Acta Math., 171 (1993), 231–261

$\mathbf{HP}^2$-bundles and elliptic homology

by

MATTHIAS KRECK and STEPHAN STOLZ(1)

Max-Planck-Institut für Mathematik
Bonn, Germany

University of Notre Dame
Notre Dame, IN, U.S.A.

1. Introduction

The (universal) elliptic genus [L1] is a ring homomorphism

$$\phi: \Omega_*^{\mathrm{SO}} \to M_* = \mathbf{Z}\left[\tfrac{1}{2}\right][\delta, \varepsilon]$$

from the oriented bordism ring to the graded polynomial ring M_*. Here $\delta=\phi(\mathbf{CP}^2)$ and $\varepsilon=\phi(\mathbf{HP}^2)$, where $\mathbf{CP}^2$ (resp. $\mathbf{HP}^2$) is the complex (resp. quaternionic) projective plane (an introduction and background information on elliptic genera can be found in [HBJ], [L1], [O2], [Se], [W]). The elliptic genus provides a connection between bordism theory, modular forms and quantum field theory. For, M_* can be identified with a ring of modular forms and, following Witten [W], the elliptic genus $\phi(M)$ of a spin manifold M can be interpreted as the S^1-equivariant index of an operator on the loop space on M. In fact, Witten used this interpretation to provide a heuristic proof for the rigidity of the elliptic genus. A rigorous proof along those lines was given by Taubes [T] (see also [BT]). The rigidity is equivalent to the multiplicativity of ϕ for certain fibre bundles $E \to B$ [O3]; namely, if E, B are closed oriented manifolds, the fibre F is a spin manifold and the structure group of the bundle is compact and connected then $\phi(E)=\phi(F)\phi(B)$.

The universal elliptic genus makes M_* and hence $M_*[\omega^{-1}]$ for any $\omega \in M_*$ a left module over Ω_*^{SO} (recall that $M_*[\omega^{-1}]=\varinjlim M_*$, where the connecting maps in the sequence are given by multiplication by ω). Landweber, Ravenel and Stong [LRS], [L1] showed that the functor

$$X \mapsto \Omega_*^{\mathrm{SO}}(X) \otimes_{\Omega_*^{\mathrm{SO}}} M_*[\omega^{-1}] \tag{1.1}$$

is a homology theory if $\omega=\varepsilon$ or $\omega=\delta^2-\varepsilon$. Recently Franke [Fr] proved this for a general ω of positive degree. This 8-periodic homology theory is called (odd primary) periodic elliptic homology.

(1) The second author was partially supported by NSF Grant DMS 88-02481, the Max-Planck-Institut in Bonn and the SFB in Göttingen.

In this situation one has the following obvious problems [L2]:

(a) Give a geometric description of elliptic (co)homology.

(b) Define elliptic (co)homology at the prime 2.

Recently Ochanine [O4] investigated an integral elliptic genus β defined for bordism classes of spin manifolds. In analogy with the homology theory above this suggests that the coefficients of integral elliptic homology should be isomorphic to the image of β with an appropriate element inverted.

The main result of this paper is a *geometric* definition of an (integral) homology theory which agrees with the Landweber–Ravenel–Stong theory after inverting the prime 2. As a by-product we obtain a new geometric description of KO_*-homology. The idea for these geometric constructions is to use fibre bundles with fibre the quaternionic projective plane $\mathbf{HP}^2$. This was motivated by the second author's proof of the Gromov–Lawson conjecture concerning the existence of positive scalar curvature metrics on simply connected spin manifolds of dimension $\geqslant 5$ [St1].

We recall that for small n the spin bordism group Ω_n^{Spin} is as follows (cf. [Mi]): $\Omega_1^{\mathrm{Spin}}\cong\mathbf{Z}/2$ is generated by S^1 (with the non-trivial spin structure), the square of S^1 is a generator of $\Omega_2^{\mathrm{Spin}}\cong\mathbf{Z}/2$ and the Kummer surface K (a 4-manifold with signature 16) is a generator of $\Omega_4^{\mathrm{Spin}}\cong\mathbf{Z}$. The group $\Omega_8^{\mathrm{Spin}}\cong\mathbf{Z}\oplus\mathbf{Z}$ is generated by $\mathbf{HP}^2$ and a manifold B (for 'Bott'), characterized by $\widehat{A}(B)=1$, $\mathrm{sign}(B)=0$. The other groups Ω_n^{Spin} are zero for $n\leqslant 8$.

For a space X let $\Omega_n^{\mathrm{Spin}}(X)$ be the bordism group of n-dimensional closed spin manifolds together with maps to the space X. Let $T_n(X)$ be the subgroup of the $\Omega_n^{\mathrm{Spin}}(X)$ consisting of bordism classes $[E, fp]$, where $p: E\to B$ is an $\mathbf{HP}^2$-bundle over a closed spin manifold B of dimension $n-8$ and f is a map from B to X. Here an $\mathbf{HP}^2$-bundle is a fibre bundle with fibre $\mathbf{HP}^2$ and structure group the projective symplectic group $\mathrm{PSp}(3)$ (which is the isometry group of $\mathbf{HP}^2$ with its standard metric). Let $\widetilde{T}_n(X)$ be the subgroup consisting of all bordism classes $[E, fp]$ as above with the additional assumption that $[B, f]$ is the trivial element of $\Omega_{n-8}^{\mathrm{Spin}}(X)$. Let $\mathrm{ell}_n(X)$ be the quotient of $\Omega_n^{\mathrm{Spin}}(X)$ modulo $\widetilde{T}_n(X)$.

Cartesian product of manifolds induces a multiplication

$$\mathrm{ell}_m(X)\times\mathrm{ell}_n(Y)\to\mathrm{ell}_{m+n}(X\times Y) \tag{1.2}$$

and a natural transformation

$$\Omega_*^{\mathrm{Spin}}(X)\otimes_{\Omega_*^{\mathrm{Spin}}}\mathrm{ell}_*\to\mathrm{ell}_*(X), \tag{1.3}$$

which is compatible with the multiplications on both sides. Here $\mathrm{ell}_*=\mathrm{ell}_*(\mathrm{pt})$ where pt is a point.

THEOREM A. (1) *$\mathrm{ell}_*(X)\otimes\mathbf{Z}_{(2)}$ is a multiplicative homology theory.*

(2) *The natural transformation* (1.3) *is an isomorphism after inverting* 2.

(3) *$\mathrm{ell}_*\cong\mathbf{Z}[s,k,h,b]/(2s,s^3,sk,k^2-2^2(b+2^6h))$, where s, k, b, h are the images of $[S^1]$, $[K]$, $[B]$, $[\mathbf{HP}^2]$, respectively, under the projection map $\Omega_*^{\mathrm{Spin}}\to\mathrm{ell}_*$.*

We remark that the relations in ell_* are consequences of corresponding relations in Ω_*^{Spin}. Combining Theorem A with the Landweber–Ravenel–Stong or the Franke result, that (1.1) is a homology theory, we obtain our main theorem. For an element $v\in\mathrm{ell}_q$ define

$$\mathrm{El}_n^v(X)=\mathrm{ell}_n(X)[v^{-1}]=\varinjlim_k \mathrm{ell}_{n+qk}(X),$$

where the limit is taken over the sequence of homomorphisms given by multiplication by v. In the special case $v=h$ (i.e. if v is represented by $\mathbf{HP}^2$) one has the following nice description of $\mathrm{El}_n^v(X)$:

$$\mathrm{El}_n^h(X)=\bigoplus_{k\in\mathbf{Z}}\Omega_{n+8k}^{\mathrm{Spin}}(X)/\sim,$$

where the equivalence relation $\sim$ is generated by identifying $[M,f]\in\Omega_*^{\mathrm{Spin}}(X)$ with $[E,fp]\in\Omega_{*+8}^{\mathrm{Spin}}(X)$ for every $\mathbf{HP}^2$-bundle $p\colon E\to M$ (i.e. total spaces of $\mathbf{HP}^2$-bundles are identified with their base).

THEOREM B. *The functor $\mathrm{El}_*^v(X)$, where v is any element of positive degree, is a multiplicative homology theory which agrees with the theory $\Omega_*^{\mathrm{SO}}(X)\otimes_{\Omega_*^{\mathrm{SO}}}M_*[\phi(v)^{-1}]$ of Landweber–Ravenel–Stong after inverting* 2.

Here, abusing notation, we denote by ϕ the homomorphism

$$\phi\colon \mathrm{ell}_*=\Omega_*^{\mathrm{Spin}}/\widetilde{T}_*(\mathrm{pt})\longrightarrow M_*$$

induced by the elliptic genus (note that $\widetilde{T}_*(\mathrm{pt})$ is in the kernel of the elliptic genus due to its multiplicative properties for $\mathbf{HP}^2$-bundles). We note that $\phi(k)=\phi(K)=2^4\delta$, $\phi(h)=\phi(\mathbf{HP}^2)=\varepsilon$ and $\phi(b)=\phi(B)=2^6(\delta^2-\varepsilon)$. Hence by part (3) of Theorem A the elliptic genus induces an isomorphism $\mathrm{ell}_*\left[\frac{1}{2}\right]\cong M_*$.

For the proof of Theorem B it suffices to show that $\mathrm{El}_*^v(X)\otimes\mathbf{Z}_{(2)}$ is a homology theory and that $\mathrm{El}_*^v(X)\left[\frac{1}{2}\right]$ is canonically isomorphic to the Landweber–Ravenel–Stong theory. The former is a corollary of part (1) of Theorem A (the direct limit of exact sequences is exact and hence $\mathrm{ell}_*(X)[v^{-1}]\otimes\mathbf{Z}_{(2)}$ is again a homology theory). The latter follows from the natural isomorphisms

$$\mathrm{ell}_*(X)\left[\tfrac{1}{2}\right]\cong\Omega_*^{\mathrm{Spin}}(X)\otimes_{\Omega_*^{\mathrm{Spin}}}\mathrm{ell}_*\left[\tfrac{1}{2}\right]\cong\Omega_*^{\mathrm{SO}}(X)\otimes_{\Omega_*^{\mathrm{SO}}}M_* \tag{1.4}$$

by inverting v resp. $\phi(v)$. The first isomorphism comes from part (2) of Theorem A, the second isomorphism is the tensor product of the isomorphism $\Omega_*^{\mathrm{Spin}}(X)[\frac{1}{2}]\cong\Omega_*^{\mathrm{SO}}(X)[\frac{1}{2}]$ and the isomorphism $\mathrm{ell}_*[\frac{1}{2}]\cong M_*$ induced by ϕ.

Unfortunately, our proof that the geometrically defined functor $\mathrm{El}_*^v(X)$ is a homology theory is *not* geometric in nature. In particular the proof of part (1) of Theorem A makes heavy use of stable homotopy theory. In fact, there are two key results (one concerning a splitting of $\mathrm{MSpin}\wedge\Sigma^8\,\mathrm{BPSp}(3)_+$, the other concerning maps from MSpin to connective real K-theory spectra) which are not proved in this paper, but in the homotopy theoretic companion paper 'Splitting certain MSpin-module spectra' by the second author [St2].

Remarks. (i) In §7 we show that the natural transformation (1.3) does not induce an isomorphism if we invert $h\in\mathrm{ell}_8$ on both sides. In particular, $\Omega_*^{\mathrm{Spin}}(X)\otimes_{\Omega_*^{\mathrm{Spin}}}\mathrm{ell}_*[h^{-1}]$ is *not* a homology theory. However, Hovey has shown recently that $\Omega_*^{\mathrm{Spin}}(X)\otimes_{\Omega_*^{\mathrm{Spin}}}\mathrm{ell}_*[b^{-1}]$ is isomorphic to our functor $\mathrm{El}_*^b(X)$ and thus a homology theory [Ho], indicating a delicate difference between $\mathrm{El}_*^h(X)$ and $\mathrm{El}_*^b(X)$.

(ii) The isomorphism (1.4) implies that $\mathrm{ell}_*(X)[\frac{1}{2}]$ is not a homology theory since the functor $\Omega_*^{\mathrm{SO}}(X)\otimes_{\Omega_*^{\mathrm{SO}}}M_*$ does not satisfy the conditions of the exact functor theorem which by [Ru] are also necessary conditions for such a tensor product to be a homology theory. In particular, $\mathrm{ell}_*(X)$ is not the connective homology theory corresponding to the periodic homology theory $\mathrm{El}_*(X)$. Our notation $\mathrm{El}_*(X)$ (instead of $\mathrm{Ell}_*(X)$) hopefully avoids that possible confusion.

(iii) We show in (5.2) that the 2-local spectrum el corresponding to the homology theory $\mathrm{ell}_*(X)\otimes\mathbf{Z}_{(2)}$ is homotopy equivalent to $\bigvee\Sigma^{8k}\,\mathrm{ko}$ where ko is the connected KO-theory spectrum. However, the ring spectrum structure on el correponding to the multiplication (1.2) does *not* correspond to the multiplication on $\bigvee\Sigma^{8k}\,\mathrm{ko}$ induced by the multiplication on ko. Otherwise the inclusion of the bottom ko would give a ring spectrum map $\mathrm{ko}\to\mathrm{el}$ which is impossible: the arguments in [St2, §7] showing that there is no ring spectrum map $\mathrm{ko}\to\mathrm{MSpin}$ apply since $\mathrm{MSpin}\to\mathrm{el}$ is a 10-equivalence.

(iv) By Spanier–Whitehead duality there are corresponding multiplicative *cohomology* theories $\mathrm{El}_v^*(X)$. It is a very interesting open problem to give a geometric construction of $\mathrm{El}_v^*(X)$. G. Segal has proposed a construction related to topological quantum field theory which might lead to such a geometric definition [Se].

(v) By construction of elliptic homology vector bundles with spin structure are orientable with respect to $\mathrm{El}_*^v(X)$ and $\mathrm{ell}_*(X)\otimes\mathbf{Z}_{(2)}$.

Next we consider the functor $X\mapsto\mathrm{koo}_n(X)=\Omega_n^{\mathrm{Spin}}(X)/T_n(X)$. It is shown in [St2] that $\mathrm{koo}_*(X)\otimes\mathbf{Z}_{(2)}\cong\mathrm{ko}_*(X)\otimes\mathbf{Z}_{(2)}$, where $\mathrm{ko}_*(X)$ is the connective real K-theory of X. In this paper we complete the computation of this functor by analyzing it at odd primes (the easier part) showing that $\mathrm{koo}_*(X)[\frac{1}{2}] \cong \Omega_*^{\mathrm{Spin}}(X) \otimes_{\Omega_*^{\mathrm{Spin}}} \mathrm{ko}_*(\mathrm{pt})[\frac{1}{2}]$. This leads to the following geometric description of periodic real K-theory:

THEOREM C. *There is a natural multiplicative isomorphism between* $\mathrm{koo}_*(X)[b^{-1}]$ *and* $\mathrm{KO}_*(X)$.

We note that Hopkins and Hovey proved recently that the natural transformation $\Omega_*^{\mathrm{Spin}}(X) \otimes_{\Omega_*^{\mathrm{Spin}}} \mathrm{KO}_*(\mathrm{pt}) \to \mathrm{KO}_*(X)$ is an isomorphism [HH].

