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Regularity of a boundary having a Schwarz function

by

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In his book [5], Davis discussed various interesting aspects concerning a Schwarz function. It is a holomorphic function S which is defined in a neighborhood of a real analytic arc and satisfies $S(\zeta)=\bar{\zeta}$ on the arc, where $\bar{\zeta}$ denotes the complex conjugate of ζ.

In this paper, we shall define a Schwarz function for a portion of the boundary of an arbitrary open set and show regularity of the portion of the boundary. More precisely, let Ω be an open subset of the unit disk B such that the boundary $\partial\Omega$ contains the origin 0 and let $\Gamma=(\partial\Omega)\cap B$. We call a function S defined on $\Omega\cup\Gamma$ a Schwarz function of $\Omega\cup\Gamma$ if

(i) S is holomorphic in Ω,

(ii) S is continuous on $\Omega\cup\Gamma$,

(iii) $S(\zeta)=\bar{\zeta}$ on Γ.

We shall give a classification of a boundary having a Schwarz function. The main theorem, Theorem 5.2, asserts that there are four types of the boundary if 0 is not an isolated boundary point of Ω: 0 is a regular, nonisolated degenerate, double or cusp point of the boundary. Namely, one of the following must occur for a small disk B_δ with radius $\delta>0$ and center 0:

(1) $\Omega\cap B_\delta$ is simply connected and $\Gamma\cap B_\delta$ is a regular real analytic simple arc passing through 0.

(2a) $\Gamma\cap B_\delta$ determines uniquely a regular real analytic simple arc passing through 0 and $\Gamma\cap B_\delta$ is an infinite proper subset of the arc accumulating at 0 or the whole arc. $\Omega\cap B_\delta$ is equal to $B_\delta\setminus\Gamma$.

(2b) $\Omega\cap B_\delta$ consists of two simply connected components Ω_1 and Ω_2. $(\partial\Omega_1)\cap B_\delta$ and $(\partial\Omega_2)\cap B_\delta$ are distinct regular real analytic simple arcs passing through 0. They are tangent to each other at 0.

(2c) $\Omega \cap B_\delta$ is simply connected and $\Gamma \cap B_\delta$ is a regular real analytic simple arc except for a cusp at 0. The cusp is pointing into $\Omega \cap B_\delta$. It is a very special one. There is a holomorphic function T defined on a closed disk $\overline{B_\varepsilon}$ such that

(i) T has a zero of order two at 0,

(ii) T is univalent on the closure $\bar{H}$ of a half disk $H=\{\tau \in B_\varepsilon; \operatorname{Im} \tau>0\}$,

(iii) T satisfies $\Gamma \cap B_\delta \subset T((-\varepsilon, \varepsilon))$ and $T(\bar{H}) \subset \Omega \cup \Gamma$, where

$$(-\varepsilon, \varepsilon)=\{\tau; -\varepsilon<\tau=\operatorname{Re} \tau<\varepsilon\}.$$

There are at least two applications of the main theorem. We first consider an application to quadrature domains. Let μ be a complex measure on the complex plane $\mathbf{C}$. A nonempty open set Ω in $\mathbf{C}$ is called a quadrature domain of μ if $|\mu|(\mathbf{C} \setminus \Omega)=0$ and if $\int|f|\, d|\mu|<\infty$ and

$$\int f d\mu=\int\int_{\Omega} f(z)\, dx\, dy \quad (z=x+iy)$$

for every holomorphic and integrable function f in Ω. If Ω is bounded, then $1/(z-\zeta)$ is holomorphic and integrable on Ω for every fixed $\zeta \in \mathbf{C} \setminus \Omega$. Hence the Cauchy transform $\hat{\Omega}(\zeta)=\int_\Omega 1/(z-\zeta)\, dx\, dy$ of Ω is equal to the Cauchy transform $\hat{\mu}(\zeta)=\int 1/(z-\zeta)\, d\mu(z)$ of μ on $\mathbf{C} \setminus \Omega$. Since $\hat{\Omega}(z)+\pi \bar{z}$ is holomorphic in Ω, we see that

$$S(z)=(\hat{\Omega}(z)+\pi \bar{z}-\hat{\mu}(z))/\pi$$

is the Schwarz function of $(\Omega \cap B) \cup((\partial \Omega) \cap B)$ if B and the support of μ are disjoint and if $0 \in \partial \Omega$. Applying our main theorem, we obtain a regularity theorem on the boundary of a quadrature domain. Let Ω be a bounded quadrature domain of μ such that the support of μ is contained in Ω. If we make a new domain $[\Omega]$ by adding all degenerate boundary points of Ω to Ω, then $[\Omega]$ is also a quadrature domain of μ and the boundary of $[\Omega]$ consists of a finite number of real analytic simple curves having at most a finite number of double and cusp points. Moreover, by applying our methods to the Schwarz function of the boundary of an unbounded domain, we are able to carry out the program proposed by Shapiro [10] and see that if an unbounded quadrature domain is not dense in $\mathbf{C}$, then it is obtained as a translation of an inversion of a bounded quadrature domain.

We next consider an application to free boundary problems. A typical problem is an obstacle problem and it is reduced to a problem to determine $\Gamma(u)$ of the following function u: u is a nonnegative function defined in the unit disk B such that

(i) $\Gamma(u)=(\partial\Omega(u))\cap B$ contains the origin 0, where $\Omega(u)=\{z\in B; u(z)>0\}$,

(ii) u is of class C^1 in B,

(iii) $\Delta u(z)=1$ in $\Omega(u)$ in the sense of distributions.

What can we say about regularity of $\Gamma(u)$?

We set

$$S(z)=\bar{z}-4\frac{\partial u}{\partial z}(z).$$

Then S is holomorphic in $\Omega(u)$, because S is continuous in $\Omega(u)$ and

$$\frac{\partial S}{\partial \bar{z}}(z)=1-4\frac{\partial^2 u}{\partial \bar{z}\,\partial z}(z)=1-\Delta u(z)=0$$

in $\Omega(u)$ in the sense of distributions. By (ii), S is also continuous on $\Omega(u)\cup\Gamma(u)$. Since u is nonnegative in B and $u(z)=0$ on $\Gamma(u)$,

$$\frac{\partial u}{\partial x}(z)=\frac{\partial u}{\partial y}(z)=0$$

on $\Gamma(u)$. Thus S is the Schwarz function of $\Omega(u)\cup\Gamma(u)$. We apply our main theorem and see that $\Omega=\Omega(u)$ and $\Gamma=\Gamma(u)$ satisfies one of (1) to (2c) for some B_δ if 0 is not an isolated point of $\Gamma(u)$. In this case, $\Gamma(u)\cap B_\delta$ is the whole arc for some δ if 0 is a nonisolated degenerate point and the cusp is pointed more sharply if 0 is a cusp point. Furthermore, we see that all the first derivatives of u are Lipschitz continuous on $\overline{B_\delta}$ and all the second derivatives of u are continuous up to $\Gamma(u)$, on $\Omega(u)$.

This is a very informative regularity theorem. An accurate description of the free boundary $\Gamma(u)$ in two dimensions was given by Caffarelli and Rivière [2] and [3]. They proved that (1) if 0 is not a regular point, then $B_\delta\setminus\Omega(u)$ is arranged along a straight line, more precisely, there is an increasing function θ defined on a half-open interval $[0,\delta)$ such that $\theta(0)=0$ and

$$B_\delta\setminus\Omega(u)\subset\{\zeta\in B_\delta;\ |\arg\zeta-\alpha|\leq\theta(|\zeta|)\ \text{or}\ |\arg\zeta-(\alpha+\pi)|\leq\theta(|\zeta|)\},$$

where α denotes a real number, (2) if 0 is a nonisolated degenerate point, then $\Gamma(u)\cap B_\delta$ is a real analytic simple arc and (3) the boundary of each connected component of the interior of $B_\delta\setminus\Omega(u)$ is the union of a finite number of real analytic simple arcs. Their results are fairly accurate, but there is still a possibility that an infinite number of connected components of the interior of $B\setminus\Omega(u)$ exist and cluster around 0. Our main theorem excludes the possibility. This is also true even if we replace the constant function with value 1 in (iii) by a positive real analytic function φ defined in B. The fact is quite interesting when we compare it with an example of the free boundary for the obstacle problem with C^∞-obstacle due to Schaeffer [9]: If we replace the constant function with value 1 in (iii) by some special positive C^∞-function φ defined in B, then there is a nonnegative function u satisfying (i) to (iii) such that an infinite number of connected components of the interior of $B\setminus\Omega(u)$ actually cluster around 0.

The purpose of this paper is to give a proof of the main theorem. The applications of the theorem stated above and related results will appear elsewhere.

Here we shall give a brief outline of the proof. Assume that 0 is not an isolated boundary point of Ω, let S be the Schwarz function of $\Omega\cup\Gamma$ and set $F(z)=zS(z)$. Then F is holomorphic in Ω, is continuous on $\Omega\cup\Gamma$ and satisfies $F(\zeta)=|\zeta|^2\geqslant 0$ on Γ. Take an appropriate neighborhood of 0 and restrict F, we denote it again by F, to the intersection V of Ω and the neighborhood. We shall show that there are two cases: (1) F is a one-to-one conformal mapping of V onto $B_\varepsilon\setminus E$, where $\varepsilon>0$ and E denotes a relatively closed subset of a half-open interval $[0,\varepsilon)$ or (2) for appropriately chosen branches, $\sqrt{F}$ is a one-to-one conformal mapping of V onto $B_{\sqrt{\varepsilon}}\setminus E$, where E denotes a relatively closed subset of an open interval $(-\sqrt{\varepsilon},\sqrt{\varepsilon})$.

The essential part of the proof is to show that the number of connected components of E is finite except in the degenerate case (2 a). To do so, we apply the following fundamental fact: The valence function of a holomorphic function is finite and constant in every connected component of the complement of the cluster set of the holomorphic function. In the case of (1), we take the inverse function $z=\Psi_1(w)$ of $w=F(z)$ defined in $B_\varepsilon\setminus E$ and set $Z_1(w)=\Psi_1(w)\overline{\Psi_1}(w)/w$, where $\overline{\Psi_1}(w)=\overline{\Psi_1(\bar{w})}$. We shall show, by taking appropriate ε, that the cluster set of Z_1 is contained in the union of the unit circle and a real analytic arc. We apply the above fact to Z_1 and see that the number of connected components of E is finite. It is not easy to show that the cluster set of Z_1 at $w=0$ is contained in the unit circle. Since

$$\Psi_1(w)^2/w = z^2/(zS(z)) = z/S(z),$$

it is sufficient to show that

$$\lim_{z\in\Omega,\, z\to 0} |S(z)/z| = 1. \tag{*}$$

In the case of (2), we take the inverse function Ψ_2 of $\sqrt{F}$, set $Z_2(w)=\Psi_2(w)\overline{\Psi_2}(w)/w^2$ and apply the above argument.

This paper consists of five sections. The fundamental fact mentioned above and a sufficient condition for a holomorphic function to have the square root which is a one-to-one conformal mapping is given in Section 1. To show (*), we need the Fuchs theorem and it is discussed in Section 2. We show (*) in Section 3. In Section 4, we show that the valence function of F is equal to 1 or 2 in $B_\varepsilon \setminus [0, \varepsilon)$ for appropriately chosen V and ε. The values 1 and 2 correspond to the cases (1) and (2), respectively. The final step of the proof of the main theorem is given in Section 5.

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§ 1. Cluster sets, valence functions and branch points

Cluster sets, valence functions and branch points can be defined for meromorphic functions in arbitrary open subsets of Riemann surfaces. Here, however we discuss them just for holomorphic functions defined in bounded open sets in the complex plane **C** for the sake of simplicity.

Definition 1.1. Let Ω be a bounded open set in **C**. For a holomorphic function f in Ω and a point ζ on the boundary $\partial\Omega$ of Ω, we set

$$C_f(\zeta) = \cap\{\overline{f(\Omega \cap B_\delta(\zeta))}; \delta > 0\},$$

where $\bar{E}$ for a set E in **C** denotes the closure of E in **C** and $B_\delta(\zeta)$ denotes a disk of radius δ and center ζ. We set

$$C_f = \cup\{C_f(\zeta); \zeta \in \partial\Omega\}$$

and call it the cluster set of f.

It is easy to show that $C_f(\zeta)$ and C_f are closed and C_f is not empty if Ω is not empty. For cluster sets, see e.g. Collingwood and Lohwater [4].

Definition 1.2. Let f be a holomorphic function in an open set Ω. We denote by $\nu_f(w)$ the number of solutions z in Ω to $f(z)=w$ and call ν_f the valence function of f. The number of solutions is counted according to their multiplicities and it may be infinite.

In this section we give several lemmas concerning cluster sets, valence functions and branch points. Proofs of them are not difficult and it seems that some proofs of them are known. But, for the sake of completeness, we give here all proofs of them.

LEMMA 1.1. *The valence function ν_f of a holomorphic function f is finite and constant in each connected component of* $\mathbf{C} \setminus C_f$.

Proof. Assume that $f(z_j)=w_0$ for an infinite number of z_j in $\Omega, j=1,2,\dots$. Taking a subsequence if necessary, we may assume that $\{z_j\}$ is a convergent sequence. If the limit is an interior point of Ω, then f is identically equal to a constant w_0 in a connected component D of Ω containing the limit. Hence $w_0 \in C_f(\zeta)$ for $\zeta \in \partial D$, and so $w_0 \in C_f$. If the limit is a boundary point ζ_0 of Ω, then $w_0 \in C_f(\zeta_0) \subset C_f$. Therefore $\nu_f(w)$ is finite in $\mathbf{C}\setminus C_f$.

