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Estimates for the $\bar{\partial}$-Neumann problem in pseudoconvex domains of finite type in $\mathbf{C}^2$

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§0. Introduction

The object of this paper is to construct a parametrix for the $\bar{\partial}$-Neumann problem for arbitrary bounded pseudoconvex domains in $\mathbf{C}^2$ of finite type, and to use this parametrix to obtain sharp regularity results for the associated Neumann operator and for solutions of $\bar{\partial}u=f$. As an application, we obtain an extension of the Henkin–Skoda theorem, which characterizes the zero sets of functions in the Nevanlinna class in strictly pseudoconvex domains, to pseudoconvex domains of finite type in $\mathbf{C}^2$.

The $\bar{\partial}$-Neumann problem is a boundary value problem for an elliptic system of partial differential equations. Let $\Omega\subset\mathbf{C}^n$ be a smoothly bounded domain. Let U be a neighborhood of the boundary $\partial\Omega$ and let $\varrho: U\to\mathbf{R}$ be a defining function so that

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$$\Omega \cap U = \{z \in \Omega \mid \varrho(z) > 0\}$$

and $\nabla\varrho(z) \neq 0$ for $\varrho(z)=0$. Let

$$\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

acting on (p,q)-forms on Ω. Then given a (p,q)-form on Ω, the $\bar{\partial}$-Neumann problem is to find a (p,q)-form $u=N(g)$ such that

$$\begin{aligned} \square(u) &= g \quad \text{on } \Omega; \\ u \rfloor \bar{\partial}\varrho &= 0 \quad \text{on } \bar{\partial}\Omega; \\ \bar{\partial}u \rfloor \bar{\partial}\varrho &= 0 \quad \text{on } \bar{\partial}\Omega. \end{aligned} \tag{0.1}$$

The first results for this problem were obtained by Kohn [K1,2], and proceeded by L^2 methods. Our analysis originates in the approach used by Greiner and Stein [GS] for the strongly pseudoconvex case, which reduces the problem of solving system (0.1) to the problem of inverting a pseudodifferential operator $\square^+$ on the boundary $\partial\Omega$. By finding an operator $\square^-$ such that $\square^- \square^+ \approx \square_b$, this in turn is reduced to the problem of inverting the boundary Kohn-Laplacian $\square_b$. The final parametrix for the Neumann operator N is then written as a composition of various operators, each of whose regularity properties is now well understood.

There are several significant differences between our results in this paper and the earlier of [GS], and several new difficulties had to be overcome in extending the results to the weakly pseudoconvex case. First, we give a more intrinsic formula for the pseudodifferential "Dirichlet to Neumann" operator which arises in the construction of the boundary operator $\square^+$. Second, we give a more natural interpretation for the operator $\square^-$ as the boundary operator induced by the $\bar{\partial}$-Neumann problem for the complementary domain $\mathbf{C}^2 \setminus \Omega$. It is worth mentioning that up to this point the analysis does not depend on pseudo-convexity or finite type. Third, the properties of the relative solving operators for $\square_b$ have to be understood in terms of a natural nonisotropic metric on the boundary of Ω. It is here, and in what follows, that pseudo-convexity and finite type play a crucial role. Fourth, the commutativity properties of these solving operators with respect to pseudo-differential operators had to be understood without the use of the more standard $S_{1/2,1/2}$ class of pseudodifferential operators. Fifth, certain micro-local smoothing properties of the Szegö projection had to be exploited.

We consider domains $\Omega \subset\subset \mathbf{C}^2$ which are pseudoconvex and of finite type. The main result on the construction of a parametrix for the Neumann operator N on (0,1)-forms is given by:

THEOREM 5.1. *For any integer k, there is an integer $\tilde{k}$, and there is an operator $T_k: C^\infty(\bar{\Omega})_{(0,1)} \to C^\infty(\bar{\Omega})_{(0,1)}$ which is isotropically smoothing of order k so that for $f \in C^\infty(\bar{\Omega})_{(0,1)}$,*

$$N(f) = N_a^k(f) + T_k(f)$$

where N_a^k is an operator explicitly given as a composition of operators which are either standard elliptic pseudodifferential operators, standard elliptic Poisson operators and Green's operators, or nonisotropic smoothing operators on the boundary of Ω. The precise definition of N_a^k is given in Definition 5.1.

Next, let

$$\bar{L}_1 = \frac{\partial \varrho}{\partial \bar{z}_2}\frac{\partial}{\partial \bar{z}_1} - \frac{\partial \varrho}{\partial \bar{z}_1}\frac{\partial}{\partial \bar{z}_2}.$$

This is an operator which is tangential along $\partial\Omega$. The main regularity results for the Neumann operator N on weakly pseudoconvex domains of finite type in $\mathbf{C}^2$ are then the following:

THEOREM 7.1. *Suppose N is the Neumann operator, and $q(L_1, \bar{L}_1)$ is a quadratic polynomial in L_1 and $\bar{L}_1$. Then the following operators are bounded on the indicated spaces:*

$$q(L_1, \bar{L}_1)N : L_k^p \to L_k^p, \quad 1 < p < \infty, \ k = 0, 1, 2, \ldots; \tag{7.1}$$

$$\bar{\partial}N \lrcorner \bar{\partial}\varrho : L_k^p \to L_{k+1}^p, \ 1 < p < \infty, \ k = 0, 1, 2, \ldots; \tag{7.2}$$

$$N : \Lambda_\alpha \to \Lambda_{\alpha+2/m} \cap \Gamma_{\alpha+2}, \quad \alpha > 0. \tag{7.3}$$

Here L_k^p are the usual spaces of functions or forms on Ω that are in $L^p(\Omega)$ along with all their derivatives up to order k. Λ_α is the usual isotropic Lipschitz space of exponent α on Ω, and the spaces Γ_α are appropriate non-isotropic Lipschitz spaces. There are also related results for the solutions of $\bar{\partial}u = f$, giving sharp L_k^p and Lipschitz estimates. These may be found in Section 7. We also obtain the following L^1 estimate:

THEOREM 8.1. *Suppose f is a smooth* $(0,1)$*-form in Ω. Then we have the* a priori *estimate:*

$$\|\bar{\partial}^* N(f)\|_{L^1(\partial\Omega)} \leq C[\|f\|_{L^1(\Omega)} + \|(\mu/\varrho) f \wedge \bar{\partial}\varrho\|_{L^1(\Omega)}]. \tag{8.1}$$

The quantities μ and ϱ are related to the non-isotropic geometry on $\partial\Omega$ and are defined in Section 8. This last estimate leads to our result of zeros of functions in the Nevanlinna class. It is as follows:

THEOREM 9.1. *Let $\Omega\subset\mathbf{C}^2$ be a bounded, smooth weakly pseudo-convex domain of finite type m. Let G be a holomorphic function on Ω. Then the zero variety $Z=Z(G)$ is the zero variety of a function F in the Nevanlinna class if and only if the zero variety Z satisfies the Blaschke condition.*

The $\bar\partial$-Neumann problem arises naturally from problems in several complex variables, and has been the subject of a great deal of interest and research. In the case of domains with nondegenerate Levi form, the original L^2 estimates were obtained by Kohn [K1], and these methods were further developed in Kohn and Nirenberg [KN]. A summary of the approach to the $\bar\partial$-Neumann problem in this situation via the method of *a priori* estimates is given in the monograph by Folland and Kohn [FoK]. The L^2 estimates for the $\bar\partial$-Neumann problem in the case of domains of finite type in $\mathbf{C}^2$ were obtained by Kohn in [K2].

The method of studying the $\bar\partial$-Neumann problem via reduction to operators on the boundary were first used by Garabedian and Spencer [GaS] and by Kohn and Spencer [KoS], but a parametrix for the Neumann operator in the case of strictly pseudoconvex domains was first obtained in [GS]. This work in turn was based on the analysis of Folland and Stein [FoS] for $\Box_b$ for strictly pseudo-convex domains, which utilized the idea of approximating the boundary by the Heisenberg group.

There has been considerable development in recent years in the analysis of $\bar\partial$, $\bar\partial_b$, and the Bergman and Szegö kernels for domains of finite type in $\mathbf{C}^2$, and this work has a major bearing on our present paper. In particular, we cite the papers of Bonami and Charpentier [BC1], [BC2], Christ [C], C. Fefferman and Kohn [FK], Machedon [M], McNeal [Mc], Nagel, Rosay, Stein and Wainger [NRSW].

The results of this paper were announced in [CNS], and the organization of the present paper follows that of the announcement closely.(1) The paper is organized as follows. In Section 1 we obtain a pseudodifferential operator description, up to errors of order -1 of the "Dirichlet to Neumann" operator for systems of second order elliptic operators with scalar principal symbol. In Section 2 we describe the operator $\Box$ and the associated $\bar\partial$-Neumann boundary conditions on $(0,1)$ forms for smoothly bounded

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domains in $\mathbf{C}^2$. In Section 3, we use the description of the "Dirichlet to Neumann" operators of section 1 to describe the boundary operator $\Box^+$ associated to the $\bar{\partial}$-Neumann problem. In Section 4, we show how to construct the operator $\Box^-$ coming from the $\bar{\partial}$-Neumann problem on the exterior of the domain Ω, and describe the relationship between $\Box^+$, $\Box^-$, and $\Box_b$. All of these calculations are done using pseudodifferential operator realizations of the operators, and we need to keep track of errors up to order -1. In Section 5, we construct a parametrix for the Neumann operator, and in Section 6 we establish a variety of commutation properties of the components of the parametrix. In Sections 7 and 8 we establish the various regularity properties of the Neumann operator mentioned above, and in Section 9 we carry out the Henkin–Skoda program in weakly pseudoconvex domains of finite type in $\mathbf{C}^2$. We wish to thank the referee for several useful suggestions which have been incorporated in the text.

§ 1. Dirichlet to Neumann operators for elliptic systems

The object of this section is to obtain a pseudodifferential operator description, up to errors of order -1, of the so-called "Dirichlet to Neumann operator" for second order elliptic operators with scalar principal symbol. The principal symbol of the Dirichlet to Neumann operator is well known and is of order 1, but later in this paper we shall need to know the zero order part of the symbol as well. This is the main content of this section. Our approach is similar to that of Greiner and Stein ([GS], Chapter 7), which in turn is based on the approach developed by A. Calderón, L. Hörmander, R. Seeley, and L. Boutet de Monvel. However, the presentation in this section gives a more intrinsic description of the operator than is available in [GS].

We begin by introducing appropriate notation and by recalling appropriate definitions. Let $\Omega\subset\mathbf{R}^{n+1}$ be open with $0\in\Omega$, and let

$$g=\sum_{i,j=1}^{n+1} g_{ij}(y)\,dy_i\,dy_j \tag{1.1}$$

be a smooth Riemannian metric defined on a neighborhood of $\bar{\Omega}$, the closure of Ω. Let V be a fixed finite dimensional complex vector space, and let $L(V)$ be the space of linear endomorphisms of V. We let Δ denote a second order linear partial differential operator defined on V-valued functions on Ω. We assume that Δ has the form

$$\Delta=-\sum_{i,j=1}^{n+1} g^{ij}(y)\frac{\partial^2}{\partial y_i\,\partial y_j}+\text{first and zero order operators} \tag{1.2}$$

where as usual, the smooth functions $\{g^{ij}\}$ are defined by the equations

$$\sum_{j=1}^{n+1} g_{ij}(y)g^{jk}(y)=\delta_{ik}. \tag{1.3}$$

Thus the principal symbol of Δ is scalar, and equals the principal symbol of the Laplace–Beltrami operator associated to the metric g.

Let M be a smooth hypersurface in Ω, with $0\in M\subset\Omega$. Shrinking Ω if necessary, M will divide Ω into two parts, and if we let ϱ denote a *signed* geodesic distance from M in the metric G, then

$$\Omega=M\cup\Omega^+\cup\Omega^-$$

where

$$\Omega^+=\{y\in\Omega\,|\,\varrho(y)>0\},$$
$$\Omega^-=\{y\in\Omega\,|\,\varrho(y)<0\}.$$

Intuitively, the Dirichlet to Neumann operator N^+ for the operator Δ on the hypersurface M relative to the domain Ω^+ can be described as follows. If f is an appropriate function defined on M, and if u is a "solution" of the Dirichlet problem

$$\begin{aligned} \Delta(u)&=0 \quad \text{on } \Omega^+\\ u&=f \quad \text{on } M, \end{aligned} \tag{1.4}$$

then the Dirichlet to Neumann operator N^+ applied to f is the restriction to M of the inward normal derivative of u. Of course this is not a precise definition since M is not the boundary of Ω^+. In a moment we will see how to deal with this problem, but for now it is important to note that N^+ will be a pseudodifferential operator on M and will be well defined only modulo infinitely smoothing operators. Also, with a similar definition, there is a Dirichlet to Neumann operator N^- associated to Δ on the hypersurface M relative to the domain Ω^-.

In order to write the operator Δ in a special form, and in order to make precise what we meant above by solving the Dirichlet problem, we need to introduce special coordinates appropriate to the operator Δ and the hypersurface M. The function ϱ is smooth on Ω, and we shall denote by $\partial/\partial\varrho$ the vector field which is dual to 1-form $d\varrho$ (in the metric g). It is given in coordinates by

$$\frac{\partial}{\partial\varrho}=\sum_{i=1}^{n+1}\left[\sum_{j=1}^{n+1} g^{ij}(y)\frac{\partial\varrho}{\partial y_j}(y)\right]\frac{\partial}{\partial y_i} \tag{1.5}$$

For each y sufficiently close to the origin, the integral curve to the vector field $\partial/\partial\varrho$ passing through y will intersect M in a unique point $\pi(y)$, and π is a smooth mapping. Also, we can choose a coordinate system on M near the origin 0 which is given by a mapping

$$\phi: M \to U \subset \mathbf{R}^n$$

where U is an open neighborhood of the origin in $\mathbf{R}^n$, and $\phi(0)=0$. Then after shrinking Ω again we may assume that the mapping

$$\Phi: \Omega \to U \times (-\varepsilon, +\varepsilon)$$

given by

$$\Phi(y) = (\phi(\pi(y)), \varrho(y))$$

is a diffeomorphism. If $(x_1, \ldots, x_n)$ are coordinates on the open set U, we shall, with an abuse of notation, use coordinates $(x_1, \ldots, x_n, t)$ as coordinates on Ω where $\varrho(y)=t$.

For each $t \in (-\varepsilon, \varepsilon)$ let

$$M_t = \{y \in \Omega \mid \varrho(y) = t\}.$$

Then $M_0=M$, and if $y \in \Omega$, the integral curve to $\partial/\partial\varrho$ passing through y intersects M_t in a unique point $\pi_t(y)$, where π_t is a smooth mapping and $\pi_0=\pi$. Let g_t be the restriction of the metric g to the hypersurface M_t, and then define the restriction of the operator Δ to V-valued functions defined on M_t by the formula

$$\Delta_t(f)(y) = \Delta(f \circ \pi_t)(y) \quad \text{for } y \in M_t. \tag{1.6}$$

Also define a mapping $\bar{C}: \Omega \to L(V)$ by the formula

$$\bar{C}(x, t)(\mathbf{v}) = \Delta(\varrho \mathbf{v})(x, t) \tag{1.7}$$

for each $\mathbf{v} \in V$. (Of course, if we choose a basis in the vector space V, then $\bar{C}$ is a matrix valued function.) Finally if f is a smooth function on Ω let f_t denote the restriction of f to M_t.

PROPOSITION 1.1. (1) *Δ_t is a second order elliptic operator defined on V-valued functions on M_t. The principal symbol of Δ_t is scalar, and equals the principal symbol of the Laplace–Beltrami operator associated to the metric g_t.*

(2) *If f is a smooth V-valued function on Ω, then in terms of the coordinates $(x, t)=(x_1, \ldots, x_n, t)$*

$$\Delta(f)(x, t) = -\frac{\partial^2 f}{\partial t^2}(x, t) + \bar{C}(x, t)\frac{\partial f}{\partial t}(x, t) + \Delta_t(f_t)(x, t). \tag{1.8}$$

Proof. Write

$$\phi(\pi(y)) = (\varphi_1(y), \dots, \varphi_n(y))$$

and put

$$\varrho(y) = \varphi_{n+1}(y).$$

Since ϱ was choosen as a signed geodesic distance to M, and since $\partial/\partial\varrho$ is the vector field dual to $d\varrho$, it follows that

$$\sum_{i,j=1}^{n+1} g^{ij}(y)\frac{\partial\varphi_{n+1}}{\partial x_j}(y)\frac{\partial\varphi_m}{\partial x_i}(y) = \frac{\partial}{\partial\varrho}(\varphi_m)(y) = \begin{cases} 1 & \text{if } m=n+1 \\ 0 & \text{if } m\leq n. \end{cases} \tag{1.9}$$

The proposition now follows from the chain rule since if we put $F(y)=f(\Phi(y))$, and $\Phi(y)=(x,t)$, then

$$\begin{aligned} \sum_{i,j=1}^{n+1} g^{ij}(y)\frac{\partial^2 F}{\partial y_i\partial y_j}(y) = \frac{\partial^2 f}{\partial t^2}(x,t)&+\left[\sum_{i,j=1}^{n+1} g^{ij}(y)\frac{\partial^2\varphi_{n+1}}{\partial y_i\partial y_j}(y)\right]\frac{\partial f}{\partial t}(x,t) \\ &+\sum_{l,m=1}^{n}\left[\sum_{i,j=1}^{n+1} g^{ij}(y)\frac{\partial\varphi_l}{\partial y_i}(y)\frac{\partial\varphi_m}{\partial y_j}(y)\right]\frac{\partial^2 f}{\partial x_l\partial x_m}(x,t). \end{aligned} \tag{1.10}$$

Since we can identify M_t with $M_0=M$ via the projection π, we can also think of Δ_t as a one parameter family of operators acting on V-valued functions on M. Write

$$\Delta_t = -A(x,t,D_x)+\bar{B}(x,t,D_x) \tag{1.11}$$

where $A(x,t,D_x)$ is a family of *scalar* second order operators, $-A(x,t,D_x)$ agrees to top order with the Laplace–Beltrami operator on M_t, and $\bar{B}(x,t,D_x)$ is a first order $L(V)$-valued differential operator acting on V-valued functions on M. (Note that such a decomposition is not unique.) We now make

DEFINITION 1.1.

$$\begin{aligned} &A_0 = A_0(x,D_x) = A(x,0,D_x); \\ &A_1 = A_1(x,D_x) = \frac{\partial}{\partial t}(A(x,t,D_x))|_{t=0}; \\ &\bar{B}_0 = \bar{B}_0(x,D_x) = \bar{B}(x,0,D_x); \\ &\bar{C}_0 = \bar{C}_0(x) = \bar{C}(x,0). \end{aligned} \tag{1.12}$$

The operators A_0, A_1, $\tilde{B}_0$, and $\tilde{C}_0$ are then differential operators acting on V-valued functions on M of orders 2, 2, 1, and 0 respectively. A_0 and A_1 are scalar operators, while $\tilde{B}_0$ and $\tilde{C}_0$ are $L(V)$-valued (i.e. matrix valued) operators. It is in terms of these operators that we will be able to describe the Dirichlet to Neumann operator for Δ.

We next recall the definition of Poisson-type operators $P^\pm : C_0^\infty(M) \to C^\infty(\Omega^\pm)$. (For a more complete discussion of these operators, see [GS], Chapter 7.)

DEFINITION 1.2. *A function $p^+(x,t,\xi) \in C^\infty(U \times [0,\varepsilon) \times \mathbf{R}^n)$ is a* symbol of Poisson type or order m *if it satisfies:*

(1) *$p^+(x,t,\xi)$ has compact support in the (x,t)-variables;*

(2) *For all multi-indicies α, β and non-negative integers γ, δ there is a constant $C = C_{\alpha,\beta,\gamma,\delta}$ so that*

$$\left| t^\delta \left(\frac{\partial}{\partial t}\right)^\gamma \left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{\partial}{\partial \xi}\right)^\alpha p^+(x,t,\xi) \right| \leq C(1+|\xi|)^{m-|\alpha|+\gamma-\delta}. \tag{1.13}$$

A similar definition is made for symbols $p^-(x, t\cdot\xi)$.

DEFINITION 1.3. *If $p^\pm(x,t,\xi)$ is a symbol of Poisson type of order m, the mapping $P^\pm$ defined on $C_0^\infty(U)$ given by*

$$P^\pm f(x,t) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} e^{ix\cdot\xi} p^\pm(x,t,\xi) \hat{f}(\xi)\, d\xi \tag{1.14}$$

is called an operator of Poisson type of order m.

The following result then describes the existence and regularity of solutions for the local Dirichlet problem for the operator Δ. (See [GS], Chapter 7 for a complete discussion.)

THEOREM A. *If $0 \in U_1 \subset U$ is a sufficiently small neighborhood of the origin in M, there are Poisson operators $P^\pm$ of order zero on U_1 such that, if R denotes the operator of restriction to M,*

(1) *$\Delta P^\pm$ are Poisson operators of order $-\infty$;*

(2) *$RP^\pm - I$ are pseudodifferential operators on M of order $-\infty$.*

Moreover, if $P_1^\pm$ are any other Poisson operators with properties (1) *and* (2), *then*

(3) *$P^\pm - P_1^\pm$ are Poisson operators of order $-\infty$.*

DEFINITION 1.4. *The Dirichlet to Neumann operators for Δ on the hypersurface M for the domains $\Omega^{\pm}$ are defined to be*

$$N^{\pm} = \pm R\frac{\partial}{\partial t}P^{\pm} \tag{1.15}$$

where $P^{\pm}$ are any operators of Poisson type solving the local Dirichlet problem as in Theorem A.

It of course follows from Theorem A that $N^{\pm}$ are well defined pseudodifferential operators of order 1 on M, since a different choice of Poisson operator leads to an error of a pseudodifferential operator of order $-\infty$. We are finally in a position to state the main result of this section.

THEOREM 1.2. *Modulo pseudodifferential operators of order* -1,

$$N^{\pm} = \mp(-A_0)^{1/2} - \frac{1}{4}A_0^{-1}A_1 \mp \frac{1}{2}(-A_0)^{-1/2}\bar{B}_0 + \frac{1}{2}\bar{C}_0. \tag{1.16}$$

Several remarks are now in order. First, $(-A_0)^{1/2}$ is understood to mean a pseudodifferential operator X of order 1 on M whose principal symbol is the positive square root of the principal symbol of $-A_0$, and such that X^2+A_0 is an operator of order 0. This determines X up to an error of order -1. The ambiguities in the second and third terms in (1.16) (in the definition of A_0 and in whether we take $A_0^{-1}A_1$ or $A_1A_0^{-1}$ and $(-A_0)^{1/2}\bar{B}_0$ or $\bar{B}_0(-A_0)^{1/2}$ are also errors of order -1. Finally, the fact that the first order term of $N^{\pm}$ is $\mp(-A_0)^{1/2}$ is well known, and so as remarked earlier, the main contribution of this theorem is the description of the zero order terms.

