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An explicit formula for the fourth power mean of the Riemann zeta-function

by

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1. Introduction and statement of results

In the celebrated paper [1] Atkinson exhibited an explicit formula for the mean square of the Riemann zeta-function on the critical line, which greatly enriched the theory of this most fundamental function in number theory (cf. the relevant parts of [2], [3], [10]). Our main object in the present paper is to indicate that if combined with the new developments due to Kuznetsov [6], [7] in the theory of automorphic functions, Atkinson's idea can be extended to the fourth power mean situation to yield an explicit formula for

$$I(T,\Delta)=(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\left|\zeta\left(\tfrac{1}{2}+i(T+t)\right)\right|^4 e^{-(t/\Delta)^2}\,dt \tag{1.1}$$

with $0<\Delta<T(\log T)^{-1}$. As will be shown in the separate papers [4], [5] our result (the theorem below) gives rise to several important consequences on the error term in the asymptotic formula for the unweighted mean

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 dt, \tag{1.2}$$

some of which can be regarded as genuine extensions of those deducible from Atkinson's formula (see also [3, Chapter 5]).

When reduced to essentials, Atkinson's idea consists of an effective application of Poisson's formula to the following trivial decomposition of double sums:

$$\sum_{m,n} f(m,n)=\left\{\sum_{m=n}+\sum_{m<n}+\sum_{m>n}\right\}f(m,n). \tag{1.3}$$

Translating this into the problem of determining the behaviour of (1.1), we encounter the identity

$$\sum_{k,l,m,n} f(k,l,m,n)=\left\{\sum_{kl=mn}+\sum_{kl<mn}+\sum_{kl>mn}\right\}f(k,l,m,n). \tag{1.4}$$

Then the rôle of Poisson's formula in (1.3) is played in this by the spectral expansion of $\mathrm{SL}(2,\mathbf{Z})$-automorphic functions. In fact, if quadruple sums are identified with sums over 2×2 integral matrices, the decomposition (1.4) can be written in the form

$$\sum_A f(A)=\left\{\sum_{\det A=0}+\sum_{\det A>0}+\sum_{\det A<0}\right\}f(A).$$

Here we have further

$$\sum_{\det A>0} f(A)=\sum_{n=1}^{\infty}\sum_{\det A=n} f(A)=\sum_{n=1}^{\infty}\sum_{ad=n}\sum_{b=1}^{d}\sum_{A\in\mathrm{SL}(2,\mathbf{Z})} f\left(A\begin{pmatrix} a & b\\ & d\end{pmatrix}\right), \tag{1.5}$$

and we are led to the theory of automorphic functions.

These are stated, however, only for the sake of understanding very roughly the fact that the discrete spectrum of the non-Euclidean Laplacian as well as the objects related to holomorphic cusp forms appear in our explicit formula for $I(T,\Delta)$. In practice we shall first relate $I(T,\Delta)$ to an integral of the product of four zeta-values, which is subsequently transformed into sums of Kloosterman sums. Then we shall appeal to Kuznetsov's trace formulas which amounts, if generalized as above, to the spectral decomposition applied to (1.5). Thereby will emerge a mysterious relation between the Riemann zeta-function and automorphic L-functions. But this will be shown first in a restricted range of the relevant parameters; and we shall face a crucial problem of analytic continuation. After settling this, a specialization of parameters will be undertaken, and certain involved technicalities will finish our discussion.

Now, in order to state our main result we introduce some basic concepts and results from the theory of automorphic functions, whose detailed expositions can be found in standard literature. It should be stressed here that notations and conventions will be introduced at the stages where we need them for the first time, and will be effective thereafter.

Thus, let $\{\lambda_j=\varkappa_j^2+\frac{1}{4}:\varkappa_j>0, j\geqslant 1\}\cup\{0\}$ be the discrete spectrum of the non-Euclidean Laplacian acting on the space of all non-holomorphic automorphic functions with respect to $\mathrm{SL}(2,\mathbf{Z})$. Let φ_j be the Maass wave form attached to the eigenvalue λ_j, so that $\{\varphi_j\}$ forms an orthonormal base of the subspace spanned by all cusp forms, and each φ_j is an eigenfunction of all Hecke operators $T(n)$, $n\geqslant 1$, and $T(-1)$. The latter means that we have, for $\mathrm{Im}(z)>0$,

$$(T(n)\varphi_j)(z)=\frac{1}{\sqrt{n}}\sum_{ad=n}\sum_{b=1}^{d}\varphi_j\left(\frac{az+b}{d}\right)=t_j(n)\varphi_j(z)$$

with a certain real number $t_j(n)$, and

$$(T(-1)\varphi_j)(z)=\varphi_j(-\bar{z})=\varepsilon_j\varphi_j(z)$$

with $\varepsilon_j=\pm 1$. The same numbers appear also in the Fourier expansion of φ_j:

$$\varphi_j(x+iy)=\varrho_j\sqrt{y}\sum_{n\neq 0}t_j(n)K_{i\varkappa_j}(2\pi|n|y)e(nx),$$

where $K_\nu(\cdot)$ is the K-Bessel function of order ν, and $e(x)=\exp(2\pi ix)$ as usual. The first Fourier coefficient ϱ_j is an important quantity in our discussion. With it we put

$$\alpha_j=|\varrho_j|^2(\cosh\pi\varkappa_j)^{-1}.$$

Then Kuznetsov [7] has shown that

$$\sum_{\varkappa_j\leqslant K}\alpha_j\ll K^2. \tag{1.6}$$

Also we shall need the Maass wave form L-function $H_j(s)$ attached to φ_j, which is defined by

$$H_j(s)=\sum_{n=1}^{\infty}t_j(n)n^{-s}.$$

This converges absolutely for $\mathrm{Re}(s)>2$, for we have the elementary bound

$$|t_j(n)|\leqslant\sigma_1(n), \tag{1.7}$$

where $\sigma_a(n)$ denotes the sum of the ath powers of divisors of n. The multiplicative property of $t_j(n)$ found by Hecke can be expressed, in terms of $H_j(s)$, by the identity

$$\sum_{n=1}^{\infty}\sigma_a(n)t_j(n)n^{-s}=\zeta(2s-a)^{-1}H_j(s)H_j(s-a), \tag{1.8}$$

providing $\mathrm{Re}(s)$ is sufficiently large. This is a counterpart of Ramanujan's identity: In the region of absolute convergence

$$\sum_{n=1}^{\infty}\sigma_a(n)\sigma_b(n)n^{-s}=\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)\{\zeta(2s-a-b)\}^{-1}. \tag{1.9}$$

Further, $H_j(s)$ can be continued to an entire function, and satisfies the functional equation

$$H_j(s) = 2^{2s-1}\pi^{2(s-1)}\Gamma(1+i\varkappa_j - s)\Gamma(1-i\varkappa_j - s)(\varepsilon_j \cosh(\pi\varkappa_j) - \cos(\pi s))H_j(1-s), \quad (1.10)$$

which implies in particular that uniformly for bounded s

$$H_j(s) \ll \varkappa_j^c. \quad (1.11)$$

Here and in what follows the letter c denotes generally a positive constant whose value may differ at each occurrence, and whose dependency on other constants, e.g. the size of s in the above case, will be inferred from the context, though we shall make them explicit if necessary.

Next, we turn to holomorphic cusp forms. Thus, let $\{\varphi_{j,k} : 1 \leqslant j \leqslant \vartheta(k)\}$, $\vartheta(k) = 0$ $(k < 6)$, be the orthonormal base, consisting of eigenfunctions of all Hecke operators $T_k(n)$, $n \geqslant 1$, of the Petersson unitary space of holomorphic cusp forms of weight $2k$ with respect to $\mathrm{SL}(2, \mathbf{Z})$. This means especially that we have, for $\mathrm{Im}(z) > 0$,

$$(T_k(n)\varphi_{j,k})(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \left(\frac{a}{d}\right)^k \sum_{b=1}^{d} \varphi_{j,k}\left(\frac{az+b}{d}\right) = t_{j,k}(n)\varphi_{j,k}(z)$$

with a certain real number $t_{j,k}(n)$. We define as before, the Hecke L-series $H_{j,k}(s)$ attached to $\varphi_{j,k}$ by

$$H_{j,k}(s) = \sum_{n=1}^{\infty} t_{j,k}(n) n^{-s}.$$

This converges absolutely for $\mathrm{Re}(s) > 2$, since we have

$$|t_{j,k}(n)| \leqslant \sigma_1(n). \quad (1.12)$$

As a matter of fact much more is known, but this elementary bound is sufficient for our discussion; the same can be said for (1.7). Corresponding to (1.8) we have, in the region of absolute convergence,

$$\sum_{n=1}^{\infty} \sigma_a(n) t_{j,k}(n) n^{-s} = \zeta(2s-a)^{-1} H_{j,k}(s) H_{j,k}(s-a). \quad (1.13)$$

$H_{j,k}(s)$ can be continued to an entire function, and satisfies a functional equation, which implies that uniformly for bounded s

$$H_{j,k}(s) \ll k^c, \quad (1.14)$$

where c and the implied constant depend only on s. Also, with the first Fourier coefficient $\varrho_{j,k}$ of $\varphi_{j,k}$ we put

$$\alpha_{j,k} = (2k-1)!\, 2^{-4k+2}\pi^{-2k-1}|\varrho_{j,k}|^2.$$

Then, as will be shown in the third section, we have the following analogue of (1.6):

$$\sum_{j=1}^{\vartheta(k)} \alpha_{j,k} \ll k. \tag{1.15}$$

With these preparations we may now state our main result, which with the obvious abuse of notation is embodied in the following

THEOREM. *If* $0<\Delta<T(\log T)^{-1}$, *then there exist absolute constants* $c(a,b;k,l)$ *such that*

$$\begin{aligned}
&(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty} \left|\zeta\left(\tfrac12+i(T+t)\right)\right|^4 e^{-(t/\Delta)^2}\,dt\\
&= (\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty} \mathrm{Re}\Bigg[\sum_{\substack{a,b,k,l\geqslant 0\\ ak+bl\leqslant 4}} c(a,b;k,l)\left(\frac{\Gamma^{(a)}}{\Gamma}\right)^k\left(\frac{\Gamma^{(b)}}{\Gamma}\right)^l\left(\tfrac12+i(T+t)\right)\Bigg] e^{-(t/\Delta)^2}\,dt\\
&\quad+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\zeta(\frac12+it)|^6}{|\zeta(1+2it)|^2}\Theta(t;T,\Delta)\,dt+\sum_{j=1}^{\infty}\alpha_j H_j\left(\tfrac12\right)^3\Theta(\varkappa_j;T,\Delta)\\
&\quad+\sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}\alpha_{j,k}H_{j,k}\left(\tfrac12\right)^3\Theta\left(i\left(\tfrac12-k\right);T,\Delta\right)+O(T^{-1}(\log T)^2).
\end{aligned}\tag{1.16}$$

Here the constant in the error term is absolute, and

$$\begin{aligned}
\Theta(r;T,\Delta) = \int_0^\infty (x(x+1))^{-1/2}\cos\left(T\log\left(1+\frac1x\right)\right)&\\
\times\Lambda(x,r)\exp\left(-\left(\frac{\Delta}{2}\log\left(1+\frac1x\right)\right)^2\right)dx;&
\end{aligned}$$

$$\Lambda(x,r) = \mathrm{Re}\left[x^{-1/2-ir}\left(\omega(r)+\frac{i}{\sinh(\pi r)}\right)\frac{\Gamma(\frac12+ir)^2}{\Gamma(1+2ir)}F\left(\frac12+ir,\frac12+ir;1+2ir;-\frac1x\right)\right],$$

where F *is the hypergeometric function, and* $\omega(r)$ *is the characteristic function of the set of real numbers.*

We remark that the sums and integrals in the above are all absolutely convergent. This follows from (1.6), (1.11), (1.14), (1.15) as well as the rapid decay of $\Theta(r;T,\Delta)$ with respect to r. The latter will be shown in the course of the proof. We also stress that it is possible to consider (1.1) with more general weights than Gauss' $(\Delta\sqrt{\pi})^{-1}\exp(-(t/\Delta)^2)$. But, according to experience our choice appears to be the most practical.

Now, our formula (1.16) should be compared with the following explicit formula for the mean square: For any $T,\Delta>0$

$$\begin{aligned}(\Delta\sqrt{\pi})^{-1}&\int_{-\infty}^{\infty}\left|\zeta\left(\tfrac{1}{2}+i(T+t)\right)\right|^2 e^{-(t/\Delta)^2}\,dt\\ &=(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\mathrm{Re}\left\{\frac{\Gamma'}{\Gamma}\left(\tfrac{1}{2}+i(T+t)\right)\right\}e^{-(t/\Delta)^2}\,dt+2\gamma-\log(2\pi) \qquad (1.17)\\ &\quad+4\sum_{n=1}^{\infty}d(n)\int_0^{\infty}(x(x+1))^{-1/2}\cos\left(T\log\left(1+\frac{1}{x}\right)\right)\cos(2\pi nx)\\ &\qquad\times\exp\left(-\left(\frac{1}{2}\Delta\log\left(1+\frac{1}{x}\right)\right)^2\right)dx+2\sqrt{\pi}\,\Delta^{-1}\exp\left(\left(\tfrac{1}{4}-T^2\right)\Delta^{-2}\right)\cos(T\Delta^{-2}),\end{aligned}$$

where γ is the Euler constant, and $d(n)$ is the divisor function. This can be proved by considering the degenerate case $v=+\infty$, $z=+\infty$ in (2.1) below with the aid of (1.3).

Transforming (1.17) by Voronoi's formula as Atkinson did in a somewhat different context, and integrating the result with respect to T, it can be seen that Δ may be taken to 0; thereby one gets an alternative proof of his formula mentioned above.

In these circumstances one may speculate that there might be analogues of Voronoi's formula for the sums

$$\sum_{\varkappa_j\leqslant K}\alpha_j H_j\left(\tfrac{1}{2}\right)^3,\quad \sum_{k\leqslant K}\sum_{j=1}^{\vartheta(k)}\alpha_{j,k}H_{j,k}\left(\tfrac{1}{2}\right)^3.$$

If they exist, we would be able to complete the analogy between (1.16) and (1.17). This appears to be a difficult problem; and we remark only that the arguments developed in [8], [13] may probably yield something close to our aim. There is, however, another way to enhance the analogy: We consider the asymptotical behaviour of our explicit formula and (1.17). Applying the saddle point method to each term of the sum over n in (1.17), one may show that for $T^{1/4}<\Delta<T(\log T)^{-1}$

$$\begin{aligned}(\Delta\sqrt{\pi})^{-1}&\int_{-\infty}^{\infty}\left|\zeta\left(\tfrac{1}{2}+i(T+t)\right)\right|^2 e^{-(t/\Delta)^2}\,dt \qquad (1.18)\\ &=2^{3/4}\pi^{1/4}T^{-1/4}\sum_{n=1}^{\infty}(-1)^{n-1}d(n)n^{-1/4}\sin(f(T,n))\exp\left(-\frac{\pi n}{2T}\Delta^2\right)+O(\log T),\end{aligned}$$

where the implied constant is absolute, and

$$f(T,n)=2T\,\mathrm{arcsinh}\left(\left(\frac{\pi n}{2T}\right)^{1/2}\right)+(2\pi nT+\pi^2n^2)^{1/2}-\tfrac{1}{4}\pi.$$

As a matter of fact we can relax the condition on Δ at the cost of introducing complexities into the right side, which is unnecessary for our present purpose. On the other hand an asymptotical evaluation of $\Theta(r;T,\Delta)$, which is to be carried out in the final section, gives

COROLLARY TO THE THEOREM. *If $T^{1/2}<\Delta<T(\log T)^{-1}$, then we have*

$$(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\left|\zeta\left(\tfrac{1}{2}+i(T+t)\right)\right|^4 e^{-(t/\Delta)^2}\,dt \tag{1.19}$$
$$=\frac{\pi}{\sqrt{2T}}\sum_{j=1}^{\infty}\alpha_j H_j\left(\tfrac{1}{2}\right)^3\varkappa_j^{-1/2}\sin\left(\varkappa_j\log\frac{\varkappa_j}{4eT}\right)\exp\left(-\left(\frac{\Delta\varkappa_j}{2T}\right)^2\right)+O(\log^B T),$$

where the constant B is explicitly computable, and the implied constant is absolute.