The proof will be complete if we show that ν_f is continuous in $\mathbf{C}\setminus C_f$. Let $w_0 \in \mathbf{C}\setminus C_f$ and let $\{w_j\}$ be a sequence of points in $\mathbf{C}\setminus C_f$ converging to w_0. If the inverse image $f^{-1}(\{w_j\}_{j=1}^{\infty} \cup \{w_0\})$ of $\{w_j\}_{j=1}^{\infty} \cup \{w_0\}$ is empty, then $\nu_f(w_j)=\nu_f(w_0)=0$ for every j. If it is not empty, then, by the same argument as above, we see that it is relatively compact in Ω. Surrounding it by a finite number of simple closed curves in Ω and applying the Hurwitz theorem, we see that $f(z)-w_j$ and $f(z)-w_0$ have the same number of zeros in the open set surrounded by the simple closed curves for sufficiently large j. Hence $\nu_f(w_j)=\nu_f(w_0)$ for large j and ν_f is continuous at w_0. Q.E.D.

Let $N=\{w \in \mathbf{C}; \nu_f(w)=0\}$. Then, by definition, the image $f(\Omega)$ of Ω is contained in $\mathbf{C}\setminus N$ and the image $f(D)$ of a connected component D of Ω is contained in the exterior $\mathbf{C}\setminus \bar{N}$ of N if f is not constant in D, because f is an open mapping of D if it is not constant in D. Lemma 1.1, together with this fact, has many applications. For example, we get the maximum modulus principle for bounded holomorphic functions from the lemma. We also see that if a bounded holomorphic function defined in a bounded connected open set has real boundary values, then it is constant.

Let W be a connected component of $\mathbf{C}\setminus C_f$ with the nonempty inverse image $f^{-1}(W)$. We regard $(f^{-1}(W), f|f^{-1}(W))$ as a finite unlimited covering surface of W, where $f|V$ for an open subset V of Ω denotes the restriction of f to V.

Definition 1.3. Let W be a connected component of $\mathbf{C}\setminus C_f$ with the nonempty inverse image $f^{-1}(W)$. We call $z \in f^{-1}(W)$ a branch point of f if $f'(z)=0$. The number of branch points of f in an open subset V of $f^{-1}(W)$ is counted according to their degrees of ramification, namely, it is the number of zeros of f' in V counted according to their multiplicities.

LEMMA 1.2. *Let W be a connected component of $\mathbf{C}\setminus C_f$ with the nonempty inverse image $f^{-1}(W)$ and suppose that $\nu_f(w)=\nu$ in W. Let V be a connected component of $f^{-1}(W)$. Then*

(1) $C_{f|V}=\partial W$,

(2) *the valence function $\nu_{f|V}$ is constant in W and satisfies $1 \leq \nu_{f|V} \leq \nu$,*

(3) *the number of branch points of f in V is not greater than* $2(\nu_{f|V}-1)$.

Consequently, the number of connected components of $f^{-1}(W)$ *is not greater than* ν *and the number of branch points in* $f^{-1}(W)$ *is not greater than* $2(\nu-1)$.

Proof. Let $\zeta\in\partial V$. Then $f(V\cap B_\delta(\zeta))\subset W$. Hence $C_{f|V}(\zeta)\subset\bar{W}$ for every $\zeta\in\partial V$, and so $C_{f|V}\subset\bar{W}$. Let $w\in W$. Then, by Lemma 1.1, $f^{-1}(w)$ is a finite set. Hence, for $\zeta\in\partial V$, we can find $\varrho>0$ and $\delta>0$ such that $f^{-1}(B_\varrho(w))\cap B_\delta(\zeta)=\emptyset$. This implies that $B_\varrho(w)\cap\overline{f(V\cap B_\delta(\zeta))}=\emptyset$, and so w is not contained in $C_{f|V}(\zeta)$ for any $\zeta\in\partial V$. Therefore w is not contained in $C_{f|V}$ and it follows that $C_{f|V}\subset\bar{W}\setminus W=\partial W$.

Assume that $C_{f|V}$ is a proper subset of ∂W and let $w_0\in(\partial W)\setminus C_{f|V}$. Since $C_{f|V}$ is closed, we can choose $r>0$ so that $C_{f|V}\cap B_r(w_0)=\emptyset$. Take $w_1\in W\cap B_r(w_0)$. Then w_0 and w_1 are contained in the same connected component of $\mathbf{C}\setminus C_{f|V}$. Hence, by Lemma 1.1, $\nu_{f|V}(w_0)=\nu_{f|V}(w_1)\geq 1$. Thus $w_0=f(z_0)$ for some $z_0\in V$, namely, $w_0\in f(V)\subset W$. This is a contradiction. Hence $C_{f|V}=\partial W$.

The second assertion (2) follows from (1) and Lemma 1.1.

Finally we shall give a brief proof of (3). We regard $(V,f|V)$ as a finite unlimited covering surface of W. Since f has at most a countable number of branch points in V, we can take a regular exhaustion $\{W_j\}$ of W so that there are no branch points on $V\cap f^{-1}(\partial W_j)$ for each j. For each fixed j, we slit W_j along piecewise real analytic arcs and make a simply connected domain Y such that there are no branch points on $V\cap f^{-1}(\partial Y)$. The number of branch points in $V\cap f^{-1}(W_j)$ is equal to the number of branch points in $X=V\cap f^{-1}(Y)$. We shall apply the Riemann–Hurwitz formula to a finite unlimited covering surface $(X,f|X)$ of Y. The Riemann–Hurwitz formula asserts that the number of branch points of $f|X$ in X is equal to $\nu_{f|X}e_Y-e_X$, where e_R for a Riemann surface R denotes the Euler characteristic of R. The Euler characteristic e_R is equal to $2(1-g_R)-b_R$, where g_R denotes the genus of R and b_R denotes the number of boundary components of R. Since $g_X=g_Y=0$ and $b_Y=1$, $e_X=2-b_X$ and $e_Y=1$. Since the number of the sheets of the covering is equal to $\nu_{f|X}$ and $b_Y=1$, $b_X\leq\nu_{f|X}$. Hence $\nu_{f|X}e_Y-e_X=\nu_{f|X}-(2-b_X)\leq 2(\nu_{f|X}-1)$. Thus the number of branch points in $V\cap f^{-1}(W_j)$ is not greater than $2(\nu_{f|X}-1)\leq 2(\nu_{f|V}-1)$ for any j. This completes the proof of (3). Q.E.D.

Next we shall discuss the case that $\nu_f(w)\leq 1$ in $\mathbf{C}\setminus C_f$.

LEMMA 1.3. *If*

(i) *f is not constant in any connected component of* Ω,

(ii) *C_f has no interior points,*

(iii) $\nu_f(w)\leq 1$ *in* $\mathbf{C}\setminus C_f$, *then* $f(\Omega)\subset \mathbf{C}\setminus C_f$, *namely, f is a one-to-one conformal mapping of* Ω *onto* $\{w\in\mathbf{C}\setminus C_f;\nu_f(w)=1\}$.

Proof. Assume that $w=f(z)\in C_f$ for some z in Ω. Then there is $\zeta\in\partial\Omega$ such that $w\in C_f(\zeta)$, namely, for every $\delta>0$ and every $\varrho>0$, $f(\Omega\cap B_\delta(\zeta))\cap B_\varrho(w)\neq\emptyset$. Since f is an open mapping at $z\in\Omega$ by (i), for small fixed ϱ, we can find a relatively compact neighborhood U of z such that $f(U)=B_\varrho(w)$. Take δ so that $U\cap B_\delta(\zeta)=\emptyset$. Then every value in an open set $f(\Omega\cap B_\delta(\zeta))\cap B_\varrho(w)=f(\Omega\cap B_\delta(\zeta))\cap f(U)$ is taken by f at least at two points, one in $\Omega\cap B_\delta(\zeta)$ and the other in U. Since C_f has no interior points, $f(\Omega\cap B_\delta(\zeta))\cap B_\varrho(w)\setminus C_f$ should not be empty. This contradicts (iii). Hence $f(\Omega)\subset\mathbf{C}\setminus C_f$.
Q.E.D.

If $\nu_f(w)>1$ for some w, then f may be quite complicated and it is difficult to describe f by using conformal mappings. We shall discuss here a special case which will appear in the proof of the main theorem. To do so, we prepare the following lemma which is easily verified:

LEMMA 1.4. *Let f be a holomorphic function in an open set* Ω. *If*
(i) $f(z)\neq 0$ *in* Ω,
(ii) $\int_J d\arg f(z)\equiv 0 \pmod{4\pi}$ *for every real analytic simple closed curve J in* Ω,
then $\sqrt{f(z)}$ *has a single-valued branch in* Ω.

Proof. By definition,

$$\sqrt{f(z)}=\exp\left(\frac{1}{2}\log f(z)\right)=\exp\left(\frac{1}{2}(\log|f(z)|+i\arg f(z))\right).$$

Hence $\sqrt{f(z)}$ is single-valued if $f(z)\neq 0$ in Ω and

$$\frac{1}{2}\int_J d\arg f(z)\equiv 0 \pmod{2\pi}$$

for every real analytic simple closed curve J in Ω. Q.E.D.

In what follows we write B_m for $B_m(0)$.

LEMMA 1.5. *If f satisfies* (i) *and* (ii) *of Lemma* 1.4, *together with*
(i) *f is not constant in any connected component of* Ω,
(ii) $C_f\subset[0,m)\cup\partial B_m$, *where* $[0,m)=\{w;0\leqq w=\mathrm{Re}\,w<m\}$,
(iii) $\nu_f(w)=2$ *in* $B_m\setminus[0,m)$ *and* $\nu_f(w)=0$ *in* $\mathbf{C}\setminus\overline{B_m}$,
(iv) *f has no branch points in* $f^{-1}(B_m\setminus[0,m))$, *namely,* $f'(z)\neq 0$ *in* $f^{-1}(B_m\setminus[0,m))$,

then

(1) Ω *is connected and a single-valued function* $\sqrt{f}$ *is one-to-one. The image* $\sqrt{f}(\Omega)$ *of* Ω *satisfies* $B_{\sqrt{m}}\setminus(-\sqrt{m},\sqrt{m})\subsetneqq\sqrt{f}(\Omega)\subsetneqq B_{\sqrt{m}}$, *where* $(-\sqrt{m},\sqrt{m})=\{w;-\sqrt{m}<w=\mathrm{Re}\,w<\sqrt{m}\}$,

or

(2) Ω *consists of two simply connected components and* $\sqrt{f}$ *is a one-to-one conformal mapping of* Ω *onto* $B_{\sqrt{m}}\setminus(-\sqrt{m},\sqrt{m})$ *for appropriately chosen branches.*

Remark. In (iv), we have assumed that $f'(z)\neq 0$ only in $f^{-1}(B_m\setminus[0,m))$. From the conclusion of the lemma, we see that $f'(z)\neq 0$ in the whole open set Ω.

Proof. By applying Lemma 1.4, we first define a single-valued holomorphic function $\sqrt{f}$ in Ω. By (ii), $C_{\sqrt{f}}\subset(-\sqrt{m},\sqrt{m})\cup\partial B_{\sqrt{m}}$ and, by (iii), $\nu_{\sqrt{f}}(w)+\nu_{\sqrt{f}}(-w)=2$ in $B_{\sqrt{m}}\setminus(-\sqrt{m},\sqrt{m})$ and $\nu_{\sqrt{f}}(w)=0$ in $\mathbf{C}\setminus\overline{B_{\sqrt{m}}}$.

If $\nu_{\sqrt{f}}(w)=1$ in one connected component of $B_{\sqrt{m}}\setminus(-\sqrt{m},\sqrt{m})$, then $\nu_{\sqrt{f}}(w)=1$ in the other connected component of $B_{\sqrt{m}}\setminus(-\sqrt{m},\sqrt{m})$, and so, by Lemma 1.1, $\nu_{\sqrt{f}}(w)\leq 1$ in $\mathbf{C}\setminus C_{\sqrt{f}}$ and $\{w\in\mathbf{C}\setminus C_{\sqrt{f}};\nu_{\sqrt{f}}(w)=1\}=B_{\sqrt{m}}\setminus C_{\sqrt{f}}$. By Lemma 1.3, $\sqrt{f}$ is a one-to-one conformal mapping of Ω onto $B_{\sqrt{m}}\setminus C_{\sqrt{f}}$. If $\sqrt{f}(\Omega)=B_{\sqrt{m}}$, then $\sqrt{f(z)}=0$ for some z in Ω. This contradicts (i) of Lemma 1.4. Hence (1) or (2) holds.

Assume next that $\nu_{\sqrt{f}}(w)=2$ in one connected component H of $B_{\sqrt{m}}\setminus(-\sqrt{m},\sqrt{m})$. Then $\nu_{\sqrt{f}}(w)=0$ in the other connected component of $B_{\sqrt{m}}\setminus(-\sqrt{m},\sqrt{m})$, and so $\nu_{\sqrt{f}}(w)=0$ in $\mathbf{C}\setminus(H\cup(-\sqrt{m},\sqrt{m})\cup\partial B_{\sqrt{m}})$. Since the exterior of $\mathbf{C}\setminus(H\cup(-\sqrt{m},\sqrt{m})\cup\partial B_{\sqrt{m}})$ is equal to H, by the fact stated after Lemma 1.1, $\sqrt{f}(\Omega)\subset H$, and so $(\sqrt{f})^{-1}(H)=\Omega$.

Let V be a connected component of $(\sqrt{f})^{-1}(H)$. By (iv), we can regard $(V,\sqrt{f}\,|V)$ as an unramified covering surface of a simply connected open set H. Hence, by the monodromy theorem, $\sqrt{f}\,|V$ is a homeomorphism, namely, $\sqrt{f}\,|V$ is a one-to-one conformal mapping of V onto H, see e.g. Section 3 of Chapter I of Ahlfors and Sario [1]. Thus Ω consists of two simply connected components. We redefine the branch of $\sqrt{f}$ in one of the two connected components of Ω and see that (2) holds.

Q.E.D.

§ 2. The Fuchs theorem

In his paper [6], Fuchs proved the following theorem:

THEOREM. *Let* D *be an unbounded connected open set in the complex plane* $\mathbf{C}$ *such that the boundary* ∂D *of* D *in* $\mathbf{C}$ *is not empty and let* f *be a holomorphic function in*

D satisfying

$$\limsup_{z\in D,\, z\to\zeta} |f(z)| \leq 1$$

for every ζ *on* ∂D. *Then one of the following must occur:*

(1) $|f(z)| \leq 1$ in D,

(2) f *has a pole at the point at infinity,*

(3) $(\log M(r))/\log r \to +\infty$ $(r\to+\infty)$, *where* $M(r)=\sup\{|f(z)|; z\in D \text{ and } |z|=r\}$.

