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A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry

by

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Introduction

It is the purpose of this paper to introduce and study a nonlinear elliptic system of equations imposed on a map from a Hermitian into a Riemannian manifold which seems to be more appropriate to Hermitian geometry than the harmonic map system. Thus, let X be a complex manifold with Hermitian metric $(\gamma_{\alpha\bar\beta})$ in local coordinates, N a Riemannian manifold with metric (g_{ij}) and Christoffel symbols Γ^i_{jk}. A harmonic map $f: X \to N$ then has to satisfy

$$\frac{1}{2}\frac{\partial}{\partial z^{\bar\beta}}\left(\gamma^{\alpha\bar\beta}\frac{\partial f^i}{\partial z^\alpha}\right)+\frac{1}{2}\frac{\partial}{\partial z^{\alpha}}\left(\gamma^{\alpha\bar\beta}\frac{\partial f^i}{\partial z^{\bar\beta}}\right)+\gamma^{\alpha\bar\beta}\Gamma^i_{jk}(f(z))\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar\beta}}=0, \quad i=1,\dots,\dim N \tag{H1}$$

in local coordinates. A disadvantage of this system is that, unless X is Kähler, a holomorphic map need not be harmonic. We therefore replace (H1) by

$$\gamma^{\alpha\bar\beta}\left(\frac{\partial^2 f^i}{\partial z^\alpha \partial z^{\bar\beta}}+\Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar\beta}}\right)=0, \quad i=1,\dots,\dim N. \tag{H2}$$

We point out that (H1) and (H2) are equivalent if X is Kählerian. In general, (H2) is analytically more difficult than (H1) because it neither has a divergence nor a variational structure.

A vague analogue of the difference between (H1) and (H2) is given by the two different possibilities of defining geodesics on a manifold when the connection is not the Levi-Civita connection, i.e., not compatible with the metric. One can define geodesics metrically, namely as critical points for a length or energy integral, or via the connection,

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namely as being autoparallel. As on a Hermitian non-Kählerian manifold, the canonical complex connection is not compatible with the metric, (H1) is analogous to the first possibility of defining geodesics, and (H2) to the second. We call a solution of (H2) *Hermitian harmonic.* From the preceding discussion, it is clear that a Hermitian harmonic map need not be harmonic in the ordinary sense, unless X is Kählerian.

We study the existence problem for (H2) by looking at the associated parabolic system, i.e., we take $f(z,t)\colon X\times[0,\infty)\to N$ and put $\partial f^i/\partial t$ instead of 0 on the right hand side of (H2), with given (continuous) initial values $f(z,0)=g(z)$. In order to show that a solution of this system exists for all $t>0$ and converges to a solution of (H2) as $t\to\infty$ we need to impose a negativity condition on the curvature of N. In §2, we present an example that shows that the negativity requirement on the image curvature is necessary. Namely, we observe that there is no nontrivial Hermitian harmonic map from a Hopf surface into the unit circle.

In §3, we study the Dirichlet problem associated with (H2), X now being a compact Hermitian manifold with smooth boundary. We solve the Dirichlet problem for given continuous boundary values, if N is complete and has nonpositive sectional curvature. This may be useful for obtaining existence results for noncompact domains via an exhaustion procedure.

A study of parabolic and elliptic systems with a nonlinearity as in the harmonic map problem and without variational or divergence structure has been undertaken by von Wahl [vW]. Apart from the fact that both his and our paper use stability results in a crucial manner, our arguments are rather different from his. Also, his main interest is not in the context of Riemannian manifolds, and in the harmonic map situation, he does not provide conditions that guarantee that as $t\to\infty$ a solution of the parabolic problem converges to a solution of the elliptic one. Some of our estimates are reminiscent of the ones of Al'ber [Al1, 2], Eells–Sampson [ES] and Hartman [Ht] for harmonic maps, but in other places we shall need more refined techniques.

In §4, we study applications of our existence result to complex geometry. We extend Siu's rigidity theorems [S1] to the case where the manifold M compared with the model space is only astheno-Kählerian, meaning that it carries a $(1,1)$ form ω with $\partial\bar\partial\omega^{m-2}=0$ ($m=\dim_{\mathbf{C}} M$) for which ω^m is a positive multiple of the volume form.

If $m=2$, the condition $\partial\bar\partial\omega^{m-2}=0$ is automatically satisfied. We can hence show, without using Kodaira's classification of compact complex surfaces, that a compact complex surface homotopy equivalent to a quotient of the unit ball in $\mathbf{C}^2$ is already $\pm$ biholomorphically equivalent to this quotient. Also, without either using Kodaira's results or Donaldson's theory of differentiable structures on 4-manifolds, we show that if N is a compact quotient of the unit ball in $\mathbf{C}^2$ (without singularities), and M is a 4-manifold with nontrivial fundamental group, then the connected sum of N and M cannot be homotopy equivalent to a complex surface. In any case, when compared with the theory initiated by Donaldson, we only have to make assumptions on the topological, but not on the differentiable structure here. We obtain a partial extension to higher dimensions, namely for complex manifolds M of algebraic dimension at least $\dim_{\mathbf{C}} M-2$. In complex dimension 3, the result says that if the connected sum of a nonsingular compact quotient of the unit ball in $\mathbf{C}^3$ and a compact manifold with nontrivial fundamental group can carry a complex structure at all, it certainly cannot admit any nonconstant meromorphic functions.

We plan to treat further applications in a future paper.

Background material about the analytic aspects can be found in [J1], and the geometric context is described in [J2].

Several extensions of our results are possible. For example, one can consider cases where domain and target are not compact but only complete and of finite volume, or where they may have certain singularities. The techniques necessary for such extensions are developed in our papers [JY1], [JY2], [JY3], and here we simply refer to them instead of elaborating these points any further.

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1. Hermitian harmonic maps between closed manifolds

We let X be a compact complex manifold with a Hermitian metric $(\gamma_{\alpha\bar\beta})$, $\alpha,\beta=1,\dots,m:=\dim_{\mathbf{C}} X$, in local coordinates $z=(z^1,\dots,z^m)$, and N a compact Riemannian manifold with metric (g_{ij}), $i,j=1,\dots,n:=\dim_{\mathbf{R}} N$ in local coordinates $(f^1,\dots,f^n)$.

We let $g\colon X\to N$ be a continuous map and look at the parabolic system

$$\begin{gathered} f\colon X\times[0,\infty)\to N\\ f(z,0)=g(z)\\ \gamma^{\alpha\bar\beta}\left(\frac{\partial^2 f^i(z,t)}{\partial z^\alpha\partial z^{\bar\beta}}+\Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar\beta}}\right)-\frac{\partial f^i(z,t)}{\partial t}=0,\quad i=1,\dots,n \end{gathered}\tag{P}$$

with $(\gamma^{\alpha\bar\beta})=(\gamma_{\alpha\bar\beta})^{-1}$, $\Gamma^i_{jk}=\frac12 g^{il}(g_{jl,k}+g_{kl,j}-g_{jk,l})$.

We put for abbreviation for $f\colon X\to N$

$$\sigma(f)^i:=\gamma^{\alpha\bar\beta}\frac{\partial^2 f^i}{\partial z^\alpha\partial z^{\bar\beta}}+\gamma^{\alpha\bar\beta}\Gamma^i_{jk}(f)\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar\beta}},\quad i=1,\dots,n.$$

The equation in (P) then takes the form

$$\sigma(f(z,t))-\frac{\partial}{\partial t}f(z,t)=0.$$

By linearizing and using results about linear parabolic systems and the implicit function theorem, it follows in a standard manner that (P) has a solution for small t and that the integral of existence in $[0,\infty)$ is open.

In order to show closedness and hence existence for all t, we assume that N has *nonpositive sectional curvature.*

We put

$$e(f):=\gamma^{\alpha\bar{\beta}}g_{ij}(f(z,t))\frac{\partial f^i}{\partial z^\alpha}\frac{\partial f^j}{\partial z^{\bar{\beta}}}.$$

We want to compute

$$\left(\gamma^{\delta\bar{\eta}}\frac{\partial^2}{\partial z^\delta\partial z^{\bar{\eta}}}-\frac{\partial}{\partial t}\right)e(f).$$

We may assume that at the point under consideration

$$\gamma_{\alpha\bar{\beta}}=\delta_{\alpha\beta} \tag{1}$$

$$g_{ij}=\delta_{ij},\quad g_{ij,k}=0,\quad \text{for all indices,} \tag{2}$$

by choosing appropriate local coordinates. Then denoting partial derivatives by subscripts

$$\begin{aligned}\left(\frac{\partial^2}{\partial z^\delta\partial z^{\bar{\delta}}}-\frac{\partial}{\partial t}\right)e(f)=\;& f^i_{z^\alpha z^\delta}f^i_{z^{\bar{\alpha}}z^{\bar{\delta}}}+f^i_{z^\alpha z^{\bar{\delta}}}f^i_{z^{\bar{\alpha}}z^\delta}\\
&+\gamma^{\alpha\bar{\beta}}{}_{,\delta}(f^i_{z^\alpha z^{\bar{\delta}}}f^i_{z^{\bar{\beta}}}+f^i_{z^\alpha}f^i_{z^{\bar{\beta}}z^{\bar{\delta}}})\\
&+\gamma^{\alpha\bar{\beta}}{}_{,\bar{\delta}}(f^i_{z^\alpha z^\delta}f^i_{z^{\bar{\beta}}}+f^i_{z^\alpha}f^i_{z^{\bar{\beta}}z^\delta})\\
&+\gamma^{\alpha\bar{\beta}}{}_{,\delta\bar{\delta}}f^i_{z^\alpha}f^i_{z^{\bar{\beta}}}\\
&+g_{ij,kl}f^i_{z^\alpha}f^j_{z^{\bar{\alpha}}}f^k_{z^\delta}f^l_{z^{\bar{\delta}}}\\
&+f^i_{z^\alpha z^\delta z^{\bar{\delta}}}f^i_{z^{\bar{\alpha}}}+f^i_{z^\alpha}f^i_{z^{\bar{\alpha}}z^\delta z^{\bar{\delta}}}\\
&-f^i_{z^\alpha t}f^i_{z^{\bar{\alpha}}}-f^i_{z^\alpha}f^i_{z^{\bar{\alpha}}t}.\end{aligned} \tag{3}$$

Differentiating the equation (P) for $f(z,t)$, we obtain

$$f^i_{z^\alpha z^\delta z^{\bar{\delta}}}-f^i_{z^\alpha t}=(g_{ij,kl}+g_{ik,jl}-g_{jk,il})f^j_{z^\delta}f^k_{z^{\bar{\delta}}}f^l_{z^\alpha}.$$

Changing indices to combine the terms with second derivatives of g_{ij} into a curvature term and using the Schwarz inequality to get rid of the terms with first derivatives of $\gamma^{\alpha\bar{\beta}}$, we obtain

$$\left(\frac{\partial^2}{\partial z^\delta\partial z^{\bar{\delta}}}-\frac{\partial}{\partial t}\right)e(f)\geqslant\tfrac{1}{2}|D^2f|^2-R_{ijkl}f^i_{z^\alpha}f^j_{z^\delta}f^k_{z^{\bar{\alpha}}}f^l_{z^{\bar{\delta}}}-ce(f). \tag{4}$$

Here, (R_{ijkl}) is the curvature tensor of N, and D^2f is the matrix of second derivatives of f in our local coordinates, and $c\geqslant 0$ is a constant.

