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Boundary behavior of extremal plurisubharmonic functions

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Introduction

In a celebrated paper Lempert [12] and later in a more general setting Klimek [9], Poletskiĭ [20], and Zaharjuta [27] introduced the notion of a pluricomplex Green function g for a bounded convex domain G in $\mathbf{C}^N$. Assuming that G contains the origin, this Green function with pole at 0 is given by

$$g(z):=\sup_u u(z), \quad z\in G,$$

where the supremum is taken over all plurisubharmonic functions $u: G\to[-\infty,0[$ with $u(w)\leqslant\log|w|+O(1)$ as $w\to 0$. This function is plurisubharmonic and it is continuous on $\overline{G}\setminus\{0\}$ if $g|\partial G:\equiv 0$. Lempert's results imply that also the sublevel sets $G_x=\{z\mid g(z)<x\}$, $x<0$, are convex. If

$$H_x(z):=\sup\{\operatorname{Re}\langle w,z\rangle \mid g(w)<x\}, \quad z\in\mathbf{C}^N,$$

denotes the supporting function of $G_x\subset\mathbf{R}^{2N}$, we introduce a type of directional Lelong number

$$D_G(a):=\lim_{x\uparrow 0}\frac{H_0(a)-H_x(a)}{-x}\in\,]0,\infty], \quad a\in S:=\{z\in\mathbf{C}^N \mid |z|=1\},$$

which measures the rate of approximation of ∂G by ∂G_x, $x<0$, in the direction of a. In the case that there is a biholomorphic mapping ψ of the ball $U:=\{z\in\mathbf{C}^N \mid |z|<1\}$ onto G with $\psi(0)=0$, this quantity is closely related to the notion of an angular derivative. For instance if $N=1$, D_G is bounded if and only if $|\psi'|$ is bounded on G (see [16]).

We show that the lower semicontinuous function D_G is connected with the boundary behavior of another extremal plurisubharmonic function, which has been introduced by Siciak [23], [24], [25]. We put $H:=H_0$ and consider

$$V_H(z)=\sup_u u(z), \quad z\in\mathbf{C}^N,$$

where the supremum is taken over all plurisubharmonic functions $u \leqslant H$ on $\mathbf{C}^N$ with $u(z) \leqslant \log|z| + O(1)$ as $z \to \infty$. This function is plurisubharmonic and continuous. It attains the values of H on a compact star shaped set

$$P_H := \{\lambda a \mid 0 \leqslant \lambda \leqslant 1/C_H(a), a \in S\},$$

the numbers $C_H(a)$ being in $]0, \infty]$. If $N=1$, P_H is the set of accumulation points of the Fekete–Leja points with respect to H (Siciak [23]).

THEOREM I. *There is some $C \geqslant 1$ such that $C_H \leqslant D_G \leqslant C C_H$.*

As a corollary of Theorem I we obtain:

THEOREM II. *The following assertions are equivalent.*
(i) *D_G (or C_H) is bounded.*
(ii) *There is some $C>0$ with*

$$G \subset G_x + C(-x)U, \quad x < 0.$$

(iii) *There is a plurisubharmonic function $v \leqslant H$ with $v(z) \leqslant \log|z| + O(1)$ as $z \to \infty$ which coincides with H on a neighborhood of zero.*

Using well known facts about the angular derivative of conformal mappings, for $N=1$, we have studied in [16] whether D_G is bounded. For $N \geqslant 2$, we investigate D_G by investigating C_H. If ∂G is of class $C^{1,1}$ then C_H is bounded. A maximal cone $\Gamma \subset \mathbf{C}^N$ of linearity of H with $\mathbf{R}$-linear hull $L(\Gamma)$ is called quasi-real if the maximal $\mathbf{C}$-linear subspace $L(\Gamma) \cap iL(\Gamma)$ of $L(\Gamma)$ intersects Γ only in the origin.

THEOREM III. *If G is a polyhedron and Γ' is the union of all quasi-real cones of H, then C_H is bounded on $\Gamma' \cap S$ and is infinite on the complement.*

Since the support of $(dd^c H)^N$ is the union of all maximal cones Γ of linearity of H for which $L(\Gamma) \cap iL(\Gamma) = \{0\}$, this shows that $\operatorname{supp}(dd^c H)^N \subset \Gamma'$. If $N \geqslant 2$, there are examples for which this inclusion is proper.

The results presented here have been developed from the author's investigations [16] and [17] for the case $N=1$.

The organization of the present paper is as follows. In part one we collect some facts about the functions $u(z,x) := H_x(z)$ and $v(z,C) := CV_H(z/C)$ where $z \in \mathbf{C}^N$, $x<0$, and $C>0$. These facts (in particular one due to Demailly [4]) show that both functions are members of two classes of one-parameter families of plurisubharmonic functions which are mapped onto each other by Kiselman's [7] partial Legendre transformation. We obtain $-\tilde{u}(\cdot, 1) \leqslant v(\cdot, 1) = V_H$. Applying Zaharjuta's two-constants-theorem for analytic functionals [26]–[28], we prove that $x \mapsto \sup_{z\in\mathbf{C}^N}(V_H(z)-H_x(z))$ is a convex function of $x<0$, which implies $v(\cdot,1)=V_H\leqslant -\tilde{u}(\cdot,C)$ for some $C>0$. By the "boundary behavior" of the partial Legendre transformation, this proves Theorem I.

In part two we study the boundary behavior of the pluricomplex Green function by investigating the function C_H. We obtain that the finiteness of the limit D_G is a local property of ∂G. We obtain a subordination principle. Using ideas of [18], [19], and Krivosheev [11], we prove Theorem III for polyhedra.

1. The "boundary behavior" of the partial Legendre transformation

Kiselman [7] has introduced the partial Legendre transformation for families of plurisubharmonic functions. In Kiselman [8] and Demailly [5] this transformation has been applied to study different types of Lelong numbers. We consider two classes of families (one in the range and one in the source of the transformation) having appropriate bounds and investigate how a type of directional Lelong number behaves under the transformation. It turns out that both classes have a prominent member.

Notations. For $z,w\in\mathbf{C}^N$, we write $\langle z,w\rangle:=\sum_{i=1}^N z_i\bar{w}_i$ and $|z|:=\langle z,z\rangle^{1/2}$. We put $B(R):=\{z\in\mathbf{C}^N \mid |z|\leqslant R\}$ for $R>0$, $S:=\{z\in\mathbf{C}^N \mid |z|=1\}$, $\mathbf{D}:=\{z\in\mathbf{C} \mid |z|<1\}$, $H_-:=\{z\in\mathbf{C} \mid \operatorname{Re} z<0\}$, $\mathbf{R}_+:=\{x\in\mathbf{R} \mid x\geqslant 0\}$, $\mathbf{R}_-:=-\mathbf{R}_+$. We will use the conventions $\infty/a=\infty$ and $a/\infty=0$ for each $a>0$, and $\inf\varnothing=\infty$.

The pluricomplex Green function of a bounded convex domain in $\mathbf{C}^N$. Let $G\subset\mathbf{C}^N$ be a bounded convex domain with $0\in G$. By $H\colon\mathbf{C}^N\to\mathbf{R}_+$ we denote its supporting function

$$H(z)=\sup_{w\in G}\operatorname{Re}\langle w,z\rangle.$$

According to Lempert [12], Klimek [9], Poletskiĭ [20], and Zaharjuta [27], we consider the pluricomplex Green function $g\colon G\to[-\infty,0[$ with pole at 0

$$g(z)=\sup_u u(z)$$

where the supremum is taken over all plurisubharmonic functions $u\colon G\to[-\infty,0[$ with $u(w)\leqslant\log|w|+O(1)$ as $w\to 0$. Then (see Klimek [9]) g is plurisubharmonic, maximal (i.e., $(dd^cg)^N=0$), continuous on $\overline{G}\setminus\{0\}$ (where $g|\partial G:\equiv 0$) and has logarithmic pole at 0, i.e., there is $C>0$ such that

$$-C\leqslant g(z)-\log|z|\leqslant C \quad \text{as } z\to 0.$$

1.1. THEOREM. *For each $z \in G$ we have*

$$g(z) = \inf\{\log r \mid 0 \leqslant r < 1, \exists f\colon \mathbf{D} \to G \text{ analytic}, f(0) = 0, f(r) = z\}.$$

Proof. By Lempert [13, Theorem 1], and Klimek [9, Corollary 1.7], we have $g(z) = \log\tanh k_G(z,0)$, $z \in G$, where k_G denotes the Kobayashi metric. Thus the assertion follows from the definiton of k_G (see e.g. [9]).

Essential for our analysis of the boundary behavior of g is the following result which extends a classical one of Study to several complex variables.

1.2. LEMMA. *With G also the sublevel sets*

$$G_x := \{z \in G \mid g(z) < x\}, \quad x < 0,$$

are convex.

Proof. (See Lempert [12, p. 462].) By 1.1, for each $x<0$,

$$\begin{aligned} G_x &= \{z \in G \mid \exists 0 \leqslant r < e^x, f\colon \mathbf{D} \to G \text{ analytic with } f(0) = 0, f(r) = z\} \\ &= \{f(r) \mid 0 \leqslant r < e^x, f\colon \mathbf{D} \to G \text{ analytic with } f(0) = 0\}. \end{aligned}$$

Let $a, b \in G_x$, $a \neq b$ and $0 < \lambda < 1$. Choose $f, g\colon \mathbf{D} \to G$ analytic with $f(0) = g(0) = 0$, and $0 \leqslant r_1, r_2 < e^x$ such that $f(r_1) = a$ and $g(r_2) = b$. We may assume $r_1 \leqslant r_2 \neq 0$. Since G is convex, the function $h\colon \mathbf{D} \to G$, $h(z) = \lambda f(zr_1/r_2) + (1-\lambda)g(z)$ is well defined, analytic and satisfies $h(0) = 0$. Thus $h(r_2) = \lambda a + (1-\lambda)b \in G_x$.

1.3. PROPOSITION. *The function $u\colon \mathbf{C}^N \times H_- \to \mathbf{R}_+$,*

$$u(z,\zeta) := \sup\{\operatorname{Re}\langle w, z\rangle \mid w \in G,\ g(w) < \operatorname{Re}\zeta\}, \quad z \in \mathbf{C}^N,\ \zeta \in H_-,$$

is continuous and plurisubharmonic. $u(z,\zeta)$ does not depend on $\operatorname{Im}\zeta$. There exists $M>0$ with $M^{-1}e^x|z| \leqslant u(z,x) \leqslant Me^x|z|$ for all $z \in \mathbf{C}^N$ and $x<0$. We have $\lim_{x\uparrow 0} u(z,x) = H(z)$, $z \in \mathbf{C}^N$.