In this section, we shall show a local version of the theorem and call it the Fuchs theorem.

THEOREM 2.1 (The Fuchs theorem). *Let* Ω *be an open set in* $\mathbf{C}$ *and let* ζ_0 *be a nonisolated boundary point of* Ω. *Let* f *be a holomorphic function in* Ω. *If there is a disk* $B_\delta(\zeta_0)$ *with radius* δ *and center* ζ_0 *such that*

(i) $\limsup_{z\in\Omega,\, z\to\zeta} |f(z)| \leq 1$ *for every* ζ *on* $(\partial\Omega)\cap B_\delta(\zeta_0)\setminus\{\zeta_0\}$,

(ii) $|f(z)| \leq \alpha|z-\zeta_0|^{-\beta}$ *in* $\Omega\cap B_\delta(\zeta_0)$ *for some positive constants* α *and* β,

then

$$\limsup_{z\in\Omega,\, z\to\zeta_0} |f(z)| \leq 1.$$

First we note that Theorem 2.1 is equivalent to the following Theorem 2.1′ from which the theorem due to Fuchs follows:

THEOREM 2.1′. *Let* Ω *be an unbounded open set in* $\mathbf{C}$ *such that the boundary* $\partial\Omega$ *is also unbounded and set* $B_1=B_1(0)$. *If a holomorphic function* f *in* Ω *satisfies*

(i) $\limsup_{z\in\Omega,\, z\to\zeta} |f(z)| \leq 1$ *for every* ζ *on* $(\partial\Omega)\setminus B_1$,

(ii) $|f(z)| \leq \alpha|z|^{\beta}$ *in* $\Omega\setminus B_1$ *for some positive constants* α *and* β,

then

$$\limsup_{z\in\Omega,\, z\to\infty} |f(z)| \leq 1.$$

Next we shall give a brief proof of Theorem 2.1′ for the sake of completeness. Our proof is similar to the proof due to Fuchs. We note the difference between them: Our condition (i) of Theorem 2.1′ is not for all boundary points but just for boundary points outside of the unit disk.

Proof of Theorem 2.1′. If f is constant in a connected component D of Ω, then, by (i), the modulus of the constant is not greater than 1 or $\partial D\subset B_1$. If $\partial D\subset B_1$ and D is unbounded, then $\mathbf{C}\setminus\overline{B_1}\subset D\subset\Omega$. This contradicts that ∂D is unbounded. Hence D is bounded if $\partial D\subset B_1$, and so $D\subset B_1$. Thus, to prove the theorem, we may assume that f is not constant in any connected component of Ω.

Assume that $\limsup_{z\in\Omega,\, z\to\infty}|f(z)|>1$. Since there are at most a countable number of zeros of f' in Ω, we can choose λ so that

$$\limsup_{z\in\Omega,\, z\to\infty}|f(z)|>\lambda>1 \tag{2.1}$$

and $f'(z)\neq 0$ on the level curves $\{z\in\Omega; |f(z)|=\lambda\}$. We consider the level curves outside of the unit disk and set

$$\Lambda=\{z\in\Omega\setminus\overline{B_1}; |f(z)|=\lambda\}$$

and

$$\Omega_0=\{z\in\Omega\setminus\overline{B_1}; |f(z)|>\lambda\}.$$

The set Ω_0 is nonempty, unbounded and open. From (i), it follows that

$$(\partial\Omega_0)\setminus(\partial B_1)=\Lambda.$$

Each connected component of Λ is a real analytic simple arc or a real analytic simple closed curve. We divide Λ into two parts Λ_a and Λ_c; Λ_a denotes the union of arcs in Λ and Λ_c denotes the union of closed curves in Λ. From (i), it follows that $\{(\partial\Omega)\setminus B_1\}\cap\bar{\Lambda}=\varnothing$. Hence the endpoints of arcs in Λ_a are all contained in $(\partial B_1)\cap\Omega$ and, for every $r>1$, there are at most a finite number of closed curves entirely contained in $\Lambda_c\cap B_r$, where $B_r=B_r(0)$.

We consider the following three cases:

Case 1. There is an unbounded arc in Λ_a.

Case 2. Each arc in Λ_a is bounded, but Λ_a is unbounded.

Case 3. Λ_a is bounded.

Case 1. Take an unbounded arc in Λ_a and take a fixed point p_0 on it. The point p_0 divides the arc into two portions. Let J be an unbounded portion of the arc and, for $R>|p_0|=r_0$, let J_R be the portion of J between p_0 and the first point of intersection of J with ∂B_R. Since $J\cap\partial B_1=\varnothing$, it follows that $J_R\cap\partial B_1=\varnothing$.

Let $\omega(z, E, D)$ be the harmonic measure of E with respect to a connected open set D, where E denotes a Borel subset of ∂D. The Beurling and Nevanlinna solution of the generalized Carleman–Milloux problem asserts that

$$\omega(z, \partial B_R, B_R \setminus J_R) \leq \omega(-|z|, \partial B_R, B_R \setminus [r_0, R)) \tag{2.2}$$

and

$$\omega(z, \partial B_1, B_R \setminus \overline{B_1} \setminus J_R) \leq \omega(-|z|, \partial B_1, \mathbf{C} \setminus \overline{B_1} \setminus [r_0, +\infty)) \tag{2.3}$$

for z in $B_R \setminus \overline{B_1} \setminus J_R$, where $[r_0, R) = \{z \in \mathbf{C}; r_0 \leq z = \operatorname{Re} z < R\}$, see e.g. Theorem 1 in Section 5 of Chapter IV of Nevanlinna [7].

Let $M(r) = \sup\{|f(z)|; z \in \Omega \cap \partial B_r\}$ for r with $\Omega \cap \partial B_r \neq \emptyset$. We have assumed that f is not constant in any connected component of Ω. Hence $M(r)$ is not equal to zero if it is defined. We first assume that $\Omega \cap \partial B_1 \neq \emptyset$. Set

$$h(z) = \log|f(z)| - (\log M(R))\, \omega(z, \partial B_R, B_R \setminus J_R) - (\log^+ M(1))\, \omega(z, \partial B_1, B_R \setminus \overline{B_1} \setminus J_R) - \log \lambda,$$

where $\log^+ t$ for $t > 0$ denotes $\max\{\log t, 0\}$. The function h is harmonic in $B_R \cap \Omega_0$ and

$$\limsup_{z \in B_R \cap \Omega_0,\, z \to \zeta} h(z) \leq 0$$

for every boundary point ζ of $B_R \cap \Omega_0$, because $M(R) \geq \lambda > 1$. Hence $h(z) \leq 0$ in $B_R \cap \Omega_0$ and, from (2.2) and (2.3), it follows that

$$\begin{aligned} \log|f(z)| \leq{} & (\log M(R))\, \omega(-|z|, \partial B_R, B_R \setminus [r_0, R)) \\ & + (\log^+ M(1))\, \omega(-|z|, \partial B_1, \mathbf{C} \setminus \overline{B_1} \setminus [r_0, +\infty)) + \log \lambda \end{aligned}$$

in $B_R \cap \Omega_0$.

Since $\omega(-|z|, \partial B_R, B_R \setminus [r_0, R)) \leq O(R^{-1/2})$ for a fixed z and $\log M(R) \leq O(\log R)$ by (ii), letting R tend to $+\infty$, we obtain

$$\log|f(z)| \leq (\log^+ M(1))\, \omega(-|z|, \partial B_1, \mathbf{C} \setminus \overline{B_1} \setminus [r_0, +\infty)) + \log \lambda$$

in Ω_0. The point at infinity is a regular boundary point of $\mathbf{C} \setminus \overline{B_1} \setminus [r_0, +\infty)$ with respect to the Dirichlet problem. Therefore

$$\limsup_{z \in \Omega_0,\, z \to \infty} \log|f(z)| \leq \log \lambda$$

and this contradicts (2.1).

We next consider the case that $\Omega \cap \partial B_1 = \emptyset$. In this case we just replace $\log^+ M(1)$

with zero in the definition of h, namely, we set

$$h(z) = \log|f(z)| - (\log M(R))\,\omega(z, \partial B_R, B_R \setminus J_R) - \log\lambda.$$

We obtain a contradiction by the same argument as above.

Case 2. Let $r_0 > 1$. In this case, for every $R > r_0$, we can choose a bounded arc in Λ_a such that it intersects with ∂B_R. Since the endpoint of the arc are contained in $\Omega \cap \partial B_1$, we can take a point p with $|p| = r_0$ on the arc and a portion J_R of the arc between p and the first point of intersection of the arc with ∂B_R. We choose such a J_R for each $R > r_0$, apply the same argument as in Case 1 and obtain again a contradiction.

Case 3. Take ϱ so that $\Lambda_a \subset B_\varrho$ and fix it. Set $\Omega_0(r) = \Omega_0 \cap (B_r \setminus \overline{B_\varrho})$ for $r > \varrho$. The boundary $\partial\Omega_0(r)$ of $\Omega_0(r)$ consists of a finite number of arcs on $\Omega_0 \cap (\partial B_\varrho \cup \partial B_r)$ and level arcs or curves each of which is a portion of a closed curve or the whole closed curve contained in Λ_c.

We apply the argument principle to f in each connected component of $\Omega_0(r)$ and apply the same argument given by Fuchs [6]. Then we obtain

$$\int_\varrho^R \frac{\nu(r)}{r}\,dr \leq (\beta+\gamma)\log R + (\log^+ \alpha - \gamma \log \varrho),$$

where

$$\gamma = -\frac{1}{2\pi}\int_{\Omega_0 \cap \partial B_\varrho} d\arg f(z)$$

and $\nu(r)$ denotes the number of closed level curves in Λ_c entirely contained in $B_r \setminus \overline{B_\varrho}$. Since $\nu(r)$ is a nondecreasing function of r, this inequality is valid for arbitrary $R > \varrho$ only if $\nu(r) \leq \beta + \gamma$, namely, the number of components in Λ_c which are entirely contained in $\mathbf{C} \setminus \overline{B_\varrho}$ is finite. Hence we can find $R > \varrho$ such that $\Lambda_c \subset B_R$, and so $\Lambda = \Lambda_a \cup \Lambda_c \subset B_R$.

Since $(\partial\Omega_0) \setminus (\partial B_1) = \Lambda \subset B_R$, $\mathbf{C} \setminus B_R \subset \Omega_0 \subset \Omega$. This contradicts the assumption that $\partial\Omega$ is unbounded. Q.E.D.

§ 3. Boundary behavior of a Schwarz function

In this section, we shall first give several remarks on a Schwarz function and next we shall show its specific boundary behavior.

We have defined the Schwarz function in the introduction. We again define it here and discuss it in detail.

Definition 3.1. Let Ω be an open subset of a disk $B_r(\zeta_0)$ of radius r and center ζ_0 such that $\zeta_0 \in \partial\Omega$. Set

$$\Gamma = (\partial\Omega) \cap B_r(\zeta_0).$$

A function S defined on $\Omega \cup \Gamma$ is called a Schwarz function of $\Omega \cup \Gamma$ if

(i) S is holomorphic in Ω,

(ii) S is continuous on $\Omega \cup \Gamma$,

(iii) $S(\zeta)=\bar{\zeta}$ on Γ, where $\bar{\zeta}$ denotes the complex conjugate of ζ.

If it is necessary to indicate the center ζ_0 or the disk $B_r(\zeta_0)$, we call S the Schwarz function of $\Omega \cup \Gamma$ at ζ_0 or in $B_r(\zeta_0)$, respectively.

First we note that if $\Omega \cup \Gamma$ has a Schwarz function S in $B_r(\zeta_0)$, then there are no connected components D of Ω such that $\partial D \subset B_r(\zeta_0)$. Indeed, if D is such a domain, then $\partial D \subset \Gamma$, and so $S(\zeta)=\bar{\zeta}$ on ∂D. Since two harmonic functions $S(z)$ and $\bar{z}$ have the same boundary values on ∂D, $S(z)=\bar{z}$ in D. This contradicts that $\bar{z}$ is not holomorphic in D. In particular, $\Omega \cap \partial B_\delta(\zeta_0) \neq \emptyset$ for every δ with $0<\delta<r$ if $\Omega \cup \Gamma$ has a Schwarz function in $B_r(\zeta_0)$.

Next we note that if Γ has an accumulation point in $B_r(\zeta_0)$, in particular if the center ζ_0 is an accumulation point of Γ, and if there exists a Schwarz function of $\Omega \cup \Gamma$ in $B_r(\zeta_0)$, then it is determined uniquely. If Γ is of positive capacity, then the theorem of Riesz–Lusin–Privaloff type guarantees the uniqueness, see e.g. Theorem in 7A of Sario and Nakai [8]. If Γ is of zero capacity, then a Schwarz function which is holomorphic in Ω and continuous on $\Omega \cup \Gamma = B_r(\zeta_0)$ is holomorphic in the disk $B_r(\zeta_0)$ and is determined uniquely by values on Γ, because Γ has an accumulation point in $B_r(\zeta_0)$.

To discuss the boundary behavior of a Schwarz function, we may assume that $r=1$ and $\zeta_0=0$ by the following lemma:

LEMMA 3.1. (1) *Let S be the Schwarz function of* $\Omega \cup \Gamma$ *at* ζ_0. *Then*

$$S_t(z) = S(z+\zeta_0) - \overline{\zeta_0}$$

is the Schwarz function of $(\Omega-\zeta_0) \cup (\Gamma-\zeta_0)$ *at* 0, *where* $E-\zeta_0$ *for a set* E *denotes a parallel translation* $\{z-\zeta_0; z \in E\}$ *of* E.

(2) *Let S be the Schwarz function of* $\Omega \cup \Gamma$ *in* $B_r = B_r(0)$. *Then, for a nonzero complex constant* k,

$$S_s(z) = \bar{k}S(z/k)$$

is the Schwarz function of $(k\Omega)\cup(k\Gamma)$ *in* $kB_r=B_{|k|r}$, *where* kE *for a set* E *denotes a set* $\{kz; z\in E\}$ *similar to* E.

