equivalence of their boundaries. Indeed, it was a deep theorem by C. Fefferman [Fe] that any biholomorphic map between strongly convex domains extends smoothly to a diffeomorphism (and hence, a CR equivalence) between the boundaries. (The local version of this result is due to Lempert [L1].) On the other hand, it has long been known that any CR equivalence between the boundaries can be extended to a biholomorphic map between the interiors. (In one complex variable, there are conditions on the parameterization of the boundary equivalence; given those conditions, the extension follows from Cauchy's integral formula.)

The implication of these observations is that any naturally defined object on the interior of a convex domain should correspond to some invariant object on the boundary of the domain; any description of the moduli space for convex domains should have a corresponding description of a moduli space for CR structures on the boundary of the domain. The main purpose of this paper is to draw this correlation for the case of small perturbations of the standard sphere.

4. Interpretations of the results

Relation to convex domains. The results in this paper arose from an attempt to describe the Lempert map (and the modular data for convex domains) completely in terms of analysis on the boundary of the domain. As a natural result, the particular normalizations which we have chosen lead to a rather precise correspondence between objects on the boundary and objects on the domain. This correspondence should not be lost in the analysis in the paper, and we would like to emphasize it here. Before we draw this correspondence, we should remind the reader that the normalization procedure can be interpreted as (i) fixing the underlying coordinate system, and finding a normal form for a CR structure under the action of the diffeomorphism group, or (ii) finding a canonical map from the standard sphere to the CR manifold such that the CR structure pulls back under this map to one in normal form.

Modular data, normal forms and the Riemann mapping theorem. It will be shown in this paper that if the CR structure is normalized to have only strictly positive Fourier coefficients in the deformation tensor, then it naturally corresponds to a point in the moduli space for strongly convex domains [BD1]. More precisely, if the CR structure is CR equivalent to that on the boundary of a convex domain D, then the circular model for the convex domain is obtained as follows: Let $p \in D$ be a base point, and pull back the complex structure from the domain to the indicatrix via the exponential map for the Kobayashi metric. Write the new complex structure on the indicatrix as a deformation of the standard one, and restrict it to the boundary. The indicatrix with the deformed complex structure obtained in this fashion is the circular model for the domain D, and the boundary of indicatrix with the deformed CR structure is in the normal form presented in Theorem 14.2. Moreover, the space of circular models described in [BD1] is equivalent to the space of CR structures presented in the normal form given in Theorem 14.2 which have no negative (or weight <4, according to the parameterization given in the statement of the theorem) Fourier coefficients. Since the circular model for the domain D is obtained from the Riemann mapping, obtaining the normal form for the CR structure can be viewed as constructing the circular model or the Riemann mapping completely from the CR structure on the boundary.

Notice that the normal form constructed in this way is only determined up to the choice of a base point $p \in D$, and a framing at p; this corresponds to the action of a finite dimensional group on the normal form (i.e.—the 'normal form' is only normalized up to the action of this finite dimensional group), and we will run into this same indeterminancy in our normalization procedure in this paper. It follows from these observations that the effect on the circular model of changing the base point of the domain is equivalent to the action on the normal form of this finite dimensional group.

It is the agreement of the normal form with the description of the circular models in the moduli space which leads to the following correspondence.

Kobayashi discs. In [L1], Lempert showed that the singular foliation of the domain by extremal Kobayashi discs through a base point induced a smooth foliation of the boundary by circles. In the normalization procedure on the boundary, we start with a smooth foliation by circles, and the normalization procedure can be considered to be normalizing this foliation—that is, finding a differentiably equivalent foliation by circles such that the new circles are the boundaries of extremal discs for the Kobayashi metric.

Plurisubharmonic Green's function. The normalization of the circle foliation is also equivalent to the normalization of the choice of a contact form. (Actually, the choice of a contact form also picks out a natural $\mathbf{R}^1$ action which is generated by the characteristic vector field, and in our normalization procedure, we require this to be a free S^1 action; this is slightly more structure than a differentiable foliation by circles.) On the other hand, a solution u to the homogeneous Monge–Ampère on the domain D also induces a natural contact form $i\bar{\partial}u$ on the boundary, for which the foliation by Kobayashi discs is the characteristic foliation associated to the restriction of $i\bar{\partial}u$ to the boundary. Thus, the normalized contact form is the 'gradient' of the Green's function on the boundary of the domain.

Extension results. The basic idea behind the extension results is rather simple-minded. Start with a contact structure which is invariant under a free S^1 action. Then the manifold M fibres as a principal S^1 bundle over a Riemann surface Σ, and the contact structure can be defined by a contact form η which is S^1 equivariant, and restricts to the fibres as the Maurer–Cartan form—that is, the contact form η is a connection form on the principal bundle. The principal S^1 bundle imbeds in a complex line bundle $E := M \otimes_{S^1} \mathbf{C}^1$ over Σ, and the S^1 action on $M \subset E$ imbeds in a $\mathbf{C}^*$ action on E. Construct an invariant CR structure on M by choosing any complex structure on Σ, and defining the holomorphic tangent space on M to be the horizontal lift (via η) of the holomorphic tangent space on Σ. This CR structure can be extended to define a complex structure on E in such a fashion that the holomorphic tangent vectors to the fibre directions are holomorphic on E (i.e.—if ζ is a fibre coordinate, then $\zeta\,\partial/\partial\zeta$ is holomorphic on E) and the horizontal lifts of the holomorphic tangent directions on Σ to $\mathbf{C}^*$ invariant vector fields are holomorphic. Using this complex structure, E is a holomorphic line bundle over Σ.

The extendable normal forms, then, are precisely those which can be expressed relative to the invariant CR structure via a deformation tensor which has no negative Fourier coefficients relative to the S^1 action. The trick is that the extension result is then reduced to a one complex variable result—if, when restricted to any fibre it has no negative coefficients, then it extends holomorphically to the entire fibre. It is then sufficient to show that this extension defines a deformation of the complex structure on the relatively compact component U of $(E\backslash M)$ which is integrable; by the Newlander–Nirenberg theorem, U with this deformed complex structure is an open complex manifold with the original CR manifold M as its boundary.

We should point out the philosophical correlation with the Bishop extension techniques. In [Bi], Bishop extended complex structures by finding complex discs along which to extend the structures (see also [HT]). In the current situation, we are essentially doing the same thing, where we are choosing a canonical family of discs by any of the following normalization techniques: (i) the solution to the homogeneous Monge–Ampère equation, (ii) finding the family of Kobayashi discs which all pass through a given point, (iii) using CR geometry to normalize the choice of a contact form on the boundary.

Direct imbedding methods. In the final section of this article, we indicate how to obtain a direct imbedding of the CR manifold. The technique is to use the solution operator for the $\bar{\partial}_b$ operator associated to the S^1 invariant CR structure, and the normal form of the deformed CR structure, to directly produce CR functions relative to the deformed CR structure by modifying functions which are CR relative to the S^1 invariant structure. The main idea behind this technique was implicitly used in [BD1] in the parameterization of the moduli space. However, this technique has not yet been used to its potential, and there are some interesting features which are worthwhile to point out:

(i) In general, it is difficult to write down explicit expressions for solution operators to the $\bar{\partial}_b$ equation on CR manifolds; however, in this case, it is possible to do so by comparing the given CR structure with a second CR structure which is invariant under a free S^1 action.

(ii) The expressions for the solution operators rely on two essential pieces of data: the solution operator relative to the S^1 invariant CR structure, and the solution to the homogeous Monge–Ampère equation. More precisely, associated to the solution to the homogeneous Monge–Ampère equation is a canonical volume form on the boundary (that is, the CR manifold). The CR functions for the deformed CR structure which are obtained by the above process are equivalent to those obtained by starting with the CR functions relative to the undeformed CR structure, and adding on a component which is L^2 perpendicular relative to the volume form associated to the homogeneous Monge–Ampère equation.

(iii) There are very few known examples where the Kobayshi metric can be computed explicitly. As the value of this metric is becoming increasingly apparent, this is a huge gap in the theory. In particular, while it is shown in [BD1] that there is a natural correspondence between strongly convex domains and their circular models, explicit examples of this correspondence are hard to find. This technique makes explicit how examples of the correspondence can be obtained via the 'back door'—starting with the circular model, and finding the CR imbedding functions. In simple examples, these imbedding functions can be explicitly written down.

(iv) Continuing along the lines of the last comment, the explicit maps from the circular models to domains in $\mathbf{C}^2$ define canonical representatives within the class of strongly convex domains up to biholomorphic equivalence. It would be of interest to study what properties these canonical representatives possess, and whether the real ellipsoids are among the list of these representives. (If they are, then these are likely to be the 'best' choice of canonical representatives; if not, then there is likely some other procedure for obtaining the canonical representatives.)

Relation to other results. Epstein has recently extended his work with Burns [BE] to a study of CR structures on three dimensional circle bundles. In [E], he analyses the space of three dimensional CR manifolds which admit a free S^1 action, as well as small perturbations of such structures. He shows that small perturbations of the S^1 invariant CR structure are generically nonimbeddable, but if the perturbation can be written as a deformation using only positive Fourier coefficients, then any imbedding of the S^1 invariant CR structure can be perturbed to an imbedding of the deformed structure. We believe that a combination of a sharpened version of his 'generic non-imbeddability' results and our normal form analysis could lead to a rather simple description of the imbeddable CR structures in terms of a filtration of the Hilbert space of normal forms. For example, in the case of small deformations of the sphere, we show in this paper that there is a Hilbert subspace of the space of normal forms which corresponds to those which are imbeddable as the boundaries of convex domains; then, using a stability result obtained by Lempert (see [L3]), it follows that this Hilbert subspace is precisely the space of imbeddable CR structures. In general, we expect that the set of the imbeddable normal forms will still form a Hilbert subspace, but that there will be further linear obstructions on the space of imbeddable normal forms which correspond to obstructions to imbeddability in a neighbourhood of certain special imbeddings of the S^1 invariant CR structure.

Also, in the paper cited above, Lempert [L3] studied the imbeddability of CR structures using the notion of Beltrami differentials. These Beltrami diffentials basically correspond to the Lie derivative with respect the circular action of the deformation tensor used in this paper; alternatively, they can be related to an anti-holomorphic twist tensor (see e.g. [BD2] where the anti-holomorphic twist associated to the Monge–Ampère foliation for strongly convex domains is related to the deformation tensor used in the description of the moduli space in [BD1]). His notions of inner actions and outer actions correspond to the deformation tensor having only nonnegative and nonpositive Fourier coefficients respectively. The result which we referred to in the last paragraph is a stability result for small perturbations of S^3. He established it using the elegant trick of gluing the complex manifold which the interior normal form bounds (if it does bound) to the complex manifold which the exterior normal form bounds in order to construct a compact complex manifold which is topologically $\mathbf{P}^2$ with the origin blown up, and analysing the stability of the spectrum of $\Box_b$ on the hypersurface contained in this compact complex manifold.

Finally, Cheng and Lee have also announced that they are able to obtain a transverse slice theorem for the action of the group of contact diffeomorphisms on the space of CR structures. More precisely, they have shown that given an arbitrary compact 3-dimensional strongly pseudoconvex CR manifold, there is a smooth local slice for the action of the contact diffeomorphism group on the space of CR structures in a neighbourhood of the given one. Such a result would give a family of normal forms for nearby CR structures in terms of deformations of a fixed inital CR structure.

II. Analysis on S^3

5. The geometry

Consider $S^3 \subset \mathbf{C}^2 \cong \mathbf{R}^4$. We will use coordinates (x^1, y^1, x^2, y^2) on $\mathbf{R}^4$, and the identification $z^k = x^k + iy^k$ for $\mathbf{R}^4 \cong \mathbf{C}^2$. The complexified tangent space to S^3 has a natural framing given by $e = z^{\bar{2}}\partial/\partial z^1 - z^{\bar{1}}\partial/\partial z^2$, $\bar{e}$, $T = -2\,\mathrm{Im}(z^1\partial/\partial z^1 + z^2\partial/\partial z^2)$, with dual coframing $\omega = z^2 dz^1 - z^1 dz^2$, $\bar{\omega}$, $\eta = -\,\mathrm{Im}(\bar{\partial}\log|z|^2)$. With this framing, e is a basis for the *holomorphic tangent space* $H_{(1,0)}$ to S^3 (that is, the restriction of the holomorphic tangent space $T_{(1,0)}$ for $\mathbf{C}^2$ to the sphere), and the vector field T is the generator of the circular action $(z^1, z^2) \mapsto (e^{i\theta}z^1, e^{i\theta}z^2)$ with period 2π. The fact that S^3 is strongly pseudoconvex implies that the dual form η is nondegenerate; in this case, $d\eta = i\omega\wedge\bar{\omega}$ and $\eta\wedge d\eta \neq 0$.

The above framing for S^3 is also adapted to a natural contact structure on S^3. (Recall that a *contact structure* is a co-dimension one distribution on the real tangent space which is fully non-integrable—that is, if the distribution is defined by a dual one-form, called a *contact form*, the one-form is non-degenerate; this is the odd-dimensional analogue of a symplectic structure.) In this case, the natural contact structure is defined by the real and imaginary parts of the holomorphic tangent vector e, and the associated contact form is η. The nondegeneracy condition on the contact form is that $\eta\wedge d\eta=\eta\wedge(i\omega\wedge\bar{\omega})\neq 0$. The vector field T is the *characteristic vector field* for the contact form η; that is, it is the vector field which is characterized by the conditions

(1) $T\rfloor\eta\equiv 1$,

(2) $T\rfloor d\eta=0$.