Remark. One can modify our functors describing El and KO by replacing the category of spin manifolds by a different category and $\mathbf{HP}^2$ by a closed manifold F in that category (with the action of a suitable Lie group G on it). In general our construction does not give a homology theory, but we expect this to hold in the following cases:

(1) non-oriented manifolds and F the real projective plane,

(2) oriented manifolds and F the complex projective plane,

(3) $\mathrm{BO}\langle 8\rangle$-manifolds and F the Cayley plane.

Recently the first two cases were confirmed by Rainer Jung [J]. In the third case $\mathrm{BO}\langle 8\rangle$ is the 7-connected cover of BO and a $\mathrm{BO}\langle 8\rangle$-manifold is a manifold M together with a lift $M \to \mathrm{BO}\langle 8\rangle$ of a classifying map of its tangent bundle. We note that such a lift exists if and only if the loop space of M admits a spin structure [W].

The paper is organized as follows. In §2 we discuss the Atiyah invariant α, the (universal) elliptic genus and then the Ochanine genus β, which can be viewed as a common generalization of both. Moreover, we prove multiplicativity of the Ochanine genus for fibre bundles with compact connected structure group and fibre dimension $\equiv 0, 3 \bmod 4$. In §3 we show that the kernel of β is $\widetilde{T}_*(\mathrm{pt})$ and go on to prove parts (2) and (3) of Theorem A, as well as Theorem C. In §4 we show that the kernel of α is equal to $T_*(\mathrm{pt})$ at odd primes. This section is technical and should be skipped in a first reading. The proof of part (1) of Theorem A is outlined in §5 using some facts which are proved in §6, the homotopy theoretic heart of the paper. In §7 we show that $\Omega_*^{\mathrm{Spin}}(X) \otimes_{\Omega_*^{\mathrm{Spin}}} \mathrm{ell}_*[h^{-1}]$ is not a homology theory.

We would like to thank Serge Ochanine and Rainer Jung for stimulating discussions and Mark Hovey for pointing out inaccuracies in earlier versions of this paper.

2. The elliptic genus and the Ochanine genus

In this section we discuss the (universal) elliptic genus and the Ochanine genus, a generalization of the (universal) elliptic genus. Furthermore we show in Propositions 2.7 and 2.8 that the Ochanine genus is multiplicative for suitable fibre bundles. The elliptic genus (resp. the Ochanine genus) can be thought of as an extension of the $\widehat{A}$-genus (resp. the Atiyah invariant) and hence we find it useful to discuss the $\widehat{A}$-genus first.

The $\widehat{A}$-genus is a ring homomorphism

$$\widehat{A}\colon \Omega_*^{\mathrm{SO}} \to \mathbf{Q}.$$

For a spin manifold M the $\widehat{A}$-genus $\widehat{A}(M)$ is an integer (namely the index of the Dirac operator [AS]) and its restriction to spin bordism can be factored in the form

$$\Omega_*^{\mathrm{Spin}} \xrightarrow{\alpha} \mathrm{KO}_*(\mathrm{pt}) \xrightarrow{\mathrm{ph}} \mathbf{Q}, \tag{2.1}$$

where the ring homomorphism ph ('Pontrjagin character') maps an element of $\mathrm{KO}_n(\mathrm{pt}) = \widetilde{\mathrm{KO}}(S^n)$ to the Chern character of its complexification evaluated on the fundamental class of S^n. To define α recall that for a spin manifold M^n the projection map $\pi^M\colon M \to \mathrm{pt}$ induces a *Gysin map* or *Umkehr homomorphism* $\pi_!^M\colon \mathrm{KO}(M) \to \mathrm{KO}^{-n}(\mathrm{pt}) = \mathrm{KO}_n(\mathrm{pt})$ in KO-theory [Bo, Chapter V, §6] which is constructed making use of the KO-theory Thom isomorphism for spin bundles. Then $\alpha(M) = \pi_!^M(1)$ where 1 is the multiplicative unit of $\mathrm{KO}(M)$ (i.e. the trivial real line bundle). The multiplicative properties of the Gysin map imply that α is a ring homomorphism. Note that the Gysin map is the topological index of Atiyah–Singer.

Now we turn to elliptic genera and the Ochanine genus. A *rational genus* is a ring homomorphism

$$\phi\colon \Omega_*^{\mathrm{SO}} \to \Lambda$$

from the oriented bordism ring to a commutative $\mathbf{Q}$-algebra with unit. Thom showed that $\Omega_*^{\mathrm{SO}} \otimes \mathbf{Q}$ is a polynomial algebra whose generators are the bordism classes of the even dimensional complex projective spaces $\mathbf{CP}^{2n}$. Hence a genus ϕ is determined by the formal power series

$$\log_\phi(x) = \sum_{n \geqslant 0} \frac{1}{2n+1} \phi(\mathbf{CP}^{2n}) x^{2n+1},$$

which is called the *logarithm* of ϕ. Following Ochanine [O2] a rational genus ϕ is called *elliptic* if its logarithm is an integral of the form

$$\log_\phi(x) = \int_0^x \frac{1}{\sqrt{1-2\delta t^2+\varepsilon t^4}}\, dt$$

with $\delta, \varepsilon \in \Lambda$. It turns out that if ϕ is the elliptic genus corresponding to arbitrary elements $\delta, \varepsilon \in \Lambda$ then $\delta = \phi(\mathbf{CP}^2)$ and $\varepsilon = \phi(\mathbf{HP}^2)$. It follows that an elliptic genus ϕ is completely determined by $\phi(\mathbf{CP}^2)$ and $\phi(\mathbf{HP}^2)$. The $\widehat{A}$-genus is an example of an elliptic genus with $\delta=\hat{A}(\mathbf{CP}^2)=-\frac{1}{8}$ and $\varepsilon=\hat{A}(\mathbf{HP}^2)=0$. Clearly, every elliptic genus factors through the *universal elliptic genus*

$$\phi\colon \Omega_*^{\mathrm{SO}} \to \mathbf{Q}[\delta,\varepsilon].$$

which sends $\mathbf{CP}^2$ (resp. $\mathbf{HP}^2$) to δ (resp. ε). It turns out that the image of ϕ is contained in $\mathbf{Z}\left[\frac{1}{2}\right][\delta,\varepsilon]$.

The Ochanine genus is a ring homomorphism $\beta\colon \Omega_*^{\mathrm{Spin}} \to \mathrm{KO}_*(\mathrm{pt})[\![q]\!]$ into the ring of power series with coefficients in $\mathrm{KO}_*(\mathrm{pt})$. It is a generalization of the elliptic genus in the sense that the following diagram is commutative [O4, Theorem 1]:

$$\begin{array}{ccc} \Omega_*^{\mathrm{Spin}} & \xrightarrow{\beta} & \mathrm{KO}_*(\mathrm{pt})[\![q]\!] \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{ph}} \\ \mathbf{Z}\left[\frac{1}{2}\right][\delta,\varepsilon] & \xrightarrow{i} & \mathbf{Q}[\![q]\!]\,. \end{array} \tag{2.2}$$

Here ph is the map $\mathrm{KO}_*(\mathrm{pt})\to\mathbf{Q}$ from (2.1) extended to power series, and i embeds $\mathbf{Z}\left[\frac{1}{2}\right][\delta,\varepsilon]$ as a subring in the power series ring by mapping

$$\delta \text{ to } -\frac{1}{8}-3\sum_{n>0}\Big(\sum_{\substack{d|n\\ d \text{ odd}}} d\Big)q^n \quad\text{and}\quad \varepsilon \text{ to } \sum_{n>0}\Big(\sum_{\substack{d|n\\ n/d \text{ odd}}} d^3\Big)q^n. \tag{2.3}$$

We note that $i(\delta)$ and $i(\varepsilon)$ are q-expansions at the cusp ∞ of level 2 modular forms of weight 2 (resp. 4) derived from the Weierstrass $\wp$-function (compare [Hir, Appendix 1], [Z]). Moreover, we can use the embedding i to identify $\mathbf{Z}\left[\frac{1}{2}\right][\delta,\varepsilon]$ with the level 2 modular forms with q-expansion coefficients in $\mathbf{Z}\left[\frac{1}{2}\right]$ [L2, §4].

To define the Ochanine genus let E be a real vector bundle over a space X. The total exterior (resp. symmetric) power operations are defined by

$$\lambda_t(E)=\sum_{i\geqslant 0}\lambda^i(E)\,t^i \quad \text{resp.} \quad S_t(E)=\sum_{i\geqslant 0}S^i(E)\,t^i,$$

where $\lambda^i(E)$ resp. $S^i(E)$ is the ith exterior (resp. symmetric) power of E. We denote by $\Theta_q(E)$ the following formal power series in q with coefficients in $\mathrm{KO}(X)$:

$$\Theta_q(E)=\sum_{i\geqslant 0}\Theta^i(E)\,q^i=\bigotimes_{n\geqslant 1}(\lambda_{-q^{2n-1}}(E)\otimes S_{q^{2n}}(E)). \tag{2.4}$$

This expression looks rather artificial but from the physics point of view it appears natural. According to Witten [W, p. 167] the index of the Dirac operator on a spin manifold M^n twisted by $\Theta_q(\widetilde{TM})$, where $\widetilde{TM}=TM-n$ is the reduced tangent bundle, can be interpreted as the index of a sort of twisted version of the signature operator on the free loop space ΛM. This operator has *no* finite dimensional analogue, a fact which might be relevant for the definition of elliptic *co*homology.

Θ_q is exponential in the sense that for vector bundles E, F

$$\Theta_q(E \oplus F) = \Theta_q(E) \otimes \Theta_q(F). \tag{2.5}$$

So Θ_q may be extended to virtual bundles and be considered as an exponential map

$$\Theta_q\colon \mathrm{KO}(X) \to \mathrm{KO}(X)[\![q]\!].$$

For an n-dimensional spin manifold M the *Ochanine genus* $\beta(M)$ [O4] is defined as

$$\beta(M) = \sum_{i\geqslant 0} \beta^i(M)\, q^i = \sum_{i \geqslant 0} \pi_!^M(\Theta^i(\widetilde{TM}))\, q^i \in \mathrm{KO}_n(\mathrm{pt})[\![q]\!]. \tag{2.6}$$

In a more compact notation, we write $\beta(M) = \pi_!^M(\Theta_q(\widetilde{TM}))$. It is easy to see that this definition agrees with the definition given in [O4]. We note that $\Theta^0(E)$ for any vector bundle E is the trivial real line bundle and hence $\beta^0(M) = \pi_!^M(1) = \alpha(M)$. The fact that Θ_q is exponential (2.5) plus the naturality of the Gysin map implies $\beta(M \times N) = \beta(M) \cdot \beta(N)$.

Recall that the universal elliptic genus ϕ is multiplicative for a fibre bundle $E \to B$ whose fibre is a spin manifold and whose structure group is compact and connected. It is an open problem whether the Ochanine genus β is multiplicative for such fibre bundles. As a consequence of the rigidity of the elliptic genus, we get the multiplicativity of β under some conditions.

PROPOSITION 2.7. *Let G be a compact, connected Lie group acting on a closed spin manifold F of dimension k preserving the spin structure. Assume that $k \equiv 0, 3$* mod 4 *or $G = S^1$. Then for any fibre bundle $p\colon E \to B$ over a closed spin manifold B with fibre F and structure group G we have $\beta(E) = \beta(B) \cdot \beta(F)$.*

This proposition is a consequence of a slightly more general result. To state it, first some notation: let $\mathrm{K}_G^*(\mathrm{pt})$ (resp. $\mathrm{KO}_G^*(\mathrm{pt})$) be the equivariant complex (resp. real) K-theory of the point and let $\widehat{\mathrm{K}}_G^*(\mathrm{pt})$ (resp. $\widehat{\mathrm{KO}}_G^*(\mathrm{pt})$) be the cokernel of the map from the non-equivariant to the equivariant K-theory.

PROPOSITION 2.8. *The conclusion of Proposition* 2.7 *holds if instead of assuming $k \equiv 0, 3$* mod 4 *or $G = S^1$ we assume that the complexification map $\widehat{\mathrm{KO}}_G^{-k}(\mathrm{pt}) \to \widehat{\mathrm{K}}_G^{-k}(\mathrm{pt})$ is injective.*

Proof of Proposition 2.7 (*using Proposition* 2.8). We recall from [ASe, §8] that

$$\begin{aligned}
&\mathrm{K}_G^{-k}(\mathrm{pt}) \cong R(G) \otimes \mathrm{K}^{-k}(\mathrm{pt}) \\
&\mathrm{KO}_G^{-k}(\mathrm{pt}) \cong A_G \otimes \mathrm{KO}^{-k}(\mathrm{pt}) \oplus B_G \otimes \mathrm{K}^{-k}(\mathrm{pt}) \oplus C_G \otimes \mathrm{KSp}^{-k}(\mathrm{pt}),
\end{aligned}$$

where $R(G)$ is the complex representation ring of G, A_G, B_G and C_G are direct summands of the Real representation ring described below, and KSp is symplectic K-theory. Recall that a Real G-module is is a complex G-module with an antilinear involution commuting with the G-action. The commuting field of a simple Real G-module, i.e. the set of complex linear endomorphisms which commute with the G-action and the involution, is $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. The Real representation ring $R_{\mathbf{R}}(G)$ is the free abelian group generated by the simple Real G-modules. It is isomorphic to $A_G \oplus B_G \oplus C_G$, where A_G (resp. B_G resp. C_G) is the free abelian group generated by the simple Real G-modules whose commuting fields are $\mathbf{R}$ (resp. $\mathbf{C}$ resp. $\mathbf{H}$). Moreover, these isomorphisms are such that the complexification map $\mathrm{KO}_G^{-k}(\mathrm{pt}) \to \mathrm{K}_G^{-k}(\mathrm{pt})$ corresponds to the obvious maps from $\mathrm{KO}^{-k}(\mathrm{pt})$ (resp. $\mathrm{K}^{-k}(\mathrm{pt})$ resp. $\mathrm{KSp}^{-k}(\mathrm{pt})$) into $\mathrm{K}^{-k}(\mathrm{pt})$. By Bott-periodicity $\mathrm{K}^{-k}(\mathrm{pt}) \cong \mathbf{Z}$ for k even and $\mathrm{K}^{-k}(\mathrm{pt})=0$ for k odd and

$$\mathrm{KO}^{-k}(\mathrm{pt}) \cong \mathrm{KSp}^{-k+4}(\mathrm{pt}) \cong \begin{cases} \mathbf{Z} & \text{for } k \equiv 0 \mod 4 \\ \mathbf{Z}/2 & \text{for } k \equiv 1,2 \mod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the complexification map $\mathrm{KO}^{-k}(\mathrm{pt}) \to \mathrm{K}^{-k}(\mathrm{pt})$ and the forgetful homomorphism $\mathrm{KSp}^{-k}(\mathrm{pt}) \to \mathrm{K}^{-k}(\mathrm{pt})$ are injective on the torsion free parts. This shows that the complexification map $\widehat{\mathrm{KO}}_G^{-k}(\mathrm{pt}) \to \widehat{\mathrm{K}}_G^{-k}(\mathrm{pt})$ is injective if $k \equiv 0,3 \mod 4$ or if all nontrivial simple Real G-modules have commuting field $\mathbf{C}$, which is the case for $G=S^1$. □

