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On inhomogeneous minima of indefinite binary quadratic forms

by

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1. Introduction

Let R_m denote the set of points of the ξ, η-plane defined by $-1 \leqslant \xi\eta \leqslant m$. An inhomogeneous lattice $\mathscr{L}$ is a set of points

$$\xi = \alpha x + \beta y, \quad \eta = \gamma x + \delta y \tag{1.1}$$

where (x, y) run through all numbers congruent (modulo 1) to any given numbers (x_0, y_0) respectively. $\Delta = \Delta(\mathscr{L}) = |\alpha\delta - \beta\gamma|$ is the determinant of $\mathscr{L}$. $\mathscr{L}$ is called admissible for R_m if it has no point in the interior of R_m. The critical determinant D_m of R_m is defined to be lower bound of $\Delta(\mathscr{L})$ over all admissible lattices $\mathscr{L}$. Barnes and Swinnerton-Dyer [1] have obtained the exact value of D_m for $21/11 \leqslant m \leqslant 2.1251\ldots$. For $m \geqslant 3$, Blaney [2] has proved that

$$D_m \geqslant \sqrt{(m+1)(m+9)} \tag{1.2}$$

and equality holds for infinitely many values of m.

In this paper we shall obtain exact values of D_m for $3 \leqslant m \leqslant 3.9437\ldots$ (Theorem 1). These results are better than those obtained by Blaney [2] and Dumir and Grover [3]. In Theorem 2, we shall obtain some lower bounds of D_m for $m \geqslant 4$ which are better than (1.2) above. For $m \in [3, 22/7]$, we find the first isolation i.e. if $\mathscr{L}$ is not equivalent to a special lattice $\mathscr{L}_0$, then $D_m \geqslant 4(9+7\sqrt{3})\,m/33$ (Theorem 3) and also observe that the second isolation is not possible for $3 \leqslant m \leqslant 22/7$. These results will be used by one of the authors in finding the successive minima of non homogeneous quadratic forms.

To obtain these results, we use the general theory of two dimensional inhomogeneous lattices developed in Barnes and Swinnerton-Dyer [1] (henceforth this paper will be referred to as BSD). Theorem 6 of BSD is the main weapon in their method which says that all critical lattices of R_m are of the form

$$\xi=\alpha\left(x-\frac{1}{2}\right)+\beta\left(y-\frac{1}{2}\right),\quad \eta=\gamma\left(x-\frac{1}{2}\right)+\delta\left(y-\frac{1}{2}\right).$$

Unfortunately, there is an arithmetical mistake in their paper namely in the inequality (3.11); as a consequence of this, (3.3) and (3.4) are no longer true and this, for $m\geq 3$, leads to a gap in the proof of that theorem. In section 3, we fill up this gap.

§ 2. The general method

Let $\mathscr{L}$ be an inhomogeneous lattice of determinant $\Delta(\mathscr{L})$ with no points on the co-ordinate axes. By Delauney's lemma, it has a divided cell. (A parallelogram with vertices as points of $\mathscr{L}$, area $\Delta(\mathscr{L})$ and one vertex in each of the four quadrants is a divided cell.) For such lattices an algorithm is developed in BSD for finding a new divided cell from a given one and thus obtaining, in general, a chain of divided cells $A_nB_nC_nD_n$ and integral pairs (h_n, k_n) for $-\infty<n<\infty$. (The condition that the chain does not break off is simply that $\mathscr{L}$ has no lattice vector parallel to a co-ordinate axis.) Let the points A_n, B_n, C_n, D_n be either in first, fourth, third and second quadrants respectively or in third, second, first and fourth quadrants. The non-zero integers (h_n, k_n) are defined in the following way (see Figure 1).

If lines A_nD_n and B_nC_n are parallel to ξ-axis then $h_n=k_n=-\infty$ and then $A_{n+1}B_{n+1}C_{n+1}D_{n+1}$ is not defined.

