Acta Math., 170 (1993), 151–180

Multiplicities of algebraic linear recurrences

by

HANS PETER SCHLICKEWEI

Universität Ulm
Ulm, Germany

1. Introduction

Let n be a natural number. We shall study linear recurrence sequences

$$u_{m+n}=\nu_{n-1}u_{m+n-1}+\nu_{n-2}u_{m+n-2}+\ldots+\nu_0u_m,\quad m=0,1,2,\ldots\,. \tag{1.1}$$

Here we assume that $\nu_{n-1},\ldots,\nu_0$ are elements of $\mathbf{C}$ with $\nu_0\neq 0$. We assume moreover that the initial values $u_0,\ldots,u_{n-1}$ of our sequence have $|u_{n-1}|+\ldots+|u_0|>0$. Let

$$G(z)=z^n-\nu_{n-1}z^{n-1}-\ldots-\nu_0 \tag{1.2}$$

be the companion polynomial of the recurrence (1.1) and write

$$G(z)=\prod_{i=1}^{r}(z-\alpha_i)^{\varrho_i} \tag{1.3}$$

with distinct numbers $\alpha_1,\ldots,\alpha_r$. We call n the order and r the rank of the recurrence (1.1). Before we state our results, we shall recall a few facts about linear recurrence sequences. An excellent account on this topic may be found in the introductory Chapter C of Shorey and Tijdeman [13]. In the sequel we quote some of the theorems collected there.

Let $(u_m)_{m=0}^{\infty}$ be a sequence satisfying relation (1.1) *with $\nu_0\neq 0$. For $i=1,\ldots,r$ let α_i and ϱ_i be determined by* (1.2) *and* (1.3) *where the numbers $\alpha_1,\ldots,\alpha_r$ are distinct. Then there exist uniquely determined polynomials $f_i\in\mathbf{Q}(u_0,\ldots,u_{n-1},\nu_0,\ldots,\nu_{n-1},\alpha_1,\ldots,\alpha_r)[z]$ of degree $\leqslant\varrho_i-1$ $(i=1,\ldots,r)$ such that*

$$u_m=\sum_{i=1}^{r}f_i(m)\alpha_i^m,\quad m=0,1,2,\ldots\,. \tag{1.4}$$

11–935202 *Acta Mathematica* 170. Imprimé le 30 juin 1993

Conversely, let $\alpha_1, \ldots, \alpha_r$ *be distinct complex numbers and* $\varrho_1, \ldots, \varrho_r$ *be natural numbers with* $\sum_{i=1}^{r} \varrho_i = n$. *Define* $\nu_0, \ldots, \nu_{n-1}$ *by* (1.3) *and* (1.2). *For* $i=1, \ldots, r$ *let* f_i *be a polynomial of degree less than* ϱ_i. *Then the sequence* $(u_m)_{m=0}^{\infty}$ *defined by* (1.4) *satisfies recurrence relation* (1.1).

The *a-multiplicity* $U(a)$ of a sequence $(u_m)_{m=0}^{\infty}$ is defined as the number of indices m such that

$$u_m = a. \tag{1.5}$$

We define the *multiplicity* U as

$$U = \sup_a U(a). \tag{1.6}$$

The well known theorem of Skolem–Mahler–Lech says the following.

If $(u_m)_{m=0}^{\infty}$ *is a recurrence sequence with infinite* 0*-multiplicity, then those* m *for which* $u_m = 0$ *form a finite union of arithmetic progressions after a certain stage.*

As an immediate consequence we obtain:

If a recurrence with companion polynomial (1.3) *generates a sequence with infinite* 0*-multiplicity, then* α_i/α_j *is a root of unity for some indices* i, j *with* $i \neq j$.

Therefore, we call the recurrence sequence $(u_m)_{m=0}^{\infty}$ *nondegenerate* if for each pair i, j $(1 \leqslant i, j \leqslant r)$, $i \neq j$, the ratio α_i/α_j of the roots of the companion polynomial (1.3) is not a root of unity. An easy consequence of the above quoted facts is as follows.

If $(u_m)_{m=0}^{\infty}$ *is a nondegenerate periodic linear recurrence sequence, then there exists a number* d *and a root of unity* α *such that*

$$u_m = d\alpha^m. \tag{1.7}$$

There is a large amount of articles in which multiplicities of sequences $(u_m)_{m=0}^{\infty}$ as in (1.1) are studied. They mostly deal with nondegenerate binary recurrences, i.e. recurrences of order 2. Kubota [6] proved that in case $n=2$ and if all terms u_m belong to a number field K, then U is bounded from above by a number depending only on the degree d of K. Beukers and Tijdeman [1] in this case established the bound

$$U \leqslant 100 \max\{d, 300\}. \tag{1.8}$$

For general order n there are some *partial* results on bounds for the multiplicity in the literature. E.g. there exist certain bounds for the case where one of the roots α_i of the companion polynomial is dominant. We cannot give all details here. We refer the reader to Chapters 1–4 of Shorey and Tijdeman [13] and the extensive amount of references given there.

Evertse [3] in the algebraic case and Evertse, Győry, Stewart and Tijdeman [5] in general proved:

Let $(u_m)_{m=0}^{\infty}$ *be a nonperiodic nondegenerate linear recurrence sequence in* $\mathbf{C}$. *Then there are only finitely many pairs of integers* l, m *with* $l \neq m$ *and*

$$u_l = u_m. \tag{1.9}$$

This implies in particular that a *nonperiodic nondegenerate recurrence sequence has finite multiplicity.*

I could not find in the literature any upper bound for the multiplicity of a nondegenerate nonperiodic sequence $(u_m)_{m=0}^{\infty}$ of arbitrary order n that holds without any restriction on the sequence.(1) It is the purpose of this paper to establish such a bound in the case when the terms u_m of the sequence are algebraic. Let K be a number field with

$$[K:\mathbf{Q}] = d. \tag{1.10}$$

We assume throughout the paper that

$$\nu_0, \ldots, \nu_{n-1}, u_0, \ldots, u_{n-1}, \alpha_1, \ldots, \alpha_r \in K. \tag{1.11}$$

Moreover we define $\omega = \omega(\alpha_1, \ldots, \alpha_r)$ to be the number of prime ideals in K ocurring in the decomposition of the ideals $(\alpha_1), \ldots, (\alpha_r)$. If the *polynomials* f_i *in* (1.4) *are all constant, then* a recent result of Schlickewei [10] on the number of solutions of S-unit equations implies that

$$U \leqslant (4(\omega+d)d!)^{2^{36(n+1)d!}(\omega+d)^6}. \tag{1.12}$$

We can now give a similar bound for the general case.

THEOREM 1.1. *Let* $(u_m)_{m=0}^{\infty}$ *be as in* (1.1). *Let* $\alpha_1, \ldots, \alpha_r$ *be defined by* (1.2), (1.3). *Suppose that we have* (1.10), (1.11). *Suppose that* $(u_m)_{m=0}^{\infty}$ *is nondegenerate. Then we have*

$$U(0) \leqslant (4(\omega+d)d!)^{2^{40n!\,d!}(\omega+d)^6}. \tag{1.13}$$

As for the multiplicity we obtain

THEOREM 1.2. *Let the hypotheses be the same as in Theorem* 1.1. *Assume moreover that* $(u_m)_{m=0}^{\infty}$ *is nonperiodic. Then we have*

$$U \leqslant (4(\omega+d)d!)^{2^{40(n+1)!\,d!}(\omega+d)^6}. \tag{1.14}$$

(1) Several months after this paper was written, A. J. van der Poorten and H. P. Schlickewei proved results of a similar type as those given in the current paper, applying p-adic analysis. In the meantime these results have appeared (Zeros of recurrence sequences. *Bull. Austral. Math. Soc.*, 44 (1991), 215–223). However, the method of proof of the current paper has the advantage of allowing generalizations to higher dimensions.

In view of (1.4), it is clear that in proving these reults, we have to study equations

$$\sum_{i=1}^{r} f_i(m)\alpha_i^m = 0, \tag{1.15}$$

where

$$f_i(z) \in K[z], \quad i = 1, ..., r, \tag{1.16}$$

and where $\alpha_1, ..., \alpha_r$ are nonzero elements in K such that for each pair i, j $(1 \leqslant i, j \leqslant r)$ with $i \neq j$

$$\alpha_i/\alpha_j \text{ is not a root of unity.} \tag{1.17}$$

We write

$$\deg f_i = \varrho_i - 1 \tag{1.18}$$

and

$$\sum_{i=1}^{r} (\varrho_i - 1) = k. \tag{1.19}$$

We shall derive Theorems 1.1 and 1.2 from

THEOREM 1.3. *Suppose that we have* (1.10), (1.16), (1.17), (1.18), (1.19). *Then equation* (1.15) *has not more than*

$$(4(\omega+d)d!)^{2^{40(k+r)!\,d!\,(\omega+d)^6}} \tag{1.20}$$

solutions $m \in \mathbf{Z}$.

It should be pointed out that the numerical constants in (1.13), (1.14), (1.20) are somewhat arbitrary. No particular care was taken to optimize them. Apart from the fact that this is the first general result of this type,(2) the significant feature of our theorems is that the bounds are rather uniform, as they depend only upon ω, but not upon the particular primes involved. Moreover the bounds do not depend upon the coefficients of the polynomials $f_i(m)$. With such a dependence we would be "far out of bounds", as in the proof we use an induction argument, and here we have no control at all over the coefficients that appear in an equation (1.15) in the induction hypothesis. In fact, these coefficients are the main troublemakers in our proof.

The method we apply is as follows. Let S consist of the set $M_\infty(K)$ of archimedean primes of K together with the finite primes corresponding to the prime ideals in the decomposition of the (α_i). Thus we have

$$|S| \leqslant d + \omega. \tag{1.21}$$

(2) Cf. the footnote on the previous page.

Equation (1.15) is almost an S-unit equation, as it will turn out that hypothesis (1.17) guarantees that the powers α_i^m strongly dominate the polynomials $f_i(m)$. This follows from a theorem of Dobrowolski [2]. However to get in this context *uniform* estimates, we can only compare the powers α_i^m with monomials m^l with coefficient 1. So if we write equation (1.15) as

$$a_1x_1+...+a_qx_q=0 \tag{1.22}$$

where the x_i are understood as being terms of the type $m^l\alpha_i^m$, and where the a_i correspond to the coefficients of the polynomials in (1.15), we may try to apply the method that was successful in counting solutions of S-unit equations of the shape (1.22). These were recently treated by Schlickewei [9], [10]. To count the solutions, in [9] as well as in [10] large solutions x_i of (1.22) are covered by the quantitative $\mathfrak{p}$-adic Subspace Theorem (Schlickewei [7], [8]) which in turn generalizes W. Schmidt's pioneering result [11].

There remain solutions that are small as compared with the height of the coefficients a_i. In [9] and [10], these are treated with a gap principle and so finally one gets a result that is uniform in the a_i. I was not able to establish a gap principle for (1.22) in the case where the x_i are no more S-units but are of the shape $m^l\alpha_i^m$ with monomial factors m^l of positive degree.

