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A converse to the mean value theorem for harmonic functions

by

WOLFHARD HANSEN and NIKOLAI NADIRASHVILI

Universität Bielefeld
Bielefeld, Germany

Universität Bielefeld
Bielefeld, Germany

0. Introduction

Let $U\neq\varnothing$ be a bounded domain in $\mathbf{R}^d$, $d\geqslant 1$, and for every $x\in U$ let B^x be an open ball contained in U with center x. If f is a harmonic integrable function on U then

$$f(x)=\frac{1}{\lambda(B^x)}\int_{B^x} f\,d\lambda \quad \text{for every } x\in U \tag{$*$}$$

(λ Lebesgue measure on $\mathbf{R}^d$). The converse question to what extent this restricted mean value property implies harmonicity has a long history (we are indebted to I. Netuka for valuable hints). Volterra [26] and Kellogg [20] noted first that a continuous function f on the closure $\overline{U}$ of U satisfying $(*)$ is harmonic on U. At least if U is regular there is a very elementary proof for this fact (see Burckel [7]): Let g be the difference between f and the solution of the Dirichlet problem with boundary value f. If $g\neq 0$, say $a=\sup g(U)>0$, choose $x\in\{g=a\}$ having minimal distance to the boundary. Then $(*)$ leads to an immediate contradiction. In fact, for continuous functions on $\overline{U}$ the question is settled for arbitrary harmonic spaces and arbitrary representing measures $\mu_x\neq\varepsilon_x$ for harmonic functions.

If f is bounded on U and Borel measurable the answer may be negative unless restrictions on the radius $r(x)$ of the balls B^x are imposed (Veech [23]): Let $U=]-1,1[$, $f(0)=0$, $f=-1$ on $]-1,0[$, $f=1$ on $]0,1[$, $0\notin B^x$ for $x\neq 0$ (similarly in $\mathbf{R}^d$, $d\geqslant 2$)!

There are various positive results, sometimes under restrictions on U, but always under restrictions on the function $x\mapsto r(x)$ (Feller [9], Akcoglu and Sharpe [1], Baxter [2] and [3], Heath [17], Veech [23] and [24]). For example Heath [17] showed for arbitrary U that a bounded Lebesgue measurable function on U having the restricted mean value property $(*)$ is harmonic provided that, for some $\varepsilon>0$, $\varepsilon d(x,\complement U)<r(x)<(1-\varepsilon)d(x,\complement U)$ holds for every $x\in U$. Veech [23] proved that a Lebesgue measurable function f on U

which is dominated by a harmonic function and satisfies $(*)$ is harmonic provided U is a Lipschitz domain and the radii $r(x)$ of the balls B^x are locally bounded away from zero. A survey of the history up to 1973 is contained in [22].

Even if f is continuous some additional condition is needed (see Littlewood [21]): Let $U=]0,1[$ and let (a_n) be a sequence in $]0,\frac{1}{2}[$ which is strictly decreasing to zero. Let $A=\{a_n: n\in\mathbf{N}\}$. Define $r(a_n)=a_n-a_{n+2}$, for $x\in U\setminus A$ take any B^x such that $B^x\cap A=\varnothing$. Proceeding by recurrence it is easily shown that there is a (unique) continuous function f on U such that f is (locally) affinely linear on $U\setminus A$, $f=0$ on $[a_1,1[$, $f(a_2)=1$, and $(*)$ holds (also) for the points $x\in A$. Of course, f is not harmonic on U and it is unbounded. (A similar construction in $\mathbf{R}^d$, $d\geqslant 2$, would lead to a rotation invariant continuous function f on the unit ball U (on $U\setminus\{0\}$ resp.) which oscillates at the boundary ∂U (at 0 resp.), has the restricted mean value property $(*)$, but is not harmonic.) It has been shown that this cannot happen for continuous bounded functions on the unit interval (Huckemann [19]).

To the best of our knowledge, however, even for continuous bounded functions on the unit ball in $\mathbf{R}^d$, $d\geqslant 2$, the question if the restricted mean value property implies harmonicity has been open until now.

We intend to show that, for every bounded domain $U\neq\varnothing$ in $\mathbf{R}^p$, every continuous function f on U which is bounded by some harmonic function $h\geqslant 0$ on U and satisfies $(*)$ for a family $(B^x)_{x\in U}$ is harmonic. As a by-product we shall obtain that even every Lebesgue measurable function f on U which is bounded by some harmonic function $h\geqslant 0$ on U and satisfies $(*)$ for a family $(B^x)_{x\in U}$ is harmonic provided the radii of the balls B^x, $x\in U$, are bounded away from 0 on every compact subset of U (improving [17] and [23]).

In contrast to the preceding work on the problem our proof will be given in a purely analytic way (though many parts of it are based on probabilistic ideas and could as well be expressed probabilistically). It will use the Martin compactification, in particular the minimal fine topology, and exploit properties of the Schrödinger equation $\Delta u-\delta d(\cdot,\complement U)^{-2}1_A u=0$ on U ($\delta>0$, A a suitable subset of U).

1. Main results

Before stating our main results we have to introduce some notation. Throughout this paper we fix a harmonic function $h\geqslant 1$ on U. A numerical function f on U is called *lower h-bounded, upper h-bounded, h-bounded* if there exists $c\in\mathbf{R}_+$ such that $f\geqslant -ch$, $f\leqslant ch$, $|f|\leqslant ch$, respectively. Given $x\in\mathbf{R}^d$ and $\varepsilon>0$, let

$$B_\varepsilon(x)=\{y\in\mathbf{R}^d:\|y-x\|<\varepsilon\}.$$

It will be convenient to write $|A|$ instead of $\lambda(A)$ for the Lebesgue measure of a Borel set $A\subset\mathbf{R}^d$. For every $x\in U$, let $\varrho(x)$ denote the distance $d(x,\complement U)$ of x from the (Euclidean) boundary ∂U. We write $C(U)$ for the set of all continuous real functions on U.

Given a real function r on U such that $0<r\leqslant\varrho$, we shall say that a Lebesgue measurable lower h-bounded numerical function f on U is *r-supermedian* if

$$f(x)\geqslant\frac{1}{|B_{r(x)}(x)|}\int_{B_{r(x)}(x)}f\,d\lambda$$

for every $x\in U$. An h-bounded Lebesgue measurable function f will be called *r-median* if f and $-f$ are r-supermedian or, equivalently, if

$$f(x)=\frac{1}{|B_{r(x)}(x)|}\int_{B_{r(x)}(x)}f\,d\lambda$$

for every $x\in U$. Clearly every lower h-bounded superharmonic function s on U is r-supermedian and every h-bounded harmonic function g on U is r-median.

Let us fix once and for all a Whitney decomposition $\mathcal{Q}$ for U (see [14]). $\mathcal{Q}$ is a partition of U into countably many disjoint (dyadic) cubes of the form

$$Q=\prod_{i=1}^{d}[m_i2^{-m},(m_i+1)2^{-m}[,\quad m\in\mathbf{Z},m_i\in\mathbf{Z},$$

such that, for every $Q\in\mathcal{Q}$, the diameter $\delta(Q)$ of Q satisfies

$$1\leqslant\frac{d(Q,\partial U)}{\delta(Q)}<3.$$

Then for every $Q\in\mathcal{Q}$

$$\delta(Q)\leqslant\varrho<4\delta(Q)\quad\text{on } Q.$$

The lower bound for $d(Q,\partial U)/\delta(Q)$ implies that there exists a constant $C>0$ (depending only on the dimension d) such that

$$\sup g(Q)\leqslant C\inf g(Q)$$

for every $Q\in\mathcal{Q}$ and every harmonic function $g\geqslant 0$ on U. For every $m\in\mathbf{N}=\{1,2,3,\ldots\}$ let U_m denote the interior of the union of all $Q\in\mathcal{Q}$ such that $\delta(Q)\geqslant 2^{-m}$. Note that the open sets U_m are increasing to U and that

$$2^{-m}\leqslant d(U_m,\partial U)\leqslant 4\cdot 2^{-m}.$$

Fix $0<\alpha\leqslant\frac{1}{2}$. Given a Lebesgue measurable lower h-bounded numerical function f on U, we define a function $\bar{f}_\alpha$ on U by

$$\bar{f}_\alpha(x)=\sup\{a\in\mathbf{R}:|\{f\geqslant ah\}\cap Q|\geqslant\alpha|Q|\},\quad x\in Q\in\mathcal{Q}.$$

For our purpose it would be sufficient to take $\alpha=\frac{1}{2}$. Allowing, however, that α is small we shall obtain in addition a new proof for the existence of minimal fine limits of harmonic functions and the connection with the Perron–Wiener–Brelot solution for the Martin compactification.

Note that $\bar{f}_\alpha$ is lower bounded on U, constant on each $Q\in\mathcal{Q}$, and that

$$\inf\frac{f}{h}(Q)\leqslant\bar{f}_\alpha\leqslant\sup\frac{f}{h}(Q),\quad |\{f\geqslant\bar{f}_\alpha h\}\cap Q|\geqslant\alpha|Q|.$$

If f is a Lebesgue measurable upper h-bounded function on U, we define similarly

$$\underline{f}_\alpha(x)=\inf\{a\in\mathbf{R}:|\{f\leqslant ah\}\cap Q|\geqslant\alpha|Q|\},\quad x\in Q\in\mathcal{Q}.$$

Obviously, $\underline{f}_\alpha=-\overline{(-f)}_\alpha$. Moreover, if f is an h-bounded Lebesgue measurable function on U, then $\underline{f}_\alpha\leqslant\bar{f}_\alpha$. Indeed, if $x\in Q\in\mathcal{Q}$ and $a>\bar{f}_\alpha(x)$ then $|\{f\geqslant ah\}\cap Q|<\alpha|Q|$, hence $|\{f<ah\}\cap Q|>(1-\alpha)|Q|\geqslant\alpha|Q|$, $\underline{f}_\alpha(x)\leqslant a$. If f is continuous, then of course $\underline{f}_{1/2}=\bar{f}_{1/2}$.