Otherwise h_n is the unique non-zero integer for which $A_{n+1}B_{n+1}$ is the unique lattice

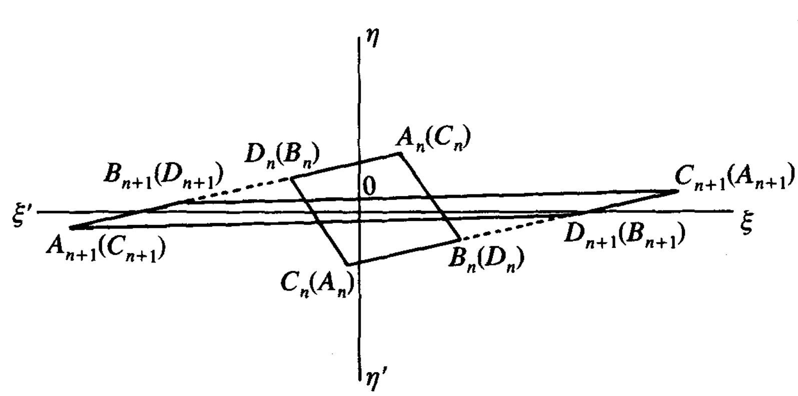

Fig. 1

step of the segment A_nD_n which intersects ξ-axis, where

$$A_{n+1}=A_n+(h_n+1)(D_n-A_n),\quad B_{n+1}=A_n+h_n(D_n-A_n).$$

Similarly C_{n+1} and D_{n+1} are defined on the line segment B_nC_n giving rise to a unique integer k_n. (In Figure 1, we have $h_n=k_n=2$.) Integers h_{n-1}, k_{n-1} are defined in the same way by considering lines A_nB_n, C_nD_n and their intersection with η-axis. Moreover h_n and k_n have the same sign.

Set $a_{n+1}=h_n+k_n$ for all n, so that a_{n+1} is integral and $|a_{n+1}|\geqslant 2$. If $h_n=k_n>0$ for each n, the lattice $\mathscr{L}$ is called a symmetrical lattice. For a symmetrical lattice, it follows from Lemma 1 of BSD that $a_n\geqslant 4$ for arbitrarily large value of $|n|$, for n of each sign.

Let $[b_1, b_2, b_3, \ldots]$ denote the continued fraction

$$b_1-\frac{1}{b_2}-\frac{1}{b_3}-\cdots$$

where the b_i's are integral and $|b_i|\geqslant 2$.

Lemma 1. *If $b_i>0$ for all i and $b_i\geqslant 4$ for some arbitrary large i, then*

$$[b_1, b_2, \ldots, b_n, b_{n+1}, \ldots]<[b_1, b_2, \ldots, b_n, b'_{n+1}, \ldots] \tag{2.1}$$

provided that $b_{n+1}<b'_{n+1}$. In particular

$$[b_1, b_2, \ldots, b_n-1]<[b_1, b_2, \ldots, b_n, \ldots]<[b_1, b_2, \ldots, b_n]. \tag{2.2}$$

This follows from Lemma 5 and its corollary of BSD.

Lemma 2. *Let $\{a_n\}_{-\infty}^{\infty}$ be a sequence associated to a symmetrical lattice $\mathscr{L}$. Let*

$$\theta_n=[a_n, a_{n-1}, a_{n-2}, \ldots],\quad \phi_n=[a_{n+1}, a_{n+2}, \ldots]$$

so that $\theta_n>1$; $\phi_n>1$ by Lemma 1 above. Then the lattice $\mathscr{L}$ is given by the set of points

$$\xi=\alpha_n\left(x-\frac{1}{2}\right)+\beta_n\left(y-\frac{1}{2}\right),\quad \eta=\gamma_n\left(x-\frac{1}{2}\right)+\delta_n\left(y-\frac{1}{2}\right) \tag{2.3}$$

where

$$\delta_n/\gamma_n=\phi_n \quad \text{and} \quad \alpha_n/\beta_n=\theta_n. \tag{2.4}$$

The quadratic form associated with $\mathscr{L}$ is given by

$$\frac{\Delta}{\theta_n\phi_n-1}|(\theta_n x+y)(x+\phi_n y)|, \quad (x,y)\equiv\left(\frac{1}{2},\frac{1}{2}\right) \pmod 1. \tag{2.5}$$

This follows from Theorem 2 of BSD.