Remark. The Bochner type inequality (4) and its derivation are the same as for ordinary harmonic maps (cf. [ES]) except that now an additional term containing first derivatives of the domain metric has to be handled by the Schwarz inequality. This accounts for the factor $\frac{1}{2}$ in (4). The curvature term remains the same.

Since the curvature of N is nonpositive, consequently

$$\left(\gamma^{\delta\bar\eta}\frac{\partial^2}{\partial z^\delta \partial z^{\bar\eta}}-\frac{\partial}{\partial t}\right)e(f)\geqslant -ce(f). \tag{5}$$

We now consider families $f(z,t,s)$ of solutions of (P), with initial values $f(z,0,s)= g(z,s)$, for $0\leqslant s\leqslant s_0$. As before, we compute (assuming again (2))

$$\begin{aligned}\left(\gamma^{\delta\bar\eta}\frac{\partial^2}{\partial z^\delta \partial z^{\bar\eta}}-\frac{\partial}{\partial t}\right)&\left(g_{ij}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\right)\\ &=2\gamma^{\delta\bar\eta}\left(g_{ij}\frac{\partial^2 f^i}{\partial z^\delta \partial s}\frac{\partial^2 f^j}{\partial z^{\bar\eta}\partial s}-\frac{1}{2}R_{ijkl}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial z^\delta}\frac{\partial f^k}{\partial s}\frac{\partial f^l}{\partial z^{\bar\eta}}\right)\geqslant 0\end{aligned} \tag{6}$$

by our curvature assumption.

Applying this with

$$f(z,t,s):=f(z,t+s)\quad \text{at } s=0,$$

we obtain with

$$k(z,t):=g_{ij}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}$$

$$\begin{aligned}\left(\gamma^{\delta\bar\eta}\frac{\partial^2}{\partial z^\delta \partial z^{\bar\eta}}-\frac{\partial}{\partial t}\right)k&=2\gamma^{\delta\bar\eta}g_{ij}f^i_{z^\delta t}f^j_{z^{\bar\eta}t}-\gamma^{\delta\bar\eta}R_{ijkl}f^i_t f^j_{z^\delta}f^k_t f^l_{z^{\bar\eta}}\\ &=2|\nabla f_t|^2-\gamma^{\delta\bar\eta}\langle R(f_t,f_{z^\delta})f_t,f_{z^{\bar\eta}}\rangle\end{aligned} \tag{7}$$

in invariant notation; here, $\langle\cdot,\cdot\rangle$ is the scalar product in TN, and ∇ is the covariant derivative in $f^{-1}TN$, and the norm comes from the metric in $f^{-1}TN\otimes T^*X$.

A consequence of (7) is

LEMMA 1. *Suppose that N has nonpositive sectional curvature. Then*

$$\sup_{z\in X} g_{ij}(f(z,t))\frac{\partial f^i}{\partial t}\frac{\partial f^j}{\partial t}$$

where f is a solution of (P), *is nonincreasing in t.*

Proof. We put $f(z,t,s)=f(z,t+s)$. Then

$$k(z,t)=g_{ij}\frac{\partial f^i}{\partial t}\frac{\partial f^j}{\partial t}.$$

Since by (7)

$$\left(\gamma^{\delta\bar{\eta}}\frac{\partial^2}{\partial z^\delta \partial z^{\bar{\eta}}}-\frac{\partial}{\partial t}\right)k \geqslant 0, \tag{8}$$

the claim follows from the maximum principle for parabolic equations. □

We let $f^0: X \to N$ be any map with bounded C^2-norm in the homotopy class of $f(\cdot, 0)$, e.g., $f(\cdot, 0)$ itself or a harmonic map homotopic to it.

Furthermore, for two homotopic maps $g_1, g_2: X \to N$, we define the homotopy distance

$$\tilde{d}(g_1, g_2)(z)$$

by choosing a homotopy

$$G: X \times [0,1] \to N,$$
$$G(z,0) = g_1(z), \quad G(z,1) = g_2(z)$$

and defining $\tilde{d}(g_1, g_2)(z)$ as the length of the unique shortest geodesic arc from $g_1(z)$ to $g_2(z)$ homotopic to $G(z,s)$, $0 \leqslant s \leqslant 1$.

We now want to compute

$$\gamma^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}}\tilde{d}^2(f(\cdot, t), f^0).$$

We have to establish some notation first. In order not to deviate from our previous conventions we continue to use a complex notation although we are going to embark upon a purely real argument.

We put, for $\alpha = 1, \ldots, m$, and similarly for $\bar{\beta} = \bar{1}, \ldots, \bar{m}$,

$$v^\alpha := v_1^\alpha \oplus v_2^\alpha := \frac{\partial}{\partial z^\alpha} f(\cdot, t) \oplus \frac{\partial}{\partial z^\alpha} f^0 \in T_{f(\cdot, t)}N \oplus T_{f^0}N.$$

We furthermore let

$$c = c_z: [0, d(f(\cdot, t), f^0)(z)] \to N$$

be the geodesic arc from $f(z,t)$ to $f^0(z)$ with $|c'| \equiv 1$, defined as before through the homotopy between $f(\cdot, t)$ and f^0,

$$e_1(z) := -c_z'(0)$$
$$e_2(z) := c_z'(\tilde{d}(f(\cdot, t), f^0)(z)),$$
$$v_i^{\alpha,\tan} := \langle v_i^\alpha, e_i \rangle e_i, \quad v_i^{\alpha,\mathrm{nor}} := v_i^\alpha - v_i^{\alpha,\tan}, \quad i = 1, 2.$$

We also note the chain rule for

$$g: X \to N, \quad \psi: N \to \mathbf{R}$$

$$\gamma^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}}\,\psi(g(z))=\gamma^{\alpha\bar{\beta}}D^2\psi\left(\frac{\partial g(z)}{\partial z^\alpha},\frac{\partial g(z)}{\partial z^{\bar{\beta}}}\right)+\langle(\operatorname{grad}\psi)\circ g(z),\sigma(g)(z)\rangle. \tag{9}$$

We then have, based on Jacobi field estimates of Karcher [Kr] and Jäger–Kaul [JäK], cf. [J1], Sections 2.2 and 2.5 and in particular (2.5.6),

$$\begin{aligned}\gamma^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}}\,\tilde{d}^2(f(\,\cdot\,,t),f^0)\geqslant \gamma^{\alpha\bar{\beta}}(&\langle v_1^{\alpha,\tan}+v_2^{\alpha,\tan},v_1^{\bar{\beta},\tan}+v_2^{\bar{\beta},\tan}\rangle\\ &+\langle v_1^{\alpha,\mathrm{nor}}-v_2^{\alpha,\mathrm{nor}},v_1^{\bar{\beta},\mathrm{nor}}-v_2^{\bar{\beta},\mathrm{nor}}\rangle)-c_1\tilde{d}(f(\,\cdot\,,t),f^0),\end{aligned} \tag{10}$$

noting that $|\sigma(f(\,\cdot\,,t))|=|(\partial f/\partial t)(\,\cdot\,,t)|$ is bounded by Lemma 1 and that $|\sigma(f^0)|$ is bounded by assumption. We also have, if the curvature of N is bounded from above by $-\mu<0$,

$$\begin{aligned}\gamma^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}}\,\tilde{d}^2(f(\,\cdot\,,t),f^0)\geqslant \mu\frac{\cosh(\mu\tilde{d}(f,f^0))}{\sinh(\mu\tilde{d}(f,f^0))}&\gamma^{\alpha\bar{\beta}}(\langle v_1^{\alpha,\mathrm{nor}},v_1^{\bar{\beta},\mathrm{nor}}\rangle+\langle v_1^{\alpha,\mathrm{nor}},v_2^{\bar{\beta},\mathrm{nor}}\rangle)\\ &-\frac{2}{\sinh}(\mu\tilde{d}(f,f^0))\gamma^{\alpha\bar{\beta}}\langle v_1^{\alpha,\mathrm{nor}},v_2^{\bar{\beta},\mathrm{nor}}\rangle-c_2,\end{aligned} \tag{11}$$

cf. [J1], formula before (2.5.6), again using that $|\sigma(f(\,\cdot\,,t))|$ and $|\sigma(f^0)|$ are bounded; these bounds of course determine the values of the constants c_1 and c_2.

We integrate (10) and then integrate the left hand by parts twice and obtain

$$\int_X e(f(\,\cdot\,,t))\leqslant c_3\int_X \tilde{d}^2(f(\,\cdot\,,t),f^0)+c_4 \tag{12}$$

where the constants depend on c_1, $|f^0|_{C^2}$, and the bounds for the second derivatives of $\gamma^{\alpha\bar{\beta}}$.

We also recall (4):

$$\left(\gamma^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}}-\frac{\partial}{\partial t}\right)e(f(\,\cdot\,,t))\geqslant -ce(f(\,\cdot\,,t))+\tfrac12|D^2f(\,\cdot\,,t)|^2. \tag{13}$$

Since

$$\left|\left(\gamma^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}}-\frac12\frac{\partial}{\partial z^\alpha}\left(\gamma^{\alpha\bar{\beta}}\frac{\partial}{\partial z^{\bar{\beta}}}\right)-\frac12\frac{\partial}{\partial z^{\bar{\beta}}}\left(\gamma^{\alpha\bar{\beta}}\frac{\partial}{\partial z^{\alpha}}\right)\right)e(f(\,\cdot\,,t))\right|\leqslant c_5|D^2f|\cdot e(f)^{1/2}, \tag{14}$$

with c_5 depending on first derivatives of $\gamma^{\alpha\bar{\beta}}$, we also have

$$\left(\frac12\frac{\partial}{\partial z^\alpha}\left(\gamma^{\alpha\bar{\beta}}\frac{\partial}{\partial z^{\bar{\beta}}}\right)+\frac12\frac{\partial}{\partial z^{\bar{\beta}}}\left(\gamma^{\alpha\bar{\beta}}\frac{\partial}{\partial z^{\alpha}}\right)-\frac{\partial}{\partial t}\right)e(f(\,\cdot\,,t))\geqslant -c_6e(f(\,\cdot\,,t)). \tag{15}$$

From (15), one obtains the pointwise bound (cf. [ES] or [J1], 3.3)

$$e(f(z,t))\leqslant c_7\sup_{t_0\leqslant\tau\leqslant t}\int_X e(f(\,\cdot\,,t)), \tag{16}$$

for some constant depending also on $(t-t_0)^{-1}$ and t_0^{-1}, with $t_0>0$.