Let $\partial^M U$ denote the Martin boundary of U, let $\partial_1^M U$ be the set of minimal points in $\partial^M U$, and let $K:(U\cup\partial^M U)\times U\to[0,\infty]$ be a Martin function for U (see [18], [8]). Let χ denote the (unique) measure on $\partial^M U$ such that $\chi(\partial^M U\setminus\partial_1^M U)=0$ and

$$h(x)=\int K(z,x)\,\chi(dz),\quad x\in U.$$

Given any measurable function φ on $\partial^M U$ such that φ^+ or φ^- is χ-integrable, we define a numerical function $H\varphi$ on U by

$$H\varphi(x)=\int K(z,x)\varphi(z)\,\chi(dz),\quad x\in U.$$

If φ is χ-integrable then $H\varphi$ is harmonic on U. We recall that the Martin compactification is h-resolutive ([8, p. 110]), i.e., that, for every bounded measurable function φ on $\partial^M U$,

$$\begin{aligned}H\varphi&=\inf\{s:s\text{ superharmonic on }U,\ \liminf_{x\to z}s(x)/h(x)\geqslant\varphi(z)\text{ for every }z\in\partial^M U\}\\&=\sup\{t:t\text{ subharmonic on }U,\ \limsup_{x\to z}t(x)/h(x)\leqslant\varphi(z)\text{ for every }z\in\partial^M U\}.\end{aligned}$$

For every $z\in\partial_1^M U$, let $\mathcal{N}_z$ denote the filter of the intersections of minimal fine neighborhoods of z with U, i.e.,

$$\mathcal{N}_z=\{V\subset U:U\setminus V\text{ is minimal thin at }z\}.$$

Recall that a subset A of U is not minimal thin at z if

$$^U R^A_{K(z,\cdot)} = K(z,\cdot)$$

(where $^U R^A_v = \inf\{s : s \text{ superharmonic } \geqslant 0 \text{ on } U,\ s \geqslant v \text{ on } A\}$), i.e., in probabilistic terms, if for every $y \in U$ the probability $^{K(z,\cdot)}R^A_1(y)$ that Brownian motion on U starting at y and conditioned by $K(z,\cdot)$ will hit A before exiting at z is equal to 1 (see [8]).

Now suppose that f is a lower h-bounded numerical function on U which is Lebesgue measurable. We extend $\bar{f}_\alpha$ to a function on $U \cup \partial^M U$ defining

$$\bar{f}_\alpha(z) = \begin{cases} \limsup_{\mathcal{N}_z} \bar{f}_\alpha, & z \in \partial^M_1 U, \\ 0, & z \in \partial^M U \setminus \partial^M_1 U. \end{cases}$$

By [8, p. 216], $\bar{f}_\alpha$ is Borel measurable on $\partial^M_1 U$. Let us note that

$$\bar{f}_\alpha(z) = \sup\{a \in \mathbf{R} : \{\bar{f}_\alpha \geqslant a\} \text{ not minimal thin at } z\}, \quad z \in \partial^M_1 U.$$

Indeed, fix $z \in \partial^M_1 U$. If $b > \limsup_{\mathcal{N}_z} \bar{f}_\alpha$ then there exists $V \in \mathcal{N}_z$ such that $\sup \bar{f}_\alpha(V) < b$, and hence $\{\bar{f}_\alpha \geqslant a\}$ is minimal thin at z for every $a \geqslant b$. Conversely, if $a \in \mathbf{R}$ such that $\{\bar{f}_\alpha \geqslant a\}$ is minimal thin at z, then $\{\bar{f}_\alpha < a\} \in \mathcal{N}_z$ and hence $\limsup_{\mathcal{N}_z} \bar{f}_\alpha \leqslant a$.

If f is an upper h-bounded Lebesgue measurable function on U, we define similarly

$$\underline{f}_\alpha(z) = \begin{cases} \liminf_{\mathcal{N}_z} \underline{f}_\alpha, & z \in \partial^M_1 U, \\ 0, & z \in \partial^M U \setminus \partial^M_1 U. \end{cases}$$

Obviously, $\underline{f}_\alpha = -\overline{(-f)}_\alpha$ on $U \cup \partial^M U$ and if $f : U \to \mathbf{R}$ is h-bounded and Lebesgue measurable, then $\underline{f}_\alpha \leqslant \bar{f}_\alpha$.

We are now ready to state our first result:

THEOREM 1.1. *Let r be a real function on U such that $0 < r \leqslant \varrho$ and let f be a lower h-bounded l.s.c. function on U which is r-supermedian. Then $f \geqslant H \bar{f}_\alpha$ for every $0 < \alpha \leqslant \frac{1}{2}$.*

The details of the proof will have to be postponed to Section 4. However, let us give a brief outline right now: To that end assume for sake of simplicity that $h = 1$. By Lusin's theorem there exists a continuous function φ on $U \cup \partial^M U$ which is nearly $\bar{f}_\alpha$ on $\partial^M U$. Fix $x \in U$ and $\varepsilon > 0$. Let t be a subharmonic function on U such that $\limsup_{y \to z} t(y) \leqslant \varphi(z)$ for every $z \in \partial^M U$ and $t(x) > H\varphi(x) - \varepsilon$. There exists $m \in \mathbf{N}$ such that $t - \varepsilon < \varphi$ on $U \setminus U_m$. Choosing a suitable superharmonic function $s \geqslant 0$ on U such that $s(x) < \varepsilon$ we obtain a subset

$$A = (U \setminus U_m) \cap \{f + s > \varphi - \varepsilon\}$$

of U such that the union of all $Q\in\mathcal{Q}$ satisfying $|A\cap Q|\geqslant\alpha|Q|$ is not minimal thin at the points $z\in\partial_1^M U$. As a consequence, for every $\delta>0$, every continuous solution $u>0$ of the Schrödinger equation $\Delta u-\delta\varrho^{-2}1_A u=0$ on U is unbounded. This fact allows us to show that starting with unit mass at x the iterated sweeping induced by the ball means on $B_{r(y)}(y)$, $y\in U$, and stopped at A does not reach the boundary of U. The (transfinite) limit measure σ is supported by A and satisfies $\sigma(f)\leqslant f(x)$, $\sigma(s)\leqslant s(x)$, and $\sigma(t)\geqslant t(x)$. Since $f+s\geqslant t-2\varepsilon$ on A, we conclude that $f(x)+\varepsilon\geqslant f(x)+s(x)\geqslant t(x)-2\varepsilon\geqslant H\varphi(x)-3\varepsilon$. Thus $f\geqslant H\bar{f}_\alpha$.

COROLLARY 1.2. *Let r be a real function on U such that $0<r\leqslant\varrho$ and let f be a continuous h-bounded function on U which is r-median. Then $f=H\bar{f}_\alpha=H\underline{f}_\alpha$ for every $0<\alpha\leqslant\frac{1}{2}$. In particular, f is harmonic.*

Proof. Since f and $-f$ are r-supermedian, Theorem 1.1 implies that $f\geqslant H\bar{f}_\alpha$ and $-f\geqslant H\overline{(-f)}_\alpha$, i.e., $f\leqslant H\underline{f}_\alpha$. Knowing $\underline{f}_\alpha\leqslant\bar{f}_\alpha$ we conclude that $f=H\underline{f}_\alpha=H\bar{f}_\alpha$. □

Surprisingly, Theorem 1.1 is strong enough to improve in addition the known results on *Lebesgue measurable* functions having the restricted mean value property (Heath [17], Veech [23]).

THEOREM 1.1′. *Let r be a real function on U such that $0<r\leqslant\varrho$ and r is bounded away from 0 on every compact subset of U. Let f be an h-bounded Lebesgue measurable function on U which is r-supermedian. Then $f\geqslant H\bar{f}_\alpha$ for every $0<\alpha\leqslant\frac{1}{2}$.*

Before giving the proof let us write down the corollary which follows from Theorem 1.1′ in the same way as Corollary 1.2 followed from Theorem 1.1.

COROLLARY 1.2′. *Let r be a real function on U such that $0<r\leqslant\varrho$ and r is bounded away from 0 on every compact subset of U. Let f be an h-bounded Lebesgue measurable function on U which is r-median. Then $f=H\bar{f}_\alpha=H\underline{f}_\alpha$ for every $0<\alpha\leqslant\frac{1}{2}$. In particular, f is harmonic.*

Proof of Theorem 1.1′. Let us fix a continuous bounded potential p on U which is strict (e.g. the potential of Lebesgue measure on U). For every $m\in\mathbf{N}$, define

$$a_m=\inf_{x\in U_m}\left\{|B_{r(x)}(x)|p(x)-\int_{B_{r(x)}(x)}p\,d\lambda\right\}.$$

Clearly, the sequence (a_m) is decreasing. We claim that $a_m>0$ for every $m\in\mathbf{N}$. Indeed, let (x_n) be a sequence in U_m such that

$$a_m=\lim_{n\to\infty}\left(|B_{r(x_n)}(x_n)|p(x_n)-\int_{B_{r(x_n)}(x_n)}p\,d\lambda\right).$$

Then there exists a subsequence (y_n) of (x_n) such that (y_n) converges to a point $x\in\overline{U}_m$ and $(r(y_n))$ converges to a real number s. By assumption on the function r we know that $s>0$. Hence by bounded convergence

$$a_m = |B_s(x)|p(x) - \int_{B_s(x)} p\,d\lambda > 0.$$

Let $c\in\mathbf{R}_+$ such that $|f|\leqslant ch$, fix $x_0\in U$ and $\varepsilon>0$. There exists $m_0\geqslant 3$ such that $x_0\in U_{m_0}=:V_{m_0}$. For every $m>m_0$ define

$$V_m = U_m \setminus U_{m-1}.$$

For every $m\geqslant m_0$ there exists a compact subset A_m of $\mathring{V}_m$ such that $f|_{A_m}$ is continuous and

$$|V_m\setminus A_m| < \varepsilon 2^{-m}\frac{a_{m+2}}{c\sup h(V_m)}.$$

We may assume without loss of generality that $x_0\in A_{m_0}$. Let A denote the union of the sets A_m, $m\geqslant m_0$. Then $U\setminus A$ is open and $f|_A$ is continuous. Let us define an h-bounded function g on U by

$$g(x) = \begin{cases} f(x)+\varepsilon p(x), & x\in A, \\ ch(x)+\varepsilon p(x), & x\in U\setminus A. \end{cases}$$

Clearly, g is l.s.c. We claim that g is r-supermedian.