LEMMA 3. *A symmetrical lattice $\mathscr{L}$ is admissible for R_m if and only if the inequalities*

$$\frac{\Delta}{m}\geqslant\frac{4(\theta_n\phi_n-1)}{(\theta_n+1)(\phi_n+1)}=\Delta_n^{+} \tag{2.6}$$

and

$$\Delta\geqslant\frac{4(\theta_n\phi_n-1)}{(\theta_n-1)(\phi_n-1)}=\Delta_n^{-} \tag{2.7}$$

hold for all n.

This is Theorem 4 of BSD.

LEMMA 4. *Let $\mathscr{L}_a, \mathscr{L}_{a,b}$ denote the symmetrical lattices corresponding to the sequences $(\overset{\times}{2a})$ and $(\overset{\times}{2a},\overset{\times}{2b})$ respectively (where $\times$ denotes infinite repitition). If these lattices are admissible for R_m and are of smallest determinant then*

$$\Delta(\mathscr{L}_a)=\max\left\{4m\sqrt{\frac{a-1}{a+1}},\,4\sqrt{\frac{a+1}{a-1}}\right\} \tag{2.8}$$

$$\Delta(\mathscr{L}_{a,b})=\max\left\{\frac{8m\sqrt{ab(ab-1)}}{2ab+a+b},\,\frac{8\sqrt{ab(ab-1)}}{2ab-a-b}\right\}. \tag{2.9}$$

These are (4.12) and (4.13) of BSD.

LEMMA 5. *If $0<D<2(k+1)$ and for any n*

$$\Delta_n^{+}\leqslant\frac{D}{k},\quad \Delta_n^{-}\leqslant D,$$

then

$$\frac{D(\theta_n-1)-4}{D(\theta_n-1)-4\theta_n}\leqslant\phi_n\leqslant\frac{4+(D/k)(\theta_n+1)}{4\theta_n-(D/k)(\theta_n+1)} \tag{2.10}$$

and

$$\left|\theta_n - \frac{2(k-1)}{2(k+1)-D}\right| \leqslant \frac{\sqrt{D^2-16k}}{2(k+1)-D}. \tag{2.11}$$

These inequalities also hold if θ_n and ϕ_n are interchanged.

This is Lemma 7 of BSD.

§ 3. Complete proof of Theorem 6 of Barnes and Swinnerton-Dyer

This theorem states:

Let $\mathscr{L}$ be a nonsymmetrical lattice of det $\Delta(\mathscr{L})$ *which is admissible for R_m $(m>1)$. Then there exist a symmetrical lattice $\mathscr{L}'$ which is R_m-admissible and*

$$\Delta(\mathscr{L}') < \Delta(\mathscr{L}). \tag{3.1}$$

To prove this, they used inequalities (3.1) to (3.4) of BSD. But (3.3) and (3.4) are no longer true because of a mistake in the inequality (3.11). We modify these inequalities and give a complete proof. We need the following lemmas:

LEMMA 6. *Let a lattice $\mathscr{L}$ of determinant Δ be admissible for the region R_m. If for some n, $\mathscr{L}$ contains a divided cell $A_nB_nC_nD_n$ with $h_n \neq k_n$ and* $\min(h_n, k_n)=h>0$, *then we have*

$$\frac{\Delta}{m+1} \geqslant \sqrt{1-\frac{2}{h}\left(\frac{m-1}{m+1}\right)+\frac{1}{h^2}} + \sqrt{1+\frac{4}{h(m+1)}} = f(m,h) \quad \text{(say)} \tag{3.2}$$

This is the correct form of the inequality (3.11) of BSD and does not yield (3.3) and (3.4).

