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A partial description of parameter space of rational maps of degree two: Part I

by

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1. Introduction

Rational maps have been much studied as dynamical systems. Many rational maps are hyperbolic, and easy to analyse. More importantly, variation of dynamics even within a one-parameter family of rational maps is usually extremely rich, and can sometimes be described in great detail. The prime example is the family of quadratic polynomials $\{z^2+a: a\in \mathbf{C}\}$ which has been the subject of a fundamental study by Douady and Hubbard [D], [D–H1], [D–H2], some of which has been reinterpreted by Thurston [T]. This is, in fact, the main motivation for the present work, which is concerned with the family of rational maps of degree two. The aim is to understand the variation of dynamics within this family.

It has been clear since the pioneering work of Fatou and Julia [F], [J] that the dynamics of a rational map is largely influenced by—and sometimes completely determined by—the dynamics of its critical points. This, in itself, is an example of a vague, but recurrent, theme in dynamical systems in general—that the variation of dynamics in some family of maps (or flows) should be determined by the movement of some (hopefully finite) set of points—which might be periodic, or homoclinic, or, as in the present example, critical. Thus, in the family of polynomials $\{z^2+a: a\in \mathbf{C}\}$, it is the behaviour of the critical point 0 which is important, and in the family of rational maps of degree two, it is the behaviour of the two critical points. The interesting dynamics of a rational map occur on its Julia set. Variation of dynamics in families of rational maps is visible even in static pictures, since the structure of the Julia set often changes radically, even up to homeomorphism, under changes in parameter. However (as always with dynamical systems), dynamics, and structure of the Julia set, are constant on hyperbolic components. The union of hyperbolic components is conjectured to be dense in the family of rational or polynomial maps of degree d, for any d.

Naívely—and I think this should be kept in mind—one would like to describe the variation of dynamics through a parameter space by the movement of critical points across the Julia set and its complement for some fixed map in the parameter space. The trouble is—as just mentioned—that the Julia set and its complement change as one moves through parameter space. But sometimes one can reognize the original Julia set, even after changes. This is what happens with the Mandelbrot set for quadratic polynomials, this being the common name for the set of a for which the Julia set of $z\mapsto z^2+a$ is connected. The Julia set of $z\mapsto z^2$ is the unit circle. Any locally connected Julia set of a quadratic polynomial, which can be assumed to be of the form $z\mapsto z^2+a$, is obtained in a unique way by making extra identifications on the circle. Furthermore, the identifications can be completely described. This means that the dynamics can be completely described. Classical arguments show that a connected Julia set of a hyperbolic rational map is locally connected, and, thanks to Yoccoz, this is now known to be true for a very large class of quadratic polynomials. (It is not always true, though—there is a non-locally connected example in [D], and Douady has other examples.) The variation in the set of identifications can be related to the position of the critical point relative to them, and thus, as we shall summarise in 1.10, to the position of a in the Mandelbrot set. The Mandelbrot set itself (according to Thurston's interpretation—see the survey of Chapter 1) can be described—almost—as a quotient of the unit disc. If one moves out of the Mandelbrot set, the Julia set breaks up into a totally disconnected set, and any sensible identification with the unit circle is lost.

There is another instance of movement in parameter space leading to extra identifications on some fixed Julia set. This comes from the concept of mating, due to Douady and Hubbard, and again, described in more detail in Chapter 1. The idea is to glue together two Julia sets of polynomials of degree d (for any d) to obtain (up to homeomorphism) the Julia set of a degree d rational map, with corresponding dynamics. Wittner [W] introduced the idea of captures. Again, more details are given in Chapter 1, but the idea is to take the Julia set of a hyperbolic polynomial and to make certain types of modifications to obtain certain rational maps. Both these constructions are very important, because they describe a large class of rational maps. Indeed, in his thesis [W], Wittner gives a conjectural description of the family

$$z\mapsto\frac{1+az}{z^2}$$

which has critical point 0 of period two, and a conjectural description of a proper subset of the family

$$z \mapsto \frac{(z-a)(z-1)}{z^2} \quad (a \neq 0)$$

which has critical point 0 of period three. The descriptions are in terms of matings and captures. (In fact, the descriptions can be proved to be valid, at least as much as the combinatorial description of the Mandelbrot set is currently known to be valid.)

The rough idea of the current work is as follows. The reader is warned not to try to interpret the idea too precisely at this stage. A much fuller explanation is given in Chapter 1, which surveys the most important results I use, and also summarises the whole of the current work as far as I have got, comprising a background paper, the current paper, the sequel (in which the main results so far are proved) with brief comments on a further sequel. The family of degree two rational maps is essentially of complex dimension two. For various reasons, we restrict attention to subvarieties in which one critical point is held periodic of fixed period. We have already mentioned the subvarieties corresponding to periods 1, 2 and 3. Period 1 gives the family of polynomials. The exact definitions are given in Chapter 1. Fix a subvariety V, and $m>0$ such that each map in V has a critical point, always called c_1, and varying continuously, of period m. Each map also has a second critical point, also varying continuously, and called c_2. Then V contains at least one polynomial f_0. Then c_2 is fixed by f_0. Under movement in the subvariety, the full orbit of c_1 moves, and c_2 also moves, identifying countably often with points in the full orbit of c_1. Each identification point in V is contained in a hyperbolic component. We consider the complement C in V of the union of these hyperbolic components. We also consider the complement C' in $\bar{\mathbf{C}}$ of the full orbit of c_1 under f_0. We would like to find identifications between, not C and C' themselves, but their universal covers. As it stands, neither universal cover exists. However, we modify C' to a manifold with boundary, which we call U. The universal cover $\tilde{U}$ is then topologically a disc with some boundary. Geometrically, $\tilde{U}$ is given the structure of a subset of the Poincaré disc with geodesic boundary. (The geometry is not intended to relate to the geometry of the subvariety.) A geodesic lamination is defined on $\tilde{U}$. A map of $\bar{\mathbf{C}}$ is associated to each leaf of the lamination, and to many of the complementary regions, which are called gaps. These maps are all two-to-one after omitting exceptional points. Thus, $\tilde{U}$ is some sort of parameter space for a family of maps.

It is too large to be associated to C. First, we have to restrict to some open subset of $\tilde{U}$, called $\tilde{U}_{ad}$, by describing its boundary. The boundary appears to identify with punctures in the subvariety V. Without going into details, consider the family

$$z \mapsto \frac{(z-a)(z-1)}{z^2} \quad (a \neq 0).$$

$a=0$ gives a Möbius transformation, whose cube is the identity. Letting $a \to \infty$ and renormalising gives $z \mapsto 1/z$, whose square is the identity. This is significant. Some idea of why it is given by the Admissable Boundary Theorem of 1.17.

Now identifications have to be made on $\tilde{U}_{ad}$, which is still too large. The first step is to collapse each lamination leaf to a point, and to collapse to a point any gap which does not have an associated map. The quotient space Q is still, in some sense, a parameter space for a family of maps. At this stage, each uncollapsed gap from $\tilde{U}_{ad}$, with its associated map, is associated to a hyperbolic component in C. Each collapsed leaf from $\partial \tilde{U} \cap \tilde{U}_{ad}$ is also associated with a hyperbolic component—or rather, with a boundary point of one—but this time, with one in $V \setminus C$. The associations are surjective, but many-to-one. (So far, I have made no attempt to extend the association beyond hyperbolic components, but I hope it might be possible to do so.) This is all right, since the aim is to relate Q to some sort of covering space of V. It seems that V should be related to the quotient of Q by the increasing orbits of a sequence of group actions, and that each group should be thought of as a lamination-preserving group acting on $\tilde{U}_{ad}$, rather like a Fuchsian group. See 1.18 for slightly more detail.

This work has been developing slowly over several years, and it still incomplete. During this time, many people have been extremely helpful to me with their attention, suggestions, corrections and encouragement. Among others, I should like to thank my colleagues at Liverpool, especially my Ph.D. student D. Ahmadi, and also A. Douady, M. Herman, C. McMullen, M. Shishikura, Tan Lei, W. Thurston, and especially both referees.

Chapter 1. Survey and statement of results

This paper is the first of a series studying hyperbolic rational maps of degree two with a view to a better understanding of the decomposition of parameter space into hyperbolic components and the complement of these. This chapter of this paper is extended to review background material, explain the results, and establish some notations which will be used throughout the series.

Here is an index for this chapter.

Here is an index for the remaining chapters of this paper.

In what follows, it is suggested that any sentences between the words "Index section" and the end of a numbered section should be omitted on a first reading, since they are not necessary for the understanding of the results.

1.1. Rational maps and hyperbolic rational maps

We consider rational maps of degree two with numbered critical points. We denote the critical points of f by $c_1(f)$, $c_2(f)$, or by c_1, c_2 if no confusion can arise. Formally, we let RM_2 denote the set of triples (f, c_1, c_2) modulo the equivalence $(f, c_1, c_2) \simeq (\tau f \tau^{-1}, \tau c_1, \tau c_2)$ whenever τ is a Möbius transformation. We denote the equivalence class of (f, c_1, c_2) by $[f, c_1, c_2]$. Then RM_2 is an orbifold of complex dimension two, with one singularity at the point $[z\mapsto 1/z^2, 0, \infty]$. We shall usually refer to an element of RM_2 simply as f.

A rational map is *hyperbolic* if the forward orbit of each critical point converges to an attractive periodic orbit. In that case, the Julia set $J(f)$ is given by

$$J(f)=\{x\colon \{f^n(x)\} \text{ does not converge to an attractive periodic orbit}\}.$$

These maps have always been of key importance ([B], [F], [J], [D], [D–H1], [D–H2], [M–S–S]). A hyperbolic map is automatically *J-stable*, that is, there is a neighbourhood U of $J(f)$ such that for all g sufficiently near f, $J(g)\subset g^{-1}U\subset U$, and there is a homeomorphism $\chi_g\colon U\to \mathbf{C}$ with $\chi_g\circ f=g\circ\chi_g$ on $f^{-1}U$, and $\chi_g(J(f))=J(g)$. The set of hyperbolic maps is open, and any two maps in a connected component of hyperbolic maps are topologically conjugate in neighbourhoods of their Julia sets. It is a longstanding conjecture that the union of hyperbolic components is dense in RM_2 (or any other suitable space), which is why it might be important to obtain information about their positions. (See [M–S–S], where it is shown that J-stable components are dense.)

1.2. Critically finite branched coverings and their types

If $f\colon \bar{\mathbf{C}}\to\bar{\mathbf{C}}$ is any branched covering, let the *postcritical set* $X(f)$ be defined by

$$X(f)=\{f^n c\colon c \text{ critical},\ n>0\}.$$

Then f is *critically finite* if $\#(X(f)<+\infty$.

It is well-known that, if H is a hyperbolic component, H usually contains a critically finite map. See [McM], for instance. In [R], we showed that each hyperbolic component in RM_2, with one exception, contained a critically finite map, which was unique. (No originality was claimed for this particular point.) The exception was the hyperbolic component containing z^2+a for all large a. In so doing, we classified the hyperbolic components into types I, II, III and IV, with type I being the exceptional component. In view of this, we classify certain branched coverings of $\bar{\mathbf{C}}$ into types II, III and IV. If f is a critically finite hyperbolic rational map of degree two, then f must be type II, III or IV, and lie in a hyperbolic component of the same type. Conversely, any critically finite rational map which is type II, III or IV is hyperbolic.

So now let f be a branched covering with critical points c_1, c_2, and let c_1 have period m under f. Then f is:

type II if $f^{m-q}c_1=c_2$ for some $0<q<m$ (so that $m\geqslant 2$),

type III if $f^q c_2=c_1$ for some $q>1$, but c_2 is not periodic (so that $m\geqslant 2$),

type IV if $f^n c_2 \neq c_1$ for any $n\geqslant 0$, but c_2 has some period p under f.

1.3. Types of hyperbolic components: index section

We give, here, the classification of hyperbolic components into types. (See [R] for more justification.) In fact, we define the type of a hyperbolic map, such that all maps in a hyperbolic component have the same type. The definition is consistent with that for critically finite maps. Let $f\in RM_2$ be hyperbolic. Then we denote by $U_1=U_1(f)$, $U_2=U_2(f)$ the components of $\bar{\mathbf{C}}\setminus J(f)$ which contain c_1, c_2. One of U_1, U_2 must be periodic. Without loss of generality, we always assume U_1 is periodic of period m. Then f is:

type I if $U_1=U_2$, in which case $U_1=\bar{\mathbf{C}}\setminus J(f)$ and $J(f)$ is a Cantor set, and f lies in the same hyperbolic component as $z\mapsto z^2+a$, for all large a ([R], for instance),

type II if $U_1\neq U_2$, but $U_2=f^{m-q}U_1$ for some $0<q<m$,

type III if $U_2\neq f^n U_1$ for any $n\geqslant 0$, but $f^q U_2=U_1$ for some $q\geqslant 2$,

type IV if $f^n U_2\neq U_1$ for any $n\geqslant 0$, but U_2 for some period p under f.

1.4. Equivalence of critically finite branched coverings

Thurston [T] defined a homotopy-type equivalence relation for critically finite branched coverings. One form of this definition is as follows.

f and g are *equivalent*, written $f\simeq g$, if there exists an orientation-preserving homeomorphism φ, and a path $\{g_t: t\in[0,1]\}$ through critically finite branched coverings such that $X(g_t)=X(g)$ for all $t\in[0,1]$, $\varphi\circ f\circ\varphi^{-1}=g_0$, and $g=g_1$.

This is simply the natural type of homotopy equivalence for critically finite branched coverings. We also write $f\simeq_\varphi g$ in the above situation, but this is not an equivalence relation.

Index section. There is another form of the definition of equivalence which is often more useful. $f\simeq g$ if there is a path $\{f_t: t\in[0,1]\}$ through critically finite branched coverings with $f_0=f$, $f_1=g$, and $X(f_t)$ varies isotopically.

If $\{f_t\}$ is as above, and $\{\varphi_t\}$ is an isotopy with $\varphi_1=$identity and $\varphi_t(X(g))=X(f_t)$, then $f\simeq_{\varphi_0^{-1}} g$ and $\varphi=\varphi_0^{-1}$, $g_t=\varphi_0^{-1}\circ f_t\circ\varphi_t$ satisfy the first definition. Similarly, given $\{g_t\}$, φ as in the first definition, we can find an isotopy $\{\varphi_t: t\in[0,1]\}$ with $\varphi_1=$identity,

$\varphi=\varphi_0^{-1}$, and pass to the second definition by taking $f_t=\varphi_t\circ g_t\circ\varphi_t^{-1}$. We shall write

$$(f, Y_0)\simeq(g, Y_1) \quad \text{or} \quad (f, Y_0)\simeq_\varphi(g, Y_1),$$

if $X(f)\subset Y_0$, $X(g)\subset Y_1$, $f(Y_0)\subset Y_0$, $g(Y_1)\subset Y_1$, $\varphi(Y_0)=Y_1$ with $\varphi\circ f=g\circ\varphi$ on Y_0, and $g_t(Y_1)\subset Y_1$ for all t, for $\{g_t\}$ as in the first definition of equivalence.

A second, equivalent definition of $(f, Y_0)\simeq(g, Y_1)$ is: there exists a path $\{f_t: t\in[0,1]\}$ through critically finite branched coverings with $f=f_0$, $g=f_1$, $X(f_t)\subset Y_t$, $f_t(Y_t)\subset Y_t$, where Y_t varies isotopically between Y_0 and Y_1.

If we use the second definition in each case, it becomes a triviality that if $f\simeq g$, then

$$(f, f^{-n}X(f))\simeq(g, g^{-n}X(g)), \quad \text{for each } n\geqslant 0.$$

1.5. Equivalence and semiconjugacy

We remark that, for critically finite branched coverings f and g, if $g\simeq_\varphi f$, then there is a well-defined homeomorphism which can be written $f^{-n}\circ\varphi\circ g^n=\Psi_n$. If, in addition, f is hyperbolic rational, and g is suitably defined near $X(g)$, then $\lim_{n\to\infty}\Psi_n=\Psi$ exists, and satisfies $\Psi\circ g=f\circ\Psi$. This type of argument, obtaining a semiconjugacy from some sort of homotopy equivalence under some hyperbolicity assumption on one of the maps, is common, and presumably old. (See [Fr] for the more difficult case involving Anosov diffeomorphisms on nilmanifolds, for example.) In the case when f is hyperbolic rational, we give details of the procedure in Chapter 4. In the sequel to this paper, we shall use some of the same ideas in the case when f is expanding with respect to a semi-metric only. It is common to extend the procedure to give information about the conjugacy classes of f and g, as we shall do in this paper, particularly in Chapter 4, when f is hyperbolic rational. The information it gives about maps up to semiconjugacy, or conjugacy, is one reason for the importance of the homotopy-type equivalence for critically finite branched coverings.

1.6. Equivalence to a rational map

The main (but related) reason for the importance of the homotopy-type equivalence for critically finite branched coverings of $\bar{\mathbb{C}}$ is, however, the reason for which it was introduced. Thurston [T] gave a necessary and sufficient condition for a critically finite branched covering to be equivalent to a rational map. If an equivalence class contains a hyperbolic rational map f, then the other rational maps in the equivalence class are those conjugate to f by Möbius transformations. (See [D–H3] for a version of the proof of Thurston's theorem.) This condition will be very important in these papers. The condition is redundant for f, if f is a degree two branched covering with critical points c_1, c_2 and $fc_1=c_1$, $f^pc_2=c_2$ for some $p\geq 1$. Thus, such an f is equivalent to a degree two polynomial. (See [L] for a proof of the redundancy.) In degree two, Thurston's condition can be simplified using work of Levy [L], Tan Lei [TL] and, more recently, Shishikura [S]. In a sequel to this paper, we shall refine this condition further.

Index section. Thurston's original condition will never be used directly in these papers (although it was used once in the background paper [R]) but it should, perhaps, be recorded here.

A simple path γ in $\bar{\mathbf{C}}\setminus X(f)$ is *peripheral* if it bounds a disc containing at most one point of $X(f)$. Let

$$f^*\gamma=\sum_{i=1}^{r}\delta_i/n_i,$$

where δ_i are the nontrivial components of $f^{-1}\gamma$ with $f|\delta_i$ of degree n_i. Then we can extend f^* so that

$$f^*\left(\sum_{i=1}^{n}x_i\gamma_i\right)=\sum_{i=1}^{n}x_if^*(\gamma_i)$$

if $x_i\in R$ and γ_i are isotopically distinct in $\bar{\mathbf{C}}\setminus X(f)$ for $1\leq i\leq n$. The first hypothesis for Thurston's necessary and sufficient condition for equivalence of f to a (unique) rational map is that an orbifold associated with the set $X(f)$ be hyperbolic, which only ever fails to happen if $\#(X(f))\leq 4$, and if $\#(X(f))\leq 3$, f is trivially equivalent to a unique rational map. But if $\#(X(f))=4$ and the associated orbifold is not hyperbolic, f is equivalent either to a rational map, or to a map of the form $x\mapsto A\mathbf{x}+\mathbf{b}\colon \mathbf{R}^2/{\sim}\to\mathbf{R}^2/{\sim}$, where $\sim$ is the equivalence relation with classes $\{\pm\mathbf{x}+\mathbf{n}:\mathbf{n}\in\mathbf{Z}^2\}$ $(\mathbf{x}\in\mathbf{R}^2)$ and A is a 2×2 matrix with integer coefficients and determinant >1. (See [T] 16.6 for some of this.) If this happens, f has precisely two critical values, and neither is periodic. Whenever f is not of this form up to equivalence, Thurston's necessary and sufficient condition for equivalence of f to a (unique) rational map becomes:

(A) *There do not exist γ_i and x_i $(1\leq i\leq n)$ and $\lambda\geq 1$ such that*

$$f^*\left(\sum_{i=1}^{n}x_i\gamma_i\right)=\lambda\sum_{i=1}^{n}x_i\gamma_i.$$

If f is degree two, the condition can be simplified to:

(B) *There do not exist nonperipheral* γ_i *and* γ_i' $(1\leqslant i\leqslant n)$ *such that* γ_i *and* γ_i' *are isotopic in* $\bar{\mathbf{C}}\setminus X(f)$, γ_i' *is a component of* $f^{-1}(\gamma_{i+1})$ (*with* $\gamma_{r+1}=\gamma_1$) *and* $f|\gamma_i'$ *is a homeomorphism.*

If $\{\gamma_1, \ldots, \gamma_r\}$ exists as in (B), then it is called a *Levy cycle*. A proof that (A) implies (B) was given in Levy's thesis [L], but there is an obscurity in the proof. For an alternative, see Tan Lei's thesis [TL]. For the moment, use the notation γ_i' as it is used in (B). Note that necessity of condition (B) is clear: if $\gamma_i'=\gamma_{i,1}$ and, similarly, $\gamma_{i,n+1}=\gamma_{i,n}'$, then standard hyperbolicity results show that $\lim_{n\to\infty}$ diameter$(\gamma_{i,n})=0$ if f is rational hyperbolic. This is impossible, since γ_i is nonperipheral.

If f is degree two, and type II, III or IV, then (B) can be refined to:

(C) *There does not exist a Levy cycle* $\gamma_1, \ldots, \gamma_r$ $(r\geqslant 2)$ *such that only one component* C *of* $\bar{\mathbf{C}}\setminus\bigcup_{i=1}^r \gamma_i$ *is not a disc, and such that* $f|C'$ *is a homeomorphism onto* C, *where* C' *is a component of* $f^{-1}C$ *and bounded by* γ_i' $(1\leqslant i\leqslant r)$.

I first proved that (B) implies (C) by an over-complicated method, but a much simpler proof was found by Tan Lei [TL]. (A slightly simplified statement of the result has been given here.) Now Shishikura [S] can prove that (A) implies (C) directly. In a sequel to this paper, we shall further refine, and use, condition (A) in the case when f is degree two and of type II, III, or IV.

Conditions (B) and (C) make sense even if $X(f)$ is infinite, although we then have to consider the possibility that the γ_i have non-transversal intersections, and take C as a component of $(\bar{\mathbf{C}}\setminus\bigcup_{i=1}^r \gamma_i)$. It is then easy to see that conditions (B), (C) are satisfied if f is hyperbolic rational, or, more generally, if $X(f)\subset\bar{\mathbf{C}}\setminus J(f)$ (when f can have parabolic basins).

1.7. The map σ_β

If ζ: $[a, b]\to\bar{\mathbf{C}}$ is any path, we define $\bar{\zeta}$: $[a, b]\to\bar{\mathbf{C}}$ by $\bar{\zeta}(t)=\zeta(a+b-t)$. If β: $[a, b]\to\bar{\mathbf{C}}$ is any simple path we take σ_β to be a homeomorphism which is identity outside a neighbourhood of $\beta([a, b])$, and such that $\sigma_\beta(\beta(a))=\beta(b)$. We take this neighbourhood as small as the circumstances require. We shall often be interested in compositions such as $\sigma_\beta\circ f$, where f is a critically finite branched covering with $\beta(a)$, $\beta(b)\in X(f)$, and $\beta((a, b))\cap X(f)=\varnothing$. In such a case, $\sigma_\beta\circ f$ is well-defined up to equivalence if the neighbourhood mentioned above avoids $X(f)\setminus\{\beta(a), \beta(b)\}$. Given the importance of maps such as σ_β in describing isotopy classes of homeomorphisms on a punctured two-sphere [M], it is not surprising that they should be of importance in describing equivalence classes of critically finite branched coverings. The assumption that β is simple is unnecessary.

Index section. If β: $[a, b]\to\bar{\mathbf{C}}$ is not simple, but there exist $a=t_0<t_1<\ldots<t_n=b$ such that $\beta_i=\beta|[t_{i-1}, t_i]$ is simple $(1\leq i\leq n)$, then we can define

$$\sigma_\beta = \sigma_{\beta_n}\circ\ldots\circ\sigma_{\beta_1},$$

and we still have $\sigma_\beta(\beta(a))=\beta(b)$. Again, σ_β can be taken as the identity outside an arbitrarily small neighbourhood of $\beta([a,b])$. If f is a critically finite branched covering with $\beta(a), \beta(b)\in X(f)$ and $\beta((a,b))\cap(X(f)\setminus\{\beta(a),\beta(b)\})=\emptyset$, then the equivalence class of $\sigma_\beta\circ f$ is independent of the precise definition of σ_β. Similarly, for any path β, we can define $\sigma_\beta=\lim_{n\to\infty}\varphi_{\beta_n}$, where $\beta=\lim_{n\to\infty}\beta_n$, and β_n has only finitely many self-intersections.

In many cases, we actually have $\sigma_\beta\circ f\simeq\sigma_{\beta'}\circ f$ for a simple path β': $[a,b]\to\bar{\mathbf{C}}$ with $\beta'(a)=\beta(a)$, $\beta'(b)=\beta(b)$, and $\beta'([a,b])$ contained in a small neighbourhood of $\beta([a, b])$. For suppose in the above that $n=2$, and $\beta(a)\notin\beta([t_1, b])$, $\beta(b)\notin\beta([a, t_1])$. Choose $u<t_1$ so that $\beta([u, t_1])\cap\beta((t_1,b])=\emptyset$, and let

$$\beta_1' = \bar{\beta}|[a, u].$$

Then let

$$\beta' = \begin{cases}\beta_1 & \text{on } [a, t_1],\\ \sigma_{\beta_1'}\circ\beta_2 = \beta_2' & \text{on } [t_1', b].\end{cases}$$

Then β' is simple, and $\sigma_\beta, \sigma_{\beta_2'}\circ\sigma_{\beta_1}, \sigma_{\beta'}$ are all isotopic via an isotopy φ_t with $\varphi_t(\beta(a))=\beta(b)$ for all t and φ_t=identity outside a small neighbourhood of $\beta([a,b])$.

1.8. The first result

The first result of this work is as follows.

POLYNOMIAL-AND-PATH THEOREM. *Let f be a critically finite hyperbolic degree two rational map with $c_1(f)$ of period $m\geq 2$. Then there exists a polynomial f_0 of the form $z\mapsto z^2+a$ with 0 of period m under f_0, and there exists a simple path γ: $[0,1]\to\bar{\mathbf{C}}$, with $\gamma(0)=\infty$, such that the following hold.*

If f is type III, $f \simeq \sigma_\gamma \circ f_0$, *where* $\gamma(1)=x$, $f^q(x)=0$, *but x is not periodic.*

If f is type II, $f \simeq \sigma_{\bar\eta} \circ \sigma_\gamma \circ f_0$, *where* $\gamma(1)=f_0^{m-q+1}(0)$, *for some* $0<q<m$, $\eta(1)=f_0^{m-q}(0)$, *and* $f_0 \circ \eta = \gamma$.

If f is type IV, $f \simeq \sigma_{\bar\eta} \circ \sigma_\gamma \circ f_0$, *where $\gamma(1)$ has period p under f_0, is either in the Julia set of f_0 or equal to ∞,* $\eta(1)=f_0^{p-1}\gamma(1)$ *and* $f_0 \circ \eta = \gamma$.

In all cases, $\gamma((0,1))$ *is disjoint from the forward orbits under f_0 of ∞, $\gamma(1)$, 0, and, in the equivalence, the critical point c_1 corresponds to* 0.

The main drawback of this result is that the hypothesis that f be a rational map—or equivalent to one—is probably completely unnecessary. This is certainly so if f is type III. In that case, the description of rational maps up to equivalence given by the theorem is rather empty. However, the result is amenable to refinement, and that, essentially, is the substance of all subsequent work in this paper (and its sequels).

In this theorem, we could arrange that $\gamma([0,1])$ was contained in the complement of the full orbit of 0 under f_0. If $\bar{\mathbf{C}} \setminus \bigcup_{i \in \mathbf{Z}} f_0^i(0)$ were a submanifold, we could take its universal cover, and regard $(\gamma, \gamma(1))$ as a point in the universal cover. Then perhaps we could find a submanifold of the universal cover which consisted of all $(\gamma, \gamma(1))$ for which the corresponding branched covering $\sigma_{\bar\eta} \circ \sigma_\gamma \circ f_0$ was equivalent to a rational map. Then perhaps we could find identifications on the submanifold, giving parameter space up to homeomorphism, together with its decomposition into hyperbolic components. Of course, doing exactly this is impossible, but this is the idea of the programme, on which some progress has been made.