Proof. Since $(z,w) \mapsto \operatorname{Re}\langle \bar{z}, w\rangle$ is continuous and pluriharmonic, it follows from Demailly [4, Corollary 6.12], that u is continuous and plurisubharmonic. Since g has a logarithmic pole at 0, there is $C \in \mathbf{R}$ with $\log|w| - C \leqslant g(w) \leqslant \log|w| + C$, $w \in G$, and thus

$$u(z,x) \leqslant \sup_{g(w)<x} |w||z| \leqslant e^{C+x}|z|, \quad z \in \mathbf{C}^N,\ x < 0.$$

On the other hand, for each $z \in \mathbf{C}^N$ and $x<0$, there is $w \in \partial G_x$ with $u(z,x) \geqslant |w||z| = e^{g(w)-C}|z| = e^{x-C}|z|$.

Another proof for the plurisubharmonicity of u can be obtained from the following representation, which holds by 1.1,

$$u(z,\zeta)=\sup_f \sup_{|\sigma|<\exp\operatorname{Re}\zeta} \operatorname{Re}\langle f(\sigma),z\rangle, \quad z\in\mathbf{C}^N,\ \zeta\in H_-,$$

where the first supremum is taken over all analytic functions $f\colon\mathbf{D}\to G$ with $f(0)=0$. Clearly, u is continuous in z. On the other hand for each f and z, the function $\zeta\mapsto \sup\{\operatorname{Re}\langle f(\sigma),z\rangle \mid |\sigma|<\exp\operatorname{Re}\zeta\}$ is plurisubharmonic, hence convex in $\operatorname{Re}\zeta$. Thus, as in Demailly [4, Corollary 6.12], we conclude that u is continuous.

1.4. *Remark.* For each $x<0$, we have $\overline{G}_x=\{z\in G \mid g(z)\leqslant x\}$.

Proof. By 1.3, in particular the function u is continuous and $x\mapsto u(z,x)$ is strictly increasing in $x<0$, hence $u(z,x)=\inf_{x<y<0} u(z,y)$ for all $x<0$ and $z\in\mathbf{C}^N$. This gives

$$\overline{G}_x=\bigcap_{x<y<0}\overline{G}_y=\bigcap_{x<y<0} G_y=\{z\in G \mid g(z)\leqslant x\}.$$

1.5. THEOREM. *Let $u\colon \mathbf{C}^N\times H_-\to\mathbf{R}_+$ be a plurisubharmonic function such that $u(z,\zeta)=u(z,\operatorname{Re}\zeta)$ for all $z\in\mathbf{C}^N$ and $\zeta\in H_-$. Assume that there is $M>0$ such that $u(z,x)\leqslant Me^x|z|$, $z\in\mathbf{C}^N$, $x<0$. The hypotheses imply that $x\mapsto u(z,x)$, $x<0$, is nondecreasing and convex. For $z\in\mathbf{C}^N$, we put $u(z,0):=\lim_{x\uparrow 0} u(z,x)\in\mathbf{R}_+$ and*

$$D_u(z):=\lim_{x\uparrow 0}\frac{u(z,0)-u(z,x)}{-x}=\sup_{x<0}\frac{u(z,0)-u(z,x)}{-x}\in[0,\infty].$$

Then the function $v\colon \mathbf{C}^N\times\,]0,\infty[\,\to\mathbf{R}_+$,

$$v(z,C):=-\tilde{u}(z,C):=\inf_{x<0}(u(z,x)-xC), \quad z\in\mathbf{C}^N,\ C>0,$$

is upper semicontinuous, is plurisubharmonic in z, is concave and nondecreasing in C ($\tilde{u}$ is called the "partial Legendre transform" of u). Moreover,

(i) *$v(z,C)\leqslant\min\{C\log^+(M|z|/C)+C, M|z|\}$ for all $z\in\mathbf{C}^N$ and $C>0$;*

(ii) *$v(z,\infty):=\lim_{C\to\infty} v(z,C)=u(z,0)$ for all $z\in\mathbf{C}^N$;*

(iii) *$D_u(z)=\inf\{C>0 \mid v(z,C)=v(z,\infty)\}$ for all $z\in\mathbf{C}^N$, i.e., $v(z,C)=v(z,\infty)$ if and only if $D_u(z)\leqslant C$.*

If $u(\cdot,x)$ is positively homogeneous for each $x<0$ (i.e. $u(tz,x)=tu(z,x)$, $t\geqslant 0$, $z\in\mathbf{C}^N$), then $v(z,C)=Cv(z/C,1)$ for all $z\in\mathbf{C}^N$ and $C>0$.

Proof. The convexity and monotonicity of $x\mapsto u(z,x)$ follows, since $\zeta\mapsto u(z,\zeta)$ is subharmonic and does not depend on $\operatorname{Im}\zeta$. By Kiselman [7, Theorem 4.1], v is upper semicontinuous, plurisubharmonic in z and concave in C.

Direct calculation shows that

$$v(z,C)\leqslant \inf_{x<0}(Me^x|z|-xC)=\begin{cases} C\log(M|z|/C)+C, & M|z|\geqslant C,\\ M|z|, & M|z|\leqslant C,\end{cases}$$

which equals the right hand side of (i). (ii) is a well known property of the Legendre transformation. To prove (iii), fix $z\in\mathbf{C}^N$ and note that for $C>0$ we have: $D_u(z)\leqslant C \Leftrightarrow u(z,0)\leqslant u(z,x)-Cx$ for all $x<0 \Leftrightarrow v(z,\infty)=u(z,0)\leqslant v(z,C)$.

1.6. *Notation and remark.* For u from 1.3, we denote the lower semicontinuous function D_u by D_G. We note that $D_G|S>0$ and that $v(z,1)-\log^+|z|=-\tilde{u}(z,1)-\log^+|z|$, $z\in\mathbf{C}^N$, is bounded.

In contrast to the case of one complex variable, D_G may be bounded on S even if ∂G is not of class C^1:

1.7. *Example.* Let $H^0:\mathbf{C}^N\to\mathbf{R}_+$ be a norm on $\mathbf{C}^N$, i.e., H^0 is convex, $H^0(\lambda z)=|\lambda|H^0(z)$ for all $\lambda\in\mathbf{C}$ and $z\in\mathbf{C}^N$, and $H^0(z)=0$ if and only if $z=0$. Then $g(z)=\log H^0(z)$, $z\in G:=\{z\in\mathbf{C}^N\,|\,H^0(z)<1\}$, is the pluricomplex Green function of G. Let $\phi:\mathbf{R}_-\to\mathbf{R}_-$ be convex with $\phi(0)=0$, and $\phi(x)\leqslant x+m$, $x\leqslant 0$, for some $m\geqslant 0$. If H is the supporting function of G, then $u(z,\zeta):=e^{\phi(\operatorname{Re}\zeta)}H(z)$ is continuous, plurisubharmonic, does not depend on $\operatorname{Im}\zeta$, and satisfies $u(z,x)\leqslant e^m(\max_{a\in S}H(a))e^x|z|$ for all $z\in\mathbf{C}^N$ and $x<0$. For all $z\in\mathbf{C}^N$,

$$D_u(z)=\lim_{x\uparrow 0}\frac{1-e^{\phi(x)}}{-x}H(z)=\phi'(0)H(z).$$

If $\phi=\mathrm{id}$, the function u coincides with the function u from 1.3 and thus $D_G=H$.

Proof. By Klimek [10, Theorem 5.1.6 and Lemma 6.1.3], $\log H^0$ is plurisubharmonic and the pluricomplex Green function of G. Since ϕ is convex and increasing (the latter fact follows from the upper bound for ϕ), the function $\zeta\mapsto\phi(\operatorname{Re}\zeta)$ is subharmonic on H_-. Since also H is a norm on $\mathbf{C}^N$, hence $(z,\zeta)\mapsto\phi(\operatorname{Re}\zeta)+\log H(z)$ is plurisubharmonic on $\mathbf{C}^N\times H_-$. Thus $u(z,\zeta)=\exp(\phi(\operatorname{Re}\zeta)+\log H(z))$ is plurisubharmonic.

Further examples are given in [16] ($N=1$) and in section two.

Siciak's extremal function. Let $\mathcal{L}$ denote the set of all plurisubharmonic functions u on $\mathbf{C}^N$ with $u(z)\leqslant\log^+|z|+C_u$, $z\in\mathbf{C}^N$, for some $C_u>0$. Fix the supporting function $H:\mathbf{C}^N\to\mathbf{R}_+$ of a bounded convex domain in $\mathbf{C}^N$ containing the origin. We introduce Siciak's extremal function $v:\mathbf{C}^N\times\,]0,\infty[\,\to\mathbf{R}_+$ with respect to H (see Siciak [25]),

$$v(z,C):=\sup_u u(z),\quad z\in\mathbf{C}^N,\ C>0,$$

where the supremum is taken over all plurisubharmonic functions u with $u/C\in\mathcal{L}$ and $u\leqslant H$. Note that $v(z,C)=Cv(z/C,1)$ for all $z\in\mathbf{C}^N$ and $C>0$, and that $v(\cdot,\infty):=\sup_{C>0}v(\cdot,C)\leqslant H$ is positively homogeneous.

1.8. LEMMA. *Let $u\in\mathcal{L}$ with $u\leqslant H$. Then*

$$u(z)\leqslant \log^+(|z|/R)+\max_{|w|\leqslant R} H(w),\quad z\in\mathbf{C}^N,\ R>0.$$

Proof. (Siciak [25, Lemma 3.4].) Fix $R>0$. By the hypothesis we have

$$u\leqslant \max_{|z|\leqslant R} H(z)=:M\quad \text{on } B(R).$$

Furthermore $u-M\in\mathcal{L}$. Hence $u-M$ is dominated by the pluricomplex Green function of $\mathbf{C}^N\setminus B(R)$ with logarithmic pole at infinity, i.e., $u(z)-M\leqslant\log^+(|z|/R)$, $z\in\mathbf{C}^N$.