Noting that

$$\tilde{d}^2(f(\cdot,\tau),f^0)\leqslant 2\tilde{d}^2(f(\cdot,t),f(\cdot,\tau))+2\tilde{d}^2(f(\cdot,t),f^0)$$

and

$$\tilde{d}^2(f(\cdot,t),f(\cdot,\tau))\leqslant |t-\tau|\sup_{\tau\leqslant s\leqslant t}\left|\frac{\partial f}{\partial t}(\cdot,s)\right|\leqslant c_8|t-\tau|$$

by Lemma 1, we obtain from (12) and (16), with $|df(z,t)|:=e(f(z,t))^{1/2}$,

$$|df(z,t)|\leqslant c_9\left(\int_X \tilde{d}^2(f(\cdot,t),f^0)\right)^{1/2}+c_9' \tag{17}$$

and then also

$$|df(z,t)|\leqslant c_{10}\sup_{w\in X}\tilde{d}(f(\cdot,t),f^0)(w)+c_{10}'. \tag{18}$$

LEMMA 2. *Suppose again that N has nonpositive sectional curvature. Then a solution of* (P) *exists for all $t\geqslant 0$.*

Proof. We already observed that the set of those t up to which a solution exists is open and nonempty.

Furthermore, (18) implies in conjunction with Lemma 1

$$|df(z,t)|\leqslant c(1+t)$$

for some constant c.

Since we also have a bound on $|(\partial f/\partial t)(z,t)|$ by Lemma 1, linear parabolic regularity yields $C^{2,\alpha}$-estimates for a solution of (P).

This implies closedness and hence global existence. □

We now want to study the question whether $f(\cdot,t_n)$ converges smoothly to some map in the same homotopy class, at least for some sequence $t_n\to\infty$.

We let $z_0\in X$ be a point where

$$\tilde{d}^2(f(\cdot,t),f^0)$$

attains its minimum.

From (10), we have

$$\gamma^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha\partial z^{\bar{\beta}}}\tilde{d}^2(f(\cdot,t),f^0)\geqslant -c_1\tilde{d}(f(\cdot,t),f^0), \tag{19}$$

and applying the maximum principle on $X\backslash B(z_0,R)$ and on $B(z_0,R)$, for $R>0$, we obtain

$$\sup_{z\in X}\tilde{d}^2(f(\cdot,t),f^0)(z)\leqslant\sup_{\partial B(z_0,R)}\tilde{d}^2(f(\cdot,t),f^0)+c(R)\sup_{z\in X}\tilde{d}(f(\cdot,t),f^0),\tag{20}$$

for some constant depending on R and the geometry of X. Now

$$\sup_{\partial B(z_0,R)}\tilde{d}^2(f(\cdot,t),f^0)\leqslant\tilde{d}^2(f(\cdot,t),f^0)(z_0)\\ +2R\sup_{z\in B(z_0,R)}\tilde{d}(f(\cdot,t),f^0)(z)(|df(z,t)|+|df^0(z)|).\tag{21}$$

Using (20), (21), (18), we obtain for an appropriate choice of $R>0$,

$$\sup_{z\in X}\tilde{d}^2(f(\cdot,t),f^0)(z)\leqslant\inf_{z\in X}\tilde{d}^2(f(\cdot,t),f^0)(z)+c_{11}\sup_{z\in X}\tilde{d}(f(\cdot,t),f^0)(z)\tag{22}$$

where c_{11} depends on $|f^0|_{C^2}$, a bound for $|\partial f/\partial t|$, and the geometry of X (through $c(R)$).

Before we study the general existence problem, we treat two—not mutually exclusive—cases, which are easier to handle:

Case 1. N has negative curvature.

Let $-\mu<0$ be an upper curvature bound.

We want to estimate $d(f(x,t),f(y,t))$ for $x,y\in X$. From (22), we see that for lifts to universal covers, $\tilde{f}$, $\tilde{f}^0$ and any $z\in$ a fundamental domain of X, $\tilde{f}(z,t)$ is contained in $B(\tilde{f}^0(z),R_2)\backslash B(\tilde{f}^0(z),R_1)$, where the ratio R_2/R_1 of the radii is uniformly bounded. We define $v_1^{\alpha,\mathrm{nor}}(z)$, $v_1^{\bar{\beta},\mathrm{nor}}(z)$, $\alpha=1,\dots,m$, $\beta=1,\dots,m$, as above as that component of the resp. derivative of $f(z,t)$ that is normal to the geodesic from $f(z,t)$ to $f^0(z)$ defined by the homotopy between $f(\cdot,t)$ and f^0. We then have

$$d(\tilde{f}(x,t),\tilde{f}(y,t))\leqslant\int_x^y(\gamma^{\alpha\bar{\beta}}\langle v_1^{\alpha,\mathrm{nor}},v_1^{\bar{\beta},\mathrm{nor}}\rangle(z))^{1/2}\,dz+c_{12},$$

where z runs on the shortest geodesic from x and y and c_{12} depends on the above ratio R_2/R_1 and on $d(f^0(x),f^0(y))$; actually $c_{12}=2R_2/R_1+d(f^0(x),f^0(y))$ will do.

Hölder's inequality yields

$$d^2(\tilde{f}(x,t),\tilde{f}(y,t))\leqslant 2d(x,y)\int_x^y\gamma^{\alpha\bar{\beta}}\langle v_1^{\alpha,\mathrm{nor}},v_1^{\bar{\beta},\mathrm{nor}}\rangle(z)\,dz+2c_{12}^2.\tag{24}$$

Then, identifying X with a fundamental domain,

$$\int_X d^2(\tilde{f}(x,t),\tilde{f}(y,t))\,dy\leqslant c_{13}\int_X d(x,y)^{2-2m}(\gamma^{\alpha\bar{\beta}}\langle v_1^{\alpha,\mathrm{nor}},v_1^{\bar{\beta},\mathrm{nor}}\rangle(y))\,dy+c_{14}\tag{25}$$

by introducing polar coordinates centered at x ($2m=\dim_{\mathbf{R}} X$); c_{13} and c_{14} depend on the geometry of X.

From (25)

$$\int_X \int_X d^2(\tilde{f}(x,t), \tilde{f}(y,t))\, dy\, dx \leqslant c_{15} \int_X \gamma^{\alpha\bar{\beta}} \langle v_1^{\alpha,\mathrm{nor}}, v_1^{\bar{\beta},\mathrm{nor}} \rangle (y)\, dy + c_{16}. \tag{26}$$

We now return to (11). On the left hand side, we may write

$$\tilde{d}(f(\cdot,t), f^0) - \inf_{z \in X} \tilde{d}(f(\cdot,t), f^0)(z).$$

By (22), this quantity is bounded by a lower order term.

We then integrate (11) over X and integrate the left hand by parts twice. We obtain

$$\int_X \gamma^{\alpha\bar{\beta}} \langle v_1^{\alpha,\mathrm{nor}}, v_1^{\bar{\beta},\mathrm{nor}} \rangle (y)\, dy \leqslant c_{17}, \tag{27}$$

with c_{17} depending on the energy of f^0 and μ, and also on the constant c_{11} of (22) and on bounds for the second derivatives of $\gamma^{\alpha\bar{\beta}}$.

Combining (26) and (27),

$$\int_X \int_X d^2(\tilde{f}(x,t), \tilde{f}(y,t))\, dy\, dx \leqslant c_{18}.$$

In particular, there exists some $x_0 \in X$ with putting

$$p := \tilde{f}(x_0, t),$$

$$\int_X d^2(\tilde{f}(x,t), p)\, dx \leqslant c_{19} \quad \left(= \frac{c_{18}}{\mathrm{Vol}(X)} \right). \tag{28}$$

Returning to (12), we conclude

$$\int_X e(f(x,t))\, dx \leqslant c_{20},$$

and finally from (16)

$$e(f(z,t)) \leqslant c_{21},$$

for all $z \in X$, $t \geqslant t_0 > 0$. Having fixed $t_0 > 0$ sufficiently small, the constant c_{21} is independent of $t \geqslant t_0$.

Since by Lemma 1, also $|\partial f/\partial t|$ is bounded independently of t, standard results about linear parabolic equations imply $C^{2,\alpha}$ bounds for a solution of (P), again independent of t, and hence global existence.

Moreover, there exists a sequence $t_n \to \infty$, for which $f(\cdot, t_n)$ converges to a smooth map f_∞ in the same homotopy class.

Case 2. This case is the following:

N, as always, has nonpositive sectional curvature, our initial map g is smooth, and we have

$$e(g^*TN) \neq 0,$$

where e denotes the Euler class.

We have the following simple topological result.

LEMMA 3. *Let M, N be compact differentiable manifolds, $g: M \to N$ smooth, and*

$$e(g^*TN) \neq 0.$$

Then for any continuous $h: M \to N$, homotopic to g, there exists some $x_0 \in M$ with

$$g(x_0) = h(x_0).$$

Proof. Let $H: M \times [0,1] \to N$ be a smooth homotopy with $H(z,0) = g(z)$, $H(z,1) = h(z)$ for all $z \in M$.

We now suppose $g(x) \neq h(x)$ for all $x \in M$. We may then parametrize the homotopy H in such a way that $(\partial H/\partial t)(x,t)|_{t=0} \neq 0$ for all x. Then $(\partial H/\partial t)(x,t)|_{t=0} \neq 0$ is a nowhere vanishing cross section of g^*TN. Consequently

$$e(g^*TN) = 0,$$

where e denotes the Euler class, cf. [St].

This contradiction proves the claim. □

We apply Lemma 3 to $f(\cdot, t)$ and f^0. Then

$$\inf_{z \in X} \tilde{d}^2(f(\cdot, t), f^0)(z) = 0,$$

and from (22)

$$\sup_{z \in X} \tilde{d}(f(\cdot, t), f^0)(z) \leqslant c_{11}.$$

(18) then yields a bound for $e(f(z,t))$, independent of t, and since $|(\partial f/\partial t)(z,t)|$ is also bounded by Lemma 1, we get global existence and convergence of $f(\cdot, t_n)$ to a smooth map in the same homotopy class for some sequence $t_n \to \infty$ as before in Case 1.

We can now address the existence question.