Next, we consider S^3 from the point of view of a principal bundle. The characteristic vector field T generates a circular action on S^3, with quotient space S^2; that is, S^3 admits the fibration $S^1\to S^3\to S^2$, called the *Hopf fibration.* In this picture, the orbits of the S^1 action are the intersections of complex lines through the origin in $\mathbf{C}^2$ with the unit sphere S^3, and the orbit space is the space of complex lines through the origin, $\mathbf{P}^1\cong S^2$.

An algebraic geometric interpretation of this bundle is as follows. The punctured complex plane $\mathbf{C}^2\setminus\{0\}$ fibres as a punctured complex line bundle over the space of complex lines through the origin in $\mathbf{C}^2$—that is, $\mathbf{P}^1\cong S^2$; this fibration is given by a point $p\in\mathbf{C}^2\setminus\{0\}$ mapping to the complex line through the origin which it defines. This is a holomorphic fibration (the quotient map is holomorphic), and it identifies $\mathbf{C}^2\setminus\{0\}$ with a punctured holomorphic line bundle over $\mathbf{P}^1$; for obvious reasons, this is called the *tautological line bundle* E over $\mathbf{P}^1$, or more precisely, it is the complement of the zero section of E.

A norm on $\mathbf{C}^2$ is the square root of a strongly convex function of the form $h=e^H|z|^2$, where H is a function which is constant along the lines through the origin. (In particular, H respects the above fibration, and defines a function on $\mathbf{P}^1$.) The sub-level sets of the norm are strongly convex circular domains (domains which are invariant under the circular action), and the sub-level set corresponding to the value 1 is the *indicatrix* for the norm. The norm on $\mathbf{C}^2$ defines a norm on the tautological line bundle E, and the level set for the value 1 corresponds to the bundle of unitary vectors in the tautological line bundle.

The imaginary part of the one form

$$-\operatorname{Im}(\bar{\partial}\log h)=-\operatorname{Im}(\bar{\partial}\log|z|^2+\bar{\partial}H)=\eta-\operatorname{Im}(\bar{\partial}H)$$

restricts to the level set $h=1$ to define a contact form whose characteristic vector field is again the generator of the circular action. On the tautological line bundle, the form $\bar{\partial}\log h$ is a connection form. (More precisely, the form $\partial\log h$ is the connection form; a tangent vector to E is horizontal if it is annihilated by $\partial\log h$). This connection form restricts to the $U(1)$ bundle of unitary frames (the level set $h=1$) as $-i\eta+\bar{\partial}H$.

The relevance of the above discussion is as follows. When $H\equiv 0$, the level set $h=1$ corresponds to the unit sphere in $\mathbf{C}^2$. In this case we will at various times interpret the one form $i\bar{\partial}\log h=\eta$ as (1) a contact form on the level set $h=1$ (in order to use contact geometry to normalize the CR structure on the boundary), (2) dual to a circular action (in order to use Fourier analysis in the normalization procedures), (3) a connection form on the $U(1)$ (or S^1) bundle of unitary frames over P^1 (in order to define horizontal lifts of frames from $\mathbf{P}^1$, or S^1 invariant lifts), and (4) the restriction of a connection form on E to the bundle of unitary frames (in order to define extensions of CR deformations to deformations of the complex structure on E). The nondegeneracy of η (where $d\eta = i\omega\wedge\bar{\omega}$) can be variously interpreted as (1) the strong pseudoconvexity of the CR structure on S^3, (2) the nondegeneracy of the contact form, (3) the negativity of the curvature form of the line bundle E (and the negativity of the line bundle), and (4) the fact that $d\eta$ descends to $\mathbf{P}^1$ to define a symplectic structure on P^1 (the curvature form defines a positive Kähler form on $\mathbf{P}^1$). Under these various guises, changing the norm h corresponds to (1) changing the indicatrix, (2) changing the norm on the tautological line bundle, (3) changing the connection form on the tautological bundle E (or the splitting into horizontal and vertical directions), (4) changing the contact structure (notice that the fibration of $\mathbf{C}^2\setminus\{0\}$ over $\mathbf{P}^1$ defines a natural identification—or diffeomorphism—between any two indicatrices, and we may equivalently be considering ourselves to always be working on the standard S^3 and simply changing the contact structure, or the connection form), and (5) changing the symplectic form on $\mathbf{P}^1$ (the curvature form).

6. The operators

The vector field T which generates the circular action induces a natural splitting of the complexified cotangent bundle

$$T^*_{\mathbf{C}}(S^3) = \mathbf{C}\cdot\eta \oplus H^{(1,0)}(S^3) \oplus H^{(0,1)}(S^3). \tag{6.1}$$

Using this splitting, the boundary Cauchy–Riemann operator acting on forms, denoted by $\bar{\partial}_b$, becomes well-defined, and on functions, it is defined by the formula

$$\bar{\partial}_b u = \bar{e}(u)\,\bar{\omega}.$$

It extends to define the $(0,1)$ part of a Hermitian connection on the holomorphic tangent bundle to S^3; furthermore, the $(1,0)$ part of the associated Hermitian connection is naturally denoted by ∂_b, where the metric is the induced metric coming from the imbedding $S^3\subset\mathbf{C}^2$. The adjoint operator to $\bar{\partial}_b$ is denoted by $\bar{\partial}_b^*$, which on $(0,1)$ forms is given by the formula

$$\bar{\partial}_b^*(v\bar{\omega}) = -e(v).$$

For basic facts about these operators, and the operators $\Box_b = \bar{\partial}_b^*\bar{\partial}_b + \bar{\partial}_b\bar{\partial}_b^*$, and its conjugate $\bar{\Box}_b = \partial_b^*\partial_b + \partial_b\partial_b^*$, one may consult [FS], [BD1].

There is a real variable analogue of these operators. Define a partial connection by

$$\hat{d}=\bar{\partial}_b+\partial_b,$$

and the associated sub-Laplacian by

$$\hat{\Delta}=\hat{d}^*\hat{d}+\hat{d}\hat{d}^*.$$

In terms of the framing for S^3 given above, this operator, acting on functions, may be written

$$\hat{\Delta}(f)=-(e+\bar{e})(e+\bar{e})(f)-J(e+\bar{e})J(e+\bar{e})(f),$$

where

$$e+\bar{e}=x^2\frac{\partial}{\partial x^1}-y^2\frac{\partial}{\partial y^1}-x^1\frac{\partial}{\partial x^2}+y^1\frac{\partial}{\partial y^2}$$

and

$$J(e+\bar{e})=i(e-\bar{e})=x^2\frac{\partial}{\partial y^1}+y^2\frac{\partial}{\partial x^1}-x^1\frac{\partial}{\partial y^2}-y^1\frac{\partial}{\partial x^2}.$$

(J is the complex structure tensor for $\mathbf{C}^2$.) The operator $\hat{\Delta}$ may be thought of as a 'horizontal' Laplacian—the associated self-adjoint operator to the horizontal partial derivative $\hat{d}=\bar{\partial}_b+\partial_b=d \pmod{\eta}$.

Using this horizontal Laplacian, the operator $\Box_b$ on functions may be expressed as

$$\Box_b=-e\bar{e}=\tfrac{1}{4}\hat{\Delta}+\tfrac{1}{2}iT.$$

and its conjugate as

$$\overline{\Box}_b=\tfrac{1}{4}\hat{\Delta}-\tfrac{1}{2}iT=\Box_b-iT. \tag{6.2}$$

Let G be the Green's operator associated to $\Box_b$. (This operator will be discussed more fully in the next section.) Then the commutation relations

$$[T,\hat{\Delta}]=[G,\hat{\Delta}]=[G,T]=[\overline{G},\hat{\Delta}]=[\overline{G},T]=0 \tag{6.3}$$

hold, and the fact that

$$\overline{\Box}_b\Box_b=\tfrac{1}{16}\hat{\Delta}\hat{\Delta}+\tfrac{1}{4}TT \tag{6.4}$$

is a real operator implies that $G\overline{G}$ is a real operator.

7. Hodge theory

The spaces Γ^k used in this paper are the weighted (or anisotropic) Sobolev spaces which we refer to as Folland–Stein spaces. (For basic facts about these spaces, and the properties of the various operators, see [Fo], [FS]. Most of the estimates work equally well for the weighted L^p spaces, and the Hölder spaces; however, in the case of the Hölder spaces, the estimates break down when we try to project onto the subspace of functions which have only positive Fourier coefficients.) The norms are equivariant with respect to the circular action $(z^1, z^2) \mapsto (e^{i\theta} z^1, e^{i\theta} z^2)$ on $S^3 \subset \mathbf{C}^2$, and more generally, under the action of the unitary group. Under the circular action, the space of L^2 functions decomposes into invariant subspaces; the components of a function in these invariant subspaces will be known as its *Fourier components*, or *Fourier coefficients.* Under the action of the unitary group, the space of L^2 functions on S^3 further decomposes into the invariant subspaces $B_{m,n}$, where m is the 'holomorphic' degree, and n the 'conjugate holomorphic degree' of the function. (For a full analysis of this decomposition into invariant subspaces in the present context, one should refer to [Fo].) The projection operators onto the various invariant subspaces are bounded in the weighted Sobolev norms. The two projections of particular importance in this paper are the Szegö projection, and the projection onto the subspace having only positive Fourier components.

The function space norms may be extended to norms on the spaces of sections of various bundles, such as $\Gamma^k(\bigwedge^{(0,1)}(S^3))$, in the standard way. In this case, the norms on the sections are equivalent to the norms on the coefficients, when the sections are expressed relative to the coframing $\eta, \omega, \bar{\omega}$ and its dual framing.

At various times, the symbol Γ^k will contain subscripts; these subscripts will refer to those elements in the Γ^k space which have only components which lie in some invariant subspace. For example, Γ^k_+, Γ^k_0, Γ^k_- refer to those elements with only strictly positive, zero, and strictly negative Fourier coefficients respectively, and Γ^k_m will refer to the mth coefficient or to those elements in the imth eigenspace of the operator T. (Notice that $\Gamma^k_0(S^3)$ corresponds to functions which are invariant under the S^1 action, and hence descend to functions on the quotient space $\mathbf{P}^1$.) The space $\Gamma^k_{0,\mathrm{Re}}$ will refer to the subspace of real valued functions which are invariant under the circular action—that is, real valued functions having only zero Fourier coefficients. Finally, $\Gamma^k_\perp$ will refer to the subspace which is L^2 orthogonal to the CR functions. Similarly, if we subscript a function in an analogous manner, it will refer to the L^2 projection of the function onto the corresponding subspace.

We have the following $\bar{\partial}_b$ Hodge theory for S^3.

THEOREM 7.1 (Folland–Stein). *On S^3, there exist integral operators S (Szegö projection onto the CR functions), G (the canonical solution operator for $\Box_b$) and Q (the projection of the space of $(0,1)$ forms onto the kernel of $\bar{\partial}_b^*$) with the following properties (the operator* $\operatorname{Avg} u$ *takes the average value of the function—or is its L^2 projection onto the constants):*

For a function u,

(1) $u=G\Box_b u+Su=\Box_b Gu+Su=(G\Box_b u)_-+(G\Box_b u)_0+\mathrm{Avg}(u)+(G\Box_b u)_++(Su)_+$,

(2) $\ker G=\{u\,|\,\bar\partial_b u=0\}$,

(3) $u=\bar G\,\bar\Box_b u+\bar S u=\bar\Box_b\bar G u+\bar S u=(\bar G\,\bar\Box_b u)_-+(\bar S u)_-+(\bar G\,\bar\Box_b u)_0+\mathrm{Avg}(u)+(\bar G\,\bar\Box_b u)_+$,

(4) $\ker\bar G=\{u\,|\,\partial_b u=0\}$,

(5) $u=(G\bar G\,\bar\Box_b\Box_b u)+(Su)_++(\bar S u)_-+\mathrm{Avg}(u)$.

For a $(0,1)$ form ϕ,

(1) $\phi=\bar\partial_b G\bar\partial_b^*\phi+Q(\phi)$,

(2) $Q\circ\bar\partial_b=0$,

(3) $Q(if\bar\omega)=i\bar S(f)\bar\omega$.

Furthermore, the operators are bounded operators between the following spaces:

(1) $G\colon\Gamma^k(\bigwedge^{(0,q)}(S^3))\to\Gamma^{k+2}(\bigwedge^{(0,q)}(S^3))$, $q=0,1$,

(2) $S\colon\Gamma^k(S^3)\to\Gamma^k(S^3)$,

(3) $Q\colon\Gamma^k(\bigwedge^{(0,1)}(S^3))\to\Gamma^k(\bigwedge^{(0,1)}(S^3))$.

Proof. The basic estimates for this result are contained in [FS]. In the case of the Heisenberg group, everything has been worked out quite explicitly in [GS]; a similar approach could be applied to the case of the sphere (see e.g. [Ge]). For more general imbeddable three dimensional CR manifolds, one can proceed as in [BG]; the basic facts that are needed in this context are that $\mathcal{V}$—operators of order 0 are bounded on L^2, and that G and S are $\mathcal{V}$—operators of order -2 and 0 respectively (see e.g. [BE]). $\Box$

Remark 7.2. The appropriate generalization of this fact to higher dimensions (in the context of this paper) is that there exists a bounded homotopy operator

$$P\colon\Gamma^k(\textstyle\bigwedge^{(0,q)}(S^{2n+1}))\to\Gamma^{k+1}(\bigwedge^{(0,q-1)}(S^{2n+1})),\quad 0<q\leqslant n,$$

such that for $\phi\in\Gamma^k(\bigwedge^{(0,q)}(S^{2n+1}))$ $(0<q<n)$,

$$\phi=\bar\partial_b P\phi+P\bar\partial_b\phi.$$

III. The diffeomorphism group

8. Diffeomorphisms of S^3

Our aim in this section is to identify a natural linear structure on the space of diffeomorphisms of S^3 which are sufficiently close to the identity. We will do this by identifying small diffeomorphisms with vector fields which are tangent to S^3.