Proof of the Theorem 1.2. The roles of Ω^+ and Ω^- can be interchanged by simply reversing the sign of the function ϱ. This has the effect of changing the signs of A_1 and $\bar{B}$. Thus we shall only make our calculations for the case of Ω^+. We shall follow the arguments of [GS], Chapter 7, using the calculus of pseudodifferential operators, and thus we shall be somewhat brief. According to Theorem A, we can calculate N^+ by using *any* operator of Poisson type which solves the local Dirichlet problem, and we construct one such operator. We begin by calculating the asymptotic development of the symbol of a fundamental solution E to the operator Δ. We let this fundamental solution act on distributions supported on M, and this gives us an operator of Poisson type E_b which (roughly) satisfies $\Delta E_b(f)=0$ on Ω^+. (Precisely, $\Delta\circ E_b$ is an operator of Poisson type of order $-\infty$.) If we let $E_{b,0}(f)$ denote the restriction of $E_b(f)$ to M, then the operator $P(f)=E_b\circ[E_{b,0}]^{-1}$ is an (approximate) Poisson kernel for the operator Δ in the sense of Theorem A. Then N^+ will be given by $R\,\partial P/\partial t$. We now proceed to give a sketch of the necessary computations.

According to Proposition 1.1 and the discussion following it, if we use the coordinates $(x,t)=(x_1,\dots,x_n,t)$ on Ω, we can write

$$\begin{aligned}\Delta = -\frac{\partial^2}{\partial t^2} - \sum_{i,j=1}^{n}(a_0^{ij}(x)+ta_1^{ij}+O(t^2))\frac{\partial^2}{\partial x_i\,\partial x_j}\\ +\sum_{j=1}^{n}\bar{b}_j(x,t)\frac{\partial}{\partial x_j}+\bar{C}(x,t)\frac{\partial}{\partial t}+\bar{d}(x,t)\end{aligned} \tag{1.17}$$

where $\bar{b}_j$, $\bar{C}$, and $\bar{d}$ take values in $L(V)$ (i.e. are matrix valued). If we let $(\xi_1,\dots,\xi_n)$ be dual variables to $(x_1,\dots,x_n)$, and η be the dual variable to t, the symbol of Δ is

$$\begin{aligned}\sigma(\Delta) &= \eta^2+\sum_{i,j=1}^{n}(a_0^{ij}(x)+ta_1^{ij}(x)+O(t^2))\xi_i\xi_j+i\left[\sum_{j=1}^{n}\bar{b}_j(x,t)\xi_j+\bar{C}(x,t)\eta\right]+\bar{d}(x,t)\\ &= d_0+d_1+d_2\end{aligned} \tag{1.18}$$

where d_j is homogeneous in ξ and η of degree $2-j$. Thus

$$\begin{aligned}d_0 &= \eta^2+\sum_{i,j=1}^{n}(a_0^{ij}(x)+ta_1^{ij}(x)+O(t^2))\xi_i\xi_j\\ d_1 &= i\left[\sum_{j=1}^{n}\bar{b}_j(x,t)\xi_j+\bar{C}(x,t)\eta\right];\\ d_2 &= \bar{d}(x,t).\end{aligned} \tag{1.19}$$

In equation (1.18) , the coefficients are related to the operators A_0, A_1, $\bar{B}_0$ and $\bar{C}_0$ by the equations

$$\begin{aligned}A_0(x,D_x) &= \sum_{i,j=1}^{n}a_0^{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j};\\ A_1(x,D_x) &= \sum_{i,j=1}^{n}a_1^{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j};\\ \bar{B}_0(x,D_x) &= \sum_{j=1}^{n}\bar{b}_j(x,0)\frac{\partial}{\partial x_j};\\ \bar{C}_0(x) &= \bar{C}(x,0).\end{aligned} \tag{1.20}$$

Let e be the symbol of a fundamental solution E for Δ, and let $e=e_0+e_1+\dots$ be its asymptotic expansion where e_j is homogeneous of degree $-2-j$. The Kohn–Nirenberg formula for composition of pseudodifferential operators then gives:

$$\begin{aligned} e_0 &= (d_0)^{-1} \\ e_1 &= -(d_0)^{-2}d_1 - i(d_0)^{-3}\left[\sum_{j=1}^{n}(d_0)_{\xi_j}(d_0)_{x_j}+(d_0)_\eta(d_0)_t\right] \end{aligned} \tag{1.21}$$

where $(d_0)_t$, for example, means the derivative of d_0 with respect to t. We put

$$\begin{aligned} D = D(x,t,\xi) &= \sqrt{\Sigma_{i,j=1}^{n}(a_0^{ij}(x)+ta_1^{ij}(x)+O(t^2))\xi_i\xi_j} \\ b(x,t,\xi) &= \sum_{j=1}^{n} b_j(x,t)\xi_j \end{aligned} \tag{1.22}$$

so that

$$\begin{aligned} d_0 &= \eta^2+D^2 \\ d_1 &= i[b(x,t,\xi)+\bar{C}(x,t)\eta]. \end{aligned} \tag{1.23}$$

As in [GS], we want to see how the fundamental solution acts on distributions supported on $\mathbf{R}^n=\{t=0\}$, and for this we need to evaluate

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{it\eta}e_j(x,t,\xi,\eta)\,d\eta$$

for $j=0,1,2,\dots$. We have

$$\begin{aligned} e_0 &= (\eta^2+D^2)^{-1} \\ e_1 &= -i\left[\frac{b(x,t,\xi)+\bar{C}(x,t)\eta}{(\eta^2+D^2)^2}+\frac{\Sigma_{j=1}^{n}(d_0)_{\xi_j}(d_0)_{x_j}+[\Sigma_{i,j=1}^{n}a_1^{ij}(x)\xi_i\xi_j+O(t)](2\eta)}{(\eta^2+D^2)^3}\right]. \end{aligned} \tag{1.24}$$

Now the following calculations are easy to establish:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{it\eta}\,d\eta}{\eta^2+D^2}=\frac{1}{2D}e^{-Dt};$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{it\eta}\,d\eta}{(\eta^2+D^2)^2}=\left[\frac{1+Dt}{4D^3}\right]e^{-Dt};$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{it\eta}\eta\,d\eta}{(\eta^2+D^2)^2}=i\left[\frac{t}{4D}\right]e^{-Dt};$$
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{it\eta}\,d\eta}{(\eta^2+D^2)^3}=\left[\frac{t^2D^2+3tD+3}{16D^5}\right]e^{-Dt};$$
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{it\eta}\eta\,d\eta}{(\eta^2+D^2)^3}=\left[\frac{Dt+D^2t^2}{16D^4}\right]e^{-Dt};$$

From this it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\eta t}e_0(x,t,\xi,\eta)\,d\eta=\frac{1}{2D}e^{-Dt},$$

and

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\eta t}e_1(x,t,\xi,\eta)\,d\eta &= -ib(x,t,\xi)\left[\frac{1+Dt}{4D^3}\right]e^{-Dt}+\tilde{C}(x,t)\left[\frac{t}{4D}\right]e^{-Dt}\\ &\quad -i\sum_{j=1}^{n}(d_0)_{\xi_j}(d_0)_{x_j}\left[\frac{t^2D^2+3tD+3}{16D^5}\right]e^{-Dt}\\ &\quad +\left[\sum_{i,j=1}^{n}a_1^{ij}(x)\xi_i\xi_j+O(t)\right]\left[\frac{t+t^2D}{8D^2}\right]e^{-Dt}.\end{aligned}\tag{1.25}$$

In particular, if we let

$$D_0(x,\xi)=\sqrt{\Sigma_{i,j=1}^{n}a_0^{ij}(x)\xi_i\xi_j}\,,$$

then when we let $t=0$, we find that the symbol $e_{b,0}(x,\xi)$ of the pseudodifferential operator $E_{b,0}$ has, modulo terms of order less than -2, the asymptotic development

$$e_{b,0}(x,\xi)=\frac{1}{2D_0(x,\xi)}-i\left[\frac{b(x,0,\xi)}{4D_0(x,\xi)^3}+\frac{3\Sigma_{j=1}^{n}(D_0)_{\xi_j}(D_0)_{x_j}}{16D_0(x,\xi)^5}\right].\tag{1.26}$$

Again using the calculus of pseudodifferential operators, we find that the inverse of the elliptic pseudodifferential operator $E_{b,0}$ has symbol

$$[e_{b,0}]^{-1}(x,\xi)=2D_0(x,\xi)+iD_0(x,\xi)^{-1}\left[b(x,0,\xi)+\sum_{j=1}^{n}(D_0)_{\xi_j}(D_0)_{x_j}\right]\tag{1.27}$$

modulo terms of order -1.

To calculate the Dirichlet to Neumann operator, we now need to compute $E_b\circ[E_{b,0}]^{-1}$, take the derivative with respect to t, and then set $t=0$. The result of this computation is that the symbol of the Dirichlet to Neumann operator, modulo terms of order -1, is

$$\begin{aligned}&-D_0(x,\xi)-\frac{i}{2}D_0(x,\xi)^{-1}\sum_{j=1}^{n}(D_0)_{\xi_j}(D_0)_{x_j}\\&-\frac{1}{4}A_0(x,\xi)^{-1}A_1(x,\xi)+\frac{1}{2}\bar{C}(x,0)-\frac{i}{2}b(x,0,\xi)D_0(x,\xi)^{-1}.\end{aligned}\tag{1.28}$$

Since the symbol of the operator $(-A_0(x,D_x))^{1/2}$ is

$$D_0(x,\xi)+\frac{i}{2}D_0(x,\xi)^{-1}\sum_{j=1}^{n}(D_0)_{\xi_j}(D_0)_{x_j},$$

up to errors of order -1, this finally shows that the operator N^+ has the desired form, and completes the proof.

§2. $\square$ on (0,1)-forms and the $\bar\partial$-Neumann conditions for domains in $\mathbf{C}^2$

The object of this section is to describe the operator

$$\square=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial$$

and the associated $\bar\partial$-Neumann conditions on (0,1)-forms for smoothly bounded domains in $\mathbf{C}^2$. Thus let g be a smooth Hermitian metric on $\mathbf{C}^2$, and let $\Omega\subset\mathbf{C}^2$ be a domain with C^∞ boundary $\partial\Omega$. There is an open neighborhood U of $\partial\Omega$ such that if ϱ denotes a signed geodesic distance in the metric g to $\partial\Omega$, then

$$\begin{aligned}&\Omega\cap U=\{z\in U\,|\,\varrho(z)>0\};\\&\nabla\varrho(z)\neq0\quad\text{for } z\in U.\end{aligned}\tag{2.1}$$

We choose a smooth orthonormal basis for (0,1)-forms on U, given by $\bar\omega_1$ and $\bar\omega_2$, where

$$\bar\omega_2=\sqrt{2}\,\bar\partial\varrho.\tag{2.2}$$

We let $\bar L_1$ and $\bar L_2$ be the dual basis of antiholomorphic vector fields on U. Then L_1 and $\bar L_1$ are tangential on $\partial\Omega$, and in fact on the set U we have

$$\begin{aligned}&L_1(\varrho)=\bar L_1(\varrho)=0;\\&L_2(\varrho)=\bar L_2(\varrho)=\frac{1}{\sqrt{2}}.\end{aligned}\tag{2.3}$$

Hence if we define a real vector field T by

$$T=\frac{1}{2i}(L_2-\bar L_2) \tag{2.4}$$

then on the set U we have

$$T(\varrho)=0 \tag{2.5}$$

so T is also tangential on $\partial\Omega$, and the vector fields $\mathrm{Re}(L_1)$, $\mathrm{Im}(L_1)$ and T span the real tangent space to $\partial\Omega$ at every point of $\partial\Omega$. If $\partial/\partial\varrho$ is the vector field dual to the one form $d\varrho$ then it is easy to see that

$$L_2=\frac{1}{\sqrt{2}}\frac{\partial}{\partial\varrho}+iT. \tag{2.6}$$

If f is a smooth function on U then

$$\bar\partial f=\bar L_1(f)\bar\omega_1+\bar L_2(f)\bar\omega_2. \tag{2.7}$$

If u is a $(0,1)$-form on U then we can write $u=u_1\bar\omega_1+u_2\bar\omega_2$, and

$$\bar\partial u=(\bar L_1(u_2)-\bar L_2(u_1)+su_1)\bar\omega_1\wedge\bar\omega_2 \tag{2.8}$$

where the scalar function s is defined by the equation

$$\bar\partial\bar\omega_1=s\bar\omega_1\wedge\bar\omega_2. \tag{2.9}$$

(Note that since $\bar\omega_2=\sqrt{2}\,\bar\partial\varrho$, $\bar\partial\bar\omega_2=0$.)

We next want to compute the formal adjoints $\bar\partial^*$ of the operators $\bar\partial$, relative to the metric g. Let dV be the volume element induced by g. Then there are scalar functions h_1 and h_2 such that for $\varphi,\psi\in C_0^\infty(U)$ we have

$$\int_U \varphi\overline{(\bar L_j\psi)}\,dV=\int_U(-L_j+h_j)\varphi\bar\psi\,dV \tag{2.10}$$

for $j=1,2$. Thus if $u=u_1\bar\omega_1+u_2\bar\omega_2$ we have

$$\bar\partial^*u=(-L_1+h_1)u_1+(-L_2+h_2)u_2 \tag{2.11}$$

and

$$\bar\partial^*(v\bar\omega_1\wedge\bar\omega_2)=(L_2-h_2+\bar s)v\bar\omega_1+(-L_1+h_1)v\bar\omega_2. \tag{2.12}$$

We can now state the $\bar\partial$-Neumann problem. The operator $\Box$ is defined to be

$$\Box=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial \tag{2.13}$$

on (0, 1)-forms, and the $\bar\partial$-*Neumann problem* is the following boundary value problem:

$$\begin{aligned}\Box u&=f \quad \text{on } \Omega;\\ u\rfloor\bar\partial\varrho&=0 \quad \text{on } \partial\Omega\\ \bar\partial u\rfloor\bar\partial\varrho&=0 \quad \text{on } \partial\Omega.\end{aligned} \tag{2.14}$$

The object of this section is to calculate $\Box$ and the two boundary conditions of (2.14) in the given coordinates on the open set U. Now if $u=u_1\bar\omega_1+u_2\bar\omega_2$ then since $\bar\partial\varrho=(1/\sqrt{2})\bar\omega_2$, $u\rfloor\bar\partial\varrho=(1/\sqrt{2})u_2$ and the first boundary conditions is just

$$u_2=0 \quad \text{on } \partial\Omega. \tag{2.15}$$

Since $\bar\partial u=(\bar L_1(u_2)-\bar L_2(u_1)+su_1)\bar\omega_1\wedge\bar\omega_2$, the second boundary condition amounts to requiring that $\bar L_1(u_2)-\bar L_2(u_1)+su_1=0$ on $\partial\Omega$. But if u already satisfies the first boundary condition, then since $\bar L_1$ is a tangential operator $\bar L_1(u_2)=0$ on $\partial\Omega$. Hence in the presence of the first boundary condition, the second boundary condition is just

$$\bar L_2(u_1)-su_1=0 \quad \text{on } \partial\Omega. \tag{2.16}$$

The computation of $\Box$ is an algebraic exercise. In the coordinates given by $\bar\omega_1$ and $\bar\omega_2$, $\Box$ is a matrix valued differential operator. To describe it we first define

$$\begin{aligned}
\Box_0&=\begin{bmatrix}-\bar L_1L_1-L_2\bar L_2 & 0\\ 0 & \bar L_1L_1-L_2\bar L_2\end{bmatrix}\\
\Box_1&=\begin{bmatrix}0 & [L_2,\bar L_1]\\ [L_1,\bar L_2] & 0\end{bmatrix}\\
\Box_2&=\begin{bmatrix}sL_2+(h_2-\bar s)\bar L_2 & 0\\ 0 & h_2\bar L_2\end{bmatrix}\\
\Box_3&=\begin{bmatrix}h_1\bar L_1 & \bar s\bar L_1\\ -sL_1 & h_1\bar L_1\end{bmatrix}\\
\Box_4&=\begin{bmatrix}\bar L_1(h_1)+L_2(s)-sh_2+|s|^2 & \bar L_1(h_2)\\ \bar L_2(h_1)-L_1(s)+sh_1 & \bar L_2(h_2)\end{bmatrix}.
\end{aligned} \tag{2.17}$$

Note that $\Box_0$ is a second order operator, $\Box_1$, $\Box_2$, and $\Box_3$ are first order operators, and

$\Box_4$ is a zero order operator. Also $\Box_3$ only involves differentiation with respect to the tangential operators L_1 and $\bar{L}_1$. Finally note that $\Box_1$ is also a tangential operator by equation (2.3).

The first main result of this section is now the following description of the operator $\Box$ and the $\bar{\partial}$-Neumann boundary conditions in terms of the coordinates introduced above.

PROPOSITION 2.1. *On the open set U with coordinates given by $\bar{\omega}_1$ and $\bar{\omega}_2$ we have*

(1) $\Box=\Box_0+\Box_1+\Box_2+\Box_3+\Box_4$;

(2) *The two $\bar{\partial}$-Neumann boundary conditions are equivalent with*

(i) $u_2=0$ *on* $\partial\Omega$,

(ii) $\bar{L}_2(u_1)-su_1=0$ *on* $\partial\Omega$;

(3) *The operator $\Box$ is a second order elliptic system with scalar principal symbol equal to the principal symbol of the Laplace–Beltrami operator associated to the Hermitian metric $\frac{1}{2}g$.*

Proof. (1) is the result of straightforward algebraic calculations using equations (2.7), (2.8), (2.11), (2.12), and (2.13). We omit these calculations, but see [GS], Chapter 6, for example, for further details. We have already proved (2), and so it only remains to prove (3).

The second order part of $\Box$ is the operator $\Box_0$, and modulo the diagonal first order operator

$$\begin{bmatrix} [L_1,\bar{L}_1] & 0 \\ 0 & [L_2,\bar{L}_2] \end{bmatrix}$$

$\Box_0$ is the scalar second order operator

$$-(L_1\bar{L}_1+L_2\bar{L}_2).$$

If we write $L_1=\frac{1}{2}(X_1-iY_1)$, and $L_2=\frac{1}{2}(X_2-iY_2)$ with X_j and Y_j real vector fields, then X_1, X_2, Y_1, and Y_2 are orthogonal, and by the Pythagorean theorem,

$$\left\|\frac{1}{\sqrt{2}}X_j\right\|=\left\|\frac{1}{\sqrt{2}}Y_j\right\|=1.$$

But modulo operators of order -1

$$-L_1\bar{L}_1-L_2\bar{L}_2=-\frac{1}{2}[(\sqrt{2}X_1)^2+(\sqrt{2}Y_1)^2+(\sqrt{2}X_2)^2+(\sqrt{2}Y_2)^2] \tag{2.18}$$

which agrees to top order with the Laplace–Beltrami operator for the metric $\frac{1}{2}g$. This completes the proof.

We want to be able to apply the results of Section 1 to the operator $2\Box$. Thus we want to write

$$2\Box = -\frac{\partial^2}{\partial\varrho^2}+\bar{C}\frac{\partial}{\partial\varrho}+\Box_\varrho \tag{2.19}$$

as in equation (1.18), where $\Box_\varrho$ acts tangentially, and then we want to write

$$\Box_\varrho = -A+\bar{B} \tag{2.20}$$

as in equation (1.11) where A is a scalar second order operator, and $\bar{B}$ is a first order operator.

Using equation (2.6) and its conjugate, we see that

$$\begin{aligned}\Box_0 &= -\frac{1}{2}\frac{\partial^2}{\partial\varrho^2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\frac{i}{\sqrt{2}}\left[\frac{\partial}{\partial\varrho},T\right]\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}-\begin{bmatrix}T^2+\bar{L}_1L_1 & 0\\ 0 & T^2+L_1\bar{L}_1\end{bmatrix};\\ \Box_2 &= \begin{bmatrix}(s-\bar{s}+h_2)/\sqrt{2} & 0\\ 0 & h_2/\sqrt{2}\end{bmatrix}\frac{\partial}{\partial\varrho}+i\begin{bmatrix}s+\bar{s}-h_2 & 0\\ 0 & -h_2\end{bmatrix}T.\end{aligned} \tag{2.21}$$

Thus we see that

$$\begin{aligned}2\Box = &-\frac{\partial^2}{\partial\varrho^2}+\begin{bmatrix}\sqrt{2}(s-\bar{s}+h_2) & 0\\ 0 & \sqrt{2}h_2\end{bmatrix}\frac{\partial}{\partial\varrho}\\ &-2(T^2+\bar{L}_1L_1)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\ &-2\begin{bmatrix}0 & 0\\ 0 & [L_1,\bar{L}_1]\end{bmatrix}+\frac{2i}{\sqrt{2}}\left[\frac{\partial}{\partial\varrho},T\right]\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}+2i\begin{bmatrix}s+\bar{s}-h_2 & 0\\ 0 & -h_2\end{bmatrix}T\\ &+2\Box_1+2\Box_3+2\Box_4.\end{aligned} \tag{2.22}$$

We can expand the vector fields T, L_1, and $\bar{L}_1$ in Taylor series in ϱ. Thus we write

$$\begin{aligned}T &= T^0+\varrho T^1+O(\varrho^2);\\ L_1 &= L_1^0+\varrho L_1^1+O(\varrho^2);\\ \bar{L}_1 &= \bar{L}_1^0+\varrho\bar{L}_1^1+O(\varrho^2).\end{aligned} \tag{2.23}$$

It follows that

$$\left[\frac{\partial}{\partial\varrho}, T\right] = T^1 + O(\varrho), \tag{2.24}$$

and hence

$$T^2+\bar{L}_1L_1 = (T^0)^2+\bar{L}_1^0L_1^0+(2T^0T^1+\bar{L}_1^0L_1^1+\bar{L}_1^1L_1^0+[T^1,T^0])\varrho+O(\varrho^2). \tag{2.25}$$

From these calculations, we can now calculate the operators A_0, A_1, $\bar{B}_0$ and $\bar{C}_0$ of Definition 1.1 for the operator $2\Box$.

PROPOSITION 2.2. *For the operator $2\Box$ on the domain Ω we have:*

$$A_0 = 2(T^0)^2+2\bar{L}_1^0L_1^0;$$

$$A_1 = 4T^1T^0+2\bar{L}_1^0L_1^1+2\bar{L}_1^1L_1^0-2[T^1,T^0];$$

$$\bar{B}_0 = -2\begin{bmatrix} 0 & 0 \\ 0 & [L_1^0,\bar{L}_1^0] \end{bmatrix} + \sqrt{2}\, iT^1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 2i\begin{bmatrix} s+\bar{s}-h_2 & 0 \\ 0 & -h_2 \end{bmatrix} T^0$$
$$+2\Box_1^0+2\Box_3^0+2\Box_4^0;$$

$$\bar{C}_0 = \begin{bmatrix} \sqrt{2}(s-\bar{s}+h_2) & 0 \\ 0 & \sqrt{2}h_2 \end{bmatrix}$$

where of course $\Box_j^0$ denotes the operator $\Box_j$ restricted to the boundary, i.e. to the set $\{\varrho=0\}$.