1.9. PROPOSITION. *The function $v\colon \mathbf{C}^N\times\,]0,\infty[\,\to\mathbf{R}_+$ is continuous. v is plurisubharmonic in the first argument, and concave and nondecreasing in the second one. For $M:=\max_{z\in S} H(z)$, we have*

$$v(z,C)\leqslant \min\{C\log^+(M|z|/C)+C, M|z|\},\quad z\in\mathbf{C}^N,\ C>0.$$

$v(\cdot\,,1)$ is maximal (i.e., solves $(dd^c v(\cdot\,,1))^N=0$) on the complement of the set

$$P_H:=\{z\in\mathbf{C}^N\mid v(z,1)=H(z)\}.$$

Proof. By 1.8 and the proof of 1.5, all functions $u\in\mathcal{L}$ with $u\leqslant H$ have the uniform bound

$$u(z)\leqslant \inf_{0<R\leqslant|z|}(\log(|z|/R)+RM)=\min\{\log^+(M|z|)+1, M|z|\},\quad z\in\mathbf{C}^N.$$

This shows $v(\cdot\,,1)^*\in\mathcal{L}$ for $v(z,1)^*=\limsup_{\zeta\to z} v(\zeta,1)$, $z\in\mathbf{C}^N$. Since H is upper semicontinuous, also $v(\cdot\,,1)^*\leqslant H$. Thus, the upper envelope $v(\cdot\,,1)$ is plurisubharmonic. Since H is uniformly continuous on $\mathbf{C}^N$, by the usual smoothing procedure for the plurisubharmonic function $v(\cdot\,,1)$, we get smooth functions $u_n\in\mathcal{L}$ with $v(\cdot\,,1)\leqslant u_n\leqslant H+1/n$, $n\in\mathbf{N}$. Thus $v(\cdot\,,1)=\sup_{n\in\mathbf{N}}(u_n-1/n)$ is lower semicontinuous (see Siciak [25, Proposition 2.12]). Hence, $v(\cdot\,,1)$ is continuous. Since $v(z,C)=Cv(z/C,1)$ for all $z\in\mathbf{C}^N$ and $C>0$, v is continuous, too. Clearly for all $C>0$,

$$v(z,C)\leqslant C\min\{\log^+(M|z|/C)+1, M|z/C|\},\quad z\in\mathbf{C}^N.$$

If $C,D>0$ and $0<\lambda<1$, it follows immediately from the definition of $v(\cdot\,,\lambda C+(1-\lambda)D)$, that $\lambda v(\cdot\,,C)+(1-\lambda)v(\cdot\,,D)\leqslant v(\cdot\,,\lambda C+(1-\lambda)D)$. The maximality of $v(\cdot\,,1)$ outside P_H follows by standard arguments solving locally an appropriate Dirichlet problem for the complex Monge–Ampère equation (see Bedford and Taylor [2, Corollary 9.2]).

1.10. THEOREM. *Let* $v\colon \mathbf{C}^N\times\,]0,\infty[\,\to\mathbf{R}_+$ *be upper semicontinuous and such that* $v(z,C)$ *is plurisubharmonic in* $z\in\mathbf{C}^N$ *and concave and nondecreasing in* $C>0$. *Assume that there is some* $M>0$ *such that*

$$v(z,C)\leqslant \min\{C\log^+(M|z|/C)+C, M|z|\},\quad z\in\mathbf{C}^N,\ C>0.$$

For $z\in\mathbf{C}^N$ *we put* $v(z,\infty):=\lim_{C\to\infty} v(z,C)\in\mathbf{R}_+$ *and*

$$C_v(z):=\inf\{C>0\mid v(z,C)=v(z,\infty)\}\in[0,\infty].$$

Then the function $u\colon \mathbf{C}^N\times H_-\to\mathbf{R}_+$ *(the "partial Legendre transform" of* $-v$*),*

$$u(z,\zeta):=(\widetilde{-v})(z,\operatorname{Re}\zeta):=\sup_{C>0}(v(z,C)+\operatorname{Re}\zeta C),\quad z\in\mathbf{C}^N,\ \zeta\in H_-,$$

is plurisubharmonic and does not depend on $\operatorname{Im}\zeta$. *Moreover*

(i) $u(z,x)\leqslant Me^x|z|$ *for all* $z\in\mathbf{C}^N$ *and* $x<0$,

(ii) $\lim_{x\uparrow 0} u(z,x)=v(z,\infty)$ *for all* $z\in\mathbf{C}^N$,

(iii) $C_v(z)=D_u(z)$ *for all* $z\in\mathbf{C}^N$ *(see 1.5).*

If $v(z,C)=Cv(z/C,1)$ *for all* $z\in\mathbf{C}^N$ *and* $C>0$, *then* $u(\cdot,x)$ *is positively homogeneous for each* $x<0$.

Proof. By Kiselman [7, Theorem 4.2], u is plurisubharmonic. By the hypothesis and by the proof of 1.5, we get for all $z\in\mathbf{C}^N$ and $x<0$,

$$u(z,x)\leqslant \sup_{C>0}\big(\inf_{y<0}(Me^y|z|-yC)+xC\big)=Me^x|z|.$$

This gives (i). (ii) is a well known property of the Legendre transformation. (iii) follows from the definition of C_v, from 1.5 (iii) and the following remark.

Remark. By well known properties of the Legendre transformation (see e.g. Rockafellar [21]), we have for u as in 1.5 and for v as in 1.10,

$$u(z,x)=\sup_{C>0}\big(\inf_{y<0}(u(z,y)-yC)+xC\big),\quad z\in\mathbf{C}^N,\ x<0,$$
$$v(z,C)=\inf_{x<0}\big(\sup_{D>0}(v(z,D)+xD)-xC\big),\quad z\in\mathbf{C}^N,\ C>0.$$

1.11. *Notation.* For v from 1.9, we write $C_H:=C_v$.

1.12. *Example.* Let $H\colon\mathbf{C}^N\to\mathbf{R}_+$ be a norm on $\mathbf{C}^N$ (see 1.7), and let v be Siciak's function with respect to H. Then $v(\cdot,1)=H$ on the set $G^0:=\{z\in\mathbf{C}^N\mid H(z)\leqslant 1\}$ and $v(\cdot,1)=1+\log H$ on $\mathbf{C}^N\setminus G^0$. In particular $C_H=H$.

Proof. Let u be the function which equals H on G^0 and equals $1+\log H$ on $\mathbf{C}^N\setminus G^0$. By the hypotheses, $\log H$ is plurisubharmonic and $\log H(z)-\log|z|$ is bounded on $\mathbf{C}^N$. Thus u belongs to $\mathcal{L}$, is bounded by H, and thus bounded by $v(\cdot,1)$. On the other hand, applying the maximum principle to the restrictions to the complex lines through the origin, we get for each $0<\varepsilon<1$ that $(1-\varepsilon)v(\cdot,1)\leqslant u$ on $\mathbf{C}^N\setminus G^0$.

The relation between D_G and C_H. Up to the end of the chapter, let G be a bounded convex domain in $\mathbf{C}^N$ containing the origin and with supporting function H.

1.13. PROPOSITION. *If v is Siciak's function with respect to H, then*

$$v(z,\infty)=\lim_{C\to\infty} v(z,C)=\lim_{C\to\infty} Cv(z/C,1)=H(z),\quad z\in\mathbf{C}^N.$$

Proof. Choose $u\colon \mathbf{C}^N\times H_-\to\mathbf{R}_+$ according to 1.3. Then by 1.3 and 1.5, for each $C>0$, we have $-\tilde{u}(\cdot,C)/C\in\mathcal{L}$ and $\sup_{C>0}-\tilde{u}(\cdot,C)=u(\cdot,0)=H$. Thus $H\leqslant v(\cdot,\infty)\leqslant H$.

1.14. THEOREM. *For u and v as in* 1.3 *and* 1.9, *respectively, we have*

$$\inf_{x<0}(u(\cdot,x)-x)\leqslant v(\cdot,1)\quad \textit{and}\quad C_H\leqslant D_G.$$

Proof. By the reasoning of the previous proof, we have $-\tilde{u}(\cdot,C)\leqslant v(\cdot,C)$ for all $C>0$, and moreover $D_G=D_u=C_{-\tilde{u}}\geqslant C_v=C_H$.

In 1.20 we will prove a converse of 1.14. In view of 1.13, the function C_H has the following interpretation:

1.15. *Remark.* If v is Siciak's function with respect to H, then the set of contact P_H (see 1.9) satisfies

$$P_H=\{\lambda a\mid a\in S, 0\leqslant\lambda\leqslant 1/C_H(a)\}$$

and is a compact set star shaped with respect to the origin. We have $\{z\in\mathbf{C}^N\mid v(z,C)=H(z)\}=CP_H$ for each $C>0$. There is $R_0>0$ such that for all $R\geqslant R_0$ (see Siciak [25]),

$$v(\cdot,1)=V_{E,H}:=\sup\{u(z)\mid u\in\mathcal{L}\text{ with } u\leqslant H \text{ on } E:=B(R)\}.$$

For $N=1$ the set P_H coincides with the set $E^*(H)$ of all accumulation points of the Fekete–Leja points of E with respect to H. (This follows from Siciak [23, (3.12), Theorem 2.1], and from the following observation: If $a\in\mathbf{C}\setminus\{0\}$ and $v(\cdot,1)$ is harmonic in a neighborhood of a then $v(a,1)<H(a)$, since otherwise, subtracting the linear function $z\mapsto(\partial v(a,1)/\partial z)z+(\partial v(a,1)/\partial\bar{z})\bar{z}$, we may assume that $v(\cdot,1)$ has the local expansion $v(z,1)=\mathrm{Re}(c(z-a)^n)+O(|z-a|^{n+1})$ for some $n\geqslant 2$ and $c\in\mathbf{C}\setminus\{0\}$; but this would not be

compatible with the property $v(\lambda z,1)\geqslant\lambda v(z,1)$ for all $0\leqslant\lambda\leqslant 1$ and $z\in\mathbf{C}$, which holds by the definition of Siciak's function.) This shows that P_H is the set of all accumulation points of the $(n+1)$-tuples $(z_0^{(n)},\dots,z_n^{(n)})$, $n\in\mathbf{N}$, of numbers in $\mathbf{C}$ which maximize the value of

$$\Bigl(\prod_{0\leqslant i<k\leqslant n}|z_i^{(n)}-z_k^{(n)}|\Bigr)\exp\Bigl(-n\sum_{k=0}^{n}H(z_k^{(n)})\Bigr).$$

Proof. $0\in P_H$ because $H\geqslant 0=H(0)$. For $z=\lambda a$, $a\in S$, $\lambda>0$, we have $|z|\leqslant 1/C_H(z/|z|)$ $\Leftrightarrow C_H(z)\leqslant 1\Leftrightarrow v(z,1)=H(z)$.

Put $m:=\min_{z\in S}H(z)>0$, $M:=\max_{z\in S}H(z)$, and choose $R_0\geqslant\max\{1,1/m\}$ with $M\leqslant R_0 m-\log R_0$. Let $R\geqslant R_0$. Then

$$\inf_{s\geqslant R}(sm-\log s)=Rm-\log R.$$

For the plurisubharmonic function $V_{E,H}$, by 1.8, we get for all $|z|\geqslant R$,

$$V_{E,H}(z)\leqslant\log|z|+M\leqslant|z|m\leqslant H(z).$$

This shows that $V_{E,H}\leqslant v(\cdot,1)$. Obviously $v(\cdot,1)\leqslant V_{E,H}$.