For such equations I can give a counting argument only in the special case $a_1=...=a_q=1$. (In fact, this is done in Theorem 1.4 below.) However, there is a device to deal with this situation. We just have to take q solutions $\mathbf{x}_1,...,\mathbf{x}_q$ of (1.22). Their determinant will be zero and this is an equation we can handle. With this respect the author owes credit to Evertse, Győry, Stewart and Tijdeman [4]. The determinant argument was picked up from this paper.

The equation $\det(\mathbf{x}_1,...,\mathbf{x}_q)=0$ then will be treated with the quantitative $\mathfrak{p}$-adic Subspace Theorem. The main problem consists in finding a way back from this equation in $\mathbf{x}_1,...,\mathbf{x}_q$ to a relation in a *single* vector $\mathbf{x}$. This is done in Lemma 4.1 (Sections 4 and 5).

It should be pointed out that the method developed here certainly can also be applied to S-unit equations. So, we have now an argument to count the number of solutions of S-unit equations that does not need a gap principle for the small solutions. The price we have to pay, however, is that we get bounds for (1.22) that involve the parameter $q!$ instead of the q we obtain with the gap principle (cf. the bounds in (1.12) and in (1.14)).

Nevertheless, the main burden of the proof has to be carried by the quantitative $\mathfrak{p}$-adic version of W. Schmidt's Subspace Theorem. In that context we give a result that is crucial in our proof and that might be of independent interest.

We denote by $M(K)$ the set of places of K. For $v\in M(K)$ let $|\ |_v$ be the associated absolute value, normalized such that on $\mathbf{Q}$ we have $|\ |_v=|\ |$ (standard absolute value) if v is archimedean, whereas for v nonarchimedean $|p|_v = p^{-1}$ if v lies above the rational prime p. Writing d_v for the local degree $[K_v : \mathbf{Q}_v]$ we put $\| \ \|_v = | \ |_v^{d_v/d}$.

Let S be a finite subset of $M(K)$ containing the set of infinite places $M_\infty(K)$. We call an element $x \in K$ an S-integer if

$$\|x\|_v \leqslant 1 \quad \text{for each } v \notin S. \tag{1.23}$$

Given $\mathbf{x} = (x_1, \ldots, x_{l+1}) \in K^{l+1}$ we define for $v \in M(K)$

$$|\mathbf{x}|_v = \begin{cases} (|x_1|_v^2 + \ldots + |x_{l+1}|_v^2)^{1/2} & \text{if } v \text{ is archimedean} \\ \max_{1 \leqslant i \leqslant l+1} |x_i|_v & \text{if } v \text{ is nonarchimedean} \end{cases}$$

and put $\|\mathbf{x}\|_v = |\mathbf{x}|_v^{d_v/d}$. We define the height as

$$H(\mathbf{x}) = \prod_{v \in M(K)} \|\mathbf{x}\|_v \tag{1.24}$$

and the S-height as

$$H_S(\mathbf{x}) = \prod_{v \in S} \|\mathbf{x}\|_v. \tag{1.25}$$

We are interested in solutions in S-integers of the equation

$$x_1 + \ldots + x_{l+1} = 0. \tag{1.26}$$

THEOREM 1.4. *Suppose $l \geqslant 2$. Assume that S has cardinality s. Let $\delta > 0$. Then the set of solutions of equation* (1.26) *in S-integers $x_1, \ldots, x_{l+1}$ satisfying*

$$\prod_{v \in S} \|x_1\|_v \cdots \|x_{l+1}\|_v < H_S(\mathbf{x})^{1-\delta} \tag{1.27}$$

is contained in the union of not more than

$$t_0 = 2(l+1)^s \left[(4sd!)^{2^{37ld!} s^6 \delta^{-2}} \right] \tag{1.28}$$

proper subspaces $V_1, \ldots, V_{t_0}$ of the l-dimensional linear space V defined by equation (1.26).

I wrote this paper in fall 1989, when I was visiting the department of mathematics of the University of Colorado. I would like to thank the department, and in particular Pat and Wolfgang Schmidt for their hospitality. It made my stay in Boulder most enjoyable.

2. The quantitative Subspace Theorem

As indicated in the introduction, the basic tool in proving our results is the quantitative Subspace Theorem in diophantine approximations. If $L(\mathbf{x})=L(x_1,\dots,x_l)$ is a linear form with coefficient vector $\boldsymbol{\alpha}\in K^l$ we define

$$\|L\|_v=\|\boldsymbol{\alpha}\|_v \quad \text{and} \quad H(L)=H(\boldsymbol{\alpha}).$$

The following lemma is the main theorem of [8], except that in [8] we require K to be a normal extension of $\mathbf{Q}$. In [10] (Corollary 2.2) we derived a slightly more general version, where K is allowed to be an arbitrary number field. For the convenience of the reader this will be quoted now.

LEMMA 2.1 (Quantitative Subspace Theorem [10]). *Let K be a number field of degree d. Let S be a finite subset of $M(K)$ of cardinality s. Suppose that for each $v\in S$ we are given $l\geqslant 2$ linearly independent linear forms $L_1^{(v)},\dots,L_l^{(v)}$ in l variables with coefficients in K. Let $0<\delta<1$. Consider the inequality*

$$\prod_{v\in S}\prod_{i=1}^{l}\frac{\|L_i^{(v)}(\boldsymbol{\beta})\|_v}{\|L_i^{(v)}\|_v\|\boldsymbol{\beta}\|_v}<H(\boldsymbol{\beta})^{-l-\delta}. \tag{2.1}$$

There exist proper subspaces $T_1,\dots,T_{t_1}$ of K^l with

$$t_1=\left[(4sd!)^{2^{35ld!}s^6\delta^{-2}}\right] \tag{2.2}$$

such that every solution $\boldsymbol{\beta}\in K^l$ of (2.1) either lies in the union $\bigcup_{i=1}^{t_1}T_i$ or satisfies

$$H(\boldsymbol{\beta})<\max\{(l!)^{9/\delta},H(L_i^{(v)})^{9ls(d!)^2/\delta}\ (v\in S;i=1,\dots,l)\}. \tag{2.3}$$

Let U be the l dimensional subspace of K^{l+1} defined by the equation

$$x_1+x_2+\dots+x_{l+1}=0. \tag{2.4}$$

To study equation (2.4), we apply Lemma 2.1 and obtain

LEMMA 2.2. *Let $S\subset M(K)$ be as in Theorem 1.4. Let $0<\delta\leqslant 1$. There exist proper subspaces $U_1,\dots,U_{t_2}$ of U with*

$$t_2=(l+1)^s\left[(4sd!)^{2^{35ld!}s^6 4\delta^{-2}}\right] \tag{2.5}$$

with the following property. Every solution $\mathbf{x}=(x_1,\dots,x_{l+1})\in K^{l+1}$ of (2.4) satisfying

$$\prod_{v\in S}\|x_1\|_v\dots\|x_{l+1}\|_v\leqslant H_S(\mathbf{x})^{1-\delta}, \tag{2.6}$$

$$x_i \text{ is an } S\text{-integer for each } i,\ 1 \leqslant i \leqslant l+1, \tag{2.7}$$

either *is contained in the union* $\bigcup_{i=1}^{t_2} T_i$ or *has*

$$H(\mathbf{x}) \leqslant l^{18ls(d!)^2/\delta}. \tag{2.8}$$

Proof. We consider the linear forms $L_1(\mathbf{x}')=x_1, \ldots, L_l(\mathbf{x}')=x_l$, $L_{l+1}(\mathbf{x}')=x_1+\ldots+x_l$ in $\mathbf{x}'=(x_1, \ldots, x_l)$. Notice that any solution of (2.4) satisfies

$$L_{l+1}(\mathbf{x}') = -x_{l+1}.$$

In view of (2.6) we obtain

$$\prod_{v\in S} \|L_1(\mathbf{x}')\|_v \ldots \|L_{l+1}(\mathbf{x}')\|_v \leqslant H_S(\mathbf{x})^{1-\delta}. \tag{2.9}$$

We now divide the solutions $\mathbf{x}$ of (2.4) into classes. Given $v\in S$ and $\mathbf{x}$ with (2.4) define i_v by

$$\|x_{i_v}\|_v = \max_{1\leqslant i\leqslant l+1} \{\|x_i\|_v\}. \tag{2.10}$$

Let $\mathfrak{C}=\mathfrak{C}((i_v)_{v\in s})$ be the set of solutions $\mathbf{x}$ giving rise to the tuple $(i_v)_{v\in S}$. There are not more than

$$(l+1)^s \tag{2.11}$$

classes $\mathfrak{C}$.

We restrict ourselves to solutions $\mathbf{x}$ in a fixed class $\mathfrak{C}$. Write $I_v=\{1, \ldots, l+1\}\setminus\{i_v\}$ $(v\in S)$. Then the definition of our forms L_i and (2.9) imply

$$\prod_{v\in S} \prod_{i\in I_v} \frac{\|L_i(\mathbf{x}')\|_v}{\|\mathbf{x}\|_v} \leqslant H_S(\mathbf{x})^{-l-\delta}. \tag{2.12}$$

Notice that in view of (2.4) we have

$$\|\mathbf{x}\|_v = \|\mathbf{x}'\|_v \quad \text{for } v \text{ nonarchimedean} \tag{2.13}$$

and

$$\|\mathbf{x}'\|_v \leqslant \|\mathbf{x}\|_v \leqslant (2l)^{d_v/2d} \|\mathbf{x}'\|_v \quad \text{for } v \text{ archimedean.} \tag{2.14}$$

For our forms L_i we obtain

$$\|L_i\|_v = 1, \quad i = 1, \ldots, l,\ v \in S \tag{2.15}$$

whereas

$$\|L_{l+1}\|_v = 1 \quad \text{for } v \in S,\ v \text{ nonarchimedean} \tag{2.16}$$

$$\|L_{l+1}\|_v = l^{d_v/2d} \quad \text{for } v \text{ archimedean.} \tag{2.17}$$

Combining (2.12)–(2.17) we see that any solution $\mathbf{x}$ in our class $\mathfrak{C}$ satisfies

$$\prod_{v\in S}\prod_{i\in I_v}\frac{\|L_i(\mathbf{x}')\|_v}{\|L_i\|_v\|\mathbf{x}'\|_v} \leqslant (2l)^{l/2} H_S(\mathbf{x}')^{-l-\delta}. \tag{2.18}$$

On the other hand (2.15), (2.16), (2.17) imply that

$$\max_{1\leqslant i\leqslant l+1} H(L_i) = l^{1/2}. \tag{2.19}$$

If we assume that

$$H(\mathbf{x}')^{\delta/2} > (2l)^{l/2} \tag{2.20}$$

we get from (2.18)

$$\prod_{v\in S}\prod_{i\in I_v}\frac{\|L_i(\mathbf{x}')\|_v}{\|L_i\|_v\|\mathbf{x}'\|_v} < H_S(\mathbf{x}')^{-l-\delta/2}. \tag{2.21}$$

Since our points $\mathbf{x}'$ have S-integers as components, we have $H_S(\mathbf{x}')\geqslant H(\mathbf{x}')$. Thus we may apply Lemma 2.1. In conjunction with (2.19), (2.20) we may conclude that there are

$$t_3 = \left[(4sd!)^{2^{35ld!}s^6 4\delta^{-2}}\right]$$

proper subspaces $T_1, \dots, T_{t_3}$ of K^l containing the solutions $\mathbf{x}'$ of (2.21) with

$$H(\mathbf{x}') > \max\{(2l)^{l/2}, (l!)^{18/\delta}, l^{18ls(d!)^2/\delta}\} = l^{18ls(d!)^2/\delta}. \tag{2.22}$$

The subspaces $T_1, \dots, T_{t_3}$ yield proper subspaces $U_1, \dots, U_{t_3}$ of the solution space U. Allowing the factor $(l+1)^s$ from (2.11) for the number of classes $\mathfrak{C}$, the assertion follows.