The case $m=1$ is explicitly excluded from the theorem. However, the work of Douady and Hubbard implies that every map $f_a: z \mapsto z^2+a$ with 0 periodic under f_a is equivalent to a map of the form

$$\sigma_{\bar\eta} \circ \sigma_\gamma \circ f_0,$$

where $\gamma: [0,1] \to \bar{\mathbf{C}}$ is a path from ∞ to the unit circle to a periodic point (under f_0) on the unit circle, and γ intersects the circle only at this point. Also, as in the theorem, $f_0 \circ \eta = \gamma$. Thus, the theorem is a generalisation of part of Douady and Hubbard's work, using Thurston's notion of equivalence.

The construction described in the theorem also generalises the concepts of matings and captures. In the type IV case, if the path γ does not intersect $J(f_0)$ except at $\gamma(1)$, then $\sigma_{\bar\eta} \circ \sigma_\gamma \circ f_0$ is a mating up to equivalence, in the sense of Douady and Hubbard. In the type III case, if the path γ crosses $J(f_0)$ exactly once, then $\sigma_\gamma \circ f_0$ is a capture up to equivalence, in the sense of Wittner [W], and all matings and captures arise, up to equivalence, in this way.

1.9. One-complex-parameter families

Since it becomes possible to visualise decompositions in a space of complex dimension one, we shall consider, for each $m \geq 1$, the algebraic variety in RM_2 consisting of all $[f, c_1, c_2]$ with c_1 of period m under f. For $m \leq 3$ (at least), this variety is irreducible. For $m=1$, the variety can be identified with $\mathbf{C}$, or with $\{g_a\colon a \in \mathbf{C}\}$, where $g_a(z)=z^2+a$. The motivation for the current work is that hyperbolic components and their positions in this family are well understood, thanks to the work of Douady and Hubbard ([D–H1], [D–H2]) and the very illuminating interpretation of their work by Thurston [T] who used laminations. (This interpretation will be summarised in a minute.) Their work led to a complete understanding of the *combinatorial* structure of the Mandelbrot set $\{g_a\colon g_a^n(0) \not\to \infty\}$, which is the complement, in the family $\{g_a\}$, of a single type I hyperbolic component intersected with the family $\{g_a\}$.

1.10. Outline of Thurston's laminations

The idea of the present work is to produce invariant laminations which describe the dynamics of degree two rational maps, and parameter laminations describing decompositions of parameter space, following Thurston's approach for the family $\{g_a\colon g_a(z)=z^2+a,\ a \in \mathbf{C}\}$. So, first it is necessary to summarise Thurston's approach [T]. All the necessary theory is contained in Thurston's preprint, although some statements given here may not be spelt out there. For the original approach, one should look in Douady and Hubbard's papers ([D], [D–H1], [D–H2]) and also at Lavaurs' note [La].

A *lamination* L on the unit disc $\mathring{D}=\{z\colon |z|<1\}$ is a closed set of disjoint geodesics —called *leaves*—with endpoints on $S^1=\{z\colon |z|=1\}$. We can use either the Euclidean or Poincaré metric on the disc. A *gap* is a component of $\mathring{D} \setminus \cup L$, and a gap G is *finite-sided* if $\bar{G}$ contains only finitely many leaves. (I am grateful to D. Ahmadi for pointing out that, according to Thurston's definition, gaps are the closures of complementary components of the lamination. But for generalisations, it will be useful to stick to the variant given here.) An equivalence relation $\approx_L$ on $\bar{\mathbf{C}}$—which, of course, contains $\mathring{D}$—is defined by two elements being equivalent if they are both in $\bar{l}$ for some $l \in L$, or both in $\bar{G}$ for some finite-sided gap G.

A lamination L is *invariant* (under $z \mapsto z^2$) if, whenever $l \in L$ has endpoints z_1, z_2,

(a) if $z_1 \neq -z_2$, $l^2 \in L$, where l^2 has endpoints z_1^2, z_2^2,

(b) we have $-l \in L$, where $-l$ has endpoints $-z_1$, $-z_2$,

(c) we have $l_1 \in L$ for at least one $l_1 \in L$ with $l_1^2=l$.

If L is invariant, and we also stipulate that

(d) 0 is in an infinite sided gap of L,

then L has exactly two longest leaves of the form $\pm l$. Then $\mu=l^2$ is called the *minor leaf of L*, and L is the unique lamination satisfying (a) to (d) with minor leaf μ. A leaf, μ, is the minor leaf of some (unique) lamination satisfying (a) to (d) if and only if:

(1) the endpoints of μ are $e^{2\pi ir}$, $e^{2\pi it}$, where r, t are odd denominator rationals of the same period under $x\mapsto 2x$ mod 1 (we say *μ has endpoints r, t*),

(2) the set $\{\mu^{2^n}: n\geq 0\}$ is a finite lamination in which μ is the image of the longest leaf (and, in fact, is the shortest leaf).

For each odd denominator rational r in (0, 1), there is a unique minor leaf μ_r with endpoint at r. The closure of the set of minor leaves satisfying 1 and 2 is a lamination which Thurston calls the *Quadratic Minor Lamination*, or *QML*. (Since it is a lamination, it is easy to compute the leaves μ_r: the pairs of endpoints are $\{1/3, 2/3\}$, $\{1/7, 2/7\}$, $\{3/7, 4/7\}$, $\{5/7, 6/7\}$ ....)

We write L_r for the invariant lamination satisfying (a) to (d) with minor leaf μ_r, and we write $\simeq_r=\simeq_L$. Then there is a critically finite branched covering $s_r: \bar{\mathbf{C}}\to\bar{\mathbf{C}}$ with the following properties.

(1) The critical points of s_r are 0, ∞, with $s_r(\infty)=\infty$, and 0 is of the same period under s_r as r under $x\mapsto 2x$ mod 1.

(2) For $|z|\geq 1$, $s_r(z)=z^2$.

(3) The lamination L_r is invariant under s_r, that is, $s_rL_r=L_r=s_r^{-1}L_r$.

(4) For all $z\in G$, $s_r^{mn}(z)\to 0$ as $n\to\infty$, where m is the period of 0 under s_r, and G is the gap of L_r containing 0, and s_r^m is topologically conjugate, near 0, to $z\mapsto z^2$.

The map s_r induces a branched cover $[s_r]$: $\bar{\mathbf{C}}/\simeq_r\to\bar{\mathbf{C}}/\simeq_r$ which is uniquely determined up to topological conjugacy by properties (1) to (4). Then $\bar{\mathbf{C}}/\simeq_r$ is homeomorphic to $\bar{\mathbf{C}}$, and $[s_r]$ is topologically conjugate to a unique critically finite g_a, where $g_a(z)=z^2+a$. By abuse of notation, we write $g_a=g_r$. Then $g_r\simeq s_r\simeq[s_r]$. Every g_a for which the critical point 0 is periodic arises in this way, for a unique μ_r (giving two corresponding odd denominator rationals) except for $g_0(z)=z^2$, which corresponds to the empty lamination on $\mathring{D}$.

For each odd denominator rational r in (0, 1) there is a gap G_r of *QML* with $\mu_r\subset\bar{G}_r$ and μ_r separating G_r from 0. We have $G_r=G_t$ only if t is an endpoint of μ_r. There is also a gap G_0 which contains 0. These are *all* the infinite-sided gaps. Let

$$M=\{a\in\mathbf{C}: g_a^n(0)\to\infty \text{ as } n\to\infty\}.$$

There is a continuous surjection Φ: $\mathbf{C}\to\mathbf{C}/\simeq_{QML}$ such that Φ is a holomorphic bijection of $\mathbf{C}\setminus M$ (which is a single hyperbolic component in the family $\{g_a\}$) onto $(\mathbf{C}\setminus D)/\simeq_{QML}$ (which is effectively just $\bar{\mathbf{C}}\setminus D$, since each equivalence class in this set contains just one point), and Φ is a homeomorphism of the hyperbolic component of g_r onto $G_r/\simeq_{QML}$, which is effectively just G_r. It is conjectured that Φ is a homeomorphism. This is equivalent to the conjecture that M is locally connected.

1.11. More general laminations: invariant and parameter

In this work, a *lamination* L is a closed set of nonintersecting complete simple geodesics—or *leaves*—on a surface R. The nature of R depends on whether L is an *invariant* or *parameter* lamination. If L is an *invariant* lamination, $\bar{\mathbf{C}}\setminus\{z: |z|=1\}\subset R\subset\bar{\mathbf{C}}$, R is open and $\bar{\mathbf{C}}\setminus R$ is infinite, so that the universal cover of R is the unit disc, and R is endowed with a complete Poincaré metric. If L is a *parameter* lamination, $R\subset\{z: |z|<1\}=\mathring{D}$, R is endowed with the Poincaré metric of $\mathring{D}$—so that, if $R\neq\mathring{D}$, R is not complete—and the boundary of R in $\mathring{D}$ consists of complete geodesics.

A *gap* of L is a component G of $R\setminus\bigcup L$. A *side* of G is a leaf l of L which intersects the boundary of a component of $G\setminus\{z: |z|=1\}$. (We use this formulation because if R is not simply connected, not all components of ∂G may be sides of G.) *Finite-sided* gaps will be of some importance. The equivalence relation $\simeq_L$ on $\bar{\mathbf{C}}$ (if L is invariant) or on R (if L is parameter) is the smallest closed equivalence relation such that $x\simeq_L y$ whenever $x, y\in l$, for $l\in L$, or x, $y\in G$ for a finite sided gap G of L.

There is a parameter lamination $\mathcal{L}_r$ for each odd denominator rational r in $(0, 1)$ with $\mathcal{L}_r=\mathcal{L}_t$ if and only if r, t are the endpoints of μ_r. Thurston's lamination QML can be regarded as $\mathcal{L}_0$. As might be expected, the parameter laminations parametrise the invariant laminations. In analogy with the results concerning QML, if r runs through the rationals of period m under $x\mapsto 2x$ mod 1, then the invariant laminations parametrised by $\mathcal{L}_r$ describe up to topological conjugacy those hyperbolic critically finite degree two rational maps $[f, c_1, c_2]$ with c_1 of period m under f. Thus, $\mathcal{L}_r$ gives information about the parameter space of such $[f, c_1, c_2]$, although it is as yet unclear to me exactly how much information $\mathcal{L}_r$ gives about this.

1.12. Objects associated to invariant laminations

Let an odd denominator rational r in $(0, 1)$ be fixed. We are going to describe a Cantor set $K_r\subset\bar{\mathbf{C}}$, a lamination $\bar{L}_r$ on $\bar{\mathbf{C}}\setminus K_r$ (which is, in fact, invariant) and a branched covering $\bar{s}_r: \bar{\mathbf{C}}\to\bar{\mathbf{C}}$ which preserves K_r and $\bar{L}_r$. We use the lamination L_r on $\{z: |z|<1\}$

and the branched covering s_r of 1.10. Let $\Phi=\Phi_r:\bar{\mathbf{C}}\to\bar{\mathbf{C}}$ be a continuous orientation-preserving subjection with the following properties.

(1) The sets 0, ∞, $\{z:|z|<1\}=\mathring{D}$, $\{z:|z|=1\}=S^1$, $\{z:|z|>1\}$ are preserved by Φ.

(2) For all z, $\Phi(-z)=-\Phi(z)$.

(3a) If μ_r does not bound a finite-sided gap of L_r, then $\Phi^{-1}(x)$ is a point unless $z\in\bar{l}$, for l in the full s_r-orbit of μ_r, when $\Phi^{-1}(\bar{l})$ is a topological rectangle with two sides in S^1 and two sides in $\mathring{D}$.

(3b) If μ_r bounds an n-sided gap G_1 of L_r, then $\Phi^{-1}(x)$ is a point unless $x\in\bar{G}$, for G in the full s_r-orbit of G_1, when $\Phi^{-1}(\bar{G})$ is a topological $2n$-gon with n sides in S^1 and n sides in $\mathring{D}$.

Let

$$K_r=\overline{\{x\in S^1:\Phi^{-1}(x)\text{ is a point}\}}.$$

Then K_r is a Cantor set with $K_r=-K_r$. Let

$$\bar{L}_r=\{l_1:\ l_1 \text{ is a component of } \partial(\Phi^{-1}(l))\cap\mathring{D}, l\in L_r\}.$$

Then we can choose Φ so that in addition to 1 to 3:

(4) $\bar{L}_r$ is a lamination on $\bar{\mathbf{C}}\setminus K_r$.

There is a branched covering $\bar{s}_r:\bar{\mathbf{C}}\to\bar{\mathbf{C}}$ such that:

(a) $\Phi\circ\bar{s}_r=s_r\circ\Phi$,

(b) $\bar{s}_r(-z)=\bar{s}_r(z)$ for all z,

(c) $\bar{s}_r\bar{L}_r=\bar{L}_r=\bar{s}_r^{-1}\bar{L}_r$.

In particular, 0 and ∞ are the critical point of $\bar{s}_r$, $\bar{s}_r\infty=\infty$, and 0 has the same period under both s_r, $\bar{s}_r$. Here is a sketch of $\bar{L}_{1/7}$. (See Diagram 1.)

1.13. Invariant laminations

An *invariant lamination* is a lamination on $\bar{\mathbf{C}}\setminus K_r$ with certain additional properties, for some odd denominator rational r in $(0,1)$—or on $\bar{\mathbf{C}}\setminus S^1$, which is the case for Thurston's invariant laminations. To start with, we need to outline definitions of inverse and forward images of geodesics under certain branched coverings. These definitions will be made precise in Chapter 6. Throughout this section, we fix r, and write $K=K_r$, $\bar{L}=\bar{L}_r$, $\Phi=\Phi_r$, $\bar{s}=\bar{s}_r$. Let l be a simple directed geodesic on $\bar{\mathbf{C}}\setminus K$ with $\bar{s}(0)\notin l$. If $\beta:[0,1]\to\bar{\mathbf{C}}$ is a path with $\beta(0)=\infty$ and $\beta(1)\notin l$, then $(\sigma_\beta\circ\bar{s})^*(l)$ is defined to be the two geodesics $\pm l_1$ which are the *straightenings* of the components of $(\sigma_\beta\circ\bar{s})^{-1}(l)$. (A straightening of a path is homotopic to it in a restricted sense.) If $\beta(1)\in l$, then

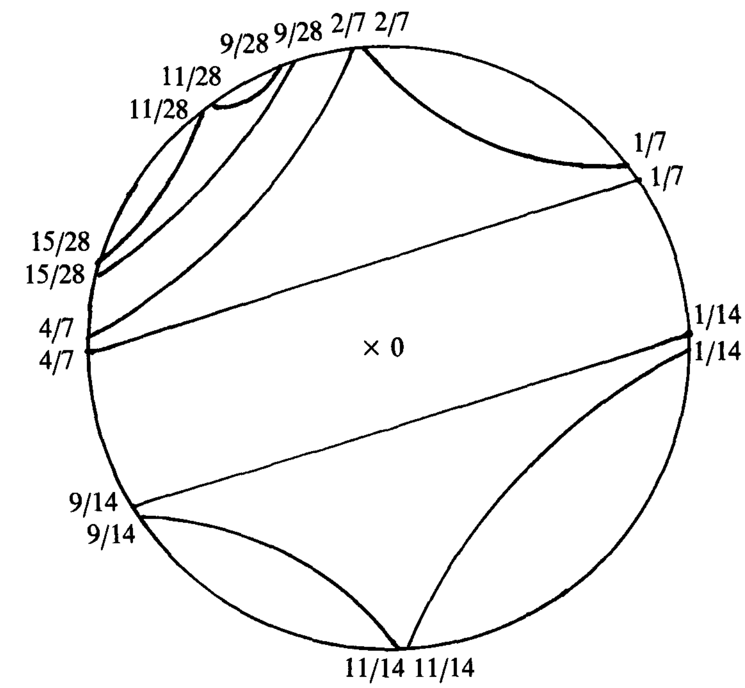

Diagram 1.

$(\sigma_\beta \circ \bar{s})^*(l)$ is the four geodesics which are the straightenings of the components of $(\sigma_\beta \circ \bar{s})^*(\{l', l''\})$, where l', l'', are paths bounding a disc containing l. In this case, the four geodesics from $(\sigma_\beta \circ \bar{s})^*(l)$ bound a topological rectangle G_∞ containing ∞. These definitions can be made for some other branched coverings besides $\sigma_\beta \circ \bar{s}$. If L is a lamination, then so is

$$(\sigma_\beta \circ \bar{s})^* L = \{l_1 \colon l_1 \in (\sigma_\beta \circ \bar{s})^* l, \text{ some } l \in L\}.$$

Let $\beta \colon [0,1] \to \bar{\mathbf{C}} \setminus K$ satisfy $\beta(0) = \infty$, $\beta((0,1)) \cap \{\beta(0), \beta(1)\} = \emptyset$, and either $\beta^{-1}(\bigcup \bar{L}_r) = \emptyset$, or $\beta^{-1}(\bigcup \bar{L}_r) = \{t\}$, for some $t \in [0,1]$. Then a lamination L on $\bar{\mathbf{C}} \setminus K$ is *invariant* or *β-invariant* if

$$(\sigma_\beta \circ \bar{s})^* L = L,$$

and a couple of other less important conditions hold to be given in Chapter 6 (see 6.9).

1.14. Lamination maps

Given a β-invariant lamination L, there is a *lamination map* $\varrho_L \colon \bar{\mathbf{C}} \to \bar{\mathbf{C}}$. In particular, it will turn out that the conditions imply that ϱ_L is a critically finite branched covering with $L=\varrho_L L=\varrho_L^{-1} L$ whenever $\beta(1)\notin\bigcup L$, and that ϱ_L always preserves $\simeq_L$, so that $[\varrho_L]\colon \bar{\mathbf{C}}/\simeq_L\to\bar{\mathbf{C}}/\simeq_L$ is well-defined. ϱ_L is uniquely defined up to equivalence, and $[\varrho_L]$ is uniquely defined up to topological conjugacy, whenever $\beta(1)\notin\bigcup L$ and the gap G_∞ of L containing ∞ is simply connected. The precise definitions are given in Chapter 6 (see 6.11).

1.15. Results on invariant laminations

These results extend the Polynomial-and-Path Theorem of 1.8. In both results, let f be a hyperbolic critically finite degree two rational map with critical point c_1 of period m.

LAMINATION MAP EQUIVALENCE THEOREM. *There is an odd denominator rational r in $(0,1)$ of period m under $x\mapsto 2x \bmod 1$ and a path $\beta\colon[0,1]\to\bar{\mathbf{C}}\setminus K_r$ with $\beta(0)=\infty$, a β-invariant lamination L on $\bar{\mathbf{C}}\setminus K_r$ with simply connected gaps G_0, G_∞ containing $0,\infty$, and a lamination map ϱ_L which is a critically finite branch covering, such that*

$$f\simeq_\Psi \varrho_L,$$

with $\Psi(c_1)=0$.

LAMINATION MAP CONJUGACY THEOREM. *Let L,ϱ_L have the properties as in the Lamination Map Equivalence Theorem. Then there exists $\Psi\colon\bar{\mathbf{C}}\to\bar{\mathbf{C}}$ and a family $\{\Psi_t\colon t\in[0,\infty)\}$ of homeomorphisms such that $\Psi=\lim_{t\to+\infty}\Psi_t$, $\Psi^{-1}(z)$ is a $\simeq_L$-equivalence class for all z, and*

$$\Psi\circ\varrho_L=f\circ\Psi.$$

In particular, $\bar{\mathbf{C}}/\simeq_L$ is a sphere, and Ψ induces a conjugacy between f and $[\varrho_L]$. Moreover, $\simeq_L$ coincides with the (a priori smaller) equivalence relation $\simeq$ generated by $z_1\simeq z_2$ if $z_1,z_2\in\bar{l}$ for some $l\in L$, or $z_1,z_2\in\bar{G}$ for a finite-sided gap G of L.

1.16. Minor gaps and parameter laminations

The last results in the present paper concern the construction of the parameter lamination. We write $K=K_r$, $\bar{L}=\bar{L}_r$, $\bar{s}=\bar{s}_r$ for some odd denominator rational r in $(0,1)$. (See 1.12.) Let $U=U_r$ be the component of $\bar{\mathbf{C}}\setminus(K\cup(\bigcup L))$ which contains ∞. Let $\pi\colon\mathring{D}\to\bar{\mathbf{C}}\setminus K$ be a holomorphic covering map with $\pi(0)=\infty$, so that geodesics in $\bar{\mathbf{C}}\setminus K$ lift to geodesics in $\mathring{D}$. Let $\tilde{U}$ denote the closure in $D=\{z\colon|z|\leq 1\}$ of the component of

$\pi^{-1}(U)$ containing 0. Then $\tilde{U}$ is topologically a closed disc, and $\partial\tilde{U}\setminus\partial D$ consists of inverse images under π of geodesics in $\bar{L}$.

If L is any invariant lamination on $\bar{\mathbf{C}}\setminus K$, then leaves of L in U lift to geodesics in $\tilde{U}$, which are either entirely in $\partial\tilde{U}$ or entirely in Interior $\tilde{U}$. (See the precise definition of invariance in 6.9.) Our parameter lamination will be a lamination on $\tilde{U}$, including geodesics in $\partial\tilde{U}$. Let L be a β-invariant lamination on $\bar{\mathbf{C}}\setminus K$. We are going to define the *minor gap* of L if $\beta(1)\notin\bigcup L$, and, in almost all cases, the *minor leaf* of L. If $\beta([0,1])\subset U$, let $\tilde{\beta}\colon[0,1]\to\tilde{U}$ be the lift of β under π with $\tilde{\beta}(0)=0$. Let G (if it exists) be the gap of L with $\beta(1)\in G$. Let

$$t_0=\sup\{t\colon\beta(t)\in\bigcup L\},$$

and let μ be the leaf of L with $\beta(t_0)\in\mu$. Let $\tilde{\mu}$ be the lift of μ with $\tilde{\beta}(t_0)\in\tilde{\mu}$, and let $\tilde{G}$ be the lift of the component of $G\cap U$ with $\tilde{\beta}(1)\in\tilde{G}$ (if $\tilde{G}$ exists). Then $\tilde{\mu}$ or μ will be referred to indiscriminately as the *minor leaf* of L, and $\tilde{G}$ or G will be referred to as the *minor gap* of L. If $\beta([0,1])\not\subset U$, let $t_0=\beta^{-1}(\bigcup\bar{L}_r)$, and let μ be the leaf of $\bar{L}_r$ with $\beta(t_0)\in\mu$. Let $\tilde{\beta}\colon[0,t_0]\to\tilde{U}$ be the lift of β with $\tilde{\beta}(0)=0$, and let $\tilde{\mu}\subset\partial\tilde{U}$ be the lift of μ with $\tilde{\beta}(t_0)\in\tilde{\mu}$. Again, $\tilde{\mu}$ or μ will be referred to as the *minor leaf* of L. Thus, the minor leaf of L always exists unless $\beta([0,1])\cap(\bigcup L)=\varnothing$—which is quite rare.

In the following theorem, we use the term *primitive*, which will be defined in 7.3, but it is not an important restriction. In the Lamination Map Equivalence Theorem (1.15), L can be chosen to be primitive. The theorem will be proved in Chapter 7. The point of the theorem is to allow the definition of parameter laminations given afterwards.

PARAMETER LAMINATIONS THEOREM. *For $i=1$ or 2, let L_i be a primitive β-invariant lamination, and let Z_i denote the minor leaf of closure of the minor gap of L_i in $\tilde{U}$. Let ϱ_{L_i} be critically finite. Then if $Z_1\cap Z_2\neq\varnothing$, there are no transversal intersections between L_1 and L_2.*

Definition of parameter laminations. Because of the Parameter Laminations Theorem, the following set $\mathcal{L}=\mathcal{L}_r$ is a well-defined lamination on $\tilde{U}$:

$\mathcal{L}$ is the closure of the set of leaves l, such that either $l=\tilde{\mu}$ or $l\in\partial\tilde{G}$, where $\tilde{\mu}$ (or $\tilde{G}$) is the minor leaf (or gap) in $\tilde{U}$ of a primitive invariant lamination L in $\bar{\mathbf{C}}\setminus K$, with ϱ_L critically finite.

Roughly speaking, infinite sided gaps of the parameter lamination are combinatorial analogues of hyperbolic components in some appropriate parameter space. Of course, I hope that the parameter laminations give topological information about the corresponding parameter spaces. Here is an abbreviated statement of a theorem about the gaps of $\mathcal{L}$, which will be proved in the sequel to this paper, if the current proof can be cut down somewhat.

PARAMETER GAPS THEOREM. *Let $\mathcal{L}$ be a parameter lamination. Then any side of a gap of $\mathcal{L}$ is also a side of a minor gap of some invariant lamination, and projects to an eventually periodic leaf in this invariant lamination.*

1.17. Admissible points and their boundary

A leaf μ in $\mathcal{L}$ is *admissible* if there is a neighbourhood V of μ in $\tilde{U}$ such that, whenever L is a primitive invariant lamination with minor gap intersecting V, ϱ_L is equivalent to a rational map. A point in Interior($\tilde{U}$) is *admissible* if it is either in an admissible leaf of $\mathcal{L}$, or in a gap of $\mathcal{L}$ with at least one admissible side. A point in $\partial\tilde{U}$ is *admissible* if it has a neighbourhood whose intersection with Interior(U) consists of admissible points. The set of admissible points in $\tilde{U}$ is open by definition, and is denoted by $\tilde{U}_{ad}$. The point 0 is admissible. The component of $\tilde{U}_{ad}$ containing 0 is denoted by $\tilde{U}_0$. Let $\mathcal{L}=\mathcal{L}_r$, where r is of period m under $x\mapsto 2x \bmod 1$. Our hope is that some quotient of $\tilde{U}_{ad}/\simeq_{\mathcal{L}}$—or, better still, of $\tilde{U}_0/\simeq_{\mathcal{L}}$—is a continuous image of—or, better still, homeomorphic to—

$$\{[f, c_1, c_2]\in RM_2\colon c_1 \text{ is period } m \text{ under} f, f \text{ is not hyperbolic of type II or III}\}.$$

Since $\mathcal{L}_0=QML$, this would be analogous to the result that $D/\simeq_{QML}$ is a continuous image of

$$\{g_a\colon g_a(z)=z^2+a,\ g_a^n(0)\not\to\infty\}.$$

The Admissibility Proposition below—which is a strengthening of the Invariant Laminations Theorem, or the Polynomial-and-Path Theorem—is a step towards this.

ADMISSIBILITY PROPOSITION. *Let f be a critically finite degree two rational map with critical point c_1 of period m. Then there is r of period m under $x\mapsto 2x$* mod 1, *such that the following hold for $\tilde{U}_{ad}$ defined relative to $U=U_r$.*

(a) *If f is type* IV, *there is a primitive invariant lamination L with minor gap intersecting $\tilde{U}_{ad}$, such that $f\simeq\varrho_L$.*

(b) *If f is type* II *or* III, *there is a primitive invariant lamination L with minor leaf μ in $\tilde{U}_{ad}$, and $f\simeq\varrho_L$.*

It would be useful to have $\tilde{U}_0$ instead of $\tilde{U}_{ad}$ in the Admissibility Proposition. The proofs of the Polynomial-and-Path Theorem, and the Improved Polynomial-and-Path Theorem (see Chapters 2 and 3 of this paper) have been chosen with this in mind. We now give the results we have on the boundary $\partial\tilde{U}_{ad}$ of $\tilde{U}_{ad}$ as a subset of $\tilde{U}$. These results enable one to compute boundary leaves, but, hopefully, they can be neatened.

