Acta Math., 166 (1991), 223–261

On isotropy irreducible Riemannian manifolds

by

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Introduction

A connected Riemannian manifold (M, g) is said to be isotropy irreducible if for each point $p \in M$ the isotropy group H_p, i.e. all isometries of g fixing p, acts irreducibly on T_pM via its isotropy representation. This class of manifolds is of great interest since they have a number of geometric properties which follow immediately from the definition. By Schur's lemma the metric g is unique up to scaling among all metrics with the same isometry group. By the same argument, the Ricci tensor of g must be proportional to g, i.e. g is an Einstein metric. Furthermore, according to a theorem of Takahashi [Ta], every eigenspace of the Laplace operator of (M, g) with eigenvalue $\lambda \neq 0$ and of dimension $k+1$ gives rise to an isometric minimal immersion into $S^k(r)$ with $r^2 = \dim M/\lambda$, by using the eigenfunctions as coordinates (see Li [L] and § 6 of this paper for further properties of these minimal immersions). By a theorem of D. Bleecker [Bl], these metrics can also be characterised as being the only metrics which are critical points for every natural functional on the space of metrics of volume 1 on a given manifold.

From the definition it follows easily that the isometry group of g must act transitively on M. Hence (M, g) is also a Riemannian homogeneous space. Conversely, we can define a connected effective homogeneous space G/H to be isotropy irreducible if H is compact and Ad_H acts irreducibly on $\mathfrak{g}/\mathfrak{h}$. Given an isotropy irreducible homogeneous space G/H, there exists a G-invariant metric g, unique up to scaling, such that (M, g) is isotropy irreducible in the first sense. But if we start with a Riemannian manifold (M, g) which is isotropy irreducible, it can give rise to several

(1) The first author acknowledges partial support from the Natural Sciences and Engineering Research Council of Canada.
(2) The second author acknowledges partial support from the National Science Foundation.

isotropy irreducible homogeneous spaces G/H since G does not have to be the full isometry group of g. The aim of this paper is to classify the Riemannian manifolds as well as the homogeneous spaces which are isotropy irreducible.

If the identity component H_0 of H also acts irreducibly on $\mathfrak{g}/\mathfrak{h}$, then G/H is called a strongly isotropy irreducible homogeneous space, and similarly for strongly isotropy irreducible Riemannian manifolds.

The most important examples of strongly isotropy irreducible homogeneous spaces are the irreducible symmetric spaces, classified by E. Cartan [C 1,2] in 1926. The non-symmetric strongly isotropy irreducible homogeneous spaces were classified independently by O. V. Manturov [Ma 1,2,3] in 1961, by J. A. Wolf [Wo 1] in 1968, and by M. Krämer [K] in 1975, but Wolf in addition studied many of their geometric properties. Both the classification of Manturov and Wolf contain some omissions (see the correction to [Wo 1]), but the classification of Krämer is complete. For an a priori proof of this classification for quotients of the classical groups, see [WZ 2].

Generalizing a theorem of Wolf [Wo 1, Theorem 1.1] it was shown in [Be, 7.46] that a non-compact isotropy irreducible homogeneous space G/H is either flat or is a symmetric space of non-compact type. Furthermore, one easily shows that if M is an isotropy irreducible Riemannian manifold, then its universal cover is also isotropy irreducible, and with the product metric, $M\times\cdots\times M$ is also isotropy irreducible. Hence we will first assume that M is compact, de Rham irreducible and simply connected and we will prove:

THEOREM A. *Let G/H be a compact, simply connected, effective Riemannian homogeneous space which is de Rham irreducible and let G_0, H_0 be the* id-*component of G, H. Then $G/H=G_0/H_0$, and if G/H is isotropy irreducible but not strongly isotropy irreducible, then G_0/H_0 is listed in Table* I *and* II. *Conversely, for every entry in Table* I *and* II *there exists in general several isotropy irreducible G/H.*