LEMMA 2.3. *Let K be a number field of degree d. Let $D\geqslant 1$. Then the number of one-dimensional subspaces of K having a basis vector $\mathbf{x}$ which satisfies*

$$H(\mathbf{x}) \leqslant D \tag{2.23}$$

is bounded by

$$2^{(2d+7)(l-1)} D^{2d(l-1)}. \tag{2.24}$$

This is essentially Lemma 5.1 of [10]. It is proved there using Lemma 8B of Chapter 1 of W. Schmidt [12].

Proof of Theorem 1.4. We combine Lemma 2.2 with Lemma 2.3. By Lemma 2.3 the solutions of (2.4), (2.6), (2.7) with $H(\mathbf{x})\leqslant l^{18ls(d!)^2/\delta}$ are contained in the union of not more than

$$2^{(2d+7)(l-1)} l^{36l^2 ds(d!)^2/\delta}$$

proper subspaces of U. Using the bound (2.5) for the number of subspaces covering the large solutions, we see that every solution of (2.4), (2.6), (2.7) is contained in the union of not more than

$$(l+1)^s\left[(4sd!)^{2^{35ld!}s^6 4\delta^{-2}}\right]+2^{(2d+7)(l-1)}l^{36l^2ds(d!)^2/\delta}$$

proper subspaces of U. But this is smaller than

$$2(l+1)^s\left[(4sd!)^{2^{37ld!}s^6\delta^{-2}}\right]=t_0.$$

3. On heights

LEMMA 3.1. *Let K be a number field of degree $D>1$. Suppose that $\alpha\in K^*$ is not a root of unity. Then*

$$H(1,\alpha)>\left(1+\frac{1}{1200}\left(\frac{\log\log D}{\log D}\right)^3\right)^{1/D}. \tag{3.1}$$

This is the main result of Dobrowolski [2].

LEMMA 3.2. *Let α be an algebraic number of degree d. Assume that α is not a root of unity. Then*

$$H(1,\alpha)>(1+2^{-14}d^{-1})^{1/10d}. \tag{3.2}$$

Proof. This follows at once from Lemma 3.1, upon noting that for $\alpha\in\mathbf{Q}$ we have $H(1,\alpha)\geqslant 2$ and upon taking $D=10d$ otherwise.

LEMMA 3.3. *Let α be an algebraic number of degree $\leqslant d$ that is not a root of unity. Let m_1,m_2,m_3,k be integers with $m_1\cdot m_2\cdot m_3\neq 0$. Then*

$$H\left(1,\left(\frac{m_1}{m_2}\right)^k\alpha^{m_3}\right)\geqslant(1+2^{-14}d^{-1})^{|m_3|/10d}|m_1|^{-|k|}|m_2|^{-|k|}. \tag{3.3}$$

Proof. Let T be the subset of $M(K)$ such that $\|\alpha\|_v>1$ for $v\in T$. Then we have

$$\begin{aligned}H\left(1,\left(\frac{m_1}{m_2}\right)^k\alpha^{m_3}\right)&\geqslant\prod_{v\in T}\left\|\left(\frac{m_1}{m_2}\right)^k\alpha^{m_3}\right\|_v=\prod_{v\in T}\left\|\left(\frac{m_1}{m_2}\right)^k\right\|_v H(1,\alpha^{m_3})\\&=\prod_{v\in T}\left\|\left(\frac{m_1}{m_2}\right)^k\right\|_v H(1,\alpha^{-m_3})\\&=\prod_{v\in T}\left\|\left(\frac{m_1}{m_2}\right)^k\right\|_v (H(1,\alpha))^{|m_3|}.\end{aligned} \tag{3.4}$$

However, for any subset R of $M(K)$ we get

$$\prod_{v\in R}\left\|\left(\frac{m_1}{m_2}\right)^k\right\|_v \leqslant \prod_{v\in M(K)}\max\{1,\|m_1^k\|_v\}\cdot\prod_{v\in M(K)}\max\{1,\|m_2^k\|_v\} = |m_1|^{|k|}\cdot|m_2|^{|k|}.$$

Thus

$$\prod_{v\in T}\left\|\left(\frac{m_1}{m_2}\right)^k\right\|_v^{-1} = \prod_{v\in m(K)\setminus T}\left\|\left(\frac{m_1}{m_2}\right)^k\right\|_v \leqslant |m_1m_2|^{|k|}$$

and we infer from (3.4) that

$$H\left(1,\left(\frac{m_1}{m_2}\right)^k\alpha^{m_3}\right) \geqslant |m_1m_2|^{-|k|}(H(1,\alpha))^{|m_3|}. \tag{3.5}$$

Using Lemma 3.2, we obtain the assertion.

Let $r>1$ be a natural number. *Suppose we are given nonzero numbers $\alpha_1,\ldots,\alpha_r$ in a number field K of degree d, such that for at least one pair i,j with $i\neq j$ $(1\leqslant i,j\leqslant r)$ α_i/α_j is not a root of unity.* Let $\varrho_1,\ldots,\varrho_r$ be natural numbers with

$$\varrho_1+\ldots+\varrho_r = k+r. \tag{3.6}$$

Suppose that for each i $(1\leqslant i\leqslant r)$ we have a sequence of σ_i nonnegative integers $k_{i1},\ldots,k_{i\sigma_i}$ with

$$0\leqslant k_{i1} < k_{i2} < \ldots < k_{i\sigma_i} = \varrho_i - 1. \tag{3.7}$$

Given an integer m, define the vector $\mathbf{x}^{(m)}$ by

$$\mathbf{x}^{(m)} = \left(m^{k_{11}}\alpha_1^m,\ldots,m^{k_{1\sigma_1}}\alpha_1^m,\ldots,m^{k_{r1}}\alpha_r^m,\ldots,m^{k_{r\sigma_r}}\alpha_r^m\right). \tag{3.8}$$

Thus $\mathbf{x}^{(m)}$ lies in $(\sigma_1+\ldots+\sigma_r)$-dimensional space. Write

$$q = \sigma_1+\ldots+\sigma_r. \tag{3.9}$$

Then (3.6) and (3.7) imply

$$q\leqslant k+r. \tag{3.10}$$

Consider a sequence of integers $m_1,\ldots,m_{q-1}$, m having

$$m_i\neq 0,\ i=1,\ldots,q-1,\quad m_1<0,\quad m>0,\quad m_1\leqslant\ldots\leqslant m_{q-1}\leqslant m. \tag{3.11}$$

Put

$$q! = l. \tag{3.12}$$

Let $\mathbf{z}$ be the vector in l-dimensional space whose components are the summands in the Laplace expansion of the $q\times q$-determinant with rows $\mathbf{x}^{(m_1)},\ldots,\mathbf{x}^{(m_{q-1})},\mathbf{x}^{(m)}$.

LEMMA 3.4. *Suppose $r>1$. Assume that* (3.11) *is satisfied. Write*

$$M=\max\{|m_1|,|m|\}. \tag{3.13}$$

Then we have

$$H(\mathbf{z})\geqslant M^{-4k}(1+2^{-14}d^{-1})^{M/10d}. \tag{3.14}$$

Proof. We assume without loss of generality that α_1/α_r is not a root of unity. Writing $\mathbf{z}=(z_1,...,z_l)$ we get upon choosing any two components z_i, z_j of $\mathbf{z}$ with $i\neq j$

$$H(\mathbf{z})\geqslant H(z_i,z_j)=H(1,z_j/z_i).$$

Let z_i be the product of the elements in the main diagonal of the matrix with rows $\mathbf{x}^{(m_1)},...,\mathbf{x}^{(m_{q-1})},\mathbf{x}^{(m)}$. For z_j we choose the product of the elements in the main diagonal from row 2 down to row $q-1$ multiplied with the element $m^{k_{11}}\alpha_1^m$ in the lower left corner and the element $m_1^{k_{r\sigma_r}}\alpha_r^{m_1}$ in the top right corner. Then we get

$$\frac{z_j}{z_i}=\left(\frac{\alpha_1}{\alpha_r}\right)^{m_1-m}\cdot\left(\frac{m_1}{m}\right)^{k_{11}-k_{r\sigma_r}}.$$

We infer from (3.6) and (3.7) that $|k_{11}-k_{r\sigma_r}|\leqslant 2k$. Moreover (3.11) and (3.13) imply $|m_1-m|\geqslant M$. The assertion now follows from Lemma 3.3 applied to the vector $(1,z_j/z_i)$.

LEMMA 3.5. *Suppose $r>1$. Let $\mathbf{z}$ be defined as above with integers $m_1,...,m_{q-1},m$ satisfying* (3.11). *Let S be the union of the set of infinite primes of K with the set of finite primes dividing at least one of $\alpha_1,...,\alpha_r$. Suppose that*

$$m>2^{25}d^3k^3l^2. \tag{3.15}$$

Then we have

$$\prod_{v\in S}\|z_1\|_v\cdots\|z_l\|_v<H(\mathbf{z})^{1/2}.$$

Proof. Using (3.6)–(3.11) and the definition of $\mathbf{z}$ we obtain

$$\prod_{v\in S}\|z_1\|_v\cdots\|z_l\|_v\leqslant M^{k^2l},$$

where M is defined in (3.13). On the other hand, Lemma 3.4 implies that

$$H(\mathbf{z})^{1/2}\geqslant M^{-2k}(1+2^{-14}d^{-1})^{M/20d}.$$

Therefore (3.16) is satisfied if

$$\frac{M}{20d}\log(1+2^{-14}d^{-1})>2k^2l\log M.$$

This in turn holds certainly true if

$$m>2^{20}d^2k^2l\log m$$

and so in particular if we have (3.15), and the lemma follows.