§3. The boundary operator $\Box^+$ for the $\bar{\partial}$-Neumann problem

Let $\Omega\subset\mathbf{C}^2$ be a domain with C^∞ boundary. We shall sometimes write $\Omega^+=\Omega$, and $\Omega^-=\mathbf{C}^2\setminus\Omega$. Recall that the $\bar{\partial}$-Neumann problem for $(0,1)$-forms on Ω consists of finding a $(0,1)$-form u such that:

$$\begin{aligned} \Box u &= f \quad \text{on } \Omega; \\ u \lrcorner \bar{\partial}\varrho &= 0 \quad \text{on } \partial\Omega; \\ \bar{\partial}u \lrcorner \bar{\partial}\varrho &= 0 \quad \text{on } \partial\Omega; \end{aligned} \tag{3.1}$$

where f is a given $(0,1)$-form. It is well known that this boundary value problem can be reduced to the problem of inverting a certain scalar pseudodifferential boundary operator on $\partial\Omega$. The object of this section is to explicitly describe this reduction and to calculate the resulting boundary operator $\Box^+$.

The reduction to the boundary is accomplished through the use of two operators associated to the domain Ω and the operator $\Box$, namely a Poisson operator P and a Green's operator G. If we let R denote the operator of restriction to the boundary, then the operator P maps (0, 1)-forms on the boundary $\partial\Omega$ to (0, 1)-forms on Ω and has the property, that modulo C^∞ errors,

$$\begin{aligned} &\Box \circ P = 0 \quad \text{on } \Omega; \\ &R \circ P = I \quad \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

The Green's operator G has the property that modulo C^∞ errors,

$$\begin{aligned} &\Box \circ G = I \quad \text{on } \Omega; \\ &R \circ G = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

We now try to find a solution to problem (3.1) of the form

$$u = P(u_b) + G(f) \tag{3.4}$$

where u_b is a (0, 1)-form on the boundary to be determined. (Actually the solution we derive in this section is exact only modulo C^∞ error terms. The more precise result is dealt with in Section 5 below.) It follows from the defining properties of P and G that, modulo C^∞ error terms,

$$\Box(u) = f \tag{3.5}$$

for *any* u of the form given in (3.4). Thus we want to determine u_b so that the two $\bar\partial$-Neumann boundary conditions are satisfied.

We now turn to the problem of defining the boundary operator $\Box^+$, whose invertibility gives us u_b.

For this we must use results about the $\bar\partial_b$ complex on $\partial\Omega$. We denote by $\mathscr{B}^{0,0}$ the space of smooth functions on $\partial\Omega$. The space of smooth (0, 1)-forms on $\partial\Omega$, which we denote by $\mathscr{B}^{0,1}$ can be identified with the restriction to $\partial\Omega$ of all smooth (0, 1)-forms on $\bar\Omega$ which satisfy the first of the $\bar\partial$-Neumann conditions, i.e. for which $u_2|_{\partial\Omega}=0$ (see [FK], p. 86 for details). Then the operator $\bar\partial_b$ carries elements of $\mathscr{B}^{0,0}$ to sections of $\mathscr{B}^{0,1}$. The correspondence $u_1 \leftrightarrow u_1\bar\omega_1$ clearly allows us to identify elements of $\mathscr{B}^{0,1}$ with functions on $\partial\Omega$, and the L^2 completion of $\mathscr{B}^{0,1}$ with $L^2(\partial\Omega)$.

We shall define the basic boundary operator $\Box^+$, a first order pseudodifferential operator, mapping $\mathscr{B}^{0,1}$ to $\mathscr{B}^{0,1}$ as follows. Let u be a smooth $(0,1)$-form on Ω that satisfies the first boundary condition, and let u_b be its restriction to the boundary. Now form

$$\Box^+ u_b = \bar\partial P(u_b) \rfloor \bar\partial \varrho|_{\partial\Omega},$$

where P is the Poisson kernel given in Section 1. It is to be noted that $\bar\partial P \rfloor \bar\partial\varrho$ satisfies the first boundary condition by its very definition. Thus $u_b \to \Box^+ u_b$ is a well-defined mapping on $\mathscr{B}^{0,1}$ to itself. Since we shall have that essentially u is given by $u = P(u_b) + G(f)$ (where G is a Green's operator as in (3.3)), the determination of u_b (and hence of u) is reduced to inverting the equation $\Box^+ u_b = -\bar\partial G(f) \rfloor \bar\partial\varrho|_{\partial\Omega}$. Since the bundle $\mathscr{B}^{0,1}$ is one-dimensional, we can therefore realize $\Box^+$ as a scalar operator.

We shall now describe this scalar realization of $\Box^+$.

As in Section 2, in a neighborhood U of $\partial\Omega$, we choose a basis $\{\bar\omega_1, \bar\omega_2\}$ for the $(0,1)$-forms. Then if u_b is a $(0,1)$-form on $\partial\Omega$ we can write it as:

$$u_b = u_1\bar\omega_1 + u_2\bar\omega_2 \tag{3.6}$$

where u_1 and u_2 are smooth functions on $\partial\Omega$. Since $P(u_b)$ is a $(0,1)$-form on Ω, on the set $\Omega \cap U$ we can write

$$P(u_b) = v_1\bar\omega_1 + v_2\bar\omega_2, \tag{3.7}$$

where v_1 and v_2 are (smooth) functions on $\Omega \cap U$. We have

$$u_j = v_j \quad \text{on } \partial\Omega, \ \text{for } j = 1, 2 \tag{3.8}$$

since $R \circ P = I$.

Recall from Section 2 that $\bar\omega_2 = \sqrt{2}\,\bar\partial\varrho$. The first $\bar\partial$-Neumann condition then gives

$$0 = R(u \rfloor \bar\partial\varrho) = u_b \rfloor \bar\partial\varrho + R(G(f)) \rfloor \bar\partial\varrho = \sqrt{2}\, u_2 \tag{3.9}$$

since $R \circ G = 0$, and so the first $\bar\partial$-Neumann boundary condition is equivalent to

$$u_2 = 0. \tag{3.10}$$

Hence by (3.8) this implies that

$$v_2 = 0 \quad \text{on } \partial\Omega. \tag{3.11}$$

Next,

$$\bar\partial u = \bar\partial P(u_b) + \bar\partial G(f) \tag{3.12}$$

and so the second $\bar\partial$-Neumann boundary condition is equivalent to

$$R(\bar\partial P(u_b) \lrcorner \bar\partial \varrho) = -R(\bar\partial G(f) \lrcorner \bar\partial \varrho). \tag{3.13}$$

Now according to equation (2.8)

$$\bar\partial P(u_b) = (\bar L_1(v_2) - \bar L_2(v_1) + s v_1))\, \bar\omega_1 \wedge \bar\omega_2, \tag{3.14}$$

and hence

$$\bar\partial P(u_b) \lrcorner \bar\partial \varrho = \frac{1}{\sqrt{2}} (\bar L_1(v_2) - \bar L_2(v_1) + s v_1)\, \bar\omega_1. \tag{3.15}$$

Now if u is to satisfy the first boundary condition, u_2 and hence v_2 is zero on $\partial\Omega$. On the other hand, the operator $\bar L_1$ is tangential, and hence

$$\bar L_1(v_2) = 0 \quad \text{on } \partial\Omega. \tag{3.16}$$

Since $u_1 = v_1$ on $\partial\Omega$, the second $\bar\partial$-Neumann boundary condition (in the presence of the first) is equivalent to the following equation for u_1:

$$\frac{1}{\sqrt{2}} R(\bar L_2(v_1) - s v_1)\, \bar\omega_1 = R(\bar\partial G(f) \lrcorner \bar\partial \varrho). \tag{3.17}$$

Near $\partial\Omega$, using the coordinates $\{\bar\omega_1, \bar\omega_2\}$, G can be written as a matrix operator

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \tag{3.18}$$

so that if $f = f_1 \bar\omega_1 + f_2 \bar\omega_2$, then

$$\begin{aligned} G(f) &= (G_{11}(f_1) + G_{12}(f_2))\, \bar\omega_1 + (G_{21}(f_1) + G_{22}(f_2))\, \bar\omega_2 \\ &= G_1(f)\, \bar\omega_1 + G_2(f)\, \bar\omega_2. \end{aligned} \tag{3.19}$$

It follows from equations (2.6) and (2.8) that

$$\bar\partial G(f) \lrcorner \bar\partial \varrho = \frac{1}{\sqrt{2}} [\bar L_1(G_2(f)) - \bar L_2(G_1(f)) + s G_1(f)]\, \bar\omega_1. \tag{3.20}$$

Thus in equation (3.17), both sides are scalar multiples of $\bar\omega_1$.

Now recall from equation (2.6) that

$$L_2 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial \varrho} + iT.$$

Hence equation (3.17) is equivalent to

$$\left[\frac{1}{2} R \frac{\partial}{\partial \varrho} P(u_b)_1 - \frac{i}{\sqrt{2}} T(u_1) - \frac{s}{\sqrt{2}} u_1\right] \bar{\omega}_1 = R(\bar{\partial} G(f) \lrcorner \bar{\partial} \varrho), \tag{3.21}$$

where $P(u_b)_1$ denotes the first component of the Poisson operator applied to u_b. But

$$R \frac{\partial}{\partial \varrho} P(u_b)_1 = N^+(u_b)_1 \tag{3.22}$$

where N^+ is the Dirichlet to Neumann operator associated with the domain Ω and the operator $\Box$ (or $2\Box$) studied in Section 1 and $N^+(u_b)_1$ is the first component. For a scalar function φ on $\partial\Omega$ we shall write, with an abuse of notation

$$N^+(\varphi) = N^+(u_b)_1 \tag{3.23}$$

where

$$u_b = \varphi \bar{\omega}_1. \tag{3.24}$$

The above calculations shows that $\Box^+$ can be realized as the scalar operator

$$\Box^+ = \frac{1}{2} N^+ - \frac{i}{\sqrt{2}} T - \frac{s}{\sqrt{2}}.$$

We summarize our discussion up to this point as follows.

PROPOSITION 3.1. *The $\bar{\partial}$-Neumann problem* (3.1) *is equivalent (modulo C^∞ error terms) to the problem of solving*

$$\Box^+(u_b)\,\bar{\omega}_1 = R(\bar{\partial} G(f) \lrcorner \bar{\partial} \varrho),$$

or

$$\Box^+(u_b) = \frac{1}{\sqrt{2}} [\bar{L}_1(G_2(f)) - \bar{L}_2(G_1(f)) + sG_1(f)]_{\partial\Omega};$$

i.e. the $\bar{\partial}$-Neumann problem is equivalent to the problem of inverting the operator $\Box^+$.

In order to calculate N^+ and $\Box^+$, we use Proposition 2.2. Note that we really want to apply the operator N^+ to a form on the boundary of the form $\varphi\bar{\omega}_1$, and then we want to calculate the coefficient of $\bar{\omega}_1$ of the result. Thus using Proposition 2.2 we see that the coefficient of $\bar{\omega}_1$ of $\bar{B}_0(\varphi\bar{\omega}_1)$ is:

$$\begin{aligned}\bar{B}_0(\varphi\bar{\omega}_1)_1 = \sqrt{2}\, iT^1(\varphi)+2i(s+\bar{s}-h_2)\, T^0(\varphi)+h_1\,\bar{L}_1(\varphi)\\ +(\bar{L}_1(h_1)+L_2(s)-sh_2+|s|^2)\,\varphi.\end{aligned} \tag{3.25}$$

Similarly, the ω_1 coefficient of $\bar{C}_0(\varphi\bar{\omega}_1)$ is

$$\bar{C}_0(\varphi\bar{\omega}_1)_1 = \sqrt{2}\,(s-\bar{s}+h_2)\,\varphi. \tag{3.26}$$

Also from Proposition 2.2 we have:

$$\begin{aligned}A_0(\varphi\bar{\omega}_1) &= [2(T^0)^2+2\bar{L}_1^0 L_1^0](\varphi)\,\bar{\omega}_1;\\ A_1(\varphi\bar{\omega}_1) &= [4T^1T^0+2\bar{L}_1^0 L_1^1+2\bar{L}_1^1 L_1^0-2[T^1,T^0]](\varphi)\,\bar{\omega}_1.\end{aligned} \tag{3.27}$$

Recall from Theorem 1 that, modulo pseudodifferential operators of order -1

$$N^+ = -(-A_0)^{1/2}-\frac{1}{4}(A_0)^{-1}A_1-\frac{1}{2}(-A_0)^{-1/2}\bar{B}_0+\frac{1}{2}\bar{C}_0. \tag{3.28}$$

Thus if we write

$$\Theta = -(T^0)^2-\bar{L}_1^0 L_1^0 \tag{3.29}$$

then

$$\begin{aligned}2\square^+ &= -\sqrt{2}\,\Theta^{1/2}+\frac{1}{2}\Theta^{-1}\Big[T^1T^0+\frac{1}{2}\bar{L}_1^0L_1^1+\frac{1}{2}\bar{L}_1^1L_1^0-\frac{1}{2}[T^1,T^0]\Big]\\ &\quad -\frac{1}{2\sqrt{2}}\Theta^{-1/2}[\sqrt{2}\,iT^1+2i(s+\bar{s}-h_2)\,T^0+h_1\bar{L}_1\\ &\quad +(\bar{L}_1(h_1)+L_2(s)-sh_2+|s|^2)]\\ &\quad +\frac{1}{\sqrt{2}}(s-\bar{s}+h_2)-\sqrt{2}\,iT^0-\sqrt{2}\,s\\ &= -\sqrt{2}\,[\Theta^{1/2}+iT^0]+\frac{1}{2}[\Theta^{-1}T^0-i\Theta^{-1/2}]\,T^1\\ &\quad -\frac{1}{\sqrt{2}}(s+\bar{s}-h_2)\,[i\Theta^{-1/2}T^0+I]\\ &\quad +E_1L_1^0+E_2\bar{L}_1^0+E_3.\end{aligned} \tag{3.30}$$

where E_1, E_2, and E_3 are pseudodifferential operators of order -1. Finally, if we note that $[\Theta^{-1}, T^0]T^1$ is also a pseudodifferential operator of order -1, we see that we have proved the following:

PROPOSITION 3.2.

$$\Box^+ = \frac{-1}{\sqrt{2}}[[-(T^0)^2-\bar{L}_1^0 L_1^0]^{1/2}+iT^0][I+E_0]+E_1 L_1^0+E_2\bar{L}_1^0+E_3,$$

where the operators E_j are pseudodifferential operators of order -1.

So far we have dealt exclusively with the $\bar{\partial}$-Neumann problem on the domain Ω. However, we may also consider the $\bar{\partial}$-Neumann problem on the complementary domain $\Omega^-=\mathbf{C}^2\setminus\Omega$. This problem also gives rise to a scalar boundary pseudodifferential operator which we call $\Box^-$. Then a calculation similar to the one above gives us

PROPOSITION 3.3.

$$\Box^- = \frac{1}{\sqrt{2}}[-[-(T^0)^2-\bar{L}_1^0 L_1^0]^{1/2}+iT^0][I+\tilde{E}_0]+\tilde{E}_1 L_1^0+\tilde{E}_2\bar{L}_1^0+\tilde{E}_3,$$

where the operators $\tilde{E}_j$ are pseudodifferential operators of order -1.

We shall need to consider the compositions $\Box^+\circ\Box^-$ and $\Box^-\circ\Box^+$, and for this we need:

PROPOSITION 3.4. *Let* $\Theta=-(T^0)^2-\bar{L}_1^0 L_1^0$. *Then*

$$[\Theta^{1/2}, T^0] = F_1 L_1^0+F_2\bar{L}_1^0+F_3,$$

where F_j are pseudodifferential operators of order zero.

Proof. It follows from the Kohn–Nirenberg formula for compositions of pseudodifferential operators that, since Θ is elliptic, modulo pseudodifferential operators of order zero

$$[\Theta^{1/2}, T^0] = \frac{1}{2}\Theta^{-1/2}[\Theta, T^0].$$

But

$$\begin{aligned}[\Theta, T^0] &= [-(T^0)^2-\bar{L}_1^0 L_1^0, T^0] = -[\bar{L}_1^0 L_1^0, T^0]\\ &= \tilde{F}_1 L_1^0+\tilde{F}_2\bar{L}_1^0,\end{aligned}$$

where $\tilde{F}_j$ is a pseudodifferential operator of order one. This completes the proof.

It now follows immediately from Propositions 3.2, 3.3, and 3.4 that we have:

PROPOSITION 3.5. *There are pseudodifferential operators F_j and $\tilde{F}_j$ of order zero for $j=4,5,6$ so that*

$$\Box^+ \circ \Box^- = -\frac{1}{2}\bar{L}_1^0 L_1^0 + F_4 L_1^0 + F_5 \bar{L}_1^0 + F_6;$$

and

$$\Box^- \circ \Box^+ = -\frac{1}{2}\bar{L}_1^0 L_1^0 + \tilde{F}_4 L_1^0 + \tilde{F}_5 \bar{L}_1^0 + \tilde{F}_6;$$

§ 4. Invertibility of $\Box^+$ and $\Box^-$ and their relation with $\Box_b$

We now turn to the problem of inverting the operator $\Box^+$, which, as we have seen, is equivalent to the $\bar{\partial}$-Neumann problem. The object of this section is to show how this can be done, and to establish the connection between the operators $\Box^+$ and $\Box^-$ and the operator $\Box_b$ which arises from the boundary $\bar{\partial}_b$ complex. The inversion is done in two steps. The first step is to invert the operator $\Box^+$ away from its characteristic variety, and this is done using standard pseudodifferential operators. The second step involving inverting the operator near its characteristic variety requires a much more stringent hypothesis on the domain Ω than we have used so far, and so in the rest of this paper, unless otherwise indicated, we make the following standing hypothesis:

Ω is a bounded, pseudoconvex domain of finite type m.

We shall use the condition of pseudoconvexity in the following way. Let $\bar{L}_1, L_1$ and T be the tangential vector fields on $\partial\Omega$ that were defined in Section 2, equations (2.3) and (2.4). These vector fields span the complexified tangent space to $\partial\Omega$ at every point, and since $[L_1, \bar{L}_1]$ is again a complex tangential vector field, we can write

$$[L_1, \bar{L}_1] = \frac{1}{i}\lambda T + \alpha L_1 + \beta L_2, \tag{4.1}$$

where λ, α, and β are smooth functions on $\partial\Omega$. Pseudoconvexity of the doman Ω is then equivalent to the condition

$$\lambda \geqslant 0. \tag{4.2}$$

The condition that $\partial\Omega$ is of finite type m is equivalent to the condition that for the function Λ_m defined in equation (4.9) below, we have

$$\Lambda_m \neq 0. \tag{4.3}$$

We now begin the study of the invertibility of the operator $\Box^+$. To begin with, note that in general the operators $\Box^\pm$ are not elliptic. We work locally in a neighborhood U of a fixed point $p \in \partial\Omega$, and we choose real coordinates (x_1, x_2, x_3, x_4) centered at p so that

$$\partial\Omega \cap U = \{x_4 = 0\}, \tag{4.4}$$

and in terms of the coordinates $x=(x_1, x_2, x_3)$ on $\partial\Omega$,

$$\begin{aligned} T^0 &= \frac{\partial}{\partial x_3}, \\ L_1^0 &= \frac{1}{2}\left[\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right] + O(x). \end{aligned} \tag{4.5}$$

(This is possible since if we write $L_1^0 = \frac{1}{2}(X_1 - iX_2)$, then $\{X_1, X_2, T^0\}$ are three linenarly independent vector fields in the variables x_1, x_2, x_3.) The symbols of these operators are thus

$$\begin{aligned} \sigma(L_1^0) &= \frac{1}{2}(i\xi_1 + \xi_2) + O(x), \\ \sigma(T^0) &= i\xi_3. \end{aligned} \tag{4.6}$$

It follows from Propositions 3.2 and 3.3 that modulo symbols of order zero, the symbols of the operators $\Box^\pm$ are then given by

$$\begin{aligned} \sigma(\Box^+) &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{1}{4}(\xi_1^2+\xi_2^2)+\xi_3^2} - \xi_3\right] + O(x), \\ \sigma(\Box^-) &= \frac{1}{\sqrt{2}}\left[-\sqrt{\frac{1}{4}(\xi_1^2+\xi_2^2)+\xi_3^2} - \xi_3\right] + O(x). \end{aligned} \tag{4.7}$$

Thus for example, when $x=0, \sigma(\Box^+)$ vanishes when $\xi_1=\xi_2=0$ and $\xi_3 \geq 0$, while $\sigma(\Box^-)$ vanishes when $\xi_1=\xi_2=0$ and $\xi_3 \leq 0$.

We can use the standard theory of pseudodifferential operators to invert the operators $\Box^\pm$ away from their characteristic varieties. Thus we make the following

DEFINITION 4.1. *Γ^+ is a pseudodifferential operator of order zero whose principal symbol equals* 1 *on the set*

$$\{\sigma([-(T^0)^2-\bar{L}_1^0L_1^0]^{1/2})<\frac{1}{4}\sigma(-iT^0)\},$$

and whose principal symbol equals 0 *on the set*

$$\{\sigma([-(T^0)^2-\bar{L}_1^0L_1^0]^{1/2})>\frac{1}{2}\sigma(-iT^0)\}.$$

Similarly, Γ^- is a pseudodifferential operator of order zero whose principal symbol equals 1 *on the set*

$$\{\sigma([-(T^0)^2-\bar{L}_1^0L_1^0]^{1/2})<-\frac{1}{4}\sigma(-iT^0)\},$$

and whose principal symbol equals 0 *on the set*

$$\{\sigma([-(T^0)^2-\bar{L}_1^0L_1^0]^{1/2})>-\frac{1}{2}\sigma(-iT^0)\}.$$

Moreover, we can assume that the operators $\Gamma^\pm$ are essentially self adjoint in the sense that $\Gamma^\pm-(\Gamma^\pm)^*$ are pseudodifferential operators of order $-\infty$.

PROPOSITION 4.1. *There exist pseudodifferential operators $Q^\pm$ and $\tilde{Q}^\pm$ of order* -1 *so that*

$$\Box^\pm Q^\pm = I-\Gamma^\pm,\ and\ \tilde{Q}^\pm\Box^\pm = I-\Gamma^\pm$$

modulo pseudodifferential operators of order $-\infty$.

Proof. This follows from standard pseudodifferential operator constructions since the operators $\Box^\pm$ are elliptic away from their characteristic varieties, in view of Propositions 3.2 and 3.3.

To invert the operator $\Box^+$ near its characteristic variety we shall need to work with the class of NIS operators of smoothing degree k. Here NIS stands for "non-isotropic smoothing", and this class was introduced and studied in [NRSW], §§4–6. In order to define this class of operators, we first need to recall from [NRSW] the definition of the non-isotropic metric on $\partial\Omega$ which is naturally induced by the complex structure in $\mathbf{C}^2$.