Definition. We call a solution $f: X \to N$, X Hermitian, N Riemannian, of

$$\gamma^{\alpha\bar{\beta}} \frac{\partial^2 f^i}{\partial z^\alpha \partial z^{\bar{\beta}}} + \gamma^{\alpha\bar{\beta}} \Gamma^i_{jk} \frac{\partial f^j}{\partial z^\alpha} \frac{\partial f^k}{\partial z^{\bar{\beta}}} = 0, \quad i = 1, \ldots, n, \tag{E}$$

Hermitian harmonic.

THEOREM 1. *Let X be a compact Hermitian manifold. Let N be a compact Riemannian manifold of negative sectional curvature. Let $g: X \to N$ be continuous, and suppose that g is not homotopic to a map onto a closed geodesic of N. Then there exists a map*

$$f: X \to N$$

homotopic to g and satisfying

$$\gamma^{\alpha\bar{\beta}} \left(\frac{\partial^2 f^i}{\partial z^\alpha \partial z^{\bar{\beta}}} + \Gamma^i_{jk} \frac{\partial f^j}{\partial z^\alpha} \frac{\partial f^k}{\partial z^{\bar{\beta}}} \right) = 0, \quad i = 1, \ldots, n.$$

Proof. The assumptions mean that we are in the situation of Case 1. As noted there, for some sequence $k_n \to \infty$, $f(x, t_n)$ converges to a smooth map $f(x)$ in the same homotopy class. We have to show that f is Hermitian harmonic. Putting $s = t$ in (7), we obtain

$$\left(\gamma^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} - \frac{\partial}{\partial t} \right) \left(g_{ij} \frac{\partial f^i}{\partial t} \frac{\partial f^j}{\partial t} \right) = 2|\nabla f_t|^2 - \gamma^{\alpha\bar{\beta}} \langle R(f_t, f_{z^\alpha}) f_t, f_{z^{\bar{\beta}}} \rangle.$$

Since

$$g_{ij} \frac{\partial f^i}{\partial t} \frac{\partial f^j}{\partial t} \geqslant 0,$$

the maximum principle implies that both terms on the right hand side converge to zero as $t \to \infty$. Therefore,

$$v(x) := \lim_{t_n \to \infty} \frac{\partial f}{\partial t}(x, t_n)$$

is a parallel section of TN along $f(X)$. The assumptions that N has negative curvature and that f cannot map M onto a closed geodesic then imply $v \equiv 0$. Hence

$$\sigma(f(x)) = \lim_{t_n \to \infty} \sigma(f(x, t_n)) = \lim_{t_n \to \infty} \frac{\partial f}{\partial t}(x, t_n) = 0,$$

and f is Hermitian harmonic. □

Remark. In the case where g is homotopic to a constant map, of course g is homotopic to a Hermitian harmonic map, namely a constant one. In this case also the global existence and convergence become easy, since in this case

$$d^2(f(z, t), p),$$

for any $p \in N$, is a globally defined smooth subsolution of

$$\left(\gamma^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} - \frac{\partial}{\partial t} \right).$$

THEOREM 2. *Let X be a compact Hermitian manifold. Let N be a compact Riemannian manifold of nonpositive sectional curvature. Let $g: X \to N$ be smooth and $e(g^*TN) \neq 0$, where e is the Euler class.*

Then there again exists a Hermitian harmonic map f homotopic to g.

Proof. Using the analysis of Case 2, the proof is similar to the one of Theorem 1. From the proof of Lemma 3, we see that f^*TN cannot have a nonzero cross-section, in particular no nontrivial parallel section. This finishes the proof. □

Remark. Similarly, we can show existence if $\chi(N) \neq 0$, and $g: X \to N$ is continuous with $g^*: H^n(N, \mathbf{Z}) \to H^n(X, \mathbf{Z})$ injective ($n = \dim N$).

Namely, we may assume that g is smooth, and since $e(TN) = \chi(N)\omega_N$, where ω_N is a generator of $H^n(N, \mathbf{Z})$, we then have by functoriality

$$e(g^*TN) = g^*(e(TN)) = \chi(N) g^*(\omega_N) \neq 0$$

by our assumptions, and Theorem 2 applies.

We now return to the general case of a nonpositively curved target N.

In (10), we replace $\tilde{d}^2(f(\cdot, t), f^0)$ on the left hand side by

$$\tilde{d}^2(f(\cdot, t), f^0) - \inf_{z \in X} \tilde{d}^2(f(\cdot, t), f^0)(z).$$

(22) implies that this quantity is bounded by $c_{11} \sup_X \tilde{d}(f(\cdot, t), f^0)$. We then integrate the left hand side by parts twice and obtain

$$\int_X e(f(z, t))\, dz \leqslant c_{22} \sup_{z \in X} \tilde{d}(f(\cdot, t), f^0)(z) + c_{23}, \tag{29}$$

and using (16) as above then

$$|df(z, t)| \leqslant c_{24} \Big(\sup_{w \in X} \tilde{d}(f(\cdot, t), f^0)(w) \Big)^{1/2} + c_{25}. \tag{30}$$

Consequently, for any $z_1, z_2 \in X$,

$$d(\tilde{f}(z_1, t), \tilde{f}(z_2, t)) \leqslant c_{26} \Big(\sup_{w \in X} \tilde{d}(f(\cdot, t), f^0)(w) \Big)^{1/2} + c_{23}. \tag{31}$$

Now suppose that for some sequence $t_n \to \infty$,

$$\tilde{d}(f(\cdot, t_n), f^0)(w) \to \infty \tag{32}$$

for some w and hence by (22) for all $w \in X$.

For any two $z_1, z_2 \in X$, we look at the geodesics γ_1^n, γ_2^n from $f^0(z_1)$ resp. $f^0(z_2)$ to $f(z_1, t_n)$ resp. $f(z_2, t_n)$, as always in the homotopy class determined by the homotopy between f^0 and $f(\cdot, t)$. We parametrize each of these geodesics by arclength on some interval $[0, T_n]$, with $\gamma_i^n(0) = f^0(z_i)$, and $\gamma_i^n(T_n) = f(z_i, t_n)$ $(i=1,2)$. Actually, T_n should also carry an index $i=1,2$, but on account of (22), this will be inessential for the sequel. By (32), $T_n \to \infty$. After selection of a subsequence, γ_1^n and γ_2^n converge to geodesic rays γ_1 and γ_2, resp.

Since N has nonpositive sectional curvature,

$$d(\gamma_1^n(\tau), \gamma_2^n(\tau)),$$

where the distance is always measured in some fixed homotopy class of arcs connecting $\gamma_1^n(\tau)$ and $\gamma_2^n(\tau)$ (alternatively, we lift things to universal covers), is a convex function of τ. Since by (30), $d(\gamma_1^n(T_n), \gamma_2^n(T_n)) \leqslant c_{28}(T_n)^{1/2}$, for large T_n, this convexity implies that for any fixed $\tau \geqslant 0$

$$\begin{aligned}\lim_{n\to\infty} d(\gamma_1^n(\tau), \gamma_2^n(\tau)) &\leqslant \lim_{n\to\infty} \left\{ \left(1 - \frac{\tau}{T_n}\right) d(\gamma_1^n(0), \gamma_2^n(0)) + \frac{\tau}{T_n} c_{28} \sqrt{T_n} \right\} \\ &= d(\gamma_1^n(0), \gamma_2^n(0)).\end{aligned}$$

Therefore, the limiting rays γ_1, γ_2 satisfy

$$d(\gamma_1(\tau), \gamma_2(\tau)) \leqslant d(\gamma_1(0), \gamma_2(0)) \quad \text{for all } \tau \geqslant 0. \tag{33}$$

We let f^0 be a harmonic map homotopic to $f(\cdot, t)$, and put

$$f^\tau(z_1) := \gamma_1(\tau) \quad \text{for } z_1 \in X,\ \tau \geqslant 0.$$

Differentiating (33), we obtain

$$e(f^\tau(z)) \leqslant e(f^0(z)) \quad \text{for all } z \in X. \tag{34}$$

Since f^0 as a harmonic map is energy minimizing, we conclude

$$e(f^\tau(z)) = e(f^0(z)) \quad \text{for all } z \in X.$$

In particular, each f^τ is harmonic and, by the uniqueness theorem of Al'ber and Hartman (the argument is given in Theorem 4 below), satisfies the same estimates on its C^2-norm as f^0.

The preceding construction implies that, if (32) holds, for each t_n (after selection of a subsequence), we can find a harmonic map f^n of the same energy as f^0 with

$$\tilde{d}(f(\cdot, t_n), f^n) \leqslant c_{29}(\tilde{d}(f(\cdot, t_n), f^0))^{1/2} + c_{30}. \tag{35}$$

We can repeat the procedure with f^n in place of f^0. After a finite number of iterations, we either obtain for each n a harmonic map $\widehat{f^n}$ homotopic to f^0, satisfying the same estimates as f^0, and with

$$\tilde{d}(f(\cdot,t_n),\widehat{f^n})\leqslant \text{const.} \tag{36}$$

independent of t_n, or N is flat. Namely, in each iteration step, we generate at least one more flat direction, cf. the argument of Theorem 4 below. Of course, if N is flat, we may assume that N is a torus, by lifting to finite covers, and then we can also trivially find a harmonic map $\widehat{f^n}$, homotopic to f^0, and satisfying (30).

We may apply the reasoning leading to (18) with the variable map $\widehat{f^n}$ instead of the fixed map f^0 and obtain

$$|df(z,t_n)|\leqslant c_{10}\sup_{w\in X}\tilde{d}(f(\,\cdot\,,t_n),\widehat{f^n})(w)+c_{10}'\leqslant \text{const.}, \tag{37}$$

by (36).

Estimate (37), combined with the reasoning of the proof of Theorem 1, yields

THEOREM 3. *Let X be a compact Hermitian manifold, N a compact Riemannian manifold of nonpositive sectional curvature. Let $g\colon X\to N$ be continuous, and suppose g is not homotopic to a map $\tilde{g}\colon X\to N$ for which there is a nontrivial parallel section of $\tilde{g}^{-1}(TN)$.*

Then g is homotopic to a Hermitian harmonic map $f\colon X\to N$.

We can also study the uniqueness question. We should remark that the statement and proof of Theorem 4 below apply as well to harmonic as to Hermitian harmonic maps.

THEOREM 4. *Suppose N has nonpositive sectional curvature. Let f_0, f_1 be homotopic Hermitian harmonic maps. Then f_0 and f_1 can be joined by a parallel family f_s, $0\leqslant s\leqslant 1$, of Hermitian harmonic maps, and*

$$g_{ij}(f_s(x))\frac{\partial f_s^i}{\partial s}\frac{\partial f_s^j}{\partial s}$$

is independent of s.

Also, for any $v\in T_zX$,

$$\left\langle R\left(df(v),\frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial s},df(v)\right\rangle\equiv 0.$$

If N has negative sectional curvature, and if f_0 and f_1 are not maps onto points or closed geodesics, then $f_0=f_1$.