4. Subspaces again

We will now apply the facts proved in Sections 2 and 3 to the equation

$$\sum_{i=1}^{r} f_i(m)\alpha_i^m = 0. \tag{4.1}$$

We assume here that

$$r > 1. \tag{4.2}$$

Throughout this section, in contrast with (1.17), we will only suppose that *there exists a pair i, j with $i \neq j$ such that α_i/α_j is not a root of unity. We assume moreover throughout this section that the number $N(0)$ of solutions $m \in \mathbf{Z}$ of (4.1) is finite.*

To derive our upper bound (1.20) for $N(0)$, we may assume that $N(0) \geqslant 5$. Then, there exists an integer m_0 with the following property.

$$\begin{array}{l}\text{The set of solutions } m < m_0 \text{ of (4.1) has cardinality } \geqslant \tfrac{1}{3}N(0), \text{ and} \\ \text{the set of solutions } m > m_0 \text{ of (4.1) also has cardinality } \geqslant \tfrac{1}{3}N(0).\end{array} \tag{4.3}$$

We may rewrite (4.1) as

$$\sum_{i=1}^{r} f_i(m+m_0)\alpha_i^{m+m_0} = 0.$$

Therefore, putting $g_i(m) = \alpha_i^{m_0} f_i(m+m_0)$ we get an equation

$$\sum_{i=1}^{r} g_i(m)\alpha_i^m = 0 \tag{4.4}$$

that is of the same shape as (4.1). However, writing $N'(0)$ for the number of solutions $m \in \mathbf{Z}$ of (4.4) we clearly have $N'(0) = N(0)$, but (4.3) is now replaced by

$$\begin{array}{l}\text{the set of solutions } m < 0 \text{ of (4.4) has cardinality } \geqslant \tfrac{1}{3}N'(0), \text{ and} \\ \text{the set of solutions } m > 0 \text{ of (4.4) also has cardinality } \geqslant \tfrac{1}{3}N'(0).\end{array}$$

So it will suffice to study equation (4.4), where we have $m_0 = 0$. In detail (4.4) reads as

$$b_{11}m^{k_{11}}\alpha_1^m + \ldots + b_{1\sigma_1}m^{k_{1\sigma_1}}\alpha_1^m + \ldots + b_{r1}m^{k_{r1}}\alpha_r^m + \ldots + b_{r\sigma_r}m^{k_{r\sigma_r}}\alpha_r^m = 0, \tag{4.5}$$

where $b_{11}, \ldots, b_{1\sigma_1}, \ldots, b_{r1}, \ldots, b_{r\sigma_r}$ are certain nonzero numbers in K and where the σ_i and the k_{ij} are as in (3.7). We treat the equation (4.5) as an equation in quasi S-units, where S is defined in Lemma 3.5. In view of our reduction (4.3), it will suffice to estimate either the cardinality of the set of solutions with $m<0$ or the cardinality of the set of solutions with $m>0$.

Let $\mathbf{x}^{(m)}=(m^{k_{11}}\alpha_1^m,\dots,m^{k_{1\sigma_1}}\alpha_1^m,\dots,m^{k_{r1}}\alpha_r^m,\dots,m^{k_{r\sigma_r}}\alpha_r^m)$ be the solution vector of (4.5) corresponding to m. Recall the definition of q in (3.9). The solutions $\mathbf{x}^{(m)}$ of (4.5) are contained in a subspace V of K^q of dimension $q-1$. Therefore any q solutions $\mathbf{x}^{(m_1)},\dots,\mathbf{x}^{(m_{q-1})},\mathbf{x}^{(m)}$ of (4.5) are linearly dependent. Given $\mathbf{x}^{(m_1)},\dots,x^{(m_{q-1})},\mathbf{x}^{(m)}$ define $\mathbf{z}=(z_1,\dots,z_l)$ as in Section 3. Then we get

$$z_1+\dots+z_l=0. \tag{4.6}$$

Let

$$M_0=M_0(\alpha_1,\dots,\alpha_r)=\text{ set of prime ideals in } K \text{ occurring in the decomposition of any of the ideals } (\alpha_i),\ i=1,\dots,r. \tag{4.7}$$

Put

$$S=M_\infty(K)\cup M_0. \tag{4.8}$$

If K has r_1 real embeddings and r_2 pairs of complex embeddings, then $|M_\infty(K)|=r_1+r_2$. Consequently we get

$$|S|=r_1+r_2+\omega, \tag{4.9}$$

where $\omega=\omega(\alpha_1,\dots,\alpha_r)$ is the cardinality of the set M_0 defined in (4.7). If we study equation (4.6) with S as in (4.8) and ask for solutions $z_1,\dots,z_l$ satisfying the analogue of condition (1.27) in Theorem 1.4 with $\delta=\frac{1}{2}$, with $l+1$ replaced by l and l having the value $l=q!$, we see that the number of such solutions is bounded by

$$t_4=2(q!)^{\omega+r_1+r_2}(4(r_1+r_2+\omega)d!)^{2^{37q!\,d!}4(r_1+r_2+\omega)^6}. \tag{4.10}$$

LEMMA 4.1. *Suppose $r\geqslant 2$ and $q\geqslant 3$. There exist*

$$t_5=qt_4+2^{25}d^3k^3l^2 \tag{4.11}$$

nonzero vectors $(d_{11}^{(j)},\dots,d_{1\sigma_1}^{(j)},\dots,d_{r1}^{(j)},\dots,d_{r\sigma_r}^{(j)})\in K^q$ $(1\leqslant j\leqslant t_5)$, each not proportional to the coefficient vector $(b_{11},\dots,b_{1\sigma_1},\dots,b_{r1},\dots,b_{r\sigma_r})$ in (4.5) with the following property. Either each solution $\mathbf{x}^{(m)}$ of (4.5) with $m<0$ or each solution $\mathbf{x}^{(m)}$ of (4.5) with $m>0$ satisfies at least one of the equations

$$d_{11}^{(i)}m^{k_{11}}\alpha_1^m+\dots+d_{1\sigma_1}^{(i)}m^{k_{1\sigma_1}}\alpha_1^m+\dots+d_{r1}^{(i)}m^{k_{r1}}\alpha_r^m+\dots+d_{r\sigma_r}^{(i)}m^{k_{r\sigma_r}}\alpha_r^m=0,\quad i=1,\dots,t_5. \tag{4.12}$$

Proof. To simplify the notation we write (4.5) as

$$a_1x_1+\dots+a_qx_q=0, \tag{4.13}$$

where $a_1\cdot\ldots\cdot a_q\neq 0$. Given q solutions $\mathbf{x}^{(m_1)},\ldots,\mathbf{x}^{(m_{q-1})},\mathbf{x}^{(m)}$ of (4.13) it is clear that we have

$$\begin{vmatrix} x^{(m_1)} & \ldots & x_q^{(m_1)} \\ \vdots & & \vdots \\ x_1^{(m_{q-1})} & \ldots & x_q^{(m_{q-1})} \\ x_1^{(m)} & \ldots & x_q^{(m)} \end{vmatrix} = 0.$$

We expand this determinant with respect to the last row and get

$$x_1^{(m)}\begin{vmatrix} x_2^{(m_1)} & \ldots & x_q^{(m_1)} \\ \vdots & & \vdots \\ x_2^{(m_{q-1})} & \ldots & x_q^{(m_{q-1})} \end{vmatrix} - + \ldots + (-1)^{q-1}x_q^{(m)}\begin{vmatrix} x_1^{(m_1)} & \ldots & x_{q-1}^{(m_1)} \\ \vdots & & \vdots \\ x_1^{(m_{q-1})} & \ldots & x_{q-1}^{(m_{q-1})} \end{vmatrix} = 0. \tag{4.14}$$

The idea of proof is now as follows. Suppose we have fixed solutions $\mathbf{x}^{(m_1)},\ldots,\mathbf{x}^{(m_{q-1})}$ of (4.5). Expanding the determinant factors in (4.14) we obtain an equation

$$\sum_{i=1}^{q} x_i^{(m)} \sum_{\tau_i} \pm x_{\tau_i(1)}^{(m_{q-1})} \ldots x_{\tau_i(q-1)}^{(m_1)} = 0, \tag{4.15}$$

where, for each i $(1\leqslant i\leqslant q)$, τ_i runs through the bijections between $\{1,\ldots,q-1\}$ and $\{1,\ldots,q\}\setminus\{i\}$. Assume that our superscripts $m_1,\ldots,m_{q-1},m$ satisfy conditions (3.11), (3.15) of Lemma 3.5. We will see later on that then all such $\mathbf{x}^{(m_1)},\ldots,\mathbf{x}^{(m_{q-1})},\mathbf{x}^{(m)}$ are also solutions of at least one equation out of a fixed set of not more than t_4 equations of type

$$\sum_{i=1}^{q} x_i^{(m)} \sum_{\tau_i} d_{i,\tau_i(1),\ldots,\tau_i(q-1)} x_{\tau_i(1)}^{(m_{q-1})} \ldots x_{\tau_i(q-1)}^{(m_1)} = 0, \quad 1\leqslant j\leqslant t_4. \tag{4.16}$$

Our goal is to prove, that we may pick $\mathbf{x}^{(m_1)},\ldots,\mathbf{x}^{(m_{q-1})}$ such that for each j $(1\leqslant j\leqslant t_4)$ the coefficient vector $(a_1,\ldots,a_q)$ in (4.13) and the coefficient vector

$$\left(\sum_{\tau_1} d^{(j)}_{1,\tau_1(1),\ldots,\tau_1(q-1)} x_{\tau_1(1)}^{(m_{q-1})} \ldots x_{\tau_1(q-1)}^{(m_1)}, \ldots, \sum_{\tau_q} d^{(j)}_{q,\tau_q(1),\ldots,\tau_q(q-1)} x_{\tau_q(1)}^{(m_{q-1})} \ldots x_{\tau_q(q-1)}^{(m_1)}\right)$$

in 4.16 are linearly independent. Actually there may occur situations, where we cannot find such $\mathbf{x}^{(m_1)},\ldots,\mathbf{x}^{(m_{q-1})}$. This is the reason, why the assertion of the lemma is slightly more complicated.