Proof of Proposition 2.8. The following argument follows closely Segal's argument in his proof that the rigidity of an elliptic genus implies its multiplicativity for fibre bundles [Se, §3]. For the fibre bundle $p: E \to B$ we have $TE \cong p^*TB \oplus T_F$, where T_F is the tangent bundle along the fibres. By the functoriality of the transfer $\pi_!^E = \pi_!^B p_!$ and hence

$$\beta(E) = \pi_!^E(\Theta_q(\widetilde{TE})) = \pi_!^B p_!(p^*\Theta_q(\widetilde{TB}) \cdot \Theta_q(\widetilde{T}_F)) = \pi_!^B(\Theta_q(\widetilde{TB}) \cdot p_!\Theta_q(\widetilde{T}_F)).$$

Now $p_!\Theta_q(\widetilde{T}_F)$ is an element of $\mathrm{KO}^{-k}(B)[\![q]\!]$ whose augmentation is $\pi_!^F(\Theta_q(\widetilde{TF})) = \beta(F) \in \mathrm{KO}^{-k}(\mathrm{pt})[\![q]\!]$ (compare the bundle $p: E \to B$ to $\pi^F: F \to \mathrm{pt}$). It suffices to show that $p_!\Theta_q(\widetilde{T}_F)$ is in the image of $(\pi^B)^*: \mathrm{KO}^{-k}(\mathrm{pt})[\![q]\!] \to \mathrm{KO}^{-k}(B)[\![q]\!]$ since this implies $p_!\Theta_q(\widetilde{T}_F) = (\pi^B)^*(\beta(F))$ and hence

$$\pi_!^B(\Theta^i(\widetilde{TB}) \cdot p_!\Theta_q(\widetilde{T}_F)) = \pi_!^B(\Theta_q(\widetilde{TB}) \cdot (\pi^B)^*(\beta(F))) = \beta(B) \cdot \beta(F).$$

To prove that $p_!\Theta_q(\widetilde{T}_F)$ is in the image of $(\pi^B)^*$ it suffices to prove the corresponding statement for τ, the tangent bundle along the fibres of $\pi: EG \times_G F \to EG \times_G \mathrm{pt} = BG$, the universal bundle with fibre F and structure group G.

CLAIM 2.9. *$\pi_! \Theta_q(\widetilde{\tau})$ is in the image of* $(\pi^{BG})^*\colon \mathrm{KO}^{-k}(\mathrm{pt})[\![q]\!] \to \mathrm{KO}^{-k}(BG)[\![q]\!]$.

To prove the claim consider the commutative diagram

$$\begin{array}{ccc} \mathrm{KO}_G(F) & \longrightarrow & \mathrm{KO}(EG \times_G F) \\ \Big\downarrow \pi_!^F & & \Big\downarrow \pi_! \\ \mathrm{KO}_G^{-k}(\mathrm{pt}) & \longrightarrow & \mathrm{KO}^{-k}(EG \times_G \mathrm{pt}) \end{array}$$

where the horizontal maps take a G-vector bundle over a G-space X to the associated vector bundle over the Borel construction $EG \times_G X$. The equivariant tangent bundle $TF \in \mathrm{KO}_G(F)$ maps to $\tau \in \mathrm{KO}(EG \times_G F)$ and hence $\pi_!^F(\Theta^i(\widetilde{TF})) \in \mathrm{KO}_G^{-k}(\mathrm{pt})$ maps to $\pi_! \Theta^i(\widetilde{\tau})$. Hence it suffices to show that the *equivariant Ochanine genus* $\beta_G^i(F) = \pi_!^F(\Theta^i(\widetilde{TF})) \in \mathrm{KO}_G^{-k}(\mathrm{pt})$ is in the image of $\mathrm{KO}^{-k}(\mathrm{pt}) \to \mathrm{KO}_G^{-k}(\mathrm{pt})$. Our assumption concerning the injectivity of $\widehat{\mathrm{KO}}_G^{-k}(\mathrm{pt}) \to \widehat{\mathrm{K}}_G^{-k}(\mathrm{pt})$ means that it is sufficient to prove the corresponding statement in complex K-theory. For k odd this is trivially true, for k even $\mathrm{K}_G^{-k}(\mathrm{pt})$ can be identified with the complex representation ring RG and via the Atiyah–Singer index theorem $\pi_!^F(\Theta^i(\widetilde{TF})) \in \mathrm{K}_G^{-k}(\mathrm{pt}) = RG$ is the equivariant index of the Dirac operator on F twisted by $\Theta^i(\widetilde{TF})$.

The Witten–Taubes rigidity theorem [T], [BT] says that this index is the trivial representation for $G = S^1$ and hence for all compact, connected Lie groups G; i.e. $\pi_!^F(\Theta^i(\widetilde{TF})) \in \mathrm{K}_G^{-k}(\mathrm{pt})$ is in the image of $\mathrm{K}^{-k}(\mathrm{pt}) \to \mathrm{K}_G^{-k}(\mathrm{pt})$. □

3. Kernel and image of the Ochanine genus

In this section we study kernel and image of the Ochanine genus and prove Theorems A and C of the introduction, except part (1) of Theorem A whose homotopy theoretic proof is defered to §5 and except the determination of $\ker \alpha$ at odd primes (Proposition 3.3) which is given in §4. We begin by an analogous discussion of the ring homomorphism

$$\alpha\colon \Omega_*^{\mathrm{Spin}} \to \mathrm{KO}_*(\mathrm{pt}).$$

By Bott-periodicity,

$$\mathrm{KO}_*(\mathrm{pt}) = \mathbf{Z}[\eta, \omega, \mu, \mu^{-1}]/(2\eta, \eta^3, \eta\omega, \omega^2 - 2^2\mu) \tag{3.1}$$

where η, ω, μ are elements of degree 1, 4, 8, respectively. In fact, for the generators S^1, K, $\mathbf{HP}^2$ and B of the low-dimensional spin bordism groups (cf. §1 after Theorem A) we have

$$\alpha(S^1) = \eta, \quad \alpha(K) = \omega, \quad \alpha(B) = \mu, \quad \alpha(\mathbf{HP}^2) = 0. \tag{3.2}$$

Geometrically $\alpha(M)$ can be interpreted as the index of a family of operators associated to M parametrized by S^n [Hit, p. 39]. Using this geometric interpretation Hitchin showed that $\alpha(M)=0$ if M has a Riemannian metric of positive scalar curvature [Hit]. In particular, $\alpha(\mathbf{HP}^2)=0$ since the standard metric on $\mathbf{HP}^2$ has positive scalar curvature. More generally, total spaces of $\mathbf{HP}^2$-bundles have metrics of positive scalar curvature and hence the subgroup $T_n(\mathrm{pt})$ consisting of bordism classes of such total spaces is in the kernel of α [St1].

PROPOSITION 3.3. $\ker\alpha=T_n(\mathrm{pt})$

Localized at 2 this was proved by the second author in his work on the Gromov–Lawson conjecture [St1]. The proof at odd primes is easier and is provided in §4 below.

Now we turn to the Ochanine genus

$$\beta\colon\Omega_*^{\mathrm{Spin}}\to \mathrm{KO}_*(\mathrm{pt})[\![q]\!].$$

Proposition 2.7 shows that the Ochanine genus is multiplicative for $\mathbf{HP}^2$-bundles (bundles with fibre $\mathbf{HP}^2$ and structure group PSp(3)). We stress that the Witten–Taubes rigidity in this special case is *not* a deep fact, since it can be proved by writing down the equivariant elliptic genus in terms of the fixed point data which are known explicitly.

The multiplicativity for $\mathbf{HP}^2$-bundles implies in particular that the subgroup $\widetilde{T}_n(\mathrm{pt})$ of Ω_n^{Spin} (consisting of total spaces of $\mathbf{HP}^2$-bundles over a zero bordant base) is contained in the kernel of β. The converse holds, too, and it is basically a corollary of Proposition 3.3.

PROPOSITION 3.4. (1) $\ker\beta=\widetilde{T}_n(\mathrm{pt})$,

(2) $\operatorname{im}\beta\cong\mathbf{Z}[\beta(S^1),\beta(K),\beta(B),\beta(\mathbf{HP}^2)]/I$, *where I is the ideal generated by* $2\beta(S^1)$, $\beta(S^1)^3$, $\beta(S^1)\cdot\beta(K)$ *and* $\beta(K)^2-2^2(\beta(B)+2^6\beta(\mathbf{HP}^2))$.

Part (2) is a result of Ochanine [O4, Theorem 3] which he proves by studying the modular properties of $\beta(M)$ for a spin manifold M. Below we give a different proof which makes use of (3.3). We note that Proposition 3.4 implies part (3) of Theorem A.

Proof. Recall from the introduction that $\Omega_1^{\mathrm{Spin}}\cong\mathbf{Z}/2$ is generated by S^1 (with the non-trivial spin structure), $\Omega_4^{\mathrm{Spin}}\cong\mathbf{Z}$ is generated by the Kummer surface K, and $\Omega_8^{\mathrm{Spin}}\cong\mathbf{Z}\oplus\mathbf{Z}$ is generated by $\mathbf{HP}^2$ and a manifold B (for 'Bott'), characterized by $\widehat{A}(B)=1$, $\operatorname{sign}(B)=0$. There are the following obvious relations between these bordism classes:

$$2[S^1]=0,\quad [S^1]^3=0,\quad [S^1]\times[K]=0,\quad [K]^2=2^2([B]+2^6[\mathbf{HP}^2]).\tag{3.5}$$

The first three relations follow from $\Omega_1^{\mathrm{Spin}}\cong\mathbf{Z}/2$ resp. $\Omega_n^{\mathrm{Spin}}=0$ for $n=3,5$, the last relation follows from the fact that $\Omega_8^{\mathrm{Spin}}\cong\mathbf{Z}\oplus\mathbf{Z}$ is detected by $\widehat{A}$-genus and signature and the calculation

$$\widehat{A}(K\times K)=(\widehat{A}(K))^2=2^2,\quad \operatorname{sign}(K\times K)=(\operatorname{sign}(K))^2=2^8.$$

Let S_* be the subalgebra of Ω_*^{Spin} spanned by $[S^1]$, $[K]$, $[B]$, and $[\mathbf{HP}^2]$.

CLAIM. *The restriction of* $\beta\colon\Omega_*^{\mathrm{Spin}}\to \mathrm{KO}_*(\mathrm{pt})[\![q]\!]$ *to* S_* *is injective.*

To prove the claim we note that

$$\begin{aligned}\beta(S^1)&=\eta(1+\dots), & \beta(K)&=\omega(1+\dots),\\ \beta(B)&=\mu(1+\dots), & \beta(\mathbf{HP}2)&=\mu(q+\dots).\end{aligned}\tag{3.6}$$

The first three equalities follow from $\beta^0(M)=\alpha(M)$ and the information about α in (3.2). The last equality follows from diagram (2.2) using the fact that $\phi(\mathbf{HP}^2)=\varepsilon$ and $\mathrm{ph}(\mu)=1$. These equalities show that the elements $\beta(B)^q\beta(\mathbf{HP}^2)^p$ resp. $\beta(K)\beta(B)^q\beta(\mathbf{HP}^2)^p$ are linearly independent over $\mathbf{Z}$ and that the elements $\beta(S^1)^r\beta(B)^q\beta(\mathbf{HP}^2)^p$ for $r=1,2$ are linearly independent over $\mathbf{Z}/2$. Hence there are no other relations between the elements $\beta(S^1)$, $\beta(K)$, $\beta(B)$ and $\beta(\mathbf{HP}^2)$ besides the obvious ones coming from the relations (3.5).

CLAIM. $\Omega_n^{\mathrm{Spin}}=S_n\oplus\widetilde{T}_n(\mathrm{pt})$.

The proof of this claim is by induction over n. $\Omega_n^{\mathrm{Spin}}=S_n$ for $n\leqslant 9$. Now assume that the claim is true for $n<8k$ and that $[M]\in\Omega_{8k+r}^{\mathrm{Spin}}$ with $0\leqslant r<8$. Subtracting if necessary a multiple of $[B]^k[S^1]^r$ (for $r=0,1,2$) or a multiple of $[B]^k[K]$ (for $r=4$) we can assume that $\alpha(M)=0$. Thus by Proposition 3.3 M is bordant to the total space of an $\mathbf{HP}^2$-bundle over some manifold N implying $[M]\equiv[N]\times[\mathbf{HP}^2] \mod \widetilde{T}_*(\mathrm{pt})$. This proves the claim since $[N]$ and hence $[N]\times[\mathbf{HP}^2]$ are in $S_*+\widetilde{T}_*(\mathrm{pt})$ by the induction assumption.

Those two claims and the multiplicativity of the Ochanine genus for $\mathbf{HP}^2$-bundles now imply both parts of Proposition 3.4. □

Proof of part (2) *of Theorem A.* First we provide a different description of the subgroups $T_*(X)$ and $\widetilde{T}_*(X)$ which will also be useful for the proof of Theorem C, as well as for the proof of part (1) of Theorem A in §5.