Proof. We shall use a method of Al'ber [Al1, 2] and Hartman [Ht].

We let $f_s(x)$, $0\leqslant s\leqslant 1$, x fixed, be the geodesic from $f_0(x)$ to $f_1(x)$ in the homotopy class determined by the homotopy between f_0 and f_1. We let $f(x,t,s)$ be a solution of (P) with initial values $f_s(x)$, for each s, $0\leqslant s\leqslant 1$.

We recall (6), i.e.,

$$\left(\gamma^{\alpha\bar\beta}\frac{\partial^2}{\partial z^\alpha \partial x^{\bar\beta}}-\frac{\partial}{\partial t}\right)\left(g_{ij}\frac{\partial f^i}{\partial s}(x,t,s)\frac{\partial f^j}{\partial s}(x,t,s)\right)\geqslant 0. \tag{38}$$

We denote by $\tilde d$ the distance function obtained by measuring the length of geodesic arcs in the homotopy class determined by the homotopy between f_1 and f_2.

Now

$$\begin{aligned}\tilde d^2(f(x,t,s),f_0(x)) &\leqslant \sup_{0\leqslant\sigma\leqslant s} g_{ij}(f(x,t,\sigma))\frac{\partial f^i}{\partial s}(x,t,\sigma)\frac{\partial f^i}{\partial s}(x,t,\sigma)\\ &\leqslant \tilde d^2(f_s(x),f_0(x))\end{aligned} \tag{39}$$

by the maximum principle from (38).

Then (18) yields a bound for the spatial gradient of $f(x,t,s)$ independent of t, and we conclude that the solution to (P) with initial values $f_s(x)$ exists for all time and converges to some map $f(x,s)$ as $t\to\infty$. We choose $x_0\in X$ with

$$\tilde d(f_0(x_0),f_1(x_0))=\sup_{x\in X}\tilde d(f_0(x),f_1(x)).$$

By construction therefore

$$\tilde d(f_0(x_0),f_s(x_0))=\sup_{x\in X}\tilde d(f_0(x),f_s(x)).$$

From (39)

$$\tilde d(f_0(x,t,s),f_0(x_0))=\tilde d(f_s(x_0),f_0(x_0)) \tag{40}$$

and similarly

$$\tilde d(f(x_0,t,s),f_1(x_0))=\tilde d(f_s(x_0),f_1(x_0)) \tag{41}$$

(40), (41) and the choice of $f_s(x)$ imply $f(x_0,s)=f(x_0,t,s)=f_s(x_0)$ for all s.

Recalling

$$\sup_{x\in X} g_{ij}(f(x,t,s))\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\leqslant \sup_{x\in X} g_{ij}(f_s(x))\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\quad \text{for } 0\leqslant s\leqslant 1,\ 0<t<\infty,$$

we note that for all t, the supremum is attained at $x=x_0$ and is independent of t.

The strong maximum principle applied to (38) then shows that

$$g_{ij}(f(x,t,s))\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\ \cdot$$

is independent of x and t.

Since s was the arc length parameter on the geodesic $f_s(x_0)$,

$$g_{ij}(f(x_0,t,s))\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}$$

and consequently

$$g_{ij}(f(x,t,s))\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}$$

is independent of s as well. Thus, for each x and t, $f(x,t,\cdot)$ is a curve of equal length from $f_0(x)$ to $f_1(x)$. Since $f(x,0,\cdot)$ was a minimal geodesic, $f(x,0,s){=}f(x,t,s)$ for all x, t, s. In particular, $f(x,t,s)$ is independent of t, i.e.,

$$f(x,t,s)=f(x,s)=f_s(x)$$

then is Hermitian harmonic for each s.

The claims then are easy consequences of the fact that because

$$g_{ij}(f(x,t,s))\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}$$

is constant and because of the curvature assumption on N, both terms on the right hand side of (6) have to vanish.

Remark. The Hermitian harmonic map equation differs from the standard one by a linear first order term. Our method described in this paragraph works more generally if we replace the second order elliptic operator in the harmonic map equation by one which differs from it by such a linear first order term. Of course, one has to make sure that such an operator is invariantly defined. For example, one may take a vector field V on the domain X and add a term of the form (df,V), the brackets denoting evaluation of a vector field on a one-form. Actually, the difference between the Hermitian and the standard harmonic map equation can be expressed in such a manner.

2. A counterexample

We let H^m be the quotient of $\mathbf{C}^m\setminus\{0\}$ by the action of

$$w\mapsto\Lambda(w):=\lambda w$$

for some $\lambda>1$.

For $m{=}2$, H^2 is a Hopf surface.

We put

$$ds^2:=\frac{1}{r}(dr^2+r\,d\omega^2),$$

where $r{:=}|w|$, and $d\omega^2$ in the standard metric on the unit sphere S^{2m-1}. This defines a Hermitian metric on $\mathbf{C}^m\setminus\{0\}$ which passes to the quotient H^m.

THEOREM 5. *For $m \geqslant 2$, there is no nontrivial Hermitian harmonic*

$$f: H^m \to S^1,$$

where S^1 is the unit circle (parametrized by $[0, 2\pi)$).

Proof. If there would exist such a map, then by the uniqueness result of Theorem 4, it would have to be homotopically nontrivial and independent of the angle $\omega \in S^{2m-1}$, depending only on the radius r.

We consider the lift to universal covers, denoted by the same letter

$$f: \mathbf{C}^m \setminus \{0\} \to \mathbf{R}.$$

The fact that f passes to quotients means that

$$f(\lambda w) = f(w) + 2\pi$$

for all $w \in \mathbf{C}^m \setminus \{0\}$. Also, because of the uniqueness result of Theorem 4, we get for $\mu > 1$

$$f(\mu w) = f(w) + 2\pi \frac{\log \mu}{\log \lambda}.$$

This is a functional equation for the logarithm, implying that

$$f(w) = 2\pi \frac{\log w}{\log \lambda}.$$

We shall now show that for $m \geqslant 2$, $f(w) = 2\pi \log w / \log \lambda$ does not satisfy the equation for a Hermitian harmonic map. In order to derive the equation, we consider

$$\frac{\partial^2}{\partial w^\alpha \partial w^{\bar\beta}} f = \frac{\partial}{\partial w^\alpha}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial w^{\bar\beta}}\right),$$

since f depends only on r,

$$\begin{aligned} &= \frac{\partial^2 f}{\partial r^2}\frac{\partial r}{\partial w^\alpha}\frac{\partial r}{\partial w^{\bar\beta}} + \frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial w^\alpha \partial w^{\bar\beta}} \\ &= \frac{\partial^2 f}{\partial r^2}\frac{w^{\bar\alpha}}{2r}\frac{w^\beta}{2r} + \frac{\partial f}{\partial r}\left(\frac{\delta_{\alpha\beta}}{2r} - \frac{w^{\bar\alpha} w^\beta}{4r^3}\right). \end{aligned}$$

The equation for a Hermitian harmonic map then is

$$0 = r\frac{\partial^2 f}{\partial w^\alpha \partial w^{\bar\alpha}} = \frac{r}{4}\frac{\partial^2 f}{\partial r^2} + \frac{2m-1}{4}\frac{\partial f}{\partial r},$$

or equivalently

$$0=\frac{\partial^2 f}{\partial r^2}+\frac{2m-1}{r}\frac{\partial f}{\partial r}.$$

The logarithm, however, satisfies the equation

$$0=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r},$$

which is different from the previous one for $m\geqslant 2$. This completes the proof. □

One can actually even show that for $m=2$, for any Hermitian metric on the Hopf surface $H=H^2$, not just for the above radially symmetric one, there is no nontrivial Hermitian harmonic

$$f\colon H\to S^1.$$

Namely, by Lemma 7 below, such a Hermitian harmonic map would be pluriharmonic, hence harmonic w.r.t. any Hermitian metric (compatible with the complex structure), thus in particular w.r.t. ds^2 as above. This, however, was just seen to be impossible.

3. The Dirichlet problem

We now let X be a compact complex manifold with a nonempty smooth boundary ∂X. Otherwise, the assumptions on X and N and the notation are as in §1, except that N need only be complete, but not compact. For the moment, we assume that the map $g\colon X\to N$ is of class $C^{2,\alpha}$.

We look at the parabolic system

$$\begin{gathered} f\colon X\times[0,\infty)\to N \\ f(z,0)=g(z) \quad \text{for } z\in X \\ f(z,t)=g(z) \quad \text{for } z\in\partial X,\ 0\leqslant t\leqslant\infty \\ \gamma^{\alpha\bar\beta}\left(\frac{\partial^2 f^i(z,t)}{\partial z^\alpha\partial z^{\bar\beta}}-\Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar\beta}}\right)-\frac{\partial f^i(z,t)}{\partial t}=0, \quad i=1,\dots,n. \end{gathered} \tag{P$'$}$$

Again, it follows from the theory of linear parabolic systems that (P′) has a solution for small t and that the interval of existence is open. A detailed treatment of the relevant construction can be found in [Hm].

We now assume again that N has *nonpositive sectional curvature.* Since

$$\frac{\partial f}{\partial t}=0 \quad \text{on } \partial X \text{ for } t>0,$$

Lemma 1 pertains to the present situation.

In order to obtain spatial estimates, we let f^0 be any map with bounded C^2-norm and $f^0|_{\partial X}=g|_{\partial X}$, for example, g itself, or the harmonic extension of $g|_{\partial X}$. Of course, f^0 has to be homotopic to g.

(22) of §1 holds again, and the maximum principle this time implies

$$\sup_{z\in X} \tilde{d}^2(f(\cdot,t),f^0)(z)\leqslant c, \tag{1}$$

for some constant c, independent of t, since for $w\in\partial X$,

$$\tilde{d}^2(f(\cdot,t),f^0)(w)=0.$$

This then can be used to obtain interior gradient bounds as in §1.

At the boundary, we need a more refined argument.