Consider $S^3\subset\mathbf{C}^2\simeq\mathbf{R}^4$. Then the linear structure of $\mathbf{R}^4$ may be used to identify a diffeomorphism $F\colon S^3\to S^3$ given by $x\mapsto F(x)=y$ with the vector $\overrightarrow{F(x)-x}$ tangent to $\mathbf{R}^4$

and based at x. After adding an appropriate multiple λ of the radial vector field $y^i\partial/\partial y^i$ based at $F(x)$, the new vector field $\vec{y}-\vec{x}+\lambda\vec{y}$, considered as a tangent vector to $\mathbf{R}^4$ based at $\vec{x}$, is tangent to S^3. This multiple λ is given by solving the equation $\langle\vec{y}-\vec{x}+\lambda\vec{y},\vec{x}\rangle=0$ where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product. The solution λ is given by $\lambda=(1/\langle\vec{x},\vec{y}\rangle)-1$.

Conversely, given a small vector field $\vec{V}$ on S^3, we may identify a smooth map $S^3\to S^3$ by

$$\vec{x}\mapsto\frac{\vec{x}+\vec{V}_x}{\|\vec{x}+\vec{V}_x\|}.$$

If V_x is sufficiently small in the C^1 norm, then this smooth map is a diffeomorphism.

9. The integrability condition

In this section, we would like to study the extra conditions imposed on a vector field by requiring that it induce a contact diffeomorphism on S^3. This will require introducing new notation in order to write the conditions in a manageable form. For this reason, we will proceed in this section to first do the calculations, and then summarize the results at the end of the section in the form of a proposition. The proof of the proposition will consist of the calculations leading up to it.

Consider the map $S^3\to S^3$ defined by radially projecting the map $S^3\mapsto\mathbf{R}^4$ given by $(x^k,y^k)\mapsto(x^k+X^k,y^k+Y^k)$ onto the sphere. Under this map, which we will refer to as Φ, the contact form η pulls back to (here u is the Euclidean norm $\|(x+X,y+Y)\|$):

$$\begin{aligned}\Phi^*\eta&=\sum_k\left(\frac{x^k+X^k}{u}\right)d\left(\frac{y^k+Y^k}{u}\right)-\left(\frac{y^k+Y^k}{u}\right)d\left(\frac{x^k+X^k}{u}\right)\\&=\frac{1}{u^2}\{(x^k+X^k)d(y^k+Y^k)-(y^k+Y^k)d(x^k+X^k)\}\\&=\frac{1}{u^2}(\eta+(X^kdy^k-Y^kdx^k)+(x^kdY^k-y^kdX^k)+(X^kdY^k-Y^kdX^k))\\&=\frac{1}{u^2}(\eta+d(x^kY^k-y^kX^k)+2(X^kdy^k-Y^kdx^k)+(X^kdY^k-Y^kdX^k)).\end{aligned}$$

The map Φ is a contact diffeomorphism if and only if $\Phi^*(\eta)=p\cdot\eta$ for some nonvanishing function p. (Notice, in particular, that this implies that Φ is a local diffeomorphism.) Thus the condition that (X,Y) corresponds to a contact diffeomorphism is that $\Phi^*\eta=0$ (mod η); we will henceforth refer to this condition as the *integrability condition*.

At this stage, it is convenient to introduce some formalism. We shall do this twice—once using the real structure of $\mathbf{R}^4$ and a second time using the complex structure of $\mathbf{C}^2$.

Recall that the characteristic vector field T for the contact form η is defined by the conditions

(1) $T \lrcorner \eta \equiv 1$,

(2) $T \lrcorner d\eta = 0$.

(We have now restricted to S^3, where $d\eta$ is of rank 2.) For the tangent vector $V = X^k \partial/\partial x^k + Y^k \partial/\partial y^k$ write

$$V = X^0 T + V_H \quad \text{where } V_H \lrcorner \eta = 0.$$

Next, we introduce the partial exterior derivative $\hat{d}$ by $\hat{d} = d \pmod{\eta}$, where this is defined relative to the splitting of the cotangent space defined by T. Then the integrability condition on $V = X^0 T + V_H$ becomes

$$I(X^0, V_H) := \hat{d}(X^0) + (V_H \lrcorner d\eta) + (X^k \hat{d} Y^k - Y^k \hat{d} X^k) = 0. \tag{9.1}$$

The final term in this last expression can be written in a more elegant fashion by using the inner product $\langle \cdot, \cdot \rangle$ coming from $\mathbf{R}^4$ as well as the complex structure operator J defined by $J(\partial/\partial x^k) = \partial/\partial y^k$, $J(\partial/\partial y^k) = -\partial/\partial x^k$ and corresponding to the identification $\mathbf{R}^4 \cong \mathbf{C}^2$. Then

$$X^k \hat{d} Y^k - Y^k \hat{d} X^k = \langle JV, \hat{d} V \rangle. \tag{9.2}$$

Since $V = X^0 T + V_H$ and $JT = -\nu$ where ν is the outward pointing unit normal to S^3, we can use the partial connection $\hat{\nabla}$ on $T(S^3)$ corresponding to $\hat{d}$ and expand this term to

$$\begin{aligned}
\langle JV, \hat{d} V \rangle &= \langle -X^0 \nu + JV_H, \hat{d}(X^0 T + V_H) \rangle \\
&= \langle \hat{d} V_H, JV_H \rangle + \langle \hat{d}(X^0 T), JV_H \rangle - X^0 \langle \nu, \hat{d} V_H \rangle - X^0 \langle \nu, \hat{d}(X^0 T) \rangle \\
&= \langle \hat{d} V_H, JV_H \rangle + \hat{d} \langle (X^0 T), JV_H \rangle - \langle (X^0 T), \hat{d} JV_H \rangle - X^0 \langle \nu, \hat{d} V_H \rangle - X^0 \langle \nu, \hat{d}(X^0 T) \rangle \\
&= \langle \hat{d} V_H, JV_H \rangle + 0 + \langle J(X^0 T), \hat{d} V_H \rangle - X^0 \langle \nu, \hat{d} V_H \rangle - X^0 \langle \nu, \hat{d}(X^0 T) \rangle \\
&= \langle \hat{d} V_H, JV_H \rangle - 2X^0 \langle \nu, \hat{d} V_H \rangle + X^0 \langle \hat{d} \nu, X^0 T \rangle \\
&= \langle \hat{d} V_H, JV_H \rangle + 2X^0 \langle \hat{d} \nu, V_H \rangle + 0 \\
&= \langle \hat{d} V_H, JV_H \rangle - X^0 (JV_H \lrcorner d\eta).
\end{aligned}$$

(The last line follows from explicitly writing out both sides of the equation, and using the observation that $\hat{d}\nu$ is the 'shape operator' for S^3 restricted to the directions tangent to the contact distribution.) Substituting this into equation (9.1), the integrability condition becomes the vanishing of

$$I(X^0, V_H) := \hat{d}(X^0) + (V_H \lrcorner d\eta) + \langle \hat{d} V_H, JV_H \rangle - X^0 (JV_H \lrcorner d\eta). \tag{9.3}$$

Our second expression for the integrability condition will be in terms of the standard CR structure on S^3 induced by the complex structure of $\mathbf{C}^2$. First notice that the contact form $\eta = -\operatorname{Im}(z^k d\bar{z}^{\bar{k}}) = -\operatorname{Im}(\bar{\partial}\log|z|^2)$ annihilates both the holomorphic and conjugate holomorphic tangent spaces to S^3. Thus, we can write

$$V_H = Z + \bar{Z}$$

in a canonical fashion, where Z is a vector field of type $(1,0)$. Using the canonical splitting of the complexified cotangent bundle of S^3, which is induced by η and its characteristic vector field T, into

$$T^*_{\mathbf{C}} S^3 = \mathbf{C}\cdot\eta \oplus H^{(1,0)} \oplus H^{(0,1)},$$

and the fact that the integrability condition $I \in H^{(1,0)} \oplus H^{(0,1)}$ is a real form, an equivalent integrability condition is that the projection of I onto the $(0,1)$ subspace is zero. Taking note of the facts that $JZ = iZ$ and $\hat{d} = \bar{\partial}_b + \partial_b$, the complexified integrability condition becomes the vanishing of

$$I^{(0,1)}(X^0, Z) := \bar{\partial}_b(X^0) + Z \lrcorner d\eta + \langle J(Z+\bar{Z}), \bar{\partial}_b(Z+\bar{Z})\rangle - X^0(JZ \lrcorner d\eta).$$

Setting

$$(X,Y) = X^0 T + Z + \bar{Z} = X^0 T + fe + \bar{f}\bar{e},$$

the complexified integrability condition becomes the vanishing of

$$\begin{aligned} I^{(0,1)}(X^0, fe) &= \bar{\partial}_b(X^0) + if\bar{\omega} + \langle i(fe - \bar{f}\bar{e}), (\bar{\partial}_b f)e + (\bar{\partial}_b \bar{f})\bar{e}\rangle - X^0(ife \lrcorner i\omega\wedge\bar{\omega}) \\ &= \bar{\partial}_b(X^0) + if\bar{\omega} + X^0 f\bar{\omega} + \frac{i}{2}(f\bar{\partial}_b\bar{f} - \bar{f}\bar{\partial}_b f) \\ &= \bar{\partial}_b(X^0) + if\bar{\omega} + X^0 f\bar{\omega} + \frac{1}{2i}(\bar{f}\bar{\partial}_b f - f\bar{\partial}_b\bar{f}). \end{aligned}$$

We have established the following proposition.

PROPOSITION 9.4. *Let Φ denote the diffeomorphism of S^3 obtained from the vector field*

$$(X,Y) = X^0 T + V_H = X^0 T + fe + \bar{f}\bar{e}$$

by mapping the point (x,y) to the point $(x+X, y+Y)$ and radially projecting it back to the sphere. Then

$$\Phi^*\eta = \frac{1}{u^2}(\eta + d(X^0) + (V_H \lrcorner d\eta) + \langle J(X,Y), d(X,Y)\rangle)$$

and if we define the integrability tensor by

$$I(X^0, V_H) = \hat{d}(X^0) + (V_H \lrcorner d\eta) + (X^k \hat{d} Y^k - Y^k \hat{d} X^k),$$

and its complexified version by

$$I^{(0,1)}(X^0, fe) = \bar{\partial}_b(X^0) + if\bar{\omega} + X^0 f\bar{\omega} + \frac{1}{2i}(\bar{f}\bar{\partial}_b f - f\bar{\partial}_b\bar{f}),$$

then Φ is a contact diffeomorphism if and only if $I = 0$.

COROLLARY 9.5. *If the vector field* (X,Y) *is invariant under the* S^1 *action, then*

$$\Phi^*\eta=\eta+\frac{1}{u^2}(I(X^0,V_H)).$$

Proof. Since the vector field (X,Y) is invariant under the S^1 action, it defines a bundle automorphism; since the fibration is preserved by the map Φ, and η restricts to the fibres to have period equal to 2π, this property is preserved after pulling it back by the map Φ. This means that $\Phi^*\eta=\eta$ (mod $\omega,\bar{\omega}$), and the result follows. □

IV. Contact diffeomorphisms

10.

In the last chapter, we showed that we could introduce a linear structure on the space of diffeomorphisms near the identity by identifying diffeomorphisms with vector fields tangent to S^3; we can make this into a weighted Banach space structure by using the weighted Sobolev space norms on the coefficients of the vector fields. We also showed that the subset of diffeomorphisms which preserved the contact structure was a non-linear subset—those which satisfied a non-linear PDE which we referred to as the integrability condition. In this section, we would like to show that the space of solutions to this PDE is a Banach submanifold, and hence, that the space of contact diffeomorphisms inherits a natural weighted Banach space structure. The main theorem will be the following:

THEOREM 10.1. *Let* S^3 *have the standard contact structure defined by the one form* η. *Then there is a natural weighted Banach space structure on the space of contact diffeomorphisms close to the identity. In particular, there is a neighbourhood of the origin in this Banach space which can be parameterized by a single real valued function.*

We should point out the interest in this theorem. It is well known that contact diffeomorphisms can be parameterized by a single real valued function, called the generating function; moreover, one can parameterize them in such a fashion that the generating function is in some Sobolev space if and only if the diffeomorphism is in the Sobolev space with one less derivative. Theorem 10.1 asserts that one can replace the ordinary Sobolev spaces by weighted (or anisotropic) Sobolev spaces—those which involve L^2 estimates on derivatives only in those directions which are tangential to the contact distribution. In one sense, these weighted spaces are perhaps the most natural spaces in which to be working, since contact diffeomorphisms preserve the weighted spaces; on the other hand, in this instance it is absolutely essential. We will be solving the $\bar{\partial}_b$ equation later in the paper, and we would like to do so without losing derivatives. These weighted spaces (in this context, they are referred to as the Folland–Stein spaces) are precisely the spaces for which one can solve the $\bar{\partial}_b$ equation without losing derivatives.

The existence of the weighted Banach space structures on the space of contact diffeomorphisms of S^3 with its standard contact structure is really a theorem in contact geometry. Its proof could be given without reference to CR geometry by using a Hodge theory for the partial exterior derivative $\hat{d}$. However, we have used the analysis associated to the $\bar{\partial}_b$ operator in the proof because this is the 'existing technology straight off the shelf' with which we are most familiar.