To that end fix $x\in U$ and define $B^x = B_{r(x)}(x)$. If $x\in U\setminus A$ then of course

$$g(x) = ch(x)+\varepsilon p(x) \geqslant \frac{1}{|B^x|}\int_{B^x}(ch+\varepsilon p)\,d\lambda \geqslant \frac{1}{|B^x|}\int_{B^x} g\,d\lambda.$$

So assume $x\in A$. We have $x\in V_k$ for some $k\geqslant m_0$. If $k>m_0$ then $B^x\cap U_{k-3}=\varnothing$, since $\varrho\geqslant 2^{-(k-3)}$ on U_{k-3} whereas, for every $y\in B^x$, $\varrho(y)\leqslant r(x)+\varrho(x)\leqslant 2\varrho(x)<8\cdot 2^{-k}$. Therefore we obtain that

$$\begin{aligned}\int_{B^x\setminus A} ch\,d\lambda &\leqslant \sum_{m=\max(m_0,k-2)}^{\infty}\int_{V_m\setminus A_m} ch\,d\lambda \\ &\leqslant \varepsilon \sum_{m=\max(m_0,k-2)}^{\infty} 2^{-m}a_{m+2} \leqslant \frac{1}{2}\varepsilon a_k \leqslant \frac{1}{2}\varepsilon\left(|B^x|p(x) - \int_{B^x} p\,d\lambda\right).\end{aligned}$$

Hence

$$|B^x|g(x) - \int_{B^x} g\,d\lambda \geqslant |B^x|f(x) - \int_{B^x} f\,d\lambda - 2\int_{B^x\setminus A} ch\,d\lambda + \varepsilon\left(|B^x|p(x) - \int_{B^x} p\,d\lambda\right) \geqslant 0.$$

Thus g is r-supermedian and we conclude by Theorem 1.1 that $g\geqslant H\bar{f}_\alpha$. In particular, $f(x_0)+\varepsilon p(x_0)\geqslant H\bar{f}_\alpha(x_0)$. Therefore $f\geqslant H\bar{f}_\alpha$. □

Moreover, it may be interesting to note that Corollary 1.2 implies the following well known result (see e.g. [8, pp. 207, 219]):

COROLLARY 1.3. *If f is an h-bounded harmonic function on U, then $\varphi(z):=\lim_{\mathcal{N}_z} f/h$ exists for χ-a.e. $z\in\partial_1^M U$ and $f=H\varphi$.*

Proof. Obviously, $\alpha\mapsto\bar{f}_\alpha$ is decreasing and $\alpha\mapsto\underline{f}_\alpha$ is increasing. Defining $\bar{f},\underline{f}$: $\partial^M U\to\mathbf{R}$ by

$$\bar{f}(z)=\sup_{0<\alpha\leqslant\frac{1}{2}}\bar{f}_\alpha(z),\quad \underline{f}(z)=\inf_{0<\alpha\leqslant\frac{1}{2}}\underline{f}_\alpha(z)$$

we obtain by the preceding corollary that $\underline{f}\leqslant\bar{f}$ and

$$f=H\underline{f}=H\bar{f}.$$

So $\underline{f}=\bar{f}$ χ-a.e. In order to finish the proof it therefore suffices to show that, for every $z\in\partial_1^M U$,

$$\underline{f}(z)\leqslant\liminf_{\mathcal{N}_z}\frac{f}{h},\quad \limsup_{\mathcal{N}_z}\frac{f}{h}\leqslant\bar{f}(z).$$

So fix $z\in\partial_1^M U$. In order to prove that $\limsup_{\mathcal{N}_z} f/h\leqslant\bar{f}(z)$ we assume, as we may (adding a suitable multiple of h), that $f\geqslant h$. Consider $0<a<\limsup_{\mathcal{N}_z} f/h$ and take $c>1$ such that $ac^2<\limsup_{\mathcal{N}_z} f/h$. Then $\{f\geqslant ac^2h\}$ is not minimal thin at z.

Since $\delta(Q)<d(Q,\partial U)$ there exists a constant $0<\varepsilon\leqslant\sqrt{1/d}$ (depending only on c and the dimension d) such that for every harmonic function $g\geqslant0$ on U and every $x\in Q\in\mathcal{Q}$

$$\frac{1}{c}g(x)\leqslant g\leqslant cg(x)\quad\text{on } B_{\varepsilon\delta(Q)}(x).$$

Note that, for every $x\in Q\in\mathcal{Q}$,

$$\frac{|B_{\varepsilon\delta(Q)}(x)\cap Q|}{|Q|}\geqslant 2^{-d}\frac{|B_{\varepsilon\delta(Q)}(x)|}{|Q|}>\varepsilon^d=:\alpha.$$

Let $x\in\{f\geqslant ac^2h\}$. Then $x\in Q$ for some $Q\in\mathcal{Q}$ and $f/h\geqslant c^{-2}f(x)/h(x)\geqslant a$ on $B_{\varepsilon\delta(Q)}(x)$, hence $|\{f\geqslant ah\}\cap Q|\geqslant\alpha|Q|$, $\bar{f}_\alpha(x)\geqslant a$. So $\{f\geqslant ac^2h\}\subset\{\bar{f}_\alpha\geqslant a\}$ and therefore $\bar{f}_\alpha(z)\geqslant a$ since $\{f\geqslant ac^2h\}$ is not minimal thin at z. Thus $\bar{f}(z)\geqslant\bar{f}_\alpha(z)\geqslant a$, $\bar{f}(z)\geqslant\limsup_{\mathcal{N}_z} f/h$. Replacing f by $-f$ we finally obtain that $\underline{f}(z)\leqslant\liminf_{\mathcal{N}_z} f/h$ finishing the proof. □

2. Schrödinger equation with singularity at the boundary

Let us first recall some notions and basic facts. Given a relatively compact open subset $W\neq\varnothing$ of $\mathbf{R}^d$ let H_W denote the (classical) harmonic kernel for W, and let G_W denote the Green function on W such that $\Delta G_W(\cdot,y)=-\varepsilon_y$ for every $y\in W$. Suppose that $V\geqslant0$ is a bounded Borel measurable function on W. Then, for every $\varphi\in\mathcal{C}_b(\partial W)$, there exists a unique continuous solution $u\in\mathcal{C}_b(W)$ of $\Delta u-Vu=0$ such that $\lim_{x\to z}u(x)=\varphi(z)$ for every regular point $z\in\partial W$. The mapping $\varphi\mapsto u$ defines a kernel $H_W^{\Delta-V}$. We have

$$H_W^{\Delta-V}=(I+K_W^V)^{-1}\circ H_W$$

where

$$K_W^V u(x)=\int_W G_W(x,y)u(y)V(y)\,\lambda(dy),\quad x\in W.$$

(For further details see e.g. [5] or [15].)

LEMMA 2.1. *Let $\varepsilon>0$, $a,b>0$ and let $W_0=\left]-\frac{1}{2},\frac{1}{2}\right[^d$, $W=(1+\varepsilon)W_0$. Then there exists a constant $\gamma<1$ such that for every $r>0$, for every Borel measurable function $0\leqslant V\leqslant br^{-2}$ on rW satisfying $\int_{rW_0}V\,d\lambda\geqslant ar^{d-2}$, for every bounded harmonic function $g\geqslant 0$ on rW, and every continuous solution $0\leqslant u\leqslant g$ of $\Delta u-Vu=0$ on rW the inequality $u\leqslant\gamma g$ holds on rW_0.*

Proof. It suffices to consider the case $r=1$. By [15], there exists $c>1$ such that

$$\sup u(W_0)\leqslant c\inf u(W_0)$$

for every Borel function $0\leqslant V\leqslant b$ on W and every continuous solution $u\geqslant 0$ of $\Delta u-Vu=0$ on W. Define

$$\eta:=\inf\{G_W(x,y):x,y\in W_0\}$$

and

$$\gamma:=\left(1+\frac{a\eta}{c}\right)^{-1}.$$

Finally, let $0\leqslant V\leqslant b$ be a Borel function on W, let $g\geqslant 0$ be a bounded harmonic function on W, and let $u\geqslant 0$ be a continuous solution of $\Delta u-Vu=0$ on W. Then, for every $x\in W_0$,

$$K_W^V u(x):=\int_W u(y)G_W(x,y)V(y)\,\lambda(dy)\geqslant\frac{u(x)}{c}\eta\int_{W_0}V\,d\lambda\geqslant\frac{a\eta}{c}u(x),$$

hence

$$\left(1+\frac{a\eta}{c}\right)u(x)\leqslant u(x)+K_W^V u(x)\leqslant g(x),$$

i.e., $u(x)\leqslant\gamma g(x)$. (The inequality $u+K_W^V u\leqslant g$ follows from the fact that $u+K_W^V u$ is harmonic on W and $\liminf_{x\to z}(g-(u+K_W^V u))(x)=\liminf_{x\to z}(g-u)(x)\geqslant 0$ for every $z\in\partial W$.) $\square$

LEMMA 2.2. *Let $V \geqslant 0$ be a locally bounded Borel measurable function on U, let $L_m = \Delta - 1_{U_m} V$, $m \in \mathbf{N}$, and define $K_m f = \int_{U_m} f(y) G_U(\cdot, y) V(y)\, \lambda(dy)$ ($m \in \mathbf{N}$, $f \geqslant 0$ Borel measurable). Then, for every $m \in \mathbf{N}$, there exists a unique Borel function $g_m \geqslant 0$ on U such that $g_m + K_m g_m = h$. The functions g_m are continuous and satisfy $L_m g_m = 0$ on U. The sequence (g_m) is decreasing. Moreover, $\lim_{m\to\infty} g_m = 0$ if and only if there is no h-bounded continuous solution $u > 0$ of the Schrödinger equation $\Delta u - Vu = 0$ on U.*

Proof. Fix $m \in \mathbf{N}$ and, for every $k \in \mathbf{N}$, define

$$g_{mk} = H_{U_k}^{L_m} h.$$

Then the sequence $(g_{mk})_{k \in \mathbf{N}}$ is decreasing and

$$g_m := \lim_{k\to\infty} g_{mk}$$

is a continuous positive function on U satisfying $L_m g_m = 0$. Since $G_{U_k}(\cdot, y) = G_U(\cdot, y) - H_{U_k} G_U(\cdot, y)$, we know that

$$g_{mk} + K_m g_{mk} - H_{U_k} K_m g_{mk} = h$$

for every $k \in \mathbf{N}$. Moreover

$$\lim_{k\to\infty} H_{U_k} K_m g_{mk} = 0,$$

since $K_m g_{mk} \leqslant K_m h$ for every $m \in \mathbf{N}$, $K_m h$ is a potential on U, and hence the harmonic minorant $\lim_{k\to\infty} H_{U_k} K_m h$ of $K_m h$ is zero. Thus

$$g_m + K_m g_m = h.$$

The uniqueness of g_m follows from a general domination principle (see e.g. [16]).