ADMISSIBLE BOUNDARY THEOREM. *Let μ be a leaf in $\partial\tilde{U}_{ad}$. Then μ is either a side of the minor gap $G(L)$ of a primitive invariant lamination L, or $\mu=\lim_{n\to\infty} G(L_n)$ for $G(L_n)$ the minor gap of a primitive invariant lamination L_n, where $G(L)$, $G(L_n)\subset\tilde{U}\setminus\tilde{U}_{ad}$ and $\varrho=\varrho_L$ or ϱ_{L_n} has the following properties.*

The period m of ∞ under ϱ is less than or equal to the period of r under $x\mapsto 2x$ mod 1. *There is a finite set A of leaves in L (or L_n) satisfying: $(\bigcup A)^-$ is connected, non-contractible in $\bar{\mathbf{C}}\setminus X(\varrho)$ and invariant under $\bar{s}_*$ and $\bar{s}_*^m l=l$ for all $l\in A$.*

1.18. Identifications between branched coverings

Many leaves in our parameter lamination represent the same map. I plan to give results on identifications between leaves in a later paper. This work is in its early stages, but there are some observations which are encouraging for further progress. These observations are best expressed for the moment in terms of the branched coverings

$$\sigma_\beta\circ f_0$$

and

$$\sigma_\zeta\circ\sigma_\beta\circ f_0,$$

which first made their appearance in the Polynomial-and-Path Theorem. Recall that f_0 is a polynomial of the form $z\mapsto z^2+a$, where the critical point 0 is of period m. For fixed m, there are only finitely many choices for f_0. Many different β might give the same branched covering, and (since ζ is defined in terms of β) we ask when the branched coverings corresponding to β_1, β_2 are equivalent to a fixed critically finite branched covering g. The answer turns out to be that this happens precisely when β_1, β_2 are in the same orbit under a certain group action—and this group action depends only on the polynomial f_0, not on the equivalence class of g. This is perhaps a little surprising when g is of type IV—when β_1, β_2 are usually paths between ∞ and periodic points in the Julia set of f_0. Actually, in the type IV case we shall have to start by modifying the paths. We leave this aside for the moment, and concentrate, for simplicity on g of type

III. The group G which acts is actually an inductive limit (in a sense yet to be explained) of isotopy groups, but if N is the least integer for which $g^N(c)$ is periodic, for the nonperiodic critical point c of g, and if $g \simeq \sigma_{\beta_i} \circ f_0$, then the equivalence class of g depends only on the isotopy class of the arc β_i between points of $f_0^{-N}X(f_0)$ in $\bar{\mathbb{C}} \setminus f_0^{-N}X(f_0)$. Then let G be the group of homeomorphisms φ fixing $X(f_0)$, and leaving $f^{-N}X(f_0)$ invariant, up to isotopy fixing $f_0^{-N}X(f_0)$, with

$$(f_0, f_0^{-N}X(f_0)) \simeq_{\varphi} (\sigma_\alpha \circ f_0, f_0^{-N}X(f_0)),$$

for some closed loop α (depending on φ) based at ∞, but otherwise not intersecting $f_0^{-N}X(f_0)$. Then G acts on arcs between ∞ and other points in $f_0^{-N}X(f_0)$ by

$$\varphi \cdot \beta = \sigma_{\bar{\alpha}} \circ \varphi \circ \beta.$$

In the type IV case, it will also be possible to consider isotopy classes relative to $f_0^{-N}X(f_0)$ for some N.

1.19. Matings, captures and admissibility

Given two odd denominator rationals r and t in $(0, 1)$, the *mating of* s_r *and* s_t, which is denoted by $s_r \amalg s_t$, is defined by

$$s_r \amalg s_t(z) = \begin{cases} s_r(z) & \text{for} \quad |z| \leq 1, \\ (s_t(z^{-1}))^{-1} & \text{for} \quad |z| \geq 1. \end{cases}$$

The term *mating of* s_r *and* s_t will also be used to denote any map equivalent to this one. For example, $s_r \amalg s_t \simeq s_t \amalg s_r$. (The critical points have to be interchanged to obtain the equivalence). Matings were introduced by Douady and Hubbard (not exactly in this terminology). Let

$$L_t^{-1} = \{l^{-1} : l \in L_t\},$$

where

$$l^{-1} = \{z^{-1} : z \in l\}.$$

Then $s_r \amalg s_t$ preserves $L_r \cup L_t^{-1}$. We can find a primitive invariant lamination $L_{r,t}$ with $\varrho_L \simeq s_r \amalg s_t$, where $L_{r,t}$ is a follows. We can choose $\Phi = \Phi_r$ so that $L'_{r,t}$ is a set of geodesics, where

$$L'_{r,t}=\bar{L}_r\cup\{l: l \text{ is a geodesic in } \{z:|z|>1\} \text{ and in the boundary of the convex hull of } \Phi^{-1}(\bar{l}_1\cap S^1) \text{ for some leaf } l_1 \text{ in } L_t^{-1}\}.$$

Then $L_{r,t}$ can be obtained from $L'_{r,t}$ by removing some isolated geodesics. We omit the details. Let G be the gap of $L_{r,t}$ containing $\Phi^{-1}((s_t(0))^{-1}$. Then $L_{r,t}$ is β-invariant, where β: $[0,1]\to\bar{\mathbf{C}}$ satisfies $\beta(0)=\infty$, $\beta([0,1])\cap S^1=\emptyset$, $\beta(1)\in G$. Thus, G is the minor gap in $\bar{\mathbf{C}}\setminus K_r$. For fixed r, let G_t denote the lifted minor gap in $\tilde{U}$.

If μ,μ' are leaves of QML, we define an ordering by: $\mu<\mu'$ if and only if μ separates μ' from 0. Then $s_r \amalg s_t$ is equivalent to a rational map if and only if there does not exist an odd denominator rational number q with $\mu_q\le\mu_r$, $\mu_{1-q}\le\mu_t$ [TL]. This happens if and only if $G_t\subset\tilde{U}_0$, where $\tilde{U}_0$ is as in 1.17.

Now let r be an odd denominator rational in $(0,1)$, and let x_0 satisfy $\bar{s}_r^n(x_0)=0$ for some $n>0$, with $x_0\neq\bar{s}_r(0)$. Let β: $[0,1]\to\bar{\mathbf{C}}\setminus K_r$ be some simple path with $\beta(0)=\infty$, $\beta(1)=x_0$, $\beta^{-1}(S^1)=\{u_1\}=\beta^{-1}(\Phi(\exp 2\pi it))$ for some rational t, and $\beta^{-1}(\bigcup\bar{L}_r)=\{u_2\}$, where $0<u_1<u_2<1$. If x_0 is non-periodic under $\bar{s}_r$, $\sigma_\beta\circ\bar{s}_r$ is of type III. If x_0 is periodic, let ζ: $[0,1]\to\bar{\mathbf{C}}\setminus K_r$ be the unique path with $\bar{s}_r\circ\zeta=\beta$, and $\zeta(1)$ periodic under $\bar{s}_r$. Then $\sigma_\zeta\circ\sigma_\beta\circ\bar{s}_r$ is type II. Such maps $\sigma_\beta\circ\bar{s}_r$, $\sigma_\zeta\circ\sigma_\beta\circ\bar{s}_r$ were studied (up to quivalence) by B. Wittner in his thesis [W]. He called them *captures*. They are determined up to equivalence by r, x_0, t, or simply by r, x_0 if μ_r is minimal in the ordering on QML. We shall see in Chapter 3 how to define a β-invariant lamination $L=L_{r,x_0,t}$ (which can be modified to be primitive) with $\varrho_L\simeq\sigma_\beta\circ\bar{s}_r$ or $\sigma_\zeta\circ\sigma_\beta\circ\bar{s}_r$. In fact, we shall do this without the restriction that $\beta^{-1}(S^1)$ is one point.

A similar result to that for matings holds. If μ_r is not minimal in the ordering on QML, the capture associated to r, x_0, t is equivalent to a rational map if and only if there does not exist q with $\mu_q\le\mu_r$, $\mu_q\le\mu_t$. Here, μ_t is any leaf of QML with endpoint at $\exp(2\pi it)$, if such a leaf exists, and if there is no such leaf, $\mu_t=\exp(2\pi it)$, and then $\mu_q\le\mu_t$ means μ_q separates $\exp(2\pi it)=\mu_t$ from 0. If μ_r is minimal, then the same captures are equivalent to rational maps, and in addition the captures associated to r, x_0, r are equivalent to rational maps, if x_0 is $\bar{s}_r^i 0$ for the unique i with $1<i\le r$ such that $\exp(2\pi ir)$ is in the boundary of the gap containing $\bar{s}_r^i(0)$.

The minor leaf of $L=L_{r,x_0,t}$ is in $\tilde{U}_0$ precisely when ϱ_L is equivalent to a rational map. In summary, the minor leaves in $\tilde{U}$ of $L_{r,t}$ and $L_{r,x_0,t}$ are in $\tilde{U}_0$ precisely when they are not separated from 0 by the minor leaf $\tilde{\mu}_{1-q}$ of $L_{r,1-q}$ in $\tilde{U}$, where μ_q is minimal; with $\mu_q\le\mu_r$. We write $\tilde{U}_{0,mc}$ (matings and captures) for the smallest connected set in $\tilde{U}_0$ consisting of leaves and gaps, and containing all minor leaves of the $L_{r,t}, L_{r,x_0,t}$ in $\tilde{U}_0$.

1.20. Tuning

Let f be a degree two critically finite branched covering of type III or IV with critical point of c of period m. Let g be equivalent to a degree two polynomial. Then the *tuning of f by g round c*, denoted $f\vdash_c g$, is defined up to equivalence as follows. (This terminology comes from Douady and Hubbard. For interval maps, the concept is due to Milnor and Thurston [M–T].) Up to equivalence, we can assume that there is a disc D containing c such that $f^m D = D$. We can also assume that $gD=D$, that there is a fixed critical point of g outside D, and that the orbit of the other critical point of g is inside D. Then define $h=f\vdash_c g$ by:

$$h = f \quad \text{outside } \bigcup f^i D,$$

$$h^m = g \quad \text{on } D,$$

$$h|f^i D \text{ is a homeomorphism for } 1 \leqslant i < m.$$

For example, $s_r \amalg s_t \simeq s_r \vdash_\infty s_t$. (See 1.19.) It is a triviality from Thurston's necessary and sufficient condition that if $f\vdash_c g$ is equivalent to a rational map, then so is f. If $m=1$ and f is equivalent to a rational map (hence to a polynomial) then $f\vdash_c g$ may not be. The precise situation in that case was given in 1.19, since $f\vdash_c g$ is then a mating. However, we have the following, which will be proved in Chapter 8.

TUNING PROPOSITION. *If f is a critically finite branched covering with critical point c of period $m>1$, and $f\vdash_c g$ has a Levy cycle* (1.6) *then so does f. Hence, if f is equivalent to a rational map, so is $f\vdash_c g$.*

Tuning of laminations. Let L be a β-invariant lamination for which ϱ_L is type IV, and the minor gap $G=G(L)$ in $\tilde{U}$ projects to a simply-connected gap in U. Then if t is any odd denominator rational in $(0, 1)$, we can find a primitive β_t-invariant lamination L_t containing L such that:

(1) $\varrho_{L_t} \simeq \varrho_L \vdash_\infty s_t$,

(2) the minor gap of L_t is contained in G.

In fact, there is a continuous map from G to $\mathring{D}$ which maps $\bigcup \mathscr{L}$ to $\bigcup QML$, and such that the inverse image of any leaf of QML is either a leaf of $\mathscr{L}$ or a finite-sided gap together with its sides. The details will be given in Chapter 3.

1.21. Examples of admissible points relative to a particular parameter lamination

We are now going to consider the parameter lamination $\mathscr{L}_{1/7}$ and its relation to rational maps $[f, c_1, c_2]$ with c_1 of period 3. That is, we consider

$$\{[g_a, 0, 2a/(a+1)]: a \in \hat{\mathbf{C}}, a \neq 0\},$$

where

$$g_a(z) = (z-a)(z-1)/z^2,$$

so that $g_a(0)=\infty$, $g_a(\infty)=1$, $g_a(1)=0$.

We give a very simple picture of a simple decomposition of the parameter space $\mathbf{C}\setminus\{0\}$. (This family was considered by Ben Wittner in his thesis [W], which contains much better and more detailed pictures.)

For $a=\pm 1$, the critical points of g_a are in the same periodic orbit, and B_a is a (very) rough sketch of the hyperbolic component of g_a intersected with the family $\{g_{a'}\}$. I do not know if the boundaries of $B_{\pm 1}$ are topological arcs as drawn, but $B_{\pm 1}$ do have three special accessible boundary points 0, α, $\bar{\alpha}$ in common, as drawn.

Recall that in 1.19, we produced minor gaps of laminations containing $\bar{L}_r$ associated with matings $s_r \amalg s_t$. We also identified those minor gaps contained in $\tilde{U}_0$ (defining $\tilde{U}_0$ relative to $\mathcal{L}_r$). We did the same for captures. It can be shown that all hyperbolic components in $B_\alpha, B_{\bar{\alpha}}$ are types III and IV, and that in $B_{\bar{\alpha}}$, all critically finite hyperbolic type IV maps are matings $s_{1/7} \amalg s_t$ up to equivalence, whereas the type III maps are captures $\sigma_\beta \circ s_{1/7}$ up to equivalence. Conversely, all type III maps $\sigma_\beta \circ s_{1/7}$ and matings $s_{1/7} \amalg s_t$ are represented in $B_{\bar{\alpha}}$. Similar results hold for B_α. It can be shown that

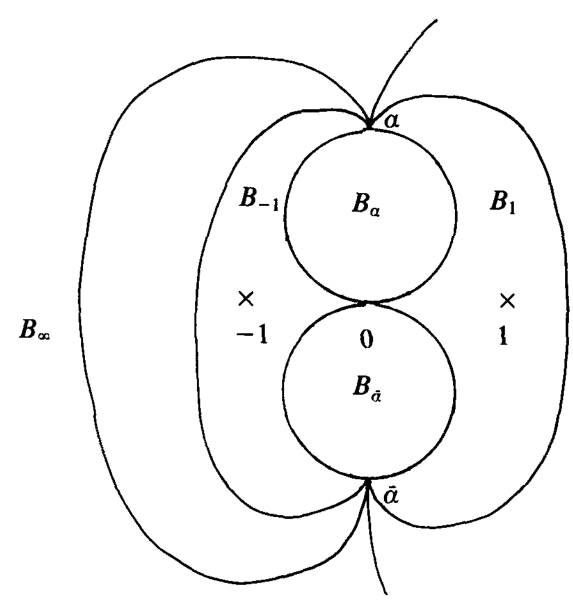

Diagram 2.

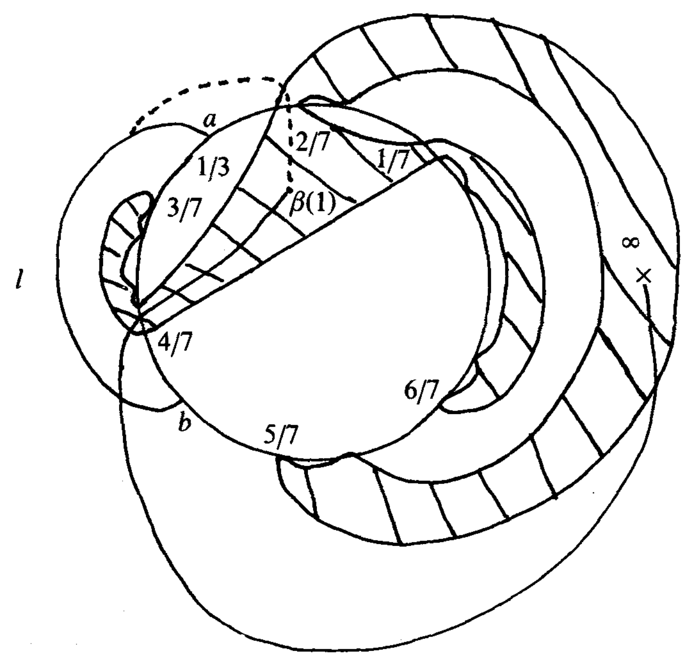

Diagram 3.

$$\{a \in B_{\bar{a}} : g_a \text{ is not hyperbolic type III}\}$$

maps continuously onto $(\tilde{U}_{0,mc})^{-}/\sim$, where $\sim$ is the lamination equivalence relation for $\mathscr{L}_{1/7}$.

The set B_∞ is the intersection with $\{g_a\}$ of a single type IV hyperbolic component in which all maps have an attractive fixed point. For large a, the boundary of B_∞ is asymptotic to

$$\{a : |-1 \pm 2/\sqrt{-a}\,| = 1\}.$$

There is a in B_∞ such that g_a is (up to conjugacy) the unique polynomial $z \mapsto z^2+b$ with 0 of period 3 and b real. There is a lamination L_0 with minor gap $\tilde{G}(L_0)$ and $\varrho_{L_0} \simeq g_a$, which is β-invariant, and $G(L_0)=\pi(\tilde{G}(L_0))$ very roughly as shown in Diagram 3.

Note that $\tilde{G}(L_0)$ intersects, but is not contained in $\tilde{U}_{0,mc}$. For any mating $s_{3/7} \amalg s_t$, one can find an invariant lamination L with minor leaf in $\tilde{G}(L_0)$ and $\varrho_L \simeq s_{3/7} \amalg s_t$. There is precisely one boundary leaf of $\tilde{U}_0$ in $G(L_0)$. It is the minor leaf of an invariant lamination L with $\varrho_L \simeq s_{3/7} \amalg s_{1/3}$. It has S^1-crossings in $\Phi^{-1}(\exp 2\pi i a_n)$ where a_n is the sequence

$$\ldots 1/56,\ 53/56,\ 1/14,\ 11/14,\ 2/7,\ 1/7,\ 25/28,\ 1/28,\ 109/112,\ 1/112 \ldots$$

The segment of the leaf between 2/7 and 1/7 lies in $\mathring{D}$. Now let l be any leaf of L_0—as drawn in Diagram 3—with endpoints $\Phi^{-1}(\exp 2\pi i a)$, $\Phi^{-1}(\exp 2\pi i b)$ for odd denominator rationals a, b with $a \in (2/7, 1/3)$ and $b \in (2/3, 5/7)$. We claim that there is a leaf μ_l with an endpoint at $\Phi^{-1}(\exp 2\pi i b)$, such that μ_l is in $\mathscr{L}_{1/7}$ and is the minor leaf of an invariant lamination $L(l)$, which is β_l-invariant, where β_l is the extension of β by the dotted line approximately as shown in Diagram 3. β_l crosses μ_l near its endpoint. Then the lifted leaf $\tilde{\mu}_l$ is in $\tilde{U}_0$. If we take μ_l with endpoint at $\Phi^{-1}(\exp 2\pi i(2/3))$, however, we obtain a boundary leaf of $\tilde{U}_0$ which has S^1-crossings in $\Phi^{-1}(\exp 2\pi i a_n)$ $(n \geq 1)$, where a_1 corresponds to the crossing nearest $\Phi^{-1}(\exp 2\pi i(2/3))$ and $\{a_n\}$ is the sequence

$$2/3,\ 9/28,\ 11/28,\ 9/14,\ 1/14,\ 9/112,\ 11/112,\ 51/56,\ 1/56\ldots$$

Let ϱ_l denote the lamination map of $L(l)$. Then ϱ_l need not be equivalent to a mating even if ϱ_l is type IV. For example, if l has endpoints $\Phi^{-1}(\exp 2\pi i(12/17))$, $\Phi^{-1}(\exp 2\pi i 5/17)$, then ϱ_l is not equivalent to a mating.

Chapter 2. Proof of the Polynomial-and-Path Theorem

2.1. Proof of the theorem for type III maps

We recall that this theorem was stated in 1.8. We start by proving the theorem if f is type III, so that c_1 has period m under f, c_2 is not in the forward orbit of c_1, but $f^q c_2 = c_1$. Note that we certainly do not use the hypothesis that f be equivalent to a rational map in the type III case.

Let $\nu: [0,1] \to \bar{\mathbf{C}}$ be any simple path with $\nu(0)=c_2$, $\nu(1)=fc_2$ and

$$\nu((0,1)) \cap \{f^i c_2, f^i c_1 : i \geq 0\} = \emptyset.$$

Then $\sigma_{\bar{\nu}} \circ f$ is a branched covering with critical points c_1, c_2 with c_1 of period m and c_2 fixed. So (as stated in 1.6) Thurston's theorem [T] about equivalence to a rational map implies there is a polynomial f_0 of the form $z \mapsto z^2+a$ such that

$$\sigma_{\bar{\nu}} \circ f \simeq_{\varphi} f_0,$$

for some homeomorphism φ, and $\varphi(c_2) \in f_0^{-q}(X(f_0))$. Then there is ψ with

$$(\sigma_{\bar{\nu}} \circ f, X(f)) \simeq_{\psi} (f_0, Y)$$

with $Y \subset f_0^{-q}(X(f_0))$ (see 1.7). Put $\gamma = \psi \circ \nu$. Then

$$f \simeq \sigma_\nu \circ \sigma_{\bar\nu} \circ f \simeq_\psi \sigma_\gamma \circ f_0$$

as required.

2.2. Idea of the proof in general

Let f be of type II or IV with $c_1(f)$ of period m. (We can also treat f of type III by a similar method.) Then we can complete the proof of the Polynomial-and-Path Theorem if the following *equivalence properties* hold, where f', f_0, x, x_0, $Y(f')$, $Y(f_0)$, ξ, α, ν satisfy the *point properties* and *path properties* listed below.

Equivalence properties. (a) $f \simeq \sigma_\xi \circ \sigma_\alpha \circ f'$.
(b) $(\sigma_{\bar\nu} \circ f', Y(f') \setminus \{f'(c_2)\}) \simeq (f_0, Y(f_0))$.

Point properties. The critical points $c_1(f')$, $c_1(f_0)$ have period m under f', f_0 respectively, and the points x, x_0 have period p under f', f_0 respectively, if f is type IV. (If f is not type IV, x, x_0 are not defined.) f_0 is a polynomial with $c_2(f_0)=\infty$, but $c_2(f')$ and $f'(c_2(f'))$ are distinct.

$$Y(f') = \begin{cases} \{c_2, f'c_2\} \cup \{f'^i c_1 : 0 \leq i < m\} & \text{if } f \text{ is type II,} \\ \{c_2, f'c_2\} \cup \{f'^i c_1 : 0 \leq i < m\} \cup \{f'^i x : 0 \leq i < p\} & \text{if } f \text{ is type IV,} \end{cases}$$

and $Y(f_0)$ is similarly defined. All points listed as in $Y(f')$ are distinct, and all points listed in $Y(f_0)$ except $c_2 = f_0(c_2) = \infty$ are distinct.

Path properties. $\xi, \alpha, \nu \colon [0,1] \to \bar{\mathbf{C}}$ are simple.
$(\xi((0,1)) \cup \alpha((0,1)) \cup \nu((0,1))) \cap Y(f') = \varnothing$.
$f' \circ \xi = \alpha$, $\xi(0) = \nu(0) = c_2(f')$, $\alpha(0) = \nu(1) = f'(c_2(f'))$, $\xi(1) = (f')^{m-q} c_1$ if f is type II, $\xi(1) = x$ if f is type IV.

Here is a proof of the theorem, if these properties hold. Note that only equivalence property (a) really refers to f. Although f is referred to later, we can speak of "type II" and "type IV" point and path properties, which are relevant to the proof when f is type II and type IV respectively. Even without equivalence property (a), we have

$$\begin{aligned} \sigma_\xi \circ \sigma_\alpha \circ f' &\simeq \sigma_\xi \circ \sigma_\nu \circ \sigma_{\bar\nu} \circ \sigma_\alpha \circ \sigma_\nu \circ \sigma_{\bar\nu} \circ f' \\ &\simeq \sigma_\xi \circ \sigma_\nu \circ \sigma_\delta \circ \sigma_{\bar\nu} \circ f' \\ &\simeq \sigma_\xi \circ \sigma_\delta \circ \sigma_{\bar\nu} \circ f', \end{aligned}$$

where $\delta=\sigma_{\bar{\nu}}\circ\alpha$, so that $\delta(0)=c_2(f')$, $\delta(1)=\alpha(1)$. Then

$$\sigma_{\bar{\nu}}\circ f'\circ\xi=\sigma_{\bar{\nu}}\circ\alpha=\delta.$$

So

$$\sigma_{\xi}\circ\sigma_{\delta}\circ\sigma_{\bar{\nu}}\circ f'\simeq_{\psi}\sigma_{\bar{\eta}}\circ\sigma_{\gamma}\circ f_0,$$

where $\psi(\delta)=\gamma$ and η is isotopic to $\psi\circ\xi$ via an isotopy which leaves $Y(f_0)$ fixed, and $f_0\circ\eta=\gamma$. If equivalence property (a) also holds, we have

$$f\simeq\sigma_{\bar{\eta}}\circ\sigma_{\gamma}\circ f_0,$$

where the properties required for the Polynomial-and-Path Theorem are satisfied by η,γ, and so the theorem is proved.

It is not hard to find f',x,ξ,α satisfying the appropriate conditions, as we shall see in this chapter. If ν is any simple path with endpoints $c_2, f'c_2$, then $\sigma_{\bar{\nu}}\circ f'$ is equivalent to some polynomial f_0. Probably, the stronger equivalence property (b) relating $\sigma_{\bar{\nu}}\circ f'$ and f_0 can be proved in the abstract, for an appropriate x_0. But the approach used in this paper will involve a path f_t in RM_2 from $f=f_1$ to a polynomial f_0. Then f' will be f_t for some $t\in(0,1]$. It is hoped that this approach will lead to a stronger result (though it has not, yet).

2.3. A reduction in the proof

Let $f\in RM_2$ be of type II or IV. Then we can find f', f_0, x, x_0, $Y(f')$, $Y(f_0)$, ξ, α, ν satisfying the equivalence, point and path properties of 2.2—and hence can prove the Polynomial-and-Path Theorem—if there exists a path f_t ($t\in[0,1]$) in RM_2 from a polynomial f_0 to $f_1=f$ and a path x_t in $\bar{\mathbb{C}}$ such that (1) to (3) hold

(1) For some t_1 in $(0,1]$ and for some ξ_{t_1}, α_{t_1}, and for $f'=f_{t_1}$, $x=x_{t_1}$, $\xi=\xi_{t_1}$, $\alpha=\alpha_{t_1}$, all the equivalence, point and path properties which refer only to f, f', ξ, α, x hold.

(2) The point properties hold with f', x replaced by f_t, x_t, for any $t\in(0,1]$, and the point properties hold for f_0, x_0.

(3) $Y(f_t)$ varies isotopically for $t\in(0,t_1]$ and $\lim_{t\to 0}Y(f_t)\setminus\{f_tc_2\}=Y(f_0)$.

To see that (1) to (3) suffice, we claim, first, that for $t\in(0,1]$ there are ξ_t, α_t such that all the equivalence, point and path properties which refer only to f, f', ξ, α, x hold with $f'=f_t$, $\xi=\xi_t$, $\alpha=\alpha_t$, $x=x_t$. For let φ_t ($t\in(0,t_1]$) be an isotopy of $\bar{\mathbb{C}}$ with $\varphi_t(Y(f_{t_1}))=Y(f_t)$ and $\varphi_{t_1}=$identity. Let $\alpha_t=\varphi_t\circ\alpha_{t_1}$ and let ξ_t be isotopic to $\varphi_t\circ\xi_{t_1}$ via an isotopy fixing $Y(f_t)$ with $f_t \circ \xi_t = \alpha_t$. Then the required properties hold, since

$$\sigma_{\xi_t} \circ \sigma_{\alpha_t} \circ f_t \simeq_{\varphi_t^{-1}} \sigma_{\xi} \circ \sigma_{\alpha} \circ f_{t_1},$$

if we write $\xi = \xi_{t_1}$, $\alpha = \alpha_{t_1}$.

Similarly, if for some $s \in (0, t_1]$ there is v_s such that the equivalence and path properties hold for $f' = f_s$, $v = v_s$, $\xi = \xi_s$, $\alpha = \alpha_s$, then we can find $\{v_t\}$ such that they hold for all $t \in (0, t_1]$, with $f' = f_t$, $v = v_t$, $\xi = \xi_t$, $\alpha = \alpha_t$. For we take $v_t = \varphi_t \circ \varphi_s^{-1} \circ v_s$. But if s is very close to 0, we can find $v_s: [0,1] \to \bar{\mathbf{C}}$ with $v_s(0) = c_2(f_s)$, $v_s(1) = f_s(c_2(f_s))$, $v_s([0,1])$ very close to $c_2(f_0)$ and $v(f_s) \setminus \{f_s(c_2(f_s))\}$ very close to $v(f_0)$. Then the appropriate equivalence and path properties must hold for f_s, v_s.

2.4. A path to a polynomial exists

Let W be any irreducible component of the algebraic variety in RM_2 consisting of all $[f, c_1, c_2]$ with c_1 of period $m > 1$ under f. Then W is path-connected. The following lemma is obviously necessary for the proposed method of 2.2.