LEMMA 4. *There exist $\delta_0>0$ and $R_0>0$ with the following property:*

If f is a solution of (P) *for $0\leqslant t\leqslant T$ and if for some t_0, $0<t_0\leqslant T$, $f(B(x_0,R),t_0)\subset B(p,\delta)$, $x_0\in X$, $B(x_0,R_0)\subset X$, $0<\delta\leqslant\delta_0$, for some R, $0<R\leqslant R_0$, $p\in N$, $(B(q,r):=\{q':d(q,q')\leqslant r\}$, d being the distance function of the manifold containing q), then*

$$|\operatorname{grad} f(x_0,t_0)|\leqslant \frac{c\delta}{R} \quad (\text{grad } \textit{denotes the spatial gradient}), \tag{2}$$

where δ_0, R_0 and c depend on the geometry of X and N and on

$$\left|\frac{\partial f(x,t_0)}{\partial t}\right|_{L^\infty(B(x_0,R))}.$$

Proof. We shall use ideas of E. Heinz [Hz] and of [JK]:

We put

$$\mu:=\max_{x\in B(x_0,R)}(R-d(x,x_0))\,|\operatorname{grad} f(x,t_0)|.$$

There exists $x_1\in B(x_0,R)$ with

$$\mu=(R-d(x_1,x_0))\,|\operatorname{grad} f(x_1,t_0)|$$

and

$$|\operatorname{grad} f(x_0,t_0)|\leqslant\frac{\mu}{R}. \tag{3}$$

We put

$$d:=R-d(x_1,x_0).$$

We choose local coordinates near $q=f(x_1,t_0)$ with

$$g_{ij}(q)=\delta_{ij},\quad g_{ij,k}(q)=0\quad \text{for all } i,j,k. \tag{4}$$

In these coordinates,

$$\gamma^{\alpha\bar\beta}\frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}} f^i(x,t_0) = -\gamma^{\alpha\bar\beta}\Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar\beta}} + \frac{\partial f^i(x,t_0)}{\partial t}, \quad i=1,\dots,n.$$

Thus, since the Γ^i_{jk} are obtained from the first derivatives of g_{ij},

$$\left|\gamma^{\alpha\bar\beta}\frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}} f^i(x,t_0)\right| \leqslant c_1\delta|\operatorname{grad} f(x,t_0)|^2 + c_2. \tag{5}$$

c_2 of course depends on the L^∞-norm of $\partial f/\partial t$.

We now choose local coordinates at x_1 with

$$\gamma_{\alpha\bar\beta}(x_1) = \delta_{\alpha\beta}. \tag{6}$$

We also abbreviate

$$\varphi(x) := f(x,t_0) \quad \text{and} \quad \varphi'(x) := \varphi(x) - \varphi(x_1).$$

We choose a linear function l (linear w.r.t. the coordinates) with

$$|l(x)| \leqslant |x-x_1| \quad \text{and} \quad \langle \operatorname{grad} l(x_1), \operatorname{grad}\varphi(x_1)\rangle = |\operatorname{grad}\varphi(x_1)|$$

and put, for $\varrho>0$, $\varrho<d$,

$$a(x) := l(x)(|x-x_1|^{-2m} - \varrho^{-2m})$$

(the absolute value again being taken w.r.t. the coordinates); we also let

$$D_\varrho := \{|x-x_1| \leqslant \varrho\}.$$

We compute, for $0<\varepsilon<\varrho$, with $\gamma := \det\gamma_{\alpha\bar\beta}$

$$\begin{aligned}\int_{D_\varrho\setminus D_\varepsilon} &\left(a\gamma^{\alpha\bar\beta}\frac{\partial^2}{\partial z^\alpha\partial z^{\bar\beta}}\varphi' - \varphi'\gamma^{\alpha\bar\beta}\frac{\partial^2}{\partial z^\alpha\partial z^{\bar\beta}}a\right)\gamma\, dz^1 \dots dz^{\bar m} \\ &= \int_{D_\varrho\setminus D_\varepsilon} -\frac{\partial}{\partial z^\alpha}(\gamma^{\alpha\bar\beta}\gamma)a\frac{\partial}{\partial z^{\bar\beta}}\varphi' + \frac{\partial}{\partial z^{\bar\beta}}(\gamma^{\alpha\bar\beta}\gamma)\varphi'\frac{\partial}{\partial z^\alpha}a \\ &\quad + \int_{\partial(D_\varrho\setminus D_\varepsilon)} \langle a \operatorname{grad}\varphi' - \varphi'\operatorname{grad} a, d\vec\sigma\rangle.\end{aligned} \tag{7}$$

Now

$$\int_{D_\varrho}\left|a\gamma^{\alpha\bar\beta}\frac{\partial^2}{\partial z^\alpha\partial z^{\bar\beta}}\varphi\right|\gamma \leqslant \int_{D_\varrho}\frac{|\gamma^{\alpha\bar\beta}(\partial^2/\partial z^\alpha\partial z^{\bar\beta})\varphi'|}{|x-x_1|^{2m-1}}\gamma,$$

since $l(x) \leqslant |x-x_1|$,

$$\int_{D_\varrho} \left| \varphi' \gamma^{\alpha\bar\beta} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}} a \right| \gamma \leqslant \int_{D_\varrho} \frac{|\varphi'|}{|x-x_1|^{2m}}$$

because of (6), the Lipschitz continuity of $\gamma^{\alpha\bar\beta}$ and the fact that

$$\frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}} a = 0, \quad a|_{\partial D_\varrho} = 0,$$

$$\int_{\partial D_\varrho} |\langle \varphi' \operatorname{grad} a, d\vec\sigma \rangle| \leqslant \frac{2m}{\varrho^{2m}} \int_{\partial D_\varrho} |\varphi'|.$$

Moreover,

$$\lim_{\varepsilon \to 0} \left(\int_{\partial D_\varepsilon} \langle a \operatorname{grad} \varphi' - \varphi' \operatorname{grad} a, d\vec\sigma \rangle \right) = \omega_{2m} |\operatorname{grad} \varphi(x_1)|.$$

We conclude

$$\begin{aligned} \omega_{2m} |\operatorname{grad} \varphi(x_1)| = \frac{2m}{\varrho^{2m}} \int_{|x-x_1|=\varrho} |\varphi(x) - \varphi(x_1)| + \int_{D_\varrho} \frac{|\gamma^{\alpha\bar\beta} (\partial^2/\partial z^\alpha \partial z^{\bar\beta}) \varphi(x)|}{|x-x_1|^{2m-1}} \\ + c'' \int_{D_\varrho} \frac{|\varphi(x) - \varphi(x_1)|}{|x-x_1|^{2m}} + c''' \int_{D_\varrho} \frac{|\operatorname{grad} \varphi(x)|}{|x-x_1|^{2m-1}}. \end{aligned} \tag{8}$$

We put

$$\varrho = d\theta, \quad 0 < \theta \leqslant 1.$$

Then, from (8), (5)

$$\begin{aligned} \frac{\mu}{d} = |\operatorname{grad} \varphi(x_1)| \leqslant \frac{c_4}{d^{2m} \theta^{2m}} \int_{|x-x_1|=d\theta} |\varphi(x) - \varphi(x_1)| \\ + c_5 \delta \int_{|x-x_1| \leqslant d\theta} \frac{|\operatorname{grad} \varphi(x)|^2}{|x-x_1|^{2m-1}} + c_6\, d\theta + c_7 \int_{|x-x_1| \leqslant d\theta} \frac{|\operatorname{grad} \varphi(x)|}{|x-x_1|^{2m-1}}. \end{aligned} \tag{9}$$

By definition of μ and d, for $x \in D_{d\theta}$

$$|\operatorname{grad} \varphi(x)| \leqslant \frac{\mu}{d(1-\theta)},$$

consequently from (9)

$$\frac{\mu}{d} \leqslant \frac{2c_4\delta}{d\theta} + \frac{c_8 \delta \mu^2 \theta}{d(1-\theta)^2} + c_6\, d\theta + c_7 \frac{\theta}{1-\theta} \mu$$

or, assuming $\theta \leqslant \frac{1}{2}$,

$$\mu \leqslant \frac{2c_4\delta}{\theta} + c_9 \theta \mu^2 + c_{10} \theta R^2 + c_{11} \theta \mu R. \tag{10}$$

This holds for all θ with $0<\theta\leqslant\frac{1}{2}$.

If we choose δ_0 sufficiently small, we can find some $\lambda>0$ with the property that whenever $0<\delta\leqslant\delta_0$

$$\frac{2c_4}{\lambda}+c_9\lambda\delta\leqslant\tfrac{1}{2}.$$

Then either $\mu<2\lambda\delta$, or there exists $\theta\leqslant\frac{1}{2}$ with

$$\theta=\frac{\lambda\delta}{\mu}.$$

Using this θ in (10), we get

$$\mu^2\leqslant 2c_{10}\lambda\delta R^2+c_{11}\lambda\delta\mu R,$$

whence

$$\mu\leqslant c_{12}(\delta+\delta^{1/2})R.$$

This and (3) would imply

$$|\operatorname{grad} f(x_0,t_0)|\leqslant c_{12}(\delta+\delta^{1/2}).$$

Since this then would have to hold for all δ, $0<\delta\leqslant\delta_0$ (by just shrinking R accordingly), we conclude that in the above alternative, the first case has to occur, i.e.,

$$\mu<2\lambda\delta.$$

In conjunction with (3), this implies (2). □

We can now prove a gradient bound at the boundary:

LEMMA 5. *Let f be a solution of* (P′), *where N is simply connected and nonpositively curved. Then for $z\in\partial X$, $t\geqslant t_0>0$*

$$|\operatorname{grad} f(z,t)|\leqslant \text{const.}\quad (\operatorname{grad} \textit{ denotes the spatial gradient}),$$

where the constant depends on the geometry of X and N and the initial and boundary values g, and on t_0.

Proof. We shall use arguments from [HKW] and [JK].

Lemma 4 gives interior gradient bounds, and it consequently suffices to show that if $d(z_0,\partial X)=R$, we have

$$\max_{d(z,z_0)\leqslant R} d(f(z,t),f(z_0,t))\leqslant cR \tag{11}$$

or equivalently, if $d(z_1, z_0)=R$, $z_1 \in \partial X$, $d(z_2, z_0) \leqslant R$, $z_2 \in X$, we have

$$d(f(z_1,t), f(z_2,t)) \leqslant cR, \tag{12}$$

for some constant c depending only on X, N, g.

We may assume that R is smaller than the injectivity radius of X. We can then lift $f(\cdot, t)|_{B(z_0,R)}$ to a map into the universal cover $\tilde{N}$ of N. We denote the lifted map again by $f(z,t)$.

We may obviously assume

$$f(z_1,t) \neq f(z_2,t).$$

We fix some $\tau > 0$. Since N has nonpositive sectional curvature and is complete, any two points in $\tilde{N}$ can be joined by a unique geodesic arc. We continue the geodesic arc from $f(z_2,t)$ to $f(z_1,t)$ beyond $f(z_1,t)$ until we reach a distance τ from $f(z_1,t)$. The corresponding point is denoted by $q=q(z_2)$. Because of the nonpositivity of the curvature of N again, the squared distance function from q, $d^2(\cdot, q)$ is (strictly) convex. From the chain rule, as f satisfies (P′)

$$\left(\gamma^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} - \frac{\partial}{\partial t}\right) d^2(f(z,t), q)) \geqslant 0. \tag{13}$$

There exists some fixed $R_0 > 0$, smaller than the injectivity radius of X, with the property that for every $z_1 \in \partial X$,

$$X' := X'(z_1) := \{z \in X : \operatorname{dist}(z_1, z) \leqslant R_0\}$$

is homeomorphic to a ball.