Again we stress that this condition on Δ is only for the sake of simplicity in the result, and one may consider smaller Δ as well.

A feature of (1.19) which one cannot miss to observe is that it shows clearly that each value of $\zeta\left(\frac{1}{2}+it\right)$, though on average, is related to all eigenvalues of the non-Euclidean Laplacian over $\mathrm{SL}(2,\mathbf{Z})$ in much the same way as prime numbers do to all complex zeros of $\zeta(s)$. To find a direct explanation of this fact seems to be a deep problem, and the solution to it will certainly bring us an image of the zeta-function as a wave in an extended sense.

Finally, it should be mentioned that the present article has some aspects common with Zavorotnyi's work [16], in which he has obtained an asymptotic formula for (1.2) with an error $O(T^{2/3+\varepsilon})$ for any fixed $\varepsilon>0$ (cf. [4]). His argument depends, however, on Kuznetsov's work [8] in an essential manner; and because of his use of a smoothing device which resembles the reflection principle (cf. [2, p. 122]) it appears that his argument may not be improved so as to yield a result as explicit as ours. We stress that our argument is independent of [8].

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2. Reduction to sums of Kloosterman sums

We begin by relating $I(T,\Delta)$ to a meromorphic function of four complex variables.

Let D_+, D_- be the domains of $\mathbf{C}^4$ where all four variables have real parts larger than, less than one, respectively. We put, for $(u,v,w,z)\in D_+$ and arbitrary $\Delta>0$,

$$Y(u,v,w,z)=-i(\Delta\sqrt{\pi})^{-1}\int_{(0)}\zeta(u+t)\zeta(v+t)\zeta(w-t)\zeta(z-t)\exp\left(\left(\frac{t}{\Delta}\right)^2\right)dt \tag{2.1}$$

where the path is $\mathrm{Re}(t)=0$. The omission of Δ on the left side is for the sake of notational simplicity; similar abbreviations will be employed at various places in the sequel.

Moving the path in (2.1) to the right appropriately, we see that Y is a meromorphic function over the entire $\mathbf{C}^4$. Then, taking (u,v,w,z) to D_- and shifting the path back to the original, we get the following continuation to D_-:

$$\begin{aligned} Y(u,v,w,z) = (\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty} \zeta(u+it)\zeta(v+it)\zeta(w-it)\zeta(z-it)\exp\Big(-\Big(\frac{t}{\Delta}\Big)^2\Big)\,dt \\ +2\sqrt{\pi}\,\Delta^{-1}\Big\{\zeta(v-u+1)\zeta(u+w-1)\zeta(u+z-1)\exp\Big(\Big(\frac{u-1}{\Delta}\Big)^2\Big) \\ +\zeta(u-v+1)\zeta(v+w-1)\zeta(v+z-1)\exp\Big(\Big(\frac{v-1}{\Delta}\Big)^2\Big) \\ +\zeta(z-w+1)\zeta(u+w-1)\zeta(v+w-1)\exp\Big(\Big(\frac{w-1}{\Delta}\Big)^2\Big) \\ +\zeta(w-z+1)\zeta(u+z-1)\zeta(v+z-1)\exp\Big(\Big(\frac{z-1}{\Delta}\Big)^2\Big)\Big\}. \end{aligned}$$

In this expression the integral is obviously regular throughout D_-; also the member in the braces is regular at the points (u,u,w,w) with $u+w\neq 2$. The latter can be confirmed easily by using the Laurent expansion of $\zeta(s)$ at $s=1$. In particular Y is regular at the point

$$P_T = \big(\tfrac12+iT, \tfrac12+iT, \tfrac12-iT, \tfrac12-iT\big)$$

for any real T, and we have

$$I(T,\Delta) = Y(P_T) - 2\sqrt{\pi}\,\Delta^{-1}\,\mathrm{Re}\Big\{\big(\gamma-\log(2\pi)+\big(\tfrac12+iT\big)\Delta^{-2}\big)\exp\Big(\Big(\frac{\frac12+iT}{\Delta}\Big)^2\Big)\Big\}. \tag{2.2}$$

Because of this identity we seek for some other way of continuing Y from D_+ to the vicinity of P_T. To this end we note that in D_+

$$Y(u,v,w,z) = \sum_{k,l,m,n=1}^{\infty} k^{-u}l^{-v}m^{-w}n^{-z}\exp\Big(-\Big(\frac{\Delta}{2}\log\frac{mn}{kl}\Big)^2\Big).$$

Hence (1.4) comes into play. We then have the decomposition, in D_+,

$$Y(u,v,w,z) = Y_0(u,v,w,z)+Y_1(u,v,w,z)+Y_1(w,z,u,v), \tag{2.3}$$

where Y_0 and Y_1 correspond to the parts with $kl=mn$ and $kl<mn$, respectively. Ramanujan's identity (1.9) gives

$$Y_0(u,v,w,z) = \zeta(u+w)\zeta(u+z)\zeta(v+w)\zeta(v+z)\{\zeta(u+v+w+z)\}^{-1}, \tag{2.4}$$

which is obviously meromorphic over $\mathbf{C}^4$. As for Y_1 we have, in D_+,

$$Y_1(u,v,w,z) = \sum_{m,n=1}^{\infty} m^{-u-w}\sigma_{u-v}(m)\sigma_{w-z}(m+n)W\Big(\frac{n}{m},w\Big) \tag{2.5}$$

with

$$W(x,\eta)=(1+x)^{-\eta}\exp\left(-\left(\tfrac{1}{2}\Delta\log(1+x)\right)^2\right).$$

This suggests to us that we should view Y_1 as a result of convolving Fourier coefficients of the Eisenstein series for $\mathrm{SL}(2,\mathbf{Z})$. In fact, a general theory of such convolutions has been already investigated by Kuznetsov [8], and one may appeal to his results, which is exactly what Zavorotnyi [16] did in a somewhat different context. However, as we have already indicated, if we follow their argument it appears quite difficult for us to keep computations explicit at all stages, which is imperative for us; and moreover the reconstruction of details which [8] lacks is equally difficult (on this matter see [14]). Therefore we shall take a different route, though we start from the same observation as [8, (106)].

We thus go back to the very reason that the divisor-sum functions appear in the Fourier expansion of the Eisenstein series. This is embodied in another identity due to Ramanujan: For $\mathrm{Re}(\xi)<0$

$$\sigma_\xi(n)=\zeta(1-\xi)\sum_{l=1}^{\infty}l^{\xi-1}c_l(n), \tag{2.6}$$

where $c_l(n)$ is the Ramanujan sum

$$\sum_{\substack{h=1\\(h,l)=1}}^{l} e\left(\frac{h}{l}n\right).$$

We regard (2.6) as an expansion over additive characters of the multiplicative function $\sigma_\xi(n)$. Then (2.6) provides a means of separating the variables m and n in the factor $\sigma_{w-z}(m+n)$ on the right side of (2.5). Thus, the combination of (2.5) and (2.6) yields, for those $(u,v,w,z)\in D_+$ such that $\mathrm{Re}(z)>\mathrm{Re}(w)+1$,

$$\begin{aligned}Y_1(u,v,w,z)=\zeta(z-w+1)\sum_{l=1}^{\infty}l^{w-z-1}\sum_{\substack{h=1\\(h,l)=1}}^{l}\sum_{m=1}^{\infty}m^{-u-w}\sigma_{u-v}(m)e\left(\frac{h}{l}m\right)\\ \times\sum_{n=1}^{\infty}e\left(\frac{h}{l}n\right)W\left(\frac{n}{m},w\right),\end{aligned} \tag{2.7}$$

in which the right side converges absolutely.

Then we need further to separate the varibles m and n in the last W-factor. To this end we introduce the Mellin transform of $W(x,\eta)$:

$$W^*(s,\eta)=\int_0^{\infty}y^{s-1}(1+y)^{-\eta}\exp\left(-\left(\tfrac{1}{2}\Delta\log(1+y)\right)^2\right)dy,\quad \mathrm{Re}(s)>0. \tag{2.8}$$

This is meromorphic in s, having simple poles at non-positive integers, and entire in η. The former assertion can be proved easily by performing partial integration many times, which gives at the same time the rapid decay of W^* as a function of s. In other words we have, for any fixed $A, B>0$,

$$W^*(s,\eta) \ll |s|^{-A} \tag{2.9}$$

as s tends to infinity in the strip $|\mathrm{Re}(s)|<B$; here the implied constant depends on η, Δ, A, B. Also we should note for the sake of a later prupose that the Beta-integral formula implies

$$W^*(s,\eta) = (\Delta\sqrt{\pi})^{-1}\Gamma(s)\int_{-\infty}^{\infty} \frac{\Gamma(\eta-s+it)}{\Gamma(\eta+it)} \exp\left(-\left(\frac{t}{\Delta}\right)^2\right) dt, \tag{2.10}$$

providing $\mathrm{Re}(\eta)>\mathrm{Re}(s)$. Further, by Mellin's inversion formula we have, for any complex η and $x>0$,

$$W(x,\eta) = \frac{1}{2\pi i}\int_{(\alpha)} W^*(s,\eta)x^{-s}\, ds, \tag{2.11}$$

where $\alpha>0$ is arbitrary.

At this step we introduce a subdomain of D_+: For $\alpha>1$ we put

$$D(\alpha) = \{(u,v,w,z)\in D_+ : \mathrm{Re}(z) > \mathrm{Re}(w)+1, \mathrm{Re}(w)>\alpha\}.$$

Then, by (2.7) and (2.11) we have in $D(\alpha)$

$$\begin{aligned} Y_1(u,v,w,z) = \zeta(z-w+1)\sum_{l=1}^{\infty} l^{w-z-1} \sum_{\substack{h=1\\(h,l)=1}}^{l} \frac{1}{2\pi i}\int_{(\alpha)} D\left(u+w-s, u-v; e\left(\frac{h}{l}\right)\right) \\ \times\zeta\left(s, e\left(\frac{h}{l}\right)\right) W^*(s,w)\, ds, \end{aligned} \tag{2.12}$$

where

$$D\left(s,\xi; e\left(\frac{h}{l}\right)\right) = \sum_{n=1}^{\infty} \sigma_\xi(n) e\left(\frac{h}{l}n\right) n^{-s},$$
$$\zeta\left(s, e\left(\frac{h}{l}\right)\right) = \sum_{n=1}^{\infty} e\left(\frac{h}{l}n\right) n^{-s},$$

and the right side of (2.12) converges absolutely (cf. (2.9)). We are going to shift the path in (2.12) to the right. For this sake we recollect some analytical properties of the D-function: If $\xi\neq 0$ and $(h,l)=1$, $D(s,\xi; e(h/l))$ has simple poles at $s=1$ and $1+\xi$ with residues $l^{\xi-1}\zeta(1-\xi)$ and $l^{-\xi-1}\zeta(1+\xi)$, respectively; and there is no other singularity. Also we have the functional equation

$$\begin{aligned} D\left(s,\xi;e\left(\frac{h}{l}\right)\right) &= 2(2\pi)^{2s-2-\xi}l^{\xi-2s+1}\Gamma(1-s)\Gamma(1+\xi-s) \\ \times\left\{D\left(1-s,-\xi;e\left(\frac{\bar{h}}{l}\right)\right)\cos\left(\frac{\pi}{2}\xi\right)\right. &\left.-D\left(1-s,-\xi;e\left(-\frac{\bar{h}}{l}\right)\right)\cos\left(\pi\left(s-\frac{\xi}{2}\right)\right)\right\}, \end{aligned} \tag{2.13}$$

where $h\bar{h}\equiv 1 \pmod{l}$. This implies in particular that $D(s,\xi;e(h/l))$ is of polynomial order with respect to s in any fixed vertical strip. In fact, these can be proved easily by expressing $D(s,\xi;e(h/l))$ in terms of Hurwitz zeta-functions.

Now we introduce another domain:

$$E(\beta)=\{(u,v,w,z):\mathrm{Re}(u+w)<\beta,\mathrm{Re}(v+w)<\beta,\mathrm{Re}(u+v+w+z)>3\beta\},$$

where $\beta>0$ is to be taken sufficiently large, though in the discussion below $\beta=5$ will be adequate. We should remark that

$$D(\alpha)\cap E(\beta)\neq\varnothing,\quad \beta>\alpha+1.$$

We work, for a moment, in this joint domain. Then we have, on noting (2.9) as well as the facts on the D-function mentioned above,

$$\begin{aligned} &\sum_{\substack{h=1\\(h,l)=1}}^{l}\frac{1}{2\pi i}\left\{\int_{(\alpha)}-\int_{(\beta)}\right\}D\left(u+w-s,u-v;e\left(\frac{h}{l}\right)\right)\zeta\left(s,e\left(\frac{h}{l}\right)\right)W^*(s,w)\,ds \\ &\quad= l^{u-v-1}\zeta(v-u+1)W^*(u+w-1,w)\sum_{n=1}^{\infty}c_l(n)n^{1-u-w} \\ &\quad\quad+l^{v-u-1}\zeta(u-v+1)W^*(v+w-1,w)\sum_{n=1}^{\infty}c_l(n)n^{1-v-w}, \end{aligned} \tag{2.14}$$

where $\beta>\alpha+1$. In the integral over the line $\mathrm{Re}(s)=\beta$ we have $\mathrm{Re}(u+w-s)<0$ and $\mathrm{Re}(v+w-s)<0$, so that we may replace $D(u+w-s,u-v;e(h/l))$ by the absolutely convergent Dirichlet series inferred from (2.13). This yields, after some rearrangement,

$$\begin{aligned} &\sum_{\substack{h=1\\(h,l)=1}}^{l}\frac{1}{2\pi i}\int_{(\beta)}D\left(u+w-s,u-v;e\left(\frac{h}{l}\right)\right)\zeta\left(s,e\left(\frac{h}{l}\right)\right)W^*(s,w)\,ds \\ &= 2(2\pi)^{w-z-1}l^{z-w}\sum_{m,n=1}^{\infty}m^{\frac{1}{2}(1-u-v-w-z)}n^{\frac{1}{2}(u+w-v-z-1)}\sigma_{v-u}(n) \\ &\times\left\{\cos\left(\frac{\pi}{2}(u-v)\right)\psi\left(\frac{4\pi\sqrt{mn}}{l};u,v,w,z\right)S(m,n;l)-\varphi\left(\frac{4\pi\sqrt{mn}}{l};u,v,w,z\right)S(m,-n;l)\right\}. \end{aligned} \tag{2.15}$$

Here $S(m,n;l)$ is the Kloosterman sum

$$\sum_{\substack{h=1\\(h,l)=1}}^{l} e\Big(\frac{1}{l}(mh+n\bar{h})\Big),$$

and

$$\begin{aligned}\psi(x;u,v,w,z)=\frac{1}{2\pi i}\int_{(\beta)}\Big(\frac{x}{2}\Big)^{u+v+w+z-1-2s}&\\ \times\Gamma(s+1-u-w)\Gamma(s+1-v-w)W^*(s,w)\,ds,&\end{aligned} \tag{2.16}$$

$$\begin{aligned}\varphi(x;u,v,w,z)=\frac{1}{2\pi i}\int_{(\beta)}\Big(\frac{x}{2}\Big)^{u+v+w+z-1-2s}\cos\big(\pi\big(w+\tfrac{1}{2}(u+v)-s\big)\big)&\\ \times\Gamma(s+1-u-w)\Gamma(s+1-v-w)W^*(s,w)\,ds.&\end{aligned} \tag{2.17}$$

Note that the last integral converges absolutely by virtue of (2.9).