LEMMA. *For some $[f, c_1, c_2] \in W$, $fc_2 = c_2$, that is, f is a polynomial up to conjugation by a Möbius transformation.*

Proof. Let

$$W_1 = \{[f, c_1, c_2, x] : [f, c_1, c_2] \in W, fx = x\}.$$

Then W_1 is a finite branched cover of W, and an algbraic variety. If W does not contain a polynomial, there is a holomorphic function defined on $W_1 \setminus \{\text{singularities}\}$ by

$$[f, c_1, c_2, x] \mapsto 1/f'(x).$$

In fact, this function must map into the unit disc. For if $|f'(x)| < 1$ for some f, then the hyperbolic component H of f contains polynomials (see [R], for instance), and $\{[g, c_1, c_2] \in H : g^m c_1 = c_1\}$ contains a polynomial, and is contained in W. Now there is a compact Riemann surface W_2 such that $W_1 \setminus \{\text{singularities}\}$ can be identified with $W_2 \setminus \{\text{finitely many points}\}$. But any bounded holomorphic function on $W_2 \setminus \{\text{finitely many points}\}$ extends to a bounded holomorphic function on W_2, and must be constant. So the original holomorphic function on W_1 must be constant, and equal to λ, for some $|\lambda| < 1$. So W_1 coincides with an irreducible component of

$$\{[f, c_1, c_2, x] : [f, c_1, c_2] \in RM_2, f(x) = x, f'(x) = \lambda\} = W_3.$$

But we can identify $W_3 \setminus \{\text{finitely many points}\}$ with

$$\{[f_{a,b}, 0, \infty, 1]: a, b \in \mathbf{C} \setminus \{-1\}, f'_{a,b}(1) = \lambda\},$$

where

$$f_{a,b}(z) = (1+b)(z^2+a)/((1+a)(z^2+b)),$$

so that

$$f'_{a,b}(1) = 2(b-a)/((1+a)(1+b)).$$

Thus $f_{a,b} \in W$ for

$$(1+a)(1+b) = 2\lambda(b-a),$$

and for such (a, b), since λ is assumed fixed, $f^n_{a,b}(0)$ is a nontrivial function of a of degree 2^n-1 for $n \geq 1$. So we have the required contradiction. (I should like to thank the referee for pointing out an error at this point.)

2.5. Proof of the theorem for f of type II

Let W be as in 2.4. Let f_t ($t \in [0, 1]$) be any path in W such that $f=f_1$, f_0 is a polynomial, and all points listed in $Y(f_t)=\{f^i_t c_1, f^j_t c_2: 0 \leq i < m, j=0, 1\}$ are distinct for $t \in [0, 1]$. We know by 2.4 that f_t exists. Then for t near 1 and some $r_t > 0$, there is φ_t: $\{z: |z| < r_t^{1/2}\} \to \bar{\mathbf{C}}$ which is holomorphic and injective and such that

$$\varphi_t(0) = f^{m-q+1}_t(c_1),$$

$$\varphi_t(r_t \zeta_t) = f_t(c_2) \quad \text{for some} \quad |\zeta_t| = 1,$$

$$\varphi_t(z^2) = f^m_t \circ \varphi_t(z).$$

Define α_t by $\alpha_t(u)=\varphi_t((1-u) r_t \zeta_t)$ and ξ_t by $f_t \circ \xi_t = \alpha_t$, $\xi_t(0)=c_2$. Then

$$f_1 \simeq \sigma_{\xi_t} \circ \sigma_{\alpha_t} \circ f_t$$

for t sufficiently near 1. So we can take t_1 to be any $t<1$ sufficiently near 1, and $f_{t_1}, \alpha_{t_1}, \xi_{t_1}$ satisfy the equivalence, point and path properties of 2.2, as required in 2.3.

2.16. Proof of the theorem for f of type IV

Again, let W be as in 2.4. We can choose a path f_t in W from $f=f_1$ to a polynomial f_0 such that $f_t \neq f_1$ for $t<1$ near 1. We can assume $(f^p_t)'(x) \neq 1$ for any point x with $f^p_t x = x$. Then we can find a continuous path x_t with $x_1=c_2(f)$ and x_t of period p under f_t, and f_t, x_t satisfy the properties of 2.2 for all $t\in[0,1)$. Now for t near 1 but <1 there exists a continuous injective φ_t: $\{z: |z|\leq 1\}\to\mathbf{C}$ such that φ_t is holomorphic on $\{z: |z|<1\}$ and

$$\varphi_t(0)=x_t, \quad \varphi_t(1)=c_2(f_t),$$

$$\varphi_t(\lambda_t z)=f_t^p\circ\varphi_t(z),$$

where $\lambda_t=(f_t^p)'(x_t)$, $|\lambda_t|<1$.

Then let ξ_t: $[0,1]\to\bar{\mathbf{C}}$ be defined by $\xi_t(u)=\varphi_t(1-u)$ and $\alpha_t=f_t\circ\xi_t$. Then

$$f_1\simeq\sigma_{\xi_t}\circ\sigma_{\alpha_t}\circ f_t,$$

and ξ_t, α_t satisfy the equivalence and path properties for $t<1$ near 1. So we can take t_1 to be any $t<1$ sufficiently near 1 and the requirements of 2.3 are met.

Chapter 3. Proof of the Lamination Equivalence Theorem

3.1. An improved theorem

We start by stating a modification of the Polynomial-and-Path Theorem for type IV maps, for which I have two proofs, but—surprisingly—neither of them short.

IMPROVED POLYNOMIAL-AND-PATH THEOREM. *Let f be a type IV critically finite degree two rational map with (as usual) critical points c_1 of period m and c_2 of period p. Then if f_0 is as in the Polynomials-and-Path Theorem with 0 of period m, one of the following possibilities occurs.*

(a) *In the equivalence*

$$f\simeq\sigma_{\bar{\eta}}\circ\sigma_{\gamma}\circ f_0,$$

we can choose γ so that $\gamma(1)$ is not in $\partial f_0^i\,U_1(f_0)$, for any $i\geq 0$,

(b) *For some critically finite map g and polynomial h,*

$$f\simeq g\vdash h,$$

with c_2 in the tuned orbit, and

$$g\simeq\sigma_{\bar{\eta}'}\circ\sigma_{\gamma'}\circ f_0,$$

where η', γ' have the properties of η, γ in the Polynomial-and-Path Theorem, and $\gamma'(1)$ is not in $\partial f_0^i\,U_1(f_0)$ for any $i\geq 0$.

3.2. Discussion of the proof of the Improved Theorem

There is nothing to prove unless either $p|m$ (including $p=m$) or $m|p$ with $p\neq m$. For all but one periodic point in $\partial f_0^i U_1(f_0)$ have period strictly greater than m and divisible by m, and the remaining point has period either equal to m or dividing m. We deal with the case $p|m$ here, and with $m|p$ at the end of Chapter 5. So now suppose $p|m$, and an equivalence

$$f\simeq \sigma_{\bar{\eta}}\circ\sigma_{\gamma}\circ f_0$$

has been obtained which does not satisfy (a) of the Improved Theorem, i.e. $\gamma(1)\in\partial f_0^i U_1(f_0)$ for some $i\geq 0$. Suppose also that this was obtained using paths $\{f_t\}$, $\{x_t\}$ as in 2.3, with $f_1=f$ and $\lim_{t\to 1} x_t=c_2(f)$. We now show that we could perhaps have made a different choice $\{x_t'\}$ rather than $\{x_t\}$, and hence x_0' rather than x_0.

For let $\psi_1\colon\{z\colon|z|\leq 1\}\to\overline{U_2(f)}$ be continuous and holomorphic on $\{z\colon|z|<1\}$ and such that

$$\psi_1(0)=c_2(f),$$

$$\psi_1(z^2)=f^p\circ\psi_1(z).$$

Then $f^p\circ\psi_1(1)=\psi_1(1)$, so that $\psi_1(1)$ has period p_1 under f, for some $p_1|p$. Define $\xi\colon[0,1]\to\bar{\mathbf{C}}$ by $\xi(t)=\psi_1(t)$, and let $\alpha=f\circ\xi$. Then, depending on whether $p_1=p$ or $p_1<p$,

$$f\simeq\sigma_{\xi}\circ\sigma_{\alpha}\circ f\quad\text{or}\quad f\simeq(\sigma_{\xi}\circ\sigma_{\alpha}\circ f)\vdash_{c_2}h,$$

for some polynomial h of the form $0\mapsto z^2+a$ with 0 of period p/p_1 under h. In the latter case, the attractive basins of h have a common fixed boundary point. Now we can assume the path $\{f_t\}$ satisfies

$$(f_t^{p_1})'(z)\neq 0 \text{ or } 1\quad\text{when}\quad f_t^{p_1}(z)=z,\quad\text{for}\quad t<1.$$

Now we can take $x_1'=\psi_1(1)$, and extend to a path $\{x_t'\}$ with x_t' of period p_1 under f_t, for all t. Then, for all t, the orbits of x_t, x_t' are disjoint under f_t, since this is true for $t=1$. Then f_t, ξ, α satisfy the requirements outlined in 2.3 to give

$$\sigma_{\xi}\circ\sigma_{\alpha}\circ f\simeq\sigma_{\bar{\eta}'}\circ\sigma_{\gamma'}\circ f_0,$$

where $\eta'\colon[0,1]\to\bar{\mathbf{C}}$ is simple, $\eta'(0)=\infty$, $\eta'(1)=x_0'$, and $\gamma'=f_0\circ\eta'$. Since x_0' is not in the orbit of x_0, and has period $\leq m$, it cannot be in the boundary of the attractive basin of $f_0^i(0)$ for any $i\geq 0$ (since the only point of period $\leq m$ in $\partial f_0^i U_1$ is in the orbit of x_0), so we are done.

3.3. Proof of the Lamination Equivalence Theorem: Type III

Let f be critically finite type III, such that

$$f \simeq \sigma_\gamma \circ f_0$$

is the equivalence of the Polynomial-and-Path Theorem. Then, for some odd denominator rational r,

$$f_0 \simeq s_r,$$

so that

$$(f_0, f_0^i\gamma(1): i \geqslant 0) \simeq_\psi (\bar{s}_r, X)$$

for some X. Then we can choose a simple $\beta: [0, 1] \to \bar{\mathbf{C}} \setminus K_r$ such that $\psi \circ \gamma$ and β are isotopic via an isotopy fixing X, and $\beta^{-1}(\cup \bar{L}_r) = \{t\}$ for some $t \in (0, 1)$. Let l_0 be the leaf of $\bar{L}_r$ with $\beta(t) \in l_0$. We have

$$f \simeq \sigma_\beta \circ \bar{s}_r.$$

Let L be defined by

$$\cup L = ((\bar{L}_r \setminus \bigcup_{i \geqslant 1} (\bar{s}_r)^{*i} l_0) \cup \bigcup \{((\sigma_\beta \circ \bar{s}_r)^*)^i l_0 : i \geqslant 0\})^-,$$

where $\bar{s}_r^*$ and $(\sigma_\beta \circ \bar{s}_r)^*$ are as in 1.13. Let G be the gap of $\bar{L}_r$ containing $\beta(1)$. Then L is a β-invariant lamination which has $\bar{s}_r^j G$ as a gap for $j \geqslant 0$, and $\infty \in (\sigma_\beta \circ \bar{s}_r)^* G$. Hence, the lamination map ϱ_L (see 1.14) is uniquely defined up to equivalence, and

$$\varrho_L \simeq \sigma_\beta \circ \bar{s}_r \simeq f.$$

3.4. Proof of the Lamination Equivalence Theorem: Type II

Let f be critically finite type II, such that

$$f \simeq \sigma_{\bar{\eta}} \circ \sigma_\gamma \circ f_0$$

is the equivalence of the Polynomial-and-Path Theorem. Then as in 3.3, if

$$f_0 \simeq_\psi \bar{s}_r,$$

we can find a simple $\beta: [0, 1] \to \bar{\mathbf{C}} \setminus K_r$ such that $\beta^{-1}(\cup \bar{L}_r) = \{t\}$, $\beta(t) \in l_0$, $\beta(1) \in G$, where

G is a gap of $\bar{L}_r$, $\bar{s}_r \circ \zeta = \beta$ with $\zeta(1) = \bar{s}_r^{m-1}\beta(1)$ and

$$f \simeq \sigma_\zeta \circ \sigma_\beta \circ \bar{s}_r.$$

Let

$$L_0 = \{l \in \bar{L}_r : l \subset \partial \bar{s}_r^i G, i \geq 0\}.$$

Then L_0 is a lamination. For $n \geq 1$, let

$$L_{n+1} = (\sigma_\beta \circ \bar{s}_r)^* L_n.$$

Then for every leaf of L_n, there is a simply-connected gap of L_n which has l in its boundary and is contained in a simply-connected gap of L_{n+1}. Also, for $n \geq 1$, the gap containing ∞ also contains $\bar{s}_r^{m-q}(0)$, and the gaps of L_n containing $\bar{s}_r^i(0)$ $(0 \leq i < m)$ are all simply-connected and distinct. Then

$$L = \lim_{n \to \infty} L_n$$

exists and is β-invariant, and

$$\varrho_L \simeq \sigma_\zeta \circ \sigma_\beta \circ \bar{s}_r \simeq f.$$

3.5. Proof of the Lamination Equivalence Theorem: Type IV(a)

Let f be critically finite type IV, and let

$$f \simeq \sigma_{\bar{\eta}} \circ \sigma_\gamma \circ f_0$$

be an equivalence as in the Improved Theorem part (a). (See 3.1.) Let

$$\psi \circ \bar{s}_r = f_0 \circ \psi,$$

where, in the previous notation, $\psi = \varphi \circ \Phi_r$. Thus, $\psi = \lim_{t \to 1} \psi_t$, where $\{\psi_t : t \in [0, 1)\}$ is a continuous path of homeomorphisms with $\psi_0 =$ identity. Then $\psi^{-1}(\gamma(1))$ contains a set D, where D is either a point of K_r, the closure of a leaf of $\bar{L}_r$, or the boundary of a finite-sided gap of $\bar{L}_r$, and D has the following properties. It can be chosen not to intersect the boundary of any gap of $\bar{L}_r$ in the full orbit of G_0 under $\bar{s}_r$ (where G_0 is the gap containing 0), and has period p under $\bar{s}_r$ (where p is the period of $\gamma(1)$ under f_0). Then (after modifying $\bar{s}_r$ if necessary) we can find a neighbourhood U of D with $SU \subset U$, for a well-defined local inverse S of $\bar{s}_r^p$ with $SD = D$, and with $\bar{s}_r^i U$ $(0 \leq i < p)$ all disjoint. Then we can find simple paths $\beta, \zeta\colon [0,1]\to\bar{\mathbf{C}}\setminus(K_r\cup\bigcup\bar{L}_r)$ with the following properties, after modifying $\bar{s}_r$, if necessary.

(a) $f\simeq\sigma_\zeta\circ\sigma_\beta\circ\bar{s}_r$.

(b) $\beta([t,1])\subset U$, $\beta([0,1])\cap\bar{s}_r^i U=\varnothing$, $1\leq i<p$, and the sets $\beta([0,1])$, $S^i\beta([t,1))$ $(i\geq 1)$ are all disjoint, but $\bar{s}_r^p\beta(1)=\beta(t)$.

(c) $\overline{\lim}_{n\to\infty}\bigcup_{i\geq n}S^i\beta([t,1])=D$.

(d) $\bigcup_{i=1}^\infty S^i(\beta([t,1]))\subset l_0$,

where l_0 is a geodesic homotopic in $\bar{\mathbf{C}}\setminus K_r$ to the component of

$$(\sigma_\beta\circ\bar{s}_r)^{1-p}\,\bar{s}_r^{-1}(\beta([0,1])\cup\bigcup_{i=1}^\infty S^i\beta([t,1])),$$

which contains $\bigcup_{i=1}^\infty S^i\beta([t,1])$.

(e) $\zeta(1)=\bar{s}_r^{p-1}\beta(t)$, and $\bar{s}_r\circ\zeta(u)=\beta(tu)$ for all $u\in[0,1]$.

Then let

$$L_1=\bigcup_{i=1}^p((\sigma_\beta\circ\bar{s}_r)^{*i}\,l_0.$$

Then L_1 is a finite lamination, all of whose leaves bound finite-sided gaps, including a gap G_1 containing l_0 such that $\bar{s}_{r*}^{-1}\bigcup\partial G_1)=(\sigma_\beta\circ\bar{s}_r)^*l_0$. Then for $n\geq 1$, let

$$L_{n+1}=(\sigma_\beta\circ\bar{s}_r)^*\,L_n,$$

and let

$$L=(\lim_{n\to\infty}L_n)\cup\bar{L}_r.$$

Then L is a β-invariant lamination with ∞ in a simply-connected gap G_∞ of period p under $\bar{s}_*$, and

$$\varrho_L\simeq\sigma_\zeta\circ\sigma_\beta\circ\bar{s}_r\simeq f.$$

3.6. Tuning laminations and the proof of the Lamination Equivalence Theorem for Type IV(b)

Let L be a β-invariant lamination with ϱ_L type IV, and such that the gap G_∞ containing ∞ is simply connected. We now show how to produce, for each odd denominator rational q in $(0,1)$, an invariant lamination $L(q)$ with $L\subset L(q)$, with the minor gap of $L(q)$ contained in the minor gap of L, and

$$\varrho_{L(q)}\simeq\varrho_L\vdash_\infty s_q.$$

Let G denote the minor gap of L in $\tilde{U}$, and let F be a lift of ϱ_L^p with $F(G)=G$, where p is the period of ∞ under ϱ_L. Then if ∂G denotes the boundary of G in $D=\{z: |z|\leq 1\}$, which contains $\tilde{U}$, we can extend F (uniquely) to ∂G. Then $F|\partial G$ is a degree two orientation-preserving covering. So there is a unique monotone $\varphi: \partial G \to \{z: |z|=1\}$ such that $\varphi(F(z))=(\varphi(z))^2$ for all $z \in \partial G$. Then we can extend φ to a map $\varphi: \bar{G} \to D$ with the following properties (where L_q is the lamination of 1.10).

(a) If $l \in L_q$ is not in the boundary of a finite-sided gap of L_q, then $\varphi^{-1}(l)$ is either a geodesic, or bounded by ∂G and two geodesics in G.

(b) If Δ is a finite-sided gap of L_q with n sides, then $\varphi^{-1}(\Delta)$ is bounded by a subset of ∂G with n components, and n geodesics in G, and $\partial\varphi^{-1}\Delta=\varphi^{-1}\partial\Delta$.

(c) Otherwise, $\varphi^{-1}(z)$ is a point for $z \in G$.

Now let $\pi: \mathring{D} \to \bar{\mathbf{C}} \setminus K_r$ be the quotient map, and let

$$L(q)' = \{\pi(l_1): l_1 \text{ is a geodesic in } G \text{ which is in } \partial\varphi^{-1}(l) \text{ for } l \text{ a leaf of } L_q \text{ or in } \partial\varphi^{-1}\Delta \text{ for } \Delta \text{ a finite-sided gap of } L_q\}.$$

Here, ∂ denotes a boundary of a subspace of G. Let $\mu_q'=\pi(\mu_q'')$, where μ_q'' is the geodesic in $\varphi^{-1}(\mu_q) \cap G$ which is nearest to $\varphi^{-1}(s_q(0))$. Let $\alpha: [0,1] \to \pi(G)$ be a simple path with

$$\alpha(0)=\beta(1), \quad \alpha([0,1]) \cap \mu_q' = \{\alpha(t)\}, \quad \alpha(1)=\pi\circ\varphi^{-1}(\bar{s}_q 0).$$

Let $\gamma: [0,1] \to \bar{\mathbf{C}} \setminus K_r$ be a path which is homotopic in $\bar{\mathbf{C}} \setminus K_r$ to $\beta * \alpha$, where

$$\beta*\alpha(t) = \begin{cases} \beta(2t) & \text{if } t \leq 1/2, \\ \alpha(2t-1) & \text{if } t \geq 1/2. \end{cases}$$

Then

$$(\sigma_\gamma \circ \bar{s}_r)^* L = (\sigma_\beta \circ \bar{s}_r)^* L = L,$$

and

$$((\sigma_\gamma \circ \bar{s}_r)^*)^p L(q)' \cap \pi(G) = L(q)'.$$

Then put

$$L(q) = L \cup \bigcup_{n=0}^{\infty} ((\sigma_\gamma \circ \bar{s}_r)^*)^n L(q)'.$$

Then $(\sigma_\gamma \circ \bar{s}_r)^* L(q) = L(q)$, and there exists $\eta: [0,1] \to \bar{\mathbf{C}}$ with $\bar{s}_r \circ \eta = \gamma$, and

$$\varrho_L \vdash_\infty s_q \simeq \sigma_{\bar{\eta}} \circ \sigma_\gamma \circ \bar{s}_r \simeq \varrho_{L(q)}.$$

Chapter 4. Equivalence and conjugacy

4.1

We start by giving details of the general semiconjugacy result mentioned in 1.5.

SEMICONJUGACY PROPOSITION. *Let f be a critically finite rational map, and let g be a critically finite branched covering with* $f \simeq_{\Psi_0} g$, *and let* Ψ_0 *be a topological conjugacy between f and g in neighbourhoods of orbits of periodic critical points of* $X(f)$, $X(g)$. *Then there is a continuous* $\Psi: \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ *such that* $\Psi = \lim_{t \to +\infty} \Psi_t$, Ψ_t *is a homeomorphism for* $t \in [0, \infty)$, Ψ *is a homeomorphism of some neighbourhood of the forward orbits of periodic critical points in* $X(f)$ *onto a neighbourhood of the corresponding points in* $X(g)$, *and* $\Psi \circ g = f \circ \Psi$.

Proof. Let $U(f)$, $U(g)$ be neighbourhoods of the forward orbits in $X(f)$, $X(g)$ of periodic critical points of f, g and let the orientation-preserving homeomorphism Ψ_0, and f_t $(t \in [0, 1])$ be such that

$$f_0 = \Psi_0 \circ g \circ \Psi_0^{-1}, \quad f_1 = f,$$

$$X(f_t) = X(f) \text{ for all } t, \quad \text{and} \quad f_t = f \text{ on } U(f) \text{ for all } t, \tag{1}$$

$$\Psi_0(U(g)) = U(f).$$

These exist, since $f \simeq g$. Then we can define a path Ψ_t $(t \in [0, 1])$ by

$$f_t \circ \Psi_t = \Psi_0 \circ g. \tag{2}$$

Then

$$f \circ \Psi_1 = \Psi_0 \circ g,$$

$$\Psi_t = \Psi_0 \quad \text{on} \quad U(g), \tag{3}$$

$$\Psi_1(g^{-1}U(g)) = f^{-1}U(f).$$

Then we can define Ψ_{t+n} $(n \in \mathbf{N}, n \geq 1, t \in [0, 1])$ by

$$f^n \circ \Psi_{t+n} = \Psi_t \circ g^n, \tag{4}$$

with, in addition,

$$\begin{aligned}\Psi_{t+n} &= \Psi_0 \quad \text{on } U(g) \text{ for all } t,\ n,\\ \Psi_{t+n} &= \Psi_n \quad \text{on } g^{-n}U(g) \text{ for all } n, \text{ and } t\in[0,1].\end{aligned} \tag{5}$$

Since $f^n\circ f\circ\Psi_{n+1}=f^{n+1}\circ\Psi_{n+1}=\Psi_0\circ g^{n+1}=\Psi_0\circ g^n\circ g=f^n\circ\Psi_n\circ g$, and $f\circ\Psi_{n+1}=\Psi_n\circ g$ on $U(g)$, we have

$$f\circ\Psi_{n+1}=\Psi_n\circ g \quad \text{for all } n. \tag{6}$$

By (6), to prove the proposition, it suffices to prove $\{\Psi_n\}$ is uniformly convergent. In fact, we show that for some $C>0$ and $0<\lambda<1$,

$$d(\Psi_{n+t}x, \Psi_n x)\leqslant C\lambda^n \quad \text{for all } n\in\mathbf{N} \text{ and } t\in[0,1], \tag{7}$$

where d denotes the spherical metric on $\bar{\mathbf{C}}$. By (5), we only have to prove this for $x\notin g^{-n}U(g)$. Choose C, λ so that

$$\sup\{d(S\Psi_u y, S\Psi_0 y)\colon y\notin U(g),\ u\in[0,1]\}\leqslant C\lambda^n, \tag{8}$$

where S is any local inverse of f^n defined on the set $\{\Psi_u y\colon u\in[0,1]\}$. Note that this set does not intersect $U(f)$. Then (7) follows immediately from (8), because $\Psi_n x=Sg^n x$, $\Psi_{t+n}x=S\Psi_t g^n x$, for some local inverse S of f^n, and $g^n x\notin U(g)$.

4.2

We are especially interested in applying the Semiconjugacy Proposition in the case when $g=\varrho_L$ is the lamination map of a β-invariant lamination L. For the rest of this chapter (except 4.7, and even there, if desired), f is critically finite rational, $g=\varrho_L$ and $\Psi\circ g=f\circ\Psi$. We are going to prove the Conjugacy Theorem modulo two results about invariant laminations. We need to map $\simeq_L$-classes to points (see 1.11). The following lemma is relevant.

LEMMA. *Let f, g, Ψ be as in* 4.1. *Let $\gamma\colon[0,1]\to\bar{\mathbf{C}}$ with $\gamma\cap U(g)=\varnothing$. If γ_n satisfies $g^n\circ\gamma_n=\gamma$ and δ is a limit of a subsequence of the sequence of sets $\{\gamma_n([0,1])\}$, then $\Psi(\delta)$ is a point.*

Proof. It suffices to show

$$\lim_{n\to\infty}\operatorname{diameter}\Psi_n(\gamma_n)=0,$$

where Ψ_n is as in 4.1. But $\Psi_n\gamma_n=S\Psi_0\gamma$, where S is a local inverse of f^n, and $\Psi_0\gamma\cap U(f)=\emptyset$. The result follows.

4.3

Lemma 4.2 can be applied to show that if l is a leaf in L with a compact subset l_1 such that $l\subset\bigcup_n\varrho_L^{-n}l_1$, and $\varrho_L\simeq f$ as in 4.1, then $\Psi(l)$ is a point. Such a leaf l is called *segment-periodic*, and any component of $\varrho_L^{-n}l$ is called *segment-preperiodic*. Then $\Psi(l)$ is a point if l is segment-preperiodic. Then $\Psi(l)$ is a point for all $l\in L$ if the following lemma is true. It will be proved in Chapter 6.

PERIODIC LEAF LEMMA. *If L is β-invariant, and $\beta(1)$ is in a gap of L, then segment-preperiodic leaves are dense in L.*

4.4

LEMMA. *If the Periodic Leaf Lemma holds, then Ψ maps $\simeq_L$-equivalence classes to points.*

Proof. We have already seen (4.2, 4.3) that $\Psi(l)$ is a point for all $l\in L$. Since Ψ is continuous, it suffices to show that $\Psi(G)$ is a point whenever G is a finite-sided gap, so that $\partial G=\bar{l}_1\cup\ldots\cup\bar{l}_n$ for $l_i\in L$ $(1\leq i\leq n)$. Since Ψ is a limit of homeomorphisms, $\partial\Psi(G)\subset\Psi(\partial G)=\bigcup_{i=1}^n\Psi(l_i)$. So $\Psi(G)$ is either all of $\bar{\mathbb{C}}$ or a point. But $\Psi(G)\cap U(f)=\emptyset$, because $\Psi_n(G)\cap U(f)=\emptyset$ for all n. So $\Psi(G)$ is a point.

4.5

We want to show that $\Psi^{-1}(z)$ is always a single $\simeq_L$-class. The following lemma is useful. For the rest of this section, we write $K_r=K$, $\bar{s}_r=\bar{s}$, $\Phi_r=\Phi$.

LEMMA. *$\bar{l}\cap K\neq\emptyset$ for all $l\in L$, if $\Psi(l)$ is a point for all $l\in L$.*

Proof. If $\bar{l}\cap K=\emptyset$ then l crosses only finitely many components I_i $(1\leq i\leq k)$ of $S^1\setminus K$. Either l is a loop, or, for each $\delta>0$ there are distinct segments of l crossing some I_i in the same direction, with crossing points distance $<\delta$ apart in the spherical metric. Choose two such segments, and join along I_i to get a closed loop $\gamma(\delta)$. If l is a loop, put $\gamma(\delta)=l$ for all $\delta>0$. In either case, $\gamma(\delta)$ separates some of the J_i $(1\leq i\leq k)$ where these are the components of $S^1\setminus\bigcup_{i=1}^k I_i$. Because $\Psi(l')$ is a point for all $l'\in L$,

$$\lim_{\delta\to 0}\text{ diameter }\Psi(\gamma(\delta))=0.$$

Then, taking limits, $\Psi^{-1}(\Psi(l))$ intersects K in an open subset, whose image under ϱ_L^n, for some n, contains K. We deduce that $\Psi^{-1}(\Psi(l))$ contains K. We deduced this from the assumption that $\bar{l}\cap K=\emptyset$. It follows that $\Psi(\bigcup L\cup K)$ is a single point z. Then $\Psi^{-1}(z)$ also contains all but one gap of L, since $\Psi^{-1}(\bar{\mathbb{C}}\setminus\{z\})$ is connected. Then $\#(X(\varrho_L))=\#(X(f))=2$, so that $X(\varrho_L)\subset U(\varrho_L)$. But then we have a contradiction, since $\varrho_L|U(\varrho_L)$ should be a homeomoporhism. So $\bar{l}\cap K\neq\emptyset$ for all l, as required.