We start by considering the equation

$$z_1+\ldots+z_l=0 \tag{4.17}$$

with $l=q!$ and study solutions of (4.17) that satisfy the hypotheses of Theorem 1.4 with $\delta=\frac{1}{2}$. Theorem 1.4 says that there exist t_4 vectors $(d_1^{(j)},\ldots,d_l^{(j)})$ $(1\leqslant j\leqslant t_4)$ each of which is linearly independent of the coefficient vector $(1, \dots, 1)$ in (4.17) such that any solution of (4.17) under consideration has for at least one j $(1 \leqslant j \leqslant t_4)$

$$d_1^{(j)} z_1 + \dots + d_l^{(j)} z_l = 0. \tag{4.18}$$

It is clear that we may suppose here moreover that for each j $(1 \leqslant j \leqslant t_4)$ there exists a λ with

$$1 \leqslant \lambda \leqslant l \quad \text{having} \quad d_\lambda^{(j)} = 0.$$

We fix the vectors $(d_1^{(j)}, \dots, d_l^{(j)})$ once and forever and relabel them as $(d_{i_1, \dots, i_q}^{(j)})$ where the subscripts $i_1, \dots, i_q$ run through the different permutations of $1, \dots, q$. Before we start studying whether we can find appropriate vectors $\mathbf{x}^{(m_1)}, \dots, \mathbf{x}^{(m_{q-1})}$ such that (4.16) and (4.13) are independent equations, we need some further notation. We denote by $(d_{i_1, \dots, i_q}^{(0)})$ an arbitrary fixed vector among $(d_{i_1, \dots, i_q}^{(j)})$ $(1 \leqslant j \leqslant t_4)$. Let $\mathbf{x}^{(m_1^{(0)})}, \dots, \mathbf{x}^{(m_{q-1}^{(0)})}$ be solutions of (4.5) with $m_1^{(0)}, \dots, m_{q-1}^{(0)}$ to be specified later. Given i with $1 \leqslant i \leqslant q$ we introduce the operator $D_i^{(0)}$ by writing

$$\begin{aligned}
&\sum_{\tau_i} d^{(0)}_{i, \tau_i(1), \dots, \tau_i(q-1)} x_{\tau_i(1)}^{(m_{q-1}^{(0)})} \cdots x_{\tau_i(q-1)}^{(m_1^{(0)})} \\
&= D_i^{(0)} \begin{pmatrix} x_1^{(1)} & \cdots & x_{i-1}^{(1)} & x_{i+1}^{(1)} & \cdots & x_q^{(1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_1^{(q-1)} & \cdots & x_{i-1}^{(q-1)} & x_{i+1}^{(q-1)} & \cdots & x_q^{(q-1)} \end{pmatrix} \\
&= D_i^{(0)} \begin{pmatrix} x_1^{(1)} & \cdots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-1)} & \cdots & x_q^{(q-1)} \end{pmatrix}_i ,
\end{aligned} \tag{4.19}$$

where for the sake of simplicity, in the matrices we have used the superscript (j) instead of $(m_j^{(0)})$ $(1 \leqslant j \leqslant q-1)$. The subscript i at the last matrix in (4.19) indicates that the ith column is deleted. Moreover, given i with $1 \leqslant i \leqslant q$ and j with $j \neq i$, $1 \leqslant j \leqslant q$ we write τ_{ij} for a bijection between $\{1, \dots, q-2\}$ and $\{1, \dots, q\} \backslash \{i, j\}$. We define the operator $D_{i,j}^{(0)}$ by

$$\begin{aligned}
&\sum_{\tau_{ij}} d^{(0)}_{i, j, \tau_{ij}(1), \dots, \tau_{ij}(q-2)} x_{\tau_{ij}(1)}^{(m_{q-2}^{(0)})} \cdots x_{\tau_{ij}(q-2)}^{(m_1^{(0)})} \\
&= D_{i,j}^{(0)} \begin{pmatrix} x_1^{(1)} & \cdots & x_{j-1}^{(1)} & x_{j+1}^{(1)} & \cdots & x_{i-1}^{(1)} & x_{i+1}^{(1)} & \cdots & x_q^{(1)} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_1^{(q-2)} & \cdots & x_{j-1}^{(q-2)} & x_{j+1}^{(q-2)} & \cdots & x_{i-1}^{(q-2)} & x_{i+1}^{(q-2)} & \cdots & x_q^{(q-2)} \end{pmatrix} \\
&= D_{i,j}^{(0)} \begin{pmatrix} x_1^{(1)} & \cdots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-2)} & \cdots & x_q^{(q-2)} \end{pmatrix}_{i,j} ,
\end{aligned} \tag{4.20}$$

where the summation in (4.20) goes over all bijections between $\{1, ..., q-2\}$ and $\{1, ..., q\} \setminus \{i, j\}$. The subscript i, j at the matrix in (4.20) indicates that the ith and the jth column are deleted. In a similar manner we define for pairwise different numbers $i_1, ..., i_n$ in $\{1, ..., q\}$

$$D^{(0)}_{i_1,\ldots,i_n} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-n)} & \ldots & x_q^{(q-n)} \end{pmatrix}_{i_1,\ldots,i_n} . \tag{4.21}$$

We shall first study equation (4.16) for a single coefficient vector $(d^{(0)}_{i,\tau_i(1),\ldots,\tau_i(q-1)})_{1\leqslant i\leqslant q}$ arising from Theorem 1.4.

Remember that $(d^{(0)}_{i,\tau_i(1),\ldots,\tau_i(q-1)})_{1\leqslant i\leqslant q}$ and the coefficient vector $(1, ..., 1)$ in (4.17) are linearly independent. Therefore we may suppose without loss of generality that either

(i) *there exists an* i $(1\leqslant i\leqslant q)$ *such that* $d^{(0)}_{i,\tau_i(1),\ldots,\tau_i(q-1)}=0$ *for all bijections* τ_i *from* $\{1, ..., q-1\}$ *onto* $\{1, ..., q\}\setminus\{i\}$, *but* $(d^{(0)}_{i,\tau_i(1),\ldots,\tau_i(q-1)})_{1\leqslant i\leqslant q}\neq(0, ..., 0)$,

or

(ii) *for all* i $(1\leqslant i\leqslant q)$ *there exists* τ_i *such that* $d^{(0)}_{i,\tau_i(1),\ldots,\tau_i(q-1)}\neq 0$, *but there exists an* i_0 $(1\leqslant i_0\leqslant q)$ *and* τ_{i_0} *having* $d_{i_0,\tau_{i_0}(1),\ldots,\tau_{i_0}(q-1)}=0$.

We first treat alternative (i). This will be also a warm-up for alternative (ii). Assume without loss of generality that

$$d^{(0)}_{q,\tau_q(1),\ldots,\tau_q(q-1)}=0 \quad \text{for all } \tau_q, \tag{4.22}$$

and

$$d^{(0)}_{1,2,3,\ldots,q}\neq 0. \tag{4.23}$$

Let $m_1^{(0)}$ be any nonzero integer such that (4.5) is satisfied and let $\mathbf{x}^{(m_1^{(0)})}$ be the corresponding solution vector. Since $m_1^{(0)}\neq 0$ we have $x_q^{(m_1^{(0)})}\neq 0$. We next want to find a suitable value $m_2^{(0)}$. We distinguish two alternatives:

Either any solution $\mathbf{x}^{(m)}$ of (4.5) with $m\neq 0$, $m\geqslant m_1^{(0)}$ has

$$d^{(0)}_{1,2,3,\ldots,q-2,q-1,q}\, x_q^{(m_1^{(0)})} x_{q-1}^{(m)} + d^{(0)}_{1,2,\ldots,q-2,q,q-1}\, x_{q-1}^{(m_1^{(0)})} x_q^{(m)} = 0. \tag{4.24}$$

Our choice of $m_1^{(0)}$ and (4.23) imply that (4.24) is a nontrivial equation for $\mathbf{x}^{(m)}$ that is certainly independent of (4.13), since $q\geqslant 3$.

Or there exists $m\geqslant m_1^{(0)}$, $m\neq 0$ such that (4.24) does not hold. We may then choose

$$m_2^{(0)}\geqslant m_1^{(0)},\ m_2^{(0)}\neq 0 \quad \text{such that (4.24) does not hold for } m=m_2^{(0)}. \tag{4.25}$$

Now suppose that for $f<q-1$ we could choose nonzero integers $m_1^{(0)}, ..., m_f^{(0)}$ having

$$m_1^{(0)}\leqslant m_2^{(0)}\leqslant \ldots \leqslant m_f^{(0)} \tag{4.26}$$

and

$$D^{(0)}_{1,2,\dots,q-i}\begin{pmatrix} x_1^{(1)} & \dots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(i)} & \dots & x_q^{(i)} \end{pmatrix}_{1,2,\dots,q-i} \neq 0, \quad i=1,\dots,f. \tag{4.27}$$

As for a possible value $m^{(0)}_{f+1}$, we again distinguish two cases.

Either we have

$$\sum_{\psi} d^{(0)}_{1,2,\dots,q-f-1,\psi(1),\dots,\psi(f+1)} x^{(m_f^{(0)})}_{\psi(2)}, \dots, x^{(m_1^{(0)})}_{\psi(f+1)} x^{(m)}_{\psi(1)} = 0 \tag{4.28}$$

for all $m \geqslant m_f^{(0)}$, $m \neq 0$, where ψ runs through the bijections from $\{1,\dots,f+1\}$ onto $\{q-f,\dots,q\}$. (4.28) is an equation in $\mathbf{x}^{(m)}$, that implies only the components $x^{(m)}_{q-f}$, $\dots, x_q^{(m)}$. Since $f<q-1$, in fact not all components of $\mathbf{x}^{(m)}$ do occur. However the coefficient of $x^{(m)}_{q-f}$ in (4.28) is

$$D^{(0)}_{1,2,\dots,q-f}\begin{pmatrix} x_1^{(1)} & \dots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(f)} & \dots & x_q^{(f)} \end{pmatrix}_{1,2,\dots,q-f}.$$

By our choice of $m_1^{(0)},\dots,m_f^{(0)}$ and by (4.27) this coefficient is nonzero. Therfore (4.28) is an equation for $\mathbf{x}^{(m)}$ that is independent of (4.13).

Or there exists $m^{(0)}_{f+1} \geqslant m_f^{(0)}$, $m^{(0)}_{f+1} \neq 0$ such that (4.28) does not hold true for $m = m^{(0)}_{f+1}$. Then we may continue the procedure. It will stop at least if we reach $f=q-1$. In that case, given $m_1^{(0)},\dots,m^{(0)}_{q-1}$, (4.27) will hold true for $i=1,\dots,q-1$. But then we are in a comfortable position: with this choice of $m_1^{(0)},\dots,m^{(0)}_{q-1}$ we look at the corresponding equation (4.16). Here $x_1^{(m)}$ will have a nonzero coefficient, whereas by our assumption in alternative (i) $x_q^{(m)}$ will have coefficient equal to zero and so equation (4.16) will be independent of equation (4.13).

5. Proof of Lemma 4.1 (continued)

After the warmup in Section 4 with alternative (i), we will now treat the more fun alternative (ii). To illustrate the method, it seems to be appropriate to first do things backwards and after that only to start with the construction of parameters $m_i^{(0)}$. In dealing with alternative (ii) we may suppose without loss of generality that

$$d^{(0)}_{1,2,\dots,q-1,q} \neq 0 \quad \text{but} \quad d^{(0)}_{1,2,\dots,q-2,q,q-1} = 0. \tag{5.1}$$

In fact this simply means that on the one hand we assume in alternative (ii) that $i_0=1$, and that on the other hand we reorder, if necessary, the variables with subscripts $2, 3, \ldots, q$.

Using the notation introduced in (4.19)–(4.21), equation (4.16) reads as

$$\sum_{i=1}^{q} x_i^{(m)} D_i^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-1)} & \ldots & x_q^{(q-1)} \end{pmatrix}_i = 0. \tag{5.2}$$

(Here we assume for the time being that solutions $\mathbf{x}^{(m_1^{(0)})}, \ldots, \mathbf{x}^{(m_{q-1}^{(0)})}$ have been chosen.) Now (5.2) is a linear equation in $\mathbf{x}^{(m)}$, and we distinguish two possibilities:

Either equations (5.2) and (4.13) are independent. Then we are in good shape.