Proof. (1) If $z\in(\Omega\cup\Gamma)-\zeta_0$, then $z+\zeta_0\in\Omega\cup\Gamma$, and so S_t is holomorphic in $\Omega-\zeta_0$ and continuous on $(\Omega-\zeta_0)\cup(\Gamma-\zeta_0)$. For $\zeta\in\Gamma-\zeta_0$, $S_t(\zeta)=S(\zeta+\zeta_0)-\overline{\zeta_0}=(\bar{\zeta}+\overline{\zeta_0})-\overline{\zeta_0}=\bar{\zeta}$, and so S_t is the Schwarz function of $(\Omega-\zeta_0)\cup(\Gamma-\zeta_0)$.

(2) S_s is holomorphic in $k\Omega$, is continuous on $(k\Omega)\cup(k\Gamma)$ and $S_s(\zeta)=\bar{k}S(\zeta/k)=\bar{k}(\overline{\zeta/k})=\bar{\zeta}$ on $k\Gamma$. Thus S_s is the Schwarz function of $(k\Omega)\cup(k\Gamma)$. Q.E.D.

Remark. More generally, let T be a one-to-one conformal mapping of $B_r(\zeta_0)$ into $\mathbf{C}$ and let $\bar{T}(w)=\overline{T(\bar{w})}$. Then $S_T=\bar{T}\circ S\circ T^{-1}$ is the Schwarz function of

$$(T(\Omega)\cap B_\delta(T(\zeta_0)))\cup(T(\Gamma)\cap B_\delta(T(\zeta_0)))$$

in $B_\delta(T(\zeta_0))$ for some δ if S is the Schwarz function of $\Omega\cup\Gamma$ in $B_r(\zeta_0)$.

We shall show

PROPOSITION 3.2. *Let* S *be the Schwarz function of* $\Omega\cup\Gamma$ *in* $B_1=B_1(0)$ *and assume that* 0 *is an accumulation point of* Γ, *in other words,* 0 *is a nonisolated boundary point of* Ω. *Then*

$$\lim_{z\in\Omega,\, z\to 0} |S(z)/z| = 1. \tag{3.1}$$

Remark. If $\Omega=B_1\setminus\{0\}$, then $\Gamma=\{0\}$ and the Schwarz function S of $\Omega\cup\Gamma$ is just a holomorphic function S in B_1 satisfying $S(0)=0$. Hence, in the proposition, it is necessary to assume that 0 is a nonisolated boundary point of Ω.

The proposition follows from the following key lemma and the Fuchs theorem:

LEMMA 3.3. *Let* S *and* Ω *be as in Proposition* 3.2. *Then there is a disk* $B_\delta=B_\delta(0)$ *such that*

$$|S(z)|>|z|/5 \quad \text{in } \Omega\cap B_\delta.$$

Proof of Proposition 3.2. First we consider $S(z)/z$. The function is holomorphic in Ω and satisfies

$$\lim_{z\in\Omega,\, z\to\zeta} |S(z)/z| = |\bar{\zeta}/\zeta| = 1$$

for every ζ on $\Gamma\setminus\{0\}=(\partial\Omega)\cap B_1\setminus\{0\}$. Since S is bounded in $\Omega\cap B_\delta$ for δ with $0<\delta<1$,

$$|S(z)/z|\leq\alpha|z|^{-1} \quad \text{in } \Omega\cap B_\delta$$

for some $\alpha>0$. Applying the Fuchs theorem, Theorem 2.1, we obtain

$$\limsup_{z\in\Omega,\, z\to 0}|S(z)/z|\leq 1. \tag{3.2}$$

Next we consider $z/S(z)$. If Lemma 3.3 is true, it follows that $z/S(z)$ is holomorphic and satisfies

$$|z/S(z)|<5 \quad \text{in } \Omega\cap B_\delta.$$

We again apply the Fuchs theorem and obtain

$$\limsup_{z\in\Omega,\, z\to 0}|z/S(z)|\leq 1. \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$1\leq\liminf_{z\in\Omega,\, z\to 0}|S(z)/z|\leq\limsup_{z\in\Omega,\, z\to 0}|S(z)/z|\leq 1$$

and (3.1) holds. Q.E.D.

Proof of Lemma 3.3. Let c be a complex number with $|c|<1$ and let $F_c(z)=z(S(z)-cz)$. We shall first show that F_c is not constant in any connected component of Ω. Assume that F_c is identically equal to a constant k in a connected component D of Ω. Then $|\zeta|^2-c\zeta^2=|\zeta|^2(1-ce^{i2\theta})=k$ on $(\partial D)\cap B_1$, where $\zeta=|\zeta|e^{i\theta}$. If $k=0$, then $\zeta=0$ and this contradicts that 0 is not an isolated point of $\partial\Omega$. If $k\neq 0$, then the line $\{w; \arg w=\arg k\}$ and the circle $\{w; |w-1|=|c|\}$ cross at most in two points. Hence $e^{i2\theta}$ has at most two solutions to the above equation, and so $\zeta=|\zeta|e^{i\theta}$ has at most four solutions if $c\neq 0$. This is again a contradiction. Hence $c=0$ and $|\zeta|^2=k$. This means that $(\partial D)\cap B_1$ is contained in a circle with center 0. If $(\partial D)\cap B_1$ is not the whole circle, then $D=B_1\setminus((\partial D)\cap B_1)$ and 0 is not the boundary point of Ω, a contradiction. Hence $(\partial D)\cap B_1$ is the whole circle and there are two possibilities: D is a disk surrounded by the circle or D is an annulus surrounded by the circle and the unit circle. In the latter case, there is another connected component of Ω inside of the circle, because 0 is a boundary point of Ω. Hence, in both cases, there is a connected component of Ω whose closure is entirely contained in B_1. This is a contradiction as mentioned after Definition

3.1. Thus we have proved that F_c is not constant in any connected component of Ω if $|c|<1$.

Now we consider the case $c=0$. By the above argument, $F_0(z)=zS(z)$ is not constant in any connected component of Ω. We choose r so that $0<r<1$ and $F_0(z)\neq 0$ on $C=\Omega\cap\partial B_r$, and fix it. We note that $F_0(z)$ tends to $\zeta\bar{\zeta}=|\zeta|^2=r^2$ as $z\in C$ tends to $\zeta\in\Gamma\cap\partial B_r$. In particular, $\inf\{|F_0(z)|; z\in C\}>0$ and F_0 is continuous on the closure $\bar{C}$ of C. The set C consists of at most a countable number of open arcs C_j on ∂B_r if $C\neq\partial B_r$.

We shall next discuss how to choose a small positive number ε. First choose ε so that

$$0<\varepsilon<1/5. \tag{3.4}$$

Since $F_c(z)=z(S(z)-cz)$ converges uniformly to $F_0(z)$ on $\bar{C}$ as c tends to 0, we can choose ε so that

$$m=\inf\{|F_c(z)|; z\in C \text{ and } |c|\leq\varepsilon\}>0. \tag{3.5}$$

By virtue of (3.5), the integral $\int_{C_j} d\arg F_c$ can be defined for each open arc C_j of C by

$$\int_{C_j} d\arg F_c=\lim_{\eta_1,\eta_2>0\to 0}\int_{\theta_{j1}+\eta_1}^{\theta_{j2}-\eta_2} d\arg F_c(re^{i\theta}),$$

where $C_j=\{re^{i\theta}; \theta_{j1}<\theta<\theta_{j2}\}$. A detailed discussion will show that $\int_{C_j}|d\arg F_c|$ is finite, but here we do not use the fact. We note that $\int_{C_j} d\arg F_c$ is well-defined and it is finite. Since $F_0(\zeta)=r^2$ for $\zeta\in\bar{C}\setminus C\subset\Gamma\cap\partial B_r$, for every small $\eta>0$ we can find a compact subset K of C such that $F_0(C\setminus K)\subset B_\eta(r^2)$. Let $\eta=r^2/5$. Since $|F_c(z)-F_0(z)|=|cz^2|\leq\varepsilon r^2<r^2/5$ by (3.4), $F_c(C\setminus K)\subset B_{(2/5)r^2}(r^2)\subset\{w\in\mathbf{C}; \operatorname{Re} w>0\}$ for every c with $|c|\leq\varepsilon$. Hence there exist only a finite number of C_j such that $\int_{C_j} d\arg F_c\geq\pi$ for some c with $|c|\leq\varepsilon$. We set

$$\iota_c=\sum{}'\int_{C_j} d\arg F_c,$$

where Σ' denotes the sum of terms satisfying $\int_{C_j} d\arg F_c\geq\pi$. We again note that F_c converges uniformly to F_0 on $\bar{C}$ as c tends to 0 and see that $\int_{C_j} d\arg F_c$ converges to $\int_{C_j} d\arg F_0$ for every fixed j. Thus we can choose ε so that

$$\iota=\sup\{\iota_c; |c|\leq\varepsilon\}<+\infty. \tag{3.6}$$

Now we choose ε so that ε satisfies (3.4), (3.5) and (3.6), and fix it.

Next take λ so that $0<\lambda<m$, where m is defined by (3.5), and set

$$\Lambda_{\lambda,c}=\{z\in\Omega\cap B_r;\ |F_c(z)|=\lambda\}.$$

Since $F_c(z)$ converges uniformly to 0 on $\{c\in\mathbf{C};|c|\leqslant\varepsilon\}$ as z tends to 0, we can find $\varrho=\varrho(\lambda)>0$ such that

$$\overline{B_\varrho}\cap\{z\in\Omega\cap B_r;\ |F_c(z)|\geqslant\lambda\}=\varnothing$$

for every c with $|c|\leqslant\varepsilon$.

Now take a regular exhaustion $\{\Omega_n\}$ of Ω. The boundary $\partial\Omega_n$ of Ω_n consists of a finite number of real analytic simple closed curves. Since $F_0(\zeta)=|\zeta|^2$ on Γ, we can find a neighborhood $U(\zeta)$ of $\zeta\in\Gamma\cap\overline{B_r}\setminus\{0\}$ such that $U(\zeta)\subset B_{(1/3)|\zeta|}(\zeta)$ and $F_0(\Omega\cap U(\zeta))\subset B_{\varepsilon|\zeta|^2}(|\zeta|^2)$. For $z\in\Omega\cap U(\zeta)$,

$$|F_c(z)-F_0(z)|=|cz^2|\leqslant\varepsilon(1+1/3)^2|\zeta|^2<2\varepsilon|\zeta|^2.$$

Hence $F_c(\Omega\cap U(\zeta))\subset B_{3\varepsilon|\zeta|^2}(|\zeta|^2)$ for $\zeta\in\Gamma\cap\overline{B_r}\setminus\{0\}$. Set

$$V=\cup\{B_{3\varepsilon t^2}(t^2);\ 0<t\leqslant r\}$$

and

$$U=\cup\{U(\zeta);\zeta\in\Gamma\cap\overline{B_r}\setminus\{0\}\}.$$

Then $F_c(\Omega\cap U)\subset V$. Now take Ω_n so that $(\Omega\setminus\Omega_n)\cap(\overline{B_r}\setminus B_\varrho)\subset U$. Let

$$\Omega_{n,\lambda,c}=\{z\in\Omega_n\cap B_r;|F_c(z)|>\lambda\}.$$

We shall apply the argument principle to F_c in each connected component of $\Omega_{n,\lambda,c}$:

$$\int_{\partial\Omega_{n,\lambda,c}} d\arg F_c=0.$$

We note that $\Omega_{n,\lambda,c}\subset B_r\setminus\overline{B_\varrho}$, F_c is holomorphic on $\overline{\Omega_{n,\lambda,c}}$ and $|F_c(z)|\geqslant\lambda$ on $\partial\Omega_{n,\lambda,c}$. We may assume that $\partial\Omega_{n,\lambda,c}$ consists of a finite number of piecewise real analytic simple closed curves. Each component of $\partial\Omega_{n,\lambda,c}$ consists of portions of C, $\Lambda_{\lambda,c}$ and $\partial\Omega_n$. Since $\lambda<m$ and $|F_c(z)|\geqslant m$ on C by (3.5), $C\cap\Lambda_{\lambda,c}=\varnothing$. We divide $\partial\Omega_{n,\lambda,c}$ into two parts. The first is the union of simple closed curves contained entirely in C, $\Lambda_{\lambda,c}$ or $\partial\Omega_n$ and the second is the union of simple closed curves which consist of portions of both $C\cup\Lambda_{\lambda,c}$ and $\partial\Omega_n$.

If a curve J of $\partial\Omega_{n,\lambda,c}$ is entirely contained in C, then $J=C=\partial B_r$ and $\int_J d\arg F_c=\int_{\partial B_r} d\arg F_c$. If a curve J is entirely contained in $\Lambda_{\lambda,c}$, then $\int_J d\arg F_c$ is equal to a positive integer multiple of -2π. If a curve J is entirely contained in $\partial\Omega_n$, then it is entirely contained in $\Omega\cap U$, because $(\partial\Omega_n)\cap(\overline{B_r}\setminus B_\varrho)\subset U$. Hence

$$F_c(J)\subset F_c(\Omega\cap U)\subset V\subset\{w\in\mathbf{C};\operatorname{Re}w>0\},$$

and so $\int_J d\arg F_c=0$.

Now we discuss the nontrivial and final case: a curve J of $\partial\Omega_{n,\lambda,c}$ consists of portions of both $C\cup\Lambda_{\lambda,c}$ and $\partial\Omega_n$. We express J as the union of an even number of arcs J_j, $j=1,2,\dots,2l$, such that $J_j\subset\partial\Omega_n$ for odd j and $J_j\subset C$ or $J_j\subset\Lambda_{\lambda,c}$ for even j. Since $F_c(J_j)\subset V\subset\{w\in\mathbf{C};\operatorname{Re}w>0\}$ for odd j, we can find, for every j, an integer a_j such that

$$\left|\sum_{k=1}^{j}\int_{J_k} d\arg F_c-2\pi a_j\right|<\frac{\pi}{2}.$$

Hence we can express $\int_J d\arg F_c$ as $2\pi a_{2l}=2\pi\sum_{j=1}^{2l}(a_j-a_{j-1})$, where $a_0=0$. If $a_j>a_{j-1}$, then j is even and $J_j\subset C$. Further it follows that $\int_{J_j} d\arg F_c\geqslant\pi$ and $a_j-a_{j-1}<(1/\pi)\int_{J_j} d\arg F_c$ in this case. Hence

$$\int_J d\arg F_c\leqslant 2\pi\sum_{a_j>a_{j-1}}(a_j-a_{j-1})<2\sum{}'\int_{J_j} d\arg F_c,$$

where Σ' denotes the sum of terms satisfying $\int_{J_j} d\arg F_c\geqslant\pi$. We note that each J_j is a connected component of $\Omega_n\cap C$.