4.6

From now on, we assume, in addition to the standing hypotheses for f, ϱ_L with $f\simeq\varrho_L$, that the gap G_0 of L containing 0 is simply connected, and that $\varrho_L^{im}z\to 0$ as $i\to\infty$ for all $z\in G_0$, and that a similar property holds for the gap G_∞ containing ∞ if ∞ is periodic under ϱ_L. Then Ψ is a homeomorphism of either $\bigcup_{i=0}^{\infty}\varrho_L^{-i}(G_0\cup G_\infty)$ or $\bigcup_{i=0}^{\infty}\varrho_L^{-i}G_0$ onto $\bar{\mathbb{C}}\setminus J(f)$, depending on whether or not ∞ is periodic under ϱ_L. If we assume the Finite-sided Gap Lemma below, then $\Psi^{-1}(z)$ can only be non-singleton if it is a union of $\simeq_L$-classes, each of which intersects K. The Finite-sided Gap Lemma will be proved in Chapter 6.

FINITE-SIDED GAP LEMMA. *If L is invariant, then any gap G which is not in the full orbit of G_0—or of G_∞ if ∞ is periodic—is finite-sided.*

4.7

Now here is the key to proving that $\Psi^{-1}(z)$ is a single $\simeq_L$-class, under the assumption that ϱ_L is type II, III or IV, that is, the equivalent rational map f is hyperbolic. The lemma simply concerns an expanding map g on a metric space.

LEMMA. *Let (X,d) be compact metric, and $g\colon X\to X$ expanding. Let $f\colon Y\to Y$ and $\psi\colon X\to Y$ be continuous, with $\psi\circ g=f\circ\psi$. Then either there is N such that $\psi^{-1}(y)$ always has $\leq N$ elements, or some $\psi^{-1}(y)$ contains x, x' with $x\neq x'$ but $g(x)=g(x')$.*

Proof. Suppose there is no bound on the number of elements in $\psi^{-1}(y)$ $(y\in Y)$. Since g is expanding, there are $\delta_0>0$ and $\lambda>1$ such that $d(gx_1,gx_2)\geq\lambda d(x_1,x_2)$ whenever $d(x_1,x_2)\leq\delta_0$. Since ψ is continuous, any limit of a sequence $\psi^{-1}(y_n)$ must be contained in some $\psi^{-1}(y)$. Then, given $\delta_0>\varepsilon>0$, choose N so that X is covered by N $\varepsilon/2$-balls. Choose $y_\varepsilon\in Y$ so that $\psi^{-1}(y_\varepsilon)$ has $\geq N$ elements. Choose $\delta>0$, and $x_i\in\psi^{-1}(y_\varepsilon)$ $(1\leq i\leq N)$ so that, if $i\neq j$, $d(x_i,x_j)\geq\delta$. Choose n so that $\lambda^n\delta>\delta_0$. Then for some $i\neq j$, $d(g^nx_i,g^nx_j)<\varepsilon$. Choose $0\leq k<n$ so that $d(g^lx_i,g^lx_j)<\varepsilon$ for $k<l\leq n$, but $d(g^kx_i,g^kx_j)\geq\varepsilon$. Then k must exist,

because otherwise $d(g^l x_i, g^l x_j) < \varepsilon < \delta_0$ for $0 \leq l \leq n$, and $d(g^n x_i, g^n x_j) \geq \lambda^n d(x_i, x_j) \geq \lambda^n \delta > \delta_0$, giving a contradiction. Then $d(g^k x_i g^k x_j) > \delta_0$, because otherwise $d(g^{k+1} x_i, g^{k+1} x_j) \geq \lambda\varepsilon$. Now $g^k x_i, g^k x_j \in \psi^{-1}(f^k y_\varepsilon) = \psi^{-1}(z_\varepsilon)$, writing $z_\varepsilon = f^k y_\varepsilon$. Taking a limit set of the sets $\psi^{-1}(z_\varepsilon)$ as $\varepsilon \to 0$, we obtain a set $\psi^{-1}(y)$ containing points x, x' with $d(x, x') \geq \delta_0$ but $g(x) = g(x')$, as required.

4.8

I think the following proposition can be made to work if f is critically finite rational but not hyperbolic, but the proof would be more involved. We continue with the assumptions made in 4.6, 4.7.

PROPOSITION. *$\Psi^{-1}(z)$ is a single $\simeq_L$-class for each z.*

Proof. If $\Psi^{-1}(z)$ is not a single $\simeq_L$-class, it must be a union of infinitely many, since it is connected. Also, by 4.5, 4.6, each such set must have infinite intersection with K. So $\Psi: K \to J(f)$, with $\Psi \circ \bar{s} = f \circ \Psi$, is not boundedly finite-to-one. We can find a metric on K for which $\bar{s}$ is expanding, since $s: z \mapsto z^2$ is expanding, $\Phi: K \to S^1$ satisfies $\Phi \circ \bar{s} = s \circ \Phi$ and $\Phi^{-1}(z)$ is non-singleton for z in at most two eventually periodic full s-orbits. So, by 4.7, we can find $z, -z \in K$ with $\Psi(z) = \Psi(-z)$. Now, for any $w \in \bar{\mathbb{C}}$,

$$\Psi^{-1} f^{-1}(w) = \varrho_L^{-1} \Psi^{-1}(w) = A \cup -A,$$

for a closed connected set A, and then $f^{-1}(w) = \{\Psi(A), \Psi(-A)\}$. Hence $f^{-1}(w)$ is a point if $\Psi(w_1) = \Psi(-w_1)$ for some $w_1 \in \varrho_L^{-1} \Psi^{-1}(w)$. So $f^{-1} \Psi_{\varrho_L}(z) = f^{-1} \Psi \bar{s}(z)$ is a point. Then $\Psi \bar{s}(z)$ must be a critical point of f, and hence in $U(f)$. But this is impossible, because z and $\bar{s}z$ are in K, and $\Psi(K) \cap U(f) = \varnothing$.

4.9

The following completes the proof of the Lamination Conjugacy Theorem, modulo the Periodic Leaf Lemma and the Finite-sided Gap Lemma.

LEMMA. *Under the previous hypotheses, $\simeq_L$ coincides with the smallest (not a priori closed) equivalence relation $\simeq$ such that $z_1 \simeq z_2$ if either $z_1, z_2 \in \bar{l}$ for some $l \in L$ or $z_1, z_2 \in \bar{G}$ for some finite-sided gap G of L.*

Proof. We clearly have $\simeq \subset \simeq_L$. So it suffices to show that $\simeq$ is closed. So it suffices to show that if $x_n \simeq y_n$, $x_n \to x$ and $y_n \to y$, then $x \simeq y$. Let N be the maximal number of elements of K in an $\simeq_L$-class. By pasing to a subsequence, we may assume there is t, with $2 \leq t \leq N$ and that there are $x_{n,i}$ $(1 \leq i \leq t)$, $l_{n,i}$ $(2 \leq i \leq t)$ such that the following hold: $x_{n,1}=x_n$, $x_{n,t}=y_n$, $x_{n,i-1}$, $x_{n,i} \in \bar{l}_{n,i}$ $(2 \leq i \leq t)$, $x_{n,i} \in K$ $(2 \leq i \leq t-1)$, $\lim_{n\to\infty} x_{n,i}=z_i$ $(1 \leq i \leq t)$ with $z_1=x$, $z_t=y$, and $\lim_{n\to\infty} \bar{l}_{n,i}$ exists in the Hausdorff topology $(2 \leq i \leq t)$. Then $\lim_{n\to\infty} \bar{l}_{n,i}$ is a union of leaves of L—possibly infinite, but $(\lim_{n\to\infty} \bar{l}_{n,i}) \cap K$ has $\leq N$ elements. So $z_{i-1} \simeq z_i$ $(2 \leq i \leq t)$ and $z_1 \simeq z_t$, that is $x \simeq y$.

Chapter 5. Rays

5.0. Contents

This chapter ends with the completion of the proof of the Improved Polynomial-and-Path Theorem for f of type IV, modulo the Tuning Proposition. The key for that is 5.9, the Endpoint Theorem.

Throughout this chapter, we consider rational maps in W, where W is an irreducible variety in RM_2 consisting of $[f, c_1, c_2]$ with c_1 of period m under f. It might be worth pointing out that it follows from the descriptions in [R] (for example) that if H is a hyperbolic component, $H \cap W$ is connected. As in the introduction, we use U_1, U_2 to denote the components of the Julia set complement containing c_1, c_2. (U_2 might not exist.)

5.1. Rays

Let $f \in W$. If $c_2 \neq f^n c_1$ for any $n \geq 0$, and $0 \leq i < m$, there is $0 < r \leq 1$ and a unique holomorphic injective map $\varphi = \varphi_{f,i}$: $\{z: |z| < r\} \to \bar{\mathbb{C}}$ such that

$$\varphi(0) = f^i c_1,$$

$$\varphi(z^2) = f^m \varphi(z).$$

If $c_2 \notin f^n U_1$ for any $n \geq 0$, we can take $r=1$, and then we call the set

$$\varphi(\{\varrho e^{2\pi i\alpha}: \varrho \in (0,1)\})$$

the *ray of argument* α *in* $f^i U_1$.

If $c_2 \in f^n U_1$ for some $n \geq 0$ but $c_2 \neq f^n c_1$, then $f^j c_2 \in f^i U_1$ for some $0 < j \leq m$, and then we can ensure that $f^j c_2 \in \operatorname{Im} \varphi$. Consider the connected component C of

$$\bigcup_{n\geq 0,\, k\geq 0} f^{-mn} \varphi(\{\varrho e^{2\pi i\alpha 2^k}: \varrho \in (0,r)\})$$

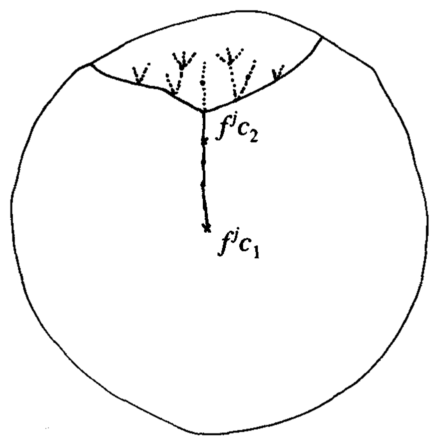

Diagram 4.

which contains $\varphi(\{\varrho e^{2\pi i\alpha}: \varrho\in(0,r)\})$. If $\varphi^{-1}(f^jc_2)\neq\varrho e^{2\pi i\alpha 2^k}$ for any $\varrho\in(0,r)$, $k\geq 0$, this is the *ray of argument α in* f^iU_1. If $\varphi^{-1}(f^jc_2)=\varrho e^{2\pi i\alpha 2^k}$ for some $\varrho\in(0,r)$, $k\geq 0$ (there is only one pair (ϱ,k) unless α is rational with odd denominator, when there is an infinity of values of k) then the *ray of argument α in* f^iU_1 is $\partial V\cap f^iU_1$, where V is the component of $f^iU_1\setminus C$ which contains f^ic_1. Thus, in this case, the ray of argument α is topologically a Y. See Diagram 4.

Note that the ray is invariant under f^{mp}, if α has period p under $x\mapsto 2x$ mod 1. This definition coincides with the previous one if $c_2\notin f^nU_1$ for $n\geq 0$.

5.2. Endpoints

A ray of rational argument has an *endpoint* if the ray is topologically a line, and one or two endpoints (but, actually, two) if the ray is topologically a Y. If $c_2\notin f^nU_1$ for $n\geq 0$, the endpoint is defined to be $\lim_{r\to 1}\varphi(re^{2\pi i\alpha})$, which does exist. This is proved, and endpoints are defined in general, using a standard argument which originates (I think) in Douady and Hubbard's work. In general, the endpoint, or set of two endpoints of a ray, is defined as $\bar{R}\setminus(R\cup\{f^ic_1\})$ if R is the ray in f^iU_1 of argument α. To see that this consists of one or two points, we can assume without loss of generality that α is of odd denominator and of period p under $x\mapsto 2x$ mod 1. Then let B' be the image under φ of an open ball which contains $\varphi(\varrho' e^{2\pi i\alpha})$ for $\varrho^{2^p}\leq\varrho'\leq\varrho$, and such that $B'\subset\operatorname{Im}\varphi$, and B' is disjoint from all its forward images except $f^{mp}B'$. Let B be another ball with $\bar{B}\subset B'$, $f^{mp}B\cap B\neq\varnothing$, and also containing $\varphi(\varrho' e^{2\pi i\alpha})$ for $\varrho^{2p}\leq\varrho'\leq\varrho$. Then

$$\bar{R}\setminus(R\cup\{f^ic_1\})\subset\lim_{N\to\infty}\left(\bigcup_{n=N}^{\infty} f^{-mpn}B\cap R\right).$$

All the components of $f^{-mpn}B$ are balls up to bounded distortion, by the Koebe distortion theorem [Du], and have diameters converging to 0 as $n \to \infty$, since any ball intersects only two others. It follows that the endpoints of R are well-defined and fixed by f^{mp}.

5.3. Rays, endpoints and centres in hyperbolic components

Now let H be a type II hyperbolic component, with $f^q c_2 \in U_1$, but $f^q c_2 \neq c_1$. Then $\varphi_{f,0} = \varphi_f$, (5.1) exists with $f^q c_2 \in \operatorname{Im} \varphi_f$ and $v(f) = \varphi_f^{-1}(f^q c_2)$ is well-defined. It was shown in [R] that

$$v: \{f \in H \cap W: f^q c_2 \neq c_1\} \to \{z: 0 < |z| < 1\}$$

is a degree three covering, except when H contains $z \mapsto 1/z^2$. For that exception, v is a homeomorphism. A component of $v^{-1}(\{re^{2\pi i a}: r \in (0,1)\})$ is called a *ray of argument a in $H \cap W$*. We call the unique critically finite map in $H \cap W$ the *centre*. It was shown in [R] that if $R = \{f_r: r \in (0,1)\}$ is a ray of argument a with $v(f_r) = re^{2\pi i a}$, for a an odd denominator rational in $(0, 1)$, then $f_1 = \lim_{r \to 1} f_r$ exists. We call f_1 the *endpoint* of R. It was also shown in [R] 6.12 that the ray of argument a in $U_1(f_1)$ ends at a parabolic point in the boundary of a parabolic basin containing $f_1^q c_2$, and that the parabolic basin has period mp. Note that, from the definition in 5.1, rays of argument a in $U_1(f)$ (for a rational) are topologically lines unless f lies on a ray of argument $2^k a$ in a type II component, for some $k \geq 0$.

5.4

The following lemma uses the Tuning Proposition, which will be proved in Chapter 8, at one point.

LEMMA. *Let $m > 1$. Let $f \in W$ be either hyperbolic of type* III *or* IV, *or have a cycle of parabolic basins. Then two rays of rational arguments a, a' from $\cup f^i U_1(f)$ can only have the same endpoint if $a = a' = 0$.*

Proof. If f is as described, then $\varphi_{f,i}$ is defined on $\{z: |z| < 1\}$ and extends continuously to $\{z: |z| = 1\}$. We aim to show, first, that for each i, all rays in $f^i U_1(f)$ have distinct endpoints. So fix i. Let

$L = \{l: l$ is a geodesic in the boundary of the convex hull of $\varphi^{-1}(x)$, some $x \in \partial f^i U_1(f)\}$.

Then L is a z^2-invariant lamination in the sense of [T], as described in 1.10. (If we took $m=1$, this is exactly how z^2-invariant laminations were produced in [T].) By the classification of z^2-invariant laminations, L contains L_r for some odd denominator rational r. We can change the definition of f in U_2 to produce a critically finite branched covering f_0, and we can change the definition of f in $\bigcup_{j=1}^{m} f^j U_1$ to produce $g \simeq f_0 \vdash_{c_1} s_r$ (see 1.20) such that $g^m(\varphi(L_r)) = \varphi(L_r)$ and g admits a Levy cycle. Then by the Tuning Proposition, so does f_0. Then so does f, which is impossible (see 1.6).

So all rays in $f^j U_1$ have distinct endpoints, for any $j \geqslant 0$. Now suppose $i \neq j$, and that rays of arguments α, β in $f^i U_1$, $f^j U_1$ have a common endpoint. Then if one of α, β—say α—is odd denominator rational, rays of argument β and $2^k\beta$ have a common endpoint for any k such that $2^k\alpha = \alpha \bmod 1$. Then by the first paragraph, $2^k\beta = \beta \bmod 1$. So if one of α, β is odd denominator rational, they both are, and of the same period under $x \mapsto 2x \bmod 1$. If both of them are even denominator rational, by applying f^n for suitable n, we see that $2^k\alpha$ has an odd denominator if and only if $2^k\beta$ does.

So now, if there are any common endpoints at all, there is $0<j<m$ with $2j \leqslant m$ and odd denominator rationals α, β of the same period p such that the rays of arguments α in U_1 and β in $f^j U_1$ have a common endpoint. Now the proof is completed by Lemma 5.5 below, which is stated separately in the form in which the result will be needed later. The map f of 5.5 can be taken equivalent to the map f or f_0 of this lemma, and the arc γ of 5.5 can be derived from the identifications between ∂U_1 and $\partial f^j U_1$ of this lemma.

5.5

LEMMA. *Let f be a critically finite degree two orientation-preserving branched covering with critical points c_1, c_2, and c_1 of period m. Let $c_1 \in U$, where $\bar{U}$ is a closed topological disc with $f^i\bar{U}$ $(0 \leqslant i < m)$ all disjoint, and $f^m U = U$. Let there exist a homeomorphism $\varphi_i\colon \{z\colon |z| \leqslant 1\} \to f^i U$ $(1 \leqslant i \leqslant m)$ with $\varphi_{i+1}(z) = f \circ \varphi_i(z)$ $(1 \leqslant i < m)$, $\varphi_1(z^2) = f \circ \varphi_m(z)$ for all $|z| \leqslant 1$. Let γ be an arc in $\bar{\mathbf{C}} \setminus \bigcup_{i \geqslant 0} f^i \bar{U}$, but with endpoints at $\varphi_m(e^{2\pi i\alpha})$, $\varphi_j(e^{2\pi i\beta})$, where $0<j<2j\leqslant m$, α, β are of period p under $s\colon x \mapsto 2x \bmod 1$, γ is of period t under f, and $\{f^i\gamma\colon 0 \leqslant i < t\}$ have disjoint interiors. Then $p=1$ and $t=m$.*

Remark. Note that f does not have to be equivalent to a rational map.

Proof. We regard s as a map on $\mathbf{R}/\mathbf{Z}$, and α, β as elements of $\mathbf{R}/\mathbf{Z}$. We start by showing that $p \leqslant 2$, and hence that if $p \neq 1$, α, β have the same orbit $\{1/3, 2/3\}$ under s. So suppose that $p \geqslant 3$.

If $2j<m$, we claim there are disjoint intervals $I(\alpha), I(\beta)$ on $\mathbf{R}/\mathbf{Z}$ which contain respectively all points $s^i\alpha$, $s^i\beta$ $(i\geq 0)$. For there are arcs $f^{j+mk}(\gamma)$ attached to $f^j\bar{U}$ at points $\varphi_j(e^{2\pi i 2^k\alpha})$ $(k\geq 0)$ and arcs $f^{km}(\gamma)$ attached to $f^j\bar{U}$ at points $\varphi_j(e^{2\pi i 2^k\beta})$ $(k\geq 0)$. The first set of arcs have second endpoints in $f^{2j}\bar{U}$ and the second set have second endpoints in $\bar{U}$. The existence of $I(\alpha)$, $I(\beta)$ follows, since $p\geq 3$, and $\bigcup_{k\geq 0} f^{j+km}\gamma \cup f^{2j}\bar{U}$ must be contained in a single component of $\bar{\mathbf{C}}\setminus(\bigcup_{k\geq 0} f^{mk}\gamma \cup \bar{U} \cup f^j\bar{U})$. In particular, α and β have disjoint orbits.

If $2j=m$, we claim there is an interval I on $\mathbf{R}/\mathbf{Z}$ which contains all points $s^i\alpha$, $s^i\beta$ $(i\geq 0)$ and satisfying $I\cap(I+1/2)=\varnothing$. For there are arcs of the form $f^i\gamma$ joining all points $\varphi_m(e^{2\pi i 2^k\alpha})$, $\varphi_m(e^{2\pi i 2^k\beta})$ in ∂U to points in ∂f^jU. Let V be the non-periodic component of $f^{-1}f^{j+1}U$. Then there is another set of arcs, disjoint from the first set, joining all points $\varphi_m(e^{2\pi i(2^k\alpha+1/2)})$ to points in ∂V. As in the case $2j<m$, the existence of I follows.

Now let I be an interval on $\mathbf{R}/\mathbf{Z}$ with endpoints of the form $s^i\alpha$, and containing all points $s^i\alpha$, $i\geq 0$, and with $I\cap(I+1/2)=\varnothing$. (If this is not true for α, it is true for β.) Now the endpoints of I are of the form $q/(2^p-1)$, for integers q. Then each component of $I\setminus\{s^i\alpha\colon i\geq 0\}$ is mapped homeomorphically by s to a component of $\mathbf{R}/\mathbf{Z}\setminus\{s^i\alpha\colon i\geq 0\}$. It follows that the components of $I\setminus\{s^i\alpha\colon i\geq 0\}$ have lengths $2^k/(2^p-1)$, $0\leq k\leq p-2$, and I has length $(2^{p-1}-1)/(2^p-1)$. Applying this if $2j<m$ (with α replaced by β if necessary), we obtain that one of $I(\alpha)$, $I(\beta)$ has length $(2^{p-1}-1)/(2^p-1)$. Then, since the two intervals are separated by intervals of length $\geq 1/(2^p-1)$, we obtain that the other intervals from $I(\alpha)$, $I(\beta)$ has length $\leq(2^{p-1}-2)/(2^p-1)$, which is impossible, by the above argument. So $p\leq 2$ if $2j<m$. Applying the argument if $2j=m$, we see that α, β have the same orbit, and s preserves the cyclic order of the points $s^i\alpha$ $(i\geq 0)$. Thus, the cyclic order of the points $s^i\beta$ is the same as for the points $s^i\alpha$. This is impossible, since $f^{km}\gamma$ joins $\varphi_m(e^{2\pi i 2^k\alpha})$ to $\varphi_j(e^{2\pi i 2^k\beta})$. So, again $p\leq 2$.

Now we show that $p=1$. So suppose $p=2$. Note that f^{mp} maps any arc of the form $f^k\gamma$ to another such arc with the same endpoints. So f^{mp} preserves the cyclic order of arcs $f^k\gamma$ with endpoints at $\varphi_m(e^{2\pi i\alpha})$. So f^{mp} must fix all such arcs, and since all such arcs are in the same orbit under f, no two have the same endpoints. If f^l maps one such arc to another, neither endpoint is fixed. But if three arcs of the form $f^k\gamma$ have an endpoint at $\varphi_m(e^{2\pi i\alpha})$, the map f^l between two of them must fix $\varphi_m(e^{2\pi i\alpha})$. So there are exactly two arcs of the form $f^k\gamma$ with endpoints at $\varphi_m(e^{2\pi i\alpha})$, and γ has period and oriented period $2m$. Then at least one component, δ, of $\overline{\bigcup_{i\geq 0} f^i\gamma}$ bounds a disc D in $\bar{C}\setminus\bigcup_{i\geq 0} f^i\bar{U}$. If the period of δ under f is u, inductively we find $\delta_0=\delta$, $\delta_1\ldots\delta_u=\delta$ such that $\delta_i\subset\overline{\bigcup_{k\geq 0} f^k\gamma}$, δ_i bounds a disc $D_i\subset\bar{\mathbf{C}}\setminus\bigcup_{k\geq 0} f^k\bar{U}$ with $f\delta_{i+1}=\delta_i$, $fD_{i+1}=D_i$. (We use the fact that D_i contains at most one critical value of f.) Since $f^u|\delta$ is a homeomorphism, $f|\delta_{i+1}$ is a

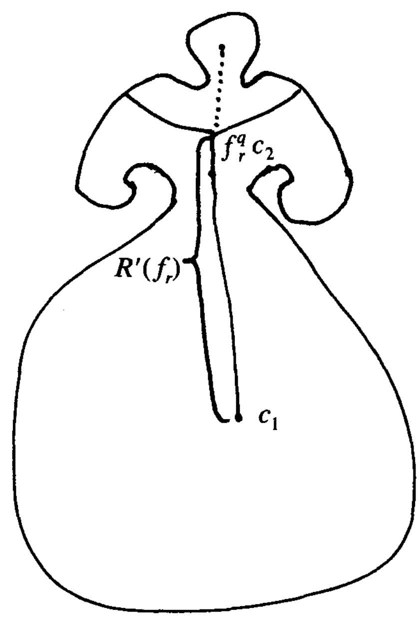

Diagram 5.

homeomorphism. Hence $f|D_{i+1}$ is a homeomorphism, since D_i contains at most one critical value of f, and thus none. So $f^u|D$ is a homeomorphism, with $f^uD=D$. Now we can assume $\gamma\subset\delta$. If $f^m\gamma\subset\delta$, then $\bar{D}\cup\bar{U}\cup f^j\bar{U}$ is a one-holed torus, which is impossible. If $f^m\gamma\not\subset\delta$, then $\gamma, f^k\gamma, f^{2k}\gamma$ are all distinct for either $k=j$ or $k=j+m$, and in δ, and $2j<m$. Now $f^m\colon\delta\to f^m\delta$ preserves orientation. This is impossible, since $f^{2j}U$ is contained in a component of $\bar{\mathbf{C}}\setminus(\bar{D}\cup f^m\bar{D}\cup\bar{U}\cup f^j\bar{U})$ and is attached to both $f^{2k}\gamma$ and $f^{2k+m}\gamma$. So $p=1$, as required.

5.6

In what follows, we obtain information about a type II hyperbolic component H from an endpoint of a ray in H. This will be used more than once. We collect the results in the following lemma.

LEMMA. *Let H be a type* II *hyperbolic component with centre f_0, with $f_0^{m-q}c_1=c_2$, $f^qc_2=c_1$. Let $\{f_r\colon r\in(0,1)\}$ be a ray of argument α in $H\cap W$, with $v(f_r)=re^{2\pi i\alpha}$, and with endpoint f_1. Let $R(f_r)$ be the ray of argument α for f_r in $U_1(f_r)$. Let $R'(f_r)$ be the component of $R(f_r)\setminus f_r^{q-m}(c_2(f_r))$ whose closure contains $c_1(f_r))$, as in Diagram 5. Then*

(a) $$R(f_1)\subset\overline{\lim}_{r\to1}R'(f_r),$$

(b) $\overline{\lim}_{r\to 1} R'(f_r)\setminus\overline{R(f_1)}$ *is contained in the parabolic basin of* f_1 *which contains* $f^q(c_2)$.

(c) f_0 *is uniquely determined up to equivalence by* f_1 *and* q. *In fact,*

$$f_0 \simeq \sigma_{\tilde{\gamma}} \circ \sigma_{\eta} \circ f_1,$$

where $\eta(0)=f_1^{m-q}(c_1)$, $\eta(1)=c_2$, $\gamma=f_1\circ\eta$ *and* $\mathrm{Im}(\eta)\subset f_1^{m-q}\overline{R(f_1)}\cup U_2(f_1)$.