LEMMA 7. (a) *For fixed $h \geqslant 1$, the function $f(m, h)$ is a decreasing function of m.*
(b) *$f(3, h)$ is a decreasing function of h for $h \geqslant 2$,*
(c) *$f(3.5, h)$ is an increasing function of h for $h \geqslant 6$,*
(d) *$f(4, h)$ is an increasing function of h for $h \geqslant 4$,*
(e) *For fixed $m>5$, $f(m, h)$ is an increasing function of h for $h \geqslant 3$.*

Proof. (a) is trivial.
Let now $h \geqslant 2$ and m be a fixed number $\geqslant 3$. We have

$$2\frac{\partial}{\partial h}f(m,h)=\left(1-\frac{2}{h}\left(\frac{m-1}{m+1}\right)+\frac{1}{h^2}\right)^{-1/2}\left(\frac{2}{h^2}\left(\frac{m-1}{m+1}\right)-\frac{2}{h^3}\right)$$
$$-\frac{4}{(m+1)h^2}\left(1+\frac{4}{(m+1)h}\right)^{-1/2} \quad \ldots(3.3)$$

Since here $h(m-1)\geqslant m+1$, on simplifying (3.3) we see that $\partial f(m,h)/\partial h$ will be positive if and only if

$$(m+1)(m-3)h^3-2(m-1)^2h^2+(m-1)(m-5)h+4(m+1)>0. \quad \ldots(3.4)$$

For a fixed m, the l.h.s. of (3.4) becomes a polynomial in h. A simple calculation shows that (b), (c), (d) and (e) are satisfied.

LEMMA 8. *Let $\mathscr{L}$ be an R_m-admissible lattice.*
(a) *If any pair (h_n, k_n) is negative or infinite then*

$$\Delta\geqslant 2(m+1). \quad \ldots(3.5)$$

(b) *If for any n, $h_n>0$, $k_n>0$, $h_n\neq k_n$ then*
(i) *for $h_n=1$ or $k_n=1$*

$$\Delta\geqslant\sqrt{m+1}\,(2+\sqrt{m+5}). \quad \ldots(3.6)$$

(ii) *For $h_n\geqslant 2$, $k_n\geqslant 2$ we have*

$$\Delta\geqslant C_m(m+1) \quad \ldots(3.7)$$

where C_m is a constant given by

$$C_m=\begin{cases}2 & \text{for} \quad m\leqslant 3\\ 1.9894 & \text{for} \quad 3<m\leqslant 3.5\\ 1.9686 & \text{for} \quad 3.5<m\leqslant 4\\ 1.9184 & \text{for} \quad 4<m\leqslant 5\\ 1.8251 & \text{for} \quad 5<m\leqslant 7\\ 1.6181 & \text{for} \quad 7<m\leqslant 23\\ 1.5 & \text{for} \quad m>23.\end{cases} \quad (3.8)$$

Proof. (3.5) and (3.6) are inequalities (3.1) and (3.2) respectively of BSD.
For $m\leqslant 3$, $h\geqslant 2$ by Lemmas 6 and 7(a), (b) we have

$$\frac{\Delta}{m+1}\geqslant f(m,h)\geqslant f(3,h)=\sqrt{1-\frac{1}{h}+\frac{1}{h^2}}+\sqrt{1+\frac{1}{h}}\geqslant 2.$$

For $3<m\leqslant 3.5$ and $h\geqslant 2$, by Lemmas 6 and 7(a), (c), we have

$$\frac{\Delta}{m+1}\geqslant f(m,h)\geqslant f(3.5,h)$$
$$\geqslant \min(f(3.5,6),f(3.5,5),f(3.5,4),f(3.5,3),f(3.5,2))$$
$$=1.9894\ldots.$$

The proof is similar for $3.5<m\leqslant 23$.