Or equations (5.2) and (4.13) are linearly dependent. In that case we obtain, comparing the coefficients of $x_1^{(m)}$ and $x_q^{(m)}$ in (5.2) and (4.13)

$$\pm a_q D_1^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-1)} & \ldots & x_q^{(q-1)} \end{pmatrix}_1 \pm a_1 D_q^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-1)} & \ldots & x_q^{(q-1)} \end{pmatrix}_q = 0. \tag{5.3}$$

Now (5.3) is a linear equation in $\mathbf{x}^{(m_{q-1}^{(0)})}$. Again we distinguish two alternatives:

Either equation (5.3) for $\mathbf{x}^{(m_{q-1}^{(0)})}$ and equation (4.13) are independent—it will turn out, that then we are in good shape.

Or (5.3) and (4.13) are linearly dependent. In this case we compare the coefficients of $x_1^{(m_{q-1}^{(0)})}$ and of $x_2^{(m_{q-1}^{(0)})}$ in (5.3) and in (4.13). The coefficient of $x_1^{(m_{q-1}^{(0)})}$ in (5.3) is

$$\pm a_1 D_{q,1}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-2)} & \ldots & x_q^{(q-2)} \end{pmatrix}_{q1}, \tag{5.4}$$

whereas $x_2^{(m_{q-1}^{(0)})}$ has the coefficient

$$\pm a_q D_{1,2}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-2)} & \ldots & x_q^{(q-2)} \end{pmatrix}_{12} \pm a_1 D_{q,2}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-2)} & \ldots & x_q^{(q-2)} \end{pmatrix}_{q2}. \tag{5.5}$$

Combining (5.4) and (5.5) with (4.13), we get

$$\begin{aligned} &\pm a_2 a_1 D_{q,1}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-2)} & \ldots & x_q^{(q-2)} \end{pmatrix}_{q1} \pm a_1 a_q D_{1,2}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-2)} & \ldots & x_q^{(q-2)} \end{pmatrix}_{12} \\ &\pm a_1^2 D_{q,2}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-2)} & \ldots & x_q^{(q-2)} \end{pmatrix}_{q2} = 0. \end{aligned} \tag{5.6}$$

Now, (5.6) may be interpreted as an equation in $x_1^{(m_{q-2}^{(0)})}, \ldots, x_q^{(m_{q-2}^{(0)})}$. Again there are two possibilities:

Either equations (5.6) and (4.13) are independent. It will turn out again, that then our problems are settled.

Or (5.6) and (4.13) are proportional. In this case we compare the coefficients of $x_2^{(m_{q-2}^{(0)})}$ and $x_3^{(m_{q-2}^{(0)})}$ in (5.6) and (4.13). $x_2^{(m_{q-2}^{(0)})}$ has in (5.6) the coefficient

$$\pm a_2 a_1 D_{q,1,2}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-3)} & \ldots & x_q^{(q-3)} \end{pmatrix}_{q12} . \tag{5.7}$$

The coefficient of $x_3^{(m_{q-2}^{(0)})}$ in (5.6) is

$$\begin{aligned} &\pm a_2 a_1 D_{q,1,3}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-3)} & \ldots & x_q^{(q-3)} \end{pmatrix}_{q13} \pm a_1 a_q D_{1,2,3}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-3)} & \ldots & x_q^{(q-3)} \end{pmatrix}_{123} \\ &\pm a_1^2 D_{q,2,3}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-3)} & \ldots & x_q^{(q-3)} \end{pmatrix}_{q23} . \end{aligned} \tag{5.8}$$

Since equations (5.6) and (4.13) are proportional, we get using (5.7) and (5.8)

$$\begin{aligned} &\pm a_3 a_2 a_1 D_{q,1,2}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-3)} & \ldots & x_q^{(q-3)} \end{pmatrix}_{q12} \pm a_2^2 a_1 D_{q,1,3}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-3)} & \ldots & x_q^{(q-3)} \end{pmatrix}_{q13} \\ &\pm a_2 a_1 a_q D_{1,2,3}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-3)} & \ldots & x_q^{(q-3)} \end{pmatrix}_{123} \pm a_2 a_1^2 D_{q,2,3}^{(0)} \begin{pmatrix} x_1^{(1)} & \ldots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-3)} & \ldots & x_q^{(q-3)} \end{pmatrix}_{q23} = 0. \end{aligned} \tag{5.9}$$

Let us review what we have done so far. In a first step we went from (5.2) to (5.3). The second step led from (5.3) to (5.6) and in the third step we reached (5.9) starting with (5.6).

After i steps with $1 \leqslant i \leqslant q-2$ we obtain an equation

$$\begin{aligned}
&\pm a_i a_{i-1} \dots a_1 D^{(0)}_{q,1,2,\dots,i-1} \begin{pmatrix} x_1^{(1)} & \dots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-i)} & \dots & x_q^{(q-i)} \end{pmatrix}_{q,1,2,\dots,i-1} \\
&\pm a_{i-1} a_{i-2} \dots a_1 a_q D^{(0)}_{1,2,\dots,i} \begin{pmatrix} x_1^{(1)} & \dots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-i)} & \dots & x_q^{(q-i)} \end{pmatrix}_{1,2,3,\dots,i} \\
&+\sum_{j=1}^{i-1} \pm a_{i-1} \dots a_{j+1} a_j^2 a_{j-1} \dots a_1 \\
&\times D^{(0)}_{q,1,2,\dots,j-1,j+1,\dots,i} \begin{pmatrix} x_1^{(1)} & \dots & x_q^{(1)} \\ \vdots & & \vdots \\ x_1^{(q-i)} & \dots & x_q^{(q-i)} \end{pmatrix}_{q,1,2,\dots,j-1,j+1,\dots,i} = 0.
\end{aligned} \tag{5.10}$$

For $i=q-2$, (5.10) is an equation satisfied by $\mathbf{x}^{(m_2^{(0)})}$. However the component $x_q^{(m_2^{(0)})}$ can only occur in the summand

$$\begin{aligned}
&\pm a_{q-3} a_{q-4} \dots a_1 a_q D^{(0)}_{1,2,\dots,q-2} \begin{pmatrix} x_1^{(1)} & \dots & x_q^{(1)} \\ \vdots & & \vdots \\ x_q^{(2)} & \dots & x_q^{(2)} \end{pmatrix}_{1,2,\dots,q-2} \\
&= \pm a_{q-3} a_{q-4} \dots a_1 a_q d^{(0)}_{1,2,\dots,q-1,q} x_q^{(m_1^{(0)})} x_{q-1}^{(m_2^{(0)})} \\
&\quad \pm a_{q-3} \dots a_1 a_q d^{(0)}_{1,2,\dots,q-2,q,q-1} x_{q-1}^{(m_1^{(0)})} x_q^{(m_2^{(0)})}
\end{aligned} \tag{5.11}$$

(cf. (4.19)–(4.21)). But fortunately we have (5.1). Therefore the coefficient of $x_q^{(m_2^{(0)})}$ in (5.11) vanishes. We conclude that (5.10) for $i=q-2$ gives an equation between the components $x_1^{(m_2^{(0)})}, \dots, x_{q-1}^{(m_2^{(0)})}$ of $\mathbf{x}^{(m_2^{(0)})}$. Therefore this equation *either* is independent of (4.13) *or* it is trivial. In the latter case we may conclude that in particular the coefficient of $x_{q-1}^{(m_2^{(0)})}$ in (5.10) equals zero. So we obtain

$$\begin{aligned}
&\pm a_{q-3} \dots a_1 a_q d^{(0)}_{1,2,\dots,q-2,q-1,q}\, x_q^{(m_1^{(0)})} \pm a_{q-2} \dots a_1 d^{(0)}_{q,1,\dots,q-3,q-1,q-2}\, x_{q-2}^{(m_1^{(0)})} \\
&+\sum_{j=1}^{q-3} \pm a_{q-3} \dots a_{j+1} a_j^2 a_{j-1} \dots a_1 d^{(0)}_{q,1,\dots,j-1,j+1,\dots,q-2,q-1,j}\, x_j^{(m_1^{(0)})} = 0.
\end{aligned} \tag{5.12}$$

But in view of (5.1) and since $a_1 \dots a_q \neq 0$, the coefficient of $x_q^{(m_1^{(0)})}$ in (5.12) is nonzero. Moreover (5.12) is an equation that involves only $x_1^{(m_1^{(0)})}, \dots, x_{q-2}^{(m_1^{(0)})}, x_q^{(m_1^{(0)})}$. Therefore (5.12) finally is independent of (4.13).

We are now in a position to roll back and to make the appropriate choice of the parameters $m_i^{(0)}$.

We begin with equation (5.12). *Either* all solutions $\mathbf{x}^{(m)}$ of (4.13) also satisfy (5.12). Since (5.12) and (4.13) are independent, Lemma 4.1 follows at once. *Or* we may choose $m_1^{(0)} \neq 0$ such that $\mathbf{x}^{(m_1^{(0)})}$ is a solution of (4.13) but not of (5.12). With this value of $m_1^{(0)}$ we enter (5.10). *Either* for all $m \geqslant m_1^{(0)}$, $m \neq 0$ equation (5.10) with $i = q-2$ holds true with $\mathbf{x}^{(m_1^{(0)})}$ and $\mathbf{x}^{(m)}$. The definition of $m_1^{(0)}$ implies that this equation is independent of equation (4.13). *Or* there exists an $m_2^{(0)} \geqslant m_1^{(0)}$, $m_2^{(0)} \neq 0$ such that $\mathbf{x}^{(m_2^{(0)})}$ is a solution of (4.13) but (5.12) is not satisfied for $i = q-2$ with $\mathbf{x}^{(m_1^{(0)})}$ and $\mathbf{x}^{(m_2^{(0)})}$.

Now suppose that for $f < q-1$ we could choose nonzero integers $m_1^{(0)}, \ldots, m_f^{(0)}$ having

$$m_1^{(0)} \leqslant \ldots \leqslant m_f^{(0)} \tag{5.13}$$

such that

$$\text{equation (5.12) does not hold for } \mathbf{x}^{(m_1^{(0)})} \tag{5.14}$$

and such that for each j $(2 \leqslant j \leqslant f)$

$$\begin{gathered} \text{equation (5.10) is not satisfied for } i = q-j \\ \text{with the solutions } \mathbf{x}^{(m_1^{(0)})}, \ldots, \mathbf{x}^{(m_j^{(0)})} \text{ of (4.13).} \end{gathered} \tag{5.15}$$

Then for a possible value $m_{f+1}^{(0)}$ we distinguish two cases. *Either* for all $m \geqslant m_f^{(0)}$, $m \neq 0$ with $\mathbf{x}^{(m)}$ satisfying (4.13) the tuple $\mathbf{x}^{(m_1^{(0)})}, \ldots, x^{(m_f^{(0)})}, \mathbf{x}^{(m)}$ gives a solution of (5.10) for $i = q-f-1$. The definition of $m_1^{(0)}, \ldots, m_f^{(0)}$ implies that this is an equation in $\mathbf{x}^{(m)}$ that is independent of (4.13). *Or* there exists $m_{f+1}^{(0)} \geqslant m_f^{(0)}$, $m_{f+1}^{(0)} \neq 0$ such that $\mathbf{x}^{(m_{f+1}^{(0)})}$ is a solution of (4.13) but the tuple $\mathbf{x}^{(m_1^{(0)})}, \ldots, \mathbf{x}^{(m_{f+1}^{(0)})}$ does not solve (5.10) for $i = q-f-1$.