Write $L_1=\frac{1}{2}(X_1-iX_2)$. For every k-tuple of integers $(i_1,\dots,i_k)$ with $i_j\in\{1,2\}$ define smooth functions $\lambda_{i_1,\dots,i_k}$, $\alpha^1_{i_1,\dots,i_k}$, $\alpha^2_{i_1,\dots,i_k}$ on $\partial\Omega$ by the equation

$$[X_{i_k},[\dots,[X_{i_2},X_{i_1}]\dots]]=\lambda_{i_1,\dots,i_k}T^0+\alpha^1_{i_1,\dots,i_k}X_1+\alpha^2_{i_1,\dots,i_k}X_2. \tag{4.8}$$

For each integer $l \geq 2$ define a smooth function Λ_l on $\partial\Omega$ by the equation

$$\Lambda_l(x) = \left[\sum |\lambda_{i_1, \dots, i_k}(x)|^2 \right]^{1/2}, \tag{4.9}$$

where the sum is taken over all k-tuples with $2 \leq k \leq l$. Finally set

$$\Lambda(x, \delta) = \sum_{j=2}^{m} \Lambda_j(x) \delta^j. \tag{4.10}$$

DEFINITION 4.2. (1) *For* $x, y \in \partial\Omega$ *set*

$$\begin{aligned} d(x, y) = \inf\{\delta > 0 | \ & \textit{there exists a continuous piecewise smooth map} \\ & \varphi: [0, 1] \to \partial\Omega \textit{ with } \varphi(0) = x, \varphi(1) = y, \textit{ and almost everywhere} \\ & \varphi'(t) = \alpha_1(t) X_1 + \alpha_2(t) X_2 \textit{ with } |\alpha_1(t)| < \delta, |\alpha_2(t)| < \delta\}. \end{aligned}$$

(2) *For* $x \in \partial\Omega$ *and* $\delta > 0$, *set*

$$B(x, \delta) = \{y \in \partial\Omega | d(x, y) < \delta\}.$$

(3) *Let* σ *be the induced volume measure on* $\partial\Omega$, *and set*

$$V_x(\delta) = \sigma(B(x, \delta))$$

and

$$V(x, y) = V_x(d(x, y)).$$

The following result summarizes some of the basic properties of the functions d, Λ and V_x:

THEOREM B. *The function* $d: \partial\Omega \times \partial\Omega \to \mathbf{R}$ *is a metric, and there are constants* C_1, C_2 *and* A *so that for all* $x, y \in \partial\Omega$:

(1) $B(x, \delta) \cap B(y, \delta) \neq \varnothing \Rightarrow B(x, \delta) \subset B(y, A\delta)$;

(2) $C_1 \delta^2 \Lambda(x, \delta) \leq V_x(\delta) \leq C_2 \delta^2 \Lambda(x, \delta)$;

(3) *If* $d(x, y) \leq \delta$ *then*

$$C_1 \leq \frac{\Lambda(x, \delta)}{\Lambda(y, \delta)} \leq C_2.$$

See [NRSW], [BDN], or [C] for further details about the metric d.

DEFINITION 4.3. *A smooth function $\varphi \in C_0^\infty(\partial\Omega)$ is a bump function supported on $B(x,\delta)$ if φ has compact support in $B(x,\delta)$.*

We are now in a position to define NIS operators. Let

$$T(f)=\int_{\partial\Omega} T(x,y)f(y)\,d\sigma(y)$$

where $T(x,y)$ is a distribution on $\partial\Omega\times\partial\Omega$.

DEFINITION 4.4. *T is an* NIS operator of order s *if T is C^∞ away from the diagonal, and if there exists a family T_ε of operators given by*

$$T_\varepsilon f(x)=\int_{\partial\Omega} T_\varepsilon(x,y)f(y)\,d\sigma(y)$$

such that:

(1) *$T_\varepsilon(f)\to T(f)$ as $\varepsilon\to 0$ whenever $f\in C_0^\infty(\partial\Omega)$.*

(2) *$T_\varepsilon\in C^\infty(\partial\Omega\times\partial\Omega)$.*

(3) *There exist constant C_{kl} so that for all ε,*

$$|X_x^I X_y^J T_\varepsilon(x,y)|\le C_{kl}\frac{d(x,y)^{s-k-l}}{V(x,y)};$$

where $|I|=k$, $|J|=l$.

(4) *For each l there is an integer N_l and a constant C_l so that whenever φ is a bump function supported on $B(x,\delta)$, then for all ε, and all I so that $|I|=l$*

$$|(X^I T_\varepsilon(\varphi))(x)|\le C_l\delta^{-l+s}\sup_y\sum_{|J|\le N_l}\delta^{|J|}|X^J\varphi(y)|.$$

(5) *The above conditions also hold for the operator T^*, i.e. the operator with kernel $T^*(x,y)=\overline{T(y,x)}$.*

We have used the notation $X^I=X_{i_1}X_{i_2}\dots X_{i_k}$ where $I=(i_1,\dots,i_k)$ with $i_j\in\{1,2\}$, and $|I|=k$. X_x^I indicates differentiation with respect to the x variables.

The main results that we shall need about the class of NIS operators are contained in the following result (see [NRSW] for details):

THEOREM C. (1) *If T_j is an NIS operator of order s_j for $j=1,2$ and if $s_1+s_2<4$ then $T_1\circ T_2$ is an NIS operator of order s_1+s_2.*

(2) *If T is an NIS operator of order s, then T maps the nonisotropic Sobolev space NL^p_k boundedly to NL^p_{k+s} whenever $k\geq 0$ and $k+s\geq 0$.*

(3) *If T is an NIS operator of order s, then T maps the nonisotropic Lipschitz space Γ_α boundedly to $\Gamma_{\alpha+s}$ whenever $k>0$ and $k+s>0$.*

See [NRSW], §6 for the precise definitions of NL^p_k and Γ_α. Note that in [NRSW] we have used the notation L^p_k for nonisotropic Sobolev spaces instead of NL^p_k. In this paper we shall reserve L^p_k for the isotropic Sobolev spaces. Note that $NL^p_0=L^p$, the standard isotropic Sobolev space L^2_k is contained in NL^2_k, and $NL^2_{mk}\subset L^2_k$.

We now turn to the problem of inverting the operator $\Box^+$ near its characteristic variety.

We can describe the operator $\bar\partial_b$ and its adjoint $\bar\partial_b^*$ as follows (as before, we always use the identification $u\leftrightarrow u\bar\omega_1$). If f is a smooth function on $\partial\Omega$ and if F is *any* smooth extension of f to $\bar\Omega$, then

$$\bar\partial_b(f)=\bar\partial(F)|_{\partial\Omega}=\bar L_1(F)|_{\partial\Omega}\,\bar\omega_1=\bar L_1(f)\,\bar\omega_1\leftrightarrow\bar L_1(f). \tag{4.11}$$

Next, $\mathscr{B}^{(0,0)}$ and $\mathscr{B}^{(0,1)}$ are pre-Hilbert spaces. In fact there is a smooth, strictly positive function W on $\partial\Omega$ so that if $d\sigma$ denotes Euclidean surface area measure on $\partial\Omega$, then for $f,g\in\mathscr{B}^{(0,0)}$

$$(f,g)_0=\int_{\partial\Omega} f(\zeta)\,\overline{g(\zeta)}\,W(\zeta)\,d\sigma(\zeta);$$

and for Θ, $\Psi\in\mathscr{B}^{(0,1)}$ with $\Theta=\theta\bar\omega_1$ and $\Psi=\psi\bar\omega_1$,

$$(\Theta,\Psi)_1=\int_{\partial\Omega}\theta(\zeta)\,\overline{\psi(\zeta)}\,W(\zeta)\,d\sigma(\zeta).$$

Now if $u\leftrightarrow u\bar\omega_1\in\mathscr{B}^{0,1}$ then the adjoint $\bar\partial_b^*(u)$ is computed using this inner product, and integration by parts shows that there is a smooth function h so that

$$\bar\partial_b^*(u)=(-L_1+h)(u). \tag{4.12}$$

In fact we can say more. Denote by $\mathcal{M}_\eta$ the operation of multiplication by a function η; i.e.

$$\mathcal{M}_\eta(f)=\eta f. \tag{4.13}$$

Then we have

PROPOSITION 4.2. *There is a smooth real valued non-vanishing function η on $\partial\Omega$ such that*

$$\bar{L}_1^* = -\mathcal{M}_{\eta^{-1}} L_1 \mathcal{M}_\eta.$$

Proof. We begin by working near a fixed point on $\partial\Omega$. After a translation and rotation, we may assume that there is a neighborhood V of $0 \in \partial\Omega$ and a smooth function $h(x, y, t)$ such that

$$\partial\Omega \cap V = \{(z_1, z_2) \in V \mid \Im(z_2) = h(\Re(z_1), \Im(z_1), \Re(z_2))\}.$$

We can identify $\partial\Omega \cap V$ with $\mathbf{R}^3$ via the identification

$$\mathbf{R}^3 \ni (x, y, t) \leftrightarrow (x+iy, t+ih(x, y, t)) \in \partial\Omega.$$

Moreover, Euclidean surface area measure on $\partial\Omega \cap V$ is just

$$g(x, y, t)\, dx \wedge dy \wedge dt = \sqrt{1+|\nabla h(x, y, t)|^2}\, dx \wedge dy \wedge dt.$$

Near 0, the function

$$\varrho_1 = h\left(\frac{z_1+\bar{z}_1}{2}, \frac{z_1-\bar{z}_1}{2i}, \frac{z_2+\bar{z}_2}{2}\right) - \frac{z_2-\bar{z}_2}{2i}$$

is also a defining function for $\partial\Omega$, and hence

$$\varrho = \varphi\varrho_1$$

where φ is a positive real function. Observe that if we write $y_2 = \Im(z_2)$, then on $\partial\Omega$,

$$\varphi = -\frac{\partial\varrho}{\partial y_2}.$$

But since

$$\varrho(x+iy, t+ih(x, y, t)) = 0,$$

it follows easily that

$$\left|\frac{\partial\varrho}{\partial y_2}(x+iy, t+ih(x+iy, t))\right| \sqrt{1+|\nabla h(x, y, t)|^2} = |\nabla\varrho(x+iy, t+ih(x, y, t))|.$$

Our first object is to see what form the operator $\bar{\partial}_b$ takes in these local coordinates. We shall write $z = x+iy$, and $h(x, y, t) = h(z, t)$. Then it is easy to check that

$$\bar{L}_1 = \frac{\partial\varrho}{\partial\bar{z}_2}\frac{\partial}{\partial\bar{z}_1} - \frac{\partial\varrho}{\partial\bar{z}_1}\frac{\partial}{\partial\bar{z}_2} \leftrightarrow \frac{\varphi(z, t)}{2}\left[\left(\frac{\partial h}{\partial t}(z, t) - i\right)\frac{\partial}{\partial\bar{z}} - \frac{\partial h}{\partial\bar{z}}(z, t)\frac{\partial}{\partial t}\right].$$

Similarly,

$$L_1 = \frac{\partial\varrho}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial\varrho}{\partial z_1}\frac{\partial}{\partial z_2} \leftrightarrow \frac{\varphi(z,t)}{2}\left[\left(\frac{\partial h}{\partial t}(z,t)+i\right)\frac{\partial}{\partial z} - \frac{\partial h}{\partial z}(z,t)\frac{\partial}{\partial t}\right].$$

Let $\eta=\frac{1}{2}\varphi Wg$, and let $\mathcal{M}_\eta$ denote the operator of multiplication by η. Note that by the computation above,

$$\eta = \frac{1}{2}W|\nabla\varrho|,$$

and hence is a globally defined positive function on $\partial\Omega$.

If (say) Θ has small support near zero, we compute

$$\begin{aligned}(\bar{L}_1(f),\Theta)_1 &= \int_{\mathbf{C}\times\mathbf{R}} \frac{\varphi(z,t)}{2}\left[\left(\frac{\partial h}{\partial t}-i\right)\frac{\partial f}{\partial \bar z} - \frac{\partial h}{\partial \bar z}\frac{\partial f}{\partial t}\right](z,t)\,\overline{\theta(z,t)}\,W(z,t)\,g(z,t)\,dx\wedge dy\wedge dt\\ &= \int_{\mathbf{C}\times\mathbf{R}} \left[\left(\frac{\partial h}{\partial t}-i\right)\frac{\partial f}{\partial \bar z} - \frac{\partial h}{\partial \bar z}\frac{\partial f}{\partial t}\right](z,t)\,\overline{\mathcal{M}_\eta\theta(z,t)}\,dx\wedge dy\wedge dt\\ &= -\int_{\mathbf{C}\times\mathbf{R}} f(z,t)\,\overline{\left[\left(\frac{\partial h}{\partial t}+i\right)\frac{\partial}{\partial z} - \frac{\partial h}{\partial z}\frac{\partial}{\partial t}\right](\mathcal{M}_\eta\theta)}\;dx\wedge dy\wedge dt,\end{aligned}$$

since

$$\operatorname{div}\left[\left(\frac{\partial h}{\partial t}-i\right)\frac{\partial}{\partial \bar z} - \frac{\partial h}{\partial \bar z}\frac{\partial}{\partial t}\right] = 0.$$

Hence

$$\begin{aligned}(\bar{L}_1(f),\Theta)_1 &= -\int_{\mathbf{C}\times\mathbf{R}} f(z,t)\,\overline{\mathcal{M}_{\eta^{-1}}L_1\mathcal{M}_\eta\theta}\,W(z,t)\,g(z,t)\,dx\wedge dy\wedge dt\\ &= -(f,\mathcal{M}_{\eta^{-1}}L_1\mathcal{M}_\eta\Theta)_0.\end{aligned}$$

A general Θ can be written as a sum of forms with small support by using a partition of unity, and this then completes the proof.

We consider the operator $\bar\partial_b$ mapping functions on $\partial\Omega$ to one-forms (i.e. elements of $\mathcal{B}^{0,1}$) and the adjoint operator $\bar\partial_b^*$ mapping elements of $\mathcal{B}^{0,1}$ to functions on $\partial\Omega$. We denote by S_0 the Szegö projection operator—i.e. the orthogonal projection of $L^2(\partial\Omega)$ onto the null space of $\bar\partial_b$. Similarly we denote by S_1 the orthogonal projection operator on the null space of $\bar\partial_b^*$. Since by our choice of basis $\bar\omega_1$ and $\bar\omega_2$ and the definition of $\mathcal{B}^{0,1}$, we can identify $\mathcal{B}^{0,1}$ with functions on $\partial\Omega$ and its L^2 completion with $L^2(\partial\Omega)$, which allows us to realize S_1 as an orthogonal projection on $L^2(\partial\Omega)$. Finally, we define the operator

$$\Box_b^1 = \bar{\partial}_b \bar{\partial}_b^* : \mathscr{B}^{0,1} \to \mathscr{B}^{0,1}, \tag{4.14}$$

which we can also identify with a differential operator on scalar-valued functions on $\partial\Omega$.

It follows immediately from Proposition 4.2 that we have the following:

COROLLARY 4.3. *There is a smooth real valued non-vanishing function η on $\partial\Omega$ so that*

$$\bar{S}_0 = \mathcal{M}_{\eta^{-1}} S_1 \mathcal{M}_\eta.$$

LEMMA 4.4. *If we identify $\Box_b^1$ with a scalar operator, then*

$$\Box^+\Box^- = \frac{1}{2}\Box_b^1 + L_1^0 F_1 + \bar{L}_1^0 F_2 + F_3, \tag{4.15}$$

where F_j are pseudodifferential operators of order zero for $j=1,2,3$.

Proof. From Proposition 3.5 we see that $\Box^+\Box^- = -\frac{1}{2}\bar{L}_1^0 L_1^0 +$errors, and from equations (4.11) and (4.12) we see that $\bar{\partial}_b\bar{\partial}_b^* = -\bar{L}_1^0 L_1^0 +$errors, where "errors" indicate a term of the form $L_1^0 F_1 + \bar{L}_1^0 F_2 + F_3$. This completes the proof.

The class of NIS operators is designed to describe the Szegö projection and the relative fundamental solution for $\bar{\partial}_b^*$.

THEOREM 4.5. (1) *If K is an NIS operator of order m, and if $\mathcal{M}_\eta$ denotes the operator of multiplication by η as above, then the commutator $[K, \mathcal{M}_\eta]$ is an NIS operator of order $m+1$.*

(2) *The operators S_0 and S_1 are NIS operators of order zero, and $\bar{S}_0 - S_1$ is an NIS operator of order -1.*

(3) *There are operators K_0: $\mathscr{B}^{0,0} \to \mathscr{B}^{0,1}$ and K_1: $\mathscr{B}^{0,0} \to \mathscr{B}^{0,0}$ such that $K_0 = K_1^*$ and*

$$\begin{aligned} &\bar{\partial}_b K_1 = (K_1)^* \bar{\partial}_b^* = I - S_1; \quad K_1 S_1 = 0; \quad S_0 K_1 = 0; \\ &\bar{\partial}_b^* K_0 = (K_0)^* \bar{\partial}_b = I - S_0; \quad K_0 S_0 = 0; \quad S_1 K_0 = 0. \end{aligned} \tag{4.16}$$

Moreover, if we identify the form $u_1 \bar{\omega}_1 \in \mathscr{B}^{0,1}$ with the function $u_1 \in \mathscr{B}^{0,0}$, we can regard K_0 and K_1 as scalar operators, and they are then NIS operators of order 1.

(4) *If we define* $K=K_0K_1$, *then* K *is an NIS operator of order* 2 *and*

$$\Box_b^1 K = K\Box_b^1 = I - S_1. \tag{4.17}$$

Proof. The proof of (1) is in [NRSW], p. 134. The proof that S_0 is an NIS operator of order zero is contained in [NRSW], § 5. The existence of K_0 and K_1 and the proof that their kernels satisfy the appropriate size estimates is contained in Christ [C]. To show that in fact they are NIS operators, i.e. that they have the appropriate size when applied to bump functions, one uses the same kind of homogeneity arguments used in [NRSW] to deal with the operator S_0. The fact that $S_1-\bar{S}_0$ is an NIS operator smoothing of order 1 follows from part (1) of the theorem, and Corollary 4.3. Thus the proof of parts (2) and (3) of the theorem will be complete when we show that $K_1=K_0^*$.

Now

$$\begin{aligned} K_1 &= K_1 - S_0 K_1 = (I-S_0)K_1 = K_0^*\bar{\partial}_b K_1 \\ &= K_0^*(I-S_1) = K_0^* - K_0^* S_1 = K_0^* \end{aligned} \tag{4.18}$$

so

$$K_1 = K_0^*. \tag{4.19}$$

Finally, to prove part (4), note that

$$\begin{aligned} \Box_b^1 K &= \bar{\partial}_b \bar{\partial}_b^* K_0 K_1 \\ &= \bar{\partial}_b (I-S_0) K_1 \\ &= \bar{\partial}_b K_1 \\ &= I - S_1, \end{aligned} \tag{4.20}$$

while

$$\begin{aligned} K\Box_b^1 &= K_0 K_1 \bar{\partial}_b \bar{\partial}_b^* \\ &= K_0 (I-S_0) \bar{\partial}_b^* \\ &= K_0 \bar{\partial}_b^* \\ &= I - S_1. \end{aligned} \tag{4.21}$$

This completes the proof.

In order to construct an approximate inverse for $\Box^+$ we need one further result.

LEMMA 4.6. *The operators* $S_1\Gamma^+$ *and* Γ^+S_1 *are infinitely smoothing operators* (*i.e. their distribution kernels on* $\partial\Omega\times\partial\Omega$ *are infinitely differentiable*).

Proof. This is based on ideas of Kohn contained in [K3], Theorem 1.18. However, since the statement of Lemma 4.6 does not appear in [K3], we sketch the main ideas in the proof.

The operators $S_1\Gamma^+$ and Γ^+S_1 are essentially adjoints of each other so it suffices to show that Γ^+S_1 is infinitely smoothing. This is equivalent to showing that $\overline{\Gamma^+S_1}$ is infinitely smoothing. But, by Corollary 4.3,

$$\overline{\Gamma^+S_1}=\Gamma^-\tilde{S}_1=\Gamma^-\mathcal{M}_\eta S_0\mathcal{M}_{\eta^{-1}},$$

and the operator $\Gamma^-\mathcal{M}_\eta$ has the same properties as the operator Γ^-. Thus our main objective is to prove the estimates

$$\|\Gamma^-S_0(u)\|_s\leq C_s\|u\|_0 \tag{4.22}$$

for every $s>0$. Here $\|\cdot\|_s$ is the norm in the standard Sobolev space L^2_k. Suppose for a moment that this is established.

Then, since $S_0\Gamma^--(\Gamma^-S_0)^*=S_0\Gamma^--S_0(\Gamma^-)^*$ is an infinitely smoothing operator, it follows by duality that for every $t>0$ we would have

$$\|S_0\Gamma^-(u)\|_0\leq C_t\|u\|_{-t}. \tag{4.23}$$

On the other hand, the operator Γ^- is bounded on all the isotropic Sobolev spaces, while the operator S_0 is bounded from the Sobolev space L^2_{mk} to L^2_k by Theorem 4.5, (1) and Theorem C, (2) and the remarks following Theorem C. Hence we have

$$\|S_0\Gamma^-(u)\|_k\leq C_k\|u\|_{mk}. \tag{4.24}$$

By interpolation, this would imply that for any $s>0$,

$$\|S_0\Gamma^-(u)\|_s\leq C_s\|u\|_0, \tag{4.25}$$

which is the analogue of equation (4.24). Again by duality, we would have for any $t>0$,

$$\|\Gamma^-S_0(u)\|_0\leq C_t\|u\|_{-t}, \tag{4.26}$$

and a final interpolation argument between equations (4.23) and (4.25), and between equations (4.22) and (4.26) shows that the two operators are indeed infinitely smoothing. Thus it remains to establish equation (4.22).

This in turn would follow from the inequality

$$\|\Gamma^-(v)\|_s \leq C_s[\|\bar{\partial}_b(v)\|_{s-1/m}+\|v\|_0], \tag{4.27}$$

since we could then apply (4.27) to $v=S_0(u)$, and use the fact that $\bar{\partial}_b S_0=0$. But equation (4.27) follows as in the proof of Theorem 1.18 in [K3]. By compactness of $\partial\Omega$ it sufficies to prove only a localized version of (4.27). One observes that in a sufficiently small neighborhood of the point p, the symbol of the operator Γ^- vanishes on the set $\{\xi_3 \geq -\sqrt{\xi_1^2+\xi_2^2}\}$. But as in [K3], we have that for f with compact support

$$\begin{aligned}\|L_1^0 f\|^2 &= \|\bar{L}_1^0 f\|^2+(f,[L_1^0,\bar{L}_1^0]f)+O(\|f\|^2+\|f\|\,\|L_1^0 f\|)\\ &= \|\bar{L}_1^0 f\|^2+\left(f,\frac{1}{i}\lambda T^0 f\right)+O(\|f\|^2+\|f\|\,\|L_1^0 f\|).\end{aligned} \tag{4.28}$$

But

$$\sigma\left(\frac{1}{i}\lambda T^0\right)=\lambda\xi_3, \tag{4.29}$$

and so is negative where the symbol of Γ^- is supported. Thus as in [K3], we can estimate $\|L_1^0(\Gamma^+ v)\|$ in terms of $\|\bar{L}_1^0(\Gamma^+ v)\|$, and then the finite type hypothesis gives us equation (4.27). This completes the sketch of the proof.