Notation. By A_H we denote the space of all entire functions f on $\mathbf{C}^N$, i.e., $f\in A(\mathbf{C}^N)$, satisfying the estimate

$$|f(z)|\leqslant C\exp(\eta H(\bar z)),\quad z\in\mathbf{C}^N,$$

for some $C>0$ and some $0<\eta<1$. Let $u\colon\mathbf{C}^N\times\,]-\infty,0[\,\to\mathbf{R}$ be the continuous function from Proposition 1.3. Recall that $u(z,x)$ is positively homogeneous in $z\in\mathbf{C}^N$ and is strictly increasing in $x<0$ with $\lim_{x\uparrow 0}u(z,x)=H(z)$ for each $z\in\mathbf{C}^N$. The space A_H can also be written as

$$A_H=\bigcup_{x<0}\{f\in A(\mathbf{C}^N)\mid\|f\|_x<\infty\},$$
$$\|f\|_x:=\sup_{z\in\mathbf{C}^N}|f(z)|\exp(-u(\bar z,x)).$$

The following result is essentially Zaharjuta's two-constants-theorem for analytic functionals (see [26, Theorem 4.1], [27], and [28, Theorem II.1.1]). Since [26]–[28] are not accessible very well, we will give an independent proof of the following theorem.

1.16. THEOREM. *If u is from 1.3, then for each $f\in A_H\setminus\{0\}$ the nonincreasing function $x\mapsto\log\|f\|_x$, $x<0$ (with values in $\mathbf{R}\cup\{\infty\}$), is convex.*

The proof of Theorem 1.16 requires some preparations.

Notation. For all $z, w \in \mathbf{C}^N$ we write $z \cdot w := \langle z, \bar{w} \rangle$. If Ω is a bounded subset of $\mathbf{C}^N$ which contains the origin in its interior, then

$$\Omega' := \{z \in \mathbf{C}^N \mid z \cdot w \neq 1 \text{ for all } w \in \Omega\}$$

is of the same type. Ω' is closed (open) if Ω is open (closed). Since G is convex, we have $G''=G$. For further information on G' and the history of this notion, we refer to Andersson [1].

By $U: \mathbf{C}^N \to [0, \infty[$, we denote the pluricomplex Green function with pole at infinity of the compact (and not pluripolar) set G', i.e.,

$$U(z) = \sup_u u(z), \quad z \in \mathbf{C}^N,$$

where the supremum is taken over all $u \in \mathcal{L}$ with $u \leqslant 0$ on G' (see Klimek [10] for elementary properties and for the history of this Green function). By the following theorem, the properties of the Green function g will imply that U is continuous, hence plurisubharmonic, and that $U(z)>0$ if and only if $z \in \mathbf{C}^N \setminus G'$.

1.17. THEOREM. *If we put $g(w):=0$ for all $w \in \mathbf{C}^N \setminus G$, the following formula holds:*

$$U(z) = -\inf_{z \cdot w = 1} g(w), \quad z \in \mathbf{C}^N.$$

In particular, we get for the level sets $\{z \in \mathbf{C}^N \mid U(z) \leqslant -x\} = G'_x$ for all $x<0$.

Proof. In Lempert [14, Theorem 5.1 and equation (5.4)], the formula for U is proven for the case that G is strictly convex and has a smooth real analytic boundary (see the remark on p. 884 of [14] for the domain $D:=G'$). Here, strict convexity means that for each $z \in \partial G$, there is a ball which contains G and which is tangent to ∂G in z.

In the general case we choose a sequence G_j, $j \in \mathbf{N}$, of strictly convex domains with smooth real analytic boundary such that $0 \in G_j \subset G_{j+1}$ for all $j \in \mathbf{N}$ and with $G = \bigcup_{j \in \mathbf{N}} G_j$. This can be done by applying the usual smoothing procedure to the gauge function of G (see Schneider [22, Theorem 3.3.1]) but with a real analytic kernel (see Cegrell and Sadullaev [3, Theorem 1.2]). If we add $\varepsilon |z|$ with sufficiently small $\varepsilon>0$, this yields gauge functions of convex domains with the desired properties.

We have $G_j' \supset G_{j+1}'$, $j \in \mathbf{N}$, and $G' = \bigcap_{j \in \mathbf{N}} G_j'$. If g_j and U_j, $j \in \mathbf{N}$, denote the corresponding Green functions (extended to functions on $\mathbf{C}^N$), we know that $g = \inf_{j \in \mathbf{N}} g_j$ and $U = \sup_{j \in \mathbf{N}} U_j$ on $\mathbf{C}^N$ (see Klimek [10, Corollary 6.1.2 and Corollary 5.1.2]). We thus get for all $z \in \mathbf{C}^N$,

$$U(z) = \sup_{j \in \mathbf{N}} \sup_{z \cdot w = 1} (-g_j(w)) = \sup_{z \cdot w = 1} \sup_{j \in \mathbf{N}} (-g_j(w)) = -\inf_{z \cdot w = 1} g(w).$$

If $x<0$ and $z \in \mathbf{C}^N$, then $U(z) \leqslant -x \Leftrightarrow g(w) \geqslant x$ for all $z \cdot w = 1 \Leftrightarrow z \cdot w \neq 1$ for all $w \in G_x \Leftrightarrow z \in G_x{}'$.

1.18. LEMMA. *Let $x_0<0$ and let u be plurisubharmonic on $\{z\in\mathbf{C}^N \mid U(z)<-x_0\}$. Then the function*

$$\tau\colon x\mapsto \max_{U(z)\leqslant -x} u(z), \quad x_0<x<0,$$

is convex.

Proof. Since U is a continuous exhausting function for $\{z\in\mathbf{C}^N \mid U(z)<-x_0\}$ and is a maximal plurisubharmonic function on $\{z\in\mathbf{C}^N \mid 0<U(z)<-x_0\}$ (see Bedford and Taylor [2, Corollary 9.2]), the assertion follows from Demailly [4, Corollary 6.12]. For sake of completeness we give a direct proof here. Let $x_0<x_1<x_2<x_3<0$. For $0<U(z)<-x_0$ define

$$\begin{aligned}\omega(z) &:= \tau(x_1)\frac{x_3+U(z)}{x_3-x_1}+\tau(x_3)\frac{-U(z)-x_1}{x_3-x_1}\\ &= \frac{\tau(x_1)-\tau(x_3)}{x_3-x_1}U(z)+\frac{\tau(x_1)x_3-\tau(x_3)x_1}{x_3-x_1}.\end{aligned}$$

Since $\tau(x_1)\geqslant\tau(x_3)$, the function ω is a continuous maximal plurisubharmonic function with $u\leqslant\omega$ on the boundary of $\{z\in\mathbf{C}^N \mid -x_3\leqslant U(z)\leqslant -x_1\}$. Thus by the minimum principle of Bedford and Taylor (see Klimek [10, Corollary 3.7.5]), also $u(z)\leqslant\omega(z)$ for all $-x_3\leqslant U(z)\leqslant -x_1$. In particular

$$\tau(x_2)=\max_{U(z)\leqslant -x_2} u(z)\leqslant \max_{U(z)=-x_2}\omega(z)=\tau(x_1)\frac{x_3-x_2}{x_3-x_1}+\tau(x_3)\frac{x_2-x_1}{x_3-x_1}.$$

This proves the convexity of τ.

Notation. Let G' be defined as above. We note that for the compact sets $G_x{}'$, $x<0$, the following holds: $G'\subset G_{x_2}{}'\subset \overline{G}_{x_1}{}'=\operatorname{int} G_{x_1}{}'$ (the interior of $G_{x_1}{}'$), whenever $x_1<x_2<0$, and $G'=\bigcap_{x<0} G_x{}'$. By $A(G')$ we denote the space of all germs of analytic functions on G', i.e.,

$$A(G')=\bigcup_{x<0} A^\infty(\overline{G}_x{}').$$

Here $A^\infty(\overline{G}_x{}')$ denotes the space of all bounded analytic functions on $\overline{G}_x{}'$. To check bounds for the functions in $A(G')$ we introduce the norms

$$\|f\|_x := \sup_{z\in\overline{G}_x{}'} |f(z)|, \quad f\in A^\infty(\overline{G}_x{}'),\ x<0.$$

The space $A(G)'$ of all analytic functionals on $A(\mathbf{C}^N)$ which are carried by some compact subset of G, can be written as

$$A(G)'=\bigcup_{x<0}\{\mu\in A(\mathbf{C}^N)' \mid |\mu|_x^*<\infty\},$$

$$|\mu|_x^* := \sup\{|\mu(h)| \mid h \in A(\mathbf{C}^N) \text{ with } \sup_{z\in G_x} |h(z)| \leqslant 1\}, \quad x<0.$$

Proof of Theorem 1.16. By the Laplace transformation (see e.g. Hörmander [6, 4.5])

$$\mathcal{T}_L: A(G)' \to A_H, \quad \mathcal{T}_L(\mu)(z) := \mu(\exp(z\cdot w)),\ z\in\mathbf{C}^N,$$

and by the Fantappiè transformation (Martineau [15, Theorem 2.2])

$$\mathcal{T}_F: A(G)' \to A(G'), \quad \mathcal{T}_F(\mu)(z) := \mu\left(\frac{1}{1-z\cdot w}\right),\ z\in G',$$

the space $A(G)'$ can be identified with A_H and $A(G')$. To be more precise, for all $x<y<0$ there is $C>0$ such that

(a) $\|\mathcal{T}_L(\mu)\|_x \leqslant |\mu|_x^*$ and $\|\mathcal{T}_F(\mu)\|_y \leqslant C|\mu|_x^*$ for all $\mu\in A(G)'$ with $|\mu|_x^*<\infty$;

(b) $|\mathcal{T}_L^{-1}f|_y^* \leqslant C\|f\|_x$ and $|\mathcal{T}_F^{-1}g|_y^* \leqslant C\|g\|_x$ for all $f\in A_H$ with $\|f\|_x<\infty$ and all $g\in A^\infty(\bar{G}_x{}')$.