This theorem completely describes the simply connected, compact, isotropy irreducible and de Rham irreducible Riemannian manifolds which are not strongly isotropy irreducible. In most cases one can also easily read off the possibilities for G and H from the other columns in Table I and II. χ_0 describes the isotropy representation of H_0 on $\mathfrak{g}/\mathfrak{h}$ and enables one to determine the action of $\hat{H}/H_0$ on $\mathfrak{g}/\mathfrak{h}$, where $\hat{H}$ is the full isotropy group of the metric (see § 3 for details). The entry $(\hat{H}/H_0)_{\min}$ is a subgroup of minimal order (not necessarily unique) which is needed to make G_0/H_0 isotropy irreducible. In a few cases there are some finite group problems involved in trying to determine all possibilities for G, which we do not try to resolve. E.g. for the biinvariant

No.	G_0	H_0	χ_0	$\hat{H}/H_0$	$(\hat{H}/H_0)_{\min}$	$N_{G_0}(H_0)/H_0Z(G_0)$
1	G_0 compact, simple, centerless; all roots of $\mathfrak{g}_0$ same length	maximal torus T	root space decomposition	$W \rtimes D$ D diagram automorphisms of $\mathfrak{g}_0$	?	W
2	$SU(nk)/\mathbf{Z}_{nk}$ $n\geq 3, k\geq 2$	$S(U(k)\times\cdots\times U(k))/\mathbf{Z}_{nk}$ n factors	$\sum_{i<j}[\mathrm{id}\widehat{\otimes}\cdots\widehat{\otimes}\mu_k\widehat{\otimes}\cdots\widehat{\otimes}\mu_k^*\widehat{\otimes}\cdots\widehat{\otimes}\mathrm{id}]_{\mathbf{R}}$	$(\mathbf{Z}_2)\times S_n$	?	S_n
3	$SO(nk)/\mathbf{Z}_2$ $n\geq 3, k\geq 3$ even	$[SO(k)\times\cdots\times SO(k)]/\mathbf{Z}_2$ n factors	$\sum_{i<j}[\mathrm{id}\widehat{\otimes}\cdots\widehat{\otimes}\varrho_k\widehat{\otimes}\cdots\widehat{\otimes}\varrho_k\widehat{\otimes}\cdots\widehat{\otimes}\mathrm{id}]$	$(\mathbf{Z}_2)^n \rtimes S_n$	?	$(\mathbf{Z}_2)^{n-1}\rtimes S_n$
4	$Sp(nk)/\mathbf{Z}_2$ $n\geq 3, k\geq 1$	$[Sp(k)\times\cdots\times Sp(k)]/\mathbf{Z}_2$ n factors	$\sum_{i<j}[\mathrm{id}\widehat{\otimes}\cdots\widehat{\otimes}\nu_{2k}\widehat{\otimes}\cdots\widehat{\otimes}\nu_{2k}\widehat{\otimes}\cdots\widehat{\otimes}\mathrm{id}]$	S_n	?	S_n
5	F_4	Spin(8) $H_0\subset$ Spin(9) $\subset F_4$	$\overset{1}{\circ}-\circ<^{\circ}_{\circ}\oplus\circ-\circ<^{\circ\,1}_{\circ}\oplus\circ-\circ<^{\circ}_{\circ\,1}$	S_3 diagram automorphisms of $\mathfrak{h}$	$\mathbf{Z}_3\subset S_3$	S_3
6	$E_6/\mathbf{Z}_3$	$[\mathrm{Spin}(8)\times U(1)\times U(1)]/(\mathbf{Z}_4\times\mathbf{Z}_2)$ $H_0\subset[\mathrm{Spin}(10)\times U(1)]/\triangle\mathbf{Z}_4$ $\subset E_6/\mathbf{Z}_3$	$[\overset{1}{\circ}-\circ<^{\circ}_{\circ}\widehat{\otimes}\overset{2}{t}\widehat{\otimes}\overset{0}{t}]_{\mathbf{R}}$ $\oplus[\circ-\circ<^{\circ\,1}_{\circ}\widehat{\otimes}\overset{1}{t}\widehat{\otimes}\overset{1}{t}]_{\mathbf{R}}$ $\oplus[\circ-\circ<^{\circ}_{\circ\,1}\widehat{\otimes}\overset{-1}{t}\widehat{\otimes}\overset{1}{t}]_{\mathbf{R}}$	$S_3\times\mathbf{Z}_2$	$\mathbf{Z}_3\subset S_3$	S_3 diagonal in $S_3\times\mathbf{Z}_2$