We are finally in a position to write down a parametrix for the operator $\Box^+$:

THEOREM 4.7. *The operator $\Box^- K\Gamma^+ + Q^+$ is a right parametrix for $\Box^+$ in the sense that*

$$\Box^+(\Box^- K\Gamma^+ + Q^+) = I+E,$$

where

$$E=\sum_{j=1}^{3} E_j^0 K_j \bar{E}_j^0 + E^{-\infty},$$

where K_j are NIS operators smoothing of order 1, E_j^0 and $\bar{E}_j^0$ are standard pseudodifferential operators of order zero, and $E^{-\infty}$ is an infinitely smoothing operator. Similarly, $\Gamma^+ K\Box^- + \tilde{Q}^+$ is a left parametrix for $\Box^+$ in the sense that

$$(\Gamma^+ K\Box^- + \tilde{Q}^+)\Box^+ = I+\tilde{E},$$

where $\tilde{E}$ is as above.

Proof. It follows from Proposition 4.1, Lemma 4.4, and equation (4.17) that

$$\Box^+(\Box^- K\Gamma^+ + Q^+) = I - S_1\Gamma^+ + F_1 L_1^0 K\Gamma^+ + F_2\bar{L}_1^0 K\Gamma^+ + F_3 K\Gamma^+ + E^{-\infty},$$

where $E^{-\infty}$ stands for an infinitely smoothing operator, and each F_j, is a standard pseudodifferential operator of order zero. But $L_1^0 K$ and $\bar{L}_1^0 K$ are NIS operators smoothing of order 1, and since Theorem 4.5, part (1) and Lemma 4.6 shows that $S_1\Gamma^+ = E^{-\infty}$, this completes the proof of the first identity. The second identity for the left parametrix is proved in exactly the same way.

These parametrices give inverses for $\Box^+$ up to an error which are smoothing of order 1. In the usual way, we can iterate the argument to obtain parametrices in which the errors are smoothing of any desired finite order. Thus if E is an operator, define an operator

$$E_k = -E + E^2 - E^3 + \ldots + (-1)^k E^k.$$

Then

$$(I+E_k)(I+E) = I + (-1)^k E^{k+1}.$$

COROLLARY 4.8. *Each of the following operators is a product of* $k+1$ *NIS operators which are smoothing of order* 1 *and standard pseudodifferential operators of order* 0*:*

$$\Box^+(\Box^- K\Gamma^+ + Q^+)(I+E_k) - I;$$

$$(I+\tilde{E}_k)(\Gamma^+ K\Box^- + \tilde{Q}^+)\Box^+ - I.$$

§ 5. A parametrix for the $\bar{\partial}$-Neumann problem

The object of this section is to write down an explicit formula which gives a parametrix for the $\bar{\partial}$-Neumann problem. This perhaps requires a word of explanation. As we have seen in Section 2, the $\bar{\partial}$-Neumann problem on a bounded domain Ω is the boundary value problem:

$$\begin{aligned} \Box u &= f \quad \text{on } \Omega; \\ u \lrcorner \bar{\partial}\varrho &= 0 \quad \text{on } \partial\Omega; \\ \bar{\partial}u \lrcorner \bar{\partial}\varrho &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

If the domain Ω is smoothly bounded, pseudoconvex, and of finite type, there is a unique solution

$$u = N(f)$$

for given smooth data f, and the L^2 regularity of the Neumann operator N was proved by Kohn in [K2]. Our object in this section is to find an explicit approximation N_a to the operator N so that $N - N_a$ is a smoothing operator of arbitrary but fixed high order.

To do this, let us fix a Poisson operator P and a Green's operator G for the elliptic system $\Box$ on the domain Ω. Thus if R is the restriction operator to the boundary $\partial\Omega$, the operators P, G, R satisfy

$$P: C^\infty(\partial\Omega)_{(0,1)} \to C^\infty(\bar{\Omega})_{(0,1)}$$

$$G: C^\infty(\bar{\Omega})_{(0,1)} \to C^\infty(\bar{\Omega})_{(0,1)}$$

$$R: C^\infty(\bar{\Omega})_{(0,1)} \to C^\infty(\partial\Omega)_{(0,1)}$$

and

$$\begin{aligned} \Box \circ P &= T_1 \\ R \circ P &= I + T_2 \\ \Box \circ G &= I + T_3 \\ R \circ G &= T_4 \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} T_1&: C^\infty(\partial\Omega)_{(0,1)} \to C^\infty(\bar{\Omega})_{(0,1)}; \\ T_2&: C^\infty(\partial\Omega)_{(0,1)} \to C^\infty(\partial\Omega)_{(0,1)}; \\ T_3&: C^\infty(\bar{\Omega})_{(0,1)} \to C^\infty(\bar{\Omega})_{(0,1)}; \\ T_4&: C^\infty(\bar{\Omega})_{(0,1)} \to C^\infty(\partial\Omega)_{(0,1)} \end{aligned} \tag{5.3}$$

are infinitely smoothing operators. For the existence of such operators, see for example [GS].

DEFINITION 5.1. *For* $f \in C^\infty(\bar{\Omega})_{(0,1)}$ *set*

$$N_a^k(f) = G(f) + P[[(\Box^- K\Gamma^+ + Q^+)(I + E_k) R(\bar{L}_2 - s) G_1(f)]\,\bar{\omega}_1], \tag{5.4}$$

where $G(f) = G_1(f)\bar{\omega}_1 + G_2(f)\bar{\omega}_2$, *and where* E_k *is the operator from Corollary* 4.8, *and is smoothing of order* k.

The main result of this section is now

THEOREM 5.1. *For any integer k, there is an integer $\bar{k}$, and there is an operator $T_k: C^\infty(\bar{\Omega})_{(0,1)} \to C^\infty(\bar{\Omega})_{(0,1)}$ which is isotropically smoothing of order k so that for $f \in C^\infty(\bar{\Omega})_{(0,1)}$,*

$$N(f) = N_a^{\bar{k}}(f) + T_k(f).$$

Remark. "A is isotropically smoothing of order k" means here that the operator A defines a bounded mapping from the standard Sobolev space $L_r^p(\Omega)$ to the standard Sobolev space $L_{r+k}^p(\Omega)$ for $r \geq 0$ and $1 < p < \infty$. Note that if A is an NIS operator smoothing of order mk, then it is isotropically smoothing of order k.

To establish the relationship between the operators N and N_a, we need the following consequence of Kohn's L^2 estimates for the $\bar{\partial}$-Neumann operator:

LEMMA 5.2. *Let $\Omega \subset \mathbf{C}^2$ be a bounded, pseudoconvex domain with C^∞ boundary $\partial\Omega$ of finite type m. There exist operators*

$$A_1: C^\infty(\bar{\Omega})_{(0,1)} \to C^\infty(\bar{\Omega})_{(0,1)}$$

$$A_2: C^\infty(\partial\Omega) \to C^\infty(\bar{\Omega})_{(0,1)}$$

$$A_3: C^\infty(\partial\Omega)_{(0,1)} \to C^\infty(\bar{\Omega})_{(0,1)}$$

with the following properties:

(1) *There is a positive real number s so that the operators A_j have bounded extensions*

$$A_1: H_k(\bar{\Omega})_{(0,1)} \to H_{k-s}(\bar{\Omega})_{(0,1)}$$

$$A_2: H_k(\partial\Omega) \to H_{k-s}(\bar{\Omega})_{(0,1)}$$

$$A_3: H_k(\partial\Omega)_{(0,1)} \to H_{k-s}(\bar{\Omega})_{(0,1)}$$

for all k.

(2) *If $v, F_1 \in C^\infty(\bar{\Omega})_{(0,1)}$, $F_2 \in C^\infty(\partial\Omega)$, and $F_3 \in C^\infty(\partial\Omega)_{(0,1)}$ satisfy*

$$\begin{aligned} \square v &= F_1 \quad \text{on } \Omega \\ v \lrcorner \bar{\partial}\varrho &= F_2 \quad \text{on } \partial\Omega \\ \bar{\partial} v \lrcorner \bar{\partial}\varrho &= F_2 \quad \text{on } \partial\Omega \end{aligned} \tag{5.5}$$

then

$$v = A_1(F_1) + A_2(F_2) + A_3(F_3).$$

We note that since it is not important for our purposes, we are not concerned here with the optimal value of the number s.

Proof. Our first objective is to reduce the inhomogeneous system (5.5) to the homogeneous $\bar{\partial}$-Neumann problem (5.1). As in Section 2, we choose a neighborhood U of $\partial\Omega$ and a smooth orthonormal basis for (0, 1) forms on U given by $\bar{\omega}_1, \bar{\omega}_2$ with $\bar{\omega}_2 = \sqrt{2}\,\bar{\partial}\varrho$. Now if on U we have

$$V = g_1\bar{\omega}_1 + g_2\bar{\omega}_2$$

then as in Section 3, we see that

$$V \lrcorner \bar{\partial}\varrho|_{\partial\Omega} = \frac{1}{\sqrt{2}}\, g_2$$

and

$$\bar{\partial}V \lrcorner \bar{\partial}\varrho|_{\partial\Omega} = \frac{1}{\sqrt{2}}(\bar{L}_1(g_2) - \bar{L}_2(g_1) + sg_1)\,\bar{\omega}_1.$$

Suppose on $\partial\Omega$ we have

$$\begin{aligned} g_2 &= \sqrt{2}\,F_2 \\ g_1 &= 0 \\ \frac{\partial g_1}{\partial \varrho}\bar{\omega}_1 &= 2\bar{L}_1(F_2) - 2F_3. \end{aligned} \tag{5.6}$$

Since $\bar{L}_1$ is a tangential operator, it follows from the above and (2.6) that if equation (5.6) is satisfied, then

$$V \lrcorner \bar{\partial}\varrho|_{\partial\Omega} = F_2;$$

$$\bar{\partial}V \lrcorner \bar{\partial}\varrho|_{\partial\Omega} = F_3.$$

Now it is easy to construct operators

$$B_j\colon C^\infty(\partial\Omega)\times C^\infty(\partial\Omega)_{(0,1)} \to C^\infty(\bar{\Omega}), \quad j = 1, 2$$

so that if

$$V=B_1(F_2,F_3)\bar{\omega}_1+B_2(F_2,F_3)\bar{\omega}_2 \tag{5.7}$$

then V satisfies equation (5.6). Moreover, the operators B_j extend to bounded operators on Sobolev spaces with only finite loss.

Now suppose v satisfies (5.5) and V is defined by (5.7). If we put

$$u=v-V$$

then $u\in C^\infty(\bar{\Omega})$, and u satisfies the $\bar{\partial}$-Neumann problem (5.1) with

$$f=F_1-\Box(V).$$

The solution to (5.1) is unique, and hence if N is the Neumann operator,

$$v-V=u=N(F_1-\Box(V)).$$

By the regularity results of Kohn, $N\colon C^\infty(\bar{\Omega})_{(0,1)}\to C^\infty(\bar{\Omega})_{(0,1)}$ and N extends to a bounded mapping on Sobolev spaces with gain $1/m$. Hence we have

$$\begin{aligned} v&=B_1(F_2,F_3)\,\bar{\omega}_1+B_2(F_2,F_3)\,\bar{\omega}_2+N(F_1-\Box(B_1(F_2,F_3)\,\bar{\omega}_1+B_2(F_2,F_3)\,\bar{\omega}_2))\\ &=A_1(F_1)+A_2(F_2)+A_3(F_3), \end{aligned}$$

and it is clear that the operators A_j have the required properties. This completes the proof of the lemma.

We now turn to the proof of Theorem 5.1. For this we need to examine equations (5.2) in greater detail. If $f=f_1\bar{\omega}_1+f_2\bar{\omega}_2\in C^\infty(\bar{\Omega})_{(0,1)}$, we write

$$\begin{aligned} G(f)&=[G_{11}(f_1)+G_{12}(f_2)]\,\bar{\omega}_1+[G_{21}(f_1)+G_{22}(f_2)]\,\bar{\omega}_2\\ &=G_1(f)\bar{\omega}_1+G_2(f)\bar{\omega}_2, \end{aligned}$$

while if $u=u_1\bar{\omega}_1+u_2\bar{\omega}_2\in C^\infty(\partial\Omega)_{(0,1)}$ we write

$$P(u)=[P_{11}(u_1)+P_{12}(u_2)]\,\bar{\omega}_1+[P_{21}(u_1)+P_{22}(u_2)]\,\bar{\omega}_2.$$

It follows from equation (5.2) that the operators $R\circ P_{12}$ and $R\circ P_{21}$ are infinitely smoothing.

For $f\in C^\infty(\bar{\Omega})_{(0,1)}$, define $N_a^k(f)$ as in equation (5.4). Using equations (5.2) we first see that

$$\square(N_a^k(f)) = f + T_5(f) \tag{5.8}$$

where $T_5\colon C^\infty(\bar{\Omega})_{(0,1)} \to C^\infty(\bar{\Omega})_{(0,1)}$ is an infinitely smoothing operator.

Next,

$$\begin{aligned} R(N_a^k(f) \lrcorner \bar{\partial}\varrho) &= R \circ G(f) \lrcorner \bar{\partial}\varrho + R \circ P_{21}(\square^- K\Gamma^+ + Q^+)(I+E^k) R(\bar{L}_2 - s)\, G_1(f) \\ &= T_6(f), \end{aligned}$$

by the above remarks where T_6 is an infinitely smoothing operator.

Finally, we compute $\bar{\partial} N_a^k(f) \lrcorner \bar{\partial}\varrho$ restricted to $\partial\Omega$. From equation (3.20) we have

$$R \circ \bar{\partial} G(f) \lrcorner \bar{\partial}\varrho = \frac{1}{\sqrt{2}} R \circ [\bar{L}_1(G_2(f)) - \bar{L}_2(G_1(f)) + sG_1(f)]\, \bar{\omega}_1.$$

Since $\bar{L}_1$ is a tangential operator and $R \circ G_2$ is an infinitely smoothing operator, it follows that $R \circ \bar{L}_1(G_2(f))$ is infinitely smoothing. Thus

$$R \circ \bar{\partial} G(f) \lrcorner \bar{\partial}\varrho = -\frac{1}{\sqrt{2}} (\bar{L}_2 - s)(G_1(f))\bar{\omega}_1 + T_7(f)$$

where T_7 is an infinitely smoothing operator. Now let us write

$$u_1 = (\square^- K\Gamma^+ + Q^+)(I+E^k) R(\bar{L}_2 - s)(G_1(f)).$$

Then according to equations (3.7) and (3.14) we have

$$\begin{aligned} R \circ \bar{\partial} P(u_1 \bar{\omega}_1) \lrcorner \bar{\partial}\varrho &= \frac{1}{\sqrt{2}} R \circ [(\bar{L}_1 P_{12}(u_1) - \bar{L}_2 P_{11}(u_1) + su_1)\, \bar{\omega}_1] \\ &= \square^+(u_1) + T_8(f) \end{aligned}$$

where T_8 is an infinitely smoothing operator. But according to Corollary 4.8, we have

$$\square^+(u_1) = R(\bar{L}_2 - s)(G_1(f)) + \tilde{T}(f)$$

where $\tilde{T}$ is an operator which is a product of $k+1$ NIS operators which are smoothing of order 1, and standard pseudodifferential operators of order 0. If we take $k \geq m \cdot l$ sufficiently large, then $\tilde{T}$ is "isotropically smoothing of order l" in the sense we are using here. It now follows that

$$R \circ \bar{\partial} N_a^k(f) \lrcorner \bar{\partial}\varrho = \tilde{T}(f) + T_9(f)$$

where $\tilde{T}$ is as above, and T_9 is another infinitely smoothing operator.

We now set $v=N(f)-N_a^k(f)$, where N is the true Neumann operator. Then the computations above show that v satisfies the hypotheses of Lemma 5.2, (2), where the F_j are given in terms of the data f by operators of fixed but high order of smoothing. If we apply Lemma 5.2, we see that

$$N(f) = N_a^k(f)+\tilde{T}(f)$$

where $\tilde{T}$ is smoothing of order k. This completes the proof of Theorem 5.1.

§ 6. Commutation properties

From the results of Section 5, we see that a parametrix for the $\bar{\partial}$-Neumann problem involves two different kinds of operators. The operators $G, P, \Box^{\pm}, \Gamma^{\pm}, Q^{\pm}$ and R are all related to standard differential or pseudodifferential operators. However, the projection operators S_j and the solving operator K are NIS operators of various smoothing degrees. When we consider estimates for the $\bar{\partial}$-Neumann problem, we shall have to control the commutators of S_j and K, not only with the "good" vector fields $\bar{L}$ and L, but also with vector fields which point in "bad" directions like T. In the case that $\partial\Omega$ is strictly pseudoconvex, one can handle the commutators because the operators S_j and K are pseudodifferential operators of the class $S_{1/2,1/2}$. However in the case treated here, these operators are not standard pseudodifferential operators, and we must proceed differently. Thus the main object of this section is to develop the good commutation properties of a certain subalgebra of the algebra of NIS operators which contains the operators we are interested in.

The basic problem we face is showing that if A is an NIS operator and T is a differential operator, then the operator TA can be written as a sum of products of NIS operators with appropriate differential operators on the right. We begin by recalling what happens if the differential operator T is "good".

LEMMA 6.1. *Let A be a NIS operator of order k, and let X denote either the vector field L_1 or the vector field $\bar{L}_1$. Then there are NIS operators A_j, $j=1,2,3$ of smoothing order k so that*

$$XA = A_1L_1+A_2\bar{L}_1+A_3.$$

This follows from Lemma 4.1 and the basic facts about compositions of NIS operators in [NRSW].

We next turn to the study of the commutation properties of a differential operator

T with the projections S_0 and S_1. Recall that S_0 is the orthogonal projection of $L^2(\partial\Omega)$ to the kernel of $\bar{L}_1$, and S_1 is the orthogonal projection on the kernel of $\bar{L}_1^*$. We need a preliminary lemma:

LEMMA 6.2. *There exist vector fields M_1, M_2, and M_3 on $\partial\Omega$ so that:*

(1) *The vector fields M_1, M_2, and M_3 span the complexified tangent space at each point $p \in \partial\Omega$.*

(2) *If $S_0F=F$, then $S_0M_jF=M_jF$ for $j=1,2,3$.*

Proof. We let $M_3=\bar{L}_1$, and we try to find M_1 and M_2 of the form

$$\begin{aligned} M_1 &= \frac{\partial}{\partial z_1} + A_1\frac{\partial}{\partial \bar{z}_1} + A_2\frac{\partial}{\partial \bar{z}_2} \\ M_2 &= \frac{\partial}{\partial z_2} + B_1\frac{\partial}{\partial \bar{z}_1} + B_2\frac{\partial}{\partial \bar{z}_2} \end{aligned} \tag{6.1}$$

where A_j, B_j are smooth functions to be determined. In order that M_1 and M_2 be tangential operators, one needs

$$\begin{aligned} A_1\frac{\partial\varrho}{\partial \bar{z}_1} + A_1\frac{\partial\varrho}{\partial \bar{z}_2} &= -\frac{\partial\varrho}{\partial z_1} \\ B_1\frac{\partial\varrho}{\partial \bar{z}_1} + B_2\frac{\partial\varrho}{\partial \bar{z}_2} &= -\frac{\partial\varrho}{\partial z_2}. \end{aligned} \tag{6.2}$$

Since $\nabla\varrho \neq 0$ on $\partial\Omega$, one can choose smooth functions A_j, B_j so that (6.2) is satisfied. It is then clear that M_1, M_2, and $\bar{L}_1$ are linearly independent over $\mathbf{C}$, so that condition (1) is satisfied.

On the other hand, since $\bar{L}_1$ involves only the derivatives $\partial/\partial\bar{z}_1$ and $\partial/\partial\bar{z}_2$, it follows that $[\bar{L}_1, M_j]$ must be linear combinations of $\partial/\partial\bar{z}_1$ and $\partial/\partial\bar{z}_2$. Since these vector fields are also tangential, they must be smooth multiples of $\bar{L}_1$, which proves that for $j=1,2$ there is a smooth function φ_j so that

$$[\bar{L}_1, M_j] = \varphi_j \bar{L}_1.$$

Now suppose that $S_0F=F$. This is equivalent to saying that $\bar{L}_1F=0$. The same is then clearly true of $\bar{L}_1F$, and we also have

$$\bar{L}_1 M_j(F) = [\bar{L}_1, M_j](F) = \varphi_j \bar{L}_1(F) = 0 \quad \text{for } j=1,2.$$

This proves condition (2), and finishes the proof of the lemma.

COROLLARY 6.3. *There exist vector fields V_1, V_2, and V_3 and smooth functions ψ_1, ψ_2, and ψ_3 on $\partial\Omega$ so that:*

(1) The vector fields V_1, V_2, and V_3 span the complexified tangent space at each point $p \in \partial\Omega$.

(2) If $S_0 F=0$, then $S_0 V_j F = S_0 V_j F$ for $j=1,2,3$.

This follows from Lemma 6.2 if we let $V_j^* = M_j + \psi_j$. We now study the commutator of S_0 with a differential operator.

LEMMA 6.4. *Let T be any first order differential operator and k any positive integer. Then there exist NIS operators $A_1, \ldots, A_n$ of smoothing order $\geqslant 1$, an NIS operator A_0 of smoothing order $\geqslant 0$, differential operators $T_1, \ldots, T_n$ of order 1, and an operator E which is smoothing of order k so that*

$$[T, S_0] = \sum_{j=1}^{n} A_j T_j + A_0 + E. \tag{6.3}$$

Proof. Using Lemma 6.2, we can write

$$T = \sum_{j=1}^{3} b_j M_j.$$

Thus by Lemma 6.2, (2) we have, assuming $S_0(F)=F$,

$$\begin{aligned} T(F) &= \sum_{j=1}^{3} b_j M_j(F) \\ &= \sum_{j=1}^{3} b_j S_0(M_j(F)) \\ &= \sum_{j=1}^{3} S_0(b_j M_j(F)) + \sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}] M_j(F) \\ &= S_0 T(F) + \sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}] M_j(F) \end{aligned} \tag{6.4}$$

where $\mathcal{M}_{b_j}$ denotes the operator of multiplication by b_j. Now letting $F=S_0(f)$, we see that we have the identity of operators:

$$[T, S_0]S_0 = \sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}] S_0 M_j - \sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}][M_j, S_0]. \tag{6.5}$$

Next let $(I-S_0)F=F$, i.e. $S_0(F)=0$. Then

$$[T, S_0]F = -S_0TF.$$

This time, using Corollary 6.3 we write

$$T = \sum_{j=1}^{3} b_j V_j.$$

Then

$$\begin{aligned}[T, S_0]F &= -S_0\left(\sum_{j=1}^{3} b_j V_j\right)F \\ &= -\sum_{j=1}^{3} b_j S_0 V_j F - \sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}]\, V_j F \\ &= -\sum_{j=1}^{3} b_j S_0 \psi_j F - \sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}]\, V_j F,\end{aligned} \tag{6.6}$$

Now letting $F=(I-S_0)(f)$, we see that we have the identity of operators:

$$\begin{aligned}[T, S_0](I-S_0) = &-\sum_{j=1}^{3} b_j S_0 \psi_j (I-S_0) - \sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}]\, V_j \\ &+\sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}] S_0 V_j + \sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}][V_j, S_0].\end{aligned} \tag{6.7}$$

Now adding equations (6.5) and (6.7) we obtain

$$\begin{aligned}[T, S_0] = &\sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}]\, S_0 M_j - \sum_{j=1}^{3} b_j S_0 \psi_j (I-S_0) - \sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}]\, V_j \\ &+\sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}] S_0 V_j - \sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}][M_j, S_0] + \sum_{j=1}^{3} [S_0, \mathcal{M}_{b_j}][V_j, S_0].\end{aligned} \tag{6.8}$$

We can now iterate this identity by in effect replacing V_j by T, and inserting this in (6.8). If we do this iteration a finite number of times we ultimately obtain the desired conclusion of the lemma, and this completes the proof.