We apply Lemma 1.18 to $u=\log|f|$ for all functions $f\in A(G')\setminus\{0\}$. Then by Theorem 1.17, $A(G')$ has the following property (even with $\delta=0$ and $C=1$): For all $x_1<x_3<0$, $0<\alpha<1$, and all $\delta>0$ there is $C\geqslant 1$ such that with $x_2 := (1-\alpha)x_1+\alpha x_3$,

$$\|f\|_{x_2} \leqslant C\|f\|_{x_1-\delta}^{1-\alpha}\|f\|_{x_3-\delta}^{\alpha}, \quad \text{for all } f\in A^\infty(\bar{G}_{x_1-\delta}{}'). \tag{1}$$

We claim that A_H has the same property (1) but with "$f\in A^\infty(\bar{G}_{x_1-\delta}{}')$" replaced by "$f\in A_H$ with $\|f\|_{x_1-\delta}<\infty$" (the corresponding property for the intermediate space $A(G)'$ is Zaharjuta's two-constants-theorem): Let $x_1<x_3<0$, $0<\alpha<1$ and $\delta>0$ be arbitrary and put $x_2:=(1-\alpha)x_1+\alpha x_3$. We choose $x_i-\frac{1}{2}\delta<\tilde{x}_i<x_i$, $i=1,2$, and put $\tilde{\delta}:=\frac{1}{3}\delta$. Then $\tilde{x}_2:=(1-\alpha)\tilde{x}_1+\alpha\tilde{x}_3<x_2$ and $\tilde{x}_i-\tilde{\delta}>x_i-\delta$, $i=1,2$. For each $f\in A_H$ we denote the corresponding function in $A(G')$ by $\tilde{f}$. By (a), (b) and by (1), there are constants $C_i\geqslant 1$, $i=1,...,4$, such that the following holds: If $f\in A_H$ with $\|f\|_{x_1-\delta}<\infty$, then $\|\tilde{f}\|_{\tilde{x}_1-\tilde{\delta}} \leqslant C_1\|f\|_{x_1-\delta}<\infty$ and

$$C_2\|f\|_{x_1-\delta}^{1-\alpha}\|f\|_{x_3-\delta}^{\alpha} \geqslant C_3\|\tilde{f}\|_{\tilde{x}_1-\tilde{\delta}}^{1-\alpha}\|\tilde{f}\|_{\tilde{x}_3-\tilde{\delta}}^{\alpha} \geqslant C_4\|\tilde{f}\|_{\tilde{x}_2} \geqslant \|f\|_{x_2}.$$

This proves the property (1) for A_H. Now fix $f\in A_H\setminus\{0\}$ and put $\sigma(x):=\log\|f\|_x\in \mathbf{R}\cup\{\infty\}$, $x<0$. Let I be the interior of the set $\{x<0 \mid \sigma(x)<\infty\}$. We assume that $x_1\in I$, then $\|f\|_{x_1-\delta}<\infty$ for all sufficiently small $\delta>0$. By the homogeneity of H, also the functions $f_k:=f(\cdot/k)^k$ belong to A_H and $\|f_k\|_{x_1-\delta}=\|f\|_{x_1-\delta}^k<\infty$ for all $k\in\mathbf{N}$. Inserting these functions in (1) we get

$$\|f\|_{x_2} \leqslant C^{1/k}\|f\|_{x_1-\delta}^{1-\alpha}\|f\|_{x_3-\delta}^{\alpha}, \quad k\in\mathbf{N},$$

hence

$$\|f\|_{x_2} \leqslant \|f\|_{x_1-\delta}^{1-\alpha} \|f\|_{x_3-\delta}^{\alpha}.$$

Since this holds for arbitrary $\delta>0$, we obtain

$$\sigma(x_2) \leqslant (1-\alpha) \lim_{y\uparrow x_1} \sigma(y) + \alpha \lim_{y\uparrow x_3} \sigma(y). \tag{2}$$

The proof will be finished when we prove that the nonincreasing function σ is continuous on I. Assume that there is $x\in I$ with $\lim_{y\uparrow x}\sigma(y)>\lim_{y\downarrow x}\sigma(y)$. Since σ is nonincreasing, we may choose $y_1<x<y_3<0$ in I with $x=\frac{1}{2}(y_1+y_3)$ so close to x such that

$$(1-\alpha) \lim_{y\uparrow y_1} \sigma(y) + \alpha \lim_{y\uparrow y_3} \sigma(y) < \sigma((1-\alpha)y_1+\alpha y_3) \quad \text{for all } 0<\alpha<\tfrac{1}{2}.$$

For any such α we may consider $y_2:=(1-\alpha)y_1+\alpha y_3$ and get a contradiction to (2).

1.19. Corollary. *Let u be from* 1.3 *and v be from* 1.9. *Put*

$$\sigma(x) := \sup_{z\in\mathbf{C}^N} (v(z,1)-u(z,x)), \quad x<0.$$

Then σ is convex, nonincreasing, and $\lim_{x\uparrow 0}\sigma(x)=0$.

Proof. By 1.15 and Siciak [25, Theorem 4.12], we have

$$v(\bar{z},1) = \sup_{n\in\mathbf{N}} \sup_{p\in P_n} \frac{1}{n} \log|p(z)|, \quad z\in\mathbf{C}^N,$$

where P_n is the set of all complex polynomials p of degree at most n with $|p(z)|\leqslant \exp(nH(z))$, $z\in\mathbf{C}^N$. This gives

$$\sigma(x) = \sup_{n\in\mathbf{N}} \sup_{p\in P_n} \frac{1}{n} \sup_{w\in\mathbf{C}^N} (\log|p(w/n)| - u(w,x)), \quad x<0.$$

By 1.16, σ is convex. Clearly $\sigma(x)\geqslant 0$ for all $x<0$. On the other hand

$$\lim_{x\uparrow 0} \sigma(x) = \sup_{z\in\mathbf{C}^N} (v(z,1)-H(z)) \leqslant 0.$$

1.20. Theorem. *Let G be a bounded convex domain in $\mathbf{C}^N$ containing the origin and with supporting function H. Let u and v be from* 1.3 *and* 1.9, *respectively. Then there is $C\geqslant 1$ with $v(\cdot,1)\leqslant\inf_{x<0}(u(\cdot,x)-xC)$ and $D_G\leqslant CC_H$.*

Proof. By 1.6, $2\log|z|\leqslant\inf_{x<0}(u(z,x)-x2)+O(1)$ for $|z|\to\infty$. Hence there is $R>0$ such that

$$v(z,1) \leqslant \inf_{x<0} (u(z,x)-x2), \quad |z|\geqslant R.$$

By 1.19, there are $x_0<0$ and $C'\geqslant 1$ with $\sigma(x)\leqslant -xC'$ for all $x_0\leqslant x<0$. We choose $C\geqslant \max\{C',2\}$ such that for all $z\in B(R)$ and all $x<x_0$,

$$\frac{u(z,x_0)-u(z,x)}{x_0-x}\leqslant\frac{u(z,\frac{1}{2}x_0)-u(z,x_0)}{\frac{1}{2}x_0-x_0}\leqslant C,$$

hence $u(z,x_0)-x_0C\leqslant u(z,x)-xC$. Thus

$$v(z,1)\leqslant\inf_{x_0\leqslant x<0}(u(z,x)-xC)=\inf_{x<0}(u(z,x)-xC)=-\tilde{u}(z,C),\quad |z|\leqslant R.$$

By the definition of C_v, this gives $D_G=D_u=C_{-\tilde{u}}\leqslant CC_v=CC_H$.

1.21. THEOREM. *For a bounded convex domain G of $\mathbf{C}^N$ containing the origin and with supporting function H the following assertions are equivalent.*

(i) *C_H is bounded on S.*

(ii) *D_G is bounded on S.*

(iii) *There is some $C>0$ with*

$$G\subset G_x+C(-x)B(1),\quad x<0.$$

(iv) *There is some $v\in\mathcal{L}$ with $v\leqslant H$ and which coincides with H on some neighborhood of zero.*

Proof. (i) $\Leftrightarrow$ (ii) holds by by 1.14 and 1.20. (ii) $\Leftrightarrow$ (iii) holds by [16, Lemma 3.4]. (i) $\Leftrightarrow$ (iv) holds by the definition of C_H.

2. Investigation of C_H

In the case of one complex variable, as it has been shown in [16], much is known about the function D_G, because of its close relationship to angular derivatives of conformal mappings, which have a rich theory. In the present section we prefer to investigate C_H to derive analogous results for the case of several variables. We start with the following crucial lemma.

2.1. LEMMA. *Let H be the supporting function of a bounded convex domain in $\mathbf{C}^N$ which contains the origin. For each bounded open set $D\subset\mathbf{C}^N$ and each $\varepsilon>0$ there is $C>0$ such that the following holds: If u is plurisubharmonic on D with $u\leqslant H$ on D and $\limsup_{\zeta\to z}u(\zeta)\leqslant H(z)-\varepsilon$ for $z\in\partial D$, then there exists $U\leqslant H$ with $U/C\in\mathcal{L}$ and $u\leqslant U$ on $\overline{D}$.*

Proof. Let v be as in 1.9. Since $v(\cdot,C)$, $C>0$, and H are continuous, by 1.13 and by Dini's theorem, $\lim_{C\to\infty}v(z,C)=H(z)$ uniformly for $z\in\partial D$. Hence we can choose $C>0$

such that $v(z,C)\geqslant H(z)-\varepsilon\geqslant\limsup_{\zeta\to z}u(\zeta)$, $z\in\partial D$. We put $U(z):=\max\{u(z),v(z,C)\}$ for $z\in D$, and $U(z):=v(z,C)$ for $z\in\mathbf{C}^N\setminus D$. Then U is plurisubharmonic on $\mathbf{C}^N$ (see e.g. Klimek [10, Corollary 2.9.15]), $U\leqslant H$, $U/C\in\mathcal{L}$, and $u\leqslant U$ on $\bar{D}$.

The fact that C_H is bounded on a certain subset A of S does not change if we translate G, as long as the translated set contains the origin. The following shows that "C_H is bounded on A" is even a local property of ∂G.

2.2. Proposition. *Let H_i be the supporting function of the bounded convex domain G_i with $0\in G_i$, $i=1,2$. Let $A\subset S$ be closed. For $i=1,2$, let $\tilde{A}_i$ be the set of all $\tilde{a}\in S$ such that there are $a\in A$ and $w\in\partial G_i$ with* $\operatorname{Re}\langle w,a\rangle=H_i(a)$ *and* $\operatorname{Re}\langle w,\tilde{a}\rangle=H_i(\tilde{a})$. *We assume that $\tilde{A}:=\tilde{A}_1=\tilde{A}_2$ and that $\tilde{A}$ has an open neighborhood V in S with $H_1|V=H_2|V$. Then $C_{H_1}|A$ is bounded if and only if $C_{H_2}|A$ is bounded.*

Proof. By v_i we denote Siciak's function with respect to H_i, $i=1,2$. Assume that $v_1(\cdot,C)=H_1$ on A for some $C>0$. Put

$$L(z):=\sup\operatorname{Re}\langle w,z\rangle,\quad z\in\mathbf{C}^N,$$

where the supremum is taken over all $w\in\partial G_2$ such that there are $a\in A$ with $\operatorname{Re}\langle w,a\rangle=H_2(a)$. By the compactness of A and ∂G_2, the supremum is in fact a maximum. By the definition, we have $L\leqslant H_2$ and $L|A=H_2|A$. If $z\in S$ and $L(z)=H_2(z)$ then there are $w\in\partial G_2$ and $a\in A$ with $\operatorname{Re}\langle w,a\rangle=H_2(a)$ and $\operatorname{Re}\langle w,z\rangle=H_2(z)$. Hence $z\in\tilde{A}$, by the definition of $\tilde{A}$. This shows that $L(z)<H_2(z)$ for all $z\in S\setminus\tilde{A}$.