Given a manifold N and maps $f\colon N\to BG=\mathrm{BPSp}(3)$, $g\colon N\to X$ let $p\colon\widehat{N}\to N$ be the pull back of the fibre bundle

$$\mathbf{HP}^2\to EG\times_G\mathbf{HP}^2\xrightarrow{\pi}BG$$

via f. A spin structure on N induces a spin structure on $\widehat{N}$ and hence we can define a homomorphism

$$\Psi\colon\Omega_{n-8}^{\mathrm{Spin}}(BG\times X)\to\Omega_n^{\mathrm{Spin}}(X)\tag{3.7}$$

by mapping the bordism class of $(N, f\times g)$ to the bordism class of $(\widehat{N}, gp)$. Note that $T_n(X)$ is the image of Ψ and $\widetilde{T}_n(X)$ is the image of $\widetilde{\Psi}$, the restriction of Ψ to

$$\ker(\Omega_{n-8}^{\mathrm{Spin}}(BG\times X)\xrightarrow{(\mathrm{pr}_2)_*}\Omega_{n-8}^{\mathrm{Spin}}(X))\cong\widetilde{\Omega}_{n-8}^{\mathrm{Spin}}(BG\wedge X_+)\cong\widetilde{\Omega}_n^{\mathrm{Spin}}(\Sigma^8 BG\wedge X_+).$$

In other words, there is an exact sequence of (left) modules over Ω_*^{Spin}

$$\widetilde{\Omega}_*^{\mathrm{Spin}}(\Sigma^8 BG\wedge X_+)\xrightarrow{\widetilde{\Psi}}\Omega_*^{\mathrm{Spin}}(X)\to \mathrm{ell}_*(X)\to 0. \tag{3.8}$$

Replacing X by a point and applying the right exact functor $\Omega_*^{\mathrm{Spin}}(X)\otimes_{\Omega_*^{\mathrm{Spin}}}(\text{—})$ gives another exact sequence which maps to the first one via maps induced by Cartesian product of manifolds. Hence we get the following commutative diagram with exact rows (tensor products are tensor products over Ω_*^{Spin}):

$$\begin{tikzcd}
\Omega_*^{\mathrm{Spin}}(X)\otimes\widetilde{\Omega}_*^{\mathrm{Spin}}(\Sigma^8 BG) \arrow[r] \arrow[d] & \Omega_*^{\mathrm{Spin}}(X)\otimes\Omega_*^{\mathrm{Spin}} \arrow[r] \arrow[d] & \Omega_*^{\mathrm{Spin}}(X)\otimes\mathrm{ell}_* \arrow[r] \arrow[d] & 0 \\
\widetilde{\Omega}_*^{\mathrm{Spin}}(\Sigma^8 BG\wedge X_+) \arrow[r] & \Omega_*^{\mathrm{Spin}}(X) \arrow[r] & \mathrm{ell}_*(X) \arrow[r] & 0.
\end{tikzcd} \tag{3.9}$$

The middle vertical map is clearly an isomorphism and the vertical map on the left is an isomorphism after inverting 2 for the following reasons: after inverting 2 the integral homology of BG is concentrated in even dimensions, hence the Atiyah–Hirzebruch spectral sequence converging to $\widetilde{\Omega}_*^{\mathrm{Spin}}(\Sigma^8 BG)$ collapses and so $\widetilde{\Omega}_*^{\mathrm{Spin}}(\Sigma^8 BG)$ is a free module over Ω_*^{Spin}. This implies that $\Omega_*^{\mathrm{Spin}}(X)\otimes\widetilde{\Omega}_*^{\mathrm{Spin}}(\Sigma^8 BG)$ is a homology theory. Thus the left vertical map is a natural transformation between homology theories (with 2 inverted). It is an isomorphism for $X=\mathrm{pt}$ and hence an isomorphism for all X (cf. [CF, Theorems 18.1 and 44.1]). Thus the five lemma implies that the vertical map on the right, which is the natural transformation of part (2) of Theorem A, is an isomorphism after inverting 2. □

Proof of Theorem C. For a space X let $\mathrm{ko}_*(X)$ be the connected real K-homology of X. Then $\mathrm{ko}_n(\mathrm{pt})=\mathrm{KO}_n(\mathrm{pt})$ for $n\geqslant 0$ and hence α can be considered a ring homomorphism $\alpha\colon\Omega_*^{\mathrm{Spin}}\to\mathrm{ko}_*(\mathrm{pt})$ which is surjective by (3.1) and (3.2). Recall from (3.3) that the kernel of α is the subgroup $T_*(\mathrm{pt})$. Hence the induced map $\mathrm{koo}_*(\mathrm{pt})=\Omega_*^{\mathrm{Spin}}/T_*(\mathrm{pt})\to\mathrm{ko}_*(\mathrm{pt})$ is an isomorphism. As explained in [St2] there is a natural transformation $\mathrm{koo}_*(X)\to\mathrm{ko}_*(X)$ restricting to this isomorphism for $X=\mathrm{pt}$. Moreover, this map is an isomorphism when localized at 2 [St2, Theorem B] and is compatible with inverting b resp. μ in the domain resp. range. Hence we get a natural transformation $\mathrm{koo}_*(X)[b^{-1}]\to\mathrm{ko}_*(X)[\mu^{-1}]$ which is an isomorphism localized at 2. The range can be identified with the periodic theory $\mathrm{KO}_*(X)$ since there is a natural transformation of homology theories $\mathrm{ko}_*(X)[\mu^{-1}]\to\mathrm{KO}_*(X)[\mu^{-1}]=\mathrm{KO}_*(X)$ which is an isomorphism for $X=\mathrm{pt}$ and hence for all X.

Hence it suffices to show that $\mathrm{koo}_*(X)[b^{-1}]\otimes\mathbf{Z}\left[\frac{1}{2}\right]$ is a homology theory. We note that replacing ell_* by koo_* and BG by BG_+ in diagram (3.9) above and using the same arguments it follows that the natural transformation $\Omega_*^{\mathrm{Spin}}(X)\otimes_{\Omega_*^{\mathrm{Spin}}}\mathrm{koo}_*\to\mathrm{koo}_*(X)$ induced by Cartesian product of manifolds is an isomorphism after inverting 2. This implies that $\mathrm{koo}_*(X)[b^{-1}]\otimes\mathbf{Z}\left[\frac{1}{2}\right]$ is isomorphic to $\Omega_*^{\mathrm{Spin}}(X)\otimes_{\Omega_*^{\mathrm{Spin}}}\mathrm{KO}_*\left[\frac{1}{2}\right]$, which is a homology theory (cf. [HH, §7]). □

4. Total spaces of $\mathbf{HP}^2$-bundles at odd primes

In this section we prove Proposition 3.3, i.e. we show that the kernel of the Atiyah invariant $\alpha:\Omega_*^{\mathrm{Spin}}\to\mathrm{KO}_*(\mathrm{pt})$ is equal to the subgroup $T_*(\mathrm{pt})$ consisting of bordism classes represented by total spaces of $\mathbf{HP}^2$-bundles. The only thing left to show is that $\ker\alpha\subseteq T_*(\mathrm{pt})$ after inverting 2. We recall from [S, p. 180] that $\Omega_*^{\mathrm{Spin}}(\mathrm{pt})\left[\frac{1}{2}\right]$ is a polynomial algebra generated by elements $[M^{4n}]$ in degree $4n$ where M^{4n} is any spin manifold with

$$s_n(M^{4n})=\begin{cases}2^a & \text{if } 2n+1 \text{ is not a prime power}\\ 2^a p & \text{if } 2n+1 \text{ is a power of some prime } p.\end{cases}$$

Here $s_n(M)$ is the characteristic number $\langle s_n(TM),[M]\rangle\in\mathbf{Z}$, defined by evaluating a certain characteristic class $s_n(TM)\in\mathrm{H}^{4n}(M;\mathbf{Z})$ of the tangent bundle on the fundamental class of M. For a real vector bundle F, $s_n(F)$ (defined e.g. in [MS, §16]) is a polynomial in the Pontrjagin classes $p_i(F)$. Due to the splitting principle, it can be characterized by the following properties:

$$s_n(F\oplus F')=s_n(F)+s_n(F')\quad\text{and}\quad s_n(F)=p_1(F)^n\ \text{ if } p_i(F)=0 \text{ for } i>1. \tag{4.1}$$

We note that if E is a 3-dimensional quaternionic vector bundle then its associated projective bundle PE is a bundle with fibre $\mathbf{HP}^2$ and structure group $\mathrm{Sp}(3)$. In particular, it is an $\mathbf{HP}^2$-bundle in the sense of the introduction.

PROPOSITION 4.2. *For each $n\geqslant 2$ there exists a 3-dimensional quaternionic vector bundle E over a $(4n-8)$-dimensional spin manifold such that*

$$s_n(PE)=\begin{cases}2^{a(n)} & \text{if } 2n+1 \text{ is not a prime power}\\ 2^{a(n)}p & \text{if } 2n+1 \text{ is a power of some prime } p,\end{cases}$$

where $a(n)=2$ if $n=2^i-1$ and $a(n)=1$ otherwise.

This shows that we can choose the generators $[M^{4n}]$ of the polynomial algebra $\Omega_*^{\mathrm{Spin}}\left[\frac{1}{2}\right]$ to be in the ideal $T_*(\mathrm{pt})$ for $n\geqslant 2$. This implies $\ker\alpha\left[\frac{1}{2}\right]\subseteq T_*(\mathrm{pt})\left[\frac{1}{2}\right]$ and proves Proposition 3.3.

Proof of Proposition 4.2. For fixed $n\geqslant 2$ and $0\leqslant r\leqslant n-2$ let E_r be the 3-dimensional quaternionic vector bundle $(\gamma_1\times\gamma_2)\oplus\mathbf{H}$ over the product $\mathbf{HP}^r\times\mathbf{HP}^{n-r-2}$ of quaternionic projective spaces. Here γ_1 and γ_2 are the canonical quaternionic line bundles over the factors and $\mathbf{H}$ is the trivial quaternionic line bundle. We choose the orientation on quaternionic projective space such that $\langle y^k,[\mathbf{HP}^k]\rangle=1$, where y is the generator of $H^4(\mathbf{HP}^k;\mathbf{Z})$ whose pull-back to $\mathbf{CP}^{2k+1}$ is the square of the 2-dimensional generator.

LEMMA 4.3. *$s_n(PE_r)=-2a_r$ where $a_r=\binom{2n}{2(r+1)}-1$.*

Before we prove the lemma we will apply it to finish the proof of Proposition 4.2 by computing $\gcd s_n(P_r)$ for $0\leqslant r\leqslant n-2$. Note that

$$a_0=(2n+1)(n-1)\quad\text{and}\quad a_1=\frac{n(n-1)(2n-3)(2n-1)}{6}-1. \tag{4.4}$$

This implies that $\gcd a_r$ is not divisible by 4. On the other hand, $\gcd a_r$ is divisible by 2 if and only if $\binom{2n}{2(r+1)}$ is odd for $0\leqslant r\leqslant n-2$ which holds if and only if $n+1$ is a power of 2.

If p is an odd prime divisor of $\gcd a_r$ then (4.4) implies that p divides $2n+1$ (in the case $p=3$ observe that if 3 divides $(2n+1)(n-1)$ then it also divides $2n+1$). Note that a_r-a_{r-1} can be written in the form

$$a_r-a_{r-1}=\binom{2n}{2(r+1)}-\binom{2n}{2r}=\binom{2n+1}{2r+2}-\binom{2n+1}{2r+1}. \tag{4.5}$$

Now assume that $2n+1=p^kq$ with q prime to p. Then for $r=\frac{1}{2}(p^k-1)$ we have $a_r-a_{r-1}\not\equiv 0 \bmod p$ since $\binom{p^kq}{p^k+1}\equiv 0 \bmod p$ and $\binom{p^kq}{p^k}\not\equiv 0 \bmod p$. Hence p does not divide $\gcd a_r$, provided $q\geqslant 1$ (for $q=1$ the number $r=\frac{1}{2}(p^k-1)$ does not satisfy the condition $r\leqslant n-2$).

For $2n+1=p^k$ we claim that $\gcd a_r$ is not divisible by p^2. This is clear from (4.4) for $k=1$. For $k\geqslant 2$ it follows from $a_r-a_{r-1}\not\equiv 0 \bmod p^2$ for $r=\frac{1}{2}(p^{k-1}-1)$ which in turn follows from $\binom{p^r}{p^{r-1}+1}\equiv 0 \bmod p^2$ and $\binom{p^r}{p^{r-1}}\not\equiv 0 \bmod p^2$.

On the other hand, p divides $\gcd a_r$ for $2n+1=p^k$, since p is a divisor of a_0 and a_r-a_{r-1} for $0\leqslant r\leqslant n-2$, which follows from (4.5) and $\binom{p^k}{i}\equiv 0 \bmod p$ for all $0\leqslant i<p^k$. This finishes the proof of Proposition 4.2. $\square$

Proof of Lemma 4.3. Consider the following general situation. Let $p\colon \widehat{N}^n\to N^{n-8}$ be a fibre bundle with fibre $\mathbf{HP}^2$ and structure group $G=\mathrm{Sp}(3)$. Such a bundle is the pull-back of the fibre bundle $\pi\colon E=EG\times_G\mathbf{HP}^2\to BG$ via the classifying map $f\colon N\to BG$ of the associated principal bundle. Then the tangent bundle $T\widehat{N}$ is isomorphic to $p^*TN\oplus T_F$, where T_F is the tangent bundle along the fibres of $p\colon\widehat{N}\to N$. Hence

$$\begin{aligned}\langle s_n(T\widehat{N}),[\widehat{N}]\rangle&=\langle p^*s_n(TN)+s_n(T_F),[\widehat{N}]\rangle\\&=\langle p_!(s_n(T_F)),[N]\rangle=\langle f^*(\pi_!(s_n(\tau))),[N]\rangle.\end{aligned} \tag{4.6}$$

Here $p_!: \mathrm{H}^n(\widehat{N};\mathbf{Z}) \to \mathrm{H}^{n-8}(N;\mathbf{Z})$ (resp. $\pi_!: \mathrm{H}^n(E;\mathbf{Z}) \to \mathrm{H}^{n-8}(BG;\mathbf{Z})$) is the Gysin map (integration over the fibre) associated to p (resp. π) [Bo, Chapter V, 6.14] and τ is the tangent bundle along the fibres of $\pi: E \to BG$.

To identify τ we note that the isotropy subgroup of the G-action on $\mathbf{HP}^2$ at the point $[0,0,1] \in \mathbf{HP}^2$ is $H = \mathrm{Sp}(2) \times \mathrm{Sp}(1)$. Hence we can identify the fibre bundle

$$\mathbf{HP}^2 \to EG \times_G \mathbf{HP}^2 \xrightarrow{\pi} BG$$

with the fibre bundle

$$\mathbf{HP}^2 = G/H \to BH \xrightarrow{Bi} BG \tag{4.7}$$

induced by the inclusion $i: H \to G$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ be the decomposition of the Lie algebra of G into the Lie algebra of H and an orthogonal subspace (with respect to the Killing form). The adjoint action of G on $\mathfrak{g}$ restricts to an H-action on $\mathfrak{h}^\perp$. The associated vector bundle $EH \times_H \mathfrak{h}^\perp$ is isomorphic to τ.

Before we can calculate $s_n(\tau)$ we have to discuss the cohomology of BG. We note that the inclusions

$$T^3 = S^1 \times S^1 \times S^1 \xrightarrow{j} H = \mathrm{Sp}(2) \times \mathrm{Sp}(1) \xrightarrow{i} G = \mathrm{Sp}(3)$$

induce monomorphisms of the integral cohomology of the corresponding classifying spaces (here j is the standard inclusion of a maximal torus which maps $(z_1, z_2, z_3) \in T^3$ to the diagonal matrix with these entries). Hence we can identify $\mathrm{H}^*(BG;\mathbf{Z})$ (resp. $\mathrm{H}^*(BH;\mathbf{Z})$) with its image in $\mathrm{H}^*(BT^3;\mathbf{Z}) = \mathbf{Z}[x_1, x_2, x_3]$ (x_i are elements of degree 2), which consists of the subring of polynomials invariant under the Weyl group of G (resp. H) [Bor]. Hence

$$\mathrm{H}^*(BG;\mathbf{Z}) = \mathbf{Z}[x_1^2, x_2^2, x_3^2]^{\Sigma_3} \quad \text{and} \quad \mathrm{H}^*(BH;\mathbf{Z}) = \mathbf{Z}[x_1^2, x_2^2, x_3^2]^{\Sigma_2},$$

where the symmetric group Σ_3 acts on $\mathbf{Z}[x_1^2, x_2^2, x_3^2]$ by permuting the generators and Σ_2 is the subgroup of Σ_3 fixing x_3^2.