11. Description of the map L

In the last chapter, we expressed the condition that the vector field $(X^0T+fe+\bar{f}\bar{e})$ corresponds to a contact diffeomorphism as the vanishing of the $(0,1)$ form (we will henceforth refer to this expression simply as I):

$$I=I^{(0,1)}(X^0,fe)=\bar{\partial}_b(X^0)+if\bar{\omega}+X^0f\bar{\omega}+\frac{1}{2i}(\bar{f}\bar{\partial}_bf-f\bar{\partial}_b\bar{f}).$$

We would now like to parameterize the set of all vector fields which satisfy this integrability condition.

Let $(X^0T+fe+\bar{f}\bar{e})$ be a vector field. Then the $(1,0)$ component can be expressed as it is as fe, or, alternatively, after raising an index via the natural two form associated to the contact form, we can express it as a $(0,1)$ form. That is,

$$fe\lrcorner d\eta=fe\lrcorner i\omega\wedge\bar{\omega}=if\bar{\omega}.$$

On the other hand, for any $(0,1)$ form, we have the Hodge decomposition

$$if\bar{\omega}=\bar{\partial}_bG\bar{\partial}_b^*(if\bar{\omega})+Q(if\bar{\omega}), \tag{11.1}$$

where the operator G is the canonical solution operator associated to $\bar{\partial}_b$, and the operator Q can be taken to be defined by the equation above. (Thus, it is the orthogonal projection onto the kernel of the operator $\bar{\partial}_b^*$; see Theorem 7.1.) Define

$$p=G\bar{\partial}_b^*(if\bar{\omega}) \tag{11.2}$$

and

$$i\overline{H}\bar{\omega}=Q(if\bar{\omega}) \tag{11.3}$$

so that

$$if\bar{\omega}=\bar{\partial}_bp+i\overline{H}\bar{\omega}; \tag{11.4}$$

then p is orthogonal to the CR functions, and $\overline{H}$ is conjugate CR. Throughout the remainder of this section, the functions f, p, H will be related as above. This will lead to a considerable simplification in the calculations.

We now introduce a mapping L which is admittedly somewhat complicated. The purpose of this non-linear mapping is to construct a local Banach space diffeomorphism—the domain of which is the space of Γ^k vector fields (or diffeomorphisms), and the image of which will lie in a certain linear subspace of the range if and only if the diffeomorphism is a contact diffeomorphism. Thus, the map will induce a weighted Banach space structure on the space of contact diffeomorphisms.

Roughly speaking, the idea behind the map L is as follows. One would like to construct an isomorphism $(X^0, fe)\mapsto(I,g)$, where g is a real valued function. Then the inverse image of $(0,g)$ would be the contact diffeomorphisms, and the function g would parameterize them. Unfortunately, if the data (X^0, fe) have k derivatives, then I has only $(k-1)$ derivatives.

This issue is circumvented by breaking I into components. By Hodge theory,

$$I=\bar{\partial}_b G\bar{\partial}_b^* I+Q(I).$$

We will show that the only component which loses too many derivatives is $\bar{\partial}_b$ of the real part of $G\bar{\partial}_b^* I$. (The other components all have k derivatives.) Thus, we break I into its components, and when we invert the map, we set $X^0=G\bar{\partial}_b^* I$; then the inverse map gains back one derivative on this component. We also gain one derivative in the mapping L by choosing the real valued function g to be roughly the real part of $G\bar{\partial}_b^*(if\bar{\omega})$. The additional complications in the mapping L arise from incorporating the reality condition—it is necessary to further decompose the spaces according to their negative and positive Fourier components, and to replace the operator $\bar{\partial}_b^*\bar{\partial}_b$ by the closely related real operator $\overline{\Box}_b\Box_b$.

We now define the mapping. Using f, p, H as in equation (11.4), define the mapping L by:

$$\begin{aligned}
L(X^0, fe) &= ((G\bar{\partial}_b^* I)_-, \mathrm{Re}(G\bar{\partial}_b^* I)_0, \mathrm{Avg}(X^0),\\
&\qquad (G\bar{\partial}_b^*(if\bar{\omega}))_-, \mathrm{Re}(G\bar{\partial}_b^*(if\bar{\omega}))_0, (G\overline{G}\,\mathrm{Im}(\overline{\Box}_b\bar{\partial}_b^* I))_0,\\
&\qquad (G\overline{G}\,\mathrm{Im}(\overline{\Box}_b\bar{\partial}_b^* I))_+, Q(I)) \qquad (11.5)\\
&= ((G\bar{\partial}_b^* I)_-, \mathrm{Re}(G\bar{\partial}_b^* I)_0, \mathrm{Avg}(X^0),\\
&\qquad p_-, \mathrm{Re}(p_0), (G\overline{G}\,\mathrm{Im}(\overline{\Box}_b\bar{\partial}_b^* I))_0,\\
&\qquad (G\overline{G}\,\mathrm{Im}(\overline{\Box}_b\bar{\partial}_b^* I))_+, Q(I)) \qquad (11.6)
\end{aligned}$$

where $\mathrm{Avg}(X^0)$ is the L^2 projection of the function X^0 onto the constants.

PROPOSITION 11.7. *The vector field $X^0T+fe+\bar{f}\bar{e}$ corresponds to a contact diffeomorphism if and only if*

$$\begin{aligned} L(X^0,fe) &= (0,0,\operatorname{Avg}(X^0),(G\bar\partial_b^*(if\bar\omega))_-,\operatorname{Re}(G\bar\partial_b^*(if\bar\omega))_0,0,0,0) \\ &= (0,0,\operatorname{Avg}(X^0),p_-,\operatorname{Re}(p_0),0,0,0). \end{aligned}$$

Proof. It is clear that if $I=0$, then $L(X^0,fe)$ has the above form. It suffices to establish the converse. The integrability condition is that $I=0$. On the other hand, by the Hodge decomposition given in Theorem 7.1, the $(0,1)$ form I can be written as

$$I=\bar\partial_b G\bar\partial_b^* I+Q(I).$$

Clearly, if the image of L is as stated in the proposition, then $Q(I)=0$. It suffices to show that $G\bar\partial_b^* I=0$. The vanishing of the first two components in the image show that $(G\bar\partial_b^* I)_-=0$ and $\operatorname{Re}(G\bar\partial_b^* I)_0=0$. Next,

$$(G\bar G\operatorname{Im}(\Box_b\bar\partial_b^* I))_+=0.$$

Apply $\bar\Box_b\Box_b$ to both sides to obtain:

$$(\operatorname{Im}(\bar\Box_b\bar\partial_b^* I))_+=(h)_+$$

for some CR function h (see Theorem 7.1). Expanding the left hand side of this equation,

$$(\operatorname{Im}(\bar\Box_b\bar\partial_b^* I))_+=\frac{1}{2i}((\bar\Box_b\bar\partial_b^* I)_+-\overline{(\bar\Box_b\bar\partial_b^* I)_-})=h_+.$$

By the previous calculations, $G(\bar\partial_b^* I)_-=0$, so $\Box_b G(\bar\partial_b^* I)_-=(\bar\partial_b^* I)_-=0$. Substituting this observation into the above equation, one concludes that

$$\frac{1}{2i}(\bar\Box_b\bar\partial_b^* I)_+=h_+.$$

Since $\bar G\,\bar\Box_b$ is the identity on the space of functions with only positive Fourier coefficients (see Theorem 7.1), applying $\bar G$ to both sides of the above equation yields

$$(\bar\partial_b^* I)_+=\bar G h_+.$$

Finally, since the operator G is defined to be zero when restricted to the CR functions (Theorem 7.1), and since the operators G and $\bar G$ commute (see equation (6.4)), the above equation becomes

$$G\bar\partial_b^* I_+=G\bar G h_+=\bar G G h_+=0.$$

A similar (but substantially simpler) argument using the vanishing of the third to last factor shows that the imaginary part of the zeroth Fourier coefficient of $G\bar\partial_b^* I$ also vanishes. $\square$

PROPOSITION 11.8. *The map L is a C^1 mapping*

$$\begin{aligned} L\colon \Gamma^k_{\mathrm{Re}}(S^3)\times\Gamma^k(H_{(1,0)}(S^3))\to\Gamma^k_-(S^3)\times\Gamma^k_{\mathrm{Re},0,\perp}(S^3)\times\mathrm{Re}\times\Gamma^{k+1}_-(S^3)\\ \times\Gamma^{k+1}_{0,\mathrm{Re},\perp}(S^3)\times\Gamma^{k+1}_{0,\mathrm{Re},\perp}(S^3)\\ \times\Gamma^{k+1}_{+,\perp}(S^3)\times\Gamma^k((\operatorname{coker}\bar\partial_b)\cap H^{(0,1)}(S^3)). \end{aligned}$$

Proof. The components of the mapping L are all given by compositions of relatively well understood operators. Thus, the main point to check in the proof will be the definition of the map—that is, that the image of the map L lies in the appropriate spaces.

To this end, a routine calculation shows that I maps from $\Gamma^k_{\mathrm{Re}}(S^3)\times\Gamma^k(H_{(1,0)}(S^3))\to\Gamma^{k-1}(H^{(0,1)}(S^3))$. This fact, plus routine calculations, show that all factors of the image of L lie in the appropriate spaces except possibly the last three. For each of these components, we will have to check that the operators do not lose too many derivatives.

To check that the second and third to last factors of L lie in the appropriate spaces, it is sufficient to show that the map

$$(X^0, fe)\mapsto \mathrm{Im}\,\overline{\Box}_b\bar\partial_b^* I$$

is a mapping from $\Gamma^k_{\mathrm{Re}}(S^3)\times\Gamma^k(H_{(1,0)}(S^3))\to\Gamma^{k-3}(S^3)$. (We point out that the crucial observation here is that we are restricting to the imaginary part of the map $\overline{\Box}_b\bar\partial_b^* I$; the real part of this map actually does lose too many derivatives. In fact, the map L is as complicated as it is precisely in order to finesse this point.)

We now calculate as follows (modulo terms in Γ^{k-3}—that is, terms which do not lose too many derivatives):

$$\begin{aligned} \mathrm{Im}(\overline{\Box}_b\bar\partial_b^* I) &= \mathrm{Im}(\overline{\Box}_b\Box_b X^0)+\mathrm{Im}\Big(\overline{\Box}_b\bar\partial_b^*\frac{1}{2i}(\bar f\bar\partial_b f - f\bar\partial_b\bar f)\Big)+\Gamma^{k-3}\\ &= \mathrm{Im}\Big(\overline{\Box}_b\bar\partial_b^*\frac{1}{2i}(\bar f\bar\partial_b f - f\bar\partial_b\bar f)\Big)+\Gamma^{k-3} \end{aligned}$$

(since $\overline{\Box}_b\Box_b X^0$ is a real operator applied to a real valued function)

$$= \mathrm{Im}\Big(\frac{1}{2i}(\bar f\,\overline{\Box}_b\Box_b f - f\,\overline{\Box}_b\Box_b\bar f)\Big)+\Gamma^{k-3}.$$

Taking note of the fact that the first term in the last line is zero (it is the imaginary part of a real valued function— notice that $\overline{\Box}_b\Box_b=\Box_b\overline{\Box}_b$), we find that $\mathrm{Im}(\overline{\Box}_b\bar\partial_b^* I)$ is in Γ^{k-3} as required.

To check the regularity of the last factor of the mapping L, is sufficient to show that

$$\bar{e}\rfloor Q\circ I\colon \Gamma^k_{\mathrm{Re}}(S^3)\times\Gamma^k(H_{(1,0)}(S^3))\to\Gamma^k(S^3).$$

On the other hand, since the image of Q is the space of one-forms in the kernel of $\bar{\partial}_b^*$, the image of $\bar{e}\rfloor Q$ is the space of conjugate CR functions. Also, the operator $G\Box_b$ restricts to the space of conjugate CR functions as an isomorphism. Hence, it is also sufficient to show that

$$\Box_b\circ\bar{e}\rfloor Q\circ I\colon \Gamma^k_{\mathrm{Re}}(S^3)\times\Gamma^k(H_{(1,0)}(S^3))\to\Gamma^{k-2}(S^3).$$

To this end, calculating modulo terms in $\Gamma^k(H^{(0,1)}(S^3))$:

$$\begin{aligned}
Q\circ I&=Q\Big(if\bar{\omega}+X^0f\bar{\omega}+\frac{1}{2i}(\bar{f}\bar{\partial}_bf-f\bar{\partial}_b\bar{f})\Big)\\
&=Q\Big(\frac{1}{2i}(\bar{f}\bar{\partial}_bf-f\bar{\partial}_b\bar{f})\Big)+\Gamma^k=Q\Big(\frac{1}{2i}(\bar{f}\bar{e}f-f\bar{e}\bar{f})\bar{\omega}\Big)+\Gamma^k\\
&=\bar{S}\Big(\frac{1}{2i}(\bar{f}\bar{e}f-f\bar{e}\bar{f})\Big)\bar{\omega}+\Gamma^k
\end{aligned}$$

by Theorem 7.1, and

$$\bar{e}\rfloor Q\circ I=\bar{S}\Big(\frac{1}{2i}(\bar{f}\bar{e}f-f\bar{e}\bar{f})\Big)+\Gamma^k.$$