When $m>23$, letting $m\to\infty$ we get

$$\frac{\Delta}{m+1}\geqslant f(m,h)\geqslant 1-\frac{1}{h}+1\geqslant\frac{3}{2}\quad\text{for}\quad h\geqslant 2.$$

This completes the proof.

We now proceed to prove the Theorem 6 of BSD.

From the estimates (3.5), (3.6) and (3.7) it suffices to give a symmetrical lattice $\mathscr{L}'$ which is admissible for R_m and for which

$$\Delta(\mathscr{L}')<\min\big(C_m(m+1),\ \sqrt{m+1}\,(2+\sqrt{m+5})\big). \tag{3.9}$$

We observe that

$$C_m(m+1)<\sqrt{m+1}\,(2+\sqrt{m+5})\quad\text{for}\quad m\leqslant 8 \tag{3.10}$$

and

$$\frac{3}{2}(m+1)>\sqrt{m+1}\,(2+\sqrt{m+5})\quad\text{for}\quad m>23. \tag{3.11}$$

The r.h.s. of (3.9) is $2(m+1)$ for $1<m\leqslant 3$, and is $\sqrt{m+1}\,(2+\sqrt{m+5})$ for $m>23$. For these values of m the proof of BSD goes through.

For $3\leqslant m\leqslant 8$, take

$$\mathscr{L}'=\begin{cases}\mathscr{L}_2 & \text{for}\quad 3<m\leqslant 4\\ \mathscr{L}_{1,5} & \text{for}\quad 4<m\leqslant 5\\ \mathscr{L}_{1,3} & \text{for}\quad 5<m\leqslant 7\\ \mathscr{L}_{1,2} & \text{for}\quad 7<m\leqslant 8.\end{cases}$$

By Lemma 4, $\mathscr{L}'$ is admissible for R_m if

$$\Delta(\mathscr{L}') = \begin{cases} 4m/\sqrt{3} & \text{for } 3 \leqslant m \leqslant 4 \\ m\sqrt{5} & \text{for } 4 < m \leqslant 5 \\ 4m\sqrt{6}/5 & \text{for } 5 < m \leqslant 7 \\ 8m\sqrt{2}/7 & \text{for } 7 < m \leqslant 8. \end{cases}$$

One can easily check that $\Delta(\mathscr{L}')<C_m(m+1)$ for $3<m\leqslant 8$. For $8<m\leqslant 23$, if the minimum on the r.h.s. of (3.9) is $C_m(m+1)$, take $\mathscr{L}'=\mathscr{L}_{1,2}$; otherwise proceed as in BSD to get the required result.

§ 4. Evaluation of D_m

Here we prove:

THEOREM 1. *We have*

$$D_m = 4m/\sqrt{3} \quad \textit{for} \quad 3 \leqslant m \leqslant 6\sqrt{10}/5 \tag{4.1}$$

$$D_m = 8\sqrt{30}/5 \quad \textit{for} \quad 6\sqrt{10}/5 < m \leqslant 19/5 \tag{4.2}$$

$$D_m = 8m\sqrt{30}/19 \quad \textit{for} \quad 19/5 \leqslant m \leqslant m_0 = \frac{19(24+5\sqrt{30})}{30(3+\sqrt{30})} \tag{4.3}$$

$$D_m = \frac{8(24+5\sqrt{30})}{30+3\sqrt{30}} \quad \textit{for} \quad m_0 \leqslant m \leqslant \frac{90+17\sqrt{30}}{30+3\sqrt{30}} = 3.9437\ldots. \tag{4.4}$$