Summing up the estimations of all cases, we obtain

$$0=\int_{\partial\Omega_{n,\lambda,c}} d\arg F_c\leqslant -2\pi\nu_{n,\lambda,c}+2\sum{}'\int_{C_{n,j}} d\arg F_c,$$

where $\nu_{n,\lambda,c}$ denotes the number of components of $\partial\Omega_{n,\lambda,c}$ which are entirely contained in $\Lambda_{\lambda,c}$, $C_{n,j}$ denotes connected components of $\Omega_n\cap C$ and Σ' denotes the sum of terms satisfying $\int_{C_{n,j}} d\arg F_c\geqslant\pi$. Hence

$$\nu_{n,\lambda,c}\leqslant\frac{1}{\pi}\sum{}'\int_{C_{n,j}} d\arg F_c.$$

By letting n tend to $+\infty$, we see that

$$\nu_{n,\lambda,c}\leqslant\lim_{n\to\infty}\nu_{n,\lambda,c}\leqslant\frac{1}{\pi}\sum{}'\int_{C_j} d\arg F_c=\frac{\iota_c}{\pi}.$$

The value ι_c/π does not depend on the choice of n and λ. If F_c has a zero in $\Omega\cap B_r$, then, for sufficiently large n and sufficiently small λ, there corresponds a curve J of $\partial\Omega_{n,\lambda,c}$ which is entirely contained in $\Lambda_{\lambda,c}$. The above estimation implies that the number of zeros of F_c in $\Omega\cap B_r$ is finite.

Thus we can find a small disk B_{δ_1} such that $F_c(z)\neq 0$ in $\Omega\cap B_{\delta_1}$, namely, $S(z)\neq cz$ in $\Omega\cap B_{\delta_1}$. If δ_1 does not depend on c when c varies on $\{|c|\leq\varepsilon\}$, then $|S(z)|>\varepsilon|z|$. But δ_1 may depend on c and we need further consideration to prove the lemma.

By (3.6), we see that $\nu_{n,\lambda,c}\leq\iota/\pi$. We set

$$\nu=\sup\{\nu_{n,\lambda,c};\ n\in\mathbf{N},\ 0<\lambda<m,\ |c|\leq\varepsilon\}.$$

We take n_0, λ_0 and c_0 so that $\nu_{n_0,\lambda_0,c_0}=\nu$. We shall show that there is an $\varepsilon_1>0$ such that $\nu_{n_0,\lambda_0,c}=\nu$ for every c on $B_{\varepsilon_1}(c_0)\cap\overline{B_\varepsilon}$.

Let $\Sigma_{\lambda,c}$ be the union of $\nu_{n_0,\lambda,c}$ components of $\partial\Omega_{n_0,\lambda,c}$ which are entirely contained in $\Lambda_{\lambda,c}$. We fix c_0 and vary λ near λ_0. Then Σ_{λ,c_0} moves near Σ_{λ_0,c_0}. More precisely, let J_{λ_0} be a curve of Σ_{λ_0,c_0}. If $\lambda<\lambda_0$ (resp. $\lambda>\lambda_0$) and if λ is sufficiently close to λ_0, then there is a curve J_λ of Σ_{λ,c_0} which is contained inside of (resp. outside of) J_{λ_0} and is close to J_{λ_0}. Since the number of components of Σ_{λ_0,c_0} is finite, we can take such a λ valid for all components of Σ_{λ_0,c_0}.

We take such λ_1 and λ_2 sufficiently close to λ_0 so that $\lambda_1<\lambda_0<\lambda_2$ and fix them. Next take $\eta>0$ so small that $\eta<\lambda_2-\lambda_0$ and $\eta<\lambda_0-\lambda_1$, and fix it. Since $F_c(z)$ is a continuous function of c, we can find $\varepsilon_1>0$ such that

$$|F_c(z)|\geq\lambda_2-\eta>\lambda_0\quad\text{on }\Sigma_{\lambda_2,c_0}$$

and

$$|F_c(z)|\leq\lambda_1+\eta<\lambda_0\quad\text{on }\Sigma_{\lambda_1,c_0}$$

for every c in $B_{\varepsilon_1}(c_0)$. Let A be a doubly connected domain surrounded by J_{λ_1} and J_{λ_2} which lie near a curve J_{λ_0} of Σ_{λ_0,c_0} and are contained in Σ_{λ_1,c_0} and Σ_{λ_2,c_0}, respectively. Then, by the above inequality, there is a curve J which is contained in A and is a component of $\Sigma_{\lambda_0,c}$ for every c in $B_{\varepsilon_1}(c_0)$. Thus we have proved that $\nu_{n_0,\lambda_0,c}\geq\nu$ for every c in $B_{\varepsilon_1}(c_0)$. By the definition of ν, $\nu_{n_0,\lambda_0,c}=\nu$ for every c on $B_{\varepsilon_1}(c_0)\cap\overline{B_\varepsilon}$.

Now take $\varepsilon_2>0$ and c_1 so that $B_{\varepsilon_2}(c_1)\subset B_{\varepsilon_1}(c_0)\cap\overline{B_\varepsilon}$. By the definition of $\varrho_0=\varrho(\lambda_0)$,

$$\overline{B_{\varrho_0}}\cap\{z\in\Omega\cap B_r;\ |F_c(z)|\geq\lambda_0\}=\varnothing$$

for every c in $B_{\varepsilon_2}(c_1)$. Since $\nu_{n,\lambda,c}=\nu$ for every $n\geq n_0$ and $\lambda\leq\lambda_0, F_c(z)\neq 0$ in $\Omega\cap B_{\varrho_0}$ for every c in $B_{\varepsilon_2}(c_1)$, namely, $S(z)\neq cz$ in $\Omega\cap B_{\varrho_0}$ for every c in $B_{\varepsilon_2}(c_1)$. This implies that

$$S(z)-c_1z\neq(c-c_1)z \quad \text{in } \Omega\cap B_{\varrho_0}$$

for every c with $|c-c_1|<\varepsilon_2$. Hence

$$|S(z)-c_1z|\geq\varepsilon_2|z| \quad \text{in } \Omega\cap B_{\varrho_0}.$$

We now apply the Fuchs theorem to $z/(S(z)-c_1z)$. Since

$$\lim_{z\in\Omega,\,z\to\zeta}\left|\frac{z}{S(z)-c_1z}\right|=\left|\frac{\zeta}{\bar\zeta-c_1\zeta}\right|\leq\frac{1}{1-\varepsilon}$$

for every ζ on $(\partial\Omega)\cap B_{\varrho_0}\setminus\{0\}$ and

$$\left|\frac{z}{S(z)-c_1z}\right|\leq\frac{1}{\varepsilon_2} \quad \text{in } \Omega\cap B_{\varrho_0},$$

by the Fuchs theorem, we obtain

$$\limsup_{z\in\Omega,\,z\to0}\left|\frac{z}{S(z)-c_1z}\right|\leq\frac{1}{1-\varepsilon}.$$

Hence there exists a positive number δ such that

$$\left|\frac{z}{S(z)-c_1z}\right|\leq\frac{2}{1-\varepsilon} \quad \text{in } \Omega\cap B_\delta.$$

Since

$$((1-\varepsilon)/2)|z|\leq|S(z)-c_1z|\leq|S(z)|+\varepsilon|z|,$$

by using (3.4), we obtain

$$|S(z)|\geq\left(\frac{1}{2}-\frac{3}{2}\varepsilon\right)|z|>\frac{|z|}{5} \quad \text{in } \Omega\cap B_\delta. \qquad \text{Q.E.D.}$$

§ 4. Estimation of a valence function

Let Ω be an open subset of the unit disk B_1 such that 0 is a nonisolated boundary point of Ω and let $\Gamma=(\partial\Omega)\cap B_1$. In this section we always assume that 0 is a nonisolated point of Γ. Assume that there is a Schwarz function S of $\Omega\cup\Gamma$ in B_1 and let $F(z)=zS(z)$, which was denoted by $F_0(z)$ in the proof of Lemma 3.3. Let ϱ be a positive number such that $F(z)\neq 0$ on $\Omega\cap\overline{B_\varrho}$ and set $m=\inf\{|F(z)|; z\in\Omega\cap\partial B_\varrho\}>0$. The existence of such a ϱ is guaranteed by Lemma 3.3.

We shall discuss the cluster set and the valence function of $F|\Omega\cap B_\varrho$. Since $|F(z)|\geqq m$ on $\Omega\cap\partial B_\varrho$ and $F(z)=z\bar{z}=|z|^2\geqq 0$ on Γ, the cluster set $C_{F|\Omega\cap B_\varrho}$ of $F|\Omega\cap B_\varrho$ is contained in $\{w\in\mathbf{C}; |w|\geqq m\}\cup[0,m)$, where $[0,m)=\{w; 0\leqq w=\mathrm{Re}\,w<m\}$. Since F is not constant in any connected component of Ω and $\lim_{z\in\Omega, z\to 0}F(z)=0$, $\nu_{F|\Omega\cap B_\varrho}(w)\geqq 1$ for some w which is close to 0 and is contained in $B_m\setminus[0,m)$. By Lemma 1.1, $\nu_{F|\Omega\cap B_\varrho}$ is equal to a constant, say ν, in $B_m\setminus C_{F|\Omega\cap B_\varrho}$. Therefore $\nu=\nu_{F|\Omega\cap B_\varrho}(w)\geqq 1$ in $B_m\setminus[0,m)$. We shall show

PROPOSITION 4.1. *It follows that* $\nu=1$ *or* 2.

Proof. Let $\{\Omega_n\}$ be a regular exhaustion of Ω and let $w\in B_m\setminus[0,m)$. Since $\nu_{F|\Omega\cap B_\varrho}(w)=\nu<+\infty$, there are ν w-points of F in $\Omega_n\cap B_\varrho$ for sufficiently large n. By the argument principle, we obtain

$$\nu=\nu_{F|\Omega\cap B_\varrho}(w)=\frac{1}{2\pi}\int_{\partial(\Omega_n\cap B_\varrho)}d\arg(F(z)-w). \tag{4.1}$$

We note that $(\partial\Omega_n)\cap(\partial B_\varrho)$ consists of a finite number of points. We may assume that $\partial(\Omega_n\cap B_\varrho)$ is expressed as the union of $\Omega_n\cap\partial B_\varrho$ and $(\partial\Omega_n)\cap\overline{B_\varrho}$: $\partial(\Omega_n\cap B_\varrho)=\Omega_n\cap\partial B_\varrho+(\partial\Omega_n)\cap\overline{B_\varrho}$. As we have noticed in the proof of Lemma 3.3, $\int_{\Omega\cap\partial B_\varrho}d\arg(F(z)-w)$ is well-defined and finite, because $\inf\{|F(z)|; z\in\Omega\cap\partial B_\varrho\}=m>|w|$ and $F(z)=\varrho^2>0$ on $(\partial\Omega)\cap(\partial B_\varrho)$. Since F is continuous on $\Omega\cup\Gamma$ and $F(z)=|z|^2\geqq 0$ on Γ, we see that

$$\int_{(\Omega\setminus\Omega_n)\cap\partial B_\varrho-(\partial\Omega_n)\cap\overline{B_\varrho}}d\arg(F(z)-w)=0 \tag{4.2}$$

for large n. Hence, by dividing (4.2) by 2π and adding it to (4.1), we obtain

$$\nu=\frac{1}{2\pi}\int_{\Omega\cap\partial B_\varrho}d\arg(F(z)-w).$$

The equation holds for every w in $B_m\setminus[0,m)$. By letting w tend to 0, we obtain

$$\nu=\frac{1}{2\pi}\int_{\Omega\cap\partial B_\varrho}d\arg F(z). \tag{4.3}$$

The equation (4.3) is valid for every smaller ϱ, because $F(z)\neq 0$ in $\Omega\cap B_\varrho$ and the right-hand side of (4.3) is continuous with respect to ϱ.

Let $G(z)=S(z)/z$. Then $F(z)=z^2G(z)$, and so $\arg F(z)=2\arg z+\arg G(z)$. We set

$$A(r)=\int_{\Omega\cap\partial B_r} d\arg G(z)$$

for r with $0<r<\varrho$. Then

$$A(r)=2\pi\nu-2\int_{\Omega\cap\partial B_r} d\arg z.$$

To show that $\nu\leq 2$, we shall estimate the value of $A(r)$.

Take ε with $0<\varepsilon<1/2$. By Proposition 3.2, we can find δ with $0<\delta<\varrho$ such that

$$1-\varepsilon\leq |G(z)|\leq 1+\varepsilon \quad \text{in } \Omega\cap B_\delta. \tag{4.4}$$

Since $(\partial\arg G(z))/(\partial s)=(\partial\log|G(z)|)/(\partial r)$ along $\Omega\cap\partial B_r$, we obtain

$$A(r)=\int_{\Omega\cap\partial B_r}\frac{\partial\log|G(z)|}{\partial r}\,r\,d\theta,$$

where $z=re^{i\theta}$. Dividing by r and integrating the equality from η to δ, we obtain

$$\int_\eta^\delta\frac{A(r)}{r}\,dr=\iint_{\Omega\cap(B_\delta\setminus\overline{B_\eta})}\frac{\partial\log|G(z)|}{\partial r}\,d\theta\,dr.$$

To apply the Fubini theorem, we shall show that $(\partial\log|G(z)|)/(\partial r)$ is integrable on $\Omega\cap(B_\delta\setminus\overline{B_\eta})$ for every η with $0<\eta<\delta$. We note that $|(\partial\log|G(z)|)/(\partial r)|\leq|(\log G(z))'|=|G'|/|G|$ and $G'(z)=(F/z^2)'=F'(z)/z^2-2F(z)/z^3$. Since $|F|=|z|^2|G|\leq(1+\varepsilon)|z|^2$ in $\Omega\cap B_\delta$, we may assume that $F(\Omega\cap B_\delta)\subset B_m$. Hence $\int_{\Omega\cap B_\delta}|F'|^2\,dx\,dy\leq\nu\cdot\text{area}\,B_m<+\infty$. Thus, by using the Schwarz inequality, we see that F' is integrable on $\Omega\cap B_\delta$. Therefore $(\partial\log|G(z)|)/(\partial r)$ is integrable on $\Omega\cap(B_\delta\setminus\overline{B_\eta})$.