Given $k \in \mathbf{N}$ the sequence $(g_{mk})_{m \in \mathbf{N}}$ is decreasing. Therefore the sequence (g_m) is decreasing and $g := \lim_{m\to\infty} g_m$ is a continuous h-bounded solution of $\Delta u - Vu = 0$. On the other hand, let $0 \leqslant u \leqslant h$ be a continuous solution of $\Delta u - Vu = 0$. Then $L_m u = 1_{U \setminus U_m} V u \geqslant 0$, hence

$$u \leqslant H_{U_k}^{L_m} u \leqslant H_{U_k}^{L_m} h = g_{mk}$$

for all $k, m \in \mathbf{N}$, i.e., $u \leqslant g$.

Thus $g = 0$ if and only if there is no h-bounded continuous $u > 0$ satisfying $\Delta u - Vu = 0$ on U. $\square$

PROPOSITION 2.3. *Let A be a Borel subset of U, let $\alpha>0$, and assume that, for every $z\in\partial_1^M U$, the union D of all $Q\in\mathcal{Q}$ satisfying $|A\cap Q|\geqslant\alpha|Q|$ is not minimal thin at z. Then, for every $\delta>0$ and every solution $u>0$ of the Schrödinger equation*

$$\Delta u-\frac{\delta}{\varrho^2}1_A u=0$$

on U, the function u/h is unbounded.

Proof. Choose $m_1\in\mathbf{N}$ such that $U_{m_1}\neq\varnothing$. Suppose that $k\geqslant 1$ and that $m_k\in\mathbf{N}$ has been chosen. Let $W_k=U_{m_k}$ and let

$$\mathcal{D}_k=\{Q\in\mathcal{Q}:\delta(Q)<2^{-(m_k+2)},|A\cap Q|\geqslant\alpha|Q|\},\quad D_k=\bigcup_{Q\in\mathcal{D}_k}Q.$$

Then $\overline{D}_k\cap\overline{W}_k=\varnothing$ and $D\subset U_{m_k+2}\cup D_k$. Since $\overline{U}_{m_k+2}$ is a compact subset of U, our assumption on D shows that D_k is not minimal thin at the points $z\in\partial_1^M U$, i.e.,

$${}^{U}R^{D_k}_{K(z,\cdot)}=K(z,\cdot)$$

for all $z\in\partial_1^M U$. (It is not difficult to show that in fact $A\cap D_k$ is not minimal thin at $z\in\partial_1^M U$.) Integrating with respect to χ we obtain that

$${}^{U}R^{D_k}_{h}=h.$$

There exists a finite subset $\mathcal{E}_k$ of $\mathcal{D}_k$ such that the closure E_k of the union of all $Q\in\mathcal{E}_k$ still satisfies

$${}^{U}R^{E_k}_{h}>\tfrac{1}{2}h\quad\text{on }\overline{W}_k.$$

Finally, we may choose $m_{k+1}\in\mathbf{N}$ such that $E_k\subset U_{m_{k+1}}$,

$$d(Q,\partial U_{m_{k+1}})>\tfrac{1}{2}\delta(Q)\ \text{ for all } Q\in\mathcal{E}_k,\quad\text{and}\quad {}^{U_{m_{k+1}}}R^{E_k}_{h}>\tfrac{1}{2}h\ \text{ on }\overline{W}_k.$$

Note that for $W_{k+1}=U_{m_{k+1}}$,

$$H_{W_{k+1}\setminus E_k}(1_{E_k}h)={}^{W_{k+1}}R^{E_k}_{h}>\tfrac{1}{2}h\quad\text{on }\overline{W}_k$$

(see e.g. [4, p. 254]). Fix $\delta>0$ and define $V=\delta\varrho^{-2}1_A$. Let (g_m) be the sequence defined in Lemma 2.2 and let $v_k=g_{m_k}$. By Lemma 2.1 there exists $\gamma<1$ such that, for all natural numbers $1\leqslant k<L$,

$$\sup\frac{v_L}{h}(E_k)\leqslant\gamma\sup\frac{v_L}{h}(\overline{W}_{k+1}).$$

Fix $L\in\mathbf{N}$. Then $0\leqslant v_L\leqslant h$ and v_L is subharmonic on U, hence for all $k\in\mathbf{N}$

$$\eta_k:=\sup\frac{v_L}{h}(\overline{W}_k)=\sup\frac{v_L}{h}(\partial W_k).$$

Using again that v_L is subharmonic we conclude that, for all $1\leqslant k<L$ and $x\in\overline{W}_k$,

$$\begin{aligned} v_L(x) &\leqslant H_{W_{k+1}\setminus E_k} v_L(x) \leqslant H_{W_{k+1}\setminus E_k}[(\eta_{k+1}1_{\partial W_{k+1}}+\gamma\eta_{k+1}1_{E_k})h](x)\\ &= \eta_{k+1}[H_{W_{k+1}\setminus E_k}h(x)-(1-\gamma)H_{W_{k+1}\setminus E_k}(1_{E_k}h)(x)]\\ &\leqslant \eta_{k+1}\big[h(x)-\tfrac{1}{2}(1-\gamma)h(x)\big]=\tfrac{1}{2}(1+\gamma)\eta_{k+1}h(x). \end{aligned}$$

Hence

$$\eta_k \leqslant \tfrac{1}{2}(1+\gamma)\eta_{k+1}.$$

Since $\eta_L\leqslant 1$ we thus obtain that, for every $l\in\mathbf{N}$,

$$v_L \leqslant \big(\tfrac{1}{2}(1+\gamma)\big)^{L-l}h \quad \text{on } \overline{W}_l.$$

Therefore, $\lim_{L\to\infty} v_L=0$ on U. By Lemma 2.2 the proof is finished. □

The next two propositions are crucial for our approach. Expressed in probabilistic terms we shall meet the following situation (see the outline of the proof for Theorem 1.1 given in the first section): A subset A of U will be constructed such that Brownian motion on U will hit A with probability 1 before leaving U. We want to know that this holds as well for the iterated sweeping induced by $\varepsilon_x\mapsto(h(x)|B_{r(x)}(x)|)^{-1}h1_{B_{r(x)}(x)}\lambda$, $x\in U$. Clearly this sweeping has less chances to hit A than (h-)Brownian motion. Therefore we have to find some stopping by A which can be expressed in terms of Brownian motion (not using the function r), which occurs with a smaller probability than for our iterated sweeping, and which is nevertheless strong enough to stop Brownian motion before reaching the boundary of U. The next proposition is essential for the proof that this goal can be achieved by killing Brownian motion at a rate $\delta\varrho^{-2}$ while it passes the set A, $\delta>0$ being a (very small) constant.

PROPOSITION 2.4. *Let $\beta,\eta\in]0,1[$. Then there exists a constant $\delta>0$ such that for every ball $B=B_r(x)$, $r>0$, $x\in\mathbf{R}^d$, for every harmonic function $g\geqslant 0$ on B, and for all Borel sets $A'\subset A\subset\mathbf{R}^d$ satisfying $A'\cap B_{\eta r}(x)=\varnothing$ or $|A\cap B|\geqslant\beta|B|$ the following holds: If*

$$0\leqslant V\leqslant \delta d(\cdot,\partial B)^{-2}1_{A'}$$

is a Borel measurable function on B and if $0\leqslant u\leqslant g$ is a continuous solution of $\Delta u-Vu=0$ on B then

$$u(x)\geqslant \frac{1}{|B|}\int_{B\setminus A} u\,d\lambda.$$

Proof. It clearly suffices to consider the case where B is the unit ball. Let w be the solution of

$$\Delta w=-\varepsilon_0+\frac{1}{|B|}$$

on B with boundary values zero on ∂B. (Of course we may write down w explicitly, e.g. $w(y)=-(1/2\pi)\ln\|y\|+(\|y\|^2-1)/4|B|$ if $d=2$, but it will not be necessary to use the explicit formula.) Clearly w depends on y only through $\|y\|$. Since $(-\varepsilon_0+(1/|B|)\lambda)(B)=0$, we therefore obtain by Green's formula that the normal derivative of w at ∂B is zero. Consequently, there exists a constant $a>0$ such that

$$w(y)\leqslant \frac{a}{|B|}(1-\|y\|)^2 \quad \text{for } \eta\leqslant\|y\|\leqslant 1.$$

Choose $\eta<b<1$ such that

$$1-b^d<\tfrac{1}{2}\beta.$$

By [15] there exists a constant $c>1$ such that for every Borel measurable $0\leqslant V\leqslant 1$ on $\frac{1}{2}(1+b)B$ and every continuous solution $u\geqslant 0$ of $\Delta u-Vu=0$ on $\frac{1}{2}(1+b)B$

$$\sup_{\|y\|\leqslant b} u(y)\leqslant c\inf_{\|y\|\leqslant b} u(y).$$

Define

$$\delta=\left(\max\left(2a,\left(\frac{2}{1-b}\right)^2,\frac{4c^2}{\beta(1-\eta)^2}\int_{\eta B} w\,d\lambda\right)\right)^{-1}.$$