Furthermore, calculating modulo terms in $\Gamma^{k-2}(S^3)$:

$$\begin{aligned}
\Box_b\circ\bar{e}\rfloor Q\circ I&=\Box_b\circ\bar{S}\Big(\frac{1}{2i}(\bar{f}\bar{e}f-f\bar{e}\bar{f})\Big)+\Gamma^{k-2}\\
&=(\bar{\Box}_b+iT)\circ\bar{S}\Big(\frac{1}{2i}(\bar{f}\bar{e}f-f\bar{e}\bar{f})\Big)+\Gamma^{k-2}\\
&=iT\circ\bar{S}\Big(\frac{1}{2i}(\bar{f}\bar{e}f-f\bar{e}\bar{f})\Big)+\Gamma^{k-2}\\
&=\tfrac{1}{2}\bar{S}(T(\bar{f}\bar{e}f-f\bar{e}\bar{f}))+\Gamma^{k-2}\\
&=\tfrac{1}{2}\bar{S}(\bar{f}\bar{e}T(f)-f\bar{e}T(\bar{f}))+\Gamma^{k-2}\\
&=\tfrac{1}{2}\bar{S}(\bar{e}(\bar{f}T(f)-fT(\bar{f})))-\tfrac{1}{2}\bar{S}(\bar{e}\bar{f}T(f)-\bar{e}fT(\bar{f}))+\Gamma^{k-2}\\
&=\tfrac{1}{2}\bar{S}(\bar{e}(\bar{f}T(f)-fT(\bar{f})))+\Gamma^{k-2}\\
&=\Gamma^{k-2},
\end{aligned}$$

where the last line uses the fact that for any function u

$$\bar{S}\bar{e}(u)=\bar{e}\rfloor(\bar{S}\bar{e}(u)\cdot\bar{\omega})=\bar{e}\rfloor Q(\bar{e}(u)\cdot\bar{\omega})=\bar{e}\rfloor Q(\bar{\partial}_bu)=0,$$

by Theorem 7.1. The proposition follows. $\square$

12. The linearized map DL

The linearization of L at the origin is

$$\begin{aligned}
DL(X^0, fe) &= ((G\bar{\partial}_b^* DI)_-, \operatorname{Re}(G\bar{\partial}_b^* DI)_0, \operatorname{Avg}(X^0), (G\bar{\partial}_b^*(if\bar{\omega}))_-, \operatorname{Re}(G\bar{\partial}_b^*(if\bar{\omega}))_0,\\
&\qquad (G\overline{G}\operatorname{Im}(\overline{\Box}_b\bar{\partial}_b^* DI))_0, (G\overline{G}\operatorname{Im}(\overline{\Box}_b\bar{\partial}_b^* DI))_+, Q(DI))\\
&= ((G\bar{\partial}_b^* DI)_-, \operatorname{Re}(G\bar{\partial}_b^* DI)_0, \operatorname{Avg}(X^0), p_-, \operatorname{Re}(p_0),\\
&\qquad (G\overline{G}\operatorname{Im}(\overline{\Box}_b\bar{\partial}_b^* DI))_0, (G\overline{G}\operatorname{Im}(\overline{\Box}_b\bar{\partial}_b^* DI))_+, Q(DI)).
\end{aligned}$$

Substituting $DI(X^0, fe) = \bar{\partial}_b X^0 + if\bar{\omega} = \bar{\partial}_b X^0 + \bar{\partial}_b p + i\overline{H}\bar{\omega}$, then $\bar{\partial}_b^* DI = \Box_b(X^0+p)$, and

$$\begin{aligned}
DL(X^0, fe) &= ((G\Box_b(X^0+p))_-, \operatorname{Re}(G\Box_b(X^0+p))_0, \operatorname{Avg}(X^0), p_-, \operatorname{Re}(p_0),\\
&\qquad (G\overline{G}\operatorname{Im}(\overline{\Box}_b\Box_b(X^0+p)))_0, (G\overline{G}\operatorname{Im}(\overline{\Box}_b\Box_b(X^0+p)))_+, i\overline{H}\bar{\omega})\\
&= (X^0_- + p_-, \operatorname{Re}(G\Box_b X^0 + p)_0, \operatorname{Avg}(X^0), p_-, \operatorname{Re}(p_0), (G\overline{G}\,\overline{\Box}_b\Box_b\operatorname{Im}(p))_0,\\
&\qquad (G\overline{G}\,\overline{\Box}_b\Box_b\operatorname{Im}(p))_+, i\overline{H}\bar{\omega}),
\end{aligned}$$

where we have used the facts that X^0 is real and $\overline{\Box}_b\Box_b$ is a real operator. Continuing the calculation (and using Theorem 7.1 again):

$$\begin{aligned}
DL(X^0, fe) &= (X^0_- + p_-, X^0_0 - \operatorname{Avg}(X^0) + \operatorname{Re}(p)_0, \operatorname{Avg}(X^0),\\
&\qquad p_-, \operatorname{Re}(p)_0, (G\overline{G}\,\overline{\Box}_b\Box_b\operatorname{Im}(p))_0, (G\overline{G}\,\overline{\Box}_b\Box_b\operatorname{Im}(p))_+, i\overline{H}\bar{\omega})\\
&= (X^0_- + p_-, X^0_0 - \operatorname{Avg}(X^0) + \operatorname{Re}(p)_0, \operatorname{Avg}(X^0),\\
&\qquad p_-, \operatorname{Re}(p)_0, \operatorname{Im}(p)_0, (\operatorname{Im}(p))_+ - S(\operatorname{Im}(p))_+, i\overline{H}\bar{\omega}).
\end{aligned}$$

This map is clearly invertible. Components 4, 5, 6 and 7 uniquely determine the function p. (Recall that p is defined to be orthogonal to the CR functions.) The last component then uniquely determines f by $if\bar{\omega} = \bar{\partial}_b p + i\overline{H}\bar{\omega}$. Finally, once p is known, X^0 is uniquely determined by the first three components of the map. (Recall that X^0 is real.) We have proved the following theorem.

THEOREM 12.1. *There is a neighbourhood of the identity in the space of Γ^k contact diffeomorphisms which is parameterized by a neighbourhood of zero in the space $\Gamma^{k+1}_{\mathrm{Re}}(S^3)$ of real valued functions on S^3. The parameterization is as follows:*

$$p \mapsto L^{-1}(0, 0, -\operatorname{Avg}(p), p_-, p_0 - \operatorname{Avg}(p), 0, 0, 0). \tag{12.2}$$

COROLLARY 12.3. *The linearization of this parameterization is*

$$p \mapsto (X^0, fe) = (-p, (\bar{\partial}_b p)^{\#}) \tag{12.4}$$

where $\#: (ig\bar{\omega}) \mapsto ge$ is the operator that maps $(0,1)$ forms to their associated $(1,0)$ Hamiltonian vector field.

Proof. This follows from a straightforward calculation. $\square$

COROLLARY 12.5. *Under the parameterization in the above theorem, the linear subspace of the parameter space given by S^1 invariant functions corresponds to the contact diffeomorphisms which are S^1 equivariant—that is, the lift of symplectic diffeomorphisms of the quotient space $S^3/S^1 \cong S^2$ relative to the symplectic form $d\eta$.*

Proof. The proof follows from a careful check of the steps in the proof of the above theorem. If we restrict at the outset to diffeomorphisms which are equivariant with respect to the S^1 action, then all of the maps involved restrict to the subspaces where the data is invariant under the S^1 action. (In fact, the only nonlinear map involved is the tensor for the integrability condition, I, and it is easy to check that I is invariant under the S^1 action if the diffeomorphism is equivariant.) It follows from the proof, then, that the space of S^1 equivariant contact diffeomorphisms are parameterized (in the same fashion) by the real-valued S^1 invariant functions p. Notice in this case that p descends to a real valued function on S^2, the S^1 equivariant contact diffeomorphisms descend to diffeomorphisms on S^2 which preserve the symplectic form $d\eta$, and we are parameterizing the space of symplectomorphisms of S^2. $\square$

Remark 12.6. Notice that S^1 invariant data roughly corresponds to the lift of objects from S^2. In this vein, S^1 invariant CR structures correspond to the lift of complex structures on S^2, and S^1 equivariant contact diffeomorphisms correspond to symplectomorphisms on S^2. The action of the S^1 equivariant contact diffeomorphisms on S^1 invariant CR structures corresponds to the action of symplectomorphisms on the complex structure, and a normal form for S^1 invariant CR structures will correspond to a normal form for the complex structure on S^2. (We are considering the coordinate system on S^2 to be fixed, here.) Finally, if we consider the full space of S^1 equivariant diffeomorphisms, these will include diffeomorphisms of S^2 which change the symplectic form on S^2, and their lifts will change the contact form to a new S^1 invariant contact form. Since all complex structures on S^2 are equivalent (via some diffeomorphism), we immediately obtain that all S^1 invariant CR structures can be normalized to be the lift of the standard complex structure on $\mathbf{P}^1 \cong S^2$ via an S^1 invariant contact form, although the contact forms may be different. The choice of the contact form to use for the lifting corresponds to the choice of a norm on $\mathbf{C}^2$ (or the tautological line bundle over $\mathbf{P}^1$), and the lift of the complex structure via the contact form $\hat{\eta}$ to an S^1 invariant CR structure is CR equivalent to the CR structure obtained by restricting the complex structure on $\mathbf{C}^2$ to the circular domain defined by the norm associated to the form $\hat{\eta}$. Thus, there is a natural correspondence between the following objects: (i) circular domains in $\mathbf{C}^2$, (ii) norms on $\mathbf{C}^2$, (iii) norms on the tautological line bundle over $\mathbf{P}^1$, (iv) curvature forms (or symplectic forms) on $\mathbf{P}^1 \cong S^2$, (v) connection forms (or S^1 invariant contact forms normalized to have period 2π on the fibres) on S^3, and (vi) S^1 invariant CR structures on S^3.

V. Normal forms

13. The action of contact diffeomorphisms on the CR structure

We now determine how contact diffeomorphisms act on the space of the CR structures.

LEMMA 13.1. *If Φ is the diffeomorphism associated to the vector field $X^0T+fe+\bar{f}\bar{e}$, then*

$$\Phi^*\omega = \frac{1}{u^2}(d(f(1+iX^0)) - f^2\bar{\omega} + (1+iX^0)^2\omega).$$

Proof. The vector field

$$X^0T+fe+\bar{f}\bar{e} = 2\,\mathrm{Re}\left((iX^0z^1+fz^{\bar{2}})\frac{\partial}{\partial z^1} + (iX^0z^2-fz^{\bar{1}})\frac{\partial}{\partial z^2}\right)$$

corresponds to the contact diffeomorphism

$$\Phi(z^1,z^2) = \frac{1}{u}(z^1+Z^1, z^2+Z^2)$$

where $u=(|z^1+Z^1|^2+|z^2+Z^2|^2)^{1/2}$ and

$$(Z^1,Z^2) = ((iX^0z^1+fz^{\bar{2}}), (iX^0z^2-fz^{\bar{1}})).$$

$$\begin{aligned}
\Phi^*(\omega) &= \frac{(z^2+Z^2)}{u}d\left(\frac{(z^1+Z^1)}{u}\right) - \frac{(z^1+Z^1)}{u}d\left(\frac{(z^2+Z^2)}{u}\right)\\
&= \frac{1}{u^2}((z^2+Z^2)d(z^1+Z^1) - (z^1+Z^1)d(z^2+Z^2))\\
&= \frac{1}{u^2}\langle d((z^1+Z^1),(z^2+Z^2)), ((z^2+Z^2), -(z^1+Z^1))\rangle\\
&= \frac{1}{u^2}\langle d((1+iX^0)(z^1,z^2)+f(z^{\bar{2}},-z^{\bar{1}})), ((1+iX^0)(z^2,-z^1)+f(z^{\bar{1}},z^{\bar{2}}))\rangle\\
&= \frac{1}{u^2}(fd(1+iX^0)+(1+iX^0)df+(1+iX^0)^2(z^2dz^1-z^1dz^2)\\
&\quad -f^2(z^{\bar{2}}dz^{\bar{1}}-z^{\bar{1}}dz^{\bar{2}})+(1+iX^0)(f)(z^{\bar{1}}dz^1+z^{\bar{2}}dz^2+z^2dz^{\bar{2}}+z^1dz^{\bar{1}}))\\
&= \frac{1}{u^2}(d(f(1+iX^0))-f^2\bar{\omega}+(1+iX^0)^2\omega),
\end{aligned}$$

which is the statement in the lemma. □

THEOREM 13.2. *If the vector field $X^0T+fe+\bar{f}\bar{e}$ corresponds to the contact diffeomorphism associated to the generating function p, then the pullback CR structure $\Phi^*(\omega+\mu\bar{\omega})$ is defined by the 1-form*

$$\hat{\omega}=\omega+\frac{\bar{e}(f(1+iX^0))-f^2+\mu(1-iX^0)^2+\mu\bar{e}(\bar{f}(1-iX^0))}{e(f(1+iX^0))-\mu\bar{f}^2+(1+iX^0)^2+\mu e(\bar{f}(1-iX^0))}\bar{\omega}.$$

The action of contact diffeomorphisms on CR structures is

$$((X^0,fe),\mu)\xmapsto{A}\hat{\mu}$$

where $\hat{\mu}$ is defined by $\hat{\omega}=\omega+\hat{\mu}\bar{\omega}$.