Proof. A stronger statement than (a) is proved in [R], 6.12–6.13, and a similar method proves (b). We sketch the method here. Let φ_r be the map $\varphi_{f_r,0}$ of 5.1. For suitable $r_3(r)$, $r_5(r)$ (as in [R], 6.10) we define

$$B_r = \varphi_r(\{z: |z-r_3(r)| < r_5(r)\}.$$

Then B_r is a disc up to bounded distortion which contains $f_r^q c_2$, $f_r^{q+mp} c_2$, and such that $f_r^i B_r \cap B_r \neq \emptyset$ if and only if $i=0$ or mp. Also, there is an integer n_r and $1>\delta>0$ such that all components of $f_r^i B_r$ are discs up to bounded distortion if $i\in\mathbf{Z}$, $i\leq n_r$, and moreover $f_r^{im} B_r \subset \varphi_r(\{z: |z|<\delta\})$ for $im\geq n_r$. Let Y_r be the union of those components of $f_r^i B_r$ ($i\in\mathbf{Z}$) which intersect $R(f_r)$. Then $R(f_r)\subset Y_r$. As in [R], we can show that one component of

$$\mathrm{Interior}\,(\overline{\lim}_{r\to 1} Y_r)$$

is contained in $U_1(f_1)$, and all other components are contained in $f_1^q\, U_2(f_1)$, and the only noninterior point of $\overline{\lim}_{r\to 1} Y_r$ is the parabolic point of f_1 in $f_1^q\,\partial U_2(f_1)$.

(a) is, in fact, fairly immediate, because φ_r clearly converges on compacta to $\varphi_1=\varphi_{f_1,0}$. It is also clear that $R(f_1)$ coincides with the intersection of $\lim_{r\to 1} R'(f_r)$ with the component of Interior $(\overline{\lim}_{r\to 1} Y_r)$ which is contained in $U_1(f_1)$. Then (b) follows. Finally, for $0<r<1$,

$$f_0 \simeq \sigma_{\tilde{\gamma}_r} \circ \sigma_{\eta_r} \circ f_r,$$

where

$$\eta_r(t) = f_r^{m-q}\,\varphi_r(r(1-t)), \quad \gamma_r = f_r \circ \eta_r.$$

It follows that

$$f_0 \simeq \sigma_{\tilde{\gamma}} \circ \sigma_{\eta} \circ f_1,$$

where $\gamma=f_1\circ\eta$, and $\mathrm{Im}(\eta)\subset f_1^{m-q}\overline{R(f_1)}\cup U_2(f_1)$, giving (c).

5.7

LEMMA. *If f is a common endpoint of rays of odd denominator rational arguments α, α' in $H \cap W$, $H' \cap W$, where H, H' are type* II *hyperbolic components, then either $H=H'$ with $\alpha=\alpha'$, or $\alpha=\alpha'=0$.*

Proof. For $g \in H$ let $g^q c_2 \in U_1$, and for $g \in H'$, let $g^{q'} c_2 \in U_1$. Without loss of generality, we can assume $0<q<q'<m$. Let $q<q'$. Then, as stated in 5.3, the ray of argument α in $U_1(f)$ has endpoint at the parabolic point in $\partial f^q U_2(f)$. So the image of the ray under $f^{q'-q}$—which is actually of argument 2α in $f^{q'-q}U_1$—has endpoint in common with the ray of argument α' in U_1. So we deduce from 5.4 that $\alpha'=0$. Similarly, we deduce that $\alpha=0$, by taking the image under $f_1^{q-q'+m}$ of the ray of argument α' in U_1.

If $q=q'$, then by 5.3, rays of arguments α, α' in $U_1(f)$ have a common endpoint. So $\alpha=\alpha'$, by 5.4. But then we deduce from 5.6(c) that $H=H'$, since they have the same centre up to equivalence, and hence have the same centre. (See 1.6. It is always true that two hyperbolic critically finite rational maps are conformally conjugate if they are equivalent. This is part of Thurston's theorem, but also the Semiconjugacy Proposition 4.1 shows topological conjugacy, and a standard hyperbolicity argument shows this implies conformal conjugacy. This argument is outlined, for example, in [R], 5.1.)

See also [D–H1], part 1, Chapter 6, where this is proved for polynomials.

5.8

LEMMA. *Let α be an odd denominator rational. Then two distinct rays of argument α in $H \cap W$, for a type* II *hyperbolic component H, cannot have the same endpoint.*

Proof. Let $\{f_r: r \in (0,1)\}$ and $\{g_r: r \in (0,1)\}$ be two rays of argument α in $H \cap W$ with $v(f_r)=v(g_r)=re^{2\pi i\alpha}$, and let $h=\lim_{r\to 1} f_r=\lim_{r\to 1} g_r$, so that h has a parabolic cycle of period mp. We aim to show $f_r=g_r$ as elements of RM_2. It suffices to do this for one r, since rays coincide if they intersect at one point. We know there is a holomorphic bijection $\psi_r: \bar{\mathbf{C}} \setminus J(f_r) \to \bar{\mathbf{C}} \setminus J(g_r)$ with $\psi_r \circ f_r = g_r \circ \psi_r$. It suffices to show that ψ_r can be chosen to extend holomorphically to $\bar{\mathbf{C}}$. By a theorem of Ahlfors [A], it suffices to show that ψ_r can be chosen to extend quasiconformally to $\bar{\mathbf{C}}$. By a standard argument written out in [R], 5.1 (for example), it suffices to show ψ_r can be chosen to extend homeomorphically to $\bar{\mathbf{C}}$. Now $\psi_r(R(f_r))=R(g_r)$, where $R(f_r)$, $R(g_r)$ are the rays of argument α in $U_1(f_r)$, $U_1(g_r)$. We claim that it suffices to find neighbourhoods Y, Y' of $J(f_r)$, $J(g_r)$ and a homeomorphism $\chi_r: Y \to Y'$ with $\chi_r \circ f_r = g_r \circ \chi_r$ where both sides of the equation are defined, and $\chi_r(Y \cap R(f_r)) = Y' \cap R(g_r)$. For then we have a convergent sequence of points

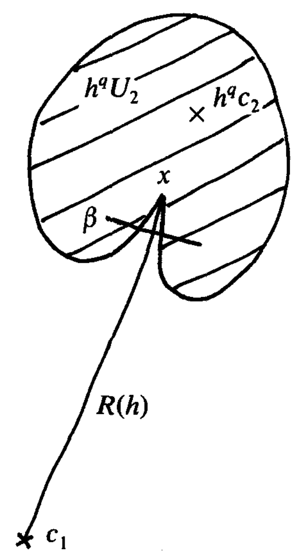

Diagram 6.

$\{z_n\}$ in $R(f_r)$ with $f^m z_{n+1}=z_n$, and $\lim_{n\to\infty}\psi_r(z_n)=\lim_{n\to\infty}\chi_r(z_n)$. Then a homotopy between ψ_r and χ_r on some compact annulus A in $U_1(f_r)$, with $(\bigcup_{n\geq 0} f^{-mn}A)\cap U_1(f_r)$ a neighbourhood of $\partial U_1(f_r)$ in $U_1(f_r)$, can be lifted under f^{mn} to a homotopy betweeen ψ_r and χ_r on $f^{-mn}A\cap U_1(f_r)$, and we see that

$$\lim_{z\to\partial U_1(f_r),\, z\in U_1(f_r)} \operatorname{dist}(\psi_r(z),\chi_r(z))=0.$$

Then ψ_r can be chosen to extend homeomorphically.

So now we have to define χ_r, for some r sufficiently near 1. Write $R^+(f_r)$, $R^-(f_r)$ for the two branches of $R(f_r)\setminus\overline{R'(f_r)}$, and similarly for g_r (with the same orientation), where $R'(f_r)$, $R'(g_r)$ are as in 5.6. Let x be the parabolic point of h in $\partial h^qU_2(h)$. So x is the endpoint of $R(h)$. Then, for some $\delta>0$, there is an arc β with the following properties.

(1) β intersects $R(h)$ in one pont, and the endpoints of β are in h^qU_2. These endpoints, and $\beta\cap R(h)$, are distance $\geqslant 3\delta$ from $J(h)$ (in the spherical metric).

(2) Let B be the neighbourhood of x which contains h^qU_2 with $\partial B\subset\partial(h^qU_2)\cup\beta$. Then the component of $h^{-mp}B$ which contains h^qU_2 is contained in B, and the corresponding component of $h^{-mp}\beta$ is distance $\geqslant 3\delta$ from β.

See Diagram 6 for the configuration. The shaded region denotes B.

Now we can find an open set B_r^+ with the following properties, for all r sufficiently near 1. We choose B_r^+ with ∂B_r^+ mostly in a small neighbourhood of $\beta\cup R'(f_r)\cup R^+(f_r)$.

(a) B_r^+ does not intersect the forward orbits of $c_1(f_r)$, $c_2(f_r)$, and $B_r^+ \cap R(f_r) = B_r^+ \cap R^+(f_r)$ is connected.

(b) $\lim_{r\to 1} \partial B_r^+ \cap B_\delta(J(f_r)) \subset \beta \cap B_\delta(J(h))$.

(c) B_r^+ contains the endpoint of $R^+(f_r)$, and $S\overline{B_r^+} \subset B_r^+$, where S is the local inverse of f_r^{mp} which fixes the endpoint of $R^+(f_r)$.

We choose B_r^- to have similar properties relative to f_r, $R^-(f_r)$, and $C_r^\pm$ to have similar properties relative to g_r, $R^\pm(g_r)$. Now we drop the suffix r, and there exist $0<\eta\le\delta$, and an integer N, such that the following statements are true for a given $\varepsilon>0$, and all r sufficiently near 1.

There exist open Y, Y' which contains η-neighbourhoods of $J(f)$, $J(g)$ respectively, and there exists a homeomorphism $\sigma\colon \bar{Y}\to\bar{Y}'$ with the following properties.

(i) $f^{-1}\bar{Y}\subset Y$, $g^{-1}\bar{Y}'\subset Y'$, and Y, Y' do not intersect the forward orbits of c_1, c_2 under f, g.

(ii) $\sigma(B^+)=C^+$, $\sigma(B^-)=C^-$, $\sigma(B^+\cap R^+(f))=C^+\cap R^+(g)$, $\sigma(B^-\cap R^-(f))=C^-\cap R^-(g)$, and $\partial B^\pm\cap\partial Y\neq\varnothing$.

(iii) $\sigma\circ f=g\circ\sigma$ on $f^{-1}(\partial Y)$.

(iv) $\mathrm{dist}(\sigma x, x)<\varepsilon$ for all $x\notin \bigcup_{i\le N} f^{-i}(B^+\cup B^-)$.

Then define $\sigma_0=\sigma$ and inductively define $\sigma_n\colon \bar{Y}\to\bar{Y}'$ ($n\ge 0$) by

$$\sigma_{n+1}=\sigma_n \quad \text{on} \quad \bar{Y}\setminus f^{-(n+1)}Y,$$

$$g\circ\sigma_{n+1}=\sigma_n\circ f \quad \text{on} \quad f^{-(n+1)}Y.$$

Then we see that, if r is sufficiently near 1, then σ_0 and σ_1 are very close on $(\bar{Y}\setminus f^{-1}Y)\cup(f^{-1}Y\setminus\bigcup_{i\le N} f^{-i}(B^+\cup B^-))$. Also, σ_0 and σ_1 have the same image $\bar{Y}'$. Since each component of $\bigcup_{i\le N} f^{-i}(B^+\cup B^-)$ is a disc, with σ_0, σ_1 close on the boundary, there is an isotopy between σ_0 and σ_1 with image in $\bar{Y}'$. This isotopy lifts to one between σ_n and σ_{n+1} on $f^{-n}\bar{Y}$. Then the diameter of the isotopy decreases to 0 as n tends to ∞, which is enough to show that σ_n converges to the required χ. By the same method, σ_n^{-1} converges, and the limit must be χ^{-1}, so χ is a homeomorphism.

5.9

We can summarize the results of 5.7, 5.8 in the following

ENDPOINT THEOREM. *If two different rays of odd denominator rational arguments α, α' in $H\cap W$, $H'\cap W$ have a common endpoint, where H, H' are type* II *hyperbolic components, then $\alpha=\alpha'=0$ and $H\neq H'$.*

5.10. Completion of the proof of the Improved Polynomial-and-Path Theorem

We recall that the Improved Theorem 3.1 remains to be proved in one case. Let s be an integer >1, and let $f\in W$ have $c_2(f)$ of period ms. We need to show that an equivalence holds for f as in the Improved Theorem. Let $\{f_t: t\in[0,1]\}$ be a path in W with $f_1=f$, f_0 a polynomial, and let $\{x_t: t\in[0,1]\}$ be a path in $\bar{\mathbf{C}}$, such that

(1) $(f_t^{ms})'(z)\neq 0$ or 1 if $f_t^{ms}(z)=z$, for any t, z, except when z is in the orbit of $c_1(f_t)$, or $t=1$, and z is in the orbit of $c_2(f_t)$,

(2) f_t is never in the closure of a ray in a type II hyperbolic component of argument α, where α has period s under $x\mapsto 2x \bmod 1$,

(3) $f_t c_2 \neq c_2$ for $t\in[0,1]$,

(4) $x_1=c_2(f_1)$, $f_t^{ms}(x_t)=x_t$.

Because rays of argument α, where α is period $s>1$ under $x\mapsto 2x \bmod 1$, always have endpoints, and because of the Endpoint Theorem, it is possible to find a path $\{f_t\}$ satisfying (1) to (4). Now Y_t varies isotopically to $t\in(0,1]$, where Y_t is the union of $\{f_t^i(x_t): 0\leq i<ms\}$ and closures of rays in $f_t^i(U_1)$ $(i\geq 0)$ of arguments of period s under $x\mapsto 2x \bmod 1$. For $t=1$, x_t is not an endpoint of a ray in Y_t. Hence the same is true for all t, in particular $t=0$. Then we can complete the proof of the Improved Theorem using $\{f_t\}$, $\{x_t\}$ as they are used in Chapter 2. See, in particular, 2.6.

Chapter 6. Invariant laminations

6.1

Throughout this chapter, $K=K_r$, $\bar{L}=\bar{L}_r$, $\Phi=\Phi_r$ and $\bar{s}=\bar{s}_r$ are as in 1.12. We also let $U=U_r$ be the component of $\bar{\mathbf{C}}\setminus(K\cup(\cup\bar{L}))$ containing ∞. We consider laminations on $\bar{\mathbf{C}}\setminus K$. Note that $z\mapsto\bar{z}^{-1}$ fixes K and S^1 pointwise, and gives an isometry of $\bar{\mathbf{C}}\setminus K$. Therefore, the components of $S^1\setminus K$ are geodesics. Throughout this chapter, let L be a lamination on $\bar{\mathbf{C}}\setminus K$ for which all leaves of $L\setminus\bar{L}_r$ are in U.

We recall that a *gap* of L is a component of $\bar{\mathbf{C}}\setminus(K\cup(\cup L))$.

A *segment* of a leaf l of L is the closure of a component of $l\setminus S^1$, also called a *leaf segment.*

A *polygon* P is a connected closed set in $\{z: |z|\leq 1\}$ or $\{z: |z|\geq 1\}$ such that $\partial P\setminus S^1\subset\cup L$.

A *gap polygon* P is a polygon which satisfies, in additon, Interior$(P)\subset G$ for some gap G of L.

A *side* of a gap G is then a leaf which intersects the boundary of at least one gap polygon with interior in G.

6.2. Straightening paths

A path $\tau: \mathbf{R} \to \bar{\mathbf{C}} \setminus K$ *can be straightened* if the lift $\tilde{\tau}: \mathbf{R} \to \mathring{D}$ has well-defined endpoints on ∂D, that is, the limits as $x \to \pm\infty$ of $\tilde{\tau}(x)$ exist on ∂D. In that case, the *straightening* of τ is the unique geodesic with a lift with the same endpoints as $\tilde{\tau}$.

6.3. Straightenings are determined by crossing

We are particularly interested in paths $\tau: \mathbf{R} \to \bar{\mathbf{C}} \setminus K$ such that either

(a) the set $\tau^{-1}(S^1)$ is not bounded above or below, consists of isolated points, and no two successive points lie in the same component of $S^1 \setminus K$,

or

(b) instead of $\tau^{-1}(S^1)$ being not bounded below, $\lim_{x \to -\infty} \tau(x)$ exists and is in K

or

(c) similarly, $\tau^{-1}(S^1)$ is not bounded above but $\lim_{x \to +\infty} \tau(x)$ exists and is in K.

We claim that any such path can be straightened, and that the corresponding geodesic is determined uniquely by the sequence of components of $S^1 \setminus K$ crossed, together with the directions of the crossings. It suffices to show that, if $\tau|[0, \infty)$ is lifted to $\tilde{\tau}$, then the Euclidean diameters of a lift of a component of $S^1 \setminus K$ intersected by $\tilde{\tau}(x)$ tends to 0 as $x \to +\infty$. So it suffices to show that, for some $\delta > 0$, if I_1, I_2 and I_3 are all distinct components of $S^1 \setminus K$, then any geodesic segment with endpoints in I_1 and I_3 and crossing I_2 in between has (Poincaré) length $\geq \delta$. But this follows from the Margulis decomposition [T2], since if δ is suitably chosen, two closed loops of length $< \delta$ in different homotopy classes must be distance $> \delta$ apart.

6.4. Straightenings of forward and backward images

Let $\varphi: \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ be a branched covering with $\varphi(K) = K = \varphi^{-1}(K)$. Then any lift of φ to the universal covering $\mathring{D}$ extends continuously to the closed disc D. Hence, if $\varphi \circ \tau$ can be straightened, the straightening $\varphi_*(\tau)$ depends only on φ and the straightening of τ.

Now let φ be a branched covering with critical points (and values) in $\bar{\mathbf{C}} \setminus K$, and let $\tau: \mathbf{R} \to \bar{\mathbf{C}} \setminus K$ be any path such that τ^{-1} (critical values of φ) is finite. If this set is empty, we define $\varphi^*\tau$ to be the straightenings of the components of $\varphi^{-1}\tau$, if these can all be straightened. If the set is $\{t_1, \ldots, t_n\}$, we define $\varphi^*\tau$ to be the straight versions of the components of $\varphi^{-1}(\tau_1 \cup \tau_2)$, if these can all be straightened, where τ_1, τ_2 are as follows: $\tau_j = \tau$ except on $\bigcup_{i=1}^{n} (t_i - \varepsilon, t_i + \varepsilon)$, these intervals are all disjoint, and

$$\tau_1([t_i-\varepsilon, t_i+\varepsilon])\cup\tau_2(t_i-\varepsilon, t_i+\varepsilon])$$

bounds a disc containing $\tau((t_i-\varepsilon, t_i+\varepsilon))$, with $\tau_1([t_i-\varepsilon, \tau_i+\varepsilon])$ on the left and $\tau_2([t_i-\varepsilon, t_i+\varepsilon])$ on the right, for some suitable orientation on τ. Once again, $\varphi^*\tau$ depends only on φ and the straightening of τ.

6.5. Conditions on the path β

From now on, let $\beta:[0,1]\to\bar{\mathbf{C}}\setminus K$ with $\beta(0)=\infty$, $\beta((0,1))\cap\{\beta(0),\beta(1)\}=\emptyset$, and either $\beta^{-1}(\bigcup\bar{L}_r)=\emptyset$, or $\beta^{-1}(\bigcup\bar{L}_r)=\{t\}$, for some $t\in[0,1]$. If $\beta(t)\in\bigcup\bar{L}_r$, let $\beta(1)$ be in the full orbit of 0 under $\bar{s}$. Thus, $\beta(1)$ is determined, in this case, by the leaf containing $\beta(t)$.

Now given our fixed lamination L, we can ensure that β also has the following properties, if we allow homotopies which keep $\beta(0)$ fixed at ∞, keep $\beta(1)$ fixed if $\beta(1)\notin U$, and allow $\beta(1)$ to move in a leaf or gap of L if $\beta(1)\in U$. We shall assume β satisfies these properties from now on. The properties (a) to (c) involve finitely many endpoints $a_1, \dots, a_q$ of intervals $I_1, \dots, I_q$ of $S^1\setminus K$, and an $\varepsilon>0$, which can be taken arbitrarily small. Let $I_i(\varepsilon)$ denote the subinterval of I_i with endpoint at a_i and of Euclidean length ε, and let $U_1, \dots, U_q$ be disjoint neighbourhoods of the $I_i(\varepsilon)$ with $U_i\cap S^1=I_i(\varepsilon)$.

(a) If c, d are adjacent points of $\beta^{-1}(S^1)$, then $\beta(c)$, $\beta(d)$ lie in different components of $S^1\setminus K$, and β is transverse to S^1.

(b) No sub-path of β can be homotoped into a leaf of L by a homotopy in $\bar{\mathbf{C}}\setminus K$ fixing the sub-path endpoints, and β is transverse to L.

If β_u is any homotopy through paths satisfying (a) and (b) such that $\beta_0=\beta$, $\beta_u(0)=\beta(0)=\infty$ for all u, $\beta_u(1)\notin\bigcup L$ for all u if $\beta(1)\notin\bigcup L$, $\beta_u(1)\in l$ for all u if $\beta(1)\in l\in L$, then, for each t, and $l\in L$ with $\beta(t)\in l$, there is $w(t,u)$, which is continuous in u, with $w(t,0)=t$ and $\beta_u(w(t,u))\in l$. Now we can state condition (c).

(c) Any two components of $\beta^{-1}(U_i)$ are separated by a point of $\beta^{-1}(S^1\setminus U_i)$. The point a_i is a limit of $I_i(\varepsilon)\cap(\bigcup L)$, and for any homotopy β_u as above and any t, l_0 such that $\beta(t)\in l_0$ and l_0 is a leaf segment with both endpoints in $S^1\setminus\bigcup_{i=1}^q I_i(\varepsilon)$,

$$\#\{t'\in[0,t]: \beta(t')\in S^1\}\leqslant\#\{t'\in[0,w(t,u)]: \beta_u(t')\in S^1\}$$

for all $u\in[0,1]$. In particular,

$$\beta([0,1])\cap(S^1\setminus\bigcup_{i=1}^q I_i(\varepsilon))\cap(\bigcup L)=\emptyset.$$

6.6. Backward images of leaves and segments under $\sigma_\beta \circ \bar{s}$

Let τ be a parametrisation of l, and let $\eta: \mathbf{R} \to \bar{\mathbf{C}} \setminus K$ be a component of $(\sigma_\beta \circ \bar{s})^{-1}\tau$ (or of $(\sigma_\beta \circ \bar{s})^{-1}(\tau_1 \cup \tau_2)$ if $\beta(1) \in l$, where τ_1, τ_2 are as in 6.4). Under the conditions on β of 6.5, and if the disc neighbourhood where $\sigma_\beta \neq$identity is taken close enough to β, points of $\eta^{-1}(S^1 \setminus (\cup_{i=1}^q I_i(\varepsilon))$ are all isolated, and we can homotope η by a homotopy which is identity outside $\eta^{-1}(\cup_i U_i)$ to satisfy condition (a), (b) or (c) of 6.3. Hence η can be straightened. We call the straightening

$$(\sigma_\beta \circ \bar{s})^* l,$$

if $\beta(1) \notin l$. If $\beta(1) \in l$, we take $(\sigma_\beta \circ \bar{s})^* l$ to be the union of the straightenings of $(\sigma_\beta \circ \bar{s})^{-1}(\tau_1 \cup \tau_2)$ and the gap they are sides of. The straightening is not affected by the choice of any changes we have made to β.

Now let l_0 be any leaf segment on l. We can assume in addition by choice of ε that the disc neighbourhood where $\sigma_\beta \neq$identity does not intersect the endpoints of l_0. The straightening of a component of $(\sigma_\beta \circ \bar{s})^{-1} l$ can be chosen so that the restriction of the homotopy to $(\sigma_\beta \circ \bar{s})^{-1} l_0$ homotopes it to a finite union of geodesic segments via a homotopy which keeps endpoints in S^1. We write

$$(\sigma_\beta \circ \bar{s})^* l_0$$

for the image of this homotopy if $\beta(1) \notin l$, which depends only on l_0, L and the homotopy class of β keeping endpoints fixed. If $\beta(1) \in l$, let τ_1 and τ_2 be defined as in 6.4, and let l_0', l_0'' be the corresponding perturbations of l_0. Then in this case, let

$$(\sigma_\beta \circ \bar{s})^* l_0$$

be the finite union of gap polygons bounded by geodesic segments obtained by homotopies of $(\sigma_\beta \circ \bar{s})^{-1} (l_0' \cup l_0'')$, keeping endpoints in S^1.

6.7. Definition of inverse images of gaps and polygons

Let

$$L_1 = (\sigma_\beta \circ \bar{s})^* L = \{l_1 : l_1 \text{ is a leaf in } (\sigma_\beta \circ \bar{s})^* l \text{ for some } l \in L\}.$$

Then L_1 is a lamination. If G is a gap of L, let

$$(\sigma_\beta \circ \bar{s})^* G$$

be the one or two gaps whose sides are the images under $(\sigma_\beta \circ \bar{s})^*$ of the sides of G, with the same orientations of the sides.

If P is a gap polygon of L, we define $(\sigma_\beta \circ \bar{s})^* P$ to be the smallest (finite) union of gap polygons satisfying

$$\partial(\sigma_\beta \circ \bar{s})^* P \cap (\bigcup L) \subset (\sigma_\beta \circ \bar{s})^* \partial(P \cap (\bigcup L)),$$

with $(\sigma_\beta \circ \bar{s})^*$ preserving the orientation of leaves round the boundary.

6.8. Forward images

Let L_1 be as in 6.7. If Y_1 if a leaf, gap, leaf segment or gap polygon of L_1, define

$$(\sigma_\beta \circ \bar{s})_* Y_1 = \bar{s}_* Y_1 = Y,$$

where Y is a leaf, gap, leaf segment or gap polygon of L with

$$Y \subset (\sigma_\beta \circ \bar{s})^* Y_1.$$

Then given a gap polygon Y_1, there might be more than one such Y, and Y might be a leaf segment, and if Y_1 is a gap with four sides all of which have the same image Y under $\bar{s}_*$ then we take $\bar{s}_* Y_1 = Y$ (since Y_1 is not in the image of $(\sigma_\beta \circ \bar{s})^*$ as so far defined) but in all other cases, $Y \mapsto Y_1$ is a well-defined function between leaves, gaps or leaf segments. If $l \in L_1$ is a leaf, then $(\sigma_\beta \circ \bar{s})_* l = \bar{s}_* l$ is indeed the straightening of $\bar{s}l$ or $(\sigma_\beta \circ \bar{s}) l$, as required in 6.4. If P is a gap polygon, $\bar{s}_* P$ is singlevalued unless $P \subset (\sigma_\beta \circ \bar{s})^* Q_1$ for one Q_1 with $Q_1 \cap \beta([0,1]) \cap S^1 \neq \emptyset$ (assuming β satisfies the conditions (a) to (c) of 6.5). In this case, all the finitely many values $Q_1, \ldots, Q_n$ of $\bar{s}_* P$ have this property.

6.9. Invariant laminations

L is β-invariant, or simply *invariant* if

$$L = (\sigma_\beta \circ \bar{s})^* L,$$

and (to simplify later statements of results) the following two properties hold.

(a) All leaves of $L \setminus \bar{L}_r$ are in U (so that the gaps of $\bar{L}_r$, apart from U, are contained in gaps of L).

(b) Recalling that m is the period of 0 under $\bar{s}$, the points $\{\bar{s}^i(0) : 0 \leq i < m\}$ are in distinct gaps of L.

If L is invariant, then

$$L = \bar{s}_* L,$$

where

$$\bar{s}_* L = \{\bar{s}_* l : l \in L\}.$$

For the rest of this chapter, let L be a β-invariant lamination.

6.10

The following lemma will be needed in the sequel to this paper, and it seems best to prove it while the notations—especially of 6.5—are remembered.

LEMMA. *The a_i (as in 6.5) are strictly preperiodic under $\bar{s}$.*

Proof. If I_i (as in 6.5) is a leaf of L and intersected by β, then an endpoint a_i cannot be periodic. So now suppose that I_i is not a leaf of L, and that an endpoint a_i has period n. Then there is a leaf segment l_2 in $(\sigma_\beta \circ \bar{s})^{*n} l_1$ with endpoint at a_i, with l_1, l_2 on opposite sides of s^1. So then there must be a gap of L containing $I_i(\varepsilon)$ (if ε is small enough), contradicting our assumptions.

6.11. Lamination maps

There is a *lamination map* $\varrho_L : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ satisfing (a) to (d) below. Recall that m is the period of 0 under $\bar{s}$. We denote by G_∞ the gap of L containing ∞ (which always exists). In particular, the conditions imply that ϱ_L is a critically finite branched covering with $L = \varrho_L L = \varrho_L^{-1} L$ whenever $\beta(1) \notin \cup L$, and that ϱ_L always preserves $\simeq_L$, so that $[\varrho_L] : \bar{\mathbf{C}}/\simeq_L \to \bar{\mathbf{C}}/\simeq_L$ is well-defined. Moreover, ϱ_L is uniquely defined up to equivalence, and $[\varrho_L]$ is uniquely defined up to topological conjugacy, whenever $\beta(1) \notin \cup L$ and the gap G_∞ of L containing ∞ is simply connected.