We may continue in this way and possibly this process stops only at $f = q-1$. Then we have $m_1^{(0)}, \ldots, m_{q-1}^{(0)}$ and the vectors $\mathbf{x}^{(m_1^{(0)})}, \ldots, \mathbf{x}^{(m_{q-1}^{(0)})}$ do not satisfy equation (5.3). Therefore in this case equation (5.2) (as equation in $\mathbf{x}^{(m)}$) is nonproportional to equation (4.13).

So far we have dealt with a construction that treats only a single coefficient vector $(d_{i_1,\ldots,i_q}^{(0)})$ arising from Theorem 1.4. However we need parameters $m_1, \ldots, m_{q-1}$ that may be of use simultaneously for all t_4 coefficient vectors we get from Theorem 1.4. For this purpose we proceed as follows. To choose m_1, coefficients $(d_{i_1,\ldots,i_q}^{(j)})$ $(1 \leqslant j \leqslant t_4)$ in alternative (i) allow any choice $m \neq 0$ where $\mathbf{x}^{(m)}$ satisfies (4.13). As for the coefficients appearing in alternative (ii) there are two possibilities.

Either there exists j_0 $(1 \leqslant j_0 \leqslant t_4)$ such that no choice of $m_1^{(j_0)}$ is possible for $(d_{i_1,\ldots,i_q}^{(j_0)})$. Then the lemma follows at once, since this even implies that all solutions $\mathbf{x}^{(m)}$ of (4.13) with $m \neq 0$ satisfy moreover one single equation which is independent of (4.13).

Or we can find for all j $(1\leqslant j\leqslant t_4)$ a parameter $m_1^{(j)}$ that fits together with $(d^{(j)}_{i_1,\dots,i_q})$ in the same way as described above for $m_1^{(0)}$ and $(d^{(0)}_{i_1,\dots,i_q})$. Then we choose for each j the parameter $m_1^{(j)}$ minimal (which is possible as our hypothesis implies that (4.13) or (4.5) respectively have only finitely many solutions) and we put

$$m_1=\max_j m_1^{(j)}. \tag{5.16}$$

Since the $m_1^{(j)}$ are chosen minimal, we may conclude that any solution $\mathbf{x}^{(m)}$ of (4.13) with $m<m_1$, $m\neq 0$ satisfies one at least of the relations (5.12). Notice that we get not more than t_4 such relations. Consequently, if in (5.16) we have $m_1>0$, then the first alternative of the lemma follows by using the definition of t_4 in (4.10). Therefore we may suppose in the sequel that

$$m_1<0. \tag{5.17}$$

Given m_1, we check whether for each j $(1\leqslant j\leqslant t_4)$ we can find $m_2^{(j)}$. If this is not possible, then there exists j_0 $(1\leqslant j_0\leqslant t_4)$ such that for each solution $\mathbf{x}^{(m)}$ with $m\geqslant m_1$, $m\neq 0$ the pair $\mathbf{x}^{(m)},\mathbf{x}^{(m_1)}$ satisfies the corresponding equation of type (5.10) or (4.28). Together with the t_4 equations coming from solutions $m<m_1$ we see that then in fact

$$t_4+1 \tag{5.18}$$

equations will suffice. Otherwise we pick for each j, $m_2^{(j)}\geqslant m_1$ and minimal with respect to our above construction and put

$$m_2=\max_j m_2^{(j)}. \tag{5.19}$$

Then solutions m with $m_1\leqslant m<m_2$ satisfy one of the t_4 relations of type (5.10) or (4.28). We may continue this procedure and we assume now, that $m_1,\dots,m_f$ with $f<q-1$ have been found. Our construction implies that if we cannot find m_{f+1}, then the solutions m satisfy one of

$$f\cdot t_4+1 \tag{5.20}$$

relations, each of type (5.10) or (4.28).

Finally, suppose we can find $m_1,\dots,m_{q-1}$. Then we are in a position to apply Theorem 1.4. In fact suppose

$$m\geqslant\max\{m_{q-1},2^{25}d^3k^3l^2\}, \tag{5.21}$$

so that in particular (3.15) holds true for m. Moreover by (5.17) we also have (3.11) for $m_1,m_2,\dots,m_{q-1},m$. By Lemma 3.5 the vector $\mathbf{z}$ whose components are the summands in the Laplace expansion of the determinant with rows $\mathbf{x}^{(m_1)},\ldots,\mathbf{x}^{(m_{q-1})},\mathbf{x}^{(m)}$ satisfies (3.16). Thus we may apply Theorem 1.4 with S as defined in Lemma 3.5 and with $\delta=\frac{1}{2}$. Consequently with our choice of $\mathbf{x}^{(m_1)},\ldots,\mathbf{x}^{(m_{q-1})}$ we see that $\mathbf{x}^{(m)}$ satisfies one of the t_4 equations of the shape (4.16). But we had fixed in advance the possible coefficients $d_{i,\tau_i(1),\ldots,\tau_i(q-1)}$ in (4.16), as we started with (4.17) and the parameter $\delta=\frac{1}{2}$. So, with our choice of $\mathbf{x}^{(m_1)},\ldots,\mathbf{x}^{(m_{q-1})}$ actually the relations (4.16) for $\mathbf{x}^{(m)}$ are independent of (4.13). We may conclude that the solutions $\mathbf{x}^{(m)}$ with m as in (5.21) satisfy apart from (4.13) one at least of t_4 relations each of which is independent of (4.13).

There still remain solutions with

$$0<m<\max\{m_{q-1},2^{25}d^3k^3l^2\}.$$

To cover these solutions as well, our construction implies that

$$(q-1)t_4+2^{25}d^3k^3l^2 \tag{5.22}$$

relations will suffice.

If we check the different alternatives we had, we may infer from (5.18), (5.20) and (5.22) that either for the range of solutions $m<0$ or for the range $m>0$

$$qt_4+2^{25}d^3k^3l^2$$

relations will suffice and Lemma 4.1 follows.

6. Proof of Theorem 1.3

We proceed by induction on r and k, where k is the sum of the degrees of the polynomials f_i as defined in (1.19).

If $r=1$, then equation (1.15) reads as

$$f(m)\alpha^m=0, \tag{6.1}$$

where $\alpha\neq0$ and f is a polynomial of degree k. So we get not more than k solutions $m\in\mathbf{Z}$ and the assertion follows.

Next suppose that $r=2$ and $k=0$. Then equation (1.15) becomes

$$a\alpha^m+b\beta^m=0, \tag{6.2}$$

where a and b are nonzero constants.

If (6.2) had more than one solution, then α/β would be a root of unity.

Before we deal with the induction step, we still have to study one particular situation that might occur for $r=2$. Denote by σ_1, σ_2 the number of nonzero coefficients occurring in the polynomials f_1, f_2 respectively. We assume now that $k>0$ and $\sigma_1+\sigma_2=2$. Then necessarily we have $\sigma_1=\sigma_2=1$ and equation (1.15) reads as

$$am^{k_1}\alpha^m + bm^{k_2}\beta^m = 0, \tag{6.3}$$

where a and b are nonzero constants and k_1, k_2 are nonnegative integers with

$$k_1+k_2=k.$$

If $k_1=k_2$, then for $m\neq 0$ (6.3) is equaivalent to (6.2). Therefore in this case (6.3) has at most 2 solutions.

We now assume $k_1>k_2$. Then for $m\neq 0$ (6.3) implies

$$\left|\frac{a}{b}\right|^2 m^{2(k_1-k_2)} - \left|\frac{\beta}{\alpha}\right|^{2m} = 0. \tag{6.4}$$

Since the function $|a/b|^2 x^{2(k_1-k_2)} - |\beta/\alpha|^{2x}$ of the real variable x has at most $2(k_1-k_2)$ stationary points, we may conclude that (6.4) has not more than $2(k_1-k_2)+1$ solutions $m\in\mathbf{Z}$. Thus for $k_1>k_2$, equation (6.3) has not more than $2k+1$ solutions.

To treat the general case, we want to apply Lemma 4.1. In proving Lemma 4.1 we need the normalization (4.4) of equation (1.15), to guarantee that at least one third of the zeros have $m<0$ and one third have $m>0$. This normalization has no impact on the degree of the polynomials, in fact we see that

$$\deg f_i = \deg g_i = \varrho_i - 1. \tag{6.5}$$

However the number of nonzero coefficients in g_i and in f_i may be quite different. So the parameters $\sigma_1, \ldots, \sigma_r$ and q in Lemma 4.1 refer to the g_i but not to the initial polynomials f_i. Instead of the initial equation (1.15) we consider now the normalized equation with parameters $r, \sigma_1, \ldots, \sigma_r$. To apply Lemma 4.1, we need moreover that equation (4.1) has only finitely many solutions. But this follows from the Skolem–Mahler–Lech Theorem, as our hypothesis in Theorem 1.1 says that for each pair, i, j $(1\leqslant i, j\leqslant r)$, $i\neq j$, α_i/α_j is not a root of unity.

We may assume that either $r=2$, $k>0$ and $\sigma_1+\sigma_2>2$ or $r>2$ and $k\geqslant 0$ since in the other cases the assertion is proved already.

The induction hypothesis is, that the assertion holds true for all equations (1.15) (never mind whether normalized or not) with parameters r', k' such that either

$$r' < r \tag{6.6}$$

or

$$r' = r \quad \text{and} \quad k' < k. \tag{6.7}$$

By Lemma 4.1 at least one third of the solutions m of the normalized equation

$$\sum_{i=1}^{r} g_i(m)\alpha_i^m = 0 \tag{6.8}$$

satisfy one at least of t_5 relations

$$\sum_{i=1}^{r} h_i^{(j)}(m)\alpha_i^m = 0, \quad 1 \leqslant j \leqslant t_5 \tag{6.9}$$

where the $h_i^{(j)}(x)$ are polynomials with

$$\deg h_i^{(j)} \leqslant \deg g_i, \quad 1 \leqslant i \leqslant r, \ 1 \leqslant j \leqslant t_5. \tag{6.10}$$

Moreover, if for some ν with $0 \leqslant \nu \leqslant \deg h_i^{(j)}$ the coefficient of x^ν in $h_i^{(j)}$ is nonzero, then x^ν has a nonzero coefficient in g_i as well.

Thus writing $\sigma_i^{(j)}$ for the number of nonzero coefficients in $h_i^{(j)}$ we obtain

$$\sigma_i^{(j)} \leqslant \sigma_i \quad \text{for all } i, j \text{ with } 1 \leqslant i \leqslant r, \ 1 \leqslant j \leqslant t_5. \tag{6.11}$$

In fact Lemma 4.1 implies that the $h_i^{(j)}$ may be chosen such that for each j $(1 \leqslant j \leqslant t_5)$ there exists an i $(1 \leqslant i \leqslant r)$ having

$$\deg h_i^{(j)} < \deg g_i \tag{6.12}$$

(we put $\deg 0 = -\infty$), which implies in particular that

$$\sigma_i^{(j)} < \sigma_i, \tag{6.13}$$

and that in fact for $\sigma_i^{(j)} = 0$ the summation in (6.9) is over

$$r' < r \tag{6.14}$$

terms.