Proof. Using the result in Lemma 13.1,

$$\begin{aligned}\Phi^*(\omega+\mu\bar{\omega})&=\frac{1}{u^2}(d(f(1+iX^0))-f^2\bar{\omega}+(1+iX^0)^2\omega\\&\quad+\mu d(\bar{f}(1-iX^0))-\mu\bar{f}^2\omega+\mu(1-iX^0)^2\bar{\omega})\\&=\frac{1}{u^2}(e(f(1+iX^0))-\mu\bar{f}^2+(1+iX^0)^2+\mu e(\bar{f}(1-iX^0)))\omega\\&\quad+\frac{1}{u^2}(\bar{e}(f(1+iX^0))-f^2+\mu(1-iX^0)^2+\mu\bar{e}(\bar{f}(1-iX^0)))\bar{\omega}\pmod{\eta}.\end{aligned}$$

It follows that the new CR structure is defined by the function $\hat{\mu}$:

$$\hat{\mu}=\frac{\bar{e}(f(1+iX^0))-f^2+\mu(1-iX^0)^2+\mu\bar{e}(\bar{f}(1-iX^0))}{e(f(1+iX^0))-\mu\bar{f}^2+(1+iX^0)^2+\mu e(\bar{f}(1-iX^0))}. \tag{13.3}$$

□

COROLLARY 13.4. *The linearization of the action of contact diffeomorphisms on CR structures at the origin is*

$$((\dot{X}^0,\dot{f}e),\dot{\mu})\xmapsto{dA}\delta\hat{\mu}=\bar{e}(\dot{f})+\dot{\mu}=-i\bar{e}\bar{e}\dot{p}+\dot{\mu}.$$

Proof. Let μ, X^0, f be small—$O(t)$—and compute $\hat{\omega}$ modulo terms $O(t^2)$.

$$\begin{aligned}\hat{\omega}&=\omega+\frac{\{\bar{e}(f)+\mu\circ\Phi\}}{\{e(f)+1+2iX^0\}}\bar{\omega}+O(t^2)\\&=\omega+(\bar{e}(f)+\mu\circ\Phi)\bar{\omega}+O(t^2)\\&=\omega+(\bar{e}(f)+\mu)\bar{\omega}+O(t^2),\end{aligned}$$

and the first variation at $(X^0,fe)=(0,0)$, $\mu=0$ is

$$\delta\hat{\mu}((\dot{X}^0,\dot{f}e),\dot{\mu})=\bar{e}(\dot{f})+\dot{\mu}. \tag{13.5}$$

□

Remark 13.6. The description of the linearization of the action of contact diffeomorphisms on the CR structures is actually more satisfying when expressed in terms of deformation tensors. In this invariant formulation, the CR structure corresponding to $\hat{\mu}$ is given by the deformation tensor

$$\hat{\phi}=\hat{\mu}\bar{\omega}\otimes e \tag{13.7}$$

and the linearization of the action becomes

$$\hat{\phi}=\phi+\bar{\partial}_b(fe) \tag{13.8}$$

$$=\phi+\bar{\partial}_b{}^{\#}\bar{\partial}_b p. \tag{13.9}$$

(See equation (2.1) for the definition of the operator #.) This formulation has the added advantage that it preserves the homogeneity of the Fourier decomposition. If the function p is S^1 invariant, then so is the associated deformation tensor; however, the coefficient function $\hat{\mu}$ is *not* S^1 invariant because it is expressed relative to a framing which is not S^1 invariant.

14. Normal forms

In this section, we present various normal forms for the deformation tensor; before stating the results, we will discuss the main ideas in the procedure. At the linearized level, we see in Corollary 13.4 that we are free to modify the form of coefficient $\hat{\mu}$ of the deformation tensor by terms of the form $-i\bar{e}\bar{e}p$, for some real valued function p. Since p is real valued, we are free to choose the negative Fourier coefficients of p in such a fashion as to normalize certain coefficients of the function $\hat{\mu}$ to be zero, and allow the positive coefficients of p to be completely determined by the negative coefficients and the condition that p is real. This results in the natural normalization that the coefficients of $\hat{\mu}$ be perpendicular to the image of $-i\bar{e}\bar{e}(p_-)$, at least at the linear level; the nonlinear version then follows from the inverse mapping theorem in Banach spaces (for some neighbourhood of the origin). Similarly, we could use the positive coefficients of p to determine the normal form for $\hat{\mu}$.

There are two points which will become readily apparent in this normalization procedure:

(1) The zeroth (or S^1 invariant) Fourier coefficient of p plays a special role. On the formal level, we are trying to normalize a complex valued function by the action of a real-valued function. While it is true that the formal result of this procedure can be written down, the answer is not as satisfying as for the other coefficients. On the other hand, since this coefficient corresponds to the S^1 equivariant contact diffeomorphisms (or bundle automorphisms which preserve the connection form η), it is natural to treat

this coefficient separately, and allow arbitrary S^1 equivariant diffeomorphisms in order to normalize the CR structure. This will result in a change in the contact form to a new S^1 invariant one, but it will enable us to normalize the corresponding complex valued Fourier coefficient of $\hat{\mu}$ to be zero. (We will see later that this coefficient of $\hat{\mu}$ will correspond to the S^1 invariant component of the CR structure, and thus descends in a natural way to define a complex structure on S^2; normalizing it to vanish is equivalent to normalizing the corresponding complex structure on $S^2 \cong \mathbf{P}^1$ to be the standard one.)

(2) Since we are normalizing the coefficient $\hat{\mu}$ of the deformation tensor relative to the framing $(e, \bar{e}, T)$ which is not S^1 invariant, the degree of homogeneity of the Fourier coefficients will be thrown off; thus, the zeroth Fourier coefficient of p will actually be normalizing the fourth Fourier coefficient of $\hat{\mu}$. This change in the homogeneity could have been avoided by referring to the deformation tensor as a whole, or choosing a better framing. However, we have chosen this approach in this paper for two basic reasons. The first is that in a paper which is already in preparation, we will deal with the higher dimensional case, and we will be forced into presenting the invariant approach there. The second reason is that we feel it is also worthwhile to present this approach. Since S^3 is parallelizable, we can do (and have presented much of it as such) all of the analysis relative to a fixed parallel framing for S^3, thus obviating the need to refer to tensorial analysis. From an analytic point of view, this eliminates much of the machinery which seems to be inherent in this problem—at least in higher dimensions.

THEOREM 14.1. *Let $\mu \in \Gamma^{k-1}(S^3)$ define a sufficiently small deformation of the standard CR structure of S^3 which is compatible with the standard contact structure. Then there is a contact diffeomorphism Φ_p parameterized by $p \in \Gamma^{k+1}_{\mathrm{Re}}(S^3)$ such that μ can be placed in the normal form $\hat{\mu} \in (\operatorname{coker}(\bar{e})^2 \cap \bigoplus_{m<4} \Gamma^{k-1}_m) \oplus (\bigoplus_{m\geqslant 4} \Gamma^{k-1}_m)$. Furthermore, the contact diffeomorphism is unique up to an S^1 equivariant contact diffeomorphism plus a preliminary automorphism of the standard CR structure.*

Proof. Consider the map

$$(\Gamma^{k+1}_{\mathrm{Re}}) \oplus \Big(\Big(\operatorname{coker}(\bar{e})^2 \cap \bigoplus_{m<4} \Gamma^{k-1}_m\Big) \oplus \Big(\bigoplus_{m\geqslant 4} \Gamma^{k-1}_m\Big)\Big) \to (\Gamma^{k-1}(S^3)) \oplus (\Gamma^{k+1}_{0,\mathrm{Re}}(S^3))$$

defined by

$$(p, \mu) \mapsto (\Phi^*_p \mu, p_0)$$

where Φ^*_p is the contact diffeomorphism corresponding to p, and p_0 is the zeroth Fourier coefficient of p. The linearization of this map at the origin is

$$(p, \mu) \mapsto (\mu - i\bar{e}\bar{e}p, p_0).$$

The linearized map is surjective, with kernel $\{(p,\mu)\,|\,\mu=p_0=0, \bar{e}\bar{e}p_-=0\}$. It is clear that the kernel of this linearized map is the set of infinitesimal contact diffeomorphisms which preserve the standard CR structure of S^3 (and which are not S^1 equivariant—these are included in the last factor of the map). More directly, p is in the kernel of the linearized map if and only if p is the restriction of the real part of a linear holomorphic function on $\mathbf{C}^2$. □

THEOREM 14.2. *Let $\mu\in\Gamma^{k-1}(S^3)$ define a sufficiently small deformation of the standard CR structure of S^3 which is compatible with the standard contact structure. Then there is a diffeomorphism of S^3 and a new S^1 invariant contact structure defined by an S^1 invariant contact form $\hat{\eta}$ such that the CR structure can be placed in the normal form $\hat{\mu}\in(\operatorname{coker}(\bar{e})^2\cap\Gamma^{k-1})\oplus(\bigoplus_{m>4}\Gamma^{k-1}_m)$, where $\hat{\mu}$ defines a deformation of the CR structure defined by $\hat{\eta}$, and the Γ^{k-1} spaces are defined relative to the contact structure defined by $\hat{\eta}$. Furthermore, the diffeomorphism is unique up to composition with a preliminary automorphism of the standard CR structure.*

Remark 14.3. (1) We should first say a word of explanation about the terminology in this and the previous theorem. There is actually a finite dimensional family of normalizing diffeomorphisms (and their corresponding normal forms) parameterized by the projection of $(p+iq)$ onto the kernel of the linearized map. Elements in the kernel of the linearized map correspond to automorphisms of the standard CR structure on S^3; thus, we may consider the normalizing diffeomorphism to be uniquely determined up to a preliminary automorphism of the standard CR structure on S^3.

(2) The fact that the normal form is only determined up to a finite dimensional family has an interesting interpretation. Elements in the kernel of the linearized map correspond to automorphisms of the standard CR structure on S^3, or equivalently, the restriction to the boundary of biholomorphic automorphisms of the standard unit ball in $\mathbf{C}^2$. In [BD1], we showed that for a bounded strongly convex domain in $\mathbf{C}^n$, there was a unique normal form for the CR structure on the boundary of the domain associated to any choice of marking for the domain—that is, for any choice of base point and holomorphic framing at that point; thus, the normalizing diffeomorphisms (or the associated normal forms) were parameterized by the choice of marking for the domain. Similarly, the biholomorphic automorphisms of the standard ball are parameterized by the markings of the ball. Thus, we may naturally consider our normal form to be normalized up to the choice of a marking of the domain which it 'bounds' (although different markings may result in the same normal form—as in the case of the standard CR structure). Alternatively, we could completely pin down the normal form for the CR structure by marking the CR manifold—choosing a base point on S^3, and specifying certain components of the second

order framing at that point.

Remark 14.4. Before beginning the proof, we should draw attention to a subtlety that is present. Theorem 14.1 normalizes the form up to an S^1 equivariant diffeomorphism. Thus, the proof of Theorem 14.2 is totally concerned with the action of S^1 equivariant diffeomorphisms. These have additional properties which will be utilized in the procedure.

(1) S^1 equivariant diffeomorphisms are bundle automorphisms, and they preserve the homogeneity of the coefficients. Thus, we may restrict our attention to its action on the S^1 equivariant part of the CR structure. (Notice that this may also be considered as the lift of a complex structure from S^2 via the connection form.)

(2) For S^1 equivariant data, the weighted Sobolev spaces agree with the unweighted Sobolev spaces; in particular, they do not depend upon the choice of S^1 equivariant contact form which is used to define the weighted spaces.

(3) We will allow the use of general S^1 equivariant diffeomorphisms to normalize the data. Thus, we will be changing the contact structure, but the new contact structure will still be invariant under the S^1 action, and it will be defined by a new contact form which is still dual to the flow of the action. Changing the contact structure is tantamount to changing the splitting of the tangent bundle to S^3 into its horizontal and vertical components.

(4) Finally, and herein lies the subtlety, we will be considering the action of the diffeomorphism on the coefficient μ which defines the deformation. If we have changed the splitting along the way (or the contact form), we will simply consider the coefficient μ to be defining the deformation relative to the new splitting. As a result, there will be many inequivalent CR structures having the same coefficient function μ, but having different contact forms; in particular, in the case that the coefficient μ of the deformation tensor is identically zero, we will be recovering strongly convex circular domains, and the contact form can be used to define a norm on the tautological line bundle over $\mathbf{P}^1$ (or a norm on $\mathbf{C}^2$) for which the set of all points of norm less than one is the corresponding circular domain.

Proof. We start with the normal form given in the previous theorem. Since we are only considering diffeomorphisms which are invariant under the S^1 action, it follows that its action on the CR deformations will preserve the homogeneity (or Fourier weighting) of the various coefficients. Thus, it is sufficient to understand the action on the zeroth Fourier coefficient. At this level, the weighted spaces are the same as the unweighted spaces. (This observation is important, and somewhat subtle; the coefficient determining the normal form for the deformation tensor will still be in the weighted Sobolev spaces, but where the weighting is defined by the new contact structure.)

We now consider the action of the S^1 invariant diffeomorphism $\Phi_{iq}\circ\Phi_p$, where Φ_p is the contact diffeomorphism induced by the real valued function p, and Φ_{iq} is the diffeomorphism induced by the vector field $2\operatorname{Re}(\bar\partial_b iq)^\# = fe+\bar f\bar e$. Since the second factor in this diffeomorphism is not a contact diffeomorphism, we must consider its action on both the contact structure and the CR structure.