Collecting (2.6), (2.12), (2.14) and (2.15), we obtain

LEMMA 1. *$Y_1(u,v,w,z)$ can be continued meromorphically to the domain $E(\beta)$, and there we have the decomposition*

$$Y_1(u,v,w,z)=Y_2(u,v,w,z)+Y_3^-(u,v,w,z)+Y_3^+(u,v,w,z). \tag{2.18}$$

Here

$$\begin{aligned}&Y_2(u,v,w,z) \qquad (2.19)\\ &=\zeta(u+z)\zeta(v+w-1)\zeta(z-w+1)\zeta(u-v+1)W^*(v+w-1,w)\{\zeta(u+z-v-w+2)\}^{-1}\\ &\quad+\zeta(v+z)\zeta(u+w-1)\zeta(z-w+1)\zeta(v-u+1)W^*(u+w-1,w)\{\zeta(v+z-u-w+2)\}^{-1},\end{aligned}$$

$$\begin{aligned}Y_3^-(u,v,w,z)&=-2(2\pi)^{w-z-1}\zeta(z-w+1)\\ &\times\sum_{m,n=1}^{\infty} m^{\frac{1}{2}(1-u-v-w-z)}n^{\frac{1}{2}(u+w-v-z-1)}\sigma_{v-u}(n)K^-_{m,n}(u,v,w,z),\end{aligned} \tag{2.20}$$

$$\begin{aligned}Y_3^+(u,v,w,z)&=2(2\pi)^{w-z-1}\zeta(z-w+1)\cos\big(\tfrac{1}{2}\pi(u-v)\big)\\ &\times\sum_{m,n=1}^{\infty} m^{\frac{1}{2}(1-u-v-w-z)}n^{\frac{1}{2}(u+w-v-z-1)}\sigma_{v-u}(n)K^+_{m,n}(u,v,w,z),\end{aligned} \tag{2.21}$$

where

$$K^-_{m,n}(u,v,w,z)=\sum_{l=1}^{\infty}\frac{1}{l}S(m,-n;l)\varphi\Big(\frac{4\pi\sqrt{mn}}{l};u,v,w,z\Big), \tag{2.22}$$

$$K^+_{m,n}(u,v,w,z)=\sum_{l=1}^{\infty}\frac{1}{l}S(m,n;l)\psi\Big(\frac{4\pi\sqrt{mn}}{l};u,v,w,z\Big), \tag{2.23}$$

and φ, ψ *are defined by* (2.16), (2.17).

In fact, as can be easily checked, the series for $Y_3^{\pm}$ converge absolutely and uniformly in $E(\beta)$. It should be remarked that we need here only trivial bounds for Kloosterman sums. This ends the reduction of our problem to sums of Kloosterman sums. What we have to undertake next is to continue $Y_3^{\pm}$ to the whole $\mathbf{C}^4$. That will be achieved in the subsequent sections.

But, before proceeding further we make here a little digression. This is to remark on the existence of an alternative argument which yields Lemma 1. As a matter of fact, in the original version of our proof of Lemma 1 we exploited a simple idea given in [9], which is, in short a twist of (1.3). For any $(a,b)=1$ we have

$$\sum_{m,n} f(m,n) = \left\{ \sum_{am=bn} + \sum_{am<bn} + \sum_{am>bn} \right\} f(m,n).$$

What is important in this is that the left side is independent of a and b. Thus, multiplying both sides by certain weights depending on a and b, and summing over all $(a,b)=1$, one may get a transformation of the left side. When f is sufficiently smooth the result of this process is essentially a sum of Kloosterman sums. A special case is, indeed, Lemma 1. In general, this argument may be regarded as a completion of van der Corput's method, and will probably be used as a means to get some non-trivial bounds for exponential sums with the aid of $\mathrm{SL}(2,\mathbf{Z})$ theory.

3. Spectral expansion

The aim of this section is to separate the variables m and n in $K_{m,n}^{\pm}$ by appealing to Kuznetsov's trace formulas.

We begin with $K_{m,n}^{-}$ which is easier than the other. To this we apply the following:

LEMMA 2. *Suppose that* $\varphi(x)$ *is, for* $x \geqslant 0$, *continuously differentiable to third order, and satisfies the conditions*

$$\varphi(0) = \varphi'(0) = 0, \tag{3.1}$$

and

$$\sum_{\nu=0}^{3} |\varphi^{(\nu)}(x)| \ll x^{-1990} \tag{3.2}$$

as x *tends to* $+\infty$. *Put*

$$\check{\varphi}(r) = 2\cosh(\pi t) \int_0^{\infty} K_{2ir}(x)\varphi(x)\,\frac{dx}{x}.$$

Then we have, for any positive integers m and n,

$$\begin{aligned}\sum_{l=1}^{\infty}\frac{1}{l}S(m,-n;l)\varphi\Big(\frac{4\pi\sqrt{mn}}{l}\Big)&=\sum_{j=1}^{\infty}\varepsilon_j\alpha_j t_j(m)t_j(n)\check{\varphi}(\varkappa_j)\\&+\frac{1}{\pi}\int_{-\infty}^{\infty}(mn)^{-ir}\sigma_{2ir}(m)\sigma_{2ir}(n)|\zeta(1+2ir)|^{-2}\check{\varphi}(r)\,dr.\end{aligned}\tag{3.3}$$

Here $\varepsilon_j,\alpha_j,t_j(n),\varkappa_j$ are all defined in the first section.

This is a corrected version of [6, Theorem 7] (see also [8, Theorem 2.7]). We note that this trace formula has been stated erroneously in some basic literature too. The proof is similar to that of [7, Theorem 2], and may be described very briefly as follows: We consider the inner product $(U_m(z,s_1),\overline{U_n(z,s_2)})$ instead of $(U_m(z,s_1),U_n(z,\bar{s}_2))$; the latter is the case treated in [7] (precise definitions of these can be found there). We then obtain, as an analogue of [7, (4.50)],

$$\begin{aligned}8\pi^{-1}\sqrt{mn}\sum_{l=1}^{\infty}l^{-2}S(m,-n;l)K_{2it}\Big(\frac{4\pi\sqrt{mn}}{l}\Big)&=\sum_{j=1}^{\infty}\varepsilon_j\alpha_j t_j(m)t_j(n)H(\varkappa_j,t)\\&+\frac{1}{\pi}\int_{-\infty}^{\infty}(mn)^{-ir}\sigma_{2ir}(m)\sigma_{2ir}(n)|\zeta(1+2ir)|^{-2}H(r,t)\,dr,\end{aligned}\tag{3.4}$$

where $|\mathrm{Im}(t)|<\frac{1}{4}$, and $H(r,t)$ is defined by [7, (4.51)]. Next, we observe that because of (3.1) and (3.2) the above $\check{\varphi}(r)$ satisfies the same condition as that for $h(r)$ in [7, Theorem 1]. In fact, this can be derived from the representation

$$\check{\varphi}(r)=\frac{1}{2\pi i}\cosh(\pi r)\int_{(3/4)}\Gamma(s+ir)\Gamma(s-ir)\varphi^*(-2s)4^s\,ds,$$

where φ^* is the Mellin transform of φ. Having these, we may proceed just as in [7, pp. 327–329], and find that (3.4) yields (3.3) with φ on the left side being replaced by

$$\tilde{\varphi}(x)=\frac{4}{\pi^2}\int_{-\infty}^{\infty}r\sinh(\pi r)K_{2ir}(x)\check{\varphi}(r)\,dr.$$

If $\varphi(x)$ has a compact support on the positive real axis, then the inversion formula for the Kontrovich–Lebedev transform (the K-Bessel transform) gives immediately $\tilde{\varphi}=\varphi$, while the general case can be handled by the usual approximation argument. This ends the proof of Lemma 2. For more details see our manuscript [15], where refinements of Lemma 2 and [7, Theorem 2] can be found.

Now, let us use Lemma 2 with $\varphi(x)=\varphi(x;u,v,w,z)$ defined by (2.17). We assume naturally that (u,v,w,z) is a point in $E(\beta)$. Then $\varphi(x;u,v,w,z)$ is obviously continuously differentiable to third order and satisfies (3.1); also we may show (3.2) by shifting the path in (2.17) to the right appropriately. Note that we need (2.9) here. Hence we have, in $E(\beta)$,

$$\begin{aligned} K_{m,n}^{-}(u,v,w,z) &= \sum_{j=1}^{\infty} \varepsilon_j \alpha_j t_j(m) t_j(n) \check{\varphi}(\varkappa_j; u,v,w,z) \\ &+ \frac{1}{\pi} \int_{-\infty}^{\infty} (mn)^{-ir} \sigma_{2ir}(m) \sigma_{2ir}(n) |\zeta(1+2ir)|^{-2} \check{\varphi}(r;u,v,w,z)\, dr \end{aligned} \tag{3.5}$$

with

$$\check{\varphi}(r;u,v,w,z) = 2\cosh(\pi r) \int_0^{\infty} K_{2ir}(x) \varphi(x;u,v,w,z) \frac{dx}{x}.$$

Inserting (2.17) into the last integral, we get a double integral. On noting that $K_{2ir}(x) \ll |\log x|$ as $x \to +0$ and $\ll \exp(-cx)$ as $x \to +\infty$, both uniformly for all real r, and that we have (2.9), we see that the double integral is absolutely convergent. Interchanging the order of integration, and computing explicitly the inner-integral, we find that

$$\begin{aligned} \check{\varphi}(r;u,v,w,z) = \frac{1}{4\pi i} \cosh(\pi r) \int_{(\beta)} & \cos\bigl(\pi\bigl(w + \tfrac{1}{2}(u+v) - s\bigr)\bigr) \Gamma\bigl(\tfrac{1}{2}(u+v+w+z-1) + ir - s\bigr) \\ \times \Gamma\bigl(\tfrac{1}{2}(u+v+w+z-1) - ir - s\bigr) & \Gamma(s+1-u-w) \Gamma(s+1-v-w) W^*(s,w)\, ds. \end{aligned} \tag{3.6}$$

Next, we shall show that the left side of (3.5) converges uniformly in any compact subset of $E(\beta)$. In fact, we have, for any fixed $A>0$,

$$\check{\varphi}(r;u,v,w,z) \ll |r|^{-A} \tag{3.7}$$

when real r tends to $\pm\infty$ while $(u,v,w,z) \in E(\beta)$ remains bounded; here the implied constant depends on the compact set to which (u,v,w,z) belongs. Combined with (1.6) and (1.7), this gives the uniformity of convergence in (3.5). To show (3.7) we move the path in (3.6) to the one consisting of the straight lines connecting the points $\beta - i\infty$, $\beta - \frac{1}{2}|r|i$, $B - \frac{1}{2}|r|i$, $B + \frac{1}{2}|r|i$, $\beta + \frac{1}{2}|r|i$ and $\beta + i\infty$, where $B>0$ is to be taken sufficiently large. If $|r|$ is larger than a constant determined solely by the compact subset of $E(\beta)$ under consideration, on this change of the path we do not encounter any singularity. Then on the new path we apply Stirling's formula and (2.9) to the integrand, getting (3.7) immediately.

Now, we insert (3.5) into (2.20). By virtue of (1.6), (1.7) and (3.7) we may interchange freely the order of summation and integration as far as $(u,v,w,z) \in E(\beta)$. We then obtain, by (1.8) and (1.9), LEMMA 3. *We have, in $E(\beta)$,*

$$Y_3^-(u,v,w,z)=Y_{3,1}^-(u,v,w,z)+Y_{3,2}^-(u,v,w,z), \tag{3.8}$$

where

$$\begin{aligned}Y_{3,1}^-&(u,v,w,z)\\ &=\frac{1}{\pi i}\int_{(0)}\zeta\big(\tfrac12(u+v+w+z-1)+\xi\big)\zeta\big(\tfrac12(u+v+w+z-1)-\xi\big)\zeta\big(\tfrac12(u+z-v-w+1)+\xi\big)\\ &\quad\times\zeta\big(\tfrac12(u+z-v-w+1)-\xi\big)\zeta\big(\tfrac12(v+z-u-w+1)+\xi\big)\zeta\big(\tfrac12(v+z-u-w+1)-\xi\big)\\ &\quad\times\{\zeta(1+2\xi)\zeta(1-2\xi)\}^{-1}\Phi(\xi;u,v,w,z)\,d\xi,\end{aligned} \tag{3.9}$$

$$\begin{aligned}Y_{3,2}^-(u,v,w,z)=\sum_{j=1}^{\infty}&\varepsilon_j\alpha_jH_j\big(\tfrac12(u+v+w+z-1)\big)H_j\big(\tfrac12(u+z-v-w+1)\big)\\ &\times H_j\big(\tfrac12(v+z-u-w+1)\big)\Phi(i\varkappa_j;u,v,w,z).\end{aligned} \tag{3.10}$$

Here

$$\Phi(\xi;u,v,w,z)=-2(2\pi)^{w-z-1}\check{\varphi}(i\xi;u,v,w,z) \tag{3.11}$$

with $\check{\varphi}$ being defined by (3.6).

We shall prove in the next section that Φ can be continued meromorphically to the entire $\mathbf{C}^5$. As a preparation we observe here that Φ exists in a fairly wide range of the five variables. In fact we may define Φ by

$$\begin{aligned}\Phi&(\xi;u,v,w,z)\\ &=i(2\pi)^{w-z-2}\cos(\pi\xi)\int_{-i\infty}^{i\infty}\cos\big(\pi\big(w+\tfrac12(u+v)-s\big)\big)\Gamma\big(\tfrac12(u+v+w+z-1)+\xi-s\big)\\ &\quad\times\Gamma\big(\tfrac12(u+v+w+z-1)-\xi-s\big)\Gamma(s+1-u-w)\Gamma(s+1-v-w)W^*(s,w)\,ds,\end{aligned} \tag{3.12}$$

where the path is curved to ensure that the poles of the first two Γ-factors in the integrand lie to the right of the path, and those of other factors are on the left of the path; we assume that ξ,u,v,w,z are such that the path can be drawn. Note that the poles of $W^*(s,w)$ are at non-positive integers. When ξ is purely imaginary and $(u,v,w,z)\in E(\beta)$, i.e., the situation in Lemma 3, the line $\mathrm{Re}(s)=\beta$ can be used as the path; so (3.11) holds under the new definition (3.12) of Φ.