7	$E_7/\mathbf{Z}_2$	$[\mathrm{Spin}(8)\times 3\,SU(2)]/\mathbf{Z}_2^3$ $H_0=[\mathrm{Spin}(8)\times\mathrm{Spin}(4)\times SU(2)]/\mathbf{Z}_2^3$ $\subset[\mathrm{Spin}(12)\times SU(2)]/\mathbf{Z}_2^2\subset E_7/\mathbf{Z}_2$	$\overset{1}{\circ}-\circ<^{\circ}_{\circ}\widehat{\otimes}\overset{1}{\circ}\widehat{\otimes}\overset{1}{\circ}\widehat{\otimes}\circ$ $\oplus\circ-\circ<^{\circ\,1}_{\circ}\widehat{\otimes}\overset{1}{\circ}\widehat{\otimes}\circ\widehat{\otimes}\overset{1}{\circ}$ $\oplus\circ-\circ<^{\circ}_{\circ\,1}\widehat{\otimes}\circ\widehat{\otimes}\overset{1}{\circ}\widehat{\otimes}\overset{1}{\circ}$	S_3	$\mathbf{Z}_3$	S_3
8	$E_7/\mathbf{Z}_2$	$7\,SU(2)/\mathbf{Z}_2^4$ $H_0=[3\,\mathrm{Spin}(4)\times SU(2)]/\mathbf{Z}_2^4$ $\subset[\mathrm{Spin}(12)\times SU(2)]/\mathbf{Z}_2^2\subset E_7/\mathbf{Z}_2$	$(1234)\oplus(1256)\oplus(3456)$ $\oplus(1357)\oplus(2457)\oplus(1467)$ $\oplus(2367)$	$GL(3,2)$	$\mathbf{Z}_7$ 7-Sylow subgroup	$GL(3,2)$
9	E_8	$[SU(5)\times SU(5)]/\triangle\mathbf{Z}_5$ maximal subgroup	$[\overset{1}{\circ}-\circ-\circ-\circ\widehat{\otimes}\circ-\circ-\overset{1}{\circ}-\circ]_{\mathbf{R}}$ $\oplus[\circ-\overset{1}{\circ}-\circ-\circ\widehat{\otimes}\overset{1}{\circ}-\circ-\circ-\circ]_{\mathbf{R}}$	$\mathbf{Z}_4$	$\mathbf{Z}_4$	$\mathbf{Z}_4$
10	E_8	$[\mathrm{Spin}(8)\times\mathrm{Spin}(8)]/\triangle(\mathbf{Z}_2\times\mathbf{Z}_2)$ $H_0\subset\mathrm{Spin}(16)/\mathbf{Z}_2\subset E_8$	$\overset{1}{\circ}-\circ<^{\circ}_{\circ}\widehat{\otimes}\overset{1}{\circ}-\circ<^{\circ}_{\circ}$ $\oplus\circ-\circ<^{\circ\,1}_{\circ}\widehat{\otimes}\circ-\circ<^{\circ\,1}_{\circ}$ $\oplus\circ-\circ<^{\circ}_{\circ\,1}\widehat{\otimes}\circ-\circ<^{\circ}_{\circ\,1}$	S_3	$\mathbf{Z}_3$	S_3
11	E_8	$[4\,SU(3)]/\triangle\mathbf{Z}_3$ $H_0\subset[E_6\times SU(3)]/\triangle\mathbf{Z}_3\subset E_8$	$[\overset{1}{\circ}-\circ\widehat{\otimes}\overset{1}{\circ}-\circ\widehat{\otimes}\circ-\circ\widehat{\otimes}\overset{1}{\circ}-\circ]_{\mathbf{R}}$ $\oplus[\circ-\circ\widehat{\otimes}\overset{1}{\circ}-\circ\widehat{\otimes}\overset{1}{\circ}-\circ\widehat{\otimes}\overset{1}{\circ}-\circ]_{\mathbf{R}}$ $\oplus[\overset{1}{\circ}-\circ\widehat{\otimes}\circ-\circ\widehat{\otimes}\overset{1}{\circ}-\circ\widehat{\otimes}\overset{1}{\circ}-\circ]_{\mathbf{R}}$ $\oplus[\overset{1}{\circ}-\circ\widehat{\otimes}\overset{1}{\circ}-\circ\widehat{\otimes}\overset{1}{\circ}-\circ\widehat{\otimes}\circ-\circ]_{\mathbf{R}}$	$S_4\times\mathbf{Z}_2$	$\mathbf{Z}_4$	$S_4\times\mathbf{Z}_2$
12	E_8	$[8\,SU(2)]/\mathbf{Z}_2^4$ $H_0=[4\,\mathrm{Spin}(4)]/\mathbf{Z}_2^4$ $\subset\mathrm{Spin}(16)/\mathbf{Z}_2\subset E_8$	$(1234)\oplus(5678)\oplus(1356)\oplus(2478)$ $\oplus(1378)\oplus(2456)\oplus(1458)\oplus(2367)$ $\oplus(1467)\oplus(2358)\oplus(1257)\oplus(3468)$ $\oplus(1268)\oplus(3457)$	$T(3,2)\rtimes GL(3,2)$	$T(3,2)\rtimes\mathbf{Z}_7$ $\mathbf{Z}_7$ 7-Sylow subgroup	$T(3,2)\rtimes GL(3,2)$