To this end, Corollary 9.5 shows that

$$\Phi_{iq}^*\eta = \frac{1}{u^2}(\eta + I(0, 2\operatorname{Re}(\bar\partial_b iq)^\#)) = \frac{1}{u^2}(\eta + 2\operatorname{Re} I^{(0,1)}(0, (\bar\partial_b iq)^\#)).$$

Similarly, Lemma 13.1 shows that

$$\Phi_{iq}^*\omega = \frac{1}{u^2}(\omega + df - f^2\bar\omega).$$

Thus, under the action of the diffeomorphism Φ_{iq}, the contact structure defined by η and the CR structure defined by $\omega+\mu\bar\omega$ are pulled back to those defined by the new forms

$$\hat\eta = (\eta + 2\operatorname{Re} I^{(0,1)}(0, (\bar\partial_b iq)^\#))$$

and

$$\hat\omega = (\omega + df - f^2\bar\omega) + (\mu\circ\Phi)(\bar\omega + d\bar f - \bar f^2\omega).$$

The linearization of these actions at the origin is given by

$$\eta \mapsto \eta + 2\operatorname{Re}(\bar\partial_b iq)^\# \lrcorner (i\omega\wedge\bar\omega) = \eta + 2\operatorname{Re}\bar\partial_b iq$$

and

$$\omega \mapsto \omega + \mu\bar\omega + \bar\partial_b f$$

or

$$\omega \mapsto \omega + (\mu + \bar e f)\bar\omega = \omega + \hat\mu\bar\omega.$$

Consider now the map

$$(\Gamma_0^{k+1}) \oplus (\operatorname{coker}(\bar e)^2 \cap \Gamma^{k-1}) \oplus \Big(\bigoplus_{m>4} \Gamma^{k-1}\Big) \to (\operatorname{coker}(\bar e)^2 \cap \Gamma^{k-1}) \oplus \Big(\bigoplus_{m\geqslant 4} \Gamma^{k-1}\Big)$$

defined by

$$(p_0 + iq_0, \mu) \mapsto (\Phi_{iq}^* \circ \Phi_p^* \mu)$$

where μ defines the CR structure in conjunction with the new contact structure defined by $\hat\eta$. The linearization of this action at the origin is

$$(p_0 + iq_0, \mu) \mapsto (\mu - i\bar e\bar e(p_0 + iq_0)).$$

Since the linearized map is surjective, the normal form for $\hat{\mu}$ follows. Furthermore, since the kernel of the linearized map is given by holomorphic automorphisms of S^2, with its standard complex structure, the normalizing diffeomorphism is determined up to a holomorphic automorphism of S^2.

We now consider the remainder of the S^1 equivariant diffeomorphism group. Since the full group is parameterized by three real valued functions (the coefficients of the vector field), and since we have already considered vector fields with arbitrary parameters in the contact directions, it suffices to consider the diffeomorphisms corresponding to vector fields of the form $X = X^0 T$. (Notice that these correspond to diffeomorphisms which simply 'rotate the fibres'—that is, they cover the identity map on S^2.) While it is true that it would have been just as simple to consider the full diffeomorphism group at once, we felt that it was more interesting to treat it in stages in order to see the effects of the various subgroups on the normalization procedure.

Consider a diffeomorphism corresponding to a vector field of the form $X = X^0 T$; denote the diffeomorphism by Φ_{X^0}. Then again by Corollaries 9.5 and 13.1,

$$\begin{aligned} \Phi^*_{X^0}\eta &= \eta + \hat{d}X^0 \\ \Phi^*_{X^0}\omega &= \frac{1}{u^2}(1+iX^0)^2\omega. \end{aligned}$$

In particular, $\Phi^*_{X^0}$ preserves the normal form for the deformation tensor; thus, it suffices to choose X^0 in such a fashion as to normalize the contact form.

To this end, we consider

$$\Phi^*_{X^0} \circ \Phi^*_{iq} \circ \Phi^*_p(\eta) = \Phi^*_{X^0} \circ \Phi^*_{iq}(\eta)$$

and its linearized action on the contact form

$$\begin{aligned} \eta \mapsto \eta + \hat{d}(X^0) + 2\,\mathrm{Re}(\bar{\partial}_b iq) &= -\,\mathrm{Im}\,\bar{\partial}(\log|z|^2 + 2q) + 2\,\mathrm{Re}\,\bar{\partial}(X^0) \\ &= -\,\mathrm{Im}\,\bar{\partial}(\log e^{2q}|z|^2) + 2\,\mathrm{Re}\,\bar{\partial}(X^0) \end{aligned}$$

where we have extended the definitions of the S^1 invariant functions q, X^0 to $\mathbf{C}^*$ invariant functions on $\mathbf{C}^2 \backslash 0$, and used the $\bar{\partial}$ operator from $\mathbf{C}^2$. Thus, X^0 is completely determined (up to a constant) by the requirement that the contact form η be the restriction to S^3 of minus the imaginary part of $\bar{\partial}\log u$ for some norm u on $\mathbf{C}^2$. Such contact forms are completely determined either by their curvature form, or by the norm (up to scale) which they induce on $\mathbf{C}^2$.

Finally, notice that the linearized map is surjective onto the space of Hermitian connection forms described above. Thus, the normalized data for a CR structure consists of a norm (up to scale) on the tautological line bundle E (or equivalently, its connection form or its indicatrix in $\mathbf{C}^2$), and a deformation tensor in normal form which describes a deformation relative to the associated contact form; conversely, any such data is the data for some CR structure. Finally, any CR structure admits only a finite dimensional set of possible 'normal forms', which differ by a preliminary automorphism of the standard CR structure. □

Remark 14.5. The normalization for the contact form in the above theorem could also have been expressed in terms of the Hodge theory for S^2. First, any S^1 invariant contact form $\hat{\eta}$ which is dual to the S^1 action differs from the standard contact form η by the pullback of a real valued form θ from S^2. Using the complex structure tensor J on S^2, we can use Hodge theory to express θ uniquely as

$$\theta = du + J\,dv$$

where u, v are real valued functions, and $J\,dv$ is co-closed. The normalization in the previous theorem is that the difference between the two contact forms is of the form $\theta = J\,dv$; such contact forms are completely determined by their curvature forms.

Remark 14.6. Before stating the next theorem, we will have to introduce the full harmonic decomposition for the sphere. Let $B_{m,n}$ denote the invariant L^2 subspaces under the $SU(2)$ action, where m represents the holomorphic degree of the subspace, and n the conjugate holomorphic degree.

We include some basic facts about the various operators on these spaces. First,

$$e\colon B_{m,n} \longmapsto \begin{cases} B_{m-1,n+1}, & m \geqslant 1, \\ 0, & m = 0, \end{cases}$$

$$\bar{e}\colon B_{m,n} \longmapsto \begin{cases} B_{m+1,n-1}, & n \geqslant 1, \\ 0, & n = 0. \end{cases}$$

Consider the operator $\bar{e}e\colon B_{m,n} \to B_{m-1,n+1} \to B_{m,n}$, $m \geqslant 1$.

$$\begin{aligned} \ker(\bar{e}e) = 0 &\implies \bar{e}e \text{ is invertible on } B_{m,n}, \ m \geqslant 1 \\ &\implies \bar{e} \text{ is surjective onto } B_{m,n}, \ m > 0. \end{aligned}$$

It follows that the operator

$$\bar{e}\bar{e}\colon B_{m,n} \xrightarrow{\text{iso}} B_{m+2,n-2}, \quad n > 1,$$

satisfies the following:

$$\ker(\bar{e}^2) = B_{m,0} \oplus B_{m,1} \subset \Big(\bigoplus_{k \geqslant -1} \Gamma_k \Big),$$
$$\operatorname{coker}(\bar{e}^2) = B_{0,n} \oplus B_{1,n} \subset \Big(\bigoplus_{k \leqslant 1} \Gamma_k \Big)$$

and

$$\bar{e}^2 \colon \Gamma_k \to \Gamma_{k+4},$$
$$\bar{e}^2 \colon \Big(\bigoplus_{k<0} \Gamma_k \Big) \to \Big(\bigoplus_{k<4} \Gamma_k \Big).$$

THEOREM 14.7. *Let $\mu \in \Gamma^{k-1}(S^3)$ define a sufficiently small deformation of the standard CR structure of S^3 which is compatible with the standard contact structure. Then there is a contact diffeomorphism of S^3 such that the CR structure can be placed in the normal form $\hat{\mu} \in \bigoplus_{l \leqslant 4} \Gamma_l^{k-1}$. Furthermore, if we additionally require that the components of μ in $B_{2,n} \oplus B_{3,n}$ are specified to be zero, then the contact diffeomorphism is unique up to an S^1 equivariant contact diffeomorphism plus an automorphism of the standard CR structure.*

Proof. Consider the map

$$\Gamma_{\mathrm{Re}}^{k+1} \times \Big(\Big(\bigoplus_{l \leqslant 4} \Gamma_l^{k-1} \Big) \setminus (B_{2,n} \cup B_{3,n}) \Big) \to \Gamma^{k-1}(S^3) \times \Gamma_{0,\mathrm{Re}}^{k+1}(S^3)$$

defined by

$$(p, \mu) \mapsto (\Phi_p^* \mu, p_0)$$

where Φ_p^* is the contact diffeomorphism corresponding to p. The linearization of this action at the origin is

$$(p, \mu) \mapsto (\mu - i\bar{e}\bar{e}p, p_0).$$

The linearized map is surjective, with kernel $\{(p,\mu) \mid \mu = p_0 = 0, \bar{e}\bar{e}p_- = 0\}$. It is clear that the kernel of this linearized map is the set of infinitesimal contact diffeomorphisms which preserve the standard CR on S^3 (and which are not S^1 equivariant—these are included in the last factor of the map). More directly, p is in the kernel of the linearized map if and only if p is the restriction of the real part of a linear holomorphic function on $\mathbf{C}^2$. □

VI. Imbedding results

15. Extension results—S^3

In this section, we will prove imbedding results for CR structures in their normal form. We will show that in general, any small perturbation of the standard CR structure on

S^3 is the strongly pseudoconcave boundary of a domain in a complex manifold. (This is a special case of a result due to Kiremidjian [K].) Our proof will use the normal form given in Theorem 14.7, and this normal form can be thought of as the *exterior normal form.* (In the proof, we will show that the deformation extends to a deformation of the complex structure on the exterior of the unit ball in $\mathbf{P}^2$.) On the other hand, if in the normal form given in Theorem 14.1, there are no negative Fourier coefficients, then the CR manifold bounds a convex domain in $\mathbf{C}^2$; this normal form can be thought of as the *interior normal form,* since the deformation extends to a deformation of the complex structure on the interior of the unit ball. The method of proof in both cases will be to explicitly write down the deformation of the complex structure on the associated complex manifold. In the latter case, the normal form will be identified with the circular model for convex domains [BD1], and we will obtain the corollary that if there exist any negative coefficients in the normal form, then the CR structure does not bound a convex domain.

THEOREM 15.1. *Let $\mu \in \Gamma^{k-1}(S^3)$ define a sufficiently small deformation of the standard CR structure of S^3. Then the CR structure imbeds as the boundary of a convex domain if and only if it can be placed in the normal form $\mu \in \bigoplus_{m \geqslant 4} \Gamma^{k-1}$. Furthermore, after composition with an S^1 equivariant diffeomorphism, the normal form agrees with the data corresponding to a point in the moduli space for marked convex domains.*

Remark 15.2. In the above theorem, the meaning of the various normalizations becomes clear. First, recall that the modular data for a convex domain is given by data on the tautological line bundle over $\mathbf{P}^1$ consisting of a norm on the line bundle (the indicatrix) and a deformation of the complex structure which is horizontal and holomorphic in the fibre directions. On the other hand, the preliminary normal form for the CR structure on the boundary of D (considered as a CR structure on S^3 via a diffeomorphism) corresponds to one which extends to define a deformation of the complex structure on the unit disc bundle in the tautological line bundle, but for which the extension does not necessarily restrict to the zero section to agree with the standard complex structure on $\mathbf{P}^1$. The secondary normalization corresponds to composing with a diffeomorphism of $\mathbf{P}^1$ so that the deformed complex structure on the zero section agrees with the standard one on $\mathbf{P}^1$, at the possible cost of changing the norm on the complex line bundle—the indicatrix. That is, the normal form for the CR structure consists of an S^1 invariant contact form, and a deformation tensor with only strictly positive Fourier coefficients. This corresponds to a point in the moduli space [BD1]. Finally, the diffeomorphism which places the data for the boundary of the convex domain D in normal form is unique up to the choice of a base point for the Kobayashi metric on D and the choice of framing at that point.

Proof. The proof of the theorem lies in expressing the normal form for the CR structure as a deformation of the standard CR structure on the sphere. Then, we can either extend the CR structure directly to produce a complex manifold for which it is the strongly pseudoconvex boundary, or we can appeal to the moduli space constructed in [BD1].

To this end, we consider the sphere as the unit circle bundle sitting inside the tautological line bundle over $\mathbf{P}^1$. Let w be a local complex coordinate for $\mathbf{P}^1$, and let ζ be a holomorphic fibre coordinate. Choose a horizontal lift $\tilde{e}$ of the holomorphic vector field $\partial/\partial w$ (using the contact form η as the connection form on the unit circle bundle). Then the new CR structure can be written as a deformation tensor

$$\phi \in \operatorname{Hom}(H_{(0,1)}, H_{(1,0)})$$

where

$$\phi = \tilde{\mu}\, d\bar{w} \otimes \tilde{e}.$$

A straightforward calculation shows that $d\bar{w}, \tilde{e}, \tilde{\mu}$ are related to $\bar{\omega}, e, \mu$ by

$$d\bar{w} = -\frac{\bar{\omega}}{\bar{\zeta}^2}, \quad \tilde{e} = -\zeta^2 e, \quad \tilde{\mu} = \frac{\bar{\zeta}^2}{\zeta^2}\mu.$$

Notice that when the deformation is expressed relative to this S^1 invariant framing, then the weight 4 terms of μ correspond to weight 0 terms of $\tilde{\mu}$. It follows that if the deformation data μ is in the normal form given in the theorem, then the coefficient $\tilde{\mu}$ in the deformation tensor has no negative Fourier components (they start at weight zero), and it may be extended as a tensor to the entire unit disc bundle over $\mathbf{P}^1$ by analytic extension, disc by disc. Although it is originally interpreted as a deformation tensor $\phi \in \operatorname{Hom}(H_{(0,1)}, H_{(1,0)})$, it may be naturally identified as an element $\phi \in \operatorname{Hom}(T_{(0,1)}, T_{(1,0)})$, or as a deformation of the full complex structure of the unit disc bundle. It is easy to check that the deformed complex structure satisfies the integrability conditions (see [BD1]), and that it extends smoothly to the zero section $\mathbf{P}^1$. (The zero Fourier coefficients correspond to a deformation of the complex structure on $\mathbf{P}^1$.) Thus, the deformed CR structure bounds a complex manifold. It is clear that it is a strongly pseudoconvex boundary.