All critical lattices are given by the symmetrical lattices corresponding to the sequences

$$(\overset{\times}{4}) \quad \textit{for} \quad 3 \leqslant m \leqslant 6\sqrt{10}/5 \tag{4.5}$$

$$(\overset{\times}{2}, \overset{\times}{12}) \quad \textit{for} \quad 6\sqrt{10}/5 \leqslant m \leqslant m_0 \tag{4.6}$$

$$({}_\infty(2,\ 12),\ 2,\ 10,\ (2,\ 12)_\infty) \quad \textit{for} \quad m_0 \leqslant m \leqslant 3.9437\ldots. \tag{4.7}$$

Proof. By Theorem 6 of BSD and Lemma 3, we need to show only that the sequences in (4.5)–(4.7) are the only sequences of positive even integers which satisfy

$$\max(\Delta_n^{-}, m\Delta_n^{+}) \leqslant D_m \quad \text{for all } n, \tag{4.8}$$

where D_m is given by (4.1)–(4.4) and that, for some n, equality holds in (4.8) for each of the given sequences in the stated range of values of m. Consider the sequence $\{a_n\}$ satisfying

$$\Delta_n^- \leqslant 8\sqrt{30}/5 \quad \text{and} \quad m\Delta_n^+ \leqslant 4m/\sqrt{3} \quad \text{for each } n. \tag{4.9}$$

The hypotheses of Lemma 5 are satisfied for each n with $D=8\sqrt{30}/5=8.7635\ldots$, $k=6\sqrt{10}/5=3.7947\ldots$ and $D/k=4/\sqrt{3}=2.3094\ldots$,. Working with sufficient accuracy to four places of decimals we have from (2.11)

$$|\theta_n-6.7676\ldots|<4.8559 \quad \text{for any } n.$$

Hence

$$1.9<\theta_n<11.6235.$$

By Lemma 1, $a_n-1<\theta_n<a_n$; and since a_n is even it follows that a_n can take only the values $2, 4, 6, \ldots, 12$.

Case (i). Suppose $a_n \geqslant 4$ for all n.

If $a_r \geqslant 4$ and $a_{r+1} \geqslant 6$, then $\theta_r>3$ and $\phi_r>5$. Also Δ_n^+ is an increasing function of θ_n, ϕ_n therefore

$$\Delta_r^+ > \frac{4(15-1)}{6\times 4} = \frac{7}{3} > \frac{D}{k}; \quad \text{a contradiction.}$$

By symmetry $a_{r-1} \geqslant 6$, $a_r \geqslant 4$ is also not possible; so that the only possibility is $(\overset{\times}{4})$.

Case (ii). Let $a_r=2$ for some r.

Then $\theta_r<[2, 12]=23/12$; and from (2.10) we get

$$\phi_r \geqslant \frac{D(\theta_r-1)-4}{D(\theta_r-1)-4\theta_r} > 11.$$

Therefore $a_{r+1}=12$ as $a_{r+1} \geqslant \phi_r$. By symmetry $a_{r-1}=12$. Further if $a_r=12$ for some r then by case (i) we must have $a_{r-1}=a_{r+1}=2$.

Thus the only sequences satisfying (4.9) are $(\overset{\times}{4})$ and $(\overset{\times}{2}, \overset{\times}{12})$. For the sequence $(\overset{\times}{4})$, $\Delta_n^+=4/\sqrt{3}$ and $\Delta_n^-=4\sqrt{3}<8\sqrt{30}/5$; and for the sequence $(\overset{\times}{2}, \overset{\times}{12})$, $\Delta_n^-=8\sqrt{30}/5$, $\Delta_n^+=8\sqrt{30}/19<4/\sqrt{3}$.

This proves (4.1) and (4.2) and establishes the assertion on critical lattices for the range $3 \leqslant m \leqslant 19/5$. The proof of (4.3) and (4.4) is similar and is left to the reader.