Now let $A'\subset A$ be Borel subsets of $\mathbf{R}^d$ such that $A'\cap\eta B=0$ or $|A\cap B|\geqslant\beta|B|$, let V be a Borel measurable function on B such that

$$0\leqslant V(y)\leqslant\frac{\delta}{(1-\|y\|)^2}1_{A'}(y)$$

for all $y\in B$, let $g\geqslant 0$ be a harmonic function on B, and let $0\leqslant u\leqslant g$ be a continuous solution of $\Delta u-Vu=0$. By Green's formula

$$-u(0)+\frac{1}{|B|}\int_B u\,d\lambda=\int_B u\Delta w=\int_B w\Delta u=\int_B wVu\,d\lambda=\int_{A'} wVu\,d\lambda.$$

(Apply Green's formula to $B'=(1-\varepsilon)B$, $w'=w-w(\partial B')$, and let $\varepsilon>0$ tend to zero. Use that $\left|\int_{\partial B'} u(\partial w/\partial n)\,d\sigma\right|\leqslant|\partial w/\partial n|(\partial B')\int_{\partial B'} u\,d\sigma$ and that $\int_{\partial B'} u\,d\sigma\leqslant\int_{\partial B'} g\,d\sigma=\sigma(\partial B')g(0)$.) Hence

$$u(0)=\frac{1}{|B|}\left(\int_B u\,d\lambda-|B|\int_{A'} wVu\,d\lambda\right).$$

Clearly,

$$\begin{aligned}|B|\int_{A'\setminus\eta B} wVu\,d\lambda &\leqslant a\int_{A'\setminus\eta B}(1-\|y\|)^2V(y)u(y)\,\lambda(dy)\\ &\leqslant a\delta\int_{A'\setminus\eta B} u\,d\lambda\leqslant\frac{1}{2}\int_A u\,d\lambda.\end{aligned}$$

Therefore
$$u(0) \geqslant \frac{1}{|B|} \int_{B \setminus A} u \, d\lambda$$
if $A' \cap \eta B = \varnothing$.

Suppose now that $|A \cap B| \geqslant \beta |B|$. Then
$$|A \cap bB| \geqslant |A \cap B| - |B \setminus bB| \geqslant (\beta - (1-b^d))|B| > \tfrac{1}{2}\beta |B|$$
and
$$V \leqslant \delta \left(\frac{2}{1-b} \right)^2 \leqslant 1 \quad \text{on } \tfrac{1}{2}(1+b)B.$$
Hence
$$\begin{aligned} |B| \int_{A' \cap \eta B} wVu \, d\lambda &\leqslant cu(0) \frac{\delta}{(1-\eta)^2} |B| \int_{\eta B} w \, d\lambda \\ &\leqslant \frac{\beta}{4c} u(0) |B| \leqslant \frac{1}{2c} u(0) \int_{A \cap bB} 1 \, d\lambda \leqslant \frac{1}{2} \int_A u \, d\lambda. \end{aligned}$$
Thus
$$|B| \int_{A'} wVu \, d\lambda = |B| \int_{A' \setminus \eta B} Vwu \, d\lambda + |B| \int_{A' \cap \eta B} Vwu \, d\lambda \leqslant \int_A u \, d\lambda$$
finishing the proof. □

Straightforward calculations show that, for $d \geqslant 2$, the condition that $A' \cap B_{\eta r} \neq \varnothing$ or $|A \cap B| \geqslant \beta |B|$ cannot be omitted: If $V = \delta 1_{\varepsilon B}$ and $u = H_B^{\Delta - V} 1$ then
$$u(0) < \frac{1}{|B|} \int_{B \setminus \varepsilon B} u \, d\lambda$$
if $\varepsilon > 0$ is sufficiently small!

The following proposition will be necessary for our application of Proposition 2.4.

PROPOSITION 2.5. *Let $\beta, \eta \in]0,1[$, let $r: U \to \mathbf{R}$ such that $0 < r \leqslant \varrho$, and let E be a Borel subset of U. Define*
$$F = \{ y \in E : |E \cap B_{r(x)}(x)| \leqslant \beta |B_{r(x)}(x)| \text{ and } \|x-y\| < \eta r(x) \text{ for some } x \in U \}.$$
Then, for every $Q \in \mathcal{Q}$,
$$|F \cap Q| \leqslant \beta \left(\frac{5\sqrt{d}}{1-\eta} \right)^d c(d) |Q|,$$
where $c(d)$ denotes the Besicovitch constant (depending only on the dimension d).

Proof. Note first that F is a Borel set (it is open relative to E). Fix $Q \in \mathcal{Q}$, let x_0 denote the center of Q, and let $Q' = x_0 + 5\sqrt{d}(Q - x_0)$. For every $y \in F \cap Q$ choose $y' \in U$ such that $|E \cap B_{r(y')}(y')| \leqslant \beta |B_{r(y')}(y')|$ and $\|y' - y\| < \eta r(y')$. Then
$$r(y') \leqslant \varrho(y') \leqslant \|y' - y\| + \varrho(y),$$

hence

$$s(y) := (1-\eta)r(y') < \varrho(y)$$

and

$$A_y := B_{s(y)}(y) \subset B_{r(y')}(y') \cap Q'.$$

By the theorem of Besicovitch ([14]) there exist disjoint sets $F_j \subset F \cap Q$, $1 \leqslant j \leqslant m \leqslant c(d)$, such that for each $j \in \{1, ..., m\}$ the balls A_y, $y \in F_j$, are disjoint and

$$F \cap Q \subset \bigcup_{j=1}^{m} \bigcup_{y \in F_j} A_y.$$

Then

$$\begin{aligned} |F \cap Q| &\leqslant \sum_{j=1}^{m} \sum_{y \in F_j} |F \cap A_y| \leqslant \sum_{j=1}^{m} \sum_{y \in F_j} |E \cap B_{r(y')}(y')| \\ &\leqslant \beta \sum_{j=1}^{m} \sum_{y \in F_j} |B_{r(y')}(y')| = \frac{\beta}{(1-\eta)^d} \sum_{j=1}^{m} \sum_{y \in F_j} |A_y| \\ &= \frac{\beta}{(1-\eta)^d} \sum_{j=1}^{m} \Big| \bigcup_{y \in F_j} A_y \Big| \leqslant \frac{\beta}{(1-\eta)^d} c(d) |Q'| = \beta \left(\frac{5\sqrt{d}}{1-\eta} \right)^d c(d) |Q|. \end{aligned}$$

□

3. Sweeping of measures

In this section we shall study transfinite sweeping of measures generated by

$$\varepsilon_x \mapsto (h(x)|B_{r(x)}(x)|)^{-1} h 1_{B_{r(x)}(x)} \lambda, \quad x \in U.$$

We have to show that stopping at suitable subsets A of U the transfinite sweeping does not reach the boundary of U. It will be just as easy to start in an abstract setting.

Let Y be a compact metrizable space and let $\mathcal{M}$ denote the set of all positive (Radon) measures on Y. Let $\mathcal{P} \subset \mathcal{C}^+(Y)$ be a convex cone such that $\mathcal{P} - \mathcal{P}$ is uniformly dense in $\mathcal{C}(Y)$. (The following considerations can easily be extended to a locally compact metrizable space Y using function cones (in the sense of [4]), but we shall not need that generality.) Let $\prec$ denote the *specific order* on $\mathcal{M}$ defined by $\mathcal{P}$, i.e., $\nu \prec \mu$ if and only if $\nu(p) \leqslant \mu(p)$ for all $p \in \mathcal{P}$. We take a sequence (p_m) in $\mathcal{P}$ that is total in $\mathcal{C}(Y)$ and choose $\varepsilon_m > 0$ such that $p_0 := \sum_{m=1}^{\infty} \varepsilon_m p_m \in \mathcal{C}(Y)$. Now suppose that $\mu, \nu \in \mathcal{M}$, $\nu \prec \mu$. Then clearly $\nu(p_0) \leqslant \mu(p_0)$. Moreover, $\nu = \mu$ if and only if $\nu(p_0) = \mu(p_0)$.

Let $\mathcal{T}$ denote the set of all specifically decreasing mappings of $\mathcal{M}$ into $\mathcal{M}$, i.e.,

$$\mathcal{T} = \{T \mid T : \mathcal{M} \to \mathcal{M},\ T\mu \prec \mu \text{ for all } \mu \in \mathcal{M}\}.$$

For every $T\in\mathcal{T}$ and $\mu\in\mathcal{M}$, let $\mathcal{M}_T(\mu)$ denote the smallest w^*-closed, T-stable subset $\mathcal{N}$ of $\mathcal{M}$ containing μ. Obviously,

$$\mathcal{M}_T(\mu)\subset\{\nu\in\mathcal{M}:\nu\prec\mu\}$$

and

$$\mathcal{M}_T(\mu)=\{\mu\}\cup\mathcal{M}_T(T\mu).$$

PROPOSITION 3.1. *For every $T\in\mathcal{T}$ and $\mu\in\mathcal{M}$, there exists a T-invariant measure $\sigma\in\mathcal{M}_T(\mu)$.*

Proof. Fix $T\in\mathcal{T}$, $\mu\in\mathcal{M}$, and define

$$a:=\inf\{\nu(p_0):\nu\in\mathcal{M}_T(\mu)\}.$$

Choose a sequence (ν_n) in $\mathcal{M}_T(\mu)$ such that $\lim_{n\to\infty}\nu_n(p_0)=a$. There exists a subsequence (ν_n') of (ν_n) which is w^*-convergent to a measure $\sigma\in\mathcal{M}$. Then $\sigma\in\mathcal{M}_T(\mu)$ and $\sigma(p_0)=\lim_{n\to\infty}\nu_n'(p_0)=a$. Since $T\sigma\in\mathcal{M}_T(\mu)$ and $T\sigma\prec\sigma$, we know that $a\leqslant(T\sigma)(p_0)\leqslant\sigma(p_0)=a$, hence $T\sigma=\sigma$. □

Actually much more is true:

PROPOSITION 3.2. *Let $T\in\mathcal{T}$ and $\mu\in\mathcal{M}$. Then for every $\nu\in\mathcal{M}_T(\mu)$,*

$$\mathcal{M}_T(\mu)=\{\varrho\in\mathcal{M}_T(\mu):\nu\prec\varrho\}\cup\mathcal{M}_T(\nu).$$

In particular, $\mathcal{M}_T(\mu)$ is totally ordered (with respect to $\prec$). There exists a unique T-invariant measure μ_T in $\mathcal{M}_T(\mu)$. It is the smallest element in $\mathcal{M}_T(\mu)$.