Let us now turn to $K_{m,n}^+$. To this we may apply [7, Theorem 2]. However, we shall take a different way, for a direct application of this trace formula causes some difficulties peculiar to our present situation. We shall use, instead, Kuznetsov's spectral decomposition [7, (7.26)] of the Kloosterman-sum zeta-function:

LEMMA 4. *For positive integers m, n we put*

$$Z_{m,n}(s) = (2\pi\sqrt{mn})^{2s-1} \sum_{l=1}^{\infty} S(m,n;l) l^{-2s}. \tag{3.13}$$

Then we have, for $\mathrm{Re}(s) > \frac{1}{2}$,

$$Z_{m,n}(s) = \frac{1}{2}\sin(\pi s) \sum_{j=1}^{\infty} \alpha_j t_j(m) t_j(n) \Gamma\left(s-\tfrac{1}{2}+i\varkappa_j\right)\Gamma\left(s-\tfrac{1}{2}-i\varkappa_j\right) \tag{3.14}$$

$$+\sum_{k=1}^{\infty} P_{m,n}(k) \frac{\Gamma(k-1+s)}{\Gamma(k+1-s)} \tag{3.15}$$

$$+\frac{1}{2\pi}\sin(\pi s) \int_{-\infty}^{\infty} (mn)^{-ir} \sigma_{2ir}(m)\sigma_{2ir}(n) |\zeta(1+2ir)|^{-2} \Gamma\left(s-\tfrac{1}{2}+ir\right)\Gamma\left(s-\tfrac{1}{2}-ir\right) dr \tag{3.16}$$

$$-\frac{1}{2\pi}\delta_{m,n}\frac{\Gamma(s)}{\Gamma(1-s)} \tag{3.17}$$

Here $\delta_{m,n}$ is the Kronecker delta, and

$$P_{m,n}(k) = (2k-1) \sum_{l=1}^{\infty} \frac{1}{l} S(m,n;l) J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{l}\right) \tag{3.18}$$

with the J-Bessel function J_{2k-1}.

The sum in (3.13) converges absolutely for $\mathrm{Re}(s) > \frac{3}{4}$ because of Weil's estimate for $S(m,n;l)$. But, the sums (3.14), (3.15) and the integral (3.16) are all absolutely convergent for $\mathrm{Re}(s) > \frac{1}{2}$. This is trivial for (3.16); and for (3.14) it follows from (1.6) and (1.7). As for (3.15) we remark that (3.18) implies

$$P_{m,n}(k) \ll \frac{1}{\Gamma(2k-1)} (2\pi\sqrt{mn})^{2k-1}, \tag{3.19}$$

where the implied constant is absolute; this is a consequence of the trivial bound for $S(m,n;l)$ and the integral representation

$$J_\nu(x) = \frac{2(\frac{1}{2}x)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^1 \cos(xt)(1-t^2)^{\nu-1/2}\, dt, \quad \mathrm{Re}(\nu) > -\tfrac{1}{2}.$$

Thus the above assertion on (3.15) follows.

Also, we remark that by virtue of the well-known identity of Petersson we have

$$P_{m,n}(k) = (-1)^k \frac{\pi}{2} \sum_{j=1}^{\vartheta(k)} \alpha_{j,k} t_{j,k}(m) t_{j,k}(n) + (-1)^{k-1} \frac{1}{2\pi}(2k-1)\delta_{m,n} \tag{3.20}$$

with the notation introduced in the first section; note that when $k<6$ the sum over j is empty. Then, (3.19) and (3.20) yield (1.15); in fact, $p_{1,1}(k)$ is infinitesimally small as k tends to $+\infty$.

Now the relation between $Z_{m,n}$ and our $K_{m,n}^{+}$ is obvious: We have, in $E(\beta)$,

$$\begin{aligned}&K_{m,n}^{+}(u,v,w,z)\\&\quad=\frac{1}{2\pi i}\int_{(\beta)}Z_{m,n}\big(\tfrac{1}{2}(u+v+w+z)-s\big)\Gamma(s+1-u-w)\Gamma(s+1-v-w)W^{*}(s,w)\,ds.\end{aligned}$$

Into this we insert (3.14)–(3.17), getting the decomposition

$$K_{m,n}^{+}(u,v,w,z)=\sum_{\nu=1}^{4}L_{m,n}^{(\nu)}(u,v,w,z), \tag{3.21}$$

where the terms are in the obvious order. Then, by (3.14) we have, after interchanging the order of summation and integration,

$$L_{m,n}^{(1)}(u,v,w,z)=\sum_{j=1}^{\infty}\alpha_j t_j(m)t_j(n)\widehat{\psi}(\varkappa_j;u,v,w,z), \tag{3.22}$$

where

$$\begin{aligned}\widehat{\psi}(r;u,v,w,z)=\frac{1}{4\pi i}\int_{(\beta)}&\sin\big(\tfrac{1}{2}\pi(u+v+w+z-2s)\big)\Gamma\big(\tfrac{1}{2}(u+v+w+z-1)+ir-s\big)\\&\times\Gamma\big(\tfrac{1}{2}(u+v+w+z-1)-ir-s\big)\Gamma(s+1-u-w)\Gamma(s+1-v-w)W^{*}(s,w)\,ds.\end{aligned} \tag{3.23}$$

This procedure is legitimate. In fact, for real r and fixed (u,v,w,z) the integral along $\mathrm{Re}(s)=\beta$ of the absolute value of the integrand in (3.23) is $O(e^{-\pi|r|/2})$, which is a consequence of Stirling's formula and (2.9). Then (3.22) is confirmed by (1.6) and (1.7). Thus we have, in particular,

$$\widehat{\psi}(r;u,v,w,z)\ll e^{-\pi|r|/2} \tag{3.24}$$

uniformly in any compact subset of $E(\beta)$.

Similarly we have

$$L_{m,n}^{(3)}(u,v,w,z)=\frac{1}{\pi}\int_{-\infty}^{\infty}(mn)^{-ir}\sigma_{2ir}(m)\sigma_{2ir}(n)|\zeta(1+2ir)|^{-2}\widehat{\psi}(r;u,v,w,z)\,dr. \tag{3.25}$$

Before considering $L_{m,n}^{(2)}$ it is expedient to introduce another function of five complex variables:

$$\begin{aligned}&\Xi(\xi;u,v,w,z)\\&\quad=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(\frac{1}{2}(u+v+w+z-1)+\xi-s)}{\Gamma(\frac{1}{2}(3-u-v-w-z)+\xi+s)}\Gamma(s+1-u-w)\Gamma(s+1-v-w)W^{*}(s,w)\,ds.\end{aligned} \tag{3.26}$$

Here the path is curved so that the poles of $\Gamma\big(\frac{1}{2}(u+v+w+z-1)+\xi-s\big)$ and those of $\Gamma(s+1-u-w)\Gamma(s+1-v-w)W^*(s,w)$ are separated to the right and the left, respectively, by the path; and ξ,u,v,w,z are such that the path can be drawn. This function will play an important rôle in our discussion below.

We then note that the combination of (2.9), (3.19) and Stirling's formula allows us to interchange the order of summation and integration in $L^{(2)}_{m,n}$, giving

$$L^{(2)}_{m,n}(u,v,w,z)=\sum_{k=1}^{\infty}P_{m,n}(k)\Xi\big(k-\tfrac{1}{2};u,v,w,z\big) \tag{3.27}$$

for $(u,v,w,z)\in E(\beta)$. In this, each Ξ-factor has the representation (3.26) with $\xi=k-\frac{1}{2}$ and the path $\mathrm{Re}(s)=\beta$, and moreover we have, for any fixed $A>0$,

$$\Xi\big(k-\tfrac{1}{2};u,v,w,z\big)\ll k^{-A},\quad k\geqslant 1 \tag{3.28}$$

uniformly in any compact subset of $E(\beta)$. To show the latter we may assume naturally that k is larger than a constant determined solely by the compact set under consideration. We then shift the path in the integral for $\Xi\big(k-\frac{1}{2};u,v,w,z\big)$ to the line $\mathrm{Re}(s)=B$ with a fixed large $B>0$. The rest of the proof is a simple application of (2.9) and Stirling's formula to the integrand.

Because of (3.28) we may modify (3.27) as follows:

$$L^{(2)}_{m,n}(u,v,w,z)=L^{(5)}_{m,n}(u,v,w,z)+L^{(6)}_{m,n}(u,v,w,z),$$

where

$$L^{(5)}_{m,n}(u,v,w,z)=\sum_{k=1}^{\infty}\left\{P_{m,n}(k)+\frac{1}{2\pi}(-1)^k(2k-1)\delta_{m,n}\right\}\Xi\big(k-\tfrac{1}{2};u,v,w,z\big)$$

and

$$L^{(6)}_{m,n}(u,v,w,z)=\frac{1}{2\pi}\delta_{m,n}\sum_{k=1}^{\infty}(-1)^{k-1}(2k-1)\Xi\big(k-\tfrac{1}{2};u,v,w,z\big). \tag{3.29}$$

Then (3.20) implies

$$L^{(5)}_{m,n}(u,v,w,z)=\frac{\pi}{2}\sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}t_{j,k}(m)t_{j,k}(n)\Xi\big(k-\tfrac{1}{2};u,v,w,z\big). \tag{3.30}$$

On the other hand we have

$$L^{(6)}_{m,n}(u,v,w,z)=-L^{(4)}_{m,n}(u,v,w,z). \tag{3.31}$$

To show this we note the identity

$$(2k-1)\frac{\Gamma(k-1+s)}{\Gamma(k+1-s)}=\frac{\Gamma(k+s)}{\Gamma(k+1-s)}+\frac{\Gamma(k-1+s)}{\Gamma(k-s)}.$$

This and (3.26) with $\xi=k-\frac{1}{2}$ and the path $\mathrm{Re}(s)=\beta$ imply that (3.29) can be written as

$$L_{m,n}^{(6)}(u,v,w,z)=\frac{1}{2\pi}\delta_{m,n}\sum_{k=1}^{\infty}(-1)^{k-1}\{F_k+F_{k-1}\}, \tag{3.32}$$

where

$$F_k=\frac{1}{2\pi i}\int_{(\beta)}\frac{\Gamma(\frac{1}{2}(u+v+w+z)+k-s)}{\Gamma(\frac{1}{2}(2-u-v-w-z)+k+s)}\Gamma(s+1-u-w)\Gamma(s+1-v-w)W^*(s,w)\,ds.$$

But we have $F_k\ll(1+k)^{-A}$ $(k\geqslant 0)$, which can be proved in just the same way as (3.28). Hence (3.32) reduces to

$$L_{m,n}^{(6)}(u,v,w,z)=\frac{1}{2\pi}\delta_{m,n}F_0,$$

which is equivalent to (3.31).

Thus we may write (3.21) as

$$K_{m,n}^{+}(u,v,w,z)=L_{m,n}^{(1)}(u,v,w,z)+L_{m,n}^{(3)}(u,v,w,z)+L_{m,n}^{(5)}(u,v,w,z).$$

We insert this into (2.21). By virtue of (1.6), (1.7), (1.12), (1.15), (3.24) and (3.28), we may interchange freely the order of summation and integration as far as $(u,v,w,z)\in E(\beta)$. Then, collecting (1.8), (1.9), (1.13), (3.22), (3.25) and (3.30), we obtain

LEMMA 5. *We have, in $E(\beta)$,*

$$Y_3^{+}(u,v,w,z)=Y_{3,1}^{+}(u,v,w,z)+Y_{3,2}^{+}(u,v,w,z)+Y_{3,3}^{+}(u,v,w,z), \tag{3.33}$$

where

$$\begin{aligned}Y_{3,1}^{+}(u,v,w,z)=\frac{1}{i\pi}\int_{(0)}&\zeta\big(\tfrac{1}{2}(u+v+w+z-1)+\xi\big)\zeta\big(\tfrac{1}{2}(u+v+w+z-1)-\xi\big)\\&\times\zeta\big(\tfrac{1}{2}(u+z-v-w+1)+\xi\big)\zeta\big(\tfrac{1}{2}(u+z-v-w+1)-\xi\big)\\&\times\zeta\big(\tfrac{1}{2}(v+z-u-w+1)+\xi\big)\zeta\big(\tfrac{1}{2}(v+z-u-w+1)-\xi\big)\\&\times\{\zeta(1+2\xi)\zeta(1-2\xi)\}^{-1}\Psi(\xi;u,v,w,z)\,d\xi,\end{aligned} \tag{3.34}$$

$$\begin{aligned}Y_{3,2}^{+}(u,v,w,z)=\sum_{j=1}^{\infty}&\alpha_jH_j\big(\tfrac{1}{2}(u+v+w+z-1)\big)H_j\big(\tfrac{1}{2}(u+z-v-w+1)\big)\\&\times H_j\big(\tfrac{1}{2}(v+z-u-w+1)\big)\Psi(i\varkappa_j;u,v,w,z),\end{aligned} \tag{3.35}$$

$$
\begin{aligned}
&Y_{3,3}^{+}(u,v,w,z)\\
&=\tfrac{1}{2}(2\pi)^{w-z}\cos\bigl(\tfrac{1}{2}\pi(u-v)\bigr)\sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}H_{j,k}\bigl(\tfrac{1}{2}(u+v+w+z-1)\bigr)\\
&\quad\times H_{j,k}\bigl(\tfrac{1}{2}(u+z-v-w+1)\bigr)H_{j,k}\bigl(\tfrac{1}{2}(v+z-u-w+1)\bigr)\Xi\bigl(k-\tfrac{1}{2};u,v,w,z\bigr).
\end{aligned}\tag{3.36}
$$

Here

$$\Psi(\xi;u,v,w,z)=2(2\pi)^{w-z-1}\cos\bigl(\tfrac{1}{2}\pi(u-v)\bigr)\widehat{\psi}(i\xi;u,v,w,z)$$

with $\widehat{\psi}$ *being defined by* (3.23).

As before, Ψ can be defined more generally by

$$
\begin{aligned}
\Psi(\xi;u,v,w,z)=-i(2\pi)^{w-z-2}\cos\bigl(\tfrac{1}{2}\pi(u-v)\bigr)&\int_{-i\infty}^{i\infty}\sin\bigl(\tfrac{1}{2}\pi(u+v+w+z-2s)\bigr)\\
&\times\Gamma\bigl(\tfrac{1}{2}(u+v+w+z-1)+\xi-s\bigr)\Gamma\bigl(\tfrac{1}{2}(u+v+w+z-1)-\xi-s\bigr)\\
&\times\Gamma(s+1-u-w)\Gamma(s+1-v-w)W^{*}(s,w)ds,
\end{aligned}\tag{3.37}
$$

where the convention about the path and the location of ξ,u,v,w,z is the same as in (3.12).

This ends the spectral expansion of $Y_1(u,v,w,z)$ when (u,v,w,z) is in $E(\beta)$. What remains for us to do is to continue analytically the above expansions to a neighbourhood of the point P_T.

4. Analytic continuation

The aim of this section is to show that the spectral expansions obtained in the preceding section can be continued to the entire $\mathbf{C}^4$, and thereby we finish the proof of the existence of Y_1 as a meromorphic function over $\mathbf{C}^4$.

By virtue of Lemmas 1, 3 and 5 our problem is equivalent to studying the analytical properties of the functions Φ, Ψ and Ξ, for the functions H_j and $H_{j,k}$ are entire. But, their definitions imply readily the relations:

$$
\begin{aligned}
\Phi(\xi;u,v,w,z)=\frac{(2\pi)^{w-z}}{4\sin(\pi\xi)}\bigl\{&\sin\bigl(\pi\bigl(\tfrac{1}{2}(z-w)+\xi\bigr)\bigr)\Xi(\xi;u,v,w,z)\\
&-\sin\bigl(\pi\bigl(\tfrac{1}{2}(z-w)-\xi\bigr)\bigr)\Xi(-\xi;u,v,w,z)\bigr\},
\end{aligned}\tag{4.1}
$$

$$\Psi(\xi;u,v,w,z)=-\frac{(2\pi)^{w-z}\cos(\frac{1}{2}\pi(u-v))}{4\sin(\pi\xi)}\{\Xi(\xi;u,v,w,z)-\Xi(-\xi;u,v,w,z)\}.\tag{4.2}$$

Hence our problem is reduced to the study of Ξ, and we are going to show that it is meromorphic over the entire $\mathbf{C}^5$. Intuitively this fact can be inferred from (3.26) by deforming the path appropriately. However, the topological situation involved here is somewhat complicated; so we employ an explicit argument to avoid any ambiguities.