We apply the Fubini theorem and make first the integration with respect to r. For fixed θ, the set $\Omega\cap(B_\delta\setminus\overline{B_\eta})\cap\{z;\arg z=\theta\}$ consists of at most a countable number of segments. At the endpoints of the segments, $\log|G(z)|$ has the value 0 except two points $\eta e^{i\theta}$ and $\delta e^{i\theta}$ if they are contained in Ω. By (4.4),

$$-2\varepsilon<\log(1-\varepsilon)\leq\log|G(z)|\leq\log(1+\varepsilon)<2\varepsilon \quad \text{in } \Omega\cap B_\delta,$$

because we have chosen ε so that $0<\varepsilon<1/2$. Hence we obtain

$$\left|\int_{\eta}^{\delta} \frac{\partial \log |G(z)|}{\partial r} dr\right| \leqslant 4\varepsilon,$$

and so

$$\left|\int_{\eta}^{\delta} \frac{A(r)}{r} dr\right| \leqslant 8\pi\varepsilon \tag{4.5}$$

for every small η.

We note that

$$A(r) = 2\pi\nu - 2\int_{\Omega\cap\partial B_r} d\arg z \geqslant 2\pi\nu - 4\pi \tag{4.6}$$

and assume that $\nu\geqslant 2$. Then $A(r)\geqslant 0$ by (4.6), and so $A(r)<2\pi$ for small r by (4.5). Hence $\nu\leqslant 2$ by (4.6). Q.E.D.

Now we define an index of $\Omega\cup\Gamma$ at 0.

Definition 4.1. Let Ω be an open subset of a disk B_r such that 0 is a nonisolated boundary point of Ω and let $\Gamma=(\partial\Omega)\cap B_r$. If there is the Schwarz function of $\Omega\cup\Gamma$ in B_r, we call ν in Proposition 4.1 the index of $\Omega\cup\Gamma$ at 0. It is equal to 1 or 2. The index at ζ_0 for an open subset Ω of a disk $B_r(\zeta_0)$ and $\Gamma=(\partial\Omega)\cap B_r(\zeta_0)$ is defined by using a parallel translation of the Schwarz function if ζ_0 is a nonisolated boundary point of Ω.

COROLLARY 4.2. *It follows that*

$$\int_J d\arg F(z) = 2\pi\nu n(J, 0)$$

for every real analytic closed curve J in $\Omega\cap B_\varrho$, where ν denotes the index of $\Omega\cup\Gamma$ at 0 and $n(J, 0)$ denotes the winding number of J with respect to 0.

Proof. We may assume that J is a simple closed curve. Since 0 is not contained in $\Omega\cap B_\varrho$, there are two cases: 0 is contained inside or outside of J.

Assume that 0 is contained in the inside D of J. For $w\in B_m\setminus[0, m)$, there are exactly ν w-points of F in $\Omega\cap B_\varrho$ as we have seen at the beginning of this section. These ν w-points of F are contained in D for $w\in B_m\setminus[0, m)$ sufficiently close to 0. Let $\{\Omega_n\}$ be a regular exhaustion of Ω. Then Ω_n contains these ν w-points of F and J for large n.

We apply the argument principle to $F-w$ in $\Omega_n \cap D$ and obtain

$$\frac{1}{2\pi}\int_{\partial(\Omega_n \cap D)} d\arg(F(z)-w) = \nu.$$

We divide $\partial(\Omega_n \cap D)$ into two parts: $\partial(\Omega_n \cap D)=(\partial\Omega_n)\cap D+J$.

Since F is continuous on $\Omega \cup \Gamma$ and $F(z)=|z|^2 \geqq 0$ on Γ, $\int_{(\partial\Omega_n)\cap D} d\arg(F(z)-w)=0$ for large n. Hence we obtain

$$\int_J d\arg(F(z)-w) = 2\pi\nu.$$

Finally, by letting w tend to 0, we obtain

$$\int_J d\arg F(z) = 2\pi\nu.$$

If 0 is contained outside of J, then ν w-points of F in $\Omega \cap B_\varrho$ are contained outside of J for $w \in B_m \setminus [0, m)$ sufficiently close to 0, and so there are no w-points of F contained in D. Hence

$$\frac{1}{2\pi}\int_{\partial(\Omega_n \cap D)} d\arg(F(z)-w) = 0$$

for Ω_n with $J \subset \Omega_n$. By the same argument as above, we see that $\int_J d\arg F(z)=0$. Q.E.D.

§ 5. Proof of the main theorem

First we prepare the following proposition:

PROPOSITION 5.1. *Let Ω be an open subset of the unit disk B_1 such that 0 is a nonisolated boundary point of Ω and let $\Gamma=(\partial\Omega)\cap B_1$. Then*

(1) *there exists a Schwarz function of $(\Omega \cap B_r)\cup(\Gamma \cap B_r)$ in B_r for some $r>0$ with index 1 at 0 if and only if there exists a function Φ_1 defined on $(\Omega \cup \Gamma)\cap B_\delta$ for some $\delta>0$ such that*

(i) *Φ_1 is holomorphic and univalent in $\Omega \cap B_\delta$,*

(ii) *Φ_1 is continuous on $(\Omega \cup \Gamma)\cap B_\delta$,*

(iii) *$\Phi_1(\zeta)=|\zeta|^2$ on $\Gamma \cap B_\delta$,*

and

(2) *there exists a Schwarz function of $(\Omega \cap B_r)\cup(\Gamma \cap B_r)$ in B_r for some $r>0$ with index 2 at 0 if and only if there exists a function Φ_2 defined on $(\Omega \cup \Gamma)\cap B_\delta$ for some $\delta>0$*

satisfying (i) *of* (1) *and*

(ii′) Φ_2^2 *is continuous on* $(\Omega\cup\Gamma)\cap B_\delta$,

(iii′) $\Phi_2^2(\zeta)=|\zeta|^2$ *on* $\Gamma\cap B_\delta$,

(iv) $\Phi_2(\Omega\cap B_\delta)\cup(-\varepsilon,\varepsilon)$ *for* $\varepsilon>0$ *contains a neighborhood of* $w=0$.

Proof. Assume that S is the Schwarz function of $(\Omega\cap B_r)\cup(\Gamma\cap B_r)$ in B_r with index 1 at 0. Take B_ϱ and m as mentioned before Proposition 4.1. Let $V=F^{-1}(B_m)\cap\Omega\cap B_\varrho$, where $F(z)=zS(z)$. Then $C_{F|V}\subset[0,m)\cup\partial B_m$ and $\nu_{F|V}(w)\leq 1$ in $\mathbf{C}\setminus C_{F|V}$. Hence, by Lemma 1.3, $\Phi_1(z)=(F|\bar{V})(z)$ is holomorphic and univalent in V, and continuous on $\bar{V}$. Since F is continuous at 0, there is a disk B_δ such that $\Omega\cap B_\delta\subset V$. By the definition of the Schwarz function, $\Phi_1(\zeta)=F(\zeta)=|\zeta|^2$ on $\Gamma\cap B_\delta$. Thus Φ_1 satisfies (i) to (iii).

Assume next that the index is equal to 2. We use the same notation as above. Then $F|\bar{V}$ is holomorphic in V, is continuous on $\bar{V}$ and does not vanish in V. From Corollary 4.2, it follows that F satisfies the condition (ii) of Lemma 1.4. Since the number of zeros of F' in $F^{-1}(B_m\setminus C_{F|V})\cap V$ is at most two by Lemma 1.2, we can take smaller ϱ and m so that $F'(z)\neq 0$ in $F^{-1}(B_m\setminus[0,m))\cap V$. Now $f=F|V$ satisfies (i) to (iv) of Lemma 1.5. Hence, by Lemma 1.5, we can take appropriate branches of $\sqrt{f(z)}$ so that $\Phi_2(z)=\sqrt{f(z)}$ is holomorphic and univalent in V. On the boundary of V, we take arbitrary branches of $\sqrt{F(z)}$ and set $\Phi_2(z)=\sqrt{F(z)}$. It also follows from Lemma 1.5 that $\Phi_2(V)\cup(-\sqrt{m},\sqrt{m})=B_{\sqrt{m}}$. Since $\Phi_2^2(z)=F(z)$ on $\bar{V}$, Φ_2^2 is continuous on $(\Omega\cup\Gamma)\cap B_\delta$ and satisfies $\Phi_2^2(\zeta)=|\zeta|^2$ on $\Gamma\cap B_\delta$. Thus Φ_2 satisfies (i), (ii′), (iii′) and (iv).

Conversely, let Φ_1 be a function defined on $(\Omega\cup\Gamma)\cap B_\delta$ satisfying (i) to (iii). Then the function S defined by $S(z)=\Phi_1(z)/z$ for $z\in(\Omega\cup\Gamma)\cap B_\delta\setminus\{0\}$ and $S(z)=0$ for $z=0$ is holomorphic in $\Omega\cap B_\delta$ and continuous on $(\Omega\cup\Gamma)\cap B_\delta\setminus\{0\}$. On $\Gamma\cap B_\delta\setminus\{0\}$, $S(\zeta)=|\zeta|^2/\zeta=\bar{\zeta}$. Hence, S is the Schwarz function of $(\Omega\cap B_\delta)\cup(\Gamma\cap B_\delta)$ in B_δ if it satisfies

$$\lim_{z\in\Omega,\,z\to 0}|S(z)|=0. \tag{5.1}$$

Take η with $0<\eta<\delta$ and we shall apply the Fuchs theorem to S in $\Omega\cap B_\eta$. If $\zeta\in\Gamma\cap B_\eta\setminus\{0\}$, then

$$\limsup_{z\in\Omega,\,z\to\zeta}|S(z)|=|\bar{\zeta}|\leq\eta.$$

Since Φ_1 is bounded in a neighborhood of 0, $|S(z)|\leq\alpha/|z|$ for some α in $\Omega\cap B_\eta$. Hence, by the Fuchs theorem,

$$\limsup_{z\in\Omega,\,z\to 0}|S(z)|=\eta. \tag{5.2}$$

The inequality (5.2) holds for every η with $0<\eta<\delta$, and so (5.1) holds. Thus S is the Schwarz function of $(\Omega\cap B_\delta)\cup(\Gamma\cap B_\delta)$ in B_δ. Since $F(z)=zS(z)=\Phi_1(z)$ and Φ_1 is univalent, the index of $(\Omega\cap B_\delta)\cup(\Gamma\cap B_\delta)$ at 0 is equal to 1. If Φ_2 satisfies (i), (ii′), (iii′) and (iv), then, by using the same argument as above, we see that the function S defined by $S(z)=(\Phi_2(z))^2/z$ for $z\in(\Omega\cup\Gamma)\cap B_\delta\setminus\{0\}$ and $S(z)=0$ for $z=0$ is the Schwarz function of $(\Omega\cap B_\delta)\cup(\Gamma\cap B_\delta)$ in B_δ with index 2 at 0. Q.E.D.

Now we shall show the main theorem.

THEOREM 5.2. *Let Ω be an open subset of the unit disk B_1 such that 0 is a nonisolated boundary point of Ω and let $\Gamma=(\partial\Omega)\cap B_1$. If there exists a Schwarz function of $\Omega\cup\Gamma$ in B_1, then, for some small $\delta>0$, one of the following must occur;* (1) *and* (2) *correspond to the index* 1 *and* 2 *of $\Omega\cup\Gamma$ at* 0, *respectively:*

(1) *$\Omega\cap B_\delta$ is simply connected and $\Gamma\cap B_\delta$ is a regular real analytic simple arc passing through* 0.

(2a) *$\Gamma\cap B_\delta$ determines uniquely a regular real analytic simple arc passing through* 0 *and $\Gamma\cap B_\delta$ is an infinite proper subset of the arc accumulating at* 0 *or the whole arc. $\Omega\cap B_\delta$ is equal to $B_\delta\setminus\Gamma$.*

(2b) *$\Omega\cap B_\delta$ consists of two simply connected components Ω_1 and Ω_2. $(\partial\Omega_1)\cap B_\delta$ and $(\partial\Omega_2)\cap B_\delta$ are distinct regular real analytic simple arcs passing through* 0. *They are tangent to each other at* 0.

(2c) *$\Omega\cap B_\delta$ is simply connected and $\Gamma\cap B_\delta$ is a regular real analytic simple arc except for a cusp at* 0. *The cusp is pointing into $\Omega\cap B_\delta$. It is a very special one. There is a holomorphic function T defined on a closed disk $\overline{B_\varepsilon}$ such that*

(i) *T has a zero of order two at* 0,

(ii) *T is univalent on the closure $\bar{H}$ of a half disk $H=\{\tau\in B_\varepsilon; \operatorname{Im}\tau>0\}$,*

(iii) *T satisfies $\Gamma\cap B_\delta\subset T((-\varepsilon,\varepsilon))$ and $T(\bar{H})\subset\Omega\cup\Gamma$.*

Conversely, if (1), (2a), (2b) *or* (2c) *holds, then $(\Omega\cap B_r)\cup(\Gamma\cap B_r)$ has the Schwarz function for some $r>0$. If* (1) *or* (2a) *holds, then the Schwarz function can be extended holomorphically onto a neighborhood of* 0. *If* (2b) *holds, then the Schwarz function can be extended from Ω_1 onto a neighborhood of* 0 *and from Ω_2 onto a neighborhood of* 0. *The two extensions are distinct in a neighborhood of* 0. *If* (2c) *holds, then the Schwarz function can not be extended onto any neighborhood of* 0.