COROLLARY 6.5. *If S_1 is the orthogonal projection onto the kernel of $\bar{L}_1^*$, if T is any first order differential operator and k is a positive integer, there exist NIS operators*

$A_1, \ldots, A_n$ of smoothing order $\geqslant 1$, an NIS operator A_0 of smoothing order $\geqslant 0$, differential operators $T_1, \ldots, T_n$ of order 1, *and an operator E which is smoothing of order k so that*

$$[T, S_1] = \sum_{j=1}^{n} A_j T_j + A_0 + E. \tag{6.9}$$

A similar identity holds for the commutator of a differential operator T with either of the operators $\bar{S}_0$ or *$\bar{S}_1$.*

Proof. The identity for the operator $\bar{S}_0$ follows by conjugating equation (6.3). Identity (6.9) then follows immediately since the operator S_1 is just the operator $\bar{S}_0$ conjugated by a multiplication operator, according to Corollary 4.3.

Now let K_1 be the relative fundamental solution operator for the operator $\bar{L}_1$, so that we have:

$$\begin{aligned} \bar{L}_1 K_1 &= I - S_1 \\ K_1 \bar{L}_1 &= I - S_0 \\ K_1 S_1 &= 0 \\ S_0 K_1 &= 0. \end{aligned} \tag{6.10}$$

LEMMA 6.6. *Let T be any first order differential operator and k a positive integer. There exist NIS operators $A_1, \ldots, A_n$ of smoothing order $\geqslant 2$, a NIS operator A_0 of smoothing order $\geqslant 1$, differential operators $T_1, \ldots, T_n$ of order* 1, *and an operator E which is smoothing of order k so that*

$$[T, K_1] = \sum_{j=1}^{n} A_j T_j + A_0 + E. \tag{6.11}$$

A similar identity holds with K_1^* or $\bar{K}_1$ in place of K_1.

Proof. Writing $\tilde{T} = [\bar{L}_1, T]$, then

$$K_1 \bar{L}_1 T K_1 - K_1 T \bar{L}_1 K_1 = K_1 \tilde{T} K_1.$$

But, $K_1 \bar{L}_1 = I - S_0, \bar{L}_1 K_1 = I - S_1$, by Theorem 4.5, so

$$(I-S_0)TK_1-K_1T(I-S_1)=K_1\tilde{T}K_1,$$

and

$$[T,K_1]=S_0TK_1-K_1TS_1+K_1\tilde{T}K_1.$$

Next,

$$S_0TK_1=TS_0K_1+[S_0,T]K_1,$$

but $S_0K_1=0$ so $S_0TK_1=[S_0,T]K_1$. Similarly, $K_1TS_1=K_1[T,S_1]$. Therefore,

$$[T,K_1]=[S_0,T]K_1-K_1[T,S_1]+K_1\tilde{T}K_1. \tag{6.12}$$

On the other hand, since $S_0K_1=0$ we have

$$S_0[T,K_1]=[S_0,T]K_1, \tag{6.13}$$

and hence according to Lemma 6.4, we have

$$\begin{aligned} S_0[T,K_1] &= \sum_{j=1}^{2} A_jT_jK_1+A_0K_1+EK_1 \\ &= \sum_{j=1}^{2} A_jK_1T_j+A_0K_1+EK_1+\sum_{j=1}^{2} A_j[T_j,K_1]. \end{aligned} \tag{6.14}$$

Now combining equations (6.12), (6.13), (6.14) and Lemma 6.4 we see that there are NIS operators $\tilde{A}_j$ of smoothing order $\geqslant 2$, $\tilde{B}_j$ of smoothing order $\geqslant 1$ and $\tilde{A}_0$ of smoothing order $\geqslant 1$, and differential operators $\tilde{T}_j$ of order 1 and an operator E smoothing of order k so that

$$[T,K_1]=\sum_{j=1}^{n}\tilde{A}_j\tilde{T}_j+\tilde{A}_0+\tilde{E}+\sum_{j=1}^{n}\tilde{B}_j[\tilde{T}_j,K_1]. \tag{6.15}$$

We can now iterate identity (6.15) by replacing $[\tilde{T},K]$ by expressions like (6.15). After a finite number of such iterations, we obtain the first conclusion of Lemma 6.5. The other identities follow in a similar manner. This completes the proof.

We now consider the subalgebra $\mathcal{A}$ of the algebra of NIS operators generated by the operators S_0, S_1, K_1, and their adjoints and complex conjugates, and by all multiplication operators $\mathcal{M}_\varphi$. It is important to note that all the NIS operators A_j of the previous lemmas actually belong to this subalgebra $\mathscr{A}$. The main result of this section deals with the commutator of an arbitrary differential operator of order m with elements of this subalgebra. In order to state this result, we must assign a formal "degree" to every element of $\mathscr{A}$. This is done in the following way:

(1) The operators S_0, S_1 and their conjugates and all multiplication operators $\mathscr{M}_\varphi$ are assigned degree 0.

(2) The operator K_1 and its adjoint and conjugate are assigned degree 1.

(3) If $U, V \in \mathscr{A}$ have degrees r and s, then the product UV is assigned degree $r+s$.

(4) If $U \in \mathscr{A}$ have degrees r and $V=\mathscr{M}_\varphi$, then the commutator $[U, V]$ is assigned degree $r+1$.

It should be pointed out that a given element of $\mathscr{A}$ might have several different representations in terms of products and commutators of the generators, and hence several different degrees might be assigned. If this happens, we agree to assign the *largest* such possible degree to the element.

Remark. It follows from the properties of NIS operators that if an operator $U \in \mathscr{A}$ has degree r in the above sense, then U is an NIS operator of smoothing order r.

We can now state our main result.

THEOREM 6.7. *Let $U \in \mathscr{A}$ have degree n, and let T be a differential operator of order m. Given any positive integer k there are elements $A_j \in \mathscr{A}$ of degree $\geqslant n+1$, $B_l \in \mathscr{A}$ of degree $\geqslant n$, differential operators T_j of order $\leqslant m$ and differential operators Q_l of order $\leqslant m-1$ so that*

$$[T, U] = \sum_j A_j T_j + \sum_l B_l Q_l + E \tag{6.16}$$

where E is smoothing of order k.

Proof. Let $U=S_0, S_1, K_1$ or K_1^*, then by the results (6.3), (6.9) and (6.11) the commutator $[T, U]$ has the expression as follows:

$$[T, U] = \sum_{j=1}^{n} A_j T_j + A_0 + E.$$

Here A_j and E are defined by $[\mathscr{M}_\varphi, U]$ or a product of $[\mathscr{M}_\varphi, U]$. The operator A_0 has the form $\mathscr{M}_\varphi U$. Now we prove the theorem by induction, first on the degree of U, and then on the order of T. We use two identities involving commutators:

$$[T, UV] = [T, U]V + U[T, V], \quad \text{and}$$

$$[T, [U, V]] = [U, [T, V]] - [V, [T, U]].$$

Note that we always have $V = \mathcal{M}_\varphi$ or $V = [\mathcal{M}_\varphi, U]$ in these two identities which allows us to apply the property (4) to gain one more formal "degree". The theorem then follows by direct computation.

Recall that we defined the operator $K = K_0 K_1 = K_1^* K_1$, and K is a relative fundamental solution for $\Box_b^1$. Now we have the following corollary:

COROLLARY 6.8. *Let $q(L_1, \bar{L}_1)$ be a quadratic polynomial in L_1 and $\bar{L}_1$, then the operator $q(L_1, \bar{L}_1)K$ extends to a bounded operator from $L_k^p(\partial\Omega)$ to itself for $1 < p < \infty$ and $k = 0, 1, 2, \ldots$.*

Proof. Since $K = K_1^* K_1$, then $K \in \mathcal{A}$ is of degree 2 by the property (3). It follows that $q(L_1, \bar{L}_1)K$ is an NIS operator of order zero. Therefore it satisfies all the properties required by the non-isotropic version of the David–Journé theorem (see [DJS]). Thus $q(L_1, \bar{L}_1)K$ maps $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$. The estimates on the kernels of these operators then also imply, by the non-isotropic version of the Calderòn–Zygmund theory, that these operators are bounded from $L^p(\partial\Omega)$ to $L^p(\partial\Omega)$, $1 < p < \infty$. Suppose that T is a differential operator of order k. We need to use the commutation properties to study $Tq(L_1, \bar{L}_1)K$. First we know that:

$$T[q(L_1, \bar{L}_1)K] = [q(L_1, \bar{L}_1)T]K + (L_1 T_1)K + (\bar{L}_1 T_2)K + T_3 K. \tag{6.17}$$

Here T_j for $j = 1, 2, 3$ are differential operators of order k. Using the result (6.16) in (6.17), we have

$$T[q(L_1, \bar{L}_1)K] = [q(L_1, \bar{L}_1)K]T + (L_1 K)\tilde{T}_1 + (\bar{L}_1 K)\tilde{T}_2 + K\tilde{T}_3 + \tilde{E},$$

where $\tilde{T}_j$ for $j = 1, 2, 3$ are differential operators of order k and $\tilde{E}$ is a differential operator of order less than k. We can pass k times differentiation to the $L_k^p(\partial\Omega)$ function and get a $L^p(\partial\Omega)$ function. Now the result follows immediately by our previous discussions.

We need one further commutation result, which will later allow us to rewrite the parametrix for the Neumann operator in a convenient way.

LEMMA 6.9.

$$[\Gamma^+, K] = \sum_i C_i D_i F_i \tag{6.18}$$

where the sum involves only finitely many terms, the D_i are pseudodifferential operators of order zero, and the C_i and F_i are NIS operators with C_i smoothing of degree 1, *and F_i smoothing of degree* 2.

Proof. First,

$$-K[\Gamma^+, \Box_b^1]K = [\Gamma^+, K] + K\Gamma^+ S_1 - S_1\Gamma^+ K.$$

According to Lemma 4.6, $S_1\Gamma^+$ and $\Gamma^+ S_1$ are infinitely smoothing. Moreover

$$[\Gamma^+, \Box_b^1] = L_1 D_1 + \bar{L}_1 D_2 + D_3$$

where D_i are order zero pseudodifferential operators. Then the lemma is proved with $F_j = -K$, $j = 1, 2, 3$, $C_1 = KL_1$, $C_2 = K\bar{L}_1$, and $C_3 = K$.

In the formula (5.4), we used $\Box^- K\Gamma^+ + Q^+$ as our "right parametrix" for $\Box^+$ which leads us to the L^p estimates for the Neumann operator N. (See §7 below.) On the other hand, we also need to put the "left parametrix" $\Gamma^+ K\Box^- + \tilde{Q}^+$ and $K\Gamma^+\Box^- + \tilde{Q}^+$ on the right to obtain estimates of Henkin–Skoda type. (See §8 below.)

PROPOSITION 6.10. *The differences*

$$\Gamma^+ K\Box^- - K\Gamma^+\Box^-, \quad \Box^- K\Gamma^+ - \Gamma^+ K\Box^-, \quad \text{and} \quad \Box^- K\Gamma^+ - K\Gamma^+\Box^-$$

are bounded operators from $\Lambda_{\alpha+1}(\partial\Omega)$ to $\Gamma_{\alpha+3-\varepsilon}(\partial\Omega)$, for $\alpha>0$ and $\varepsilon>0$.

Remark. The operators $\Box^- K\Gamma^+$, $\Gamma^+ K\Box^-$, and $K\Gamma^+\Box^-$ themselves can only map $\Lambda_{\alpha+1}(\partial\Omega)$ to $\Gamma_{\alpha+2-\varepsilon}(\partial\Omega)$.

The proof will require the following lemma:

LEMMA 6.11. *Suppose A is a standard pseudodifferential operator of order zero defined on $\partial\Omega$. Then*

$$A: \Gamma_\alpha(\partial\Omega) \to \Gamma_{\alpha-\varepsilon}(\partial\Omega),$$

for $\alpha>0$ and $\varepsilon>0$.

Proof. We first fix $0<\varepsilon<1/m$ where m is the type of the domain. Then it is known that $\Gamma_\alpha \subset \Lambda_{\alpha/m}$ for all $\alpha>0$. Since A is a standard pseudodifferential operator of order zero, then

$$A\colon \Lambda_{\alpha/m} \to \Lambda_{\alpha/m} \subset \Gamma_{\alpha-\varepsilon}, \tag{6.19}$$

if $0<\alpha<m\varepsilon/(m-1)$. Hence this lemma is true for $0<\alpha<m\varepsilon/(m-1)$. Next we consider the space $\Gamma_{1+\alpha}$ with $0<\alpha<m\varepsilon/(m-1)$. We may use the identities

$$L_1^0 A(f) = AL_1^0(f) + B(f),$$

and

$$\bar{L}_1^0 A(f) = A\bar{L}_1^0(f) + \tilde{B}(f).$$

Here B and $\tilde{B}$ are standard pseudodifferential operators at order zero. For the first term of these two identities, we have $f \in \Gamma_{1+\alpha}$ then $L_1^0(f) \in \Gamma_\alpha$ and $\bar{L}_1^0(f) \in \Gamma_\alpha$. We may apply (6.19) to get the right estimates. For the second term, since $\Gamma_{1+\alpha} \subset \Gamma_\alpha$, the estimate is obvious by applying (6.19) again. Now we may apply the interpolation theorem in [NRSW] § 6, to show the lemma is true for $0<\alpha<1+m\varepsilon/(m-1)$. We also can iterate this method to prove the lemma for general α. For a general ε, we just need to use the obvious inclusion relation between Γ_α spaces to prove the lemma.

Proof of Proposition 6.10. The estimate for the difference of

$$\Gamma^+ K \Box^- \quad \text{and} \quad K\Gamma^+ \Box^-$$

follows by Lemma 6.9, Lemma 6.11 and the result (3) of Theorem C in Section 4. Now we consider the difference of $\Box^- K\Gamma^+$ and $\Gamma^+ K \Box^-$. As we have seen in Theorem 4.7,

$$\Box^+(\Box^- K\Gamma^+ + Q^+) = I + E,$$

and

$$(\Gamma^+ K \Box^- + \tilde{Q}^+)\Box^+ = I + \tilde{E}.$$

It follows that

$$\Box^- K\Gamma^+ = \Gamma^+ K \Box^- + \tilde{E}\Box^- K\Gamma^+ + \Gamma^+ K \Box^- E + Q_1 + E^{-\infty},$$

where $Q_1 = Q^+ + \tilde{Q}^+$ is a standard pseudodifferential operator of order -1. Now let us first look at $\Gamma^+ K \Box^- E$.

We know by the proof of Theorem 4.7, and by Lemma 4.6 that

$$\Box^+(\Box^- K\Gamma^+ = Q^+) = I + F_1 L_1 K\Gamma^+ + F_2 \bar{L}_1 K\Gamma^+ + F_3 K\Gamma^+ + E^\infty.$$

Then the term Γ^+KE equals

$$\Gamma^+K\Box^-(F_1L_1K\Gamma^+ + F_2\bar{L}_1K\Gamma^+ + F_3K\Gamma^+) + E^\infty.$$

Consider,

$$\Gamma^+K\Box^-F_1L_1K\Gamma^+ = \Gamma^+KF_1L_1\Box^-K\Gamma^+ + \Gamma^+K[\Box^-, F_1L_1]K\Gamma^+.$$

We use the identity $\Box^-K\Gamma^+ = \Gamma^+K\Box^- + \tilde{E}\Box^-K\Gamma^+ + \Gamma^+K\Box^-E + Q_1 + E^\infty$, already pointed out above. This allows us to write

$$\Gamma^+K\Box^-F_1L_1K\Gamma^+ = \Gamma^+KF_1L_1\Gamma^+K\Box^-$$

plus other terms which are even more smoothing.

Now if $f\in\Lambda_{\alpha+1}$, then $\Box^-(f)\in\Lambda_\alpha\subset\Gamma_\alpha$. Since L_1K is NIS of smoothing order 1, it maps this to $\Gamma_{\alpha+1}$. By Lemma 6.11, this in turn is mapped by F_1 to $\Gamma_{\alpha+1-\varepsilon}$; again by K to $\Gamma_{\alpha+3-\varepsilon}$, and by Γ^+ to $\Gamma_{\alpha+3-\varepsilon}$. The other terms are dealt with similarly, completing the proof of the proposition.

§ 7. Estimates for the $\bar\partial$-Neumann operator

We shall now state and prove some of the estimates for the $\bar\partial$-Neumann problem that are consequences of the previous sections. We shall use the following notation. The space $L^p_k(\Omega)$ will denote the space of functions on Ω (or forms on Ω, depending on the context) that are in $L^p(\Omega)$, together with all their derivatives up to order k. That is, here we are considering the isotropic L^p_k spaces. Similarly, $\Lambda_\alpha(\Omega)$ will denote the isotropic Lipschitz (Hölder) spaces of exponent α. Also, $\Gamma_\alpha(\Omega)$ will denote the *non-isotropic* Lipschitz spaces, related to those appearing in [NRSW], and defined to consist of those functions (or forms) which belong to $\Gamma_\alpha(\partial\Omega)$, (as defined in [NRSW], § 6), and uniformly so on each of the manifolds $M_t = \{\varrho = t\}$.

THEOREM 7.1. *Suppose N is the Neumann operator, and $q(L_1, \bar{L}_1)$ is a quadratic polynomial in L_1 and $\bar{L}_1$. Then the following operators are bounded on the indicated spaces:*

$$q(L_1, \bar{L}_1)N : L^p_k \to L^p_k, \quad 1 < p < \infty, \ k = 0, 1, 2, \ldots; \tag{7.1}$$

$$\bar\partial N \lrcorner \bar\partial\varrho : L^p_k \to L^p_{k+1}, \quad 1 < p < \infty, \ k = 0, 1, 2, \ldots; \tag{7.2}$$

$$N : \Lambda_\alpha \to \Lambda_{\alpha+2/m} \cap \Gamma_{\alpha+2}, \quad \alpha > 0. \tag{7.3}$$

Proof. (i) *The estimate* (7.1). We recall first some basic facts from the theory of Besov spaces, (see e.g. [GS], Chapter 12). We let $L_k^p(\partial\Omega)$ denote the isotropic space of function on the boundary $\partial\Omega$ which together with their derivatives of order not exceeding k belong to $L^p(\partial\Omega)$. The space $B^p(\partial\Omega)$ arises as a real interpolation space between $L^p(\partial\Omega)$ and $L_1^p(\partial\Omega)$; in fact

$$B^p = [L^p, L_1^p]_{1-1/p,p}.$$

(This is equivalent with the analogue of the approximation property which characterizes B^p.) Another basic property can be stated as follows. Suppose $f \in L_1^p(\Omega)$. Then $R(f) \in B^p(\partial\Omega)$, where R denotes the operator of restriction to the boundary. This holds if $1<p<\infty$. Conversely, suppose P is any Poisson operator of order 0. Then

$$P\colon B^p(\partial\Omega) \to L_1^p(\Omega),$$

again for $1<p<\infty$.

We shall also need a slight generalization of these facts. We define a space B_k^p by

$$B_k^p = [L_k^p(\partial\Omega), L_{k+1}^p(\partial\Omega)]_{1-1/p,p}$$

for any integer $k=0,1,2,\ldots$. Then one has that $f \in L_{k+1}^p(\Omega)$ implies that $R(f) \in B_k^p$; conversely if $f \in B_k^p$, then $P(f) \in L_k^p(\Omega)$ for any Poisson operator of order 1.

We next use the approximate representation of the Neumann operator given by Theorem 5.1, and the commutation property of Γ^+ and K given in Lemma 6.6. The result is that modulo higher order terms of the same character (or terms corresponding to the elliptic problem) we have

$$N_a(f) = P\Box^- K\Gamma^+ R\bar{L}_2 G_1(f). \tag{7.4}$$

We shall need the following lemma:

LEMMA 7.2. *The operator $q(L_1, \bar{L}_1)K$ maps B_k^p to B_k^p boundedly for $1<p<\infty$, and $k=0,1,2,\ldots$.*

Proof. From the Corollary 6.8, we know that $q(L_1, \bar{L}_1)K$ maps $L_k^p(\partial\Omega)$ *to* $L_k^p(\partial\Omega)$. The desired result then is a consequence of the interpolation definition of B_k^p.

Returning to (7.4), and using the fact that $P\Box^-$ is a Poisson operator of order 1, we see that

$$q(L_1,\bar{L}_1)P\Box^- = P\Box^- q(L_1^0,\bar{L}_1^0)+P'L_1^0+P''\bar{L}_1^0+P''' \tag{7.5}$$

where $L_1^0, \bar{L}_1^0$ denote the restrictions of L_1 and $\bar{L}_1$ to the boundary, and P', P'', P''' are Poisson operators of order 1. In fact (7.5) can be verified by an easy application of the product formula for pseudodifferential operators. Thus in analyzing $q(L_1,\bar{L}_1)N_a(f)$ we are led to consider

$$P\Box^- q(L_1^0,\bar{L}_1^0)KR\bar{L}_2G_1(f) \tag{7.6}$$

together with other terms which are even better.

If $f\in L_k^p(\Omega)$ then $\bar{L}_2G_1(f)\in L_{k+1}^p(\Omega)$ by standard estimates (see, e.g. [GS]). Thus $R\bar{L}_2G_1(f)\in B_k^p$, and by the lemma, the same holds after applying $q(L_1^0,\bar{L}_1^0)K$. Finally, since $P\Box^-$ is a Poisson operator of order 1, we get that (7.6) belongs to $L_k^p(\Omega)$. This completes the proof of estimate (7.1).

(ii) *The estimate* (7.3). This comes in two parts. The first is that N maps $\Lambda_\alpha(\Omega)$ to $\Lambda_{\alpha+2/m}(\Omega)$. To prove this it suffices to prove a similar result for the principal term, i.e. for (7.4). The main point here is the following lemma:

LEMMA 7.3. *The operator K maps $\Lambda_\alpha(\partial\Omega)$ to $\Lambda_{\alpha+2/m}(\partial\Omega)$.*

In proving this lemma we may assume that the global type m is strictly larger than 2, for otherwise the result is already contained in [GS], § 14. We prove that K maps $L^\infty(\partial\Omega)$ to $\Lambda_{2/m}(\partial\Omega)$ (note that $2/m<1$). To do this it clearly suffices to show that

$$\int_{\partial\Omega}|K(x_1,y)-K(x_2,y)|\,d\sigma(y)\leq A\|x_1-x_2\|^{2/m} \tag{7.7}$$

where $K(x,y)$ is the kernel of the operator K and $\|\cdot\|$ denotes the Euclidean distance.