To calculate $s_n(\tau) = s_n(EH \times_H \mathfrak{h}^\perp)$ we describe the representation $\mathfrak{h}^\perp$ more explicitly as $\mathbf{H}^2$ with $(a,b) \in H = \mathrm{Sp}(2) \times \mathrm{Sp}(1)$ mapping a point $x \in \mathbf{H}^2$ to $ax\bar{b}$ (here $\bar{b} \in \mathbf{H}$ is the quaternionic conjugate of $b \in \mathbf{H}$ and the multiplication is the matrix product). In particular,

$$\mathfrak{h}^\perp|_{T^3} \cong H_1 \otimes H_3^{-1} \oplus H_1 \otimes H_3 \oplus H_2 \otimes H_3^{-1} \oplus H_2 \otimes H_3, \tag{4.8}$$

where H_i is the 1-dimensional complex representation of T^3 with $(z_1, z_2, z_3) \in T^3$ acting by multiplication by z_i. Hence the pull-back of τ to BT^3 is a sum of complex line bundles and it follows from (4.1) and (4.8) that

$$s_n(\tau) = (x_1 - x_3)^{2n} + (x_1 + x_3)^{2n} + (x_2 - x_3)^{2n} + (x_2 + x_3)^{2n}. \tag{4.9}$$

For the calculation of $\pi_!$ we observe that $H^*(BH;\mathbf{Z})$ is a free module over $H^*(BG;\mathbf{Z})$ with basis $\{1, x_3^2, x_3^4\}$ (this follows e.g. from the Leray–Hirsch theorem [Sw, p. 365] applied to the fibre bundle (4.7)). Hence each $s \in H^*(BH;\mathbf{Z})$ can be written uniquely in the form $s = s_0 + s_1 x_3^2 + s_2 x_3^4$ with $s_i \in H^*(BG;\mathbf{Z})$. It follows from the Serre spectral sequence description of $\pi_!$ [Bo, Chapter V, 6.14] that $\pi_!(s) = s_2$. Using this we can calculate $\pi_!(s_n(\tau))$ for small n, but it soon becomes very tedious to express $s_n(\tau)$ as a linear combination of the basis elements. In this situation the following commutative diagram is useful:

$$\begin{array}{ccc} \mathbf{Z}[x_1^2, x_2^2, x_3^2]^{\Sigma_2} & \xrightarrow{\pi_!} & \mathbf{Z}[x_1^2, x_2^2, x_3^2]^{\Sigma_3} \\ {\scriptstyle x_1^2\times}\downarrow & & \downarrow{\scriptstyle w\times} \\ \mathbf{Z}[x_1^2, x_2^2, x_3^2] & \xrightarrow{A} & \mathbf{Z}[x_1^2, x_2^2, x_3^2]\,. \end{array} \tag{4.10}$$

Here $w = A(x_1^2 x_3^4)$ and A is the anti-symmetrization map which sends a polynomial p to $\sum \operatorname{sign}(\sigma)\,\sigma(p)$, where the sum extends over all $\sigma \in \Sigma_3$ and $\operatorname{sign}(\sigma)$ is the sign of the permutation σ. To prove the commutativity of the diagram we note that all maps are module maps over $\mathbf{Z}[x_1^2, x_2^2, x_3^2]^{\Sigma_3}$. Hence it suffices to check commutativity on the elements of the basis $\{1, x_3^2, x_3^4\}$ which is a short calculation.

Now we calculate $\pi_!(s_n(\tau))$ or rather $Bk^*(\pi_!(s_n(\tau)))$, where $k\colon \mathrm{Sp}(1)\times\mathrm{Sp}(1)\to\mathrm{Sp}(3)$ is the embedding which sends (h_1, h_2) to the diagonal matrix with entries h_1, h_2, 1 in $\mathrm{Sp}(3)$.

$$\begin{aligned} &Bk^*(A(x_1^2\, s_n(\tau))) \\ &= Bk^*\Big(\sum_{\sigma\in\Sigma_3} \operatorname{sign}(\sigma)\, x_{\sigma(1)}^2 [(x_{\sigma(1)} - x_{\sigma(3)})^{2n} + (x_{\sigma(1)} + x_{\sigma(3)})^{2n}]\Big) \\ &= 2x_1^{2n+2} + x_2^2[(x_2 - x_1)^{2n} + (x_2 + x_1)^{2n}] - 2x_2^{2n+2} - x_1^2[(x_1 - x_2)^{2n} + (x_1 + x_2)^{2n}] \\ &= 2(x_1^2 - x_2^2)\Big(\sum_{i=0}^{n} x_1^{2i} x_2^{2(n-i)} - \sum_{i=0}^{n} \binom{2n}{2i} x_1^{2i} x_2^{2(n-i)}\Big) \\ &= 2x_1^2 x_2^2 (x_1^2 - x_2^2) \sum_{i=0}^{n-2}\Big(1 - \binom{2n}{2(i+1)}\Big) x_1^{2i} x_2^{2(n-i-2)}. \end{aligned}$$

With regard to the first equality we note that the terms $x_1^2(x_2 \pm x_3)^{2n}$ are in the kernel of A since they are symmetric with respect to interchanging x_2 and x_3. Since $Bk^*(w) = x_1^2 x_2^2 (x_1^2 - x_2^2)$ the above calculation and the commutative diagram (4.10) imply

$$Bk^*(\pi_!(s_n(\tau))) = \sum_{i=0}^{n-2} 2\Big(1 - \binom{2n}{2(i+1)}\Big) x_1^{2i} x_2^{2(n-i-2)}.$$

Now the statement of the lemma follows from (4.6), since the classifying map of E_r is the inclusion of $\mathbf{HP}^r \times \mathbf{HP}^{n-r-2}$ into $\mathbf{HP}^\infty \times \mathbf{HP}^\infty = B(\mathrm{Sp}(1)\times\mathrm{Sp}(1))$ composed with Bk

and $x_1^{2i}x_2^{2(n-i-2)}$ evaluated on the fundamental class $[\mathbf{HP}^r \times \mathbf{HP}^{n-r-2}]$ is one for $i=r$ and zero otherwise. □

5. Total spaces of $\mathbf{HP}^2$-bundles at the prime 2

In this section we outline the proof that $\mathrm{ell}_*(X)\otimes\mathbf{Z}_{(2)}$ is a homology theory using a splitting result (Proposition 5.1) we prove in the next section. The strategy is to produce a 2-local spectrum el and a natural isomorphism $\mathrm{ell}_*(X)\otimes\mathbf{Z}_{(2)} \to \pi_*(\mathrm{el}\wedge X_+)$. We will actually show that the spectrum el is homotopy equivalent to the wedge $\bigvee \Sigma^{8k}\,\mathrm{ko}$ of suspensions of the connective real K-theory spectrum (Corollary 5.2). Unfortunately we are unable to describe directly a map from MSpin to $\bigvee \Sigma^{8k}\,\mathrm{ko}$ which factors through the connective elliptic homology inducing an isomorphism. The difficulties are related to the fact that el and $\bigvee \Sigma^{8k}\,\mathrm{ko}$ are not homotopy equivalent as ring spectra (cf. remark (iii) of the introduction). From now on all spectra and abelian groups are *localized at the prime* 2. In particular, we write $\mathrm{ell}_*(X)$ instead of $\mathrm{ell}_*(X)\otimes\mathbf{Z}_{(2)}$.

To construct el recall from (3.8) that $\mathrm{ell}_*(X)$ fits into the exact sequence

$$\widetilde{\Omega}_*^{\mathrm{Spin}}(\Sigma^8 BG\wedge X_+) \xrightarrow{\widetilde{\Psi}} \Omega_*^{\mathrm{Spin}}(X) \to \mathrm{ell}_*(X) \to 0.$$

As in [St1, §3] the reduced transfer map $\widetilde{\Psi}$ can be identified via the Pontrjagin–Thom construction with

$$\pi_*(\mathrm{MSpin}\wedge\Sigma^8 BG\wedge X_+) \xrightarrow{(\widetilde{T}\wedge 1)_*} \pi_*(\mathrm{MSpin}\wedge X_+).$$

Here $\widetilde{T}$ is the restriction of the map $T\colon \mathrm{MSpin}\wedge\Sigma^8 BG_+ \to \mathrm{MSpin}$ of [St1, §3] to $\mathrm{MSpin}\wedge\Sigma^8 BG$.

Since $\mathrm{ell}_n(X)$ is the cokernel of the transfer map the cofibre spectrum of $\widetilde{T}$ seems to be a good candidate for the spectrum representing $\mathrm{ell}_*(X)$. But this is not the case since the map $(\widetilde{T}\wedge 1)_*$ is not injective. To overcome this difficulty we split $\mathrm{MSpin}\wedge\Sigma^8 BG$ in an appropriate way.

PROPOSITION 5.1. (1) *The spectrum* $\mathrm{MSpin}\wedge\Sigma^8 BG$ *splits as* $A\vee B$ *such that* $\widetilde{T}|_A$ *induces a monomorphism and* $\widetilde{T}|_B$ *induces the trivial map in* $\mathbf{Z}/2$*-homology.*

(2) *There is a map* $S\colon \bigvee \Sigma^{8k}\,\mathrm{ko} \to \mathrm{MSpin}$ *such that* $A\vee\bigvee \Sigma^{8k}\,\mathrm{ko} \xrightarrow{\widetilde{T}|_A\vee S} \mathrm{MSpin}$ *is a homotopy equivalence.*

Now we define the spectrum el as the cofibre spectrum of $\widetilde{T}|_A$ and denote the projection from MSpin to el by π. Part (2) implies

COROLLARY 5.2. *The composition* $\bigvee \Sigma^{8k} \mathrm{ko} \xrightarrow{S} \mathrm{MSpin} \xrightarrow{\pi} \mathrm{el}$ *is a homotopy equivalence.*

In particular, the map $\pi_*(\mathrm{MSpin} \wedge X_+) \xrightarrow{(\pi\wedge 1)_*} \pi_*(\mathrm{el} \wedge X_+)$ is surjective for all X. The relation between the homology theory corresponding to el and the functor $\mathrm{ell}_*(X)$ is described by the following diagram with exact rows:

$$\begin{array}{ccccccc}
\pi_*(A \wedge X_+) & \xrightarrow{(\widetilde{T}|_A \wedge 1)_*} & \pi_*(\mathrm{MSpin} \wedge X_+) & \xrightarrow{(\pi\wedge 1)_*} & \pi_*(\mathrm{el} \wedge X_+) & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & \\
\pi_*(\mathrm{MSpin} \wedge \Sigma^8 BG \wedge X_+) & \xrightarrow{(\widetilde{T}\wedge 1)_*} & \pi_*(\mathrm{MSpin} \wedge X_+) & \xrightarrow{\mathrm{pr}} & \mathrm{ell}_*(X) & \longrightarrow & 0.
\end{array}$$

The vertical map on the right is surjective since the projection map pr is surjective by definition of $\mathrm{ell}_*(X)$. The next proposition and a diagram chase imply that it is also injective, which proves part (1) of Theorem A.

PROPOSITION 5.3. *The composition of* $\widetilde{T}$ *and* π *is homotopic to zero.*

The rest of §5 is devoted to the proof of this proposition. The idea is to compare $\pi\colon \mathrm{MSpin} \to \mathrm{el}$ to a map $B\colon \mathrm{MSpin} \to \mathrm{KO}[\![q]\!]$ which is a homotopy theoretic version of the Ochanine genus. Here $\mathrm{KO}[\![q]\!]$ is the product of countably many copies KO_i indexed by the non-negative integers. We think of an element of $\pi_*(\mathrm{KO}[\![q]\!])$ as a formal power series of q with coefficients in $\pi_*(\mathrm{KO}) = \mathrm{KO}_*(\mathrm{pt})$ which motivates our notation. To construct B consider the projection map $\gamma\colon \mathrm{BSpin} \to \mathrm{BO}$ as an element of KO(BSpin) and define $B^i = \Phi(\Theta^i(-\gamma)) \in \widetilde{\mathrm{KO}}^0(\mathrm{MSpin})$, where $\Phi\colon \mathrm{KO}^*(\mathrm{BSpin}) \to \widetilde{\mathrm{KO}}^*(\mathrm{MSpin})$ is the KO-theory Thom isomorphism and Θ^i is the KO-theory operation defined in (2.4). Let $B\colon \mathrm{MSpin} \to \mathrm{KO}[\![q]\!]$ be the map with components $B^i\colon \mathrm{MSpin} \to \mathrm{KO}_i$.

LEMMA 5.4. *The induced map* $\Omega_*^{\mathrm{Spin}} \cong \pi_*(\mathrm{MSpin}) \xrightarrow{B_*} \pi_*(\mathrm{KO}[\![q]\!]) = \mathrm{KO}_*(\mathrm{pt})[\![q]\!]$ *is the Ochanine genus* β.

Proof. Let M^n be a spin manifold. Recall that the Pontrjagin–Thom isomorphism maps the bordism class $[M] \in \Omega_n^{\mathrm{Spin}}$ to the element of $\pi_n(\mathrm{MSpin})$ represented by the composition $S^n \xrightarrow{T} \mathrm{M}(-TM) \xrightarrow{\mathrm{Mc}} \mathrm{MSpin}$. Here $\mathrm{M}(-TM)$ is the Thom spectrum of the inverse of the tangent bundle, Mc is the map of Thom spectra induced by the classifying map $M \to \mathrm{BSpin}$ of $-TM$, and T is the Thom (collapsing) map. Hence, using naturality of Θ^i and naturality of the Thom isomorphism we get

$$B_*^i([M]) = \Phi(\Theta^i(-\gamma))\mathrm{Mc}T = \Phi(\Theta^i(\widetilde{TM}))T.$$

To identify this with $\beta^i(M) = \pi_!^M(\Theta^i(\widetilde{TM}))$ recall that the Gysin map $\pi_!$ associated to a fibre bundle $\pi\colon E \to B$ with fibre a manifold F^n is the composition

$$\pi_!\colon \mathrm{KO}^*(E) \xrightarrow{\Phi} \widetilde{\mathrm{KO}}^*(\mathrm{M}(-\tau)) \xrightarrow{T(\pi)^*} \widetilde{\mathrm{KO}}^*(\Sigma^n B_+) = \widetilde{\mathrm{KO}}^{*-n}(B_+). \tag{5.5}$$

Here τ is the tangent bundle along the fibres which is assumed to be a spin bundle, Φ is the Thom isomorphism, and $T(\pi)$ is the Thom map associated to π. Interpreting $T\colon S^n \to \mathrm{M}(-TM)$ as the Thom map $T(\pi^M)$ associated to the fibre bundle $\pi^M\colon M \to \mathrm{pt}$ we conclude

$$\Phi(\Theta^i(\widetilde{TM}))\, T(\pi^M) = \pi_!^M(\Theta^i(\widetilde{TM})) = \beta^i(M),$$

which proves the lemma. □

We recall that the multiplicativity of the Ochanine genus β for certain fibre bundles (Proposition 2.7) implies that the subgroup $\widetilde{T}_n(\mathrm{pt})$ of Ω_n^{Spin} (consisting of total spaces of $\mathbf{HP}^2$-bundles over zero-bordant bases) is in the kernel of the Ochanine genus. Homotopy theoretically, $\widetilde{T}_n(\mathrm{pt})$ is the image of the reduced transfer map $\widetilde{T}\colon \mathrm{MSpin}\wedge\Sigma^8 BG \to \mathrm{MSpin}$ on homotopy groups. This implies that the composition of $\widetilde{T}$ and B is trivial on homotopy groups. In fact, more is true:

LEMMA 5.6. *The composition* $\mathrm{MSpin}\wedge\Sigma^8 BG \xrightarrow{\widetilde{T}} \mathrm{MSpin} \xrightarrow{B} \mathrm{KO}[\![q]\!]$ *is zero homotopic.*

Proof. By construction [St1, §3], the transfer map T is the composition

$$T\colon \mathrm{MSpin}\wedge\Sigma^8 BG_+ \xrightarrow{\mathrm{id}\wedge t} \mathrm{MSpin}\wedge\mathrm{MSpin} \xrightarrow{\mu} \mathrm{MSpin}, \tag{5.7}$$

where μ is the multiplication of the ring spectrum MSpin, and t is the map

$$t\colon \Sigma^8 BG_+ \xrightarrow{T(\pi)} \mathrm{M}(-\tau) \xrightarrow{\mathrm{M}c} \mathrm{MSpin}.$$

Here $T(\pi)$ is the Thom map associated to the fibre bundle $\mathbf{HP}^2 \to EG\times_G \mathbf{HP}^2 \xrightarrow{\pi} BG$, τ is the tangent bundle along the fibres and $\mathrm{M}c$ is the map of Thom spectra induced by the classifying map of $-\tau$.