Proof. For every $\nu\in\mathcal{M}_T(\mu)$ define

$$\mathcal{D}_\nu=\{\varrho\in\mathcal{M}_T(\mu):\nu\prec\varrho\}$$

and let

$$\mathcal{N}=\{\nu\in\mathcal{M}_T(\mu):T(\mathcal{D}_\nu\setminus\{\nu\})\subset\mathcal{D}_\nu\}.$$

Since $\mathcal{D}_\nu$ is w^*-closed, we obtain that, for every $\nu\in\mathcal{N}$,

$$\mathcal{M}_T(\mu)=\mathcal{D}_\nu\cup\mathcal{M}_T(\nu)$$

and hence

$$\mathcal{D}_\nu=\{\varrho\in\mathcal{M}_T(\mu):\varrho(p_0)\geqslant\nu(p_0)\}.$$

We claim that $\mathcal{N}=\mathcal{M}_T(\mu)$. Obviously, $\mu\in\mathcal{N}$ since $\mathcal{D}_\mu=\{\mu\}$. Furthermore, $\mathcal{N}$ is T-stable. Indeed, let $\nu\in\mathcal{N}$. Since $\mathcal{M}_T(\mu)=\mathcal{D}_\nu\cup\mathcal{M}_T(\nu)$ and $\mathcal{M}_T(\nu)=\{\nu\}\cup\mathcal{M}_T(T\nu)$ we conclude that

$$\mathcal{D}_{T\nu}=\mathcal{D}_\nu\cup\{T\nu\}.$$

Hence $T(\mathcal{D}_\nu\setminus\{\nu\})\subset\mathcal{D}_\nu$ implies that $T(\mathcal{D}_{T\nu}\setminus\{T\nu\})\subset\mathcal{D}_{T\nu}$, i.e., $T\nu\in\mathcal{N}$. Finally, suppose that (ν_n) is a sequence in $\mathcal{N}$ which is w^*-convergent to $\nu\in\mathcal{M}_T(\mu)$. Consider $\varrho\in\mathcal{D}_\nu\setminus\{\nu\}$. Then $\varrho(p_0)>\nu(p_0)=\lim_{n\to\infty}\nu_n(p_0)$. If $n\in\mathbf{N}$ such that $\varrho(p_0)>\nu_n(p_0)$ then $\varrho\in\mathcal{D}_{\nu_n}\setminus\{\nu_n\}$, hence $T\varrho\in\mathcal{D}_{\nu_n}$, $\nu_n\prec T\varrho$. Therefore $\nu\prec T\varrho$, $T\varrho\in\mathcal{D}_\nu$. So $\mathcal{N}$ is w^*-closed.

Having shown that $\mathcal{N}=\mathcal{M}_T(\mu)$ we obtain that $\mathcal{M}_T(\mu)$ is totally ordered. By Proposition 3.1 there exists a T-invariant $\sigma\in\mathcal{M}_T(\mu)$. If ν is any T-invariant element in $\mathcal{M}_T(\mu)$, then $\mathcal{M}_T(\nu)=\{\nu\}$, hence $\mathcal{M}_T(\mu)=\mathcal{D}_\nu$, i.e., $\nu\prec\varrho$ for every $\varrho\in\mathcal{M}_T(\mu)$. In particular, $\nu=\sigma$ finishing the proof. □

From now on suppose that T is a kernel on Y such that $Tp\leqslant p$ for every $p\in\mathcal{P}$ and that $T:\mathcal{M}\to\mathcal{M}$ is defined by

$$(T\mu)(f)=\mu(Tf),\quad f\in\mathcal{C}(Y).$$

Then clearly $T\mu\prec\mu$ for every $\mu\in\mathcal{M}$.

PROPOSITION 3.3. *Let $s>-\infty$ be a l.s.c. function on Y such that $Ts\leqslant s$. Then $\nu(s)\leqslant\mu(s)$ for every $\nu\in\mathcal{M}_T(\mu)$.*

Proof. Consider

$$\mathcal{N}=\{\nu\in\mathcal{M}:\nu(s)\leqslant\mu(s)\}.$$

Obviously $\mu\in\mathcal{N}$. If $\nu\in\mathcal{N}$ then $(T\nu)(s)=\nu(Ts)\leqslant\nu(s)\leqslant\mu(s)$, i.e., $T\nu\in\mathcal{N}$. Finally, let (ν_n) be a sequence in $\mathcal{N}$ which is w^*-convergent to a measure $\nu\in\mathcal{M}$. Then $\nu(s)\leqslant\liminf_{n\to\infty}\nu_n(s)\leqslant\mu(s)$, hence $\nu\in\mathcal{N}$. Thus $\mathcal{M}_T(\mu)\subset\mathcal{N}$. □

Define the *base* $b(T)$ of T by

$$b(T)=\{x\in Y:T\varepsilon_x=\varepsilon_x\}.$$

Clearly, $b(T)$ is a Borel subset of Y.

LEMMA 3.4. *Every T-invariant $\sigma\in\mathcal{M}$ is supported by $b(T)$.*

Proof. Since $Tp_0(x)=(T\varepsilon_x)(p_0)$ and $T\varepsilon_x\prec\varepsilon_x$ for every $x\in Y$, we have $Tp_0\leqslant p_0$ and $b(T)=\{Tp_0=p_0\}$. Fix a T-invariant $\sigma\in\mathcal{M}$. Then $\sigma(Tp_0)=(T\sigma)(p_0)=\sigma(p_0)$, hence $\sigma(\{Tp_0<p_0\})=0$, i.e., $\sigma(\complement b(T))=0$. □

LEMMA 3.5. *Let $\mu\in\mathcal{M}$ and $\nu,\varrho\in\mathcal{M}_T(\mu)$ such that $\nu\prec\varrho$. Then $1_{b(T)}\nu\geqslant 1_{b(T)}\varrho$.*

Proof. By Proposition 3.2, $\nu\in\mathcal{M}_T(\varrho)$. Hence it suffices to consider the case $\varrho=\mu$. Let

$$\mathcal{N}=\{\nu\in\mathcal{M}_T(\mu): 1_{b(T)}\nu\geqslant 1_{b(T)}\mu\}.$$

Clearly, $\mu\in\mathcal{N}$ and $\mathcal{N}$ is T-stable. Suppose that (ν_n) is a sequence in $\mathcal{N}$ which is w^*-convergent to a measure $\nu\in\mathcal{M}_T(\mu)$. The representation $\nu_n=1_{b(T)}\mu+(\nu_n-1_{b(T)}\mu)$ shows that the sequence $(\nu_n-1_{b(T)}\mu)$ in $\mathcal{M}$ is w^*-convergent to a measure $\tau\in\mathcal{M}$ such that $\nu=1_{b(T)}\mu+\tau$. So $\nu\in\mathcal{N}$. Thus $\mathcal{N}=\mathcal{M}_T(\mu)$ finishing the proof. □

LEMMA 3.6. *Let A be a Borel subset of $b(T)$ and define a kernel T' on Y by $T'(x,\cdot)=1_{\complement A}T(x,\cdot)$ if $x\in Y\setminus b(T)$, $T(x,\cdot)=\varepsilon_x$ if $x\in b(T)$. Then $T'\in\mathcal{T}$ and, for every $\mu\in\mathcal{M}$, $1_{Y\setminus A}\mu_T=1_{Y\setminus A}\mu_{T'}$.*

Proof. Obviously, $T'\in\mathcal{T}$. Moreover, it is easily seen that $b(T')=b(T)$ (if $T(x,\cdot)\neq\varepsilon_x$ then $T'(x,\cdot)\neq\varepsilon_x$ since $T1\leqslant 1$). Fix $\mu\in\mathcal{M}$ and consider the set

$$\mathcal{N}=\{\nu\in\mathcal{M}_T(\mu): 1_{Y\setminus A}\nu=1_{Y\setminus A}\nu' \text{ for some } \nu'\in\mathcal{M}_{T'}(\mu)\}.$$

Of course, $\mu\in\mathcal{N}$. Furthermore, it follows immediately from the definitions that $1_{Y\setminus A}\nu=1_{Y\setminus A}\nu'$ implies that $1_{Y\setminus A}T\nu=1_{Y\setminus A}T'\nu'$. So $\mathcal{N}$ is T-stable.

Next let (ν_n) be a sequence in $\mathcal{N}$ which is w^*-convergent to $\nu\in\mathcal{M}_T(\mu)$. Choose $\nu_n'\in\mathcal{M}_{T'}(\mu)$ such that $1_{Y\setminus A}\nu_n=1_{Y\setminus A}\nu_n'$, $n\in\mathbf{N}$. In order to show that $\nu\in\mathcal{N}$ we may assume that (ν_n') is w^*-convergent to a measure $\nu'\in\mathcal{M}_{T'}(\mu)$. Moreover, by Proposition 3.2, we may assume without loss of generality that the sequences (ν_n) and (ν_n') are monotone with respect to specific order. Then by Lemma 3.5 the sequences $(1_A\nu_n)$ and $(1_A\nu_n')$ are monotone with respect to $\leqslant$, the usual order between measures. Hence the sequences $(1_A\nu_n)$, $(1_A\nu_n')$ are w^*-convergent to measures ϱ, ϱ', respectively, such that $\varrho(Y\setminus A)=\varrho'(Y\setminus A)=0$. So we obtain that $\nu=\varrho+\lim_{n\to\infty}1_{Y\setminus A}\nu_n$, $\nu'=\varrho'+\lim_{n\to\infty}1_{Y\setminus A}\nu_n'$, $1_{Y\setminus A}\nu=1_{Y\setminus A}\nu'$. Thus $\mathcal{N}=\mathcal{M}_T(\mu)$.