(a)
$$\varrho = \sigma_\beta \circ \bar{s} \circ \varphi,$$

where the disc neighbourhood in which $\sigma_\beta \neq$ identify intersects S^1 only in $\cup_{i=1}^q I_i(\varepsilon)$, and in at most one leaf of $\bar{L}_r$—and that only if $\beta(t) \in \bar{L}_r$ for some t. Moreover, φ is a homeomorphism, and $\varphi = \varphi_1$ where φ_t is an isotopy between $\varphi, \varphi_0 =$ identity, and for all t, $\varphi_t =$ identity on K and leaves invariant all but finitely many components of $S^1 \setminus K$.

(b) $\varrho^{-1}(l) = (\sigma_\beta \circ \bar{s})^*(l)$ for all $l \in L$, unless $\beta(1) \in l$, when $\varrho^{-1}(l) = (\sigma_\beta \circ \bar{s})^*(l) \cup G_\infty$.

Thus $\varrho l=\bar{s}_* l$ for all $l \in L$, and $\varrho L=L$. Similarly, if G is a gap of L, $\varrho^{-1}(G)=(\sigma_\beta \circ \bar{s})^* G$ and $\varrho(G)=\bar{s}_*(G)$.

(c) 0 is always a critical point for ϱ, and if $\beta(1) \notin \cup L$, ϱ is a branched covering with critical points 0, ∞, and two points from $\{\varrho^i 0, \varrho^j \infty : i,j \geq 0\}$ are equal whenever they are in the same gap of L.

(d) The map ϱ^m fixes 0 and is topologically conjugate near 0 to $z \mapsto z^2$ or $z \mapsto z^4$, depending on whether or not ∞ is distinct from all points $\bar{s}^i(0)$. If the gap G_0 containing 0 is simply connected, $\varrho^{mn} z \to 0$ as $n \to \infty$ for all $z \in G_0$. If ∞ is periodic under ϱ and G_∞ is simply connected, similar properties hold for all $z \in G_\infty$.

6.12. Thick and thin

A gap polygon is *thick* if it either has at least three leaf segments in its boundary, or contains a non-point interval $I \subset S^1$ in its boundary, with $K \cap I \neq \varnothing$. A polygon is *thick* if it contains a thick gap polygon. A polygon is *thin* if it is not thick.

6.13. The main results

The aim of this chapter is to prove the following propositions. In each result, recall that L is a β-invariant lamination, and let G_0, G_∞ be the gaps containing 0, ∞. All arguments are modelled on arguments in [T]. Since we have defined the lamination map (6.11) so that orbits under ϱ_L and $\bar{s}_*$ coincide, the results do indeed complete the proof of the Lamination Map Conjugacy Theorem, as claimed in 4.6.

ORBIT OF GAPS PROPOSITION. *There are finitely many gaps G_i $(1 \leq i \leq n)$ of L and $\bar{s}_* G_\infty$ (which may be a leaf or a gap) such that, if G is any gap, then for some $i>0$,*

$$\bar{s}_*^i G = G_j \text{ or } \bar{s}_* G_\infty.$$

for some j, $1 \leq j \leq n$.

ORBIT OF SIDES PROPOSITION. *Let G be any periodic gap of L, that is, $\bar{s}_*^i G = G$ for some $i>0$. Then there are finitely many sides l_i $(1 \leq i \leq n)$ of G such that, if l is a side of G, then for some j, and some $k>0$,*

$$\bar{s}_*^k l = l_j.$$

FINITE-SIDED GAPS PROPOSITION. *Every gap G which is not in the full backward orbit of G_0 or G_∞ (that is $\bar{s}_*^i G \neq G_0$ or G_∞ for any $i \leq 0$) is finite-sided.*

PERIODIC LEAF PROPOSITION. *Let $\bar{s}_*^{i+j}G_\infty=\bar{s}_*^iG_\infty$ for some $i\geq 0$ and $j>0$. (This is true for instance, if $\bar{s}_*G_\infty$ is a gap.) Then*

$$\{l: \bar{s}_*^{i+j}l=\bar{s}_*^i l \text{ for some } i\geq 0 \text{ and } j>0\}$$

is dense in L.

6.14. Deductions

G_∞ is 4-sided if $\bar{s}_*G_\infty$ is not a gap, and $\bar{s}_*$ is a bijection from sides of G to sides of $\bar{s}_*G$, if $G\neq G_0$ or G_∞. Hence, the Finite-sided Gaps Proposition follows from the Orbit of Gaps Proposition and the Orbit of Sides Proposition. The Periodic Leaf Proposition also follows from these, if gaps are dense in L. We claim that this is true. Given any transversal to L, and points of intersection x_1, x_2 of segments of leaves l_1, l_2 with the transversal, there must be a gap of L intersecting the transversal between the segments of l_1, l_2 unless both l_1, l_2 do not intersect S^1. So if gaps are not dense, there is an open set of such leaves which do not cross S^1, and these leaves must be contained in $\{z: |z|>1\}$ (because gaps approximate all leaves in $\bar{L}_r$). Then the set of leaves in $\{z: |z|>1\}$ which do not intersect S^1 is invariant under $\bar{s}$, and forms a lamination L'. Then $L''=\{\Phi(l)^{-1}: l\in L'\}$ (with $l^{-1}=\{z^{-1}: z\in l\}$) is z^2-invariant with gaps not dense, hence by [T] has no gaps at all and has all leaves vertical. But this gives a contradiction, because the projection of G_∞ is a gap of L''.

So it remains to prove the Orbit of Gaps Proposition and the Orbit of Sides Proposition.

6.15

The Orbit of Gaps Proposition follows immediately from the Thick Polygon Lemma below. We shall see in 6.18 that the Orbit of Sides Proposition follows from the Length Lemma and Thick Polygon Lemma below. Again, let L be a β-invariant lamination.

LENGTH LEMMA. *There exists a finite $A\subset K$ with $\bar{s}A\subset A$ such that, for any $0<\varepsilon_1$, there is $\varepsilon_2\leq\varepsilon_1$ such that the following holds.*

For any leaf segment l from L, there exists N such that, for all $n\geq N$, either diameter$(\bar{s}_*^n l)\geq\varepsilon_2$, *or $\bar{s}_*^n l$ is a component of $S^1\setminus K$ with endpoint in A, or, for some $a\in A$, $a\in\bar{C}$, where C is a component of $\bar{\mathbb{C}}\setminus(S^1\cup\bar{s}_*^n l)$ of diameter $\leq\varepsilon_2$.*

Remark. Here, diameter is with respect to the spherical metric.

THICK POLYGON LEMMA. *There are finitely many gap polygons P_i ($1\leq i\leq t$) such that, if P is any thick polygon, either $\bar{s}_*^n P\subset \bar{s}_* G_\infty$ for some $n>0$, with $\bar{s}_* G_\infty = l\in L$, or $\bar{s}_*^n P\subset\{P_i\colon 1\leq i\leq t\}$ for some $n>0$, even if $\bar{s}_*^n P$ is multivalued.*

6.16

The Thick Polygon Lemma follows from the Length Lemma. For let P be a thick polygon. We can assume $\bar{s}_*^n P$ is a finite union of *polygons* for all n. If $\bar{s}_*^n P$ is multivalued for some least $n>0$, then all values Q satisfy $Q\cap \mathrm{Im}(\beta)\cap S^1=\emptyset$, and we are done, since there are only finitely many such Q. So we can assume $\bar{s}_*^n P$ is a single polygon for all n, with $\bar{s}_*^n P\cap \mathrm{Im}(\beta)\cap S^1=\emptyset$. Then $\bar{s}_*^n P$ is always thick, and has ≥ 3 sides if P does. Then we can apply the Length Lemma to three of the leaf sides l_1, l_2, l_3 of P (if three such sides exist). For sufficiently large n, $\bar{s}_*^n l_i$ ($i=1,2,3$) satisfy the conclusions of the Length Lemma, and are sides of $\bar{s}_*^n P$, giving only finitely many possibilities for $\bar{s}_*^n P$. If a non-point component of $\partial P\cap S^1$ contains an endpoint a of a component of $S^1\setminus K$, then the same is true for $\partial\bar{s}_*^n P\cap S^1$, with a replaced by $\bar{s}^n a$. Since a is eventually periodic under $\bar{s}$, we are done.

6.17. Proof of the Length Lemma

Let β satisfy the properties of 6.5 with $\varepsilon=\varepsilon_1$. Choose $\varepsilon_2>0$ with $\varepsilon_2<\varepsilon_1$ such that:

(a) if l is a leaf segment of diameter $\leq\varepsilon_2$, the one component of $\bar{\mathbb{C}}\setminus(S^1\cup l)$ has diameter $\leq\varepsilon_2$,

(b) all components of $S^1\setminus K$ crossed by β have diameter ε_2,

(c) for some $\lambda>1$, if l is any leaf segment of diameter $\nu\leq\varepsilon_2$ and l is a component of $(\sigma_\beta\circ\bar{s})^*\bar{s}_* l$ then $\bar{s}_* l$ has diameter $\geq\lambda\nu$. (This might necessitate changing the spherical metric to an equivalent one, but that does not matter.) Let $A=\{\bar{s}^j a_i\colon j\geq 0,\ 1\leq i\leq q\}$, for a_i as in 6.5. We call a leaf segment *good* if either l has diameter $\geq\varepsilon_2$ or one component C of $\bar{\mathbb{C}}\setminus(S^1\cup l)$ has diameter $\leq\varepsilon_2$ and $\bar{C}$ contains a point a of A. Then if l is not equal to a component of $(\sigma_\beta\circ\bar{s})^*\bar{s}_* l$, $\bar{s}_* l$ is good, since the endpoints of $\bar{s}_* l$ bound an interval containing either a point of A or an interval of $S^1\setminus K$ crossed by β. If l is good and l is a component of $(\sigma_\beta\circ\bar{s})^*\bar{s}_* l$, $\bar{s}_* l$ is good. So we only need to show that given l, there is $N\geq 0$ such that $\bar{s}_*^N l$ is good. Choose a least $m\geq 0$ so that λ^m diameter$(l)\geq\varepsilon_2$. Then either there is a least $0\leq i<m$ such that $\bar{s}_*^i l\neq(\sigma_\beta\circ\bar{s})^*\bar{s}_*^{i+1} l$—in which case $\bar{s}_*^{i+1} l$ is good—or there is a least $0\leq i\leq m$ with diameter $(\bar{s}_*^i l)\geq\varepsilon_2$—in which case $\bar{s}_*^i l$ is good. So we are done.

6.18. Proof of the Orbit of Sides Proposition

Let G be a periodic gap, and let l be a side of G. Then there is a segment l_0 on l and a thick gap polygon P with Interior$(P)\subset G$ and $l_0\subset\partial P$. Then $\bar{s}_*^n P$ takes finitely many polygon values as n varies, and $\bar{s}_*^n l_0$ is always in the boundary of one of the polygons Q_n of $\bar{s}_*^n P$. If ever Q_n is thin, for a least n, then $\bar{s}_*^n P$ is multivalued, and $Q_n\cap \mathrm{Im}(\beta)\cap S^1\neq\emptyset$, for β as in 6.5 (with $\varepsilon=\varepsilon_1$) and we are done, since there are then only finitely many possibilities for such Q_n, and only two leaf segments for each such Q_n, hence only finitely many possibilities for such $\bar{s}_*^n l_0$. If Q_n is never thin, the Thick Polygon Lemma implies $Q_n\subset\{P_i\colon 1\leq i\leq t\}$ for all large n, for suitably chosen P_i. Then the Length Lemma implies $\bar{s}_*^n l_0$ is one of only finitely many leaf segments in $\partial(\cup_{i=1}^t P_i)$ for all large n, and we are done.

6.19

We complete this chapter with the definition and properties of a subset of a β-invariant lamination L which will be needed for the work on parameter laminations in Chapter 7. Let $\Omega(L)$ be the (possibly empty) closed set of leaves which have no endpoints in K and which intersect only those components of $S^1\setminus K$ which are periodic under $\bar{s}$. Note that, since $\cup\Omega(L)$ intersects only finitely many intervals of $S^1\setminus K$, it lies in a finite type subsurface C of $\bar{\mathbf{C}}\setminus K$ of the form $\bar{\mathbf{C}}\setminus(\cup_{i=1}^n I_i)$, where I_i are closed intervals of S^1 whose union contains K.

A half-leaf l^+ is *segmentwise-periodic* if $\bar{s}_*^n l_1=l_1$ for all segments l_1 on l^+, and some $n>0$.

$\Omega(L)$-Lemma. *Let $\Omega=\Omega(L)\neq\emptyset$. Then there is $N>0$ such that the following holds.*

(1) *All leaves and segments of Ω are fixed by $\bar{s}_*^N$, which orientations preserved.*

(2) *If a half-leaf l^+ of L has $\bar{l}^+\cap(\cup\Omega)\neq\emptyset$, then l^+ is a asymptotic to a half-leaf l_1^+ with $l_1\in\Omega$.*

(3) *If $\beta(1)$ is not in a periodic leaf of L, then only finitely many leaves of $L\setminus\Omega$ are asymptotic to Ω, all of them periodic under $\bar{s}_*$ and segmentwise isolated.*

Proof. (1) For any leaf segment l_1 in L, $l_1\subset(\sigma_\beta\circ\bar{s})^*\bar{s}_*l_1$, with equality if $(\sigma_\beta\circ\bar{s})^*\bar{s}_*l_1$ has intersections with S^1 only at either periodic points of K or in periodic intervals of $S^1\setminus K$. Hence $\bar{s}_*\Omega=\Omega$, and there is N such that, up to isotopy preserving K and $S^1\setminus K$, $\bar{s}_*^N l_1=l_1$ for any leaf segment l_1 in Ω, with orientation preserved. But then $\bar{s}_*^N l=l$ for any leaf l in Ω (not just up to isotopy) with orientation preserved, and $\bar{s}_*^N l_1 = l_1$ for all leaf segments l_1 in Ω (not just up to isotopy) with orientation preserved.

(2) Let l^+ be in a component V of $C \setminus (\cup \Omega)$. Fix a base point x_0 in V. If, for some $\varepsilon > 0$, there is $\Delta > 0$ such that all points in l^+ which are spherical distance less than ε from $\cup \Omega$ can be joined to x_0 by a geodesic in V of length $< \Delta$, then all such points on l^+ must be close to compact geodesics in Ω—of which there are only finitely many. But if l^+ passes sufficiently close to a compact geodesic l_0, l^+ is asymptotic to l_0.

If Δ never exists, then for some $\varepsilon > 0$ and $\Delta > 0$ the set of points in V distance $< \varepsilon$ from $\cup \Omega$ which cannot be joined to x_0 by geodesics in V of length $< \Delta$ is of the form $\cup_{i=1}^n V_i$, where each V_i is bounded by two asymptotic half-leaves from Ω and an arc in V. Once l^+ enters a V_i, it cannot exit, hence must be asymptotic to the two bounding half-leaves in Ω.

(3) Let l_1^+ be a half-leaf in $l_1 \in \Omega$, with l_1 of oriented period M under $\bar{s}_*$. Let $l_1^+ \subset l_1$ also denote lifts to $D = \{z : |z| \leq 1\}$. Let a be the endpoint of l_1^+ in ∂D, and b the other endpoint of l_1. Then $\bar{s}_*^M$ lifts to a monotone map R defined on geodesics in L asymptotic to l_1^+ sufficiently near l_1^+, with $R(l_1) = l_1$. Since $\beta(1) \notin \cup L$, R is injective. If A denotes the set of lifts of periodic sides of gaps asymptotic to l_1^+, then $R(A) = A \cap \mathrm{Image}(R)$, and $R^{-1}R(A) = A$. So strictly preperiodic sides of gaps cannot be asymptotic to l_1^+. So by the Orbit of Sides Proposition, only finitely many sides of gaps are asymptotic to l_1^+, hence only finitely many leaves are asymptotic to l_1^+, all of them segmentwise isolated.

Chapter 7. Parameter laminations

7.1

The aim of this chapter is to prove the Parameter Laminations Theorem of 1.16. We start by giving the main argument of the proof. We continue to use $K = K_r$, $\bar{L} = \bar{L}_r$, $\Phi = \Phi_r$ as in 1.12, and $U = U_r$ is the component of $\bar{\mathbf{C}} \setminus (K \cup (\cup \bar{L}))$ containing ∞, with universal cover $\tilde{U}$ contained in the disc universal cover $\mathring{D}$ of $\bar{\mathbf{C}} \setminus K$.

Proposition. *For $i = 1, 2$, let L_i be a β_i-invariant lamination, with either a minor gap in $\tilde{U}$ or a strictly preperiodic minor leaf. Let Z_i denote the closure of the minor gap of L_i in $\tilde{U}$, if this exists, and the minor leaf of L_i in $\tilde{U}$ otherwise. Suppose there is no half leaf l^+ of L_i with $\#(l^+ \cap S^1) = \infty$, and l^+ intersecting only periodic intervals of $S^1 \setminus K$. Let $Z_1 \cap Z_2 \neq \emptyset$. Then any transversal intersection between L_1 and L_2 must be isolated, and must occur between leaves l_1, l_2 which are eventually periodic, where one is* *periodic if and only if they both are. In addition, if both l_1 and l_2 are non-compact, then $\bar{s}_*^n l_1$ is of oriented period k if and only if the same is true for $\bar{s}_*^n l_2$. Furthermore, l_1 has only finitely many intersections with $\cup L_2$, and vice versa, and $\bar{s}_*^n l_1$ and $\bar{s}_*^n l_2$ intersect transversally for all n.*

Proof. Note that $Z_1 \cap Z_2$ is connected since both Z_1, Z_2 are convex. Let $\tilde{\beta}: [0,1] \to \tilde{U}$ satisfy $\tilde{\beta}(0)=0$ and $\tilde{\beta}(1) \in Z_1 \cap Z_2$. Let $\beta = \pi \circ \tilde{\beta}$. Then L_1 and L_2 are β-invariant. We can assume $\#(\beta^{-1}(S^1)) < +\infty$.

We say ϱ has the *finiteness property* if there is a finite set $\{I_1, \dots I_r\}$ of components of $S^1 \setminus K$ such that $\varrho = \bar{s}$ on $(L_1 \cup L_2) \cap S^1 \setminus (I_1 \cup \dots \cup I_r)$. We can find a branched covering ϱ with the finiteness property and such that $\varrho l = \bar{s}_* l$ for any leaf l in $L_1 \cup L_2$. In fact, we can take ϱ of the form $\sigma_\beta \circ \bar{s} \circ \varphi$, where φ is isotopic to the identity via an isotopy preserving K and all but finitely many intervals of $S^1 \setminus K$.

Now, if $l_1 \in L_1$ intersects L_2 transversally, then l_1 intersects transversally an eventually periodic leaf l_2 of L_2, where l_2 is in one of finitely many orbits, and is a side of a gap (using 6.14 and the Orbit of Sides Proposition, and the hypothesis of the present proposition, that the minor leaf is eventually periodic if there is no minor gap). If $l_1 \cap l_2 \neq \emptyset$, then $\varrho^n l_1 \cap \varrho^n l_2 \neq \emptyset$. So to complete the proof (since L_1 and L_2 are interchangeable), it suffices to show that if $l_2 \in L_2$ has oriented period k under $\bar{s}_*$, and $l_1 \cap l_2 \neq \emptyset$ with l_1 eventually periodic, then l_1 is periodic, $\bar{s}_*^k l_1 = l_1$ if l_2 is non-compact, and $\#(l_1 \cap l_2) < +\infty$. To do this, we shall show that all points of $l_1 \cap l_2$ are periodic under ϱ, and fixed by ϱ^k if l_2 is non-compact, and that only finitely many segments of l_1 (or l_2) can contain points fixed by ϱ^k. To show points are periodic, since $\varrho^k: l_2 \to l_2$ is an orientation-preserving homeomorphism, we only have to show that any point of $l_1 \cap l_2$ has finite forward orbit under ϱ, for which we can assume that l_1 is periodic. So now, to complete the proof, we only need to show that, if l_2 is non-compact, there is a finite union l_2' of segments of l_2 such that, for any $x \in l_2$, $\varrho^{kn} x \in l_2'$ for all sufficiently large n (and similarly for any periodic leaf of L_1). Now by the hypothesis of the nonexistence of certain half-leaves in L_2, there is a bound N on the number of consecutive segments of l_2 which can cross some $\varrho^{-j} I_i$, $1 \le i \le r$, $0 \le j \le k$. So since ϱ has the finiteness property, the proof is completed by Lemma 7.2.

7.2

LEMMA. *Let $l_2 \in L_2$ be a non-compact leaf of oriented period k, where L_2 is as in* 7.1. *Then there is a finite union l_2'' of segments of l_2, and a finite union l_2''' of segments which* *contains segments adjacent to* l_2'', *such that* $\bar{s}_*^k l_2''' \subset l_2''$, *and for any segment* μ *on* l_2, $\bar{s}_*^{kn}\mu \subset l_2''$ *for all sufficiently large* n.

Proof. First we show there is a segment μ on l_2 with $\bar{s}_*^k\mu = \mu$. If, for some μ, $\bar{s}_*^k\mu \neq \mu$, let μ' be the finite connected union of segments with μ, $\bar{s}_*^k\mu$ as end segments. If $\bar{s}_*^{nk}\mu'$ is ever a single segment, it is fixed by $\bar{s}_*^k$. If not, $\bar{s}_*^{kn}\mu'$ has the same number of segments for all $n \geq M$ (some M), and intersects only periodic intervals of $S^1 \setminus K$. Then $\bigcup_{n\geq M} \bar{s}_*^{nk}\mu'$ is a forbidden half-leaf. So there does exist μ with $\bar{s}_*^k\mu = \mu$. We can find a maximal finite union l_2'' of segments containing μ such that $\bar{s}_*^k l_2'' = l_2''$. For all segments on l_2'' intersect only periodic intervals of $S^1 \setminus K$. Then l_2'' has the required properties, because we can take l_2''' to be the union of l_2'' and one adjacent segment on either side, and if ν is any segment on l_2, the number of segments between μ and $\bar{s}_*^{kn}\nu$ can only decrease with n.

7.3. Definition

A β-invariant lamination L for which ϱ_L is critically finite, but not type II, is *primitive* if:

(a) no half leaf l^+ from L has $\#(l^+ \cap S^1) = \infty$ and all S^1-intersections with periodic intervals of $S^1 \setminus K$,

(b) no two finite-sided gaps have a side in common, and no leaf is approached on both sides by the same gap,

(c) if G is an infinite-sided gap of L of period n under $\bar{s}_*$, then:

(c1) if G is simply connected, at most one side of G has oriented period n, and no other periodic side of G in $L \setminus \bar{L}_r$ can also be a side of another gap,

(c2) if G is not simply connected, and $n > 1$, there is a boundary component γ of G of oriented period n, which separates G from $\bar{s}_*^i G = \varrho_L^i G$ $(0 < i < n)$ such that γ is either a compact leaf of L or a finite union of leaves without S^1-crossings, and points of K. Any side of G which is not in γ cannot also be a side of another gap.

Let G_0 be the gap of $\bar{L}_r$ containing 0. Let ϱ_L be type II. Then L is *primitive* if there exists a primitive lamination L' (with $\varrho_{L'}$ type IV) such that $L \subset L'$ and every leaf of $L' \setminus L$ is a side of some finite-sided gap G which also has a side in common with a gap in the full orbit of G_0.

7.4

Proving the Parameter Laminations Theorem for primitive laminations will be quite easy, but showing that primitivity is no real restriction takes a bit more work. We need to prove the following.

PROPOSITION. *Let L be a β-invariant lamination with ϱ_L critically finite, and such that, if $\beta(1) \in \bigcup L$, the minor leaf of L is strictly preperiodic (so that ∞ is strictly preperiodic under ϱ_L). Then there is a primitive β-invariant lamination L' (after possibly moving the endpoint of β within its minor gap) such that $(L' \setminus L) \cup (L \setminus L')$ consists of only segmentwise isolated leaves and leaves in $\bigcup_{n \geq 0} (\sigma_\beta \circ \bar{s})^{*n} \Omega(L)$, there is a one-to-one correspondence $G \mapsto G'$ between infinite-sided gaps of L, L' with $(G \setminus G') \cup (G' \setminus G)$ consisting of finite-sided gaps and isolated leaves of L, L', and $\varrho_L \simeq \varrho_{L'}$ if ϱ_L is type* II, III *or* IV.

7.5. Proof that primitivity conditions (a), (b) can be satisfied

The first step in proving 7.4, given L as in 7.4, is to find L_3 such that, although L_3 is not primitive, L_3 satisfies conditions (a), (b) in the definition of primitive (7.3) and satisfies all the other conditions satisfied by L' in 7.4 relative to L. We start by adding segmentwise isolated leaves to obtain L_1, so that L_1 satisfies all the conditions of 7.4 except primitivity, $\Omega(L_1) = \Omega(L)$, and no side of an infinite-sided gap of L_1 is asymptotic to $\Omega(L)$. So let G be a periodic infinite-sided gap of L with some sides (but necessarily only finitely many) asymptotic to $\Omega(L)$. (We know from the $\Omega(L)$-Lemma 6.19 that only periodic leaves can be asymptotic to $\Omega(L)$.) We shall add some leaves to G, and then extend invariantly to the full orbit of G, and do this to all such orbits to obtain L_1. Let $\tilde{G}$ be a lift of G to the universal cover $\mathring{D} = \{z : |z| < 1\}$. We shall add leaves to $\tilde{G}$ and then project them down. Note that, although only finitely many sides of G can be asymptotic to $\Omega(L)$, their lifts might comprise infinitely many sides of $\tilde{G}$, if G is not simply-connected. We add a geodesic l' in $\tilde{G}$ joining points a, b in ∂D, whenever the interval I bounded by a, b has the following property. I is a component of the closure of the set of points $x \in \partial D$ such that x is separated from $\tilde{G}$ by a side l of $\tilde{G}$ which projects to a side of G asymptotic to $\Omega(L)$. The projections of such leaves l' will give only finitely many leaves in G.

Now, having defined L_1, let $\Omega'(L)$ consist of all leaves l such that l contains a half-leaf l^+ with infinitely many S^1-crossings, all with periodic components of $S^1 \setminus K$. Let

$$L_2 = L_1 \setminus \bigcup_{n \geq 0} (\sigma_\beta \circ \bar{s})^{*n} \Omega'(L).$$

Then L_2 satisfies property (a). Now let L_3 be obtained from L_2 by removing all leaves which are approached on both sides by the same gap, or on both sides by finite-sided gaps. We note that each infinite-sided gap of L_1 is contained in an infinite-sided gap of L_3, and each infinite-sided gap of L_3 contains a unique infinite-sided gap of L_1. By the

Finite-sided Gaps Proposition, the forward orbit of an infinite-sided gap must contain 0 or ∞. For any two adjacent finite-sided gaps of L_2 have the same period or preperiod—so, by the Orbit of Gaps Proposition, adjacent finite-sided gaps of L_2 occur only in finite groups.

7.6

Before proceeding further with the proof of 7.4, we need to consider non-simply-connected gaps. First, we have the following

LEMMA. *Let L be a β-invariant lamination and let G be a non-simply-connected periodic gap of L of period n under $\bar{s}_*$. Then there exists a finite set Γ of disjoint compact geodesics, with $\bigcup\Gamma\subset G$ or ∂G, such that $\bar{s}_*^n\Gamma=\Gamma$. If G is finite-sided, Γ can be chosen so that each element of Γ bounds an annulus round a boundary component of G.*

Also,

$$\beta((t_1,t_2])\cap\bigcup_{i\geqslant 0}\bar{s}_*^i(\bigcup\Gamma)=\varnothing,$$

where

$$t_1=\mathrm{Min}\{t:\beta(t)\in S^1\}$$

if this exists,

$$t_1=1$$

otherwise, and

$$t_2=\mathrm{Max}\{t:\beta(t)\in\bigcup L\}.$$

Proof. If γ is any compact geodesic in G or ∂G, then γ has finitely many S^1-crossings, and γ is determined by its S^1-crossings, so the set $\{\bar{s}_*^i\gamma : i\geqslant 0\}$ is finite, and $\bar{s}_*^i\gamma\subset\bar{s}_*^iG$ or $\bar{s}_*^i\partial G$. So we can find $\Gamma'\subset\{\bar{s}_*^i\gamma : i\geqslant 0\}$ with $\bar{s}_*^n\Gamma'=\Gamma'$. The loops of Γ' may not be disjoint, but we can obtain Γ by lifting each component of $\bigcup\Gamma'$ to the universal cover $\mathring{D}$ of $\bar{\mathbb{C}}\setminus K$, taking the convex hull and projecting back down to $\bar{\mathbb{C}}\setminus K$.