We consider first the composition of t and B^i. By naturality of Θ^i, naturality of the Thom isomorphism and the construction of $\pi_!$ (see (5.5)) we get:

$$B^i t = \Phi(\Theta^i(-\gamma))\mathrm{M}cT(\pi) = \Phi(\Theta^i(\widetilde{\tau}))T(\pi) = \pi_!(\Theta^i(\widetilde{\tau})).$$

By (2.9) $\pi_!(\Theta^i(\widetilde{\tau}))$ is in the image of $\mathrm{KO}^{-8}(\mathrm{pt}) \to \mathrm{KO}^{-8}(BG)$, i.e. the restriction of $B^i t\colon \Sigma^8 BG_+ \to \mathrm{KO}$ to $\Sigma^8 BG$ is trivial.

Note that this implies that $B\widetilde{T}$, the restriction of BT to $\mathrm{MSpin}\wedge\Sigma^8 BG$, is trivial using the following fact. □

LEMMA 5.8. *There is a multiplication map* $\mu\colon \mathrm{KO}[\![q]\!]\wedge\mathrm{KO}[\![q]\!] \to \mathrm{KO}[\![q]\!]$ *such that* $B\colon \mathrm{MSpin} \to \mathrm{KO}[\![q]\!]$ *is a ring spectrum map.*

Proof. The multiplication μ on $\mathrm{KO}[\![q]\!] = \prod_{i\geq 0}\mathrm{KO}_i$ is given by 'multiplication of power series'. Its KO_s-component is the composition

$$\prod_{i\geq 0}\mathrm{KO}_i \wedge \prod_{i\geq 0}\mathrm{KO}_i \to \prod_{0\leq i\leq s}\mathrm{KO}_i \wedge \prod_{0\leq i\leq s}\mathrm{KO}_i = \bigvee_{0\leq i\leq s}\mathrm{KO}_i \wedge \bigvee_{0\leq i\leq s}\mathrm{KO}_i \to \mathrm{KO}_s, \tag{5.9}$$

where the first map is the projection map and the second map restricted to the summand $\mathrm{KO}_i \wedge \mathrm{KO}_j$ is the trivial map for $i+j\neq s$ and is the multiplication of KO (induced by the tensor product of vector bundles) for $i+j=s$.

After applying the Thom isomorphism (making use of its multiplicative properties) the proof that $B=\Phi(\Theta_q(-\gamma))$ is a ring spectrum map follows from the following facts:

(1) The multiplication on MSpin is induced by the Whitney sum.

(2) The multiplication on $\mathrm{KO}[\![q]\!]$ is induced by the tensor product.

(3) Θ_q is exponential (cf. (2.5))

This proves Lemma 5.8. □

Proof of Proposition 5.3. To show that the composition

$$g\colon \mathrm{MSpin}\wedge\Sigma^8 BG \xrightarrow{\widetilde{T}} \mathrm{MSpin} \xrightarrow{\pi} \mathrm{el}$$

is trivial we recall that el is homotopy equivalent to $\bigvee \Sigma^{8k}\,\mathrm{ko}$ by (5.2) and note that the natural map $\bigvee \Sigma^{8k}\,\mathrm{ko}\to\prod_{k\geqslant 0}\Sigma^{8k}\,\mathrm{ko}$ is a homotopy equivalence since it induces an isomorphism in homotopy. Hence g is equivalent to a sequence of maps

$$g^k\colon \mathrm{MSpin}\wedge\Sigma^8 BG\to\Sigma^{8k}\,\mathrm{ko}.$$

We note that $\Sigma^{8k}\,\mathrm{ko}$ is homotopy equivalent to $\mathrm{ko}\langle 8k\rangle$, the $(8k-1)$-connected cover of KO and hence by [St2, Theorem 5.2], which shows that for suitable spectra X the group $[X,\mathrm{ko}\langle k\rangle]$ of homotopy classes of maps from X to $\mathrm{ko}\langle k\rangle$ can be computed in terms of a pull back diagram, g^k is homotopic to zero if and only if it induces zero in $\mathbf{Z}/2$-homology (which is clear by construction of el) and the composition with the projection p_{8k} from $\Sigma^{8n}\,\mathrm{ko}=\mathrm{ko}\langle 8k\rangle$ to KO is zero-homotopic. We know from Lemma 5.6 that the composition $B\widetilde{T}$ is trivial. In particular, B factors through a map $\overline{B}\colon\mathrm{el}\to\mathrm{KO}[\![q]\!]$, and the composition

$$\mathrm{MSpin}\wedge\Sigma^8 BG\xrightarrow{g}\mathrm{el}\xrightarrow{\overline{B}}\mathrm{KO}[\![q]\!]$$

is trivial. Let $\overline{B}^i\colon\mathrm{el}\to\mathrm{KO}$ be the ith component of $\overline{B}$ (recall that $\mathrm{KO}[\![q]\!]=\prod_{i\geqslant 0}\mathrm{KO}_i$), let $\overline{B}_k$ be the restriction of $\overline{B}$ to the summand $\Sigma^{8k}\,\mathrm{ko}$ of $el=\bigvee\Sigma^{8k}\,\mathrm{ko}$ and let $\overline{B}^i_k$ be the restriction of $\overline{B}^i$ to $\Sigma^{8k}\,\mathrm{ko}$.

CLAIM 5.10. *$\overline{B}^i_k$ is trivial for $k>i$ and $\overline{B}^k_k$ is the projection map* $p_{8k}\colon\Sigma^{8k}\,\mathrm{ko}=\mathrm{ko}\langle 8k\rangle\to\mathrm{KO}$. □

Assuming the claim which will be proved in §6 and assuming inductively that g^k is trivial for $k<i$ it follows that

$$0=\overline{B}^i g=\sum_{0\leqslant k}\overline{B}^i_k g^k=\sum_{0\leqslant k\leqslant i}\overline{B}^i_k g^k=p_{8i}g^i$$

is trivial. This implies that g^i is trivial and proves Proposition 5.3. □

6. A splitting of MSpin

This section is devoted to the proof of the splitting result (5.1). As before $G=\mathrm{PSp}(3)$ and we localize everything at the prime 2. The strategy of the proof is to show first that the statement of the proposition holds on the level of homology groups. Then we use a result of [St2] to show that the splitting of $\mathrm{H}_*\mathrm{MSpin}\wedge\Sigma^8 BG$ as a comodule over the dual Steenrod algebra can be realized geometrically. The study of the maps induced by $\widetilde{T}$ and S in homology is made a lot easier by the fact that the spectra involved are 'homology ko-module spectra' which implies that their homology groups have a nice structure (they are extended $A(1)_*$-comodules).

For the convenience of the reader we begin by recalling the definitions of ring spectra and (homology) module spectra (cf. [Sw, (13.50) and (13.51)], [St2]). Then we construct the map S and state the result concerning the homology of homology ko-module spectra before calculating the maps induced by $\widetilde{T}$ and S.

A *ring spectrum* is a spectrum E with a 'product' $\mu\colon E\wedge E\to E$ and a 'unit' $\iota\colon S^0\to E$ such that the diagrams expressing the associativity of μ resp. that ι is a unit for μ are commutative up to homotopy. A map $f\colon E\to E'$ between two ring spectra is a *ring spectrum map* if the appropriate diagrams comparing the multiplication and the unit in E with those in E' are homotopy commutative.

For example, the Whitney sum of vector bundles induces a multiplication $\mathrm{MSpin}\wedge\mathrm{MSpin}\to\mathrm{MSpin}$ which makes MSpin a ring spectrum (the unit is given by the inclusion of the bottom cell). Similarly, the tensor product of vector bundles induces a product $\mathrm{ko}\wedge\mathrm{ko}\to\mathrm{ko}$ which makes ko a ring spectrum (again, the unit is given by the inclusion of the bottom cell). The KO-theory Thom class for spin bundles gives a map $D\colon\mathrm{MSpin}\to\mathrm{ko}$. The multiplicativity of the Thom class implies that D is a ring spectrum map. As shown in [St2] a ring spectrum map $\mathrm{ko}\to\mathrm{MSpin}$ doesn't exist, but there is a map $s\colon\mathrm{ko}\to\mathrm{MSpin}$ which is a right inverse to D and induces an algebra map in homology.

An E-module spectrum is a spectrum F with an action map $\alpha\colon E\wedge F\to F$ such that appropriate diagrams are commutative up to homotopy. These diagrams encode the associativity of the action and the fact that the unit acts trivially. An E-module spectrum map is a map $f\colon F\to F'$ between E-module spectra such that the diagram comparing the E-action on F with the action on F' is homotopy commutative. The simplest kind of E-module spectrum is of the form $E\wedge X$ where X is some spectrum and the E-action is given by the multiplication in E. If $f\colon X\to F$ is a map from a spectrum X to an E-module spectrum F we can form the composition

$$\hat{f}\colon E\wedge X\xrightarrow{1\wedge f}E\wedge F\xrightarrow{\alpha}F$$

which is an E-module map we call the *E-extension* of f.

For example, the transfer map $T\colon \mathrm{MSpin}\wedge\Sigma^8 BG_+\to\mathrm{MSpin}$ is (by definition) the MSpin-extension of a map $t\colon\Sigma^8 BG_+\to\mathrm{MSpin}$ (cf. (5.7) and [St1, §3]). In particular, T is an MSpin-module map. The corresponding statement holds when we remove the base point from BG_+ and replace T (resp. t) by their restrictions $\widetilde{T}$ (resp. $\tilde{t}$).

As in [St2] we generalize the notion of E-module spectra and E-module map by replacing 'homotopy commutative' by 'commutative in homology'. Such spectra (resp. maps) we call *homology E-module spectra* (resp. *homology E-module maps*). We note that we can regard every MSpin-module spectrum as a homology ko-module spectrum via the map $s\colon\mathrm{ko}\to\mathrm{MSpin}$ which looks like a ring spectrum map in homology. Moreover, every MSpin-module map such as $\widetilde{T}$ can be considered as a homology ko-module map.

Now we construct the map S. For $k\geqslant 0$ let $s_k\colon S^{8k}\to\mathrm{MSpin}$ be the map corresponding to the bordism class of the kth power of $\mathbf{HP}^2$. Let S be the ko-extension of the map $\vee s_k\colon S^{8k}\to\mathrm{MSpin}$. More explicitly, S is the composition

$$\bigvee\Sigma^{8k}\,\mathrm{ko}=\mathrm{ko}\wedge\Big(\bigvee_{k\geqslant 0}S^{8k}\Big)\xrightarrow{s\wedge\vee s_k}\mathrm{MSpin}\wedge\mathrm{MSpin}\xrightarrow{\mu}\mathrm{MSpin},\tag{6.1}$$

where μ is the multiplication map of the ring spectrum MSpin.

Proof of Claim 5.10. We note that if we identify el with $\bigvee\Sigma^{8k}\,\mathrm{ko}$ via (5.2) then $\overline{B}_k$ corresponds to the composition

$$\Sigma^{8k}\,\mathrm{ko}\xrightarrow{S|_{\Sigma^{8k}\mathrm{ko}}}\mathrm{MSpin}\xrightarrow{B}\mathrm{KO}[\![q]\!].$$

It follows that $\overline{B}_k$ is the ko-extension of $S^{8k}\xrightarrow{s_k}\mathrm{MSpin}\xrightarrow{B}\mathrm{KO}[\![q]\!]$ (regarding $\mathrm{KO}[\![q]\!]$ as homology ko-module spectrum via $\mathrm{ko}\xrightarrow{s}\mathrm{MSpin}\xrightarrow{B}\mathrm{KO}[\![q]\!]$). Recall that $s_k\in\pi_{8k}(\mathrm{MSpin})\cong\Omega_{8k}^{\mathrm{Spin}}$ is the kth power of the bordism class of $\mathbf{HP}^2$ and hence

$$Bs_k=B_*(s_k)=\beta([\mathbf{HP}^2]^k)=\mu^k(q^k+\ldots)\in\pi_{8k}(\mathrm{KO})[\![q]\!]$$

by (5.4) and (3.6). This implies that $\overline{B}{}^i_k$, the ko-extension of $S^{8k}\xrightarrow{s_k}\mathrm{MSpin}\xrightarrow{B^i}\mathrm{KO}$, is trivial for $k>i$ and that $\overline{B}{}^k_k$ is the ko-extension of $\mu^k\colon S^{8k}\to\mathrm{KO}$, which agrees with $p_{8k}\colon\Sigma^{8k}\,\mathrm{ko}=\mathrm{ko}\langle 8k\rangle\to\mathrm{KO}$ by Bott-periodicity (note that $\mathrm{ko}\xrightarrow{s}\mathrm{MSpin}\xrightarrow{B^0}\mathrm{KO}$ is the canonical projection map p_0). □

Recall that the homology of a spectrum X is a (left) comodule over the dual Steenrod algebra A_*, i.e. there is a homomorphism $\psi\colon\mathrm{H}_*X\to A_*\otimes\mathrm{H}_*X$ satisfying a 'coassociativity' condition. The Hopf algebra A_* can be described explicitly as the polynomial algebra $\mathbf{Z}/2[\zeta_1,\zeta_2,\ldots]$ with generators ζ_j of degree 2^j-1 (the ζ_j's are the conjugates of the usual generators ξ_j). The coproduct is given by the formula

$$\psi(\zeta_j)=\sum_{i=0}^{j}\zeta_i\otimes\zeta_{j-i}^{2^i}\tag{6.2}$$

(cf. [Ra, Theorem 3.1.1]).