Now since the non-isotropic distance ϱ satisfies $\varrho(x_1,x_2)\leq A\|x_1-x_2\|^{1/m}$, we can reduce the estimate of the left side of (7.7) to three integrals,

$$\int_{\varrho(x_1,y)\leq C_1\|x_1-x_2\|^{1/m}}|K(x_1,y)|\,d\sigma(y),$$

$$\int_{\varrho(x_2,y)\leq C_2\|x_1-x_2\|^{1/m}}|K(x_1,y)|\,d\sigma(y),$$

$$\int_{\varrho(x_1,y)\geq C_1\|x_1-x_2\|^{1/m}}|K(x_1,y)-K(x_2,y)|\,d\sigma(y),$$

where C is an appropriately large constant.

Now $|K(x_1,y)|$ is dominated by $A(\varrho(x_1,y))^2V(x_1,y)^{-1}$, and thus by equation (4.5) in [NRSW] we have that the first two integrals are dominated by $A\|x_1-x_2\|^{2/m}$.

For the third integral we use the fact that

$$|K(x_1,y)-K(x_2,y)|\leq A\|x_1-x_2\|\sup_x|\nabla_x K(x,y)|$$

where the supremum is taken over the line segment joining x_1 and x_2. However since $\partial\Omega$ is of type m, any derivative can be expressed in terms of products of at most m factors of L_1 or $\bar{L}_1$. Thus the integrand in the third integral is bounded by

$$A\|x_1-x_2\|\int_{\varrho(x_1,y)\geq C_1\|x_1-x_2\|^{1/m}}\frac{(\varrho(x_1,y))^{2-m}}{V(x,y)}\,d\sigma(y)$$

which since $m>2$ gives the estimate $A\|x_1-x_2\|\cdot\|x_1-x_2\|^{(2-m)/m}=A\|x_1-x_2\|^{2/m}$, by equation (4.5) in [NRSW], concluding the proof of (7.7). Thus we see that K maps $L^\infty(\partial\Omega)$ to $\Lambda_{2/m}(\partial\Omega)$.

Notice that at this stage we have only used the fact that K was an NIS operator of smoothing order 2. If we now invoke the more precise properties of K, and in particular the commutation properties in Section 6, we also see that K maps $L_1^\infty(\partial\Omega)$ (the space of bounded functions whose first derivatives are bounded) to $\Lambda_{1+1/m}(\partial\Omega)$. It then follows by the usual interpolation properties of Λ_α that K maps $\Lambda_\alpha(\partial\Omega)$ to $\Lambda_{\alpha+2/m}(\partial\Omega)$ for $0<\alpha<1$ (see e.g. [GS], § 13). Finally using the commutation properties of K again, we see that the same result holds for any non-integral α, and a last interpolation establishes the desired result for all $\alpha>0$.

With the lemma proved, we return to (7.4). If $f\in\Lambda_\alpha(\Omega)$, then $\bar{L}_2G(f)\in\Lambda_{\alpha+1}(\Omega)$, and hence $R\bar{L}_2G(f)\in\Lambda_{\alpha+1}(\partial\Omega)$. The same is true after applying the zero order standard pseudodifferential operator Γ^+ (see [GS], Lemma (13.5)), and the result is mapped to $\Lambda_{\alpha+1+2/m}(\partial\Omega)$ by K, if we use the lemma proved above. Finally the Poisson operator of order 1, $P\Box^-$ maps this to $\Lambda_{\alpha+2/m}(\Omega)$ (see [GS], § 13). The required Λ_α estimates are therefore proved.

The second part of the estimate (7.3) is that $N(f)\in\Gamma_{\alpha+2}(\Omega)$ whenever $f\in\Lambda_\alpha(\Omega)$. The main point here is contained in the following lemma:

LEMMA 7.4. *Supposte that $f\in\Gamma_\alpha(\partial\Omega)$. Then $P_1(f)\in\Gamma_\alpha(\Omega)$.*

Proof. Consider first the case $0<\alpha<1$. We shall use the ideas in proposition (6.3) of [NRSW]. For each fixed size 2^{-k}, there is given a partition of unity $1=\Sigma_j\Phi_{k,j}$ where the

$\Phi_{k,j}$ are "bump functions" on balls centered at points x_j^k of radius $\approx 2^{-k}$, having bounded overlap, with

$$|L_1\Phi_{k,j}|+|\bar{L}_1\Phi_{k,j}|\leqslant C2^k.$$

Write

$$f=\sum_{k=0}^{\infty}f_k, \quad \text{with}$$

$$f_0=\sum_j \Phi_{0,j}f(x_j^0)$$

$$f_k=\sum_j \Phi_{k,j}f(x_j^k)-\sum_j \Phi_{k-1,j}f(x_j^{k-1}), \quad k=1,2,\dots.$$

Then since $f\in\Gamma_\alpha$ it is easily seen that

$$\begin{aligned} &\|f_k\|\leqslant C2^{-k\alpha} \\ &\|L_1f_k\|_{L^\infty}+\|\bar{L}_1f_k\|_{L^\infty}\leqslant C2^k2^{-k\alpha}. \end{aligned} \tag{7.8}$$

One can also make the crude isotropic estimate

$$\|\nabla f_k\|_{L^\infty}\leqslant C2^{mk}2^{-k\alpha}. \tag{7.9}$$

Now write $F=P_1(f)$ and $F_k=P_1(f_k)$.

Since the Poisson kernel P_1 maps $L^\infty(\partial\Omega)$ to $L^\infty(\Omega)$ (see [GS], Lemma (15.34)), it follows that

$$\|F_k\|_{L^\infty(\Omega)}\leqslant C2^{-k\alpha}. \tag{7.10}$$

Moreover,

$$\begin{aligned} L_1P_1&=P_1L_1^0+P' \\ \bar{L}_1P_1&=P_1\bar{L}_1^0+P'', \end{aligned} \tag{7.11}$$

where P' and P'' are zero order Poisson operators, since L_1 and $\bar{L}_1$ are tangential. Thus

$$\|L_1F_k\|_{L^\infty}\leqslant C2^k2^{-k\alpha}+\|P'(f_k)\|_{L^\infty}.$$

However, a combination of (7.8) and (7.9) shows that

$$\|f_k\|_{\Lambda_\varepsilon}\leqslant C', \quad (\text{whenever } \varepsilon m\leqslant\alpha);$$

this together with the fact that zero order Poisson operators are bounded on Λ_α, $\alpha>0$, gives

$$\|L_1 F_k\|_{L^\infty(\Omega)}+\|\bar{L}_1 F_k\|_{L^\infty(\Omega)}\leq C2^k 2^{-k\alpha}. \tag{7.12}$$

Using the argument in proposition (6.3) of [NRSW] one sees that (7.10) together with (7.12) implies $F=\Sigma_k F_k\in\Gamma_\alpha(\Omega)$.

The commutation property (7.11) allows us to pass from $0<\alpha<1$ to $1<\alpha<2$. The result for $\alpha=1$ then follows by the interpolation property of the Γ_α spaces (Proposition (6.2) in [NRSW]). A similar argument proves the lemma for all α, $0<\alpha<\infty$.

We can now complete the proof of property (7.3) of the theorem. We require that N_a be given in a different form from that which appeared in (7.4). By Proposition 6.10, we use the form whose main term is

$$N_a(f)=P_1K\Gamma^+\Box^-R\bar{L}_2G_1(f). \tag{7.13}$$

We start with $f\in\Lambda_\alpha(\Omega)$. Then by the usual elliptic estimate,

$$\Gamma^+\Box^-R\bar{L}_2G_1(f)\in\Lambda_\alpha(\partial\Omega).$$

However as is easily seen, $\Lambda_\alpha(\partial\Omega)\subset\Gamma_\alpha(\partial\Omega)$, for all α, while K, being an NIS operator of smoothing order 2 maps $\Gamma_\alpha(\partial\Omega)$ to $\Gamma_{\alpha+2}(\partial\Omega)$ (Proposition (6.3) in [NRSW]). Thus an application of Lemma 7.3 concludes the proof for our estimates for $N(f)$ when $f\in\Lambda_\alpha$.

(iii) *The estimate* (7.2). Stripped of all the notation, this is really an elliptic estimate. In effect, $\bar{\partial}N\lrcorner\bar{\partial}\varrho$ is essentially

$$(-\bar{L}_2+s)P_1(\Box^-K\Gamma^++Q^+)R\bar{L}_2G_1.$$

However, by the symbolic calculus and the results of § 1 and § 2, we have that

$$(-\bar{L}_2+s)P_1=P_1\Box^++P',$$

where P' is a Poisson operator of order -1. We insert this in the above, and we get an expression whose main term is $P_1(R\bar{L}_2G_1)$ if we use Theorem 4.7 in § 4. This has the smoothing properties of order -1 (elliptic) operators, and gives the desired conclusion. Thus the proof of Theorem 7.1 is complete.

COROLLARY 7.5. *Suppose f is a* (0, 1) *form with $\bar{\partial}f=0$, and let u be the solution of $\bar{\partial}u=f$ given by $u=\bar{\partial}^*N(f)$. Then*

(a) $L_1(u)$ *and* $\bar{L}_1(u)\in L^p_k$ *iff* $f\in L^p_k$, $1<p<\infty$, $k=0,1,2,\ldots$;
(b) $u\in\Lambda_{\alpha+1/m}\cap\Gamma_{\alpha+1}$ *iff* $f\in\Lambda_\alpha$ *for* $\alpha>0$.

Proof. Part (a) is essentially a corollary of (7.1) of the theorem. The proof of (b) is much the same as the proof of (7.3) of the theorem. In fact, the role of Lemma 7.2 is replaced by the assertion that the operator $\bar{L}_1K$ maps $\Lambda_\alpha(\partial\Omega)$ to $\Lambda_{\alpha+1/m}(\partial\Omega)$, the proof being very similar to that of Lemma 7.2.

COROLLARY 7.6. *The Bergman projection operator is a bounded mapping from* $L^p_k(\Omega)$ *to itself,* $1<p<\infty$, $k=0,1,2,\ldots$.

Proof. One uses the identity (see Kohn [K2]) that

$$B=I-\bar{\partial}^*N\bar{\partial},$$

and then the assertion is proved in the same way as estimate (7.1) of the theorem.

§ 8. Estimates of Henkin–Skoda type

We shall now extend to pseudoconvex domains of finite type in $\mathbf{C}^2$ estimates for solutions of $\bar{\partial}u=f$ proved by Henkin and Skoda in the case of strongly pseudoconvex domains. These estimates are crucial ingredients in proving the sufficiency of the Blaschke-type condition for zeros of holomorphic functions of the Nevanlinna class in Ω, which we take up in the next section.

Recall the definition of $\Lambda(x,\delta)$ made in § 4:

$$\Lambda(x,\delta)=\sum_{j=2}^{m}\Lambda_j(x)\,\delta^j.$$

Let $h\to\mu(x,h)$ be the function inverse to $\delta\to\Lambda(x,\delta)$. Thus clearly

$$\mu(x,h)\approx\min_{2\le j\le m}\left(\frac{h}{\Lambda_j(x)}\right)^{1/j}.$$

If $\varrho(x)$ denotes the distance of $x\in\Omega$ from the boundary, then we let $\mu(x)$ be defined by

$$\mu(x)=\mu(\pi(x),\varrho(x)).$$

Thus $\varrho(x)$ is essentially the radius of the largest "normal" disc in Ω centered at x, while $\mu(x)$ is essentially the radius of the largest "tangential" disc in Ω centered at x. The basic L^1 estimate is as follows:

THEOREM 8.1. *Suppose f is a smooth* $(0,1)$ *form in* Ω. *Then we have the a priori estimate:*

$$\|\bar{\partial}^* N(f)\|_{L^1(\partial\Omega)} \leq C\left[\|f\|_{L^1(\Omega)} + \left\|\left(\frac{\mu}{\varrho}\right) f \wedge \bar{\partial}\varrho\right\|_{L^1(\Omega)}\right]. \tag{8.1}$$

The proof of Theorem 8.1 will be based on the following lemma:

LEMMA 8.2. *Suppose K_1 is an NIS operator of smoothing order* 1 *and let* $f \in L^\infty(\partial\Omega)$. *If* $F = K_1(f)$, *then*

$$|F(x_1) - F(x_2)| \leq A\mu(x_1, \|x_1 - x_2\|), \tag{8.2}$$

where $\|x_1 - x_2\|$ is the Euclidean distance between x_1 and x_2.

Several remarks are in order. One can actually show that F belongs to the nonisotropic Lipschitz space $\Gamma_1(\partial\Omega)$. The estimate (8.2) is the best *isotropic* estimate that can be made for elements of $\Gamma_1(\partial\Omega)$. Observe also that in the strongly pseudoconvex case ($m=2$), the estimate means that $F \in \Lambda_{1/2}$. Note that we have trivially

$$\|F\|_{L^\infty} \leq A\|f\|_{L^\infty}. \tag{8.2'}$$

Proof of (8.2). Let $K_1(x, y)$ be the kernel of the operator K_1. It clearly suffices to show that

$$\int_{\partial\Omega} |K_1(x_1, y) - K_1(x_2, y)|\, d\sigma(y) \leq A\mu(x_1, \|x_1 - x_2\|). \tag{8.3}$$

To do this choose γ so that $\Lambda(x_1, \gamma) = \|x_1 - x_2\|$; then of course $\gamma = \mu(x_1, \|x_1 - x_2\|)$. Notice also that $\gamma \geq c\varrho(x_1, x_2)$, because if we apply the function $\Lambda(x_1, \cdot)$ to both sides we get that this is equivalent with $\|x_1 - x_2\| \geq c\Lambda(x_1, \varrho(x_1, x_2))$, which is indeed the case. Thus for a sufficiently large constant C we can reduce the estimate (8.3) to similar estimates for the following three integrals:

(i) $$\int_{\varrho(x_1, y) \leq C\gamma} |K_1(x_1, y)|\, d\sigma(y);$$

(ii) $$\int_{\varrho(x_1, y) \leq C\gamma} |K_1(x_2, y)|\, d\sigma(y);$$

(iii) $$\int_{\varrho(x_1, y) \geq C\gamma} |K_1(x_1, y) - K_1(x_2, y)|\, d\sigma(y).$$

Since $|K(x_1, y)| \leq A\varrho(x_1, y)/V(x_1, y)$, the first integral is bounded by $A\gamma$, by the use of (4.5) in [NRSW]. A similar bound holds for the second integral, and so we now turn to the integral (iii). In order to make the calculation here we use the coordinate system appearing after (1.9) in [NRSW], which is centered at x_1, and where the "ball" centered at x_1 of "radius" $\varrho(x_1, x_2)$ is given by a box (whose dimensions are essentially $\varrho(x_1, x_2)$ and $\Lambda(x_1, \varrho(x_1, x_2))$). If $\{x_t\}_{0\leq t\leq 1}$ denotes the straight line in this coordinate system joining x_1 to x_2, then each x_t belongs to the same ball.

Now

$$K_1(x_1, y) - K_1(x_2, y) = \int_0^1 \frac{d}{dt}(K_1(x_t, y))\, dt. \tag{8.4}$$

This equals

$$\|x_1 - x_2\| \int_0^1 (T_t K_1)(x_t, y)\, dt,$$

where T_t denotes a family of vector fields, for which one can make uniform estimates (in t) on their coefficients. Now consider $(\Sigma_{k=2}^m \Lambda_k(x_t)\gamma^k)T_t$. Note first,

$$\sum_{k=2}^m \Lambda_k(x_t)\gamma^k = \Lambda(x_t, \gamma) \approx \Lambda(x_1, \gamma) = \|x_1 - x_2\|,$$

because

$$\varrho(x_t, x_1) \leq c\varrho(x_1, x_2) \leq c\gamma.$$

Moreover, by the definition of the quantities $\Lambda_k(x_t)$ we get that

$$\begin{aligned} \left|\left(\sum_{k=2}^m \Lambda_k(x_t)\gamma^k T_t\right)(K_1)(x_t, y)\right| &\leq c\sum_{k=2}^m \gamma^k \sum_{|I|\leq k} |(X^I K_1)(x_t, y)| \\ &\leq c\sum_{k=2}^m \gamma^k \frac{\varrho(x_t, y)^{1-k}}{V(x_t, y)} \\ &\leq c\sum_{k=2}^m \gamma^k \frac{\varrho(x_1, y)^{1-k}}{V(x_1, y)}. \end{aligned}$$

The next to the last inequality follows because K_1 is the kernel of an NIS operator of smoothing order 1, and the last inequality follows because $\varrho(x_t, x_1) \leq \varrho(x_1, x_2)$, and $\varrho(x_1, y) \geq C\gamma \geq (C/c)\varrho(x_1, x_2)$ for those y's under consideration, if we take C sufficiently large. We now insert these estimates into (8.4), and carry out the y integration over the range indicated in (iii). We then get the following estimate for the integral (iii), if we apply (4.6) of [NRSW]:

$$A\int_0^1 \sum_{k=2}^{m} \gamma^k \cdot \gamma^{1-k}\, dt = A\gamma \int_0^1 dt \leq A'\gamma.$$

This proves (8.3), and thus Lemma 8.2 is proved.

We now turn to the proof of Theorem 8.1 and the estimate (8.1). If we write

$$f = f_1\bar{\omega}_1 + f_2\bar{\omega}_2,$$

and

$$u = N(f) = u_1\bar{\omega}_1 + u_2\bar{\omega}_2,$$

then according to (2.11) we must estimate

$$(-L_1+h_1)u_1 + (-L_2+h_2)u_2. \tag{8.5}$$

The right side of (8.1) is essentially

$$\|f_2\|_{L^1(\Omega)} + \left\| \left(\frac{\mu}{\varrho}\right) f_1 \right\|_{L^1(\Omega)}.$$

Now the estimate for (8.5) breaks up into two parts. The first is the estimate for $(-L_2+h_2)u_2$, which in fact is elliptic. Now u_2 is, up to a better error term, $G_{22}(f_2)$; and since u_2 satisfies the Dirichlet boundary condition, we see (using the fact that $L_2 = (1/\sqrt{2})\,\partial/\partial\varrho + iT$), that $(-L_2+h_2)u_2$ becomes

$$-\frac{1}{\sqrt{2}}\frac{\partial}{\partial\varrho} G_{22}(f_2)|_{\varrho=0}.$$

The operator $\partial G_{22}/\partial\varrho|_{\varrho=0}$ maps function on Ω to functions on $\partial\Omega$ and is essentially the adjoint of the Poisson kernel. Since the Poisson integral maps $L^\infty(\partial\Omega)$ to $L^\infty(\Omega)$, we see that the operator $\partial G_{22}/\partial\varrho|_{\varrho=0}$ maps $L^1(\Omega)$ to $L^1(\partial\Omega)$. Thus we have

$$\|(-L_2+h_2)\,u_2\|_{L^1(\partial\Omega)} \leq A\|f_2\|_{L^1(\Omega)}. \tag{8.6}$$

To study $(-L_1+h_1)\,u_1$ we use the approximate Neumann operator given by (7.13). Since L_1 is tangential this gives as the main term for $(-L_1+h_1)u_1|_{\varrho=0}$ the operator

$$-L_1 K\Gamma^+ \square^- R\bar{L}_2 G_1. \tag{8.7}$$

We can now write (8.7) as the sum of two operators, $I+II$. Here

$$I=-L_1K\Gamma^+\Box^-R\bar{L}_2G_{11},\quad \text{and}\quad II=-L_1K\Gamma^+\Box^-R\bar{L}_2G_{12}.$$

Turning to I, this operator, mapping functions on Ω to functions on $\partial\Omega$, must be proved to be bounded from $L^1(\mu/\varrho,\Omega)$ to $L^1(\partial\Omega)$. Let us take the adjoint of the operator (8.7). Since $\bar{L}_2G_{11}$ is a transmission-type pseudodifferential operator of order -1, then the adjoint of $R\bar{L}_2G_{11}$ is a Poisson operator of order zero (see Boutet de Monvel [B]). Also $\Gamma^+\Box^-$ is a pseudodifferential operator (on $\partial\Omega$) of order 1, so combining this with what we just said, the adjoint of $\Gamma^+\Box^-R\bar{L}_2G_{11}$ is a Poisson operator of order 1. Finally $-L_1K$ is an NIS operator of smoothing order 1, so its adjoint is of the same kind. Altogether then the adjoint of (8.7) is the operator (mapping functions on $\partial\Omega$ to functions on Ω) of the form

$$P^{(1)}K_1, \tag{8.8}$$

where K_1 is an NIS operator smoothing of order 1, and $P^{(1)}$ is a Poisson operator of order 1.

What we must show, therefore, is that if $f\in L^\infty(\partial\Omega)$ then $(\varrho/\mu)P^{(1)}K_1(f)\in L^\infty(\Omega)$, and the indicated mapping is bounded. In view of Lemma 8.2, our theorem will be proved once we have established the following lemma:

LEMMA 8.3. *Suppose F satisfies the estimates* (8.2) *and* (8.2′). *Let $P^{(1)}$ be a Poisson operator of order* 1. *Then*

$$|P^{(1)}(F)(x,\varrho)|\leq A'\frac{\mu(x)}{\varrho}. \tag{8.9}$$

We recall the following simple facts about Poisson operators, $P^{(1)}$, of order 1, and their kernels, $P^{(1)}_\varrho(x,y)$. We have

$$P^{(1)}(F)(x)=\int_{\partial\Omega}P^{(1)}_\varrho(x,y)F(y)\,d\sigma(y).$$

Then

(a) $P^{(1)}(1)$ is a smooth function;
(b) $|P^{(1)}_\varrho(x,y)|\leq A\|x-y\|^{-4}$;
(c) $|P^{(1)}_\varrho(x,y)|\leq A\varrho^{-2}\|x-y\|^{-2}$.

The assertion (b) follows because $P^{(1)}_\varrho(x,y)$ is (uniformly in ϱ) the kernel of a pseudodifferential operator of order 1, and the dimension of $\partial\Omega$ is 3. (c) follows by the same reasoning, since $\varrho^2 P^{(1)}$ is a Poisson operator of order -1.

Now

$$\begin{aligned} P^{(1)}(F)(x,\varrho) &= \int_{\partial\Omega} P^{(1)}_{\varrho}(x,y)F(y)\,d\sigma(y) \\ &= \int_{\partial\Omega} P^{(1)}_{\varrho}(x,y)[F(y)-F(x)]\,d\sigma(y)+F(x)P^{(1)}(1). \end{aligned}$$

The last term is clearly controlled by (8.2′). The next to last integral can be written as an infinite sum

$$\sum_{k=1}^{\infty}\int_{\|x-y\|\approx 2^k\varrho} P^{(1)}_{\varrho}(x,y)[F(y)-F(x)]\,d\sigma(y)+\int_{\|x-y\|\leqslant\varrho} P^{(1)}_{\varrho}(x,y)[F(y)-F(x)]\,d\sigma(y).$$

Now by (8.2)

$$|F(y)-F(x)|\leqslant A\mu(x,\|x-y\|),$$

and since

$$\mu(x,h)\approx \min_{2\leqslant j\leqslant m}\left(\frac{h}{\Lambda_j(x)}\right)^{1/j},$$

then

$$\mu(x,\|x-y\|)\leqslant C2^{k/2}\mu(x,\varrho)\quad\text{if}\quad \|x-y\|\approx 2^k\varrho.$$

Thus if we use estimate (b) for $P^{(1)}_{\varrho}$, we get as an estimate for the sum,

$$\sum_{k=1}^{\infty} C2^{k/2}\mu(x,\varrho)\int_{\|x-y\|\approx 2^k\varrho}\|x-y\|^{-4}\,dy\leqslant C\left(\frac{\mu(x,\varrho)}{\varrho}\right)\sum_{k=1}^{\infty}2^{k/2}2^{-k}\leqslant C\left(\frac{\mu}{\varrho}\right).$$

The term

$$\int_{\|x-y\|\leqslant\varrho} P^{(1)}_{\varrho}(x,y)[F(y)-F(x)]\,d\sigma(y)$$

is estimated similarly, but here we use (c) instead of (b). This completes the proof of the lemma, and hence the estimates for I is established. The estimates for II is straightforward because G_{12} is a pseudodifferential operator of order -3. Theorem 8.1 is now proved.