THEOREM 2. *We have*

(a) $D_m \geqslant 8\sqrt{30}/5$ for $m \geqslant 6\sqrt{10}/5$ (4.10)

(b) $D_m \geqslant 4\sqrt{5}$ for $m \geqslant 4$ (4.11)

(c) $D_m \geqslant 3\sqrt{10}$ for $m \geqslant 5$. (4.12)

Equality occurs in (4.11) at $m=4$ and the critical sequence is $(\overset{\times}{2}, \overset{\times}{10})$.

Proof. (a) follows from (4.2), noting that D_m is an increasing function of m.

(b) If $\mathscr{L}$ is a non-symmetrical lattice then by Lemma 8, (3.5)–(3.8),

$$\Delta(\mathscr{L}) \geqslant \min\left(C_m(m+1), \sqrt{m+1}\,(2+\sqrt{m+5})\right) > 4\sqrt{5}.$$

Let $\mathscr{L}$ be a symmetrical lattice. By Lemma 3, it suffices to prove that

$$\max(m\Delta_n^+, \Delta_n^-) \geqslant 4\sqrt{5}. \tag{4.13}$$

Let $\{a_n\}$ be a sequence satisfying

$$\Delta_n^- \leqslant 4\sqrt{5} \quad \text{and} \quad \Delta_n^+ \leqslant 4\sqrt{5}/m \leqslant \sqrt{5} \quad \text{for each } n. \tag{4.14}$$

Working as in Theorem 1, we find that there are almost two sequences namely $(\overset{\times}{4})$ and $(\overset{\times}{2}, \overset{\times}{10})$ which may satisfy (4.14). But for the sequence $(\overset{\times}{4})$, $\Delta_n^+ = 4/\sqrt{3} > \sqrt{5}$, the only possibility left is $(\overset{\times}{2}, \overset{\times}{10})$. For this sequence $\Delta_n^+ = \sqrt{5}$ and $\Delta_n^- = 4\sqrt{5}$. This proves (b).

(c) Let $m \geqslant 5$. If $\mathscr{L}$ is a nonsymmetrial lattice then by Lemma 8, (3.5)–(3.8),

$$\Delta(\mathscr{L}) \geqslant \min\left(C_m(m+1), \sqrt{m+1}(2+\sqrt{m+5})\right) > 3\sqrt{10}.$$

If $\mathscr{L}$ is a symmetrical lattice, one can see easily that there is no sequence $\{a_n\}$ satisfying

$$\Delta_n^- < 3\sqrt{10} \quad \text{and} \quad \Delta_n^+ < 3\sqrt{10}/5.$$

This proves (c).

§ 5. Isolation of D_m

THEOREM 3. *Let* $3 \leqslant m \leqslant 3.4$. *If* $\mathscr{L}$ *is an* R_m *admissible lattice then either*

$$\Delta(\mathscr{L}) \geqslant \begin{cases} \dfrac{4(9+7\sqrt{3})\,m}{33} & \textit{for} \quad 3 \leqslant m \leqslant 22/7 \\[2ex] \dfrac{8(9+7\sqrt{3})}{21} & \textit{for} \quad 22/7 \leqslant m \leqslant 3.4, \end{cases} \tag{5.1}$$

or $\mathscr{L}$ is a symmetrical lattice corresponding to the sequence $(\overset{\times}{4})$ *Further equality holds in* (5.1) *if and only if* $3 \leqslant m \leqslant 22/7$ *and $\mathscr{L}$ is a symmetrical lattice corresponding to the sequence* $({}_\infty 4, 6, 4_\infty)$.

Proof. If $\mathscr{L}$ is a non-symmetrical lattice and is admissible for R_m, then by Lemma 8,

$$\Delta(\mathscr{L}) \geqslant \min\left(C_m(m+1), \sqrt{m+1}\,(2+\sqrt{m+5})\right)$$
$$= (1.9894\ldots)(m+1) > 4(9+7\sqrt{3})\,m/33.$$

Therefore any lattice which is admissible for R_m and for which (5.1) is false, must be a symmetrical lattice.