To this end we introduce the set

$$N=\{(\xi,u,v,w,z):\text{at least one of } \xi+\tfrac{1}{2}(u+v+w+z-1), \xi+\tfrac{1}{2}(u+z-v-w+1) \text{ and } \xi+\tfrac{1}{2}(v+z-u-w+1) \text{ is equal to a non-positive integer}\} \tag{4.3}$$

We put $N^*=\mathbf{C}^5\setminus N$, which is open and arcwise connected. This can be proved easily by connecting two points of N^* by a straight line with possible indents. If $(\xi,u,v,w,z)\in N^*$, then we can obviously draw a path which is needed in (3.26); thus Ξ is well-defined at all points of N^*. Then, by a routine argument we can show that Ξ is regular and single-valued over N^*. Namely, starting at a point of N^*, Ξ can be continued analytically to any point of N^*, and the result is always given by the representation (3.26) with a suitable choice of the path.

Having this, we confine (ξ,u,v,w,z) in the domain defined by the conditions:

$$\begin{bmatrix} (u,v,w,z)\in E(\beta), \\ \mathrm{Max}\{|\mathrm{Re}(u)|,|\mathrm{Re}(v)|,|\mathrm{Re}(w)|,|\mathrm{Re}(z)|\}<Q \\ \mathrm{Re}(\xi)>3Q, \end{bmatrix} \tag{4.4}$$

where Q is a large positive constant. Then we may use the line $\mathrm{Re}(s)=\beta$ as the path in (3.26); the above domain is a subset of N^*. On setting this, we insert (2.8) into (3.26). The resulting double integral is absolutely convergent. Interchanging the order of integration, we have

$$\Xi(\xi;u,v,w,z)=\int_0^\infty y^{-1}(1+y)^{-w}\exp\left(-\left(\tfrac{1}{2}\Delta\log(1+y)\right)^2\right)G(y;\xi;u,v,w,z)\,dy, \tag{4.5}$$

where

$$G(y;\xi;u,v,w,z) = \frac{1}{2\pi i}\int_{(\beta)}\frac{\Gamma(\frac{1}{2}(u+v+w+z-1)+\xi-s)}{\Gamma(\frac{1}{2}(3-u-v-w-z)+\xi+s)}\Gamma(s+1-u-w)\Gamma(s+1-v-w)y^s\,ds, \tag{4.6}$$

or rather, using the hypergeometric function F,

$$G(y;\xi;u,v,w,z)=\Gamma(A)\Gamma(B)\Gamma(C)^{-1}F(A,B;C;-y)y^{\xi+\frac{1}{2}(u+v+w+z-1)} \tag{4.7}$$

with

$$A=\xi+\tfrac{1}{2}(u+z-v-w+1),\quad B=\xi+\tfrac{1}{2}(v+z-u-w+1),\quad C=1+2\xi.$$

We then invoke Gauss' integral representation of the hypergeometric function: For $|y|<1$

$$F(a,b;c;y)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 x^{a-1}(1-x)^{c-a-1}(1-xy)^{-b}\,dx,$$

providing $\mathrm{Re}(c)>\mathrm{Re}(a)>0$. Thus, if $0\leqslant y<1$, we have, instead of (4.7),

$$\begin{aligned}G(y;\xi;u,v,w,z)&=\frac{\Gamma(\xi+\frac{1}{2}(v+z-u-w+1))}{\Gamma(\xi+\frac{1}{2}(v+w-u-z+1))}\,y^{\xi+\frac{1}{2}(u+v+w+z-1)}\\&\quad\times\int_0^1 x^{\xi+\frac{1}{2}(u+z-v-w-1)}(1-x)^{\xi+\frac{1}{2}(v+w-u-z-1)}(1+xy)^{-\xi+\frac{1}{2}(u+w-v-z-1)}\,dx.\end{aligned}\tag{4.8}$$

But both sides of (4.8) are obviously regular for $\mathrm{Re}(y)>0$; hence, by analytic continuation, (4.8) holds for all $y\geqslant 0$. We next transform (4.5). To this end we note that for any complex η and any positive integer P

$$\begin{aligned}&\frac{1}{\Gamma(\eta+1)}(1+x)^{\eta}\\&\qquad=\sum_{j=0}^{P-1}\frac{x^j}{\Gamma(j+1)\Gamma(\eta-j+1)}+\frac{x^P}{\Gamma(P)\Gamma(\eta-P+1)}\int_0^1(1-\theta)^{P-1}(1+x\theta)^{\eta-P}\,d\theta.\end{aligned}$$

We apply this to the last factor of the integrand of (4.8); then we get a new expression for $G(y;\xi;u,v,w,z)$, which consists of a sum with P terms and an explicit remainder term. This sum is exactly the sum of the residues of the first P poles of the integrand in (4.6) which are on the right of the path. Inserting this result on G into (4.5), we find that

$$\begin{aligned}\Xi(\xi;u,v,w,z)&=\sum_{j=0}^{P-1}(-1)^j\frac{\Gamma(j+\xi+\frac{1}{2}(u+z-v-w+1))\Gamma(j+\xi+\frac{1}{2}(v+z-u-w+1))}{\Gamma(j+1)\Gamma(j+1+2\xi)}\\&\quad\times W^*\big(j+\xi+\tfrac{1}{2}(u+v+w+z-1),w\big)+(-1)^P\frac{\Gamma(P+\xi+\frac{1}{2}(v+z-u-w+1))}{\Gamma(P)\Gamma(\xi+\frac{1}{2}(v+w-u-z+1))}\\&\quad\times\int_0^1(1-x)^{\xi+\frac{1}{2}(v+w-u-z-1)}J_P(x;\xi;u,v,w,z)\,dx\end{aligned}\tag{4.9}$$

with

$$\begin{aligned}J_P(x;\xi;u,v,w,z)&=x^{P+\xi+\frac{1}{2}(u+z-v-w-1)}\int_0^1\int_0^\infty(1-\theta)^{P-1}y^{P+\xi+\frac{1}{2}(u+v+w+z-3)}\\&\qquad\times(1+xy\theta)^{-P-\xi+\frac{1}{2}(u+w-v-z-1)}(1+y)^{-w}\exp\big(-(\tfrac{1}{2}\Delta\log(1+y))^2\big)\,dy\,d\theta.\end{aligned}$$

So far we have assumed (4.4); but we may now drop it. Indeed (4.9) yields a meromorphic continuation of Ξ to the whole $\mathbf{C}^5$. The meromorphy of the first P terms in (4.9) is obvious. In the last term of (4.9) we perform partial integration $[P/2]$ times. The resulting integral converges absolutely and uniformly in the domain which is defined solely by $\mathrm{Max}\{|\mathrm{Re}(\xi)|,|\mathrm{Re}(u)|,|\mathrm{Re}(v)|,|\mathrm{Re}(w)|,|\mathrm{Re}(z)|\}<P/6$. Since P is arbitrary, we have finished the proof of the first assertion in the following:

LEMMA 6. *$\Xi(\xi;u,v,w,z)$ is meromorphic over the entire $\mathbf{C}^5$, and regular except for the points in the set N defined in* (4.3). *Moreover, if $|\xi|$ tends to infinity in any fixed vertical strip, we have, for any fixed $A>0$,*

$$\Xi(\xi;u,v,w,z)\ll|\xi|^{-A} \tag{4.10}$$

uniformly for bounded (u,v,w,z). And the same holds when $\mathrm{Re}(\xi)$ tends to $+\infty$ in any fixed horizontal strip.

The decay property (4.10) can be proved in much the same way as (3.7); also the third assertion in the lemma can be shown by a slight modification of the proof of (3.28). So we may omit the details.

As an imediate consequence of Lemma 6 we state

COROLLARY TO LEMMA 6. *If $(\pm\xi,u,v,w,z)$ are not in N, then the relations* (4.1) *and* (4.2) *hold; thus Φ and Ψ are meromorphic over $\mathbf{C}^5$. Also, as functions of ξ, they are of rapid decay uniformly for bounded (u,v,w,z) when $|\xi|$ tends to infinity in any fixed vertical strip. In particular, $Y_{3,2}^{\pm}$ exist as meromorphic functions over the entire $\mathbf{C}^4$; and the same holds for $Y_{3,3}^{+}$.*

The assertion on $Y_{3,2}^{\pm}$ is a consequence of (1.6), (1.11) and the rapid decay of Φ and Ψ. Similarly the assertion on $Y_{3,3}^{+}$ follows from (1.14), (1.15) and the last statement in Lemma 6. We should remark here that if $(\pm\xi,u,v,w,z)\notin N$ then we can draw the paths in (3.12) and (3.37).

Now it remains for us to consider the continuation of $Y_{3,1}^{\pm}$ which are the contributions of the continuous spectrum. To this end we assume first that (u,v,w,z) is in $E(\beta)$. Then, putting

$$Y_c(u,v,w,z)=Y_{3,1}^{-}(u,v,w,z)+Y_{3,1}^{+}(u,v,w,z) \tag{4.11}$$

and

$$\begin{aligned}S(\xi;u,v,w,z)=&\zeta\big(\tfrac12(u+v+w+z-1)+\xi\big)\zeta\big(\tfrac12(u+v+w+z-1)-\xi\big)\\&\times\zeta\big(\tfrac12(u+z-v-w+1)+\xi\big)\zeta\big(\tfrac12(u+z-v-w+1)-\xi\big)\\&\times\zeta\big(\tfrac12(v+z-u-w+1)+\xi\big)\zeta\big(\tfrac12(v+z-u-w+1)-\xi\big),\end{aligned}$$

we have, by (3.9), (3.34), (4.1) and (4.2),

$$\begin{aligned}Y_c(u,v,w,z)=2i(2\pi)^{w-z-2}\int_{(0)}&(2\pi)^{2\xi}\big\{\cos\big(\tfrac12\pi(u-v)\big)-\sin\big(\pi\big(\tfrac12(z-w)+\xi\big)\big)\big\}\\&\times S(\xi;u,v,w,z)\Gamma(1-2\xi)\{\zeta(2\xi)\zeta(1+2\xi)\}^{-1}\Xi(\xi;u,v,w,z)\,d\xi.\end{aligned} \tag{4.12}$$

Here we have used the functional equation for $\zeta(s)$ to transform the factor $\zeta(1-2\xi)^{-1}$. We are going to shift the path in (4.12) to the right. The singularities of the integrand which we may encounter in this procedure are all poles, and located at

$$\tfrac{1}{2}(u+v+w+z-3), \quad \tfrac{1}{2}(u+z-v-w-1), \quad \tfrac{1}{2}(v+z-u-w-1) \tag{4.13}$$

and

$$\varrho/2, \quad n/2, \quad n=2,3,\ldots \tag{4.14}$$

with ϱ running over all complex zeros of $\zeta(s)$. In fact, when $(u,v,w,z)\in E(\beta)$, Lemma 6 implies that $\Xi(\xi;u,v,w,z)$ is regular for $\mathrm{Re}(\xi)\geqslant 0$. And the poles given in (4.13) come from $S(\xi;u,v,w,z)$; those in (4.14) from $\Gamma(1-2\xi)\zeta(2\xi)^{-1}$.

We then assume that besides $(u,v,w,z)\in E(\beta)$,

$$\mathrm{Max}\{|\mathrm{Re}(u)|,|\mathrm{Re}(v)|,|\mathrm{Re}(w)|,|\mathrm{Re}(z)|\}<R \tag{4.15}$$

with an arbitrary large positive integer R. Further, we may suppose, by an obvious reason, that the poles given in (4.13) are all simple, and do not coincide with any of those given in (4.14). Then we move the path in (4.12) to $\mathrm{Re}(\xi)=3R+\frac{1}{4}$, getting

$$Y_c(u,v,w,z)=F_-(u,v,w,z)+U(u,v,w,z)+Y_c^{(R)}(u,v,w,z). \tag{4.16}$$

Here F_- and U are the contributions of residues at the poles given in (4.13) and (4.14), respectively; $Y_c^{(R)}$ is the same as (4.12) but with the path $\mathrm{Re}(\xi)=3R+\frac{1}{4}$. By virtue of Lemma 6, F_- and U are meromorphic over $\mathbf{C}^4$, and $Y_c^{(R)}$ is regular in the domain (4.15) without the condition $(u,v,w,z)\in E(\beta)$. Since R is arbitrary, Y_c is meromorphic over $\mathbf{C}^4$. Therefore we have established the crucial

LEMMA 7. *$Y_1(u,v,w,z)$ exists as a meromorphic function over the entire $\mathbf{C}^4$, and the decomposition* (2.3) *holds throughout $\mathbf{C}^4$.*

In fact, this is a result of collecting (2.18), (2.19), Lemma 3, Lemma 5, Corollary to Lemma 6, (4.11) and the meromorphy of Y_c which has just been proved.

5. Specialization

In this section we shall finish the proof of our explicit formula for $I(T,\Delta)$. Having proved Lemma 7, it remains for us only to specialize (2.3) by setting $(u,v,w,z)=P_T$. This amounts to studying the local behaviour, around P_T, of the various components of Y_1 which have been introduced in the above discussion.

More explicitly, collecting (2.18), (3.8), (3.33), (4.11), Corollary to Lemma 6 and Lemma 7, we now have the decomposition, over $\mathbf{C}^4$,

$$Y_1(u,v,w,z)=\{Y_2+Y_c+Y_{3,2}^-+Y_{3,2}^++Y_{3,3}^+\}(u,v,w,z) \tag{5.1}$$

with the obvious abuse of notation. In this $Y_{3,2}^{\pm}$ are regular at P_T. To see this we observe that when (u,v,w,z) is near P_T the point (ir,u,v,w,z) with an arbitrary real r is not in the set N defined in (4.3); hence by (4.1) and (4.2) the functions $\Phi(ir;u,v,w,z)$ and $\Psi(ir;u,v,w,z)$ are regular at P_T for each real r. Then, as a special case of the statement on $Y_{3,2}^{\pm}$ in Corollary to Lemma 6 we can conclude that they are in fact regular at P_T. Also, we can show similarly that $Y_{3,3}^+$ is regular at P_T. Namely we may set $(u,v,w,z)=P_T$ in the series expansions (3.10), (3.35) and (3.36) without any modification, and find that

$$\begin{aligned}&\{Y_{3,2}^-+Y_{3,2}^++Y_{3,3}^+\}(P_T)\\&=\sum_{j=1}^{\infty}\alpha_jH_j\left(\tfrac12\right)^3\{\Phi+\Psi\}(i\varkappa_j;P_T)+\frac12\sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}H_{j,k}\left(\tfrac12\right)^3\Xi\left(k-\tfrac12;P_T\right).\end{aligned} \tag{5.2}$$

Note that we have dropped ε_j's, for $H_j\left(\frac12\right)=0$ if $\varepsilon_j=-1$, which is a consequence of (1.10).