As before, we lift $f|_{X'}$ to a map into $\tilde{N}$.

In order to have $\partial X'$ smooth, we may round off the corners slightly, without changing the notation.

We then solve the following linear parabolic problem:

$$h: X' \times [0, \infty) \to \mathbf{R}$$

$$\left(\gamma^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} - \frac{\partial}{\partial t}\right) h(z,t) = 0 \tag{L}$$

$$h(\cdot, t)|_{\partial X'} = d^2(f(\cdot, t), q)|_{\partial X'} \quad \text{for } t \geqslant 0 \tag{14}$$

$$h(z,0) = d^2(f(z,0), q) \quad \text{for all } z \in X'.$$

Since f has $C^{2,\alpha}$ boundary values on $\partial X' \cap \partial X$, so does h.

The maximum principle implies

$$d^2(f(z,t),q) \leqslant h(z,t) \quad \text{for all } t \geqslant 0,\ z \in X. \tag{15}$$

Now

$$\begin{aligned} d(f(z_1,t),f(z_2,t)) &= d(f(z_2,t),q) - d(f(z_1,t),q) \quad \text{by choice of } q \\ &\leqslant \frac{1}{2\tau}(d^2(f(z_2,t),q) - d^2(f(z_1,t),q)) \\ &\leqslant \frac{1}{2\tau}(h(z_2,t) - h(z_1,t)) \end{aligned} \tag{16}$$

by (14), (15), since $z_1 \in \partial X$.

Thus, (12) is reduced to a boundary Lipschitz bound for the solution of the linear problem (L). This in turn is known from the theory of linear parabolic equations, noting that h has $C^{2,\alpha}$ boundary values on $\partial X' \cap \partial X$. □

Since we have established time-independent gradient estimates, we obtain global existence of a solution of (P′) as in §1, and likewise convergence to some smooth map f with $f|_{\partial X} = g|_{\partial X}$, at least for some sequence $f(\cdot, t_n)$, $t_n \to \infty$.

THEOREM 6. *Let X be a compact complex manifold with nonempty smooth boundary ∂X and Hermitian metric $(\gamma_{\alpha\bar\beta})$.*

Let N be a complete Riemannian manifold of nonpositive sectional curvature. Let $g: X \to N$ be continuous. Then there exists a unique Hermitian harmonic map $f: X \to N$, i.e., f satisfies

$$\gamma^{\alpha\bar\beta}\Big(\frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}} f^i + \Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar\beta}}\Big) = 0, \quad i = 1, \ldots, \dim N$$

in local coordinates, with

$$f|_{\partial X} = g|_{\partial X}$$

and which is homotopic to g w.r.t. fixed boundary values.

Proof. We first assume that g is of class $C^{2,\alpha}$, and let $f(z,t)$ be a solution of (P′).

As just observed, $f(z,t)$ exists for all $t>0$, and a subsequence $f(z,t_n)$, $t_n \to \infty$, converges to a smooth map f, homotopic to g, and with $f|_{\partial X} = g|_{\partial X}$.

As in §1, proof of Theorem 1,

$$v(x) := \lim_{t \to \infty} \frac{\partial f}{\partial t}(x,t)$$

is a parallel section of $f^{-1}TN$. Since $v(x) = 0$ on ∂X as we keep the boundary values fixed, $v(x)$ vanishes identically. This implies that f is Hermitian harmonic. Also $f|_{\partial X} = g|_{\partial X}$ by construction, and f is homotopic to g.

In order to treat the case where g is only continuous, we choose a sequence g_n of $C^{2,\alpha}$ maps converging uniformly to g. For each g_n, we get a corresponding solution $f_n(x,t)$ of (P′) with Hermitian harmonic limit $f_n(x)$ by what we have already proved. By a reasoning analogous to the proof of Theorem 4, for $n, m\in\mathbf{N}$

$$\sup_{x\in X, t\geqslant 0} d(f_n(x,t), f_m(x,t)) \leqslant \sup_{\substack{(x,t)\in(\partial X,[0,\infty)) \\ \text{or } x\in X, t=0}} d(f_n(x,t), f_m(x,t)) = \sup_{x\in X} d(g_n(x), g_m(x)).$$

Consequently, $(f_n(x,t))$ forms a Cauchy sequence in the C^0-topology and thus converges to some limit $f(x,t)$. Since the maps $f_n(x,t)$ satisfy uniform interior estimates, they also solve our parabolic system, and the limit $f(x)$ for $t\to\infty$ is the limit of $f_n(x)$ for $n\to\infty$ and is Hermitian harmonic, coincides with g on ∂X and is homotopic to g.

Uniqueness follows from the proof of Theorem 4. □

4. Some rigidity theorems in Hermitian geometry

For our first applications, we formulate

Definition. Let X be an m-dimensional Hermitian manifold. X is called astheno-Kähler[2] if it carries a $(1,1)$ form ω satisfying:

(i) $\partial\bar\partial\omega^{m-2}=0$,

(ii) ω^m is a positive multiple of the volume form.

We do not know the most general condition under which a compact Hermitian manifold is astheno-Kähler.

One necessary condition, however, is immediate:

LEMMA 6. *Let X be a compact astheno-Kähler manifold. Then every holomorphic 1-form on X is closed.*

Proof. Let ω satisfy the conditions of the definition. Let φ be a holomorphic 1-form, i.e., $\bar\partial\varphi=0$. Then

$$\int \partial\varphi\wedge\bar\partial\bar\varphi\wedge\omega^{m-2} = \int \varphi\wedge\bar\varphi\wedge\partial\bar\partial\omega^{m-2} \quad \text{since } \varphi \text{ is holomorphic}$$
$$=0 \quad \text{by (i)},$$

and this implies $\partial\varphi=0$ by (ii).

(2) after the Greek word for "weak"

LEMMA 7. *Let N be a compact locally Hermitian symmetric space of noncompact type, and assume that the universal cover of N does not have the upper half plane H as a global factor. Let X be a compact astheno-Kähler manifold of dimension m.*

Let

$$f: X \to N$$

be Hermitian harmonic. Then f is pluriharmonic. If f has real rank $=2\dim_{\mathbf{C}} N$ at some point, then f is $\pm$ holomorphic.

Proof. This follows as in [S1]. Namely, from (19) and our assumption on ω, we get

$$\begin{aligned} 0 &= \int_x g_{i\bar{j}} \bar{\partial} f^i \wedge \partial f^{\bar{j}} \wedge \partial\bar{\partial}\omega^{m-2} \\ &= \int_x (R_{i\bar{j}k\bar{l}} \bar{\partial} f^i \wedge \partial f^{\bar{j}} \wedge \partial f^k \wedge \bar{\partial} f^{\bar{l}} \wedge \omega^{m-2} - g_{i\bar{j}} D' \bar{\partial} f^i \wedge D'' \partial f^{\bar{j}} \wedge \omega^{m-2}). \end{aligned}$$

The integrand on the right hand side is pointwise nonnegative, because ω^m is a positive multiple of the volume form of X, cf. [S1] or [J2], pp. 132 ff. Hence, the integrand is identically zero, and the analysis of [S1] of the curvature expression yields the claim. □

The rank condition imposed on f can be considerably relaxed, depending on the dimension and rank of N, cf. [S2]. Here, we only note the following result, obtained in the same manner as Lemma 7.

LEMMA 8. *Let N be a compact Kähler of strongly negative curvature, i.e.,*

$$R_{i\bar{j}k\bar{l}}(A^i B^{\bar{j}} - C^i D^{\bar{j}})(\overline{A^l B^{\bar{k}} - C^l D^{\bar{k}}}) > 0$$

unless the terms in brackets vanish. Let f and X be as in Lemma 8.

If the real rank of f at some point is at least 3, then f is $\pm$ holomorphic.

Remark. Actually, for the preceding results, ω^m need only be a positive multiple of the volume form on a dense subset of X.

Combining these results with Theorem 1 or Theorem 2, we obtain

THEOREM 6. *Let N be a compact locally Hermitian symmetric space of noncompact type, without the upper half plane H as a global factor of its universal cover, or let N be a compact strongly negatively curved Kähler manifold. Let X be a compact astheno-Kähler manifold.*

If X is homotopy equivalent to N, then X is $\pm$ biholomorphically equivalent to N.

Proof. By Theorem 1 there exists a Hermitian harmonic homotopy equivalence

$$f: X \to N.$$

By Lemma 7 or 8, resp., f is $\pm$ holomorphic. Since f, as a homotopy equivalence, is of degree ± 1, an easy argument shows that f has maximal rank everywhere, cf. [S1]. Thus f is $\pm$ biholomorphic. □

We note that the condition

$$\partial\bar{\partial}\omega^{m-2}=0$$

is automatically satisfied for $m=2$. Thus we obtain the following result, without having to use Kodaira's classification of compact complex surfaces.

COROLLARY 1. *Let the compact Kähler manifold N be covered by the unit ball in $\mathbf{C}^2$. Then any compact complex surface X which is homotopy equivalent to N already is $\pm$ biholomorphically equivalent to N.*

This easily follows from Theorem 3 by equipping X with a Hermitian metric and noting that N has strongly negative curvature, cf. [S1].

COROLLARY 2. *Let N be a compact complex surface with a Kähler metric of strongly negative curvature, for example a nonsingular quotient of the unit ball in $\mathbf{C}^2$. Let M be a compact manifold of four real dimensions with $\pi_1(M)\neq 0$. Then the connected sum of M and N cannot carry a complex structure.*

Proof. We proceed by contradiction and assume that the connected sum of M and N, denoted by X, carries a complex structure. We equip X with a Hermitian metric and choose a $(1,1)$ form ω for which ω^2 is a positive multiple of the volume form. There exists a map

$$f\colon X\to N$$

of degree one, obtained by collapsing M to a point. By Theorem 1, we may assume that f is harmonic. Lemma 8 implies that f is $\pm$ holomorphic.

The next lemma then yields the desired contradiction, since f maps $\pi_1(M)$ to $0\in\pi_1(N)$:

LEMMA 9. *Let X, Y be compact complex manifolds of the same dimension, and let $f\colon X\to Y$ be holomorphic and of degree ± 1. Then f is injective on the fundamental group.*

Proof. Let V be the subset of X where the Jacobian of f vanishes. V is a complex hypersurface, and f restricted to $X\setminus V$ is injective, since of degree ± 1.

If f is not injective on $\pi_1(X)$, then $V\neq\varnothing$. We let $\alpha\in\pi_1(X)$, $\alpha\neq 0$, with $f_{\#}(\alpha)=0$.