Finally, let $\nu'\in\mathcal{M}_{T'}(\mu)$ such that $1_{Y\setminus A}\mu_T=1_{Y\setminus A}\nu'$. Then

$$1_{Y\setminus A}T'\nu'=1_{Y\setminus A}T\mu_T=1_{Y\setminus A}\mu_T=1_{Y\setminus A}\nu',$$

hence $T'\nu'=\nu'$. By Proposition 3.2, $\nu'=\mu_{T'}$. □

Let us now make a suitable choice of Y and $\mathcal{P}$. We take a sequence (q_n) of continuous potentials on U having compact superharmonic support such that for every $x\in U$ and every neighborhood W of x there exist $n,m\in\mathbf{N}$ with $0\leqslant q_n-q_m\leqslant 1_W$, $(q_n-q_m)(x)>0$. There exists a metrizable compactification Y of U such that the functions q_n/h, $n\in\mathbf{N}$, can be extended to functions $\psi_n \in \mathcal{C}^+(Y)$ separating the points of Y. If U is regular then the potentials q_n (and hence the functions q_n/h) tend to 0 at ∂U, so Y is the one point compactification of U. Let $\mathcal{P} \subset \mathcal{C}^+(Y)$ be the min-stable convex cone generated by $\{1\} \cup \{\psi_n : n \in \mathbf{N}\}$. Then $\mathcal{P} - \mathcal{P}$ is uniformly dense in $\mathcal{C}(Y)$.

Given a Borel function $0 < r \leqslant \varrho$ on U and a Borel subset A of U, we define a Markov kernel $P = P_{A,r}$ on Y by

$$P(x, \cdot) = \begin{cases} (h(x)|B_{r(x)}(x)|)^{-1} h 1_{B_{r(x)}(x)} \lambda, & x \in U \setminus A, \\ \varepsilon_x, & x \in A \cup (Y \setminus U). \end{cases}$$

Then $Pp \leqslant p$ for every $p \in \mathcal{P}$, i.e., $P \in \mathcal{T}$. Note that $b(P) = A \cup (Y \setminus U)$. Define P' as in Lemma 3.6. The following proposition will the main tool for the proof of Theorem 1.1.

PROPOSITION 3.7. *Let $0 < r \leqslant \varrho$ be a Borel function on U and let A be a Borel subset of U. Assume that there exists $\alpha > 0$ such that, for every $z \in \partial_1^M U$, the union of all $Q \in \mathcal{Q}$ satisfying $|A \cap Q| \geqslant \alpha |Q|$ is not minimal thin at z. Then, for every $x \in U$, $(\varepsilon_x)_P(Y \setminus A) = 0$, $(\varepsilon_x)_P(A) = 1$.*

Proof. Choose $\beta = \frac{1}{2}\alpha \left(10\sqrt{d}\right)^{-d} c(d)^{-1}$ and define

$$A_0 = \{y \in A : \|y' - y\| < \tfrac{1}{2} r(y') \text{ and } |A \cap B_{r(y')}(y')| \leqslant \beta |B_{r(y')}(y')| \text{ for some } y' \in U\}.$$

By Proposition 2.5, $|A_0 \cap Q| \leqslant \frac{1}{2}\alpha |Q|$ for every $Q \in \mathcal{Q}$ and hence $A' := A \setminus A_0$ satisfies

$$|A' \cap Q| = |A \cap Q| - |A_0 \cap Q| \geqslant \tfrac{1}{2}\alpha |Q|$$

for every $Q \in \mathcal{Q}$ with $|A \cap Q| \geqslant \alpha |Q|$. Moreover, for every $y' \in U$,

$$A' \cap B_{r(y')/2}(y') = \varnothing \quad \text{or} \quad |A \cap B_{r(y')}(y')| \geqslant \beta |B_{r(y')}(y')|.$$

Choose $\delta > 0$ according to Proposition 2.4 $\left(\text{with } \eta = \frac{1}{2}\right)$ and take

$$V = \frac{\delta}{\varrho^2} 1_{A'}.$$

Then for every continuous solution $u > 0$ of $\Delta u - Vu = 0$ on U the function u/h is unbounded by Proposition 2.3. For every $m \in \mathbf{N}$ let $g_m \in \mathcal{C}^+(U)$ such that $g_m + K_m g_m = h$ where $K_m g_m = \int_{U_m} g_m(y) G_U(\cdot, y) V(y) \, \lambda(dy)$ (see Lemma 2.2). Then $\lim_{m \to \infty} g_m = 0$ by Lemma 2.2 whereas by Proposition 2.4, for every $m \in \mathbf{N}$ and every $y \in U$,

$$g_m(y) \geqslant \frac{1}{|B_{r(y)}(y)|} \int_{B_{r(y)}(y) \setminus A} g_m \, d\lambda,$$

i.e., $P'(g_m/h)(y) \leqslant (g_m/h)(y)$ for every $y \in U$.

Fix $x_0 \in U$ and define

$$\sigma = (\varepsilon_{x_0})_P, \quad \sigma' = (\varepsilon_{x_0})_{P'}.$$

We claim first that $\sigma'(Y \setminus U) = 0$. To that end choose a superharmonic function $s > 0$ on U such that $s(x_0) = 1$ and $\lim_{y \to z,\, y \in U} s(y) = \infty$ for every irregular boundary point $z \in \partial U$. Fix $m \in \mathbf{N}$ and define a l.s.c. function φ_m on Y by

$$\varphi_m(x) = \liminf_{y \to x,\, y \in U} \frac{g_m + s/m}{h}(y), \quad x \in Y.$$

Then $\varphi_m = (g_m + s/m)/h$ on U and hence

$$P'\varphi_m \leqslant \varphi_m.$$

So we conclude by Proposition 3.3 that

$$\sigma'(\varphi_m) \leqslant \varphi_m(x_0) \leqslant \frac{g_m(x_0) + 1/m}{h(x_0)}$$

and therefore

$$\lim_{m \to \infty} \sigma'(\varphi_m) = 0.$$

Hence the equality $\sigma'(Y \setminus U) = 0$ will be established once we have shown that $\varphi_m \geqslant 1$ on $Y \setminus U$ for every $m \in \mathbf{N}$.

To that end it suffices to show that $\liminf_{x \to z} \varphi_m(x) \geqslant 1$ for every $z \in \partial U$. Since $0 \leqslant g_m \leqslant h$ and hV is bounded on U_m, the potential $K_m g_m$ is bounded on U. Furthermore, $\lim_{x \to z} K_m g_m(x) = 0$ and hence $\lim_{x \to z} g_m(x)/h(x) = 1$ for every regular boundary point $z \in \partial U$. Now fix $z \in \partial U$ and choose a sequence (x_n) in U such that $\lim_{n \to \infty} x_n = z$ and

$$a := \liminf_{y \to z,\, y \in U} \varphi_m(y) = \lim_{n \to \infty} \frac{g_m(x_n) + s(x_n)/m}{h(x_n)}.$$

If there exists a subsequence (y_n) of (x_n) such that $\lim_{n \to \infty} g_m(y_n)/h(y_n) = 1$, then of course $a \geqslant 1$. So assume that $\limsup_{n \to \infty} g_m(x_n)/h(x_n) < 1$. Then z is an irregular boundary point by our preceding considerations. Knowing that

$$\frac{g_m}{h} + \frac{1}{h} K_m g_m = 1$$

and that $K_m g_m$ is bounded we obtain moreover that $\limsup_{n \to \infty} h(x_n) < \infty$. Since $\lim_{x \to z} s(x) = \infty$, we conclude that $\lim_{n \to \infty} s(x_n)/h(x_n) = \infty$, $a = \infty$.

Therefore $\varphi_m \geqslant 1$ on $Y \setminus U$, $\sigma'(Y \setminus U) = 0$. So $\sigma(Y \setminus U) = \sigma'(Y \setminus U) = 0$ by Lemma 3.6. Moreover, $\sigma(U \setminus A) = 0$ by Lemma 3.4. Thus $\sigma(Y \setminus A) = 0$. Since P is a Markov kernel, we have $\nu(Y) = 1$ for every $\nu \in \mathcal{M}_P(\varepsilon_{x_0})$, hence $\sigma(Y) = 1$, $\sigma(A) = 1$. $\square$

The following lemma shows that dealing with r-supermedian functions we may always assume that r is Borel measurable (cf. Veech [23]).

LEMMA 3.8. *Let r be a real function on U such that $0<r\leqslant\varrho$. Then there exist Borel measurable functions r', r'' on U such that $0<r'\leqslant r\leqslant r''\leqslant\varrho$ and such that every r-supermedian l.s.c. lower h-bounded Lebesgue measurable function on U is r'-supermedian and r''-supermedian.*

Proof. Define

$$r''(x)=\limsup_{y\to x} r(y), \quad x\in U.$$

Moreover, let $A_0=\varnothing$, $A_m=\{r\geqslant 2^{-m}\}$, $m\in\mathbf{N}$, and define

$$r'(x)=\liminf_{y\to x,\, y\in A_m} r(y), \quad x\in \bar{A}_m\setminus\bar{A}_{m-1}.$$

Clearly, $0<r'\leqslant r\leqslant r''\leqslant\varrho$ and r', r'' are Borel measurable. Moreover, for every $x\in U$, there exist $x_n', x_n''\in U$ such that $\lim x_n'=\lim x_n''=x$ and $\lim r(x_n')=r'(x)$, $\lim r(x_n'')=r''(x)$. So the proof is easily finished using Fatou's lemma. □

4. Proof of Theorem 1.1

In addition to the results of the preceding sections we shall need the following nice property of the Whitney decomposition.