It turns out that the homology of ko as A_*-comodule is closely related to the Hopf algebra $A(1)_* = A_*/(\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \dots)$ ($A(1)_*$ is the Hopf algebra dual to the subalgebra $A(1)$ of A generated by Sq^1 and Sq^2). Note that we can view A_* as a (right) $A(1)_*$-comodule by composing the coproduct (6.2) with the projection map on $A(1)_*$. Let M be a (left) $A(1)_*$-comodule. Recall that the cotensor product $A_* \square_{A(1)_*} M$ is defined by the exact sequence

$$0 \to A_* \square_{A(1)_*} M \to A_* \otimes M \xrightarrow{\psi\otimes 1 - 1\otimes\psi} A_* \otimes A(1)_* \otimes M,$$

where ψ denotes the $A(1)_*$-comodule structure maps for both A_* and M. We note that the (left) A_*-comodule structure on $A_* \otimes M$ induces a A_*-comodule structure on $A_* \square_{A(1)_*} M$. Such A_*-comodules are refered to as 'extended' $A(1)_*$-comodules. The homology of ko is an example of such a comodule: $\mathrm{H}_*\mathrm{ko} \cong A_* \square_{A(1)_*} \mathbf{Z}/2$ (cf. [Ra, p. 76]). It turns out that the homology of every homology ko-module spectrum Y is an extended $A(1)_*$-module. More precisely, the map on homology induced by the action map $\mathrm{ko} \wedge Y \to Y$ makes H_*Y a module over $\mathrm{H}_*\mathrm{ko}$. Let $\pi\colon \mathrm{H}_*Y \to \overline{\mathrm{H}_*Y}$ be the projection onto the indecomposables of this module. Note that $\overline{\mathrm{H}_*Y}$ is an $A(1)_*$-comodule since the augmentation ideal of $\mathrm{H}_*\mathrm{ko}$ is an $A(1)_*$-comodule.

PROPOSITION 6.3 [St2, §2]. *For a homology* ko*-module spectrum Y the composition*

$$\Phi_Y\colon \mathrm{H}_*Y \xrightarrow{\psi} A_* \otimes \mathrm{H}_*Y \xrightarrow{1\otimes\pi} A_* \otimes \overline{\mathrm{H}_*Y}$$

is an A_*-comodule isomorphism onto $A_* \square_{A(1)_*} \overline{\mathrm{H}_*Y}$.

As remarked above, an MSpin-module spectrum Y can be considered as a homology ko-module spectrum via the map $s\colon \mathrm{ko} \to \mathrm{MSpin}$. Then the above definition of the $A(1)_*$-comodule $\overline{\mathrm{H}_*Y}$ agrees with the definition of $\overline{\mathrm{H}_*Y}$ for the MSpin-module spectrum Y in [St1, §6] as explained in [St2, §2].

Note that a homology ko-module map $f\colon Y \to Z$ induces a $A(1)_*$-comodule map

$$\overline{f_*}\colon \overline{\mathrm{H}_*Y} \to \overline{\mathrm{H}_*Z}.$$

It is clear that the definition of the isomorphism Φ_Y is functorial, and hence we can identify the induced map $f_*\colon \mathrm{H}_*Y \to \mathrm{H}_*Z$ with the 'extended' homomorphism

$$\mathrm{id}\square\overline{f_*}\colon A_* \square_{A(1)_*} \overline{\mathrm{H}_*Y} \to A_* \square_{A(1)_*} \overline{\mathrm{H}_*Z}.$$

The following proposition is then the analogue of (5.1) on the homology level.

PROPOSITION 6.4. *The* $A(1)_*$*-comodule* $\overline{\mathrm{H}_*(\mathrm{MSpin}\wedge\Sigma^8BG)}$ *can be decomposed in the form* $A\oplus B$ *such that*

$$A\oplus\overline{\mathrm{H}_*(\mathrm{ko}\wedge(\vee S^{8k}))}\xrightarrow{\overline{\widetilde{T}}_*|_A\oplus\overline{S}}\overline{\mathrm{H}_*\mathrm{MSpin}}$$

is an isomorphism and $\overline{\widetilde{T}}_*|_B$ *is trivial.*

This result implies Proposition 5.1 since the splitting $\overline{\mathrm{H}_*(\mathrm{MSpin}\wedge\Sigma^8BG)}=A\oplus B$ is induced by a splitting of $\mathrm{MSpin}\wedge\Sigma^8BG$ as homology ko-module spectrum by [St2, Proposition 8.5].

Proof of Proposition 6.4. To make the structure of the proof transparent we subdivide it into a sequence of claims and their proofs which together imply Proposition 6.4. Recall that $\overline{\mathrm{H}_*\mathrm{MSpin}}$ is a polynomial algebra with a generator y_n in each degree $n\geqslant 8$, $n\neq 2^m\pm 1$ [St1, (9.2)].

CLAIM 1. $\mathrm{im}(\overline{S}_*)=\mathbf{Z}/2[y_8]\subset\overline{\mathrm{H}_*\mathrm{MSpin}}$

By construction, S is of the form $S=\bigvee_k \hat{s}_k$, where $\hat{s}_k\colon \mathrm{ko}\wedge S^{8k}\to\mathrm{MSpin}$ is the ko-extension of $s_k\colon S^{8k}\to\mathrm{MSpin}$. By Lemma 2.9 of [St2] the homomorphism

$$\overline{(\hat{s}_k)_*}\colon \mathrm{H}_*S^{8k}=\overline{\mathrm{H}_*(\mathrm{ko}\wedge S^{8k})}\to\overline{\mathrm{H}_*\mathrm{MSpin}}$$

agrees with

$$\mathrm{H}_*S^{8k}\xrightarrow{(s_k)_*}\mathrm{H}_*\mathrm{MSpin}\xrightarrow{p}\overline{\mathrm{H}_*\mathrm{MSpin}}$$

where p is the projection on the H_*ko-indecomposables. Hence $\overline{(\hat{s}_k)_*}$ maps the generator of $\mathrm{H}_*(S^{8k})$ to $(p(x))^k$, where $x\in\mathrm{H}_8(\mathrm{MSpin})$ is the image of $[\mathbf{HP}^2]\in\Omega_8^{\mathrm{Spin}}(\mathrm{MSpin})\cong\pi_8(\mathrm{MSpin})$ under the Hurewicz map. Note that x is non-trivial since $\mathbf{HP}^2$ has non-zero mod 2 characteristic numbers (e.g. the mod 2 Euler characteristic).

Recall from Proposition 6.3 that

$$\mathrm{H}_*(\mathrm{MSpin})\xrightarrow{\psi}A_*\otimes\mathrm{H}_*(\mathrm{MSpin})\xrightarrow{1\otimes p}A_*\otimes\overline{\mathrm{H}_*\mathrm{MSpin}} \tag{6.5}$$

is a monomorphism with image $A_*\Box_{A(1)_*}\overline{\mathrm{H}_*\mathrm{MSpin}}$. Under this composition the element x maps to $1\otimes p(x)$ since x is in the image of the Hurewicz map and hence $\psi(x)=1\otimes x$. It follows that $p(x)$ is non-zero and thus $p(x)=y_8$, the only non-trivial element in degree 8. We conclude that the image of $\overline{S}_*$ is the subalgebra $\mathbf{Z}/2[y_8]$.

CLAIM 2. $\overline{\mathrm{H}_*\mathrm{MSpin}}$ *is spanned by* $\mathrm{im}(\overline{S}_*)$ *and* $\mathrm{im}(\overline{\widetilde{T}}_*)$.

Recall that $\widetilde{T}\colon\mathrm{MSpin}\wedge\Sigma^8BG\to\mathrm{MSpin}$ is the MSpin-extension of $\tilde{t}\colon\Sigma^8BG\to\mathrm{MSpin}$. It follows that the image of $\overline{\widetilde{T}}_*$ is the ideal in $\overline{\mathrm{H}_*\mathrm{MSpin}}$ generated by the image of $\mathrm{H}_*\Sigma^8BG\xrightarrow{\tilde{t}_*}\mathrm{H}_*\mathrm{MSpin}\xrightarrow{p}\overline{\mathrm{H}_*\mathrm{MSpin}}$. According to [St1, (8.8)] the homomorphism

$$\mathrm{H}_*\Sigma^8BG_+\xrightarrow{pt_*}\overline{\mathrm{H}_*\mathrm{MSpin}}\to Q\overline{\mathrm{H}_*\mathrm{MSpin}}$$

is onto in positive degrees, where the unlabeled map is the projection on the indecomposables of the algebra $\overline{\mathrm{H}_*\mathrm{MSpin}}$. Hence, if we replace BG_+ by BG and t by its restriction $\tilde{t}$, the corresponding homomorphism is surjective in degrees >8. It follows that $\overline{\mathrm{H}_*\mathrm{MSpin}}$ is spanned (as a vector space) by the images of $\overline{S_*}$ and $\overline{\widetilde{T}_*}$.

CLAIM 3. *The intersection of* $\mathrm{im}(\overline{S_*})$ *and* $\mathrm{im}(\overline{\widetilde{T}_*})$ *is trivial.*

The idea of the proof is to find a homology ko-module map $W\colon \mathrm{MSpin} \to K$ such that $\overline{W_*}$ maps $\mathrm{im}(\overline{\widetilde{T}_*})$ trivially and $\mathrm{im}(\overline{S_*})$ injectively.

Let $w_n \in \mathrm{H}^n\mathrm{MO}$ be the element corresponding to the nth Stiefel–Whitney class under the Thom isomorphism. Consider w_n as a map $w_n\colon \mathrm{MO} \to \Sigma^n H$ into the nth suspension of the $\mathbf{Z}/2$–Eilenberg–MacLane spectrum H. Define

$$w = \prod_n w_n\colon \mathrm{MO} \to K = \prod_n \Sigma^n H$$

where n runs through the non-negative integers. Note that the Cartan formula for the Stiefel–Whitney classes imples that w is a ring spectrum map if we equip K with the 'power series multiplication' (cf. (5.9)). In particular, we can regard K as a ko-module spectrum via the ring spectrum map $\mathrm{ko} \xrightarrow{\iota} H \xrightarrow{S_H} \mathrm{MO} \xrightarrow{w} K$, where ι corresponds to the non-zero class of $\mathrm{H}^0\mathrm{ko}$ and S_H is the ring spectrum map from Proposition 6.1 in [St2]. This composition agrees with $\mathrm{ko} \xrightarrow{s} \mathrm{MSpin} \xrightarrow{\mathrm{pr}} \mathrm{MO} \xrightarrow{w} K$ by [St2, Proposition 6.7] and hence the composition $W\colon \mathrm{MSpin} \xrightarrow{\mathrm{pr}} \mathrm{MO} \xrightarrow{w} K$ is a homology ko-module map. The element $W_*(x^k)$ of $H^{8k}(K)$ is non-trivial, since the Euler characteristic of $(\mathbf{HP}^2)^k$ is odd. Hence $\overline{W_*}(y_8^k)$ is non-zero and $\overline{W_*}$ maps $\mathrm{im}(\overline{S_*}) = \mathbf{Z}/2[y_8]$ injectively.

It remains to be shown that $\mathrm{im}(\overline{\widetilde{T}_*})$ is in the kernel of $\overline{W_*}$. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{MSpin} \wedge \Sigma^8 BG \arrow[r, "\widetilde{T}"] \arrow[d, "\mathrm{pr} \wedge 1"'] & \mathrm{MSpin} \arrow[d, "\mathrm{pr}"] & \\
\mathrm{MO} \wedge \Sigma^8 BG \arrow[r, "\widetilde{T}'"] & \mathrm{MO} \arrow[r, "w"] & K
\end{tikzcd}$$

where $\widetilde{T}'$ is the MO-extension of $\Sigma^8 BG \xrightarrow{\tilde{t}} \mathrm{MSpin} \xrightarrow{\mathrm{pr}} \mathrm{MO}$. Identifying $\pi_*(\mathrm{MO})$ with the unoriented bordism ring $\mathfrak{N}_*$ the image of the induced map $\widetilde{T}'_*\colon \pi_n(\mathrm{MO} \wedge BG) \to \pi_n(\mathrm{MO})$ consists of the bordism classes represented by total spaces of $\mathbf{HP}^2$-bundles over manifolds which represent zero in $\mathfrak{N}_{n-8}$. In particular, the mod 2 Euler characteristic of such a bordism class $[M]$ is zero. It follows that $[M]$ is in the kernel of $w_*\colon \pi_n(\mathrm{MO}) \to \pi_n(K)$ since the mod 2 Euler characteristic is $w_n(M)$ evaluated on the fundamental class of M. Thus the composition $w\widetilde{T}'$ induces the zero homomorphism on homotopy. We note that this composition is an MO- and hence H-module map. The following lemma then implies that $w\widetilde{T}'$ and hence $W\widetilde{T}$ is zero homotopic which proves Claim 3.

LEMMA 6.6. *Let X and Y be H-module spectra and let $f: X \to Y$ be an H-module map. Assume that f induces the trivial map in homotopy. Then f is zero homotopic.*

We prove this lemma at the end of the section, after proving the following claim which finishes the proof of Proposition 6.4.