We choose $K\geqslant 0$ with $K\geqslant\max_{z\in S}(H_1(z)-H_2(z))$. We put

$$\eta:=\min_{z\in S\setminus V}(H_2(z)-L(z))>0\quad\text{and}\quad\varepsilon:=\frac{\eta}{\eta+K}.$$

We consider the plurisubharmonic function $v:=\varepsilon v_1(\cdot,C)+(1-\varepsilon)L$. On $\Gamma(V):=\{ta\,|\,t\geqslant 0, a\in V\}$ the hypothesis gives

$$v\leqslant\varepsilon H_1+(1-\varepsilon)H_2=H_2.$$

For $z\notin\Gamma(V)$, we get

$$\begin{aligned}v(z)&=H_2(z)+\varepsilon(v_1(z,C)-H_2(z))-(1-\varepsilon)(H_2(z)-L(z))\\&\leqslant H_2(z)+\varepsilon K|z|-(1-\varepsilon)\eta|z|=H_2(z).\end{aligned}$$

Furthermore $v=\varepsilon H_1+(1-\varepsilon)H_2=H_2$ on A. For sufficiently large $|z|$, we have $v(z)\leqslant\varepsilon v_1(z,C)+(1-\varepsilon)H_2(z)\leqslant H_2(z)-1$. By 2.1 and the definition of v_2, this shows $v_2(\cdot,C')=H_2$ on A for some $C'>0$.

The following trivial lemma is the counterpart of the subordination principle for angular derivatives.

2.3. LEMMA. *Let $G_1 \subset G_2$ be bounded convex domains in $\mathbf{C}^N$ both containing the origin. Let $A \subset S$ with $H_1|A = H_2|A$. Then $C_{H_2}|A \leqslant C_{H_1}|A$.*

Proof. With the notation of the proof of 2.2, we have $v_1(\cdot, 1) \leqslant H_1 \leqslant H_2$ hence $v_1 \leqslant v_2$. Since $v_1(\cdot, \infty) = H_1 = H_2 = v_2(\cdot, \infty)$ on A, $C_{H_2}|A \leqslant C_{H_1}|A$ follows from the definition of C_{H_i}, $i = 1, 2$.

In view of Example 1.7, for $N \geqslant 2$, smoothness of ∂G is not necessary for C_H to be bounded on S, but it is sufficient in the following sense.

2.4. LEMMA. *Let $G = \{z \in \mathbf{C}^N \mid \operatorname{dist}(z, K) < \varepsilon\}$ for some $\varepsilon > 0$ and some compact convex set $K \subset \mathbf{C}^N$ and let $0 \in G$. If H is the supporting function of G then C_H is bounded on S. In particular C_H is bounded for each bounded convex domain G containing the origin with boundary of class $C^{1,1}$.*

Proof. Note that $\bar{G} = K + B(\varepsilon)$ hence $H(z) = H_K(z) + \varepsilon|z|$, $z \in \mathbf{C}^N$, where H_K denotes the supporting function of K. We consider the plurisubharmonic function

$$u(z) := H_K(z) + \varepsilon \log|z| + \varepsilon, \quad z \in \mathbf{C}^N.$$

Then

$$\begin{aligned}
&u = H \quad \text{on } S,\\
&u(z) \leqslant H_K(z) + \varepsilon|z| = H(z) \quad \text{for } z \in \mathbf{C}^N,\\
&u(z) \leqslant H(z) - \varepsilon(|z| - \log|z| - 1) \leqslant H(z) - 1 \quad \text{for sufficiently large } |z|.
\end{aligned}$$

By 2.3, there is $C > 0$ with $v(\cdot, C) = H$ on S.

2.5. LEMMA. *Let H be the supporting function of a compact convex set in $\mathbf{C}^N$. Let $a \in \mathbf{C}^N$ be such that there is $\eta \in S$ and an unbounded domain D of $\mathbf{C}$ with $0 \in D$ and such that $z \mapsto H(a + z\eta)$, $z \in D$, is affine (i.e., convex and concave). Then there is no plurisubharmonic function $v \leqslant H$ on $\mathbf{C}^N$ with $v(a) = H(a)$ and $v < H$ outside a compact set. If in particular H is the supporting function of a bounded convex domain in $\mathbf{C}^N$ containing the origin, then $C_H(a) = \infty$.*

Proof. If there were such a function v, then $u: D \to \mathbf{R}$,

$$u(z) := v(a + z\eta) - H(a + z\eta), \quad z \in D,$$

is subharmonic, nonpositive with $u(0) = 0$. Thus $u \equiv 0$. Since D is unbounded, this contradicts the fact that v equals H only on a compact subset of $\mathbf{C}^N$.

The following lemma shows how to deal with Cartesian products.

2.6. LEMMA. *Let G_l be a bounded convex domain in $\mathbf{C}^{N_l}$ containing the origin with supporting function H_l, $l=1,\dots,n$ $\left(N=\sum_{l=1}^n N_l\right)$. Let H be the supporting function of $G:=\prod_{l=1}^n G_l$, i.e., $H(z)=\sum_{l=1}^n H_l(z_l)$, $z\in\prod_{l=1}^n \mathbf{C}^{N_l}=\mathbf{C}^N$. Then there is $C>0$ with (see 1.9)*

$$\frac{1}{n}\prod_{l=1}^{n} P_{H_l}\subset P_H\subset C\prod_{l=1}^{n} P_{H_l}$$

and

$$\frac{1}{n}C_H(z)\leqslant \max_{1\leqslant l\leqslant n} C_{H_l}(z_l)\leqslant CC_H(z),\quad z=(z_l)_{l=1,\dots,n}\in\prod_{l=1}^{n}\mathbf{C}^{N_l}=\mathbf{C}^N.$$

Proof. Let v and v_l be Siciak's function with respect to H and H_l, respectively ($l=1,\dots,n$). It follows from the definition of Siciak's function that

$$\sum_{l=1}^{n} v_l(z_l,1)\leqslant v(z,n),\quad z=(z_l)_{l=1,\dots,n}\in\mathbf{C}^N.$$

This gives $(1/n)\prod_{l=1}^n P_{H_l}\subset P_H$ and hence

$$\frac{1}{n}C_H(z)\leqslant \max_{1\leqslant l\leqslant n} C_{H_l}(z_l),\quad z=(z_l)_{l=1,\dots,n}\in\mathbf{C}^N.$$

To prove the other estimate, choose $R>0$ with $P_H\subset B(R)$ and $v(z,1)\leqslant H(z)-1$ for all $|z|\geqslant R$. Let $a=(a_1,\dots,a_n)\in P_H$ and $1\leqslant k\leqslant n$. Then

$$u_{k,a}(z_k):=v(a_1,\dots,a_{k-1},z_k,a_{k+1},\dots,a_n;1)-\sum_{l\neq k}H_l(a_l)\leqslant H_k(z_k),\quad z_k\in\mathbf{C}^{N_k},$$

is plurisubharmonic on $\mathbf{C}^{N_k}$ with $u_{k,a}(z_k)\leqslant H_k(z_k)-1$ for all $|z_k|\geqslant R$. By 2.1, there is $C_k>0$ with

$$H_k(a_k)=u_{k,a}(a_k)\leqslant v_k(a_k,C_k)\leqslant H_k(a_k),$$

hence $v_k(a_k,C_k)=H_k(a_k)$ and $a_k\in C_kP_{H_k}$. With $C:=\max_{k=1,\dots,n}C_k$, this proves the second part of the assertion.

The remaining part of this section is devoted to an investigation of the case of a polyhedron. We will prove a converse of 2.5. We refer to Rockafellar [21] for standard notations for convex sets.

Notation. In the sequel, linearity and dimension always concern the field of real numbers. If $F\subset\mathbf{C}^n$ we write $\Gamma(F):=\{ta\mid t\geqslant 0, a\in F\}$. If $\Gamma\subset\mathbf{C}^n$ is a cone (i.e., $t\Gamma\subset\Gamma$ for all $t>0$), by $L(\Gamma)\subset\mathbf{C}^n$, we denote its linear hull. By $\operatorname{int}\Gamma$ we denote the relative interior (in $L(\Gamma)$) of Γ.

A cone Γ is called "real" if $L(\Gamma)\cap iL(\Gamma)=\{0\}$ and "complex" otherwise (see Krivosheev [11]). We call it "quasi-real" if $\Gamma\cap(L(\Gamma)\cap iL(\Gamma))=\{0\}$.

Let $H\colon \mathbf{C}^n\to\mathbf{R}$ be the supporting function of a compact convex set. A convex cone Γ will be called a maximal cone of linearity for H if $\{(z,H(z))\,|\,z\in\Gamma\}$ is a face of the epigraph $\operatorname{epi} H=\{(z,y)\in\mathbf{C}^N\times\mathbf{R}\,|\,y\geqslant H(z)\}$ of H, or what is the same if $H|\Gamma$ is linear (i.e., H is convex and concave) and for each convex cone $\Gamma'\subset\mathbf{C}^n$ on which H is linear and with $\Gamma'\cap\operatorname{int}\Gamma\neq\varnothing$ we have $\Gamma'\subset\Gamma$. In this case, Γ is closed. If H is linear on a convex cone Γ, by $\tilde{\Gamma}\subset\mathbf{C}^n$, we denote the maximal cone of linearity for H with $\Gamma\cap\operatorname{int}\tilde{\Gamma}\neq\varnothing$. In this case $\operatorname{int}\Gamma\subset\operatorname{int}\tilde{\Gamma}$.