Table I. G_0/H_0 simply connected, de Rham irreducible, and rank H_0=rank G_0

No.	G_0	H_0	χ_0	$\hat{H}/H_0$	$(\hat{H}/H_0)_{\min}$	$N_{G_0}(H_0)/H_0 Z(G_0)$
1	$SO(nk)$ $n \geqslant 3, k \geqslant 3$ odd	$SO(k)\times\cdots\times SO(k)$ n factors	$\sum_{i<j}[\mathrm{id}\hat{\otimes}\cdots\hat{\otimes}\varrho_k\hat{\otimes}\cdots\hat{\otimes}\varrho_k\hat{\otimes}\cdots\hat{\otimes}\mathrm{id}]$	$(\mathbf{Z}_2)^{n-1}\ltimes S_n$	?	$(\mathbf{Z}_2)^{n-1}\ltimes S_n$ if n odd $(\mathbf{Z}_2)^{n-2}\ltimes S_n$ if n even
2 (a)	$SO(n^2)$ n odd	$SO(n)\times SO(n)$	$\Lambda^2\varrho_n\hat{\otimes}(S^2\varrho_n-\mathrm{id})\oplus(S^2\varrho_n-\mathrm{id})\hat{\otimes}\Lambda^2\varrho_n$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$
2 (b)	$\mathrm{Spin}(n^2)/\mathbf{Z}_2$ $n\equiv 0 \bmod (4)$	$SO(n)/\mathbf{Z}_2\times SO(n)/\mathbf{Z}_2$	$\Lambda^2\varrho_n\hat{\otimes}(S^2\varrho_n-\mathrm{id})\oplus(S^2\varrho_n-\mathrm{id})\hat{\otimes}\Lambda^2\varrho_n$	D_8	$\mathbf{Z}_2$	D_8
2 (c)	$SO(n^2)/\mathbf{Z}_2$ $n\geqslant 3, n\equiv 2 \bmod (4)$	$SO(n)/\mathbf{Z}_2\times SO(n)/\mathbf{Z}_2$	$\Lambda^2\varrho_n\hat{\otimes}(S^2\varrho_n-\mathrm{id})\oplus(S^2\varrho_n-\mathrm{id})\hat{\otimes}\Lambda^2\varrho_n$	D_8	$\mathbf{Z}_2$ outer	$\mathbf{Z}_2\times\mathbf{Z}_2$
3 (a)	$SO(4n^2)/\mathbf{Z}_2$, n odd	$Sp(n)/\mathbf{Z}_2\times Sp(n)/\mathbf{Z}_2$	$S^2\nu_{2n}\hat{\otimes}(\Lambda^2\nu_{2n}-\mathrm{id})\oplus(\Lambda^2\nu_{2n}-\mathrm{id})\hat{\otimes}S^2\nu_{2n}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	1
3 (b)	$\mathrm{Spin}(4n^2)/\mathbf{Z}_2$, n even	$Sp(n)/\mathbf{Z}_2\times Sp(n)/\mathbf{Z}_2$	$S^2\nu_{2n}\hat{\otimes}(\Lambda^2\nu_{2n}-\mathrm{id})\oplus(\Lambda^2\nu_{2n}-\mathrm{id})\hat{\otimes}S^2\nu_{2n}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$
4	$(K\times\cdots\times K)/\triangle Z(K)$ n factors, $n\geqslant 3$ K compact, simple, simply connected	$\triangle(K/Z(K))$	$(n-1)\,\mathrm{Ad}_K$	$S_n\times D$ D diagram automorphisms of $\mathfrak{k}$	$\mathbf{Z}_n\subset S_n$	1
5	G_0 compact, simple, simply connected	1	$\dim(G_0)\cdot\mathrm{id}$	$\Gamma\subset\mathrm{Aut}(\mathfrak{g})$ Γ acting irreducibly on $\mathfrak{g}$	–	–
6	$\mathrm{Spin}(8)$	G_2 $G_2\subset\mathrm{Spin}(7)\subset\mathrm{Spin}(8)$	$\circ\equiv\overset{1}{\bullet}\oplus\circ\equiv\overset{1}{\bullet}$	S_3	$\mathbf{Z}_3$	1
7	E_7	$SO(8)/\mathbf{Z}_2$ $H_0\subset SU(8)/\mathbf{Z}_2\subset E_7$	$\overset{2}{\circ}-\circ<^{\circ}_{\circ}\oplus\circ-\circ<^{\circ\,2}_{\circ}\oplus\circ-\circ<^{\circ}_{\circ\,2}$	$(\mathbf{Z}_2\times\mathbf{Z}_2)\ltimes S_3$	$\mathbf{Z}_3\subset S_3$	$(\mathbf{Z}_2\times\mathbf{Z}_2)\ltimes S_3$