Proof. Assume first that there is a Schwarz function of $\Omega\cup\Gamma$ in B_1 with index 1 at 0. By Proposition 5.1, there is a function Φ_1 satisfying (i) to (iii) of Proposition 5.1. Since Φ_1 is continuous at 0, by Lemmas 1.1 and 1.3, we can find a neighborhood U of 0 and $\varepsilon>0$ such that $\Phi_1|\Omega\cap U$ is a one-to-one conformal mapping of $\Omega\cap U$ onto $B_\varepsilon\setminus E$, where E is a relatively closed subset of $[0,\varepsilon)$ containing 0. Let $z=\Psi_1(w)$ be the inverse function of $\Phi_1|\Omega\cap U$ defined in $B_\varepsilon\setminus E$. Let $\overline{\Psi_1}(w)=\overline{\Psi_1(\bar{w})}$. Then $\overline{\Psi_1}$ is holomorphic in $B_\varepsilon\setminus E$, and so $Z_1(w)=\Psi_1(w)\overline{\Psi_1}(w)/w$ is holomorphic in $B_\varepsilon\setminus E$.

To apply Lemma 1.2 to Z_1, we shall determine the cluster set of Z_1. Since Ψ_1 is a bounded function, by the Fatou theorem, $\Psi_1(u+iv)$ converges as $v>0$ (resp. $v<0$) tends to 0 for almost all fixed u on $[0,\varepsilon)$. By (iii), the modulus of the limit must be equal to $\sqrt{u}$ for u on E. We denote the limit by $\sqrt{u}\,e^{i\theta_+(u)}$ (resp. $\sqrt{u}\,e^{i\theta_-(u)}$). If $\sqrt{u}\,e^{i\theta_+(u)}=\sqrt{u}\,e^{i\theta_-(u)}$ a.e. on E, then, by the generalized Painlevé theorem, Ψ_1 can be extended holomorphically onto B_ε. Since $z=0$ is a nonisolated point of Γ, $w=0$ is also a nonisolated point of E, and so the extended holomorphic function, we denote it again by Ψ_1, satisfies $|\Psi_1(u)|=\sqrt{u}$ as $u\in E$ tends to $w=0$. This contradicts that $|\Psi_1(w)|=O(|w|)$ if Ψ_1 is holomorphic at $w=0$.

Hence

$$P=\{u\in E;\ \sqrt{u}\,e^{i\theta_+(u)} \text{ and } \sqrt{u}\,e^{i\theta_-(u)} \text{ exist and } \sqrt{u}\,e^{i\theta_+(u)}\neq\sqrt{u}\,e^{i\theta_-(u)}\}$$

has positive linear measure. Since

$$\lim_{v\to+0} Z_1(u+iv)=e^{i(\theta_+(u)-\theta_-(u))}\neq 1$$

and

$$\lim_{v\to-0} Z_1(u+iv)=e^{-i(\theta_+(u)-\theta_-(u))}=1/e^{i(\theta_+(u)-\theta_-(u))}\neq 1$$

for $u\in P$, $Z_1=-1$ if Z_1 is a constant function. Now take a smaller $\varepsilon>0$ so that $\varepsilon\in P$ and $Z_1(w)\neq 1$ on $\partial B_\varepsilon\setminus\{\varepsilon\}$, and fix it. We can take such an ε, because Z_1 is not the constant function with value 1. We denote again by Z_1 the restriction of Z_1 onto $B_\varepsilon\setminus E$ and denote again by E the intersection of E and B_ε. Then $\cup\{C_{Z_1}(\zeta);\zeta\in\partial B_\varepsilon\setminus\{\varepsilon\}\}$ is a real analytic arc J which does not contain 1. The Lindelöf theorem asserts that the existence of a limit $\sqrt{u}\,e^{i\theta_+(u)}$ (resp. $\sqrt{u}\,e^{i\theta_-(u)}$) at u of the function Ψ_1 implies the existence of the limit for any non-tangential approach to u. For the Lindelöf theorem, see e.g. Theorem 2.3 of Collingwood and Lohwater [4]. In particular, $\Psi_1(u+iv)$ converges to $\sqrt{\varepsilon}\,e^{i\theta_+(\varepsilon)}$ (resp. $\sqrt{\varepsilon}\,e^{i\theta_-(\varepsilon)}$) as $u+iv\in\partial B_\varepsilon$ with $v>0$ (resp. $v<0$) tends to ε. Hence J has two endpoints $e^{i(\theta_+(\varepsilon)-\theta_-(\varepsilon))}\neq 1$ and $e^{-i(\theta_+(\varepsilon)-\theta_-(\varepsilon))}\neq 1$. If $\zeta\in E\cup\{\varepsilon\}\setminus\{0\}$, then $C_{Z_1}(\zeta)\subset\partial B_1$, because $|\Psi_1(w)|$ tends to $\sqrt{\zeta}$ as w tends to $\zeta\in E\cup\{\varepsilon\}\setminus\{0\}$.

Since

$$|Z_1(w)|^2 = \left|\frac{\Psi_1(w)\overline{\Psi_1}(w)}{w}\right|^2 = \left|\frac{\Psi_1(w)^2}{w}\right| \left|\frac{\Psi_1(\bar{w})^2}{\bar{w}}\right|$$

and

$$\frac{\Psi_1(w)^2}{w} = \frac{z^2}{\Phi_1(z)} = \frac{z^2}{F(z)} = \frac{z}{S(z)},$$

by Proposition 3.2, we see that

$$\lim_{w \in B_\varepsilon \setminus E,\, w \to 0} |Z_1(w)| = 1.$$

Hence $C_{Z_1}(0) \subset \partial B_1$, and so $C_{Z_1} \subset J \cup \partial B_1$.

Take a small disk $B_\eta(1)$ with center 1 so that $J \cap B_\eta(1) = \emptyset$. By Lemma 1.2, we see that $Z_1^{-1}(B_\eta(1) \setminus \partial B_1)$ has at most a finite number of connected components, say n. Now we shall show that E has at most $n+1$ components. Assume that E has more than $n+1$ components. Take $n+1$ disjoint disks $\{B^{(j)}\}_{j=1}^{n+1}$ with centers on $[0, \varepsilon)$ so that each $B^{(j)}$ contains at least one component of E and $\bigcup \partial B^{(j)} \subset B_\varepsilon \setminus E$. We can take $\{B^{(j)}\}$ so that $Z_1(w) \neq 1$ on $\bigcup \partial B^{(j)}$ and we can choose ξ with $0 < \xi < \eta$ so that $Z_1(\bigcup \partial B^{(j)}) \cap B_\xi(1) = \emptyset$. Since $Z_1(w) = |\Psi_1(w)|^2/w > 0$ for $w \in [0, \varepsilon) \setminus E$ and $|Z_1(w)| \to 1$ as $w \to E$, we see that, for each j, there is a point $w_j \in B^{(j)} \cap ([0, \varepsilon) \setminus E)$ such that $Z_1(w_j) \in B_\xi(1) \setminus \partial B_1$. This means that at least one connected component of $Z_1^{-1}(B_\xi(1) \setminus \partial B_1)$ intersects with $B^{(j)} \setminus E$. We have chosen ξ so that $Z_1(\bigcup \partial B^{(j)}) \cap B_\xi(1) = \emptyset$. Therefore each connected component of $Z_1^{-1}(B_\xi(1) \setminus \partial B_1)$ does not intersect with $\bigcup \partial B^{(j)}$, and so there is at least one connected component of $Z_1^{-1}(B_\xi(1) \setminus \partial B_1)$ in each $B^{(j)} \setminus E$. Thus the number of connected components of $Z_1^{-1}(B_\xi(1) \setminus \partial B_1)$ must be not less than $n+1$. This is a contradiction and we have proved that E has at most a finite number of components.

Since 0 is a nonisolated point of E, we can choose $\varepsilon_1 > 0$ so that $[0, \varepsilon_1) \subset E$. We take a smaller neighborhood U of 0 such that $\Phi_1 | \Omega \cap U$ is a one-to-one conformal mapping of $\Omega \cap U$ onto $B_{\varepsilon_1} \setminus [0, \varepsilon_1)$. Let $\sqrt{\Phi_1}$ be a one-to-one conformal mapping of $\Omega \cap U$ onto $H = \{\tau \in B_{\sqrt{\varepsilon_1}}; \operatorname{Im} \tau > 0\}$ and let $z = T(\tau)$ be its inverse function in H. Set

$$\tilde{T}(\tau) = \begin{cases} T(\tau), & \tau \in H \\ \overline{S(T(\bar{\tau}))}, & \bar{\tau} \in H. \end{cases}$$

Then $\tilde{T}$ is bounded and $\tilde{T}(\varrho + i\sigma)$ converges as $\sigma > 0$ or $\sigma < 0$ tends to 0 for almost all fixed ϱ in $(-\sqrt{\varepsilon_1}, \sqrt{\varepsilon_1})$. If the limit $\lim_{\sigma \to +0} \tilde{T}(\varrho + i\sigma) = \lim_{\sigma \to +0} T(\varrho + i\sigma)$ exists and is equal to

ζ, then it is on Γ and

$$\lim_{\sigma\to -0}\tilde{T}(\varrho+i\sigma)=\lim_{\sigma\to +0}\overline{S(T(\varrho+i\sigma))}=\overline{S(\zeta)}=\zeta.$$

Hence, by the generalized Painlevé theorem, $\tilde{T}$ can be extended holomorphically onto $B_{\sqrt{\varepsilon_1}}$ and the extension, we denote it again by $\tilde{T}$, is univalent in a neighborhood of 0, because $|\tilde{T}(\varrho)|=|\varrho|$ for $\varrho\in(-\sqrt{\varepsilon_1},\sqrt{\varepsilon_1})$. Thus we can find B_δ with center $z=0$ stated in (1). $\Gamma\cap B_\delta$ is regular analytic, because it is the image of an interval in the real axis under $\tilde{T}$.

In the case of index 2, we take $Z_2(w)=\Psi_2(w)\overline{\Psi_2}(w)/w^2$ instead of Z_1, where Ψ_2 is the inverse function of Φ_2 in (2) of Proposition 5.1. Ψ_2 is defined in $B_\varepsilon\setminus E$ and E is a relatively closed subset of $(-\varepsilon,\varepsilon)$. By the same argument as above, we see that $\lim_{v\to +0}\Psi_2(u+iv)=|u|e^{i\theta_+(u)}$ and $\lim_{v\to -0}\Psi_2(u+iv)=|u|e^{i\theta_-(u)}$ exist a.e. on E.

If $|u|e^{i\theta_+(u)}=|u|e^{i\theta_-(u)}$ a.e. on $E\cap(-\eta,\eta)$ for some $\eta>0$, then Ψ_2 can be extended holomorphically onto B_η and the extension is univalent in a neighborhood of $w=0$, because $||u|e^{i\theta_+(u)}|=|u|$ on E and $w=0$ is a nonisolated point of E. Hence this is the case (2a).

Assume next that $P\cap(-\eta,\eta)$ has positive linear measure for every η with $0<\eta<\varepsilon$, where

$$P=\{u\in E;\ |u|e^{i\theta_+(u)} \text{ and } |u|e^{i\theta_-(u)} \text{ exist and } |u|e^{i\theta_+(u)}\neq|u|e^{i\theta_-(u)}\}.$$

Since $P\cap(-\varepsilon,0)$ or $P\cap(0,\varepsilon)$ has positive linear measure, we may assume that $P\cap(0,\varepsilon)$ has positive linear measure. If $P\cap(-\varepsilon,0)$ has zero linear measure, then Ψ_2 can be extended holomorphically onto $(-\varepsilon,0)$, and so Z_2 can be also extended holomorphically onto $(-\varepsilon,0)$. We denote again the extension by Z_2. We take a smaller $\varepsilon>0$ so that $\varepsilon\in P$ and $Z_2(w)\neq 1$ on $\partial B_\varepsilon\setminus\{\varepsilon\}$. We restrict Z_2 onto $B_\varepsilon\setminus E$. If $P\cap(-\varepsilon,0)$ has positive linear measure, then take first $\varepsilon'\in P\cap(-\varepsilon,0)$ and take next $\varepsilon''\in P\cap(0,\varepsilon)$ so that $Z_2(w)\neq 1$ on $\partial B_{\varepsilon_1}(u_0)\setminus\{\varepsilon',\varepsilon''\}$, where $u_0=(\varepsilon'+\varepsilon'')/2$ and $\varepsilon_1=(\varepsilon''-\varepsilon')/2$. We denote again by Z_2 the restriction of Z_2 onto $B_{\varepsilon_1}(u_0)\setminus E$. In both cases, we see that $C_{Z_2}\subset J\cup\partial B_1$ for a real analytic arc J or the union J of two real analytic arcs such that $1\notin J$, because

$$\frac{\Psi_2(w)^2}{w^2}=\frac{z^2}{\Phi_2(z)^2}=\frac{z^2}{F(z)}=\frac{z}{S(z)}$$

and

$$\lim_{w\in B_\varepsilon\setminus E,\ w\to 0}|Z_2(w)|=1.$$

Thus, by the same argument as above, we see that E has at most a finite number of components in a neighborhood of $w=0$. Since 0 is a nonisolated boundary point of E, there is a relatively closed interval in $(-\varepsilon, \varepsilon)$ which is a component of E and contains 0. There are two possibilities: 0 is an interior point of the interval or 0 is an endpoint of the interval.