If G is finite-sided, we can start by taking γ to bound an annulus in G round one of the boundary components, as required. If $\beta(t)\in\bigcup_{i\geqslant 0}\bar{s}_*^i(\bigcup\Gamma)$ for some $t\in(t_1,t_2]$, then taking inverse images under $\sigma_\beta\circ\bar{s}$, we obtain a leaf of L intersecting $\bigcup_{i\geqslant 0}\bar{s}_*^i(\bigcup\Gamma)$. In fact, this leaf can be in $(\sigma_\beta \circ \bar{s})^*\mu$, if μ is the minor leaf of L. This gives a contradiction, as required.

7.7

The following few lemmas will be useful for proving primitivity condition (c2), when proving 7.4.

LEMMA. *Let* Γ *as in* 7.6 *consist of a single orbit under* $\bar{s}_*$. *Then no two segments of* $\cup\Gamma$ *in* $\{z: |z|>1\}$ (*or* $\{z: |z|<1\}$) *with endpoints* (x_1, y_1) *and* (x_2, y_2) *have* $\Phi_r(x_1)=\Phi_r(x_2)$ *and* $\Phi_r(y_1)=\Phi_r(y_2)$. *If*

$$B = \{\Phi_r(x): x \text{ is an endpoint of a segment from } \cup\Gamma\},$$

then B *consists of one or two orbits under* $z\mapsto z^2$, *and if* B *consists of one orbit, each loop of* Γ *intersects* S^1 *exactly twice.*

Remark. The core of this lemma is the result of Tan Lei [TL] about Levy cycles for matings of polynomials of which at least one has corresponding minor leaf with endpoints in a single orbit under $z\mapsto z^2$.

Proof of Lemma. Let $a, b \in S^1$ be such that there are segments of Γ in $\{z: |z|>1\}$ with endpoints in $\Phi_r^{-1}(a)$ and $\Phi_r^{-1}(b)$, and let q be the least integer >0 such that $\bar{s}_*^q$ preserves this set of segments. Then $\bar{s}_*^q$ must either preserve or reverse order of intersection with S^1 for this set of segments. So the set consists of only one segment if the order is preserved, and at most two if the order is reversed, since Γ consists of a single orbit under $\bar{s}_*$. (We shall see that the set consists of a single segment in this case also.) B is the orbit of the endpoints of μ_r, the minor leaf corresponding to r, and so consists of one or two orbits.

Now suppose B consists of one orbit. Then for each $a \in S^1$ for which there are segments of $\cup\Gamma$ in $\{z: |z|>1\}$ (or $\{z: |z|<1\}$) with endpoints in $\Phi_r^{-1}(a)$, there are either exactly two such segments, whose second endpoints are in different components of $S^1 \setminus K$, or there is one such segment whose orientation is reversed by $\bar{s}_*^p$ for some p. There are the same number of segments in $\{z: |z|>1\}$ and $\{z: |z|<1\}$, so the same alternative must occur for each. So there is either a loop γ and a least integer $p>0$ such that $\bar{s}_*^p\gamma=\gamma$, with $\bar{s}_*^p$ reversing order on γ, or there is a finite type surface S which is not an annulus, and a least integer $p>0$ with $\bar{s}_*^p S=S$, $\partial S\subset\cup\Gamma$, and $\bar{s}_*^p$ cyclically permutes the components of ∂S. Then we can extend $\bar{s}_*^p$ to a homeomorphism φ of γ, S. Then φ

has a fixed point. If $\varphi:\gamma\to\gamma$, then φ leaves invariant a segment of γ, hence must leave invariant two segments of γ, one in $\{z:|z|<1\}$ and one in $\{z:|z|>1\}$. Hence γ consists of exactly two segments, since all segments then have the same period p under $\bar{s}_*$, and this is only possible if there are only two segments in γ. If $\varphi:S\to S$, then φ leaves invariant a polygon P of S, and cyclically permutes the leaf segments in ∂P. These leaf segments must all be in different components of ∂S, since S has no genus. $S\setminus P$ is a disc, so φ leaves invariant another polygon P' in S with similar properties. Then P and P' must be on opposite sides of S^1, since p was chosen minimal. So then for Euler characteristic reasons (because S is a union of polygons all with at least three leaf segments as sides) $S=P\cup P'$, and we are done.

7.8

We continue with the previous notation of 7.6. Suppose $t_2<t_1$. Let

$$L_1=\{l_1: l_1 \text{ is a geodesic in } \{z:|z|>1\} \text{ with endpoints } \Phi_r(x),\ \Phi_r(y) \text{ and there is a leaf segment from } L \text{ in } \{z:|z|>1\} \text{ with endpoints } x,\ y\}.$$

Then $L_1^{-1}\{l: l^{-1}\in L_1\}$ is an invariant lamination (under $z\mapsto z^2$) in the sense of 1.10.

LEMMA. *If $t_2<t_1$ and some infinite-sided polygon P of G contains more than one segment from* $\cup\Gamma$ and Γ *consists of a single orbit under $\bar{s}_*$, then every loop of* Γ *crosses S^1 exactly twice.*

Proof. All the infinite sided polygons of G which contain segments of $\cup\Gamma$ are in $\{z:|z|>1\}$ and in the same periodic orbit under $\bar{s}_*$. Then we can define a critically finite degree two branched covering ϱ_1 which preserves $\cup\bar{s}_*^iP$ and $\cup\Gamma\setminus\cup\bar{s}_*^iP$. Then by 5.5, any arc of $\cup\Gamma\setminus\cup\bar{s}_*^iP$ has both endpoints in the same $\bar{s}_*^jP$, that is, each loop of Γ intersects S^1 exactly twice.

7.9

We continue with the notation of 7.6. Let

$$\Gamma'=\{l: l \text{ is a leaf in } \{z:|z|<1\} \text{ with endpoints } a,\ b \text{ and there is a segment } l_1 \text{ from } \Gamma \text{ in } \{z:|z|>1\} \text{ with endpoints in } \Phi_r^{-1}(a),\ \Phi_r^{-1}(b)\}.$$

Then Γ' is a forward-invariant lamination (under $z\mapsto z^2$) in the sense of 1.10. Let μ' be the minor leaf of Γ', which is a leaf of QML (see 1.10) and let μ be the corresponding segment of Γ. We recall that there is a partial ordering on leaves of QML: $\mu_1<\mu_2$ if μ_1 separates μ_2 from 0.

LEMMA. *If $t_1\leq t_2$, μ' has endpoints a, b, and $\mu''<\mu'$ has endpoints c, d, then there is a leaf of L in $\{z:|z|>1\}$ with endpoints in $\Phi_r^{-1}(c)$, $\Phi_r^{-1}(d)$. Consequently, if μ' is not isolated in QML there is a leaf l_μ of L in $\{z:|z|>1\}$ separating μ from ∞ and with endpoints in $\Phi_r^{-1}(a)$, $\Phi_r^{-1}(b)$.*

Proof. If $t_1\leq t_2$, there is a diagonal leaf segment l_0 of L in $\{z:|z|>1\}$. Let $\pm x$ be the images under Φ_r of the endpoints of l_0. Now we can assume c, d are periodic of period t (say). Let U_0 and V_0 be the intervals with endpoints x^2, c, and x^2, d respectively. Let U_1, V_1 be the preimages of these under $z\mapsto z^2$ which contain the periodic preimages of c, d. Then one of the intervals has endpoint $+x$ and the other has endpoint $-x$. Similarly, we define U_n, V_n for all n, so that U_{n+1}, V_{n+1} are preimages of U_n, V_n and U_n always contains the periodic element of $s^{-n}c$ (if $s(z)=z^2$) and similarly for V_n. Then none of the U_n, V_n, contain a, b, and they are all the same side of a, b, because the orbits of c, d are all on one side of a, b. (This is simply a fact about endpoints of leaves of QML). So only the component of $\mathrm{Im}(\beta)\setminus S^1$ containing $\beta(0)=\infty$ crosses a line joining U_n and V_n in $\{z:|z|>1\}$. Then there is a sequence of segments l_n of L such that l_{n+1} is a preimage under ϱ of l_n and the images of the endpoints of l_n under Φ_r are in U_n, V_n. Taking limits, we obtain the required leaf of L corresponding to μ''. The existence of the leaf l_μ follows, if μ' is not isolated.

7.10

We continue with the notation of 7.6.

LEMMA. *If G has period $n>1$ and is nonsimply-connected, and $\beta(1)\in G$, then there is a component δ of $\bar{\mathbf{C}}\setminus G$ which is fixed by ϱ^n and separates G from ϱ^iG $(0<i<n)$ and one of the following holds.*

(a) *δ is a compact geodesic.*

(b) *δ is a union of geodesics without S^1-crossings, and of endpoints in S^1.*

Proof. First we show that there exists δ with $\varrho^n\delta=\delta$ and separating G from ϱ^iG $(0<i<n)$. For some i, we can find a disc D_i with $\varrho^iG\subset D_i$, $\partial D_i\subset\partial\varrho^iG$ and such that D_i contains no other ϱ^jG. Then $\varrho^j\infty\notin D_i$ for $1\leq j<i+1$. So, for $0\leq j<i$, all components of $\varrho^{-j}D_i$ are discs which do not contain $\varrho\infty$ or ϱ^kG except for $k=i-j$, and all components of $\varrho^{-i}D_i$ are discs. One such component contains G, and the boundary component δ separates G from ϱ^jG $(0<j<n)$. We claim that $\varrho^n\delta=\delta$. For if D' is any component of

$\bar{\mathbb{C}}\setminus G$ with $\partial D'\neq\delta$, $D'\cap\{\varrho^i\infty : i\geq 0\}=\emptyset$, so all components of $\varrho^{-j}D'$ are discs disjoint from $\varrho^i G$ ($i\geq 0$) for all $j\geq 0$. So we must have $\delta\subset\varrho^{-n}\delta$.

Now we claim that if Γ is as in 7.6, with all loops in a single periodic orbit under $\bar{s}_*$, then some $\gamma\in\Gamma$ in G must separate $\varrho\infty$, $\varrho 0$ from $\varrho^i G$, $0<i<n$. It suffices to show that some $\gamma\in\Gamma$ in G must separate $\varrho 0$ from $\varrho^i G$, $0<i<n$. For this it suffices that some $\gamma\in\Gamma$ in G separates some $\varrho^j 0$ from $\varrho^i G$, $0<i<n$. (For then, if $j\geq 1$ is minimal, we take the preimage under $\bar{s}_*^{j-1}$.) But if not, then some $\gamma\in\Gamma$ in G bounds a disc which is disjoint from $\{\varrho^i 0, \varrho^i\infty : i\geq 0\}$. But then γ cannot be periodic under $\bar{s}_*$, giving a contradiction.

Thus, the lemma will be proved if $G\cup\partial G$ contains a loop γ from Γ and either $\gamma\subset\partial G$ or there is an annulus A in G with $\partial A\subset\gamma\cup\partial G$ or γ bounds a component of $\bar{\mathbb{C}}\setminus\bigcup\Gamma$ which is disjoint from K (because in this case $\gamma\subset\partial G$). Now let t_1, t_2 be as in 7.6. There are two cases to consider: $t_2<t_1$ and $t_1\leq t_2$.

Case $t_2<t_1$. If $\bigcup\Gamma\cap\partial G\neq\emptyset$, we are done. So now suppose $\bigcup\Gamma\cap\partial G=\emptyset$. As remarked in 7.8, for L_1 as defined there, L_1^{-1} is an invariant lamination (under $z\mapsto z^2$). If one segment l from Γ is the only one in an infinite-sided polygon P of G in $\{z: |z|>1\}$, then one component P' of $P\setminus l$ has exactly one leaf l_1 from L as a side, and l, l_1 have the same period under $\bar{s}_*$. If B (as in 7.7) consists of two orbits, each segment from Γ in $\{z: |z|<1\}$ bounds rectangular polygons on both sides. So some leaves from the orbit of l_1 can be joined by leaves of $\bar{L}_r$ to give the required δ. If B consists of one orbit, then, as in 7.7, each loop of Γ intersects S^1 exactly twice, and some infinite-sided polygon P of G contains more than one segment from $\bigcup\Gamma$. So now, by 7.8, we can assume that these last two properties hold. The leaf segment of ∂P whose oriented period is the period of P must then be a leaf of L with endpoints in K, and there must be a leaf of $\bar{L}_r$ with the same endpoints. The union of these two leaves and their endpoints is the required δ.

Case $t_1\leq t_2$. If B consists of two orbits, let l_μ be as in 7.9 (since in this case μ' is not isolated in QML). Then leaves from the orbit of l_μ can be joined by leaves of $\bar{L}_r$ to give the required δ. If B consists of a single orbit, let μ'' be the immediate predecessor of μ', and let l'' be the leaf of L corresponding to μ''. By 7.7, each loop of Γ intersects S^1 exactly twice, and hence there is a leaf of $\bar{L}_r$ with the same endpoints as l''. The union of these leaves and their endpoints is the required δ.

7.11. Proof of Proposition 7.4

We continue from the progress we made in 7.5. So let L_3 be the lamination obtained in 7.5. Essentially, we want to modify L_3 to satisfy condition (c). Suppose first that

ϱ_L—and also ϱ_{L_3}—is not type II. Then there is nothing to prove unless the gap G containing $\varrho_{L_3}\infty$ is infinite-sided, and $\varrho=\varrho_{L_3}$ is type IV and G is of some period n under ϱ. Now we have to consider two cases separately.

Case 1: *G is simply connected.* Suppose l is a side of G of oriented period $p>n$ under $\bar{s}_*$ which is also a side of a finite-sided gap G'. Then we claim that G' has no side in common with $\varrho^i G$, $0<i<n$. If G' is simply connected, we obtain this from 5.5. For if G' does have a side in common with some $\varrho^i G$, $0<i<n$, we can find γ which follows closely a minimal possible number of sides of G' from a side of G to a side of some $\varrho^i G$, $0<i<n$. Then $\{\varrho^j\gamma: 0\leqslant j<p\}$ is a set of isotopically disjoint arcs, and $\varrho^p\gamma=\gamma$ up to isotopy. This contradicts 5.5. If G' is not simply connected, and $\varrho^i G$ $(0<i<n)$ has a side in common with G', we can also contradict 5.5. For by 7.6, we can find compact geodesics in G' which bound annuli round the boundary components of G', and let $G''=G'\setminus\bigcup(\text{annuli})$. Then we can find γ' in G'' joining different boundary components of G'', and then extend γ' into the annuli to obtain γ joining G to $\varrho^i G$ (some $0<i<n$). We take γ' and γ to have the minimum possible number of intersections with S^1, and then all $\varrho^i\gamma$ $(i\geqslant 0)$ must be disjoint or equal, up to isotopy. Then γ is periodic under ϱ, up to isotopy, and for Euler characteristic reasons, the period must be (period(G)$\times$number of boundary components), which must be p.

So now obtain L_4 from L_3 by removing the full orbit of sides of G of oriented period $>n$ which are common to finite-sided gaps. So now no side of G which is not in $\bar{L}_r$ and is of oriented period $>n$ can be common to any other gap (by 5.5). Rename G as the gap containing $\beta(1)$. If G is still simply-connected, we add a leaf to G as follows. Let $\tilde{G}$ be the lift of G to the unit disc D, and F a lift of $\varrho_{L_4}^n$ to D with $F\tilde{G}=\tilde{G}$. Then there are points a, b on ∂D which are endpoints of sides of $\tilde{G}$ fixed by F, and all the finitely many sides of $\tilde{G}$ fixed by F are between a and b. Let L_5 be obtained by adding to L_4 the full orbit of the projection of this leaf. If, on the other hand, the new gap G is not simply conneted, take $L_4=L_5$.

Case 2: *G is not simply connected.* Let δ be as in 7.10. By 7.10, if any side of G which is not in δ is common to a finite-sided gap G', then G' has no side in common with $\varrho^i G$ $(0<i<n)$, and no side of G which is not in δ or $\bar{L}_r$ can be common to $\varrho^i G$ $(0<i<n)$. So now remove the full orbit of sides of G not in δ which are common to finite-sided gaps, to obtain L_4, and put $L_5=L_4$.

Now L_5 satisfies conditions (a) and (c) in the definition of primitive, but some leaves may again be approached on both sides by finite-sided gaps. Remove the full orbit of these, to obtain $L_6=L'$ with all the required properties.

Finally, if ϱ_L was type II, let $L_3' = L_3 \cup \bar{L}_r$, and work with L_3' to obtain L_5, and finally L_6. Then every leaf of $\bar{L}_r \setminus L_3$ is a side of at most one finite-sided gap of L_6. Let L' be obtained by removing from L_6 all leaves of $\bar{L}_r \setminus L_3$, and all sides in $L_6 \setminus L_3$ of finite-sided gaps of L_6 which have leaves of $\bar{L}_r \setminus L_3$ as sides. Again, L' has the required properties.

7.12. Proof of the Parameter Laminations Theorem

If leaves l_1, l_2 of L_1, L_2 intersect transversally, then we can assume they are both periodic sides of infinite-sided periodic gaps, by the definition of primitive and 7.1. Then $\varrho_{L_1}, \varrho_{L_2}$ must be type II or IV, and we can assume (by adding leaves if necessary) that they are both type IV. Let G_1, G_2 be the minor gaps in U of L_1, L_2.

Now l_1 must be approached on both sides by gaps of L_1. So, if G_1 is simply-connected, by the definition of primitive, the oriented period n of l_1 is the same as the period of G_1, and if G_1 is not simply-connected, l_1 is either compact or has no S^1-crossings.

Now we know from 7.1 that l_1, l_2 have the same period under $\bar{s}_*$ unless one of them is compact, and even in that case, periodic segments of the leaves have the same period. Let ϱ be the map of 7.1, so that ϱ preserves L_1, L_2. First we consider the case when one of G_1, G_2 is simply-connected, and we assume G_1 is. Thus G_2 and l_2 are invariant under ϱ^n. Then we can find an open subset Y of G_1, whose boundary is contained in ∂G_1 together with a single component of $l_2 \cap G_1$, and such that ϱ^n is a homeomorphism of Y onto itself. For ϱ^n fixes all components of $G_1 \cap G_2$, $G_1 \setminus G_2$, and is a homeomorphism on all but one of them, because ϱ^n has only one critical point in the forward orbit of G_1. Lifting Y, ϱ^n to $\tilde{Y}$ in $\tilde{U}$, and F with $F\tilde{Y} = \tilde{Y}$, we see that F fixes the lift of the component of $l_2 \cap G_1$ in $\partial \tilde{Y}$, and that all lifts of sides are eventually periodic. Since F is a homeomorphism on $\partial \tilde{Y}$, this implies all sides are fixed. So all sides of Y are fixed by ϱ^n. So some other side of G_1 is asymptotic to l_1, contradicting the definition of primitive.

If both G_1 and G_2 are nonsimply connected, then, by the definition of primitive, l_1, l_2 are contained in boundary components δ_1, δ_2 of G_1, G_2, where the oriented period of δ_i is the period of G_i, and δ_i is either a compact geodesic, or a finite union of leaves without S^1-crossings; and in the latter case, δ_i runs parallel to a compact geodesic in G_i. The segments of δ_i in $\{z: |z|>1\}$ have the same endpoints in S^1 as the leaves of a finite z^2-forward invariant lamination on $\{z: |z|<1\}$. The shortest of these segments is the last crossed by β_i. Therefore, if δ_1 and δ_2 intersect transversally, so do the shortest segments in $\{z: |z|>1\}$, and so do the corresponding leaves in <u>QML</u>, the lamination of minor leaves of z^2-invariant laminations [T]. But QML is a lamination, that is, leaves do not intersect. So δ_1 and δ_2 do not intersect transversally, and we are done.

Chapter 8. The Tuning Proposition

8.1

We recall that if $g: \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is a degree two critically finite branched covering of type II, III or IV which is not equivalent to a rational map, then condition (C) of 1.6 holds. This means that g has a *fixed subsurface* in the following sense. There is a homotopy $\{f_t: t \in [0, 1]\}$, through branched coverings with $X(f_t)=X(g)=X$ for all t, $f_0=g$, and, if we write $f_1=f$, there is a compact connected subsurface with boundary $Y \subset \bar{C} \setminus X$ such that $fY=Y$, $f|Y$ is a homeomorphism, and f cyclically permutes the components of ∂Y all of which are nontrivial simple loops in $\bar{\mathbf{C}} \setminus X$. Then Y is a *fixed subsurface* for g. If $Y_1 \subset Y$ is any subsurface such that all components of ∂Y_1 are nontrivial in $\bar{\mathbf{C}} \setminus X$ and fY_1 is isotopic to Y_1 in $\bar{\mathbf{C}} \setminus X$, then f must cyclically permute the components of ∂Y_1. Then Y is a *minimal fixed subsurface* if no such subsurface Y_1 exists, except with Y_1 homotopy equivalent to Y. The minimality or not of Y is clearly a condition on the isotopy class of $f|Y$. This is made precise in the following lemma. Clearly, minimal fixed subsurfaces exist when fixed subsurfaces exist.

8.2

FIXED SUBSURFACE LEMMA. *If g has a fixed subsurface, then g has a unique minimal fixed subsurface Y up to isotopy, and if f is as above, and Y has k boundary components, then either $f|Y$ is isotopic to a pseudo-Anosov homeomorphism* [F–L–P] *in the sense that it preserves transverse stable and unstable measured foliations, or f is isotopic to an isometry (for some noncomplete Poincaré metric on Y, such that boundary components are geodesics of length* 1 *and f^k is isotopic to the identity).*

Proof. We claim, first, that any two fixed subsurfaces Y_1, Y_2 for g must intersect, and that intersections cannot be removed by isotopies of either that keep X fixed. Now only one component U of $\bar{\mathbf{C}} \setminus Y_1$ contains a component of $g^{-1}U$ up to isotopy in $\bar{C} \setminus X$, and U contains both critical points. So if $Y_1 \cap Y_2 = \varnothing$, we must have $Y_2 \subset U$. But then, if γ is the component of ∂Y_2 bounding a disc in U containing Y_2, and Y_2' is the component of $g^{-1}Y_2$ isotopic to Y_2 in $\bar{\mathbf{C}} \setminus X$, and $\gamma' = g^{-1}\gamma \cap Y_2'$, we must have γ' and γ isotopic in $\bar{\mathbf{C}} \setminus X$, with $g: \gamma' \to \gamma$ preserving orientation. This gives a contradiction, because the boundary components of Y_2 are cyclically permuted by g^{-1} up to isotopy, so Y_2 would have to be a disc—which is impossible, since $Y_2 \cap X = \emptyset$, but each component of ∂Y_2 is nontrivial in $\bar{\mathbf{C}} \setminus X$. So $Y_1 \cap Y_2 \neq \emptyset$.

Now let Y be any minimal fixed subsurface for g, and choose f as in 8.1 with $fY=Y$. We classify the isotopy class of $f|Y$ following [F–L–P], where only homeomorphisms fixing boundary components were considered, but the principle is the same. Since $f|Y$ leaves invariant no proper subsurface, as already noted (8.1) there is no disjoint loop set left invariant by f up to isotopy. We know that f_* fixes a point in the Thurston compactification of Teichmüller space, where f_* is the homeomorphism of this compactification (which is a ball) which is induced by the isotopy class of f [F–L–P]. If f_* fixes a point in Teichmüller space, then $f|Y$ is isotopic to an isometry. Then Y can be given a hyperbolic structure such that all boundary components are geodesics of length one, and we can assume without loss of generality that $f|Y$ preserves this hyperbolic structure. By the Lefschetz Fixed Point Theorem, f has a fixed point y_0. Let l_i be a shortest geodesic segment from y_0 to a boundary component b_i, $1 \leq i \leq k$. Then for $i \neq j$, l_i and l_j can only intersect at y_0. In fact, any two shortest geodesic segments from y_0 to ∂Y cannot intersect. So, since $Y \setminus \bigcup_{i=1}^{k} l_i$ is a disc, l_i is unique. So $fl_i = l_{i+1}$, $fl_k = l_1$, for a suitable renumbering. Then f^k fixes all l_i, and must be the identity on Y.

Now suppose f_* does not fix a point in Teichmüller space. Since no disjoint loop set is left invariant by $f|Y$, f_* must fix a projective measured foliation in the boundary of Teichmüller space, for which all leaves are dense, and proceeding as in [F–L–P], $f|Y$ must be isotopic to a pseudo-Anosov which preserves exactly two transverse measured foliations.

Now suppose there is a second minimal fixed subsurface Y' with $Y \cap Y' \neq \emptyset$, $Y \neq Y'$. We can assume that Y and Y' have the smallest possible number of intersections, allowing movement of Y, Y' under isotopies fixing X, and without loss of generality that the set of segments of $\partial Y' \cap Y$ is preserved by f, and similarly for the set of segments of $\partial Y \cap Y'$. Then f must be isotopic to an isometry, and, up to isotopy, all segments are fixed pointwise by f^k. Then we can assume that f preserves Y, Y', $Y \cap Y'$, and that f^k fixes Y, Y' pointwise. Then let D_0, D_0' be the disc components of $\bar{\mathbf{C}} \setminus Y$, $\bar{\mathbf{C}} \setminus Y'$ which contain both critical points, and let D be the component of $D_0 \cap D_0'$ which contains both critical points. (D does exist.) Then $\partial(D_0 \setminus D)$ is fixed by f^k, and $f^k(D_0 \setminus D) = D_0 \setminus D$, since $f^i|D_0 \setminus D$ is a homeomorphism for each i, and $f^k(\partial(D_0 \setminus D)) = \partial(D_0 \setminus D)$. Then $X(f) \cap (D_0 \setminus D)$ is invariant under f^k, giving a contradiction, since every point in $X(f) = X(g)$ has a critical point in its forward orbit.

8.3. Proof of the Tuning Proposition

If f is a branched covering with periodic critical point c of period m, and g is a polynomial of the form $z \mapsto z^2+a$ with 0 periodic, recall (1.20) that $f \vdash_c g$ is another critically finite degree two branched covering, known as the tuning of f round c by g. Recall also that the Tuning Proposition states that, if $m>1$, and f is equivalent to a rational map, so is $f \vdash_c g$.

Proof. If $f \vdash_c g$ is not equivalent to a rational map, we can assume without loss of generality that a minimal fixed subsurface Y is invariant under $f \vdash_c g$, and we can assume that f, $f \vdash_c g$ are equal off $f^i D$ $(0 \leq i < m)$ where D is a disc containing c and the tuned orbit, and that $f^m D = D$. We can also assume that $Y \setminus \bigcup_{i=0}^{m-1} f^i D$ is invariant under f and $f \vdash_c g$, and that all intersections between Y and $f^i D$ $(0 \leq i < m)$ are essential. There must be intersections since, by hypothesis, f is equivalent to a rational map, and hence has no fixed subsurface. The intersection arcs are periodic. Hence, by the Fixed Subsurface Lemma, $f \vdash_c g | Y$ must be isotopic to an isometry (for some suitable metric) and $(f \vdash_c g)^k | Y$ must be isotopic to the identity, where ∂Y has k components. In fact, $k=m$. For $f \vdash_c g$ has a fixed point in a component C of $Y \setminus \bigcup_{i=0}^{m-1} f^i D$ (since $m>1$), and ∂C consists of nk components of $\partial C \cap \partial Y$, and nk components of $\partial C \cap (\bigcup_{i=0}^{m-1} f^i D)$ for some n. Then Y has Euler characteristic $\leq 1 - nk/2$. So we must have $n=1$. Now $f \vdash_c g$ acts as an order k rotation on the components of $\partial C \cap \partial Y$, hence also on the components of $\partial C \cap (\bigcup_{i=0}^{m-1} f^i \partial D)$. Hence $m | k$. Since $Y \cup (\bigcup_{i=0}^{m-1} f^i \partial D)$. has genus 0 and $m>1$, we must have $m=k$. So now we can assume $(f \vdash_c g)^m$ fixes all components of $Y \cap (f^i \partial D)$ for all i, and also all components of $\partial Y \cap D$, $\partial D \setminus \partial Y$. From this, we deduce that $(f \vdash_c g)^k$ fixes a component of $D \setminus Y$ which does not contain the critical point of $f \vdash_c g$ in D. This is impossible. So we are done.

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