Table II. G_0/H_0 simply connected, de Rham irreducible, and $\mathrm{rank}\, H_0<\mathrm{rank}\, G_0$

metric on a connected, compact, simple Lie group G one can adjoin any finite subgroup Γ of $\mathrm{Aut}(G)$ whose natural representation on $\mathfrak{g}$ is irreducible. Then $[G\ltimes\Gamma]/\Gamma$ is an isotropy irreducible homogeneous space which is not strongly isotropy irreducible, but as a Riemannian manifold it is of course isometric to G. This Riemannian manifold can also be presented as $[G\times G]/\triangle G$, in which case it becomes a strongly isotropy irreducible symmetric space.

If M is simply connected and isotropy irreducible, but not de Rham irreducible, then M is either flat or isometric to a Riemannian product $N\times\cdots\times N$, where N is isotropy irreducible and de Rham irreducible. In the flat case one can write M in many ways as an isotropy irreducible homogeneous space: $M=[\mathbf{R}^n\ltimes H]/H$, where H acts irreducibly on $\mathbf{R}^n$.

The only entry in Table I and II for which G_0 is not the full id-component of the isometry group is again a compact simple Lie group with a biinvariant metric.

In § 6 we describe how to determine the isotropy irreducible Riemannian manifolds which are not simply connected. This uses the column $N_{G_0}(H_0)/H_0Z(G_0)$ in Table I and II. There are again some finite group problems, which we will not solve completely.

In § 1 we make some general remarks about isotropy irreducible manifolds and describe the most interesting examples we obtain in our classification. In § 2 we discuss the general theory of isotropy irreducible manifolds and reduce the classification to the case where G is compact and simple. In that case the negative of the Killing form of G induces the standard homogeneous metric on G/H, which must be Einstein by the above remarks. Such Einstein metrics were classified in [WZ 1]. To select from them the candidates G_0/H_0 which might be isotropy irreducible as Riemannian manifolds, we derive a criterion on the isotropy representation χ_0 of H_0 on $\mathfrak{g}/\mathfrak{h}$ which turns out to be necessary and sufficient:

THEOREM B. *Let G_0/H_0 be a simply connected, compact, effective Riemannian homogeneous space with $H_0 \neq 1$ and G_0 simple. Then there exists a Riemannian homogeneous space G/H with G_0, H_0 the* id-*component of G, H and such that G/H is isotropy irreducible iff there exists a finite group of automorphisms of $\mathfrak{h}$ which permutes transitively the dominant weights of the* Ad_{H_0} *irreducible summands of* $\mathfrak{g}/\mathfrak{h}$.

In particular all irreducible summands of Ad_{H_0} are equivalent up to some (possibly outer) automorphism of $\mathfrak{h}$. Included is the possibility that all dominant weights are the same, in which case we can choose the finite group to be trivial. This actually occurs for the Examples 4 and 6 in Table II.