In the former case, take a smaller ε so that $(-\varepsilon, \varepsilon)\subset E$ and set $H=\{w\in B_\varepsilon; \operatorname{Im} w>0\}$. Let

$$\tilde{T}_1(w)=\begin{cases}\Psi_2(w), & w\in H\\ \overline{S(\Psi_2(\bar{w}))}, & \bar{w}\in H.\end{cases}$$

Then $\tilde{T}_1$ can be extended holomorphically onto B_ε and the extension is univalent in a neighborhood of 0. The same holds for

$$\tilde{T}_2(w)=\begin{cases}\overline{S(\Psi_2(\bar{w}))}, & w\in H\\ \Psi_2(w), & \bar{w}\in H.\end{cases}$$

We note that $\tilde{T}_1\neq\tilde{T}_2$. Indeed, if $\tilde{T}_1=\tilde{T}_2$, then $\lim_{v\to+0}\Psi_2(u+iv)=\lim_{v\to-0}\Psi_2(u+iv)$ a.e. on $(-\varepsilon, \varepsilon)$ and contradicts that $P\cap(-\varepsilon, \varepsilon)$ is of positive measure. Thus this is the case (2 b).

Now we discuss the final case. We may assume that 0 is an endpoint of an interval in the positive real axis, namely, we may assume that $[0, \varepsilon)\subset E$ and $E\cap(-\varepsilon, 0)=\emptyset$ for some $\varepsilon>0$. We apply the same argument as in the proof of the case (1) replacing $\sqrt{\Phi_1}$ with $\sqrt{\Phi_2}$. Let T be the inverse function of $\sqrt{\Phi_2}$ and let $\tilde{T}$ be its extension onto $B_{\sqrt{\varepsilon}}$. In this case, $\tilde{T}$ is not univalent in a neighborhood of 0 but $\tilde{T}$ has a zero of order two at 0, because $|\tilde{T}(\varrho)|=\varrho^2$ for $\varrho\in(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$. By the definition, $\tilde{T}$ is univalent in $H=\{\tau\in B_{\sqrt{\varepsilon}}; \operatorname{Im}\tau>0\}$. If $\tilde{T}$ is not univalent on $\bar{H}$ for every ε, then $\tilde{T}((0, \sqrt{\varepsilon}))$ and $\tilde{T}((-\sqrt{\varepsilon}, 0))$ are on the same real analytic simple arc. Since $|\tilde{T}(\varrho)|=\varrho^2$ for $\varrho\in(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$, this implies that $\tilde{T}(-\varrho)=\tilde{T}(\varrho)$ for $\varrho\in[0, \sqrt{\varepsilon})$. Hence $\lim_{v\to+0}\Psi_2(u+iv)=\lim_{v\to-0}\Psi_2(u+iv)$ a.e. on $[0, \varepsilon)$ and contradicts that $P\cap[0, \varepsilon)$ is of positive measure. Thus $\tilde{T}$ is univalent on $\bar{H}$ for some ε. This is the case (2 c).

Next we shall show the converse. Let Ω and Γ be as in the case (1). Then there is a holomorphic and univalent function T defined in a disk B_ε such that $T(0)=0$, $T((-\varepsilon, \varepsilon))\subset\Gamma$ and $T(H)\subset\Omega$, where $H=\{\tau\in B_\varepsilon; \operatorname{Im}\tau>0\}$. We note that $S(\tau)=\tau$ is the Schwarz function of $H\cup(-\varepsilon, \varepsilon)$ in B_ε. By Remark to Lemma 3.1, $S_T=\bar{T}\circ S\circ T^{-1}=\bar{T}\circ T^{-1}$ is the Schwarz function of $(\Omega\cap B_r)\cup(\Gamma\cap B_r)$ in B_r for some $r>0$.

The same argument is valid for the cases (2 a) and (2 b). The argument is also valid for the case (2 c). Indeed, T has a zero of order two at 0 in this case, but it is univalent in H. Thus $\bar{T}\circ T^{-1}$ is well-defined in $\Omega\cap B_r$ for some $r>0$ and it is the Schwarz function.

The Schwarz function can be extended from one side of the boundary to a neighborhood of 0 in the cases (1), (2 a) and (2 b), because it is expressed as $\bar{T}\circ T^{-1}$ and T^{-1} can be extended from one side of the boundary to a neighborhood of 0. If (2 c) holds, then the Schwarz function S is not univalent in $\Omega\cap B_\delta$ for any $\delta>0$, because $S(\zeta)=\bar{\zeta}$ on $\Gamma\cap B_\delta$. If S can be extended onto a neighborhood of 0, then it must be univalent in a neighborhood of 0, because $\lim_{z\to 0}|S(z)/z|=1$. Hence the Schwarz function can not be extended onto any neighborhood of 0 if (2 c) holds. Q.E.D.

Remark. In the case of (1) we have constructed a holomorphic function T, which was denoted by $\tilde{T}$, in B_ε such that $T(0)=0$, $T((-\varepsilon,\varepsilon))\subset\Gamma$ and $T(H)\subset\Omega$, where $H=\{\tau\in B_\varepsilon;\ \operatorname{Im}\tau>0\}$. The function T is canonical in the sense that T satisfies $|T(\varrho)|=|\varrho|$ on $(-\varepsilon,\varepsilon)$. The Schwarz function S can be expressed as $\bar{T}\circ T^{-1}$ by T. We note here that S can be expressed as $\bar{T}\circ T^{-1}$ by any holomorphic function T in B_ε such that $T(0)=0$, $T((-\varepsilon,\varepsilon))\subset\Gamma$ and $T(H)\subset\Omega$. The same holds for the other cases.

Definition 5.1. We call the origin 0 a regular point of Γ if (1) holds, a nonisolated degenerate point if (2 a) holds, a double point if (2 b) holds and a cusp point if (2 c) holds. We call the origin 0 a degenerate point if 0 is an isolated point of Γ or (2 a) holds.

COROLLARY 5.3. *Let S be the Schwarz function of $\Omega\cup\Gamma$ in B_1 and assume that 0 is a nonisolated point of Γ. If 0 is a regular, nonisolated degenerate or double point, then*

$$|S(z)-a_1 z|\leq\alpha|z|^2 \quad on\ (\Omega\cup\Gamma)\cap B_\delta \tag{5.3}$$

for some α and $\delta>0$, where $a_1=\bar{s}^2$, and s denotes the unit vector with initial point at 0 and tangent to Γ or the arc determined by Γ. In particular,

$$\Gamma\cap B_\delta\subset\{\zeta\in B_\delta;|\zeta-rs|\leq\alpha r^2\}\cup\{\zeta\in B_\delta;|\zeta+rs|\leq\alpha r^2\},$$

where $r=|\zeta|$.

If 0 is a cusp point, then

$$|S(z)-a_1 z|\leq\alpha|z|^{3/2} \quad on\ (\Omega\cup\Gamma)\cap B_\delta \tag{5.4}$$

for some α and $\delta>0$, where $a_1=\bar{s}^2$, and s denotes the unit vector with initial point at 0, tangent to Γ and pointing into Ω. In particular,

$$\Gamma\cap B_\delta\subset\{\zeta\in B_\delta;|\zeta+rs|\leq\alpha r^{3/2}\}$$

where $r=|\zeta|$.

Proof. Assume that 0 is a regular or nonisolated degenerate point. We denote again by S the extension of the Schwarz function onto a neighborhood of 0. Let

$$S(z) = a_1 z + a_2 z^2 + \dots$$

be the Taylor expansion of S at 0. Since

$$a_1 = S'(0) = \lim_{\zeta \in \Gamma \cap B_\delta, \zeta \to 0} S(\zeta)/\zeta = \lim \bar{\zeta}/\zeta,$$

$|a_1| = 1$ and $a_1 = \bar{s}^2$, where s denotes the unit tangent vector of Γ or of the arc determined by Γ with initial point at 0. If 0 is a double point, the same holds for another extension of the Schwarz function. The two extensions are distinct, but their first coefficients a_1 are identical, because $a_1 = \bar{s}^2$. Thus (5.3) holds in these cases. For ζ on $\Gamma \cap B_\delta$, we obtain

$$|\bar{\zeta} - \bar{s}^2 \zeta| = |S(\zeta) - a_1 \zeta| \leqslant ar^2.$$

Since

$$|\zeta - rs|\,|\zeta + rs| = |\zeta^2 - r^2 s^2| = r|\zeta - \bar{\zeta} s^2| = r|\bar{\zeta} - \zeta \bar{s}^2|,$$

we obtain

$$|\zeta - rs|\,|\zeta + rs| \leqslant ar^3,$$

and so

$$|\zeta - rs| \leqslant ar^2 \quad \text{or} \quad |\zeta + rs| \leqslant r.$$

In the latter case, it follows that $|\zeta - rs| \geqslant r$, and so

$$|\zeta + rs| \leqslant \frac{ar^3}{|\zeta - rs|} \leqslant ar^2.$$

Assume next that 0 is a cusp point. We use the notation as in the proof of Theorem 5.2. Let $\sqrt{\Phi_2}$ be a one-to-one conformal mapping of $\Omega \cup U$ onto $H = \{\tau \in B_{\sqrt{\varepsilon}}; \operatorname{Im} \tau > 0\}$ and let $z = T(\tau)$ be its inverse function defined in H. Let $\tilde{T}$ be its extension onto $B_{\sqrt{\varepsilon}}$. Since $|\tilde{T}(\varrho)| = \varrho^2$ for $\varrho \in (-\sqrt{\varepsilon}, \sqrt{\varepsilon})$, the Taylor expansion of $\tilde{T}$ at $\tau = 0$ is of the form

$$z = \tilde{T}(\tau) = -s(\tau^2 + b_3 \tau^3 + \dots),$$

where s denotes the unit tangent vector of Γ with initial point at 0 pointing into Ω. The right-hand side of the equality can be written as $-s\tau^2 f(\tau)$, where $f(\tau)$ is a holomorphic function in a neighborhood of 0 with $f(0)=1$. Hence it follows that $\sqrt{z}=\sqrt{-s}\,\tau\sqrt{f(\tau)}$, where $\sqrt{z}$ in Ω denotes an appropriately chosen branch and $\sqrt{f(\tau)}$ denotes a branch defined in a neighborhood of $\tau=0$ satisfying $\sqrt{f(0)}=1$. The function $\sqrt{f(\tau)}$ is holomorphic, and so $\sqrt{z}$ is a holomorphic function of τ, say $g(\tau)$, in a neighborhood of 0 satisfying $g(0)=0$ and $g'(0)=\sqrt{-s}$. Hence

$$\tau = g^{-1}(\sqrt{z}) = (1/\sqrt{-s})\sqrt{z}(1+c_2\sqrt{z}+c_3 z+\dots).$$

Since $\tau^2=w=\Phi_2(z)=\sqrt{zS(z)}$, we obtain

$$zS(z) = \tau^4 = \frac{z^2}{s^2}(1+4c_2\sqrt{z}+\dots),$$

and so

$$S(z) = \frac{z}{s^2}(1+4c_2\sqrt{z}+\dots).$$

Hence (5.4) holds. For ζ on $\Gamma\cap B_\delta$ with small $\delta>0$, we obtain $|\zeta-rs|\geqslant r$ in this case. Hence

$$|\zeta+rs| \leqslant \frac{\alpha r^{3/2+1}}{|\zeta-rs|} \leqslant \alpha r^{3/2}. \qquad \text{Q.E.D.}$$

We note that we also obtain (5.3) and (5.4) by using the expression $S=\bar{T}\circ T^{-1}$ of the Schwarz function. Next we show

COROLLARY 5.4. *Let S be the Schwarz function of $\Omega\cup\Gamma$ in B_1. Then both $\lim_{z\in\Omega, z\to\zeta}(S(z)-\bar{\zeta})/(z-\zeta)$ and $\lim_{z\in\Omega, z\to\zeta}S'(z)$ exist and are identical for every ζ on Γ. They are equal to $S'(\zeta)$ if ζ is an isolated point of Γ and are equal to $s(\zeta)^{-2}$ if ζ is a nonisolated point of Γ, where $s(\zeta)$ denotes the unit vector with initial point at ζ and tangent to Γ or the arc determined by Γ.*

Proof. For an isolated point ζ on Γ, the limits exist and are equal to $S'(\zeta)$, because $S(\zeta)=\bar{\zeta}$. If 0 is a nonisolated point of Γ, then, by the proof of Corollary 5.3, we see that

$$\lim_{z\in\Omega, z\to 0} S(z)/z = \lim_{z\in\Omega, z\to 0} S'(z) = s(0)^{-2}.$$

The corollary follows from equalities $(S(z)-\bar{\zeta})/(z-\zeta)=S_t(\tau)/\tau$ and $S'(z)=(dS_t/d\tau)(\tau)$, where $\tau=z-\zeta$ and $S_t(\tau)=S(\tau+\zeta)-\bar{\zeta}$ denotes the Schwarz function of $(\Omega-\zeta)\cup(\Gamma-\zeta)$ at 0. Q.E.D.

Further we shall show two corollaries which follows immediately from Theorem 5.2 and Corollary 5.4.

COROLLARY 5.5. *Let S be the Schwarz function of* $\Omega \cup \Gamma$ *in* B_1. *Set*

$$\tilde{S}(z) = \begin{cases} S(z) & \text{in } \Omega \\ \bar{z} & \text{on } B_1 \setminus \Omega. \end{cases}$$

Then $\tilde{S}$ *is a Lipschitz continuous function on* $\overline{B_\delta}$ *for every* δ *less than* 1 *and satisfies*

$$\frac{\partial}{\partial z}\tilde{S}(z) = \begin{cases} S'(z) & \text{in } \Omega \\ 0 & \text{a.e. on } B_1 \setminus \Omega, \end{cases}$$

$$\frac{\partial}{\partial \bar{z}}\tilde{S}(z) = \begin{cases} 0 & \text{in } \Omega \\ 1 & \text{a.e. on } B_1 \setminus \Omega \end{cases}$$

in the sense of distributions.

COROLLARY 5.6. *Let S be the Schwarz function of* $\Omega \cup \Gamma$ *in* B_1. *If* $B_{\delta_1} \setminus \Omega$ *has no interior points for some* $\delta_1 > 0$, *in particular, if the area of* $B_{\delta_1} \setminus \Omega$ *is equal to zero for some* $\delta_1 > 0$, *then S is holomorphic in* B_{δ_1}. *Furthermore, if* 0 *is a nonisolated point of* Γ, *there is a* δ_2 *with* $0 < \delta_2 < \delta_1$ *such that*

$$\Gamma \cap B_{\delta_2} \subset \{\zeta \in B_{\delta_2};\ |S'(\zeta)| = 1\}.$$

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