Next, we consider Y_c in the immediate neighbourhood of P_T. We return to (4.16), and move the contour in $Y_c^{(R)}$ back to the imaginary axis, while keeping (u,v,w,z) close to P_T. The poles which we encounter in this process are those given in (4.14) ($n\leqslant 6R$), and $\frac12(3-u-v-w-z)$, which is close to $\frac12$. For, other poles of $S(\xi;u,v,w,z)$ are either close to $-\frac12$ or cancelled by the zeros of the factor $\cos\left(\frac12\pi(u-v)\right)-\sin\left(\pi\left(\frac12(z-w)+\xi\right)\right)$, and moreover Lemma 6 implies that $\Xi(\xi;u,v,w,z)$ is regular for $\mathrm{Re}(\xi)\geqslant-\frac14$. We denote by $F_+(u,v,w,z)$ the contribution of the pole $\frac12(3-u-v-w-z)$. Then we have

$$Y_c^{(R)}(u,v,w,z)=F_+(u,v,w,z)-U(u,v,w,z)+Y_c^*(u,v,w,z),$$

where Y_c^* has the same expression as the right side of (4.12) but with different (u,v,w,z). Hence, by (4.16),

$$Y_c(u,v,w,z)=\{F_++F_-\}(u,v,w,z)+Y_c^*(u,v,w,z) \tag{5.3}$$

when (u,v,w,z) is close to P_T. Here we should note that Y_c^* is regular at P_T, and

$$Y_c^*(P_T)=\frac1\pi\int_{-\infty}^{\infty}\frac{|\zeta(\frac12+it)|^6}{|\zeta(1+2it)|^2}\{\Phi+\Psi\}(it;P_T)\,dt. \tag{5.4}$$

This ends the local study of the decomposition (5.1) in the vicinity of P_T. By replacing P_T by P_{-T} we get the same result as above for $Y_1(w,z,u,v)$. Then, invoking Lemma 7, we collect (2.3), (2.4), (5.1) and (5.3). This gives, for (u,v,w,z) near P_T,

$$\begin{aligned}Y(u,v,w,z)&=M(u,v,w,z)+Y_c^*(u,v,w,z)+Y_c^*(w,z;u,v)\\&\quad+\{Y_{3,2}^-+Y_{3,2}^++Y_{3,3}^+\}(u,v,w,z)+\{Y_{3,2}^-+Y_{3,2}^++Y_{3,3}^+\}(w,z,u,v),\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}M(u,v,w,z)=&\,Y_0(u,v,w,z)+Y_2(u,v,w,z)+Y_2(w,z,u,v)\\&+\{F_+ + F_-\}(u,v,w,z)+\{F_+ + F_-\}(w,z,u,v).\end{aligned}\tag{5.6}$$

We should stress here that M is regular at P_T, for we know already that in (5.5) all members except for M are regular at P_T. Hence we have, by (5.2), (5.4) and (5.5),

$$\begin{aligned}Y(P_T)=M(P_T)+2\,\mathrm{Re}\Bigg[&\sum_{j=1}^{\infty}\alpha_j H_j\big(\tfrac12\big)^3\{\Phi+\Psi\}(i\varkappa_j;P_T)\\&+\frac12\sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}H_{j,k}\big(\tfrac12\big)^3\Xi\big(k-\tfrac12;P_T\big)\\&+\frac1\pi\int_{-\infty}^{\infty}\frac{|\zeta(\frac12+it)|^6}{|\zeta(1+2it)|^2}\{\Phi+\Psi\}(it;P_T)\,dt\Bigg].\end{aligned}\tag{5.7}$$

Here we have used the fact that for real r and integral k we have $\Phi(ir;P_{-T})=\overline{\Phi(ir;P_T)}$, $\Psi(ir;P_{-T})=\overline{\Psi(ir;P_T)}$ and $\Xi\big(k-\frac12;P_{-T}\big)=\overline{\Xi\big(k-\frac12;P_T\big)}$; these are consequences of the definitions (3.12), (3.26) and (3.37).

We are now going to transform $M(P_T)$ into a closed from. For this sake we compute $F_\pm$ explicitly. We have

$$\begin{aligned}F_-(u,v,w,z)=&-(2\pi)^{w-z}\big\{\cos\big(\tfrac12\pi(u-v)\big)-\cos\big(\pi\big(z+\tfrac12(u+v)\big)\big)\big\}\zeta(u+z-1)\zeta(2-v-w)\\&\times\zeta(v+z-1)\zeta(2-u-w)\big\{\cos\big(\tfrac12\pi(u+v+w+z)\big)\zeta(4-u-v-w-z)\big\}^{-1}\\&\times\Xi\big(\tfrac12(u+v+w+z-3);u,v,w,z\big)\\&+(2\pi)^{w-z}\big\{\cos\big(\tfrac12\pi(u-v)\big)+\cos\big(\pi\big(z-w+\tfrac12(u-v)\big)\big)\big\}\\&\times\zeta(u+z-1)\zeta(v+w)\zeta(z-w)\zeta(v-u+1)\big\{\cos\big(\tfrac12\pi(u+z-v-w)\big)\zeta(2-u-z+v+w)\big\}^{-1}\\&\times\Xi\big(\tfrac12(u+z-v-w-1);u,v,w,z\big)\\&+(2\pi)^{w-z}\big\{\cos\big(\tfrac12\pi(u-v)\big)+\cos\big(\pi\big(z-w+\tfrac12(v-u)\big)\big)\big\}\\&\times\zeta(v+z-1)\zeta(u+w)\zeta(z-w)\zeta(u-v+1)\big\{\cos\big(\tfrac12\pi(v+z-u-w)\big)\zeta(2-v-z+u+w)\big\}^{-1}\\&\times\Xi\big(\tfrac12(v+z-u-w-1);u,v,w,z\big)\end{aligned}$$

and

$$\begin{aligned}F_+(u,v,w,z)=&-(2\pi)^{w-z}\big\{\cos\big(\tfrac12\pi(u-v)\big)-\cos\big(\pi\big(w+\tfrac12(u+v)\big)\big)\big\}\zeta(u+z-1)\zeta(2-v-w)\\&\times\zeta(v+z-1)\zeta(2-u-w)\big\{\cos\big(\tfrac12\pi(u+v+w+z)\big)\zeta(4-u-v-w-z)\big\}^{-1}\\&\times\Xi\big(-\tfrac12(u+v+w+z-3);u,v,w,z\big).\end{aligned}$$

In order to simplify these we note that (4.1) and (4.2) give

$$\begin{aligned}(2\pi)^{w-z}\big\{\cos\big(\tfrac12\pi(u-v)\big)+\cos\big(\pi\big(z-w+\tfrac12(u-v)\big)\big)\big\}&\sec\big(\tfrac12\pi(u+z-v-w)\big)\\\times\Xi\big(\tfrac12(u+z-v-w-1);u,v,w,z\big)&\\=4\{\Phi+\Psi\}\big(\tfrac12(u+z-v-w-1);u,v,w,z\big),&\end{aligned}$$

$$\begin{aligned}&(2\pi)^{w-z}\left\{\cos\left(\tfrac{1}{2}\pi(u-v)\right)+\cos\left(\pi\left(z-w+\tfrac{1}{2}(v-u)\right)\right)\right\}\sec\left(\tfrac{1}{2}\pi(v+z-u-w)\right)\\&\qquad\times\Xi\left(\tfrac{1}{2}(v+z-u-w-1);u,v,w,z\right)\\&\qquad=4\{\Phi+\Psi\}\left(\tfrac{1}{2}(v+z-u-w-1);u,v,w,z\right)\end{aligned}$$

as well as

$$\begin{aligned}&(2\pi)^{w-z}\sec\left(\tfrac{1}{2}\pi(u+v+w+z)\right)\\&\times\left[\left\{\cos\left(\tfrac{1}{2}\pi(u-v)\right)-\cos\left(\pi\left(z+\tfrac{1}{2}(u+v)\right)\right)\right\}\Xi\left(\tfrac{1}{2}(u+v+w+z-3);u,v,w,z\right)\right.\\&\left.+\left\{\cos\left(\tfrac{1}{2}\pi(u-v)\right)-\cos\left(\pi\left(w+\tfrac{1}{2}(u+v)\right)\right)\right\}\Xi\left(-\tfrac{1}{2}(u+v+w+z-3);u,v,w,z\right)\right]\\&\qquad=-4\{\Phi+\Psi\}\left(\tfrac{1}{2}(u+v+w+z-3);u,v,w,z\right).\end{aligned}$$

Collecting these and (2.4), (2.19), (5.6), we find that

$$M(u,v,w,z)=\sum_{j=0}^{10}M_j(u,v,w,z), \tag{5.8}$$

where

$$M_0(u,v,w,z)=\zeta(u+w)\zeta(u+z)\zeta(v+w)\zeta(v+z)\{\zeta(u+v+w+z)\}^{-1},$$

$$\begin{aligned}&M_1(u,v,w,z)\\&\quad=\zeta(u+z)\zeta(v+w-1)\zeta(z-w+1)\zeta(u-v+1)\{\zeta(u+z-v-w+2)\}^{-1}W^*(v+w-1,w),\end{aligned}$$

$$\begin{aligned}&M_2(u,v,w,z)\\&\quad=\zeta(v+z)\zeta(u+w-1)\zeta(z-w+1)\zeta(v-u+1)\{\zeta(v+z-u-w+2)\}^{-1}W^*(u+w-1,w),\end{aligned}$$

$$\begin{aligned}M_3(u,v,w,z)=&\,4\zeta(u+z-1)\zeta(v+z-1)\zeta(2-u-w)\zeta(2-v-w)\{\zeta(4-u-v-w-z)\}^{-1}\\&\times\{\Phi+\Psi\}\left(\tfrac{1}{2}(u+v+w+z-3);u,v,w,z\right),\end{aligned}$$

$$\begin{aligned}M_4(u,v,w,z)=&\,4\zeta(u+z-1)\zeta(v+w)\zeta(z-w)\zeta(v-u+1)\{\zeta(2+v+w-u-z)\}^{-1}\\&\times\{\Phi+\Psi\}\left(\tfrac{1}{2}(u+z-v-w-1);u,v,w,z\right),\end{aligned}$$

$$\begin{aligned}M_5(u,v,w,z)=&\,4\zeta(v+z-1)\zeta(u+w)\zeta(z-w)\zeta(u-v+1)\{\zeta(2+u+w-v-z)\}^{-1}\\&\times\{\Phi+\Psi\}\left(\tfrac{1}{2}(v+z-u-w-1);u,v,w,z\right),\end{aligned}$$

and for $1 \leqslant j \leqslant 5$

$$M_{5+j}(u,v,w,z) = M_j(w,z,u,v).$$

In (5.8) we set $(u,v,w,z) = P_T + (a_1,a_2,a_3,a_4)\delta$ (a vector sum) with small complex δ, and expand each term into a Laurent series in δ. The singular parts must cancel out each other, for M is regular at P_T; and the sum of the constant term is equal to $M(P_T)$, regardless of the choice of (a_1,a_2,a_3,a_4). We choose it in such a way that no singularities of any of the M_j $(0 \leqslant j \leqslant 10)$ are encountered when $|\delta|$ tends to 0. This is possible, for the exceptional a_1,a_2,a_3,a_4 satisfy a finite number of linear relations. Thus, we shall assume hereafter that $\delta \neq 0$ is small, and (a_1,a_2,a_3,a_4) is chosen as above; and we denote $(a_1,a_2,a_3,a_4)\delta$ by either (δ) or $(\delta_1,\delta_2,\delta_3,\delta_4)$.

Now, let us compute the constant terms of $M_j(P_T+(\delta))$ $(0 \leqslant j \leqslant 10)$. Since M_0 is trivial, we begin with M_1. This is not difficult. By (2.10) we have

$$M_1(P_T+(\delta)) = \zeta(\delta_1+\delta_4+1)\zeta(\delta_2+\delta_3)\zeta(\delta_4-\delta_3+1)\zeta(\delta_1-\delta_2+1)\{\zeta(2+\delta_1-\delta_2-\delta_3+\delta_4)\}^{-1}$$
$$\times \Gamma(\delta_2+\delta_3)(\Delta\sqrt{\pi})^{-1} \int_{-\infty}^{\infty} \frac{\Gamma(\frac{1}{2}-\delta_2-i(T+t))}{\Gamma(\frac{1}{2}+\delta_3-i(T+t))} e^{-(t/\Delta)^2}\, dt.$$

Hence the singularity of M_1 at P_T is of order 4. We then see that the constant term of $M_1(P_T+(\delta))$ is a linear combination of the first five coefficients of the power series in δ for the last integral. Thus the constant term in question has the form

$$(\Delta\sqrt{\pi})^{-1} \int_{-\infty}^{\infty} \sum_{\substack{a,b,k,l \geqslant 0 \\ ak+bl \leqslant 4}} d(a,b;k,l) \left(\frac{\Gamma^{(a)}}{\Gamma}\right)^k \left(\frac{\Gamma^{(b)}}{\Gamma}\right)^l \left(\tfrac{1}{2}-i(T+t)\right) e^{-(t/\Delta)^2}\, dt \tag{5.9}$$

with the obvious abuse of notation; the constants $d(a,b;k,l)$ may depend on (a_1,a_2,a_3,a_4). Apparently M_2, M_6 and M_7 can be treated in just the same way, and their constant terms have the same form as (5.9).

On the other hand M_3 is not easy. We divide this into two parts $M_{3,1}$ and $M_{3,2}$ corresponding to Φ and Ψ, respectively, in the obvious manner. We deal with $M_{3,1}$ only, for $M_{3,2}$ is quite similar. We first separate the singular part of

$$\Phi\left(\tfrac{1}{2}(u+v+w+z-3); u,v,w,z\right)$$

on the present supposition, i.e. $(u,v,w,z) = P_T + (\delta)$. We note that $\frac{1}{2}(u+v+w+z-3)$ is close to $-\frac{1}{2}$; thus we need to consider $\Phi(\xi; u,v,w,z)$ in a neighbourhood of the point $\left(-\frac{1}{2}, P_T\right) \in \mathbf{C}^5$, which is in the set N defined in (4.3). We suppose, for a moment, that $\mathrm{Re}(\xi)$ is close to $-\frac{1}{2}$, but $|\mathrm{Im}(\xi)|$ is not small. Then we can draw the path in (3.12). We move it to the line $\mathrm{Re}(s)=\frac{1}{4}$. We encounter only one pole at $s=\frac{1}{2}(u+v+w+z-1)+\xi$ which is simple; so we have

$$\Phi(\xi;u,v,w,z)=\Phi_0(\xi;u,v,w,z)+\Phi_1(\xi;u,v,w,z), \tag{5.10}$$

where

$$\Phi_0(\xi;u,v,w,z)=-(2\pi)^{w-z-1}\cos(\pi\xi)\sin\left(\pi\left(\xi+\tfrac{1}{2}(z-w)\right)\right)\Gamma(-2\xi)$$
$$\times\Gamma\left(\xi+\tfrac{1}{2}(v+z-u-w+1)\right)\Gamma\left(\xi+\tfrac{1}{2}(u+z-v-w+1)\right)W^*\left(\xi+\tfrac{1}{2}(u+v+w+z-1),w\right),$$

and Φ_1 has the same expression as (3.12) but with $\mathrm{Re}(s)=\frac{1}{4}$ as the path. We may now drop the condition on $\mathrm{Im}(\xi)$, since Φ_1 is regular in a neighbourhood of $\left(-\frac{1}{2},P_T\right)$. Then, (5.10) entails a decomposition of $M_{3,1}$. We denote by $M_{3,1}^{(0)}$ and $M_{3,1}^{(1)}$ the parts corresponding to Φ_0 and Φ_1, respectively. We have, at $(u,v,w,z)=P_T+(\delta)$,

$$\Phi_0\left(\tfrac{1}{2}(u+v+w+z-3);u,v,w,z\right)$$
$$=\tfrac{1}{4}(2\pi)^{\delta_3-\delta_4}\cos\left(\pi\left(\delta_4+\tfrac{1}{2}(\delta_1+\delta_2)\right)\right)\left(\cos\tfrac{1}{2}\pi(\delta_1+\delta_2+\delta_3+\delta_4)\right)^{-1}$$
$$\times\Gamma(\delta_2+\delta_4)\Gamma(\delta_1+\delta_4)(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\frac{\Gamma(\frac{1}{2}-\delta_1-\delta_2-\delta_4-i(T+t))}{\Gamma(\frac{1}{2}+\delta_3-i(T+t))}e^{-(t/\Delta)^2}\,dt,$$

where we have used (2.10). Hence, as in the case of M_1, $M_{3,1}^{(0)}(P_T+(\delta))$ has the constant term of the form of (5.9). We next consider $M_{3,1}^{(1)}$. Since Φ_1 is regular at $\left(-\frac{1}{2},P_T\right)$, the constant term in $M_{3,1}^{(1)}(P_T+(\delta))$ is a linear combination of the first three coefficients of the power series in δ for $\Phi_1\left(\frac{1}{2}(u+v+w+z-3);u,v,w,z\right)$ with $(u,v,w,z)=P_T+(\delta)$. We shall show that this is fairly small when T is large. For this sake we transform Φ_1 by inserting (2.10) into its defining integral representation. The resulting double integral converges absolutely. We interchange the order of integration, and set $\xi=\frac{1}{2}(u+v+w+z-3)$. We get, in the vicinity of P_T,

$$\Phi_1\left(\tfrac{1}{2}(u+v+w+z-3);u,v,w,z\right)=\tfrac{1}{2}i(2\pi)^{w-z-1}\sin\left(\tfrac{1}{2}\pi(u+v+w+z)\right)(\Delta\sqrt{\pi})^{-1}$$
$$\times\int_{-\infty}^{\infty}\frac{e^{-(t/\Delta)^2}}{\Gamma(w+it)}\int_{(1/4)}\cos\left(\pi\left(w+\tfrac{1}{2}(u+v)-s\right)\right)$$
$$\times\Gamma(s+1-u-w)\Gamma(s+1-v-w)\Gamma(u+v+w+z-2-s)\Gamma(w-s+it)(\sin\pi s)^{-1}\,ds\,dt.$$