Now act on the deformation tensor by an S^1 equivariant diffeomorphism which puts the deformation tensor in the normal form described in Theorem 14.2. Then the data for the normal form consists of an S^1 invariant contact form (which is equivalent to prescribing a norm on the tautological line bundle E over $\mathbf{P}^1$, or an indicatrix for the norm in $\mathbf{C}^2$) and a deformation tensor describing the CR structure relative to the lift of the standard complex structure on $\mathbf{P}^1$ to an S^1 invariant CR structure. This normal form corresponds to a circular model (or a point in the moduli space) precisely when the deformation tensor has only strictly positive Fourier coefficients.

Finally, notice that if in fact the CR structure is equivalent to one on the boundary of a convex domain in $\mathbf{C}^2$, then the circular model shows that the CR structure is equivalent to one in the normal form given in the statement of Theorem 14.2 with only strictly positive Fourier coefficients in the deformation tensor; this property is preserved under the action of the finite dimensional group which is not normalized by this procedure. This shows that for CR manifolds which are sufficiently small perturbations of the sphere, the CR manifold imbeds as the boundary of a domain in $\mathbf{C}^2$ if and only if the normal form has only strictly positive Fourier coefficients. □

Our next theorem is a special case of a theorem due to Kiremidjian [K]; we include it as an application of our normal form analysis.

THEOREM 15.3 (Kiremidjian). *Let $\mu \in \Gamma^{k-1}(S^3)$ define a sufficiently small deformation of the standard CR structure of S^3. Then there is a complex manifold for which this CR manifold is the strongly pseudoconcave boundary.*

Proof. The proof follows from direct construction of the manifold. The original sphere can be considered to be imbedded in $\mathbf{P}^2$; as such, it is the strongly pseudoconcave boundary of the complement of the unit ball.

Alternatively, we may proceed as follows. Consider the tautological line bundle E over S^2. By taking the one point compactification of the leaves, this sits inside a $\mathbf{P}^1$ bundle over S^2. The total space of this bundle is again $\mathbf{P}^2$ with the origin blown up—denoted $\tilde{\mathbf{P}}^2$; the blow up of the origin is the original S^2. In this interpretation, the complement of the unit ball in $\mathbf{P}^2$ is the exterior of the unit disc bundle in the $\mathbf{P}^1$ bundle over S^2. It also naturally fibres as a unit disc bundle over the hyperplane at infinity in $\mathbf{P}^2$. These considerations show that this bundle is naturally identified with the dual of the unit disc bundle associated to the tautological line bundle over S^2.

As in the previous theorem, the deformed CR structure can be expressed as

$$\phi \in \mathrm{Hom}(H_{(0,1)}, H_{(1,0)})$$

where

$$\phi = \tilde{\mu}\, d\bar{w} \otimes \tilde{e}.$$

By the normal form in Theorem 14.7, there is a contact diffeomorphism which will normalize the coefficient $\tilde{\mu}$ in the deformation tensor to have no positive Fourier coefficients. This implies that the tensor can be extended to the exterior of the unit disc bundle inside the $\mathbf{P}^1$ bundle. As before, the tensor defines a deformation of the full complex structure of the unit disc bundle over the hyperplane at infinity, which is integrable. Thus, the CR hypersurface bounds this complex manifold, and it is clear that it is a strongly pseudoconcave boundary.

Alternatively, if we dualize the bundle (or consider the S^1 bundle as an S^1 bundle over the hyperplane at infinity, and hence as the unit circle bundle in the dual to the tautological line bundle over the hyperplane at infinity), then the negative coefficients become positive coefficients, and the deformation tensor can be analytically extended as a tensor on the full unit disc bundle (by holomorphically extending it along the discs), and viewed as a deformation of the full complex structure of the unit disc bundle. Again, the deformed complex structure is integrable. It follows that the CR manifold bounds this complex manifold, and it is clear that it is strongly pseudoconcave. $\square$

16. General extension results

The results in the previous section can be easily generalized to the case where the underlying contact manifold admits a free transverse S^1 action. In this case, the natural requirements for the extension of the CR structure to a complex structure on an associated manifold is a normal form in which the CR structure can be written as a deformation of an S^1 invariant CR structure, where the deformation tensor has no negative Fourier coefficients (or alternatively, no positive ones). Finding the normal form can be viewed as finding the boundaries of a natural family of discs along which to do a Bishop type extension of the complex structure.

THEOREM 16.1. *Let M be a compact three dimensional CR manifold such that the underlying contact manifold admits a free transverse S^1 action. Suppose, further, that the CR structure admits a normal form relative to this S^1 action which has no negative Fourier coefficients. (More precisely, the given CR structure can be expressed as a deformation of an S^1 invariant CR structure with no negative Fourier components in the deformation tensor.) Then M is the strongly pseudoconvex boundary of a complex manifold.*

Proof. First, since S^1 acts freely on M, the quotient space Σ of M by the S^1 action is a smooth compact surface, and M fibres as a principal S^1 bundle over Σ. Choose an S^1 invariant contact form η on M, normalized such that the periods of the fibres of the map are 2π.

Choose a complex structure for Σ, and let w be a local holomorphic coordinate on Σ. The complex structure on Σ can be lifted to an S^1 invariant CR structure on M

by defining the horizontal lift e of $\partial/\partial w$—via the connection form η—to be a basis for the holomorphic tangent space on M. (Alternatively, e is dual to the invariant forms $\eta, dw, d\bar{w}$.)

Consider the prescribed CR structure on M to be a deformation of the S^1 invariant CR structure just constructed. Let $\tilde{e}$ be a holomorphic vector field for the deformed complex structure. Since it has the same underlying contact structure, it can be written as

$$\tilde{e} = \bar{\alpha} e - \bar{\beta} \bar{e}.$$

(The reason for taking the complex conjugates of the coefficients and for the minus sign will become apparent below.) The strong pseudoconvexity condition implies that

$$[\tilde{e}, \bar{\tilde{e}}] = [\bar{\alpha} e - \bar{\beta} \bar{e}, \alpha \bar{e} - \beta e] = (|\alpha|^2 - |\beta|^2)[e, \bar{e}] \qquad (\mathrm{mod}\ e, \bar{e})$$

is non-vanishing on M. Thus, the term $|\alpha|^2 - |\beta|^2$ is nowhere zero, and either $|\alpha| > |\beta|$ or $|\beta| > |\alpha|$. We may assume that $|\alpha| > |\beta|$. (Otherwise, by starting with the conjugate of the complex structure on Σ, we can change between the two cases above.) Since this is a global condition on the coefficients, and a basis for the holomorphic tangent space is only determined up to multiplication by a non-vanishing function, we will normalize our choice of $\tilde{e}$ by requiring that $\alpha \equiv 1$.

Associated to the S^1 principal bundle M over Σ is a complex line bundle E over Σ defined by

$$E := \mathbf{C} \otimes_{S^1} M.$$

The S^1 action on M naturally extends to an S^1 action on E, and is canonically imbedded in a $\mathbf{C}^*$ action on E. Choose a local fibre coordinate ζ; the vector field $\zeta\, \partial/\partial\zeta$ is a generator of the $\mathbf{C}^*$ action on E. Extend the S^1 invariant vector field e on $M \subset E$ to a $\mathbf{C}^*$ invariant vector field on E. Define a $\mathbf{C}^*$ invariant complex structure on E by choosing the pair $\partial/\partial\zeta, e$ to be a basis for the the holomorphic tangent space. Using this complex structure, E becomes a holomorphic line bundle over Σ, and M is the unit sphere bundle in E associated to some Hermitian metric on the holomorphic line bundle E. The form η extends to E as a connection form for this metric.

The coordinates (w, ζ) are local holomorphic coordinates for the holomorphic line bundle, and the surface Σ may be considered as the zero section of this holomorphic line bundle. Extend the basis $\tilde{e} = e - \bar{\beta}\bar{e}$ for the given CR structure on $M \subset E$ to a vector field on the unit disc bundle U of E (that is, the connected component of the complement of M which contains the zero section of E) by extending the coefficients harmonically along the fibres. Extend the CR structure from the unit sphere bundle $M \subset E$ to an almost complex structure on U by defining

$$\tilde{e}, \quad \zeta\frac{\partial}{\partial\zeta}$$

to be a basis for the almost complex structure. (Notice that when $\bar{\beta}\equiv 0$, this recovers the holomorphic structure of the complex line bundle.) This almost complex structure is integrable as long as the coefficient $\bar{\beta}$ is conjugate holomorphic in the fibre directions; that is, β has no negative Fourier coefficients.

In terms of the dual coframing for the CR structures, the S^1 invariant coframing dual to the holomorphic tangent space for the S^1 invariant CR structure on M is given by $\eta, dw, d\bar{w}$, and a coframing for the given CR structure is given by $\eta, \tilde{\omega}, \bar{\tilde{\omega}}$ where

$$\tilde{\omega} = dw + \beta\, d\bar{w}.$$

In a more invariant formulation, the given CR structure can be expressed as a deformation $\phi \in \mathrm{Hom}(H_{(0,1)}, H_{(1,0)})$ of the S^1 invariant CR structure by

$$\tilde{e} = e - \bar{\phi}(e)$$

where

$$\phi = \beta\, d\bar{w} \otimes e.$$

The deformation tensor ϕ extends to define a deformation of the complex structure on the unit disc bundle U if the coefficients of ϕ relative to an invariant framing have no negative Fourier coefficients. □

Remark 16.2. The zeroth Fourier components in the deformation tensor correspond to a deformation of the complex structure on Σ. In particular, they can be eliminated by appropriately choosing the original complex structure on Σ, or the S^1 invariant CR structure.

17. Direct imbedding results

The last section characterizes those deformations of the CR structure which arise from deforming the complex structure on a holomorphic line bundle. It follows from basic results on complex manifolds that the ring of CR functions for these deformed structures is a small perturbation of the ring of holomorphic functions for the S^1 invariant complex structure. However, it is instructive to also give a direct construction of this perturbation argument, using only the solution for the $\Box_b$ operator of the S^1 invariant CR structure. (Notice also that it can be expressed in terms of solutions for the $\bar{\partial}$ operator on tensor powers of a holomorphic line bundle over the Riemann surface Σ.)

THEOREM 17.1. *Let M be a principal S^1 bundle over a complex surface Σ with connection form η. Suppose that η defines a contact structure on M. Suppose, in addition, that M admits an S^1 invariant CR structure for which the cohomology for $\Box_b$ lies completely in the negative Fourier components. Let ϕ define a deformation of this CR structure which has only strictly positive Fourier components. Then, if ϕ is sufficiently small, M with the CR structure defined by ϕ is imbeddable as the strongly pseudoconvex boundary of a domain in a complex manifold.*

Proof. We begin by referring to the basic facts about the $\Box_b$ operator which we will be using—namely, that the Green's operator produces the canonical solution for the $\bar{\partial}_b$ equation with estimates, and it solves the equation whenever the one form is orthogonal to the kernel of $\bar{\partial}_b^*$. We will perturb the CR functions relative to the S^1 invariant CR structure to obtain CR functions for the given CR structure by an iterative procedure which involves iteratively solving for a correction term using the solution operator to the $\bar{\partial}_b$ equation in the S^1 invariant CR structure.

The iterative procedure is as follows. Let h be a CR function for the S^1 invariant CR structure, and let $u=\sum_{k=1}^{\infty} u_k$ be such that $\tilde{h}=h+u$ is the corresponding CR function for the given CR structure. Then we can solve iteratively as follows:

$$\bar{\partial}_b u_1 = \phi\circ\partial_b(h),$$
$$\bar{\partial}_b u_k = \phi\circ\partial_b(u_{k-1}), \quad k>1.$$

Notice that at each stage, the solution exists as long as the kernel of $\bar{\partial}_b^*$ is orthogonal to the positive weight Fourier components. Furthermore, we could write down the full iterative solution to this procedure as follows:

$$\begin{aligned}\tilde{h} = h+u &= \sum_{k=0}^{\infty}(\bar{\partial}_b^* G\phi\circ\partial_b)^k h \\ &= h+\bar{\partial}_b^* G\sum_{k=0}^{\infty}(\phi\circ\partial_b\bar{\partial}_b^* G)^k\phi\circ\partial_b(h).\end{aligned}$$

This sum converges as long as the operator sup-norm of $\bar{\partial}_b^* G\phi\circ\partial_b$ is less than one. Furthermore,

$$\begin{aligned}(\bar{\partial}_b-\phi\circ\partial_b)\tilde{h} &= \Box_b G\sum_{k=0}^{\infty}(\phi\circ\partial_b\bar{\partial}_b^* G)^k\phi\circ\partial_b(h) - \sum_{k=0}^{\infty}(\phi\circ\partial_b\bar{\partial}_b^* G)^k\phi\circ\partial_b(h) \\ &= (\Box_b G - I)\sum_{k=0}^{\infty}(\phi\circ\partial_b\bar{\partial}_b^* G)^k\phi\circ\partial_b(h).\end{aligned}$$

In the case that $\Box_b G=I$ on the space of positive Fourier coefficients, then the right hand side vanishes, and the iteratively defined function $\tilde{h}$ is CR relative to the given CR structure. $\Box$

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