CLAIM 4. *$\overline{\mathrm{H}_*\mathrm{MSpin} \wedge \Sigma^8 BG}$ can be decomposed in the form $A \oplus B$ such that $\bar{\bar{T}}_*|_A$ is a monomorphism and $\bar{\bar{T}}_*|_B$ is trivial.*

It suffices to show that $\bar{\bar{T}}_*$ is a split surjection on its image (take A to be the image of a split and take $B = \ker \bar{\bar{T}}_*$). The proof of this fact is parallel to the proof of Proposition 8.5 in [St1]. By Lemma 8.6 of that paper it suffices to show that $\bar{\bar{T}}_*: \overline{\mathrm{H}_*\mathrm{MSpin} \wedge \Sigma^8 BG} \to \mathrm{im}(\bar{\bar{T}})$ induces a surjection in Q_0-homology ($Q_0 = \mathrm{Sq}^1$ acts on an $A(1)_*$-comodule M as a differential and the corresponding homology groups are denoted $\mathrm{H}_*(M; Q_0)$). Moreover, it follows from results proved there that

$$\mathrm{H}_*(\mathrm{H}_*\Sigma^8 BG; Q_0) \to \mathrm{H}_*(\overline{\mathrm{H}_*\mathrm{MSpin}}; Q_0) \to Q\mathrm{H}_*(\overline{\mathrm{H}_*\mathrm{MSpin}}; Q_0)$$

is onto in degrees >8. Here the first map is induced by $\mathrm{pr}\,\tilde{t}$ and the second map is the projection on the indecomposables of the algebra $\mathrm{H}_*(\overline{\mathrm{H}_*\mathrm{MSpin}}; Q_0)$, which is a polynomial algebra with generators of degree $4n \geqslant 8$ [St1, Lemma 9.4]. Thus it follows that we can choose these generators z_{4n} to be in the image of $\mathrm{pr}\,\tilde{t}$ except for z_8. Hence the image of

$$\bar{\bar{T}}_*: \mathrm{H}_*(\overline{\mathrm{H}_*\mathrm{MSpin} \wedge \Sigma^8 BG}; Q_0) \to \mathrm{H}_*(\overline{\mathrm{H}_*\mathrm{MSpin}}; Q_0) \tag{6.7}$$

is the ideal generated by z_{4n}, $4n > 8$. On the other hand, the direct sum decomposition of $A(1)_*$-comodules $\overline{\mathrm{H}_*\mathrm{MSpin}} = \mathrm{im}(\bar{\bar{T}}_*) \oplus \mathbf{Z}/2[y_8]$ induces a corresponding decomposition of $\mathrm{H}_*(\overline{\mathrm{H}_*\mathrm{MSpin}}; Q_0)$. Comparison implies that the image of (6.7) is equal to $\mathrm{H}_*(\mathrm{im}(\bar{\bar{T}}_*); Q_0)$. □

Proof of Lemma 6.6. Let X be an H-module spectrum. Then H_*X is a module over H_*H and abusing notation we denote by $\mathrm{pr}: \mathrm{H}_*X \to \overline{\mathrm{H}_*X}$ the projection on the indecomposables. The composition $\mathrm{H}_*X \xrightarrow{\psi} A_* \otimes \mathrm{H}_*X \xrightarrow{1\otimes \mathrm{pr}} A_* \otimes \overline{\mathrm{H}_*X}$ is an isomorphism of A_*-comodules (cf. Proposition 6.7 in [St1]). In particular, H_*X is a free A_*-comodule, hence X is a (generalized) $\mathbf{Z}/2$–Eilenberg–MacLane spectrum and the Hurewicz homomorphism maps $\pi_*(X)$ isomorphically onto the primitives $P(\mathrm{H}_*X) \subset \mathrm{H}_*X$. On the other hand the above isomorphism shows that $P(\mathrm{H}_*X)$ maps isomorphically onto $\overline{\mathrm{H}_*X}$ under pr. Finally, the functoriality of these isomorphisms implies that the induced map f_* on homology is determined by the induced map on homotopy. □

7. $\Omega_*^{\mathrm{Spin}}(X)\otimes_{\Omega_*^{\mathrm{Spin}}}\mathrm{ell}_*[h^{-1}]$ is not a homology theory

In this section we show that the natural transformation

$$\Omega_*^{\mathrm{Spin}}(X)\otimes_{\Omega_*^{\mathrm{Spin}}}\mathrm{ell}_*[h^{-1}]\to \mathrm{ell}_*(X)[h^{-1}] \tag{7.1}$$

induced by the Cartesian product of manifolds (cf. (1.3)) is not injective for suitable X (it is always surjective). This implies in particular that the left hand side is not a homology theory since a natural transformation between homology theories which is an isomorphism for $X=\mathrm{pt}$ is an isomorphism for all X. In this section we localize again all $\mathbf{Z}$-modules and spectra at the prime 2.

To show that (7.1) is not injective we find (for a suitable X) an element $[M,f]\in \widetilde{\Omega}_*^{\mathrm{Spin}}(X)$ such that

(a) $[M,f]\otimes 1$ is a non-trivial element of $\Omega_*^{\mathrm{Spin}}(X)\otimes_{\Omega_*^{\mathrm{Spin}}}\mathrm{ell}_*[h^{-1}]$,

(b) $[M,f]\otimes 1$ maps to zero under (7.1).

We choose X to be a finite CW-complex such that $\widetilde{\mathrm{H}}_*(X;\mathbf{Z}/2)$ as $A(1)_*$-comodule is isomorphic to $\Sigma^r A(1)_*$ for some r (such a space exists [DM, Proposition 2.1]). Regarding $X\wedge\mathrm{MSpin}$ as an MSpin-module spectrum Proposition 6.3 gives an A_*-comodule isomorphism

$$\mathrm{H}_*(X\wedge \mathrm{MSpin})\cong A_*\square_{A(1)_*}(\widetilde{\mathrm{H}}_*(X)\otimes N), \tag{7.2}$$

where the quotient $N=\overline{\mathrm{H}_*(\mathrm{MSpin})}$ of $\mathrm{H}_*(\mathrm{MSpin})$ is a polynomial algebra with a generator y_n of degree n for each $n\geqslant 8, n\neq 2^i\pm 1$. Note that with our choice of X the $A(1)_*$-comodule $\widetilde{\mathrm{H}}_*(X)\otimes N$ is free and hence $X\wedge\mathrm{MSpin}$ is a (generalized) Eilenberg–MacLane spectrum. In particular, the Hurewicz homomorphism maps $\widetilde{\Omega}_*^{\mathrm{Spin}}(X)\cong\pi_*(X\wedge\mathrm{MSpin})$ isomorphically onto the primitive elements $P(\mathrm{H}_*(X\wedge\mathrm{MSpin}))$. Another consequence of (7.2) is the isomorphism

$$P(\mathrm{H}_*(X\wedge\mathrm{MSpin}))\cong P(\widetilde{\mathrm{H}}_*(X)\otimes\mathrm{H}_*(\mathrm{MSpin}))\xrightarrow{\cong} P(\widetilde{\mathrm{H}}_*(X)\otimes N),$$

which is given by the restriction of $1\otimes\pi$, where π is the the projection map from $\mathrm{H}_*(\mathrm{MSpin})$ to N. Let

$$h\colon \pi_*(X\wedge\mathrm{MSpin})\xrightarrow{\cong} P(\widetilde{\mathrm{H}}_*(X)\otimes N)$$

be the composition of the Hurewicz map and the isomorphism above. We sum up the discussion by saying that h is an isomorphism for our choice of X.

To study $P(\widetilde{\mathrm{H}}_*(X)\otimes N)$ we note that the image of the diagonal map $\psi\colon N\to A(1)_*\otimes N$, which is injective, is $P(A(1)_*\otimes N)$. Hence it gives an isomorphism

$$\Sigma^r N\underset{\cong}{\xrightarrow{\psi}} P(\Sigma^r A(1)_*\otimes N)\cong P(\widetilde{\mathrm{H}}_*(X)\otimes N).$$

Let $[M,f]\in\widetilde{\Omega}^{\mathrm{Spin}}_{r+11}(X)\cong\pi_{r+11}(X\wedge \mathrm{MSpin})$ be the element with $h([M,f])=\psi(\sigma^r y_{11})\in N$.

Then $[M,f]\otimes 1$ is in the kernel of the natural transformation (7.1) since by part (1) of Theorem A and Corollary 5.2

$$\mathrm{ell}_*(X)\cong\pi_*(\mathrm{el}\wedge X)\cong\pi_*\Big(\bigvee\Sigma^{8k}\,\mathrm{ko}\wedge X\Big)\cong\pi_*\Big(\bigvee_{k\geqslant 0}\Sigma^{8k+r}H\Big)$$

which is zero in degree $r+11$. The last isomorphism follows from the fact that $\mathrm{H}_*\mathrm{ko}\wedge X\cong A_*\square_{A(1)_*}\mathrm{H}_*X\cong\Sigma^r A_*$ which implies that $\mathrm{ko}\wedge X$ is homotopy equivalent to the rth suspension of the $\mathbf{Z}/2$–Eilenberg–MacLane spectrum H.

This proves (b) above. To prove (a) we have to show that for all $k\geqslant 0$ $[M,f]\times[\mathbf{HP}^2]^k$ is not in the image of the multiplication map

$$\widetilde{\Omega}^{\mathrm{Spin}}_*(X)\otimes\widetilde{T}_*(\mathrm{pt})\to\widetilde{\Omega}^{\mathrm{Spin}}_*(X).$$

To prove this we translate into stable homotopy theory and consider the following diagram:

$$\begin{array}{ccc}
\pi_*(X\wedge\mathrm{MSpin})\otimes\pi_*(\mathrm{MSpin}\wedge\Sigma^8 BG) & & \\
\big\downarrow{\scriptstyle 1\otimes\widetilde{T}_*} & & \\
\pi_*(X\wedge\mathrm{MSpin})\otimes\pi_*(\mathrm{MSpin}) & \xrightarrow{\times} & \pi_*(X\wedge\mathrm{MSpin}) \\
\big\downarrow{\scriptstyle h\otimes\bar h} & & \big\downarrow{\scriptstyle h} \\
P(\widetilde{\mathrm{H}}_*(X)\otimes N)\otimes P(N) & \xrightarrow{m\otimes 1} & P(\widetilde{\mathrm{H}}_*(X)\otimes N) \\
\big\downarrow{\scriptstyle \psi^{-1}\otimes 1} & & \big\downarrow{\scriptstyle \psi^{-1}} \\
\Sigma^r N\otimes P(N) & \xrightarrow{m} & \Sigma^r N \\
\big\downarrow{\scriptstyle p\otimes\bar p} & & \big\downarrow{\scriptstyle p} \\
\Sigma^r(N/\mathfrak{I})\otimes(P(N)/\mathfrak{I}\cap P(N)) & \xrightarrow{\bar m} & \Sigma^r(N/\mathfrak{I}).
\end{array}$$

Here $\times$ is the obvious multiplication map, $\bar h$ is the Hurewicz homomorphism followed by the projection from $P(\mathrm{H}_*(\mathrm{MSpin}))$ to $P(N)$ and m is the multiplication in N. $\mathfrak{I}$ is the ideal in $N=\mathbf{Z}/2[y_n\,|\,n\geqslant 8, n\neq 2^i\pm 1]$ generated by y_{11}^2 and y_n for $n\neq 8,11$. $\bar m$ is the map induced by m, p is the projection and $\bar p$ its restriction to $P(N)$.

By [GP, Theorem 3.2] $P(N)/\mathfrak{I}\cap P(N)\cong\mathbf{Z}/2[y_8]$ and hence it follows froms Claims 1 and 3 in the proof of Proposition 6.4 that $\bar p\bar h\widetilde{T}_*$ is trivial. This shows that $[M,f]\times[\mathbf{HP}^2]^k$ is not in the image of the multiplication map (7.2) since

$$p\psi^{-1}h([M,f]\times[\mathbf{HP}^2]^k)=\sigma^r y_{11}y_8^k\neq 0.$$

References

[ASe] ATIYAH, M. F. & SEGAL, G. B., Equivariant K-theory and completion. *J. Differential Geom.*, 3 (1969), 1–18.

[AS] ATIYAH, M. F. & SINGER, I. M., The index of elliptic operators III. *Ann. of Math.*, 87 (1968), 546–604.

[Bo] BOARDMAN, J. M., Stable homotopy theory. Mimeographed notes, Warwick, 1966.

[Bor] BOREL, A., Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math.*, 57 (1953), 115–207.

[BT] BOTT, R. & TAUBES, C. H., On the rigidity theorems of Witten. *J. Amer. Math. Soc.*, 2 (1989), 139–186.

[CF] CONNER, P. E. & FLOYD, E. E., *Differentiable Periodic Maps.* Springer-Verlag, Berlin–Heidelberg–New York, 1964.

[DM] DAVIS, D. M. & MAHOWALD, M., v_1- and v_2-periodicity in stable homotopy theory. *Amer. J. Math.*, 103 (1981), 615–659.

[Fr] FRANKE, J., On the construction of elliptic cohomology. *Math. Nachr.*, 158 (1992), 43–65.

[GP] GIAMBALVO, V. & PENGELLEY, D. J., The homology of MSpin. *Math. Proc. Cambridge Philos. Soc.*, 95 (1984), 427–436.

[HBJ] HIRZEBRUCH, F., BERGER, TH. & JUNG, R., *Manifolds and Modular Forms.* Vieweg Verlag, Braunschweig/Wiesbaden, 1992.

[Hit] HITCHIN, N., Harmonic spinors. *Adv. in Math.*, 14 (1974), 1–55.

[HH] HOPKINS, M. J. & HOVEY, M. A., Spin cobordism determines real K-theory. Preprint, 1990.

[Ho] HOVEY, M. A., Spin bordism and elliptic homology. Preprint, 1991.

[J] JUNG, R., MPI doctoral dissertation. In preparation.

[L1] LANDWEBER, P. S., Elliptic genera: an introductory overview, in *Elliptic Curves and Modular Forms in Algebraic Topology* (P. S. Landweber, ed.), pp. 1–10. Springer-Verlag, 1988.

[L2] — Elliptic cohomology and modular forms, in *Elliptic Curves and Modular Forms in Algebraic Topology* (P. S. Landweber, ed.), pp. 55–68. Springer-Verlag, 1988.

[LRS] LANDWEBER, P. S., RAVENEL, D. C. & STONG, R. E., Periodic cohomology theories defined by elliptic curves. To appear.

[Mi] MILNOR, J. W., Spin structures on manifolds. *Enseign. Math.*, 9 (1963), 198–203.

[MS] MILNOR, J. W. & STASHEFF, J. D., *Characteristic Classes.* Ann. of Math. Stud., 76. Princeton Univ. Press, 1974.

[O1] OCHANINE, S., Signature modulo 16, invariants de Kervaire généralisés et nombres caractéristiques dans la K-théorie réelle. *Mém. Soc. Math. France*, 5 (1981).

[O2] — Sur les genres multiplicatifs définis par des intégrales elliptiques. *Topology*, 26 (1987), 143–151.

[O3] — Genres elliptiques equivariants, in *Elliptic Curves and Modular Forms in Algebraic Topology* (P. S. Landweber, ed.), pp. 105–122. Springer-Verlag, 1988.

[O4] — Elliptic genera, modular forms over KO_* and the Brown–Kervaire invariant. *Math. Z.*, 206 (1991), 277–291.

[Ra] RAVENEL, D. C., *Complex Cobordism and Stable Homotopy Groups of Spheres.* Academic Press, 1986.

[Ru] RUDIAK, Y., Exactness theorems for the cohomology theories MU, BP, and $P(n)$. *Mat. Zametki*, 40 (1986), 115–126; English translation in *Math. Notes*, 40 (1986), 562–569.

[Se] SEGAL, G., Elliptic cohomology. Séminaire Bourbaki, 695 (1987–88). *Astérisque*, 161–162 (1988), 187–201.

[St1] STOLZ, S., Simply connected manifolds of positive scalar curvature. *Ann. of Math.*, 136 (1992), 511–540.
[St2] — Splitting certain MSpin-module spectra. To appear in *Topology*.
[S] STONG, R. E., *Notes on Cobordism Theory*. Mathematical Notes, 7. Princeton Univ. Press, 1968.
[Sw] SWITZER, R. M., *Algebraic Topology—Homotopy and Homology*. Springer-Verlag, Berlin–New York, 1975.
[T] TAUBES, C. H., S^1 actions and elliptic genera. *Comm. Math. Phys.*, 122 (1989), 455–526.
[W] WITTEN, E., The index of the Dirac operator in loop space, in *Elliptic Curves and Modular Forms in Algebraic Topology* (P. S. Landweber, ed.), pp. 161–181. Springer-Verlag, 1988.
[Z] ZAGIER, D., Note on the Landweber–Stong elliptic genus, in *Elliptic Curves and Modular Forms in Algebraic Topology* (P. S. Landweber, ed.), pp. 216–224. Springer-Verlag, 1988.

MATTHIAS KRECK
Fachbereich Mathematik
Universität Mainz
Saarstraße 21
6500 Mainz
Germany
E-mail address: kreck@topologie.mathematik.uni-mainz.de

STEPHAN STOLZ
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
U.S.A.
E-mail address: stolz.1@nd.edu

Received October 2, 1991
Received in revised form November 9, 1992