In this we set $(u,v,w,z)=P_T+(\delta)$, and denote the result by $\Phi_1^*(\delta)$, so that we are concerned with $(\Phi_1^*)^{(\nu)}(0)$ $(\nu=0,1,2)$. We then shift the above path $\mathrm{Re}(s)=\frac{1}{4}$ to $\mathrm{Re}(s)=\frac{5}{4}$. We encounter poles at $s=\frac{1}{2}+\delta_3-i(T-t)$, $\delta_1+\delta_2+\delta_3+\delta_4+1$ and 1. These are all simple, because of our choice of (a_1,a_2,a_3,a_4). We now have

$$\Phi_1^*(\delta)=\sum_{j=1}^{4}\Phi_{1,j}^*(\delta), \tag{5.11}$$

where

$$\Phi_{1,1}^*(\delta)=-\tfrac{1}{2}(2\pi)^{\delta_3-\delta_4}\sin\big(\tfrac{1}{2}\pi(\delta_1+\delta_2+\delta_3+\delta_4)\big)(\Delta\sqrt{\pi}\,)^{-1}\int_{-\infty}^{\infty}\frac{e^{-(t/\Delta)^2}}{\Gamma(\frac{1}{2}+\delta_3-i(T+t))}$$
$$\times\sin\big(\pi\big(i(T+t)+\tfrac{1}{2}(\delta_1+\delta_2)\big)\big)\Gamma\big(\tfrac{1}{2}-\delta_1-i(T+t)\big)\Gamma\big(\tfrac{1}{2}-\delta_2-i(T+t)\big)$$
$$\times\Gamma\big(-\tfrac{1}{2}+\delta_1+\delta_2+\delta_4+i(T+t)\big)(\cos\pi(i(T+t)-\delta_3))^{-1}\,dt,$$

$$\Phi_{1,2}^*(\delta)=\tfrac{1}{4}(2\pi)^{\delta_3-\delta_4}\big(\cos\tfrac{1}{2}\pi(\delta_1+\delta_2+\delta_3+\delta_4)\big)^{-1}\cos\big(\pi\big(\delta_4+\tfrac{1}{2}(\delta_1+\delta_2)\big)\big)$$
$$\times\Gamma(1+\delta_1+\delta_4)\Gamma(1+\delta_2+\delta_4)(\Delta\sqrt{\pi}\,)^{-1}\int_{-\infty}^{\infty}\frac{\Gamma(-\frac{1}{2}-\delta_1-\delta_2-\delta_4-i(T+t))}{\Gamma(\frac{1}{2}+\delta_3-i(T+t))}\,e^{-(t/\Delta)^2}\,dt,$$

$$\Phi_{1,3}^*(\delta)=-\tfrac{1}{4}(2\pi)^{\delta_3-\delta_4}\big(\cos\tfrac{1}{2}\pi(\delta_1+\delta_2+\delta_3+\delta_4)\big)^{-1}\cos\big(\pi\big(\delta_3+\tfrac{1}{2}(\delta_1+\delta_2)\big)\big)$$
$$\times\Gamma(1-\delta_1-\delta_3)\Gamma(1-\delta_2-\delta_3)\{\Gamma(2-\delta_1-\delta_2-\delta_3-\delta_4)\}^{-1}(\Delta\sqrt{\pi}\,)^{-1}$$
$$\times\int_{-\infty}^{\infty}\frac{e^{-(t/\delta)^2}}{-\frac{1}{2}+\delta_3-i(T+t)}\,dt,$$

$$\Phi_{1,4}^*(\delta)=\tfrac{1}{2}i(2\pi)^{\delta_3-\delta_4-1}\sin\big(\tfrac{1}{2}\pi(\delta_1+\delta_2+\delta_3+\delta_4)\big)(\Delta\sqrt{\pi}\,)^{-1}\int_{-\infty}^{\infty}\frac{e^{-(t/\Delta)^2}}{\Gamma(\frac{1}{2}+\delta_3-i(T+t))}$$
$$\times\int_{(5/4)}\cos\big(\pi\big(\delta_3+\tfrac{1}{2}(\delta_1+\delta_2)-s\big)\big)\Gamma(s-\delta_1-\delta_3)\Gamma(s-\delta_2-\delta_3)$$
$$\times\Gamma(\delta_1+\delta_2+\delta_3+\delta_4-s)\Gamma\big(\tfrac{1}{2}+\delta_3-i(T+t)-s\big)(\sin\pi s)^{-1}\,ds\,dt.$$

Then, by Stirling's formula we see easily that

$$\sum_{j=1}^{3}\Phi_{1,j}^*(\delta)\ll T^{-1+c|\delta|}+e^{-c(T/\Delta)^2}\tag{5.12}$$

uniformly for $T,\Delta>0$ and small complex δ. Similarly we have

$$\begin{aligned}\Phi_{1,4}^*(\delta)&\ll\Delta^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|T+t|)^{c|\delta|}(1+|T+t+r|)^{-5/4+c|\delta|}\\&\qquad\times\exp\big(-\tfrac{3}{2}\pi|r|-\tfrac{1}{2}\pi|T+t+r|+\tfrac{1}{2}\pi|T+t|-(t/\Delta)^2\big)\,dt\,dr\\&\ll T^{-5/4+c|\delta|}+e^{-c(T/\Delta)^2}.\end{aligned}\tag{5.13}$$

Collecting (5.11)–(5.13), and using Cauchy's integral formula with the circle $|\delta|=(\log T)^{-1}$, we find that the relevant differential coefficients of $\Phi_1^*(\delta)$ at $\delta=0$ are

$$O((\log T)^2(T^{-1}+\exp(-c(T/\Delta)^2));$$

hence the constant term of $M^{(1)}_{3,1}(P_T+(\delta))$ has the same bound. This ends the treatment of M_3. In much the same way we can deal with M_j, $j=4,5,8,9,10$, and get the same result as that on M_2.

Gathering these considerations we conclude that $M(P_T)$ is the sum of a term of the form of (5.9) and a term of the order of $(\log T)^2\times(T^{-1}+\exp(-c(T/\Delta)^2))$ uniformly for $T,\Delta>0$.

Returning to (5.7), we see that to finish the proof of our theorem it remains for us to relate $\{\Phi+\Psi\}(ir;P_T)$ and $\Xi\big(k-\frac{1}{2};P_T\big)$ to the hypergeometric function. By (4.1) and (4.2) we have, for real r,

$$\{\Phi+\Psi\}(ir;P_T)=\frac{1}{4}\left(1+\frac{i}{\sinh(\pi r)}\right)\Xi(ir;P_T)+\frac{1}{4}\left(1-\frac{i}{\sinh(\pi r)}\right)\Xi(-ir;P_T). \tag{5.14}$$

On the other hand, (4.5) with (4.7) holds obviously on the present specialization, too; and we have

$$\begin{aligned}\Xi(ir;P_T)=\frac{\Gamma(\frac{1}{2}+ir)^2}{\Gamma(1+2ir)}\int_0^\infty & y^{-1/2+ir}(1+y)^{-1/2+iT}\\ &\times\exp\big(-\big(\tfrac{1}{2}\Delta\log(1+y)\big)^2\big)F\big(\tfrac{1}{2}+ir,\tfrac{1}{2}+ir;1+2ir;-y\big)\,dy.\end{aligned} \tag{5.15}$$

Hence

$$\begin{aligned}\{\Phi+\Psi\}(ir;P_T)=\frac{1}{2}\int_0^\infty & y^{-1/2}(1+y)^{-1/2+iT}\exp\big(-\big(\tfrac{1}{2}\Delta\log(1+y)\big)^2\big)\\ &\times\mathrm{Re}\left[y^{ir}\left(1+\frac{i}{\sinh(\pi r)}\right)\frac{\Gamma(\frac{1}{2}+ir)^2}{\Gamma(1+2ir)}F\big(\tfrac{1}{2}+ir,\tfrac{1}{2}+ir;1+2ir;-y\big)\right]dy.\end{aligned}$$

Further, we observe that (5.15) holds also for $k-\frac{1}{2}$ in place of ir, and thus get a representation of $\Xi\big(k-\frac{1}{2};P_T\big)$. After replacing the variable y by $1/x$, we insert these into (5.7), and end the proof of our theorem.

6. Asymptotics

In this final section we shall study the asymptotical behaviour of our explicit formula (1.16), and prove (1.19). Obviously we may always assume that

$$0<\Delta<T(\log T)^{-1}, \tag{6.1}$$

and that Δ is sufficiently large.

Our problem is equivalent to analysing the function $\Theta(r;T,\Delta)$. We begin with the case $r=i\big(\frac{1}{2}-k\big)$ where $k\geqslant 1$ is an integer. We note first that by (2.8) we have

$$\big|W^*\big(s,\tfrac{1}{2}-iT\big)\big|\leqslant W^*\big(\mathrm{Re}(s),\tfrac{1}{2}\big)\ll\Delta^{-\mathrm{Re}(s)}, \tag{6.2}$$

providing $\mathrm{Re}(s)$ is positive and bounded. On the other hand, by (3.26) we have

$$\begin{aligned}\Theta\big(i\big(\tfrac{1}{2}-k\big);T,\Delta\big)&=(-1)^k\,\mathrm{Re}\big\{\Xi\big(k-\tfrac{1}{2};P_T\big)\big\}\\&=(-1)^k\,\mathrm{Re}\left[\frac{1}{2\pi i}\int_{(1/4)}\frac{\Gamma(k-s)}{\Gamma(k+s)}\Gamma^2(s)W^*\big(s,\tfrac{1}{2}-iT\big)\,ds\right].\end{aligned}$$

Hence, shifting the path to the right appropriately we see that for any fixed $A>0$ we have

$$\Theta\big(i\big(\tfrac{1}{2}-k\big);T,\Delta\big)\ll\begin{cases}\Delta^{-k}, & k\leqslant A\\(k\Delta)^{-A}, & k>A\end{cases}\tag{6.3}$$

uniformly in T; the implied constant depends on A at most.

We then turn to the case where r is real. By (5.14) our problem has been reduced to the asymptotical study of $\Xi(ir;P_T)$. We know already that this is of fast decay when r tends to $\pm\infty$, but what we need now is a result which is uniform in the three parameters r,T,Δ.

To get such a result we note first that we have, more precisely than (6.2),

$$W^*\big(s,\tfrac{1}{2}-iT\big)\ll\Delta^{-\mathrm{Re}(s)}\exp(-|\mathrm{Im}\,s|/T)\tag{6.4}$$

with the same condition on $\mathrm{Re}(s)$. This can be shown by turning the line of integration in (2.8) with $\eta=\frac{1}{2}-iT$ by $T^{-1}\,\mathrm{sgn}(\mathrm{Im}(s))$, and taking the absolute value of the integrand. Then we have

$$W^*\big(s,\tfrac{1}{2}-iT\big)\ll W^*\big(\mathrm{Re}(s),\tfrac{1}{2}\big)\exp(-|\mathrm{Im}\,s|/T),$$

which gives (6.4); note that we need (6.1) here. Next we set $(\xi,u,v,w,z)=(ir,P_T)$ in (3.26) with the path $\mathrm{Re}(s)=\frac{1}{4}$, and move the path to $\mathrm{Re}(s)=m$ an arbitrary positive integer. We get

$$\begin{aligned}\Xi(ir;P_T)=&\sum_{l=0}^{m-1}\frac{(-1)^l\Gamma(l+\frac{1}{2}+ir)^2}{\Gamma(l+1)\Gamma(l+1+2ir)}W^*\big(l+\tfrac{1}{2}+ir,\tfrac{1}{2}-iT\big)\\&+\frac{1}{2\pi i}\int_{(m)}\frac{\Gamma(\frac{1}{2}+ir-s)}{\Gamma(\frac{1}{2}+ir+s)}\Gamma(s)^2W^*\big(s,\tfrac{1}{2}-iT\big)\,ds.\end{aligned}$$

The estimate (6.4) implies that the sum over l is

$$\ll e^{-|r|/T}((1+|r|)\Delta)^{-1/2}\left(1+\left(\frac{|r|}{\Delta}\right)^{m-1}\right).$$

On the other hand the last integral is, by (6.4) and Stirling's formula,

$$\begin{aligned}&\ll\Delta^{-m}\int_{(m)}|s|^{2m-1}(|s+ir|\,|s-ir|)^{-m}\exp\left(-\frac{\pi}{2}(2|s|-|s+ir|+|s-ir|)-\frac{|s|}{T}\right)|ds|\\&\ll\Delta^{-m}\left\{(1+|r|)^{-2m}T^{2m}+(1+|r|)^{m-1}\exp\left(-\frac{|r|}{2T}\right)\right\},\end{aligned}$$

where the implied constant depends only on m. Taking m sufficiently large in these, we get, for any fixed $A>0$,

$$\Xi(ir;P_T)\ll|r|^{-A},\quad T(\log T)^2\leqslant|r|, \tag{6.5}$$

providing

$$T^a\leqslant\Delta\leqslant T(\log T)^{-1}, \tag{6.6}$$

where $a>0$ is an arbitrary fixed small constant, and T is larger than a constant determined solely by a and A.