We are able to prove Theorem B directly only when G and H have equal rank but without the assumption that G_0 is simple. In the unequal rank case it follows from a case by case argument. In these cases we have to determine the full isometry group of G_0/H_0, in order to see whether an isotropy irreducible G/H exists. This boils down to a problem of extending automorphisms of $\mathfrak{h}$ to $\mathfrak{g}$ and is discussed in § 3. The methods in § 3 may be of independent interest since they enable one to determine the full isometry group of the standard Riemannian homogeneous metric in many circumstances, assuming that one knows the id-component of the isometry group already. They extend results of Cartan [C 1,2] for symmetric spaces and of Wolf [Wo 1] for strongly isotropy irreducible spaces. These methods are then applied in § 4 and § 5 to the equal and unequal rank cases and finish the proof of our classification.

It would be interesting to have a direct proof of Theorem B also in the unequal rank case and without the assumption that G_0 is simple. It would imply in particular that if G_0/H_0 is a homogeneous space whose isotropy representation is equal to

$\phi\oplus\cdots\oplus\phi$ for some irreducible representation ϕ, then the universal cover of G_0/H_0 is one of Example 4 or 6 in Table II. Presently this only follows if G_0 is also simple.

In § 6 we determine which subcoverings are still isotropy irreducible and discuss two applications. In the first one we describe the isotropy irreducible homogeneous spaces G^*/H^* with G^* connected, and in the second one we study some of the geometric properties of the minimal isometric immersions as in [L].

We emphasize that, whereas strongly isotropy irreducible is a local condition, isotropy irreducible is in general a global one. In many of our examples the group G/G_0 can be very large and gives rise to many global symmetry properties of the manifold G_0/H_0, most of which are reflected in symmetry properties of the isotropy representation Ad_{H_0}. We believe that these large global symmetry groups make these examples particularly appealing and may be helpful in other geometric considerations.

It may not be a coincidence that our examples frequently occur in other geometric contexts. E.g. in the classification of homogeneous spaces of positive sectional curvature [Wa], [BB] all the examples are isotropy irreducible as homogeneous spaces (although the metrics in general are not) except for the Aloff-Wallach examples $\mathrm{SU}(3)/S^1$. Many of our examples also occur in the classification of homogeneous isoparametric hypersurfaces [TT] and in the classification of 3-symmetric or k-symmetric spaces, see e.g. [WG]. In a forthcoming paper we will also discuss the close relationship between isotropy irreducible homogeneous spaces and primitive subgroups, as defined in [Go], [GR].

We would like to thank Alex Rosa for pointing out the relationship of Example 6 in § 1 with finite geometries.

§ 1. General remarks and examples

Let (M, g) be a Riemannian manifold with isometry group $\hat{G}$ and isotropy group $\hat{H}_p$ for $p \in M$. We say that (M, g) is an *isotropy irreducible Riemannian manifold* if for every $p \in M$, $\hat{H}_p$ acts irreducibly on T_pM via its isotropy representation. This implies that $\hat{G}$ must act transitively on M. Indeed, if p lies in a principal orbit of $\hat{G}$, then $\hat{H}_p$ leaves the tangent space to the principal orbit invariant, and hence this tangent space is either all of T_pM, in which case $\hat{G}$ acts transitively, or it is trivial. But the latter case is impossible since then $\hat{G}$ is discrete, which contradicts the fact that $\hat{H}_p$ is non-trivial for each $p \in M$. Hence we can also write M as a homogeneous space $\hat{G}/\hat{H}_p$.

Conversely, given a connected homogeneous space G/H, where G acts effectively on G/H and H is compact, we say that G/H is an *isotropy irreducible homogeneous* *space* if Ad_H acts irreducibly on $\mathfrak{g}/\mathfrak{h}$. There then exists a G-invariant metric g on G/H, which is uniquely determined up to scaling by Schur's Lemma, and $(M=G/H, g)$ is an isotropy irreducible Riemannian manifold in the above sense. But G does not have to be the full isometry group of (M, g). In fact even the dimension of G could be smaller than the dimension of the full isometry group. Hence an isotropy irreducible homogeneous space gives rise to an isotropy irreducible Riemannian manifold, unique up to scaling, but an isotropy irreducible Riemannian manifold can give rise to several isotropy irreducible homogeneous spaces. Since such descriptions of the same manifold as homogeneous spaces in different ways can sometimes be useful, we try to classify all isotropy irreducible homogeneous spaces in this paper. We will achieve this modulo some finite group problems.