Remarks. (1) The operator $\bar{\partial}^*N$ has a unique extension to all forms for which the right hand side of (8.1) is finite. In fact, the same proof as (8.1) gives the weaker inequality

$$\|\bar{\partial}^*N(f)\|_{L^1(\Omega)} \leq C\left[\|f\|_{L^1(\Omega)} + \left\|\left(\frac{\mu}{\varrho}\right)f \wedge \bar{\partial}\varrho\right\|_{L^1(\Omega)}\right] \tag{8.10}$$

which holds for all smooth (0, 1) forms f on Ω. From (8.10), $\bar{\partial}^*N$ extends as a mapping from all forms in this space to $L^1(\Omega)$.

(2) We claim if $\bar{\partial}f=0$ in the sense of distributions and if $u=\bar{\partial}^*N(f)$ then $\bar{\partial}u=f$ in the sense of distributions. To see this, approximate f by a sequence of $f_\varepsilon \in C_0^\infty(\Omega)$ in the norm given by the right hand side of (8.10). Since as is known,

$$\bar{\partial}\bar{\partial}^*N(f_\varepsilon) = f_\varepsilon - N(\bar{\partial}^*\bar{\partial}(f_\varepsilon))$$

and $\bar{\partial}f_\varepsilon \to 0$ in the sense of distributions, the result follows.

(3) Note that if $\bar{\partial}f=0$, the sequence $u_\varepsilon=\bar{\partial}^*N(f_\varepsilon)$ converges to u in $L^1(\Omega)$, $\bar{\partial}u_\varepsilon \to f_\varepsilon$ in the sense of distributions and

$$\int_{\partial\Omega} |u_\varepsilon(z)|\, d\sigma(z) \leq C\left[\|f\|_{L^1(\Omega)} + \left\|\left(\frac{\mu}{\varrho}\right)f \wedge \bar{\partial}f\right\|_{L^1(\Omega)}\right]$$

uniformly in ε.

§9. Zeros of holomorphic functions of Nevanlinna class

A basic problem in complex analysis is to describe the zero varieties of certain classes of holomorphic functions in domains $\Omega \subset \mathbf{C}^n$. Let $H(\Omega)$ denote the space of all holomorphic functions on Ω. For $G \in H(\Omega)$, the zero variety of G is

$$Z(G) = \{z \in \Omega \mid G(z) = 0\}.$$

For example, when $n=1$, and the domain is the unit disc $\mathbf{D}=\{z \in \mathbf{C}^1 \mid |z| \leq 1\}$, the Nevanlinna class is

$$N(\mathbf{D}) = \left\{ f \in H(\mathbf{D}) \;\middle|\; \sup_{0<r<1} \int_{\partial D} \log^+|f_r|\, d\sigma < \infty \right\}$$

where $f_r(z)=f(rz)$. The zero variety of a holomorphic function in the unit disc is a discrete sequence of points $\{\alpha_j\}$ in $\mathbf{D}$. The zero varieties of functions in the Nevanlinna class are precisely the discrete subsets $Z=\{\alpha_j\}$ in $\mathbf{D}$ which satisfy the *Blaschke condition:*

$$\sum_j (1-|\alpha_j|) < \infty. \tag{B'}$$

When $n>1$ and the domain Ω is smoothly bounded and strongly pseudo-convex, a theorem proved independently by Henkin [H1], [H2], and by Skoda [S], characterizes the zero varieties Z of functions F in the Nevanlinna class

$$N(\Omega) = \left\{ F \in H(\Omega) \,\middle|\, \sup_{\varepsilon>0} \int_{\partial\Omega_\varepsilon} \log^+|F|\, d\sigma < \infty \right\},$$

by an analogue of the Blaschke condition:

$$\int_Z \varrho(z)\, d\sigma(z) < \infty. \tag{B}$$

(Here $\varrho(z)$ is the *positive* distance of $z \in \Omega$ to the boundary $\partial\Omega$.) $\Omega_\varepsilon=\{z \in \Omega \mid \varrho(z)>\varepsilon\}$ and $d\sigma$ is the volume element on Z.

The main object of this section is to prove the following extension of the Henkin–Skoda theorem:

THEOREM 9.1. *Let $\Omega \subset \mathbf{C}^2$ be a bounded, smooth weakly pseudo-convex domain of finite type m. Let $G \in H(\Omega)$. Then the zero variety $Z=Z(G)$ is the zero variety of a function F in the Nevanlinna class if and only if the zero variety Z satisfies the Blaschke condition* (B).

In the original work of Skoda (Théorème 3 of [S]), the sufficiency of the Blaschke condition is stated for an arbitrary complex hypersurface X of a strictly pseudoconvex domain Ω satisfying $H^2(\Omega, \mathbf{Z})=(0)$. This topological restriction is not necessary in Theorem 9.1 because of the assumption that the hypersurface is given from the beginning as the zero set of a globally defined holomorphic function. In general, a complex hypersurface in Ω could be defined as a set which, near each point of Ω, is locally a zero set of a locally defined holomorphic function with non-vanishing gradient. For pseudoconvex domains, or more generally for domains of holomorphy, the obstruction to finding a global holomorphic defining function lies in the cohomology group $H^2(\Omega, \mathbf{Z})$.

The fact that the condition (B) is a necessary condition is well-known: it is a consequence of Green's formula (see [Che], [Ma]). Thus we only need to show that condition (B) is sufficient. The main point is that using previously known ideas, the sufficiency can be reduced to the L^1 estimate given by Theorem 8.1. For this reason we shall be brief, leaving some of the details to the cited literatures. We begin by formulating the Blaschke condition in a slightly different way.

DEFINITION 9.1. *Let $\Omega\subset\mathbf{C}^2$ be a domain with smooth boundary, and let $\{\omega_1,\omega_2\}$ be a basis for the $(1,0)$ forms near $\partial\Omega$, with $\omega_2=\partial\varrho$. A positive, closed $(1,1)$ current*

$$\theta=\sum_{i,j=1}^{2}\theta_{ij}\,\omega_i\wedge\bar{\omega}_j$$

satisfies the Blaschke condition *if the following inequality holds:*

$$\mathcal{A}(\theta)=\int_{\Omega}\varrho(z)(\theta_{11}+\theta_{22})(z)<\infty. \tag{9.1}$$

Here θ_{ij} are finite measures on Ω, and θ_{ii} nonnegative.

If $G\in H(\Omega)$, and $\theta=i\partial\bar{\partial}\log|G|$, then θ is then a positive closed $(1,1)$ current on Ω which is essentially the current of integration over $Z(G)$. For such positive currents, the Blaschke condition (B) is equivalent to condition (9.1).

In the case that Ω is a bounded smooth weakly pseudo-convex domain of finite type and θ is a positive closed $(1,1)$ form which satisfies the Blaschke condition, Bonami and Charpentier [BC2] showed that the component θ_{11} satisfies a better estimate.

THEOREM D (Bonami–Charpentier). *Suppose θ satisfies the condition* (9.1). *Then the θ_{11} component of θ satisfies the* generalized Malliavin condition:

$$\int_{\Omega}\left(\frac{\mu^2(z)}{\varrho(z)}\right)\theta_{11}(z)\leqslant C\cdot\mathcal{A}(\theta)<\infty. \tag{9.2}$$

Using the fact that θ satisfies the condition (B) and Theorem D, we have the following corollary:

COROLLARY 9.2. *The θ_{12} and θ_{21} components of θ satisfy the mixed condition:*

$$\int_{\Omega}\mu(z)(|\theta_{12}|+|\theta_{21}|)(z)\leqslant C\cdot\mathcal{A}(\theta)<\infty. \tag{9.3}$$

Proof. Since the matrix $\{\theta_{ij}\}$ is nonnegative,

$$\begin{aligned}\int_\Omega \mu(z)\{|\theta_{12}(z)|+|\theta_{21}(z)|\} &\leq \int_\Omega \frac{2\varrho^{1/2}(z)\mu(z)}{\varrho^{1/2}(z)}\cdot\{\theta_{11}^{1/2}(z)\cdot\theta_{22}^{1/2}(z)\} \\ &\leq \int_\Omega \varrho(z)\theta_{22}(z)+\int_\Omega \frac{\mu^2(z)}{\varrho(z)}\theta_{11}(z) \\ &\leq C\cdot\mathscr{A}(\theta).\end{aligned}$$

The last inequality is a consequence of (9.1) and (9.2). This completes the proof.

Now let $G\in H(\Omega)$ and let $\theta=i\partial\bar{\partial}\log|G|$. Following the method of Lelong (see [L] and also [S]) to find a function $F\in N(\Omega)$ with the same zero variety, we need to solve the equation:

$$i\partial\bar{\partial}\Theta=\theta, \tag{9.4}$$

with $\Theta\in L^1(\partial\Omega)$. Then the function F determined by

$$\Theta=\log|F|$$

belongs to the Nevanlinna class, and has the same zero variety.

As a result of assumption (B), θ will satisfy (9.1) and hence also (9.2) and (9.3). To solve equation (9.4), we find a 1-form ζ which satisfies the equation

$$i\,d\zeta=\theta. \tag{9.5}$$

Then we decompose ζ into

$$\zeta=\zeta_{1,0}+\zeta_{0,1}$$

where $\zeta_{1,0}$ and $\zeta_{0,1}$ are bidegree $(1,0)$ and $(0,1)$ forms. Note that then $\zeta_{0,1}$ is $\bar{\partial}$-closed since θ is a $(1,1)$ form. We shall prove that $\zeta_{0,1}$ satisfies

$$\int_\Omega |\zeta_{0,1}(z)|\,dV(z)+\int_\Omega\left|\left(\frac{\mu(z)}{\varrho(z)}\right)\bar{\partial}\varrho\wedge\zeta_{0,1}(z)\right|dV(z)<\infty. \tag{9.6}$$

If we then let u be the solution to

$$\bar{\partial}u=\zeta_{0,1} \tag{9.7}$$

given by Theorem 8.1 and the remarks at the end of § 8, it follows that

$$\begin{aligned} i\partial\bar\partial(2i\Im u) &= i\partial\bar\partial(u-\bar u) \\ &= i(\partial\bar\partial u+\bar\partial\partial\bar u) \\ &= i(\partial\zeta_{0,1}+\bar\partial\zeta_{1,0}) \\ &= i\{(\partial+\bar\partial)\,\zeta_{0,1}+(\partial+\bar\partial)\,\zeta_{1,0}\} \\ &= i\,d\zeta=\theta. \end{aligned}$$

Thus if we put $\Theta=2i\Im u$, the function Θ satisfies equation (9.4), and according to Theorem 8.1, $\Theta\in L^1(\partial\Omega)$.

Thus we now need to solve equation (9.5) and establish inequality (9.6) for its solution. To solve (9.5) which involves only the d-operator, we need a general version of the Cartan–Poincaré lemma. We begin with

PROPOSITION 9.3. *Let $\Omega\subset\mathbf{R}^n$ be a domain. Let ω be a p-form on Ω, and X a vector field on Ω. Then if $\omega\lrcorner X$ denotes the contraction of the form with the vector field, we have the identity*

$$d\omega\lrcorner X+d(\omega\lrcorner X)=\frac{d}{ds}\,\omega(\exp(sX))|_{s=0}.$$

This is easy to check directly (and also see exercises 7–18 on p. 319, volume 1 of [Sp]).

Let Ω,ω, and X be as in Proposition 9.3, and define

$$\mathscr{F}_s(z)=\exp(sX)(z).$$

Note that $\mathscr{F}_0$ is the identity mapping.

COROLLARY 9.4. *Suppose that the diffeomorphisms $\{\mathscr{F}_s\}$ map Ω to itself for $0\leqslant s\leqslant s_0$. Then for $z\in\Omega$*

$$\omega(z)=\omega(e^{sX}(z))-d\left(\int_0^{s_0}(\omega\lrcorner X)\,(e^{sX}(z))\,ds\right)-\int_0^{s_0}d\omega\lrcorner X(e^{sX}(z))\,ds.$$

Proof. We have

$$\begin{aligned} \omega(e^{s_0X}(z))-\omega(z) &= \int_0^{s_0}\frac{d}{ds}(\omega(e^{sX}(z)))\,ds \\ &= \int_0^{s_0}d\omega\lrcorner X(e^{sX}(z))+\int_0^{s_0}d(\omega\lrcorner X)\,(e^{sX}(z))\,ds. \end{aligned}$$

This proves the corollary, and gives us a general version of the Cartan–Poincaré lemma as follows:

PROPOSITION 9.5. *Suppose Ω, ω, and X are as above and suppose $\{\mathcal{F}_s\}$ satisfies the hypotheses of the Proposition* 9.3. *If $d\omega=0$, then*

$$\omega = \mathcal{F}_{s_0}^* \omega - d\left(\int_0^{s_0} \mathcal{F}_s^*(\omega \lrcorner X)\, ds\right),$$

where $\mathcal{F}_s^*\omega$ denotes the pullback of the form ω induced by the diffeomorphism $\mathcal{F}_s$.

We now turn to the problem of solving equation (9.5). We shall first work in a small neighborhood of the boundary of Ω. Let

$$i\partial\bar{\partial} \log|G| = \theta = \sum_{i,j=1}^{2} \theta_{ij}\omega_i \wedge \bar{\omega}_j.$$

We suppose that θ satisfies equations (9.1), (9.2) and (9.3). Let

$$\Sigma_\varepsilon = \{z \in \Omega \mid -\varepsilon < \varrho(z) < \varepsilon\}.$$

Let $X=\partial/\partial\varrho$ and let

$$\mathcal{F}_s = \exp(sX).$$

Note that $\mathcal{F}_s$ maps $\Omega \cap \Sigma_\varepsilon$ to Ω for $0 \leqslant s \leqslant s_0$ if s_0 and ε are sufficiently small. Set

$$v = -\int_0^{s_0} \theta \lrcorner X(e^{sX})\, ds.$$

Then v is a real 1-form on $\Sigma_\varepsilon \cap \Omega$. Since $d\theta=0$ we have by the Cartan–Poincaré lemma

$$\theta = \mathcal{F}_{s_0}^* \theta + dv = d(\mathcal{F}_{s_0}^*(i\bar{\partial} \log|G|) + v).$$

Put

$$w = \mathcal{F}_{s_0}^*(i\bar{\partial} \log|G|) + v.$$

There is no problem in establishing estimate (9.6) for $\mathcal{F}_{s_0}^*(i\bar{\partial}\log|G|)$, since the form $i\bar{\partial}\log|G|$ is being evaluated strictly inside the domain. Thus we want to estimate v. Now since $\omega_2=\partial\varrho$ and $\bar{\omega}_2=\bar{\partial}\varrho$ (up to a constant), it follows that, up to a constant,

$$\theta \lrcorner X = \theta_{12}\omega_1 + \theta_{22}\omega_2 - \theta_{21}\bar{\omega}_1 - \theta_{22}\bar{\omega}_2.$$

Hence if we decompose v into its $(1,0)$ and $(0,1)$ parts,

$$v = v_{1,0}+v_{0,1},$$

with $v_{0,1}=\bar{v}_{1,0}$ and

$$v_{0,1} = v^1_{0,1}\bar{\omega}_1+v^2_{0,1}\bar{\omega}_2$$

then

$$v^1_{0,1}(z) = \int_0^{s_0} \theta_{12}(e^{sX}(z))\,ds.$$

In order to establish the estimate (9.6) for v, we need the following estimates for $v_{0,1}$:

$$\int_\Omega \{|v^1_{0,1}(z)|+|v^2_{0,1}(z)|\}\,dV(z) \leqslant C\mathscr{A}(\theta) < \infty, \tag{9.8}$$

and

$$\int_\Omega \frac{\mu(z)}{\varrho(z)}|v^1_{0,1}(z)|\,dV(z) \leqslant C\int_\Omega \mu(z)|\theta_{12}(z)| \leqslant C\mathscr{A}(\theta) < \infty. \tag{9.9}$$

In order to get the estimate (9.8), let us define the operator

$$T(\theta)(z) = \int_0^{s_0} \theta(e^{sX}(z))\,ds.$$

Then we have the following lemma:

LEMMA 9.6. *Let $\alpha>-1$ is a real number, then there exist a constant c independent of θ such that*

$$\int_{\Sigma_\varepsilon} [\varrho_\varepsilon(z)]^\alpha|(T\theta)(z)|\,dV(z) \leqslant c\int_{\Sigma_\varepsilon} [\varrho_\varepsilon(z)]^{\alpha+1}|\theta(z)|\,dV(z).$$

Proof. In the appropriate coordinates system, we may assume $z=(x_1,x_2,x_3,x_4)$. Then $\varrho(z)=x_4$ and $\Omega=\{z\in\mathbf{C}^2|\,x_4>0\}$. For $\alpha>-1$,

$$\begin{aligned}\int_{\Sigma_\varepsilon} [\varrho_\varepsilon(z)]^\alpha|(T\theta)(z)|\,dV(z) &\leqslant c\int_{\Sigma_\varepsilon} (x_4-\varepsilon)^\alpha\left(\int_0^\varepsilon |\theta(z+s)|\,ds\right)dV(z)\\ &\leqslant \int_0^{x_4-\varepsilon} (x_4-\varepsilon+s)^\alpha\,ds\int_{\Sigma_\varepsilon} |\theta(z)|\,dV(z)\end{aligned}$$

$$\leq c\cdot\int_{\Sigma_\varepsilon}(x_4-\varepsilon)^{\alpha+1}\,|\theta(z)|\,dV(z)$$
$$= c\cdot\int_{\Sigma_\varepsilon}[\varrho_\varepsilon(z)]^{\alpha+1}\,|\theta(z)|\,dV(z).$$

This completes the proof.

It is easy to see that the estimate (9.8) is just the case $\alpha=0$ by using a limiting argument. In order to get the estimate (9.9), we need the following lemma:

LEMMA 9.7. *Let*

$$v(z)=\int_0^{s_0}\theta(\mathcal{F}_s(z))\,ds$$

Then

$$\int_\Omega\frac{\mu(z)}{\varrho(z)}|v(z)|\,dV(z)\leq C(\Omega)\int_\Omega\mu(z)\,|\theta(z)|\,dV(z). \tag{9.10}$$

Proof. We use the coordinate system which arises in the proof of Lemma 9.6. Hence we have

$$\mathcal{F}_s(x_1,x_2,x_3,x_4)=(x_1,x_2,x_3,x_4+s).$$

Then the equation (9.10) is equivalent to

$$\int_0^{x_4}\mu(x_1,x_2,x_3,t)\frac{dt}{t}\leq C\cdot\mu(x_1,x_2,x_3,x_4). \tag{9.11}$$

To show this we need to observe that in terms of size

$$\mu(x_1,x_2,x_3,t)=\min_{2\leq k\leq m}\left(\frac{t}{\Lambda_k}\right)^{1/k}\approx\left(\sum_{k=2}^m|\Lambda_k|^{1/k}\cdot t^{-1/k}\right)^{-1}.$$

The functions Λ_k for $k=2,3,\dots,m$ have the property

$$\Lambda_k(x_1,x_2,x_3,x_4)\approx\Lambda_k(x_1,x_2,x_3,x_4+s)$$

for small s. This leads to

$$\mu(x_1,x_2,x_3,2^{-j}x_4)\leq C\cdot 2^{-j/m}\mu(x_1,x_2,x_3,x_4).$$

Once we have this, it is easy to rewrite (9.11) as follows:

$$
\begin{aligned}
\int_0^{x_4} \mu(x_1, x_2, x_3, t)\frac{dt}{t} &= \sum_{j=0}^{\infty} \int_{2^{-j}x_4}^{2^{-j+1}x_4} \mu(x_1, x_2, x_3, t)\frac{dt}{t} \\
&\leq \sum_{j=0}^{\infty} \mu(x_1, x_2, x_3, 2^{-j+1}x_4)\cdot\frac{2^{-j+1}x_4 - 2^{-j}x_4}{2^{-j}x_4} \\
&\leq C\cdot\sum_{j=0}^{\infty} 2^{-j/m}\mu(x_1, x_2, x_3, x_4)\cdot\frac{2^{-j+1}x_4}{2^{-j}x_4} \\
&\leq C\cdot\mu(x_1, x_2, x_3, x_4).
\end{aligned}
$$

Now we prove (9.10):

$$
\begin{aligned}
\int_\Omega \frac{\mu(z)}{\varrho(z)}|v(z)|\,dV(z) &= \int_\Omega \frac{\mu(z)}{\varrho(z)}\left|\int_0^{s_0} \theta(x_1, x_2, x_3, x_4+s)\,ds\right| dV(z) \\
&\leq \int_\Omega |\theta(x_1, x_2, x_3, x_4)|\left(\int_{0\leq s\leq x_4} \mu(x_1, x_2, x_3, x_4-s)\frac{ds}{x_4-s}\right) dV(z) \\
&\leq \int_\Omega |\theta(x_1, x_2, x_3, x_4)|\left(\int_{0\leq t\leq x_4} \mu(x_1, x_2, x_3, t)\frac{dt}{t}\right) dV(z).
\end{aligned}
$$

Now we may apply the result (9.11) to get

$$
\int_\Omega \frac{\mu(z)}{\varrho(z)}|v(z)|\,dV(z) \leq C\cdot\int_\Omega \mu(z)\,|\theta(z)|\,dV(z).
$$

This completes the proof of the lemma.

So far we have found a solution w to equation (9.5) which satisfies the estimate (9.6), but this solution is only defined near the boundary of the domain. We still need to patch this solution near the boundary with a solution in the interior. Let us extend the solution w to a 1-form on all of Ω by using a smooth cut-off function near the boundary of Ω. Call this extension $\tilde{w}$. Then the 2-form

$$
\theta - i\,d\tilde{w}
$$

is an exact 2-form on all of Ω since θ is exact. But we also know that this 2-form has compact support in Ω. We now apply the classical theory of harmonic integrals to conclude that

$$
\theta - i\,d\tilde{w} = i\,d\alpha
$$

where α is a one form which is smooth up to the boundary (see Chapter V of De Rham [D], Theorem 25). Thus

$$\theta = i\, d(\tilde{w}+\alpha)$$

and the form $\zeta=\tilde{w}+\alpha$ satisfies the estimates (9.6). This completes the construction of the required solution, and thus completes the proof of Theorem 9.1.

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