2.7. Lemma. *Let H be the supporting function of a compact convex polyhedron in $\mathbf{C}^N$. Let $\Gamma\subset\mathbf{C}^N$ be a maximal cone of linearity for H. Then there is an open convex cone $U\subset\mathbf{C}^N$ which contains* $\operatorname{int}\Gamma$, *there are $0\leqslant l\leqslant N$, the supporting function H' of a compact convex polyhedron in $\mathbf{C}^{N-l}$, a linear function $A\colon\mathbf{C}^l\to\mathbf{R}$ such that after a suitable $\mathbf{C}$-linear transformation of $\mathbf{C}^N$ the function H has the representation*

$$H(z)=H'(z')+A(z''),\quad z=(z',z'')\in U\subset\mathbf{C}^{N-l}\times\mathbf{C}^l.$$

$H(z)\geqslant H'(z')+A(z'')$ holds for all $z\in\mathbf{C}^N$ (we use the convention $H'=0$ and $A=0$ if $l=N$ or $l=0$, respectively). $L(\Gamma)\cap iL(\Gamma)=\{0\}\times\mathbf{C}^l\subset\mathbf{C}^{N-l}\times\mathbf{C}^l$. For $l<N$, let $P\colon\mathbf{C}^N\to\mathbf{C}^{N-l}$ denote the canonical projection. H' is linear on $P(\Gamma)$ and $L(P(\Gamma))=L(\widetilde{P(\Gamma)})$. $P(\Gamma)$ is a real cone in $\mathbf{C}^{N-l}$. If $z'\in P(\Gamma)$ and $z=(z',z'')\in\mathbf{C}^{N-l}\times\mathbf{C}^l$ we have that $z\in\Gamma$ if and only if $H(z)=H'(z')+A(z'')$.

Proof. By the hypothesis, there are $b_1,\dots,b_M\in\mathbf{C}$ with

$$H(z)=\max_{1\leqslant i\leqslant M}\operatorname{Re}\langle z,b_i\rangle,\quad z\in\mathbf{C}^N.$$

We may assume that there is $1\leqslant m\leqslant M$ with

$$F:=\{(z,H(z))\,|\,z\in\Gamma\}=\bigcap_{i=0}^{m}\{(z,y)\in\mathbf{C}^N\times\mathbf{R}\,|\,y=\operatorname{Re}\langle z,b_i\rangle\}\cap\operatorname{epi}H,$$

where all hyperplanes $y=\operatorname{Re}\langle z,b_i\rangle$ occur which support $\operatorname{epi}H$ in F. If $a\in\operatorname{int}\Gamma$, then $\operatorname{Re}\langle a,b_i\rangle<H(a)$ for all $i=m+1,\dots,M$. Otherwise the hyperplane $y=\operatorname{Re}\langle z,b_i\rangle$ would support $\operatorname{epi}H$ in all points of the face F, which is a contradiction to the choice of m. Thus $H(z)=\max_{1\leqslant i\leqslant m}\operatorname{Re}\langle z,b_i\rangle$ for all z from a neighborhood of a. Taking the union over $a\in\operatorname{int}\Gamma$ of those neighborhoods we get an open set U in $\mathbf{C}^N$ (we may assume that U is a convex cone) with $\operatorname{int}\Gamma\subset U$ and

$$H(z)=\max_{1\leqslant i\leqslant m}\operatorname{Re}\langle z,b_i\rangle,\quad z\in U.$$

We consider the maximal $\mathbf{C}$-linear subspace $E:=L(\Gamma)\cap iL(\Gamma)$ of $L(\Gamma)$ and put $l:=\dim_{\mathbf{C}}E$. Applying an appropriate $\mathbf{C}$-linear transformation of $\mathbf{C}^N$, we may assume that $E=\{0\}\times\mathbf{C}^l\subset\mathbf{C}^{N-l}\times\mathbf{C}^l$. Since H is linear on Γ, we obtain for each $a\in\operatorname{int}\Gamma$ that

$$z''\mapsto H(a+(0,z''))=H(a)+\max_{1\leqslant i\leqslant m}\sum_{k=N-l+1}^{N}\operatorname{Re} z_k\bar{b}_{i,k},$$

$z''=(z_{N-l+1},\dots,z_N)$, is affine in a zero neighborhood of $\mathbf{C}^l$. Hence we get for all $k=N-l+1,\dots,N$ and $i,j=1,\dots,N$ that $b_{i,k}=b_{j,k}=:b^k$. This gives

$$H(z)=\max_{1\leqslant i\leqslant m}\langle(z_1,\dots,z_{N-l}),(b_{i,1},\dots,b_{i,N-l})\rangle+\sum_{k=N-l+1}^{N}\operatorname{Re} z_k\bar{b}^k=:H'(z')+A(z''),$$

for all $z=(z',z'')\in U\subset\mathbf{C}^{N-l}\times\mathbf{C}^l$. Obviously $H(z)\geqslant H'(z')+A(z'')$ holds for all $z\in\mathbf{C}^N$.

For the sequel, we assume that $l<N$. First, from the representation of H, we obtain that H' is convex and concave hence linear on $P(\Gamma)$ and thus $P(\Gamma)\subset\widetilde{P(\Gamma)}$, by definition. Since $\operatorname{int}\Gamma\subset U$, we have

$$\varnothing\neq\operatorname{int}\Gamma=((\widetilde{P(\Gamma)}\times\mathbf{C}^l)\cap U)\cap\operatorname{int}\Gamma.$$

By the representation of H, H is linear on the convex cone $(\widetilde{P(\Gamma)}\times\mathbf{C}^l)\cap U$. Thus $(\widetilde{P(\Gamma)}\times\mathbf{C}^l)\cap U\subset\tilde{\Gamma}=\Gamma$. Since U is open and $\widetilde{P(\Gamma)}\times\mathbf{C}^l$ is a convex cone, we have

$$L((\widetilde{P(\Gamma)}\times\mathbf{C}^l)\cap U)=L(\widetilde{P(\Gamma)}\times\mathbf{C}^l),$$

and thus

$$L(\widetilde{P(\Gamma)})=P(L(\widetilde{P(\Gamma)})\times\mathbf{C}^l)=P(L(\widetilde{P(\Gamma)}\times\mathbf{C}^l))\subset P(L(\Gamma))=L(P(\Gamma)).$$

To argue by contradiction we assume that $L(P(\Gamma))=L(\widetilde{P(\Gamma)})$ is complex. Then there is $\eta\in L(P(\Gamma))\setminus\{0\}$ with $\mathbf{C}\eta\subset L(P(\Gamma))$. After a suitable $\mathbf{C}$-linear transformation of $\mathbf{C}^{N-l}$ we may assume that $\eta=(0,\dots,0,1)\in\mathbf{C}^{N-l}$. If we apply the previous arguments with H and Γ replaced by H' and $\widetilde{P(\Gamma)}$, we get an open cone $U'\subset\mathbf{C}^{N-l}$ with $\operatorname{int}P(\Gamma)\subset\operatorname{int}\widetilde{P(\Gamma)}\subset U'$ and some $b^{N-l}\in\mathbf{C}$ such that

$$H'(z')=H''(z_1,\dots,z_{N-l-1})+\operatorname{Re} z_{N-l}\bar{b}^{N-l},\quad z'=(z_1,\dots,z_{N-l})\in U'.$$

For the open cone $P^{-1}(U')$, we have $P^{-1}(U')\supset P^{-1}(\operatorname{int}P(\Gamma))\supset\operatorname{int}\Gamma$. Hence we may assume that $U\subset P^{-1}(U')$. This shows that H is linear on $E'\cap U$ where $E':=\{0\}\times\mathbf{C}^{l+1}$ and thus $E'\subset L(\Gamma)$. This contradicts the choice of l.

To prove the remaining part of the assertion, we choose a linear functional $B\colon \mathbf{C}^N \to \mathbf{R}$ such that its graph is a supporting hyperplane for $\operatorname{epi} H$ which touches $\operatorname{epi} H$ precisely on Γ. We show that $B(z',z'')=H'(z')+A(z'')$ for all $z'\in P(\Gamma)$ and $z''\in \mathbf{C}^l$. Because of the continuity of A, H' and B, it is enough to prove this for $z'\in \operatorname{int} P(\Gamma)=P(\operatorname{int}\Gamma)$. Let $z'\in P(\operatorname{int}\Gamma)$ and choose $w''\in \mathbf{C}^l$ with $(z',w'')\in \operatorname{int}\Gamma$. Then there is a neighborhood of $w''\in \mathbf{C}^l$ on which $H(z',\cdot)$ equals $B(z',\cdot)$ and $H'(z')+A$. Since both functions are affine on $\mathbf{C}^l$, we have $B(z',\cdot)=H'(z')+A$ on $\mathbf{C}^l$. We thus proved our claim. If now $H(z)=H'(z')+A(z'')$ with $z'\in P(\Gamma)$ and $z''\in \mathbf{C}^l$, then by the previous remark, we get $H(z)=B(z',z'')$ and hence $z=(z',z'')\in\Gamma$.

2.8. LEMMA. *Let H be the supporting function of a compact convex polyhedron in $\mathbf{C}^n$. Assume that $\Gamma\subset\mathbf{C}^n$ is a real maximal cone of linearity for H. Then for each $a\in\Gamma$ there is a continuous plurisubharmonic function $v_a\colon \mathbf{C}^n\to\mathbf{R}$ with $v_a\leqslant H$, such that $v_a(z)=H(z)$ if and only if $z=a$. Moreover, $v_a(z)$ is a continuous function of $(a,z)\in \Gamma\times\mathbf{C}^n$.*

Proof. Since Γ is real, as shown in [18, Lemma 4], there is a $\mathbf{C}$-linear orthogonal mapping $T\colon \mathbf{C}^n\to\mathbf{C}^n$ with $T(L(\Gamma))\subset\mathbf{R}^n$. Hence we may assume that $L(\Gamma)\subset\mathbf{R}^n$. By the hypothesis, there is $b\in\mathbf{C}^n$ with

$$\Gamma=\{z\in\mathbf{C}^n \mid H(z)=\operatorname{Re}\langle z,b\rangle\} \quad\text{and}\quad H(z)\geqslant \operatorname{Re}\langle z,b\rangle \text{ for all } z\in\mathbf{C}^n.$$

If we put $\tilde H:=H-\operatorname{Re}\langle\cdot,b\rangle$ then $\tilde H\geqslant 0$ and $\Gamma=\{z\in\mathbf{C}^n \mid \tilde H(z)=0\}\subset\mathbf{R}^n$. Since $\tilde H$ is the maximum of finitely many linear functions, this shows that there exists $\varepsilon>0$ with $\tilde H(z)\geqslant \varepsilon\sum_{j=1}^n |\operatorname{Im} z_j|$ for all $z\in\mathbf{C}^n$. We consider the continuous subharmonic function $h\colon \mathbf{C}\to\mathbf{R}_+$ which is harmonic on the disc $\mathbf{D}$ and which equals $\zeta\mapsto|\operatorname{Im}\zeta|$ if $|\zeta|\geqslant 1$. It is well known that

$$\max_{\zeta\in\mathbf{D}}(h(\zeta)-|\operatorname{Im}\zeta|)=h(0)=2/\pi.$$

For $a\in\Gamma\subset\mathbf{R}^n$ we define

$$h_a(z):=\sum_{j=1}^n h(z_j-a_j)-2(n-1)/\pi,\quad z\in\mathbf{C}^n.$$

By [18, Lemma 3], the plurisubharmonic function h_a satisfies $h_a(z)\leqslant\sum_{j=1}^n|\operatorname{Im} z_j|$ if $|z-a|\geqslant\sqrt{n}$, and $h_a(z)\leqslant\sum_{j=1}^n|\operatorname{Im} z_j|+2/\pi$ if $|z-a|\leqslant\sqrt{n}$, where equality holds if and only if $z=a$. We define the plurisubharmonic function

$$v_a(z):=\varepsilon(h_a(z)-2/\pi)+\operatorname{Re}\langle z,b\rangle,\quad z\in\mathbf{C}^n.$$

Then v_a does not exceed H. If $z\in\mathbf{C}^n$ then $v_a(z)=H(z)$ if and only if $z=a$.