If (M, g) is isotropy irreducible, then its universal Riemannian cover $\tilde{M}$ is also isotropy irreducible, since the isometry group of $\tilde{M}$ contains all the covering transformations, and all isometries of M lift to isometries of $\tilde{M}$. In fact, the isotropy group of $p \in M$ is isomorphic to the subgroup of the isotropy group of any $\tilde{p} \in \tilde{M}$ above p which normalizes the group of covering transformations, and the isotropy representation of one restricts to the isotropy representation of the other. But not every Riemannian subcover of an isotropy irreducible manifold is necessarily isotropy irreducible.

If we start instead with a Riemannian homogeneous space $M=G^*/H^*$ which is isotropy irreducible but not simply connected and let $\tilde{G}_0$ be the universal cover of the id-component G_0^* of G^*, then the action of G_0^* on M lifts to an action of $\tilde{G}_0$ on $\tilde{M}$ with a possible ineffective kernel $N \subset Z(\tilde{G}_0)$. The isometries in G^* which are not in G_0^* also lift to $\tilde{M}$, uniquely modulo the deckgroup. Hence these lifts, the deck group and $\tilde{G}_0/N \subset I(\tilde{M}, \tilde{g})$ generate a Lie group G which acts transitively on $\tilde{M}$ with isotropy group H. Then $\tilde{M}=G/H$ is isotropy irreducible as a homogeneous space. Clearly $\dim G = \dim G^*$, and G as well as H cannot be connected if G^*/H^* is not strongly isotropy irreducible.

Next, if N is an isotropy irreducible Riemannian manifold, then $M=N\times\cdots\times N$ (k times) with the product metric is also isotropy irreducible, since the isometry group of M contains the symmetric group S_k, which acts by interchanging the factors. Conversely it follows from the de Rham decomposition theorem that if M is isotropy irreducible, simply connected, and a Riemannian product, then M is either flat or all its de Rham factors must be isometric to each other.

Similarly in the homogeneous case. If G/H is isotropy irreducible, then the product of G/H with itself k times becomes an isotropy irreducible homogeneous space, if we adjoin to $G\times\cdots\times G$ (k times) the symmetric group S_k or a subgroup of it that still acts transitively on the factors. Conversely, if G/H is an isotropy irreducible homogeneous space which is simply connected but de Rham reducible, then it is either flat, or G_0/H_0 is the product of K/L with itself k times. To prove this last statement we need the fact, proved in § 2, that the metric on G_0/H_0 is always naturally reductive, if M is not flat. Then the de Rham decomposition theorem for naturally reductive spaces (see [KN, Theorem X.5.2]) implies that $G_0/H_0=K_1/L_1\times\cdots\times K_k/L_k$. But the factors are isometric as Riemannian manifolds and since G/H is isotropy irreducible, there exists an isometry $h_{ij}\in H$ which maps the ith factor into the jth factor. Moreover, h_{ij} normalizes G_0 and hence gives an isomorphism of K_i with K_j which carries L_i into L_j.

The flat case is somewhat special and will always be excluded. Notice though that $\mathbf{R}^n$ can be written as an isotropy irreducible homogeneous space in many ways. If we represent $\mathbf{R}^n$ as a homogeneous space, then it is of the form $[\mathbf{R}^n\ltimes H]/H$ for any closed subgroup $H\subset O(n)$ and the isotropy representation of H is given by the embedding $H\subset O(n)$. Hence $[\mathbf{R}^n\ltimes H]/H$ is isotropy irreducible iff H is an irreducible subgroup of $O(n)$. There are many such subgroups, especially since H need not be connected. Indeed H can also be a finite group.

Hence both for the homogeneous spaces and the Riemannian manifolds it will suffice to assume first that all manifolds are simply connected, de Rham irreducible and not flat. In § 6 we will discuss the non-simply connected case.

Given an isotropy irreducible $M=G/H$, since H is compact, we can write $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$, where $\mathfrak{m}$ is Ad_H invariant and can be identified