LEMMA 4.1. *Let (x_n) be a sequence in U that converges to a point $z\in\partial_1^M U$. Let (y_n) be a sequence in U such that, for each $n\in\mathbf{N}$, the points x_n, y_n are contained in the same cube $Q_n\in\mathcal{Q}$. Then (y_n) converges to z as well.*

Proof. For every $Q\in\mathcal{Q}$ with center x_Q let $\widetilde{Q}=x_Q+\frac{3}{2}(Q-x_Q)$. Then there exists a constant $C>1$ such that, for every $Q\in\mathcal{Q}$ and for every harmonic function $g\geqslant 0$ on the interior of $\widetilde{Q}$,

$$\sup g(Q)\leqslant C\inf g(Q).$$

In particular, for each $n\in\mathbf{N}$, the inequalities

$$\frac{1}{C}G_U(y_n,\tilde{z})\leqslant G_U(x_n,\tilde{z})\leqslant CG_U(y_n,\tilde{z})$$

hold for every $\tilde{z}\in\partial\widetilde{Q}_n$, hence by minimum principle for every $\tilde{z}\in U\setminus\widetilde{Q}_n$.

Given $m\in\mathbf{N}$, there exists $n_m\in\mathbf{N}$ such that $x_n\in U\setminus U_{m+3}$ for every $n\geqslant n_m$. Then $\widetilde{Q}_n\cap U_m=\varnothing$ for every $n\geqslant n_m$. In particular, the sequence (y_n) has no accumulation point in U. In order to show that $\lim y_n=z$ we may hence assume without loss of generality that (y_n) converges to a point $z'\in\partial^M U$. Suppose that the Martin kernel K is based at $x_0\in U$, i.e.,

$$K(x,\cdot)=\frac{G_U(x,\cdot)}{G_U(x,x_0)}, \quad x\in U.$$

If $x_0 \in U_m$ then the inequalities above imply that for every $n \geqslant n_m$

$$K(y_n, \cdot) \leqslant C^2 K(x_n, \cdot) \quad \text{on } U \setminus \widetilde{Q}_n,$$

hence

$$K(z', \cdot) \leqslant C^2 K(z, \cdot).$$

Since $K(z, \cdot)$ is a minimal harmonic function, we conclude that $z' = z$. □

COROLLARY 4.2. *For every $z \in \partial_1^M U$ and every neighborhood W of z, there exists a neighborhood W' of z such that every cube $Q \in \mathcal{Q}$ intersecting W' is contained in W.*

Proof. Suppose the contrary. Choose a fundamental system $\{W_n : n \in \mathbf{N}\}$ of neighborhoods of z satisfying $W_{n+1} \subset W_n$, $n \in \mathbf{N}$. Then for every $n \in \mathbf{N}$ there exists a cube $Q_n \in \mathcal{Q}$ such that $Q_n \cap W_n' \neq \varnothing$, but $Q_n \not\subset W$. Taking

$$x_n \in Q_n \cap W', \quad y_n \in Q_n \setminus W$$

we obtain a sequence (x_n) converging to z whereas of course (y_n) does not converge to z. This contradicts Lemma 4.1. □

Having in mind the outline given in Section 1 the proof of Theorem 1.1 is now easily accomplished: Let r be a real function on U such that $0 < r \leqslant \varrho$, let f be an r-supermedian l.s.c. lower h-bounded function on U, and fix $0 < \alpha \leqslant \frac{1}{2}$. We have to show that $f \geqslant H\bar{f}_\alpha$. By Lemma 3.8 we may assume that r is Borel measurable. Since multiples of h are r-supermedian and

$$H\overline{(f+ch)}_\alpha = H(\bar{f}_\alpha + c) = H\bar{f}_\alpha + ch$$

we may assume that $f \geqslant 0$.

Fix $x_0 \in U$, $a > 0$, and $\varepsilon > 0$. There exists a compact subset L of $\partial_1^M U$ such that $\bar{f}_\alpha|_L$ is continuous and $H1_{\partial^M U \setminus L}(x_0) < \varepsilon/a$. Choose $\varphi \in \mathcal{C}(U \cup \partial^M U)$ such that $\varphi = \inf(\bar{f}_\alpha, a)$ on L and $0 \leqslant \varphi \leqslant a$. There exists a subharmonic function t on U such that $\limsup_{x \to z} t(x)/h(x) \leqslant \varphi(z)$ for every $z \in \partial^M U$ and $t(x_0) > H\varphi(x_0) - \varepsilon$. Moreover, there exists a superharmonic function $s \geqslant 0$ on U such that $\liminf_{x \to z} s(x)/h(x) \geqslant a$ for every $z \in \partial^M U \setminus L$ and $s(x_0) < \varepsilon$. Fix $m \in \mathbf{N}$ such that $t/h - \varepsilon < \varphi$ on $U \setminus U_m$ and define

$$A = (U \setminus U_m) \cap \left\{ \frac{f+s}{h} \geqslant \varphi - \varepsilon \right\}.$$

Let

$$\mathcal{D} = \{Q \in \mathcal{Q} : |A \cap Q| \geqslant \alpha |Q|\}, \quad D = \bigcup_{Q \in \mathcal{D}} Q.$$

We claim that, for every $z \in \partial_1^M U$, the set D is not minimal thin at z.

Indeed, consider first $z\in L$. There exists a neighborhood W of z such that $W\cap U_m=\varnothing$ and

$$|\varphi-\varphi(z)|<\tfrac{1}{2}\varepsilon \quad \text{on } W.$$

By Corollary 4.2 we may choose a neighborhood W' of z such that $Q\subset W$ whenever the cube $Q\in\mathcal{Q}$ intersects W'. Since $\bar{f}_\alpha(z)>\varphi(z)-\frac{1}{2}\varepsilon$, we know by definition of $\bar{f}_\alpha(z)$ that the set

$$A(z):=\{x\in W'\cap U:\bar{f}_\alpha(x)>\varphi(z)-\tfrac{1}{2}\varepsilon\}$$

is not minimal thin at z. Given $x\in A(z)$ consider $Q\in\mathcal{Q}$ containing x. Then $Q\subset W$, hence $Q\subset U\setminus U_m$ and $\bar{f}_\alpha=\bar{f}_\alpha(x)>\varphi(z)-\frac{1}{2}\varepsilon>\varphi-\varepsilon$ on Q. Therefore

$$\alpha|Q|\leqslant\left|\left\{y\in Q:\frac{f(y)}{h(y)}\geqslant\bar{f}_\alpha(x)\right\}\right|\leqslant\left|\left\{y\in Q:\frac{f(y)}{h(y)}\geqslant\varphi(y)-\varepsilon\right\}\right|\leqslant|A\cap Q|,$$

$x\in Q\in\mathcal{D}$. This shows that $A(z)\subset D$. So D is not minimal thin at z.

Suppose next that $z\in\partial^M U\setminus L$. Consider a neighborhood W of z such that

$$\frac{s}{h}>a-\varepsilon \quad \text{on } W$$

and choose a corresponding neighborhood W'. Given $x\in W'$, we have $x\in Q$ for some $Q\in\mathcal{Q}$ and then $Q\subset W$. Hence $Q\subset U\setminus U_m$ and

$$\frac{f+s}{h}\geqslant\frac{s}{h}>a-\varepsilon\geqslant\varphi-\varepsilon \quad \text{on } Q\ ,$$

hence $Q\subset A$. So $Q\in\mathcal{D}$. Therefore $W'\subset D$ which shows that D is not minimal thin at z.

Define $Y,\mathcal{P}$, and $P=P_{A,r}$ as in Section 3. Let $\sigma\in\mathcal{M}_P(\varepsilon_{x_0})$ be P-invariant. Then by Proposition 3.7,

$$\sigma(Y\setminus A)=0,\quad \sigma(A)=1.$$

Let P^U denote the restriction of P on U. Suppose that v is a l.s.c. lower bounded numerical function on U such that $P^U v\leqslant v$. Then $\sigma(v)\leqslant v(x_0)$. Indeed, extend v to a l.s.c. function $w>-\infty$ on Y (e.g. taking $w=\inf v(U)$ on $Y\setminus U$). Then $Pw\leqslant w$, hence $\sigma(w)\leqslant w(x_0)$ by Proposition 3.3, i.e., $\sigma(v)\leqslant v(x_0)$.

In particular, $\sigma(f/h)\leqslant f(x_0)/h(x_0)$, $\sigma(s/h)\leqslant s(x_0)/h(x_0)$, and $\sigma(t/h)\geqslant t(x_0)/h(x_0)$. Since $(f+s)/h\geqslant\varphi-\varepsilon>(t/h)-2\varepsilon$ on A, we obtain that

$$\frac{s(x_0)+f(x_0)}{h(x_0)}\geqslant\sigma\left(\frac{f+s}{h}\right)\overset{!}{\geqslant}\sigma\left(\frac{t}{h}-2\varepsilon\right)\geqslant\frac{t(x_0)}{h(x_0)}-2\varepsilon\geqslant\frac{1}{h(x_0)}H\varphi(x_0)-3\varepsilon.$$

Since $\inf(\bar{f}_\alpha,a)\leqslant\varphi+a1_{\partial^M U\setminus L}$ on $\partial^M U$, we know that

$$H(\inf(\bar{f}_\alpha,a))(x_0)\leqslant H\varphi(x_0)+aH1_{\partial^M U\setminus L}(x_0)\leqslant H\varphi(x_0)+\varepsilon.$$

Thus

$$f(x_0) > H(\inf(\bar{f}_\alpha, a))(x_0) - 5\varepsilon h(x_0).$$

Since $\varepsilon > 0$ and $a > 0$ are arbitrary, we finally conclude that $f(x_0) \geqslant H\bar{f}_\alpha(x_0)$, and the proof is finished.

Final remark. The applicability of our method for the proof of the main results is certainly not restricted to the special case of ball means, Laplace operator, and a bounded domain in Euclidean space.

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WOLFHARD HANSEN
Fakultät für Mathematik
Universität Bielefeld
Universitätsstraße
D-33615 Bielefeld
Germany

NIKOLAI NADIRASHVILI
Fakultät für Mathematik
Universität Bielefeld
Universitätsstraße
D-33615 Bielefeld
Germany

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