Case (i). $3 \leqslant m \leqslant 22/7$.

By Lemma 3, it is enough to prove that the only sequences $\{a_n\}$ satisfying

$$\max(m\Delta_n^+, \Delta_n^-) \leqslant 4(9+7\sqrt{3})\,m/33 \quad \text{for all } n \tag{5.2}$$

are $(\overset{\times}{4})$ and $({}_\infty 4, 6, 4_\infty)$; and that equality holds in (5.2) for the second one.

Let

$$\Delta_n^+ \leqslant 4(9+7\sqrt{3})/33 = D/k = 2.5605\ldots \tag{5.3}$$

$$\Delta_n^- \leqslant 4(9+7\sqrt{3})\,m/33 \leqslant 8(9+7\sqrt{3})/21 = D = 8.0473\ldots. \tag{5.4}$$

The hypotheses of Lemma 5 are satisfied with $k=22/7$ and we get as before that only choices of a_n's can be $4, 6, 8, \ldots, 34$. If $a_n \geqslant 8$ for some n, then, by Lemma 1, $\theta_n \geqslant [8, \overset{\times}{4}] = 6+\sqrt{3}$; $\phi_n \geqslant [\overset{\times}{4}] = 2+\sqrt{3}$; so that $\Delta_n^+ > D/k$; a contradiction to (5.3). Hence $a_n \leqslant 6$ for all n.

If $a_n=6$ and also $a_{n+1}=6$, then $\theta_n > a_{n-1}=5$, $\phi_n > 5$ which gives $\Delta_n^+ > D/k$; again a contradiction. Hence $a_{n+1}=4$. By symmetry $a_{n-1}=4$.

If none of the a_n's is 6, we have the sequence $(\overset{\times}{4})$ and it is easy to verify that strict inequality holds in (5.2) for this sequence. If $a_n=6$ for some n, then by Lemma 1,

$$\theta_n = [a_n, a_{n-1}, \ldots] \geqslant [6, 4, 4, \ldots] = 4+\sqrt{3}$$

$$\phi_n = [a_{n+1}, \ldots] \geqslant [4, 4, \ldots] = 2+\sqrt{3}$$

so that $\Delta_n^+ \geqslant D/k$. Further equality holds if and only if $\{a_n\} = ({}_\infty 4, 6, 4_\infty)$. It is easy to verify that for this sequence $\max(m\Delta_n^+, \Delta_n^-) = m\Delta_n^+$ for all n. This proves case (i).

Case (ii). $22/7 < m \leqslant 3.4$.

It is enough to prove that $(\overset{\times}{4})$ is the only sequence satisfying

$$\max(m\Delta_n^+, \Delta_n^-) \leqslant 8(9+7\sqrt{3})/21 \quad \text{for all } n. \tag{5.5}$$

Let

$$\Delta_n^+ \leqslant 8(9+7\sqrt{3})/21m \leqslant 4(9+7\sqrt{3})/33 = D/k$$

$$\Delta_n^- \leqslant 8(9+7\sqrt{3})/21 = D \quad \text{and} \quad k = 22/7.$$

Then working as in case (i) we get $\{a_n\} = (\overset{\times}{4})$.

Remark. For $3 \leqslant m \leqslant 22/7$, the second isolation of D_m is not possible. We can approximate the sequence $({}_\infty 4, 6, 4_\infty)$ by the sequences of the type

$$(\overset{\times}{6}, \underbrace{4, 4, \ldots, \overset{\times}{4}}_{l})$$

where l tends to ∞. For such sequences the values of $\max_n \Delta_n^-$ and $\max_n \Delta_n^+$ are arbitrary close to those for the sequence $({}_\infty 4, 6, 4_\infty)$. Hence these sequences yield admissible symmetrical lattices of determinant arbitrary close to $D_m^{(2)}$, where $D_m^{(2)} = 4(9+7\sqrt{3})\,m/33$ is the second minimum.

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