Since V has real codimension 2, α can be represented by a loop γ with $\gamma\cap V=\varnothing$. Consequently, f is injective on γ. Since $f_{\#}(\alpha)=0$, $f(\gamma)$ bounds a disk D in Y. Since

$f(V)$ is of complex codimension at least 2, i.e., of real codimension at least 4, we may assume $D\cap f(V)=\varnothing$. Since

$$F\colon X\backslash V\to Y\backslash f(V)$$

is bijective, $f^{-1}(D)$ is a disk with boundary γ, contradicting the assumption that γ represents a nontrivial element of $\pi_1(X)$.

Remark. For Corollary 2, we neither need Kodaira's classification of compact complex surfaces nor Donaldson's theory of differentiable structures on 4-manifolds, and in any case, we only need an assumption on the topological, but not on the differentiable structure of X, the connected sum of M and N.

The preceding results can be partially extended to higher dimensions as follows:

THEOREM 7. *Let N be a compact complex manifold of dimension n, with Kähler metric of strongly negative curvature, for example a nonsingular quotient of the unit ball in $\mathbf{C}^n$. Let M be a compact manifold of $2n$ real dimensions, with $\pi_1(M)\neq 0$. For any complex structure on the connected sum of M and N, denoted by X, there cannot be any meromorphic map from X onto a compact complex manifold of (complex) dimension $n-2$. In particular, the algebraic dimension of X is at most $n-3$.*

Remark. We do not know whether X can carry any complex structure at all.

Proof. Assume $H\colon X\to Y$ is a meromorphic map from X onto a compact complex manifold Y of dimension $n-2$. Removing the points of indeterminancy of H by blowing ups, we may assume that H is holomorphic. Since H is onto, the generic fibre is a smooth compact complex surface. We denote the fibers by $C_y=H^{-1}(y)$, for any $y\in Y$. We call $y\in Y$ regular, if C_y is nonsingular, and call y singular otherwise. We note that the singular fibers may be of higher dimension than the regular ones. We equip X with a Hermitian metric. This then induces a Hermitian metric on each C_y.

We look at the map

$$g\colon X\to N$$

of degree one, obtained by collapsing M to a point. We put

$$g_y:=g|_{C_y},\quad \text{for } y\in Y.$$

We now distinguish several cases:

Case 1. For each regular fiber C_y, g_y is homotopic to a constant map. We consider the Hermitian metric on C_y as a Riemannian metric $(\gamma_{\alpha\beta})$, with real indices α, β, and consider the heat flow for the ordinary harmonic map problem:

$$
\begin{gathered}
f_y : C_y \times [0,\infty) \to N, \\
\frac{\partial f_y^i}{\partial t} = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha}\left(\gamma^{\alpha\beta}\sqrt{\gamma}\frac{\partial f_y^i}{\partial x^\beta}\right) + \gamma^{\alpha\beta}\Gamma^i_{jk}\frac{\partial f_y^j}{\partial x^\alpha}\frac{\partial f_y^k}{\partial x^\beta} \quad \text{for } i = 1, \dots, \dim_{\mathbf{R}} N, \\
\text{with real coordinates } x^\alpha, \\
f_y(x,0) = g_y(x) \quad \text{for all } x \in C_y.
\end{gathered} \tag{1}
$$

By stability of the heat flow, f_y depends smoothly on y, for y regular, and so does the limit map $f_y(\,\cdot\,,\infty)$. Each $f_y(\,\cdot\,,\infty)$ is constant, by uniqueness of harmonic maps because the image has nonpositive curvature. The constant may depend on y, however.

Although our subsequent argument will not exploit this, we note here that we may extend this convergence to the smooth part of the singular fibers by taking limits. The reason is the following:

Since g may be assumed smooth, we have a uniform bound of the energies of the maps g_y. We then get uniform estimates on the maps f_y, at least away from the singularities of the singular fibers, because we can always control the maps on a ball of radius R by the total energy and the geometry of the ball of radius $2R$, cf. [J1] for details.

In any case, for each singular fiber C, we choose a small neighborhood U and put $\Sigma := \partial U$, in such a way that $\overline{U}$ in particular intersects no other singular fiber and that C is a deformation retract of U. Since Σ intersects only regular fibers, we obtain a limiting map $f_\Sigma(\,\cdot\,,\infty)$. Since a fiber generically intersects Σ in a curve and since the limit map is constant on each fiber, the real dimension of the image of Σ under $f_\Sigma(\,\cdot\,,\infty)$ is at most $2n-2$. Since N is a $K(\pi,1)$-space, $f_\Sigma(\,\cdot\,,\infty)$ may be extended smoothly to U as a map $f_U(\,\cdot\,,\infty)$, in such a way that the real dimension of the image is at most $2n-1$. We have thus constructed a continuous map $f : X \to N$ which is homotopic to g but which cannot be surjective as its image has dimension at most $2n-1$. This is a contradiction, since g is of degree 1.

Case 2. For each regular y, g_y is homotopic to a map onto a closed geodesic of N. Since closed geodesics in N are unique in their homotopy classes, because of the negativity of the sectional curvature, each g_y then is homotopic to a map onto the same closed geodesic.

One can then use the heat flow (1) for the ordinary harmonic map problem as in Case 1 and extend the resulting map again to the singular fibers and homotop g into a map g' of lower rank, reaching a contradiction as before.

Having ruled out Case 1 and Case 2, we can apply Theorem 1 to homotop each g_y, for regular y, into a Hermitian harmonic map $f_y : C_y \to N$. We have to distinguish two further cases.

Case 3. The maps f_y have real rank $\leqslant 2$ everywhere. As in [JY3], one obtains a compact holomorphic curve Σ_y, and a holomorphic map $h_y\colon C_y \to \Sigma_y$ and a harmonic map $\varphi_y\colon \Sigma_y \to N$ with

$$f_y = \varphi_y \circ h_y.$$

If the curves Σ_y have genus 0, φ_y is constant, and the analysis of Case 1 applies. If Σ_y has genus 1, φ_y maps Σ_y onto a closed geodesic, by Preissman's theorem (cf. e.g. [J2]), and the analysis of Case 2 applies. We may therefore assume that the genus of Σ_y, for generic y, is at least 2.

If the conformal structure of Σ_y is independent of y, then all maps φ_y have the same image in N by uniqueness of harmonic maps, and we can homotop g into a map of lower rank essentially as in Case 2.

We then treat the case of varying conformal structure. We have a smooth map

$$f\colon X \backslash D \to \mathcal{M}_g$$

by mapping each C_y holomorphically onto Σ_y, where D is some divisor (possibly empty) in X, and $\mathcal{M}_g$ is the universal modular curve of genus g. Of course, strictly speaking, the universal modular curve only exists after lifting to finite covers. We therefore have to check local liftability near the branch points of $\mathcal{M}_g$. Since X is smooth, we can locally choose a fixed marking for the fundamental group of each C_y, and the image of this fundamental group under h_y can then be used to fix a local marking for the fundamental group of Σ_y. This implies local liftability.

We can thus lift to finite covers (without changing notation) so that the image $\mathcal{M}_g$ is smooth. We equip $\mathcal{M}_g$ with its Weil–Petersson metric. Since its holomorphic sectional curvature is negative by an old result of Ahlfors, we can use an argument of Kalka [Kl] to conclude that f as a smooth family of holomorphic maps from the fibers C_y is also holomorphic in the directions transverse to the fibers. We shall discuss Kalka's argument in Case 4 below in more detail. Thus,

$$f\colon X \backslash D \to \mathcal{M}_g$$

is holomorphic. Since the holomorphic sectional curvature of $\mathcal{M}_g$ has a negative upper bound (see [T]), we can use an extension of Yau's Schwarz lemma, due to Royden (Theorem 2 in [R]), to show that f extends to a holomorphic map

$$f\colon X \to \overline{\mathcal{M}}_g$$

into the stable curve compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$; details can be found in [JY4].

This means that we can associate a Riemann surface Σ_y, possibly with nodes, to each $y \in Y$, not only to the regular ones. Moreover, because N is negatively curved we can then also define harmonic maps $\varphi_y: \Sigma_y \to N$ for singular y's as limits of those for regular ones.

We thus obtain

$$g': X \to N$$

by defining $g'(z) = \varphi_y \circ h_y(z)$ for $z \in C_y$, and as before we see that on the one hand g' is of lower rank, and on the other hand homotopic to g, thus reaching a contradiction as before.

Case 4. It remains to study the case where for generic y, the Hermitian harmonic $f_y: C_y \to N$ has real rank $\geqslant 3$ at some point. Lemma 8 implies that such a f_y has to be $\pm$ holomorphic. Also f_y depends smoothly on y, by uniqueness and *a priori* estimates. We now want to display Kalka's argument [Kl] to show that such a smooth family of holomorphic maps into a negatively curved target, defined on a complex parameter space is a holomorphic family.

We look at the holomorphic map

$$H: X \to Y.$$

Let y_0 be regular, with f_{y_0} holomorphic.

We let w^1, w^2 denote local holomorphic coordinates on C_{y_0}. The Cauchy–Riemann equations on C_y, for close to y_0, then take the form

$$\frac{\partial f}{\partial \overline{w}^i} + \mu_{\bar{i}}^j \frac{\partial f}{\partial w^j} = 0, \tag{4}$$

where of course $\mu_{\bar{i}}^j(y_0) = 0$.

Putting $f = f_y$ and differentiating (4) w.r.t. $\bar{y}$, we obtain

$$\frac{\partial^2 f}{\partial \bar{y} \partial \overline{w}^i} + \mu_{\bar{i}}^j \frac{\partial^2 f}{\partial \bar{y} \partial w^j} + \frac{\partial \mu_{\bar{i}}^j}{\partial \bar{y}} \frac{\partial f}{\partial w^j} = 0. \tag{5}$$

Since H is holomorphic, $\partial \mu_{\bar{i}}^j / \partial \bar{y} = 0$, and consequently (5) implies that $\partial f / \partial \bar{y}$ is holomorphic on C_y.

$\partial f / \partial \bar{y}$ is a section of $f^{-1}TN$ which is a negative bundle as N has negative holomorphic sectional curvature and f is not constant on the fibers. As all holomorphic sections of a negative bundle vanish, $\partial f / \partial \bar{y} = 0$, and f is holomorphic in y, as claimed.

So far, f is defined only on the regular fibers, but since N has negative holomorphic sectional curvature, we may again apply the Schwarz lemma to extend f as a holomorphic map to all of X. Since the action of f on the fundamental group is the same as that of g (this follows, because the union of the singular fibers has real codimension at least 2 in X) and since N is a $K(\pi,1)$-space, f is homotopic to g, and this time Lemma 9 yields a contradiction as in the proof of Corollary 2.

In conclusion, we have reached a contradiction in every possible case, and thus a meromorphic $H: X \to Y$ as above cannot exist. This proves the result. □

Corollary 1 can obviously be extended in the same way as Corollary 2.

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