In view of (6.10)–(6.14), we may apply the induction hypothesis to (6.9). We infer from (6.10), (6.12), (6.14), denoting by $r^{(j)}$ the number of nonzero polynomials $h_i^{(j)}$ $(1 \leqslant i \leqslant r)$ that

$$r^{(j)} + \sum_{\substack{i=1 \\ h_i^{(j)} \neq 0}}^{r} \deg h_i^{(j)} \leqslant k + r - 1.$$

Therefore each equation (6.9) has not more than

$$(4(\omega+d)d!)^{2^{40(k+r-1)!\,d!}(\omega+d)^6} \tag{6.15}$$

solutions. Multiplying (6.15) with the number t_5 of equations (6.9) and allowing the factor 3, since in Lemma 4.1 we take care only of one third of the zeros of (6.8) we get not more than

$$3t_5(4(\omega+d)d!)^{2^{40(k+r-1)!\,d!}(\omega+d)^6} \tag{6.16}$$

solutions.

Remember the definition of t_4 and t_5 in (4.10) and (4.11) respectively. Using the estimates $q \leqslant k+r$ (cf. (3.10)) and $r_1+r_2 \leqslant d$ and since $l=q!$ we infer from (6.16) that

$$\begin{aligned} N(0) &\leqslant 3\cdot((k+r)\cdot 2((k+r)!)^{(\omega+d)}(4(\omega+d)d!)^{2^{37(k+r)!\,d!}4(\omega+d)^6}+2^{25}d^3k^3l^2) \\ &\quad \times (4(\omega+d)d!)^{2^{40(k+r-1)!\,d!}(\omega+d)^6} \\ &< (4(\omega+d)d!)^{2^{39(k+r)!\,d!}(\omega+d)^6}\cdot(4(\omega+d)d!)^{2^{40(k+r-1)!\,d!}(\omega+d)^6} \\ &< (4(\omega+d)d!)^{2^{40(k+r)!\,d!}(\omega+d)^6}, \end{aligned}$$

and Theorem 1.3 follows.

7. Proof of Theorems 1.1 and 1.2

Theorem 1.1 is a direct consequence of Theorem 1.3. In fact the polynomials in relation (1.15) in that case have sum of degree k with $k+r \leqslant n$ (cf. (1.4)).

We now turn to the proof of Theorem 1.2. If none of the characteristic roots α_i $(1 \leqslant i \leqslant r)$ of our recurrence is a root of unity, then equation (1.5) for $a \neq 0$ may be read as

$$-a\cdot 1^m + \sum_{i=1}^{r} f_i(m)\alpha_i^m = 0. \tag{7.1}$$

But in that case the left hand side of (7.1) is a nondegenerate recurrence sequence of order $\leqslant n+1$. Thus, we may apply Theorem 1.1 and get not more than

$$(4(\omega+d)d!)^{2^{40(n+1)!\,d!}(\omega+d)^6} \tag{7.2}$$

solutions.

So in the sequel we assume that exactly one of the characteristic roots α_i $(1 \leqslant i \leqslant r)$ is a root of unity. If $r=1$, then we get an equation

$$f(m)\alpha^m = a. \tag{7.3}$$

We infer from (7.3) that

$$|f(m)|^2=|a|^2 \tag{7.4}$$

since α is a root of unity. If f is a constant, then (7.4) may have infinitely many solutions. But in that case our recurrence sequence is periodic. If f is nonconstant, we consider the equation

$$|f(x)|^2=|a|^2$$

of the real variable x. Our hypothesis implies that the polynomial $|f(x)|^2-|a|^2$ has degree $<2n$. We infer that equation (7.4) has not more than $2n$ solutions $m\in\mathbf{Z}$, and the assertion of Theorem 1.2 follows.

Now assume that $r>1$. Then we have characteristic roots $\alpha_1,\dots,\alpha_r$ such that α_i/α_j is not a root of unity but one at least of the roots, say α_r, is a root of unity. If $\alpha_r=1$ our equation becomes

$$(f_r(m)-a)1^m+\sum_{i=1}^{r-1}f_i(m)\alpha_i^m=0. \tag{7.5}$$

Again we may apply Theorem 1.1 and we conclude that (7.5) has not more than

$$(4(\omega+d)d!)^{2^{40n!\,d!\,(\omega+d)^6}} \tag{7.6}$$

solutions.

Therefore in the remainder of the proof we may suppose that α_r is a root of unity that is different from 1 and that we have $r>1$. We have to study the equation

$$-a\cdot 1^m+f_r(m)\alpha_r^m+\sum_{i=1}^{r-1}f_i(m)\alpha_i^m=0. \tag{7.7}$$

Let c be the order of the root of unity α_r. We split equation (7.7) into c equations

$$(-a\alpha_r^{-b}+f_r(b+mc))\alpha_r^{b+mc}+\sum_{i=1}^{r-1}f_i(b+mc)\alpha_i^{b+mc}=0,\quad b=0,\dots,c-1. \tag{7.8}$$

By hypothesis, for each b $(0\leqslant b\leqslant c-1)$ the left hand side of (7.8) represents a nondegenerate recurrence sequence or it is identically zero. If there exists a b_0 with $0\leqslant b_0\leqslant c-1$ for which we get a zero sequence, then we have in particular

$$f_i(x)\equiv 0\quad\text{for } i=1,\dots,r-1$$

and

$$f_r(x)=a\alpha_r^{-b_0}.$$

Thus our initial sequence is of the shape

$$u_m = d\alpha_r^m$$

and hence it is periodic.

Consequently, we may suppose that (7.7) has only finitely many zeros. But then we are in a position to apply Lemma 4.1 to equation (7.7). In fact, since $r>1$ our hypothesis implies in particular that there exists a pair i,j $(1\leqslant i,j\leqslant r)$ with $i\neq j$ such that α_i/α_j is not a root of unity, and this was the second general assumption we needed in Lemma 4.1.

Since $r>1$, the left hand side of (7.7) is an expression with $r+1$ characteristic roots, so in the notation of Lemma 4.1 we replace r by $r+1$ and we have $q\geqslant 3$. Now the relations we obtain in Lemma 4.1 may be chosen such that for each j $(1\leqslant j\leqslant t_5)$ the coefficient vector $(d_{11}^{(j)},\dots,d_{r+1,\sigma_{r+1}}^{(j)})$ has some particular predesigned component equal to zero. We choose this component such that in the relations of Lemma 4.1 we get rid of the term corresponding to $-a\cdot 1^m$ in (7.7), and we infer from Lemma 4.1:

At least one third of the solutions $m\in\mathbf{Z}$ of (7.7) sastify one at least of t_5 relations of the type

$$\sum_{i=1}^{r} g_i^{(j)}(m)\alpha_i^m = 0, \quad 1\leqslant j\leqslant t_5 \tag{7.9}$$

where the $g_i^{(j)}$ are polynomials having

$$\deg g_i^{(j)} \leqslant \deg f_i \quad \text{for each pair } i,j \text{ with } 1\leqslant i\leqslant r, 1\leqslant j\leqslant t_5. \tag{7.10}$$

Equations (7.9) however are of the type considered in Theorem 1.3. Since

$$r+\sum_{i=1}^{r}\deg f_i\leqslant n$$

we conclude using (7.10) that for each j $(1\leqslant j\leqslant t_5)$ (7.9) has not more than

$$(4(\omega+d)d!)^{2^{40n!\,d!}(\omega+d)^6} \tag{7.11}$$

solutions. Allowing the factor t_5 for the number of relations (7.9) and the factor 3, as we may have covered only one third of the solutions we finally see that (1.5) has not more than

$$3t_5(4(\omega+d)d!)^{2^{40n!\,d!}(\omega+d)^6}$$

solutions. In our context the parameter q occurring in Lemma 4.1 is bounded by $n+1$. Thus using (4.11) we obtain

$$
\begin{aligned}
&3t_5(4(\omega+d)d!)^{2^{40n!\,d!}(\omega+d)^6}\\
&\leqslant (6(n+1)(n+1)!^{(\omega+d)}(4(\omega+d)d!)^{2^{37(n+1)!d!}4(\omega+d)^6}+3\cdot 2^{25}d^3k^4)\cdot(4(\omega+d)d!)^{2^{40n!d!}(\omega+d)^6}\\
&< (4(\omega+d)d!)^{2^{40n!\,d!}(\omega+d)^6}\cdot(4(\omega+d)d!)^{2^{39(n+1)!\,d!}(\omega+d)^6}\\
&\leqslant (4(\omega+d)d!)^{2^{40(n+1)!\,d!}(\omega+d)^6}
\end{aligned}
$$

and Theorem 1.2 follows.

References

[1] BEUKERS, F. & TIJDEMAN, R., On the multiplicities of binary complex recurrences. *Compositio Math.*, 51 (1984), 193–213.

[2] DOBROWOLSKI, E., On a question of Lehmer and the number of irreducible factors of a polynomial. *Acta Arith.*, 34 (1979), 391–401.

[3] EVERTSE, J.H., On sums of S-units and linear recurrences. *Compositio Math.*, 53 (1984), 225–244.

[4] EVERTSE, J.H., GYÖRY, K., STEWART, C.L. & TIJDEMAN, R., S-unit equations in two variables. *Invent. Math.*, 92 (1988), 461–477.

[5] — S-unit equations and their applications, in *New Advances in Transcendence Theory* (A. Baker, ed.), pp. 110–174. Cambridge University Press, 1988.

[6] KUBOTA, K.K., On a conjecture by Morgan Ward III. *Acta Arith.*, 33 (1977), 99–109.

[7] SCHLICKEWEI, H.P., An upper bound for the number of subspaces occurring in the p-adic subspace theorem in diophantine approximation. *J. Reine Angew. Math.*, 406 (1990), 44–108.

[8] — The quantitative subspace theorem for number fields. *Compositio Math.*, 82 (1992), 245–273.

[9] — An explicit upper bound for the number of solutions of the S-unit equation. *J. Reine Angew. Math.*, 406 (1990), 109–120.

[10] — S-unit equations over number fields. *Invent. Math.*, 102 (1990), 95–107.

[11] SCHMIDT, W.M., The subspace theorem in diophantine approximations. *Compositio Math.*, 69 (1989), 121–173.

[12] — *Diophantine Approximations and Diophantine Equations.* Lecture Notes in Math., 1467. Springer-Verlag, 1991.

[13] SHOREY, T.N. & TIJDEMAN, R., *Exponential Diophantine Equations.* Cambridge University Press, 1986.

HANS PETER SCHLICKEWEI
Abteilung Mathematik
Universität Ulm
Oberer Eselsberg
D-7900 Ulm
Germany

Received March 2, 1990
Received in revised form September 6, 1991