2.9. Lemma. *Let H be the supporting function of a compact convex polyhedron in $\mathbf{C}^N$. Let $\Gamma\subset\mathbf{C}^N$ be a maximal cone of linearity for H. We assume that Γ is quasi-real. Then there exists a plurisubharmonic function $v\leqslant H$ on $\mathbf{C}^N$ with $v=H$ on $\Gamma\cap B(1)$ such that $v<H$ outside some compact set.*

Proof. We apply 2.7. We use the notations of 2.7, omit the $\mathbf{C}$-linear transformation, and recall that $L(\Gamma)\cap iL(\Gamma)=\{0\}\times\mathbf{C}^l\subset\mathbf{C}^{N-l}\times\mathbf{C}^l$. Since Γ is quasi-real, we have $\Gamma\cap(\{0\}\times\mathbf{C}^l)=\{0\}$, in particular $l<N$. Hence $\Gamma\cap(P(K)\times\mathbf{C}^l)$ is bounded for the compact set $K:=\Gamma\cap B(1)$.

Fix $a\in K$ and put $a':=P(a)$. By Lemma 2.7, for $\Gamma':=\widetilde{P(\Gamma)}$ we have $L(\Gamma')=L(P(\Gamma))$. Hence by 2.7, the cone Γ' is real. We apply Lemma 2.8 to H', Γ' and a'. Let $v_{a'}\colon\mathbf{C}^{N-l}\to\mathbf{R}$ be according to 2.8 and define

$$v_a(z):=v_{a'}(z')+A(z''),\quad z=(z',z'')\in\mathbf{C}^{N-l}\times\mathbf{C}^l=\mathbf{C}^N.$$

Then v_a is plurisubharmonic on $\mathbf{C}^N$, satisfies $v_a(z)\leqslant H'(z')+A(z'')\leqslant H(z)$ for all $(z',z'')=z\in\mathbf{C}^N$. Moreover $v_a(a)=H'(a')+A(a'')=H(a)$. Vice versa, let $z\in\mathbf{C}^N$ with $v_a(z)=H(z)$. Then by the previous estimate, we get $v_{a'}(z')=H'(z')$ hence $z'=a'\in P(K)$, by 2.8. Furthermore $H'(z')+A(z'')=H(z)$ and thus $z\in\Gamma$, by 2.7. Thus we have $z\in\Gamma\cap(P(K)\times\mathbf{C}^l)$. Now define

$$v=\sup_{a\in K}v_a.$$

Then $v\leqslant H$, and by the continuity of $(a,z)\mapsto v_a(z)$, v is continuous and plurisubharmonic on $\mathbf{C}^N$. Let $z\in\mathbf{C}^N$ with $v(z)=H(z)$. By the compactness of K, there is a sequence $(a_n)_{n\in\mathbf{N}}$ in K with $\lim_{n\to\infty}a_n=a\in K$ and $H(z)=\lim_{n\to\infty}v_{a_n}(z)=v_a(z)$. Hence $z\in\Gamma\cap(P(K)\times\mathbf{C}^l)$. Thus $v<H$ outside a compact set.

2.10. Proposition. *Let H be the supporting function of a compact convex polyhedron in $\mathbf{C}^N$. Let $\Gamma':=\bigcup\Gamma$ where the union is taken over all quasi-real maximal cones Γ of linearity for H. Then there exists a plurisubharmonic function $v\leqslant H$ on $\mathbf{C}^N$ with $v<H$ outside some compact set and with $v=H$ on $\Gamma'\cap B(1)$. Since we may replace $\bigcup\Gamma$ by $\bigcup\operatorname{int}\Gamma$ in the definition of Γ', we have $\mathbf{C}^N\setminus\Gamma'=\bigcup\operatorname{int}\Gamma$ where the union is taken over maximal cones of linearity for H which are not quasi-real.*

Proof. For Γ as in the definition of Γ', choose v_Γ according to 2.9. Then $v:=\max_\Gamma v_\Gamma$ suffices. Since the relative boundary of a quasi-real closed maximal cone of linearity for H is the union of such cones (of lower dimension), we may replace $\bigcup\Gamma$ by $\bigcup\operatorname{int}\Gamma$ in the definition of Γ'.

2.11. THEOREM. *Let G be a bounded open convex polyhedron in $\mathbf{C}^N$ containing the origin and with supporting function H. Let Γ' be as in 2.10. Then C_H is bounded on $\Gamma' \cap S$ and $C_H \equiv \infty$ on $(\mathbf{C}^N \setminus \Gamma') \cap S$. The support of $(dd^c H)^N$, which is the union of all real maximal cones of linearity for H, is contained in Γ'.*

Proof. Choose v according to 2.10. Since v is upper semicontinuous, there is a ball $B(R)$, $R \geqslant 1$, and some $\varepsilon > 0$ such that $v \leqslant H - \varepsilon$ on $\partial B(R)$. By 2.10, 2.1, and by the definition of C_H, there is $C > 0$ with $C_H \leqslant C$ on $\Gamma' \cap S$. If $a \in (\mathbf{C}^N \setminus \Gamma') \cap S$, by 2.10, there is a maximal cone Γ of linearity for H with $a \in \operatorname{int} \Gamma$ and which is not quasi-real. Hence there is some $\eta \in \Gamma \setminus \{0\}$ with $\mathbf{C}\eta \subset L(\Gamma)$. It follows that the set $D := \{\zeta \in \mathbf{C} \mid a + \zeta\eta \in \operatorname{int} \Gamma\}$ is an unbounded domain which contains 0 and on which $\zeta \mapsto H(a + \zeta\eta)$ is affine. Hence $C_H(a) = \infty$, by 2.5.

The remaining assertion follows from the geometrical characterization of the support of $(dd^c H)^N$ ([18, Proposition 10], and Krivosheev [11]) and since each real cone is quasi-real.

2.12. *Remark.* If the convex polyhedron of 2.11 is the Cartesian product of convex polyhedra in $\mathbf{C}$, then Γ' and the support of $(dd^c H)^N$ coincide. For $N = 1$, this is obvious. The general case follows from 2.11, 2.6 and [19, Lemma 3.4].

If $N \geqslant 2$, it may happen that the support of $(dd^c H)^N$ is a proper subset of Γ':

2.13. *Example.* Let

$$G = \left\{ z = (z_1, z_2) = (x_1, \dots, x_4) \in \mathbf{C}^2 \;\middle|\; \sum_{i=1}^{4} |x_i| < 1 \right\}.$$

Its supporting function is $H : \mathbf{C}^2 \to \mathbf{R}$

$$H(z) = H(x_1, \dots, x_4) = \max\{\varepsilon x_i \mid \varepsilon \in \{-1, 1\}, i = 1, \dots, 4\} = \max_{i=1,\dots,4} |x_i|,$$

$z \in \mathbf{C}^2$. Then the support of $(dd^c H)^2$ is the union of all maximal cones of linearity for H (m.c.l.) of dimension one or two. Γ' (see 2.10) contains in addition some (but not all) m.c.l. of dimension three. It contains no m.c.l. of dimension four.

Proof. Note that the polar set G^0 equals

$$\{z \in \mathbf{C}^2 \mid H(z) \leqslant 1\} = \{(x_1, \dots, x_4) \in \mathbf{C}^2 \mid \max_{i=1,\dots,4} |x_i| \leqslant 1\}.$$

Of course each m.c.l. of dimension one is real. If Γ is an m.c.l. of dimension two, then $\Gamma = \Gamma([a, b])$ for some face $[a, b]$ of G^0 of dimension one and where $a, b \in \{-1, 1\}^4$ are extremal points of G^0 which differ precisely in one coordinate. One easily checks that

$L(\Gamma)\neq iL(\Gamma)$. Hence Γ is real. If Γ is an m.c.l. of dimension bigger than two, then $L(\Gamma)\cap iL(\Gamma)$ has at least the dimension two, i.e., Γ is complex. By 2.11, this gives the assertion about the support of $(dd^cH)^2$. To describe Γ', we note that if Γ is a m.c.l. of dimension four then trivially $\Gamma\cap(L(\Gamma)\cap iL(\Gamma))=\Gamma\neq\{0\}$. Hence Γ is not quasi-real and thus $\operatorname{int}\Gamma\subset\mathbf{C}^2\setminus\Gamma'$. We now consider two examples of m.c.l. Γ of dimension three. If $F:=\{z\in\mathbf{C}^2 \mid x_1=x_2=1\}\cap\partial G^0$ then F is a face of G^0 of dimension two. It is the convex hull of the extremal points $(1,1,1,1)$, $(1,1,1,-1)$, $(1,1,-1,1)$, and $(1,1,-1,-1)$. For $\Gamma:=\Gamma(F)$, we get $L(\Gamma)=\mathbf{R}(1,1)\times\mathbf{C}$ and hence $L(\Gamma)\cap iL(\Gamma)=(0,0)\times\mathbf{C}$. Obviously $\Gamma\cap((0,0)\times\mathbf{C})=\{0\}$ which shows that $\Gamma\subset\Gamma'$. If $F:=\{z\in\mathbf{C}^2 \mid x_1=x_4=1\}\cap\partial G^0$ then F is the convex hull of the extremal points $(1,1,1,1)$, $(1,1,-1,1)$, $(1,-1,1,1)$, and $(1,-1,-1,1)$. For $\Gamma:=\Gamma(F)$, we get $L(\Gamma)\cap iL(\Gamma)=\mathbf{R}(1,0,0,1)+\mathbf{R}(0,1,-1,0)$. Since $(1,0,0,1)=\frac{1}{2}(1,1,-1,1)+\frac{1}{2}(1,-1,1,1)\in\Gamma$, Γ is not quasi-real and hence $\operatorname{int}\Gamma\subset\mathbf{C}^2\setminus\Gamma'$.

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