We then move to the case

$$|r|\leqslant T(\log T)^2, \tag{6.7}$$

and assume (6.6). This time we use the expression

$$\Xi(ir;P_T)=\int_0^\infty R(y,r)y^{-1/2+ir}(1+y)^{-1/2+iT}\exp\left(-\left(\tfrac12\Delta\log(1+y)\right)^2\right)dy, \tag{6.8}$$

where

$$R(y,r)=\int_0^1(x(1-x))^{-1/2+ir}(1+yx)^{-1/2-ir}\,dx; \tag{6.9}$$

this is equivalent to (5.15). In (6.8) we replace the path by the one consisting of the segment L_1 connecting 0 and y^* and the half line L_2 connecting y^* and $+\infty e^{i\mu}$, where

$$\mu=\operatorname{sgn}(r)(T(\log T)^2)^{-1},\quad y^*=\Delta^{-1}\log T+i(1+\Delta^{-1}\log T)\tan\mu.$$

We note that on the new path we have $R(y,r)\ll1$ uniformly for all r satisfying (6.7), since $|\operatorname{Arg}(1+yx)|\leqslant\mu$ there. Then we see readily that the contribution of L_2 is $O(\exp(-c(\log T)^2))$. On the other hand we have on L_1

$$\operatorname{sgn}(r)\operatorname{Arg}y=\left|\arctan\left\{\left(1+\frac{\Delta}{\log T}\right)\tan\mu\right\}\right|\geqslant\frac{\Delta}{2T(\log T)^3}.$$

Thus the contribution of L_1 is $O(\exp(-\Delta|r|(2T\log^3T)^{-1}))$. Hence we have

$$\Xi(ir;P_T)\ll\exp\left(-\frac{\Delta|r|}{2T(\log T)^3}\right)+\exp(-c(\log T)^2),\quad |r|\leqslant T(\log T)^2. \tag{6.10}$$

The estimates (6.5) and (6.10) imply that we may now restrict ourselves to the range

$$|r|\leqslant\frac{T}{\Delta}(\log T)^5 \tag{6.11}$$

as far as we assume (6.6). To deal with this case we use again (6.8) with (6.9), but this time we interchange the order of integration: We have

$$\Xi(ir;P_T)=\int_0^1 (x(1-x))^{-1/2+ir}V(x,r)\,dx, \tag{6.12}$$

where

$$V(x,r)=\int_0^\infty y^{-1/2+ir}(1+y)^{-1/2+iT}(1+xy)^{-1/2-ir}\exp\left(-\left(\tfrac{1}{2}\Delta\log(1+y)\right)^2\right)dy. \tag{6.13}$$

First we settle two special cases. Thus, if $|r|\leqslant\log^{30}T$ we divide the last integral into two parts according to $0\leqslant y\leqslant T^{-1}\log^{30}T$ and $T^{-1}\log^{30}T<y$. These are estimated, respectively, by taking simply the absolute value of the integrand and by performing partial integration. We then find that

$$\Xi(ir;P_T)\ll T^{-1/2}(\log T)^{15},\quad |r|\leqslant(\log T)^{30}. \tag{6.14}$$

On the other hand, if r is positive, we turn the line of integration in (6.13) by $\pi/6$, and take the absolute value of the integrand. This gives

$$\Xi(ir;P_T)\ll e^{-cr}+e^{-c(\log T)^2},\quad 0<r<\frac{T}{\Delta}(\log T)^5. \tag{6.15}$$

Gathering these observations, we see that more restrictively than (6.11) we may assume

$$-\frac{T}{\Delta}(\log T)^5\leqslant r\leqslant-(\log T)^{30}. \tag{6.16}$$

For these r we compute $\Xi(ir;P_T)$ asymptotically. To this end we impose a rather drastic condition on Δ:

$$T^{1/2}\leqslant\Delta\leqslant T(\log T)^{-25}. \tag{6.17}$$

This is only for the sake of simplicity; in fact the computation may well be carried out on the assumption (6.6) only. Note that the upper bound of Δ is implied by (6.16).

Now, we apply the saddle point method to $V(x,r)$ on the condition (6.16) and (6.17). The saddle point is located at $y=y_0$:

$$y_0=2|r|(T-|r|+((T-|r|)^2+4xT|r|)^{1/2})^{-1}, \tag{6.18}$$

which is approximately $|r|/T$. This lies between 0 and $2\Delta^{-1}\log^5T$ because of (6.16). We move the path in (6.13) to the one consisting of two segments S_1,S_2 and a half line S_3:

$$\begin{aligned}S_1&=\left\{y=\lambda\left(1-\varepsilon\exp\left(\tfrac{1}{4}\pi i\right)\right);0\leqslant\lambda\leqslant y_0\right\},\\ S_2&=\left\{y=y_0\left(1+\xi\exp\left(\tfrac{1}{4}\pi i\right)\right);-\varepsilon\leqslant\xi\leqslant\varepsilon\right\},\\ S_3&=\left\{y=\lambda\left(1+\varepsilon\exp\left(\tfrac{1}{4}\pi i\right)\right);y_0\leqslant\lambda\right\},\end{aligned}$$

where ε is a small positive constant. On S_1 the integrand of $V(x,r)$ is

$$\ll \lambda^{-1/2}\exp\left\{-|r|\arctan\left(\frac{\varepsilon}{\sqrt{2}-\varepsilon}\right)+(T+|r|)\arctan\left(\frac{\varepsilon y_0}{\sqrt{2}+(\sqrt{2}-\varepsilon)y_0}\right)\right\},$$

which is, by (6.16)–(6.18),

$$\ll \lambda^{-1/2}\exp\left(-\tfrac{1}{3}\varepsilon^2|r|\right).$$

Hence S_1 contributes $O\left(\exp(-\frac{1}{3}\varepsilon^2|r|)\right)$ to $V(x,r)$ uniformly for $0\leqslant x\leqslant 1$. We consider next S_3. Here the integrand is

$$\ll \lambda^{-1/2}\exp\left\{|r|\arctan\left(\frac{\varepsilon}{\sqrt{2}+\varepsilon}\right)-T\arctan\left(\frac{\varepsilon y_0}{\sqrt{2}+(\sqrt{2}+\varepsilon)y_0}\right)-\tfrac{1}{8}(\Delta\log(1+\lambda))^2\right\}.$$

We have used the fact that on S_3

$$\log(1+y)=\frac{1}{2}\log\left\{\left(1+\lambda\left(1+\frac{\varepsilon}{\sqrt{2}}\right)\right)^2+\frac{\varepsilon^2}{2}\lambda^2\right\}+i\arctan\left(\frac{\varepsilon\lambda}{\sqrt{2}+(\sqrt{2}+\varepsilon)\lambda}\right),$$

and thus

$$\mathrm{Re}[(\log(1+y))^2]\geqslant(\log(1+\lambda))^2-\left(\frac{\varepsilon\lambda}{1+\lambda}\right)^2\geqslant\tfrac{1}{2}(\log(1+\lambda))^2,$$

providing ε is sufficiently small. Then, we see readily that S_3 contributes $O(\exp(-\frac{1}{3}\varepsilon^2|r|))$ to $V(x,r)$ uniformly for $0\leqslant x\leqslant 1$. We now have, on (6.16) and (6.17),

$$V(x,r)=V_0(x,r)+O\left(\exp\left(-\tfrac{1}{3}\varepsilon^2|r|\right)\right), \tag{6.19}$$

where $V_0(x,r)$ is the contribution of S_2, and the implied constant is absolute. By the definition we have

$$V_0(x,r)=y_0e^{\pi i/4}\int_{-\varepsilon}^{\varepsilon}g(\xi)\exp(if(\xi))\,d\xi, \tag{6.20}$$

where

$$g(\xi)=(y(1+y)(1+xy))^{-1/2}\exp\left(-\left(\tfrac{1}{2}\Delta\log(1+y)\right)^2\right),$$
$$f(\xi)=r\log y+T\log(1+y)-r\log(1+xy)$$

with $y=y_0\left(1+\xi\exp\left(\frac{1}{4}\pi i\right)\right)$. We have

$$f(\xi)=f(0)+\tfrac{1}{2}f''(0)\xi^2+\tfrac{1}{6}f'''(0)\xi^3+O(|r|\xi^4),$$

where

$$\begin{aligned} f''(0)&=i|r|\left(1-\frac{T}{|r|}\left(\frac{y_0}{1+y_0}\right)^2-\left(\frac{xy_0}{1+xy_0}\right)^2\right),\\ f'''(0)&=O(|r|). \end{aligned} \tag{6.21}$$

These imply in particular that $\exp(if(\xi))\ll\exp\left(-\frac{1}{3}\xi^2|r|\right)$ $(-\varepsilon\leqslant\xi\leqslant\varepsilon)$. Thus we may truncate the integral in (6.20) at $\xi=\pm\xi_0$, $\xi_0=|r|^{-2/5}$, so that uniformly for $0\leqslant x\leqslant 1$,

$$V_0(x,r)=V_1(x,r)+O(\exp(-c|r|^{1/5})), \tag{6.22}$$

where $V_1(x,r)$ is the part corresponding to $-\xi_0\leqslant\xi\leqslant\xi_0$, and the constant in the error term is absolute. We then note that if $|\xi|\leqslant\xi_0$ we have

$$\exp(if(\xi))=\exp\left(if(0)+\tfrac{1}{2}if''(0)\xi^2\right)\left\{1+\tfrac{1}{6}if'''(0)\xi^3+O(|r|\xi^4+|r|^2\xi^6)\right\} \tag{6.23}$$

as well as

$$g(\xi)=g(0)\left\{1+\frac{g'}{g}(0)\xi+O((1+(\Delta y_0)^2)\xi^2)\right\}, \tag{6.24}$$

where $(g'/g)(0)\ll 1+(\Delta y_0)^2$. Here the assertion on $g(\xi)$ may require a proof: We have

$$(\log(1+y))^2=(\log(1+y_0))^2+2e^{\pi i/4}\frac{y_0\xi}{1+y_0}\log(1+y_0)+O((\xi y_0)^2).$$

But

$$\frac{y_0\xi}{1+y_0}\log(1+y_0)\ll y_0^2|r|^{-2/5}\ll T^{-2}|r|^{8/5}$$

which is $O((\Delta\log T)^{-2})$ by (6.16) and (6.17). Thus

$$\exp\left(-\left(\tfrac{1}{2}\Delta\log(1+y)\right)^2\right)=\exp\left(-\left(\tfrac{1}{2}\Delta\log(1+y_0)\right)^2\right)(1+b\xi+O((\Delta y_0\xi)^2))$$

with

$$b=\frac{1}{2}e^{\pi i/4}\Delta^2\frac{y_0}{1+y_0}\log(1+y_0)\ll(\Delta y_0)^2,$$

which implies (6.24). Now, by (6.23) and (6.24) we have, for $|\xi|\leqslant\xi_0$,

$$\begin{aligned}g(\xi)\exp(if(\xi))&=g(0)\exp\left(if(0)+\tfrac{1}{2}if''(0)\xi^2\right)\\&\quad\times\left\{1+\frac{g'}{g}(0)\xi+\frac{1}{6}if'''(0)\xi^3+O((1+(\Delta y_0)^2)(1+|r|\xi^2)^2\xi^2)\right\}\end{aligned}$$

with the implied constant being absolute. This and (6.21) give

$$V_1(x,r)=e^{\pi i/4}y_0g(0)\exp(if(0))\left(\frac{2\pi}{|f''(0)|}\right)^{1/2}\{1+O(|r|^{-1}(1+(\Delta y_0)^2))\}$$

uniformly for $0\leqslant x\leqslant 1$. Thus we have

$$\begin{aligned}V_1(x,r)&=e^{\pi i/4}\left(\frac{2\pi}{T}\right)^{1/2}\exp\left\{iT\log(1+y_0)+ir\log\frac{y_0}{1+xy_0}-\left(\frac{\Delta r}{2T}\right)^2\right\}\\&\quad\times(1+O((|r|^{-1}+\Delta^{-1})\log^{15}T)).\end{aligned}$$

But, we have also

$$T\log(1+y_0)+r\log\frac{y_0}{1+xy_0}=r\log\frac{|r|}{eT}+\left(x-\frac{1}{2}\right)\frac{r^2}{T}+O\left(\frac{|r|^3}{T^2}\right).$$

These and (6.12), (6.19), (6.22) yield, after some rearrangement,

$$\begin{aligned}\Xi(ir;P_T)=\left(\frac{2\pi}{T}\right)^{1/2}&\exp\left\{\frac{\pi i}{4}+ir\log\frac{|r|}{eT}-\frac{ir^2}{2T}-\left(\frac{\Delta r}{2T}\right)^2\right\}\\ &\times\int_0^1(x(1-x))^{-1/2}\exp\left(ir\log(x(1-x))+\frac{ir^2}{T}x\right)dx \qquad (6.25)\\ &+O\left\{T^{-1/2}(|r|^{-1}+\Delta^{-1})\exp\left(-\left(\frac{\Delta r}{2T}\right)^2\right)\log^{15}T\right\},\end{aligned}$$

as far as (6.16) and (6.17) hold. To this integral we may again apply the saddle point method. The computation is, however, quite routine. So we may state only the result: It is equal to

$$\left(\frac{\pi}{|r|}\right)^{1/2}\exp\left(\frac{1}{4}\pi i-2ir\log 2+\frac{ir^2}{2T}\right)(1+O(|r|^{-1}+\Delta^{-1}\log^5 T)).$$

Inserting this into (6.25) we obtain, on (6.16) and (6.17),

$$\Xi(ir;P_T)=i2^{1/2}\pi(T|r|)^{-1/2}\exp\left(ir\log\frac{|r|}{4eT}-\left(\frac{\Delta r}{2T}\right)^2\right)\left(1+O\left(\frac{\log^{20}T}{\sqrt{|r|}}\right)\right), \qquad (6.26)$$

where the implied constant is absolute. This ends our discussion on $\Xi(ir;P_T)$ when r is real.

We can now finish the proof of (1.19). First, (1.14), (1.15) and (6.3) imply that the contribution of the holomorphic cusp forms, i.e., the double sum in (1.16), can be neglected. Next, collecting (1.6), (1.11), (5.14), (6.5), (6.10), (6.14) and (6.15), we see that the contribution of the non-holomorphic cusp forms is equal to

$$\frac{1}{2}\sum_{(\log T)^{30}\le\varkappa_j\le\frac{T}{\Delta}(\log T)^5}\alpha_jH_j\left(\tfrac{1}{2}\right)^3\mathrm{Re}\{\Xi(-i\varkappa_j;P_T)\}+O(1).$$

We should stress here that this holds on the condition (6.6) only. If $T(\log T)^{-25}<\Delta\le T(\log T)^{-1}$, then the last sum is empty. Thus we may assume that (6.17) holds, without violating the assumption in the corollary. Then we can use (6.26), and see that the above is equal to

$$\begin{aligned}&\frac{\pi}{\sqrt{2T}}\sum_{(\log T)^{30}\le\varkappa_j\le\frac{T}{\Delta}(\log T)^5}\alpha_jH_j\left(\tfrac{1}{2}\right)^3\varkappa_j^{-1/2}\sin\left(\varkappa_j\log\frac{\varkappa_j}{4eT}\right)\exp\left(-\left(\frac{\Delta\varkappa_j}{2T}\right)^2\right)\\ &\qquad+O\left\{T^{-1/2}(\log T)^{20}\sum_{\varkappa_j\le\frac{T}{\Delta}(\log T)^5}\alpha_j\left|H_j\left(\tfrac{1}{2}\right)\right|^3\varkappa_j^{-1}\right\}+O(1).\end{aligned} \qquad (6.27)$$

Here we invoke the following spectral mean value of $H_j\left(\frac{1}{2}\right)$:

$$\sum_{\varkappa_j \leqslant K} \alpha_j H_j\left(\tfrac{1}{2}\right)^4 \ll K^2 (\log K)^{20},$$

whose proof can be found in [13]. This and (1.6) yield that the error term in (6.27) is $O(\Delta^{-1}T^{1/2}(\log T)^{36})$, which ends the treatment of the sum over the discrete spectrum in (1.16). Finally, combining the above results on $\Xi(ir; P_T)$ with the classical bounds for individual values and the fourth power mean of $\zeta\left(\frac{1}{2}+it\right)$, we find that the contribution of the continuous spectrum is infinitesmally small as T tends to infinity. Thereby we conclude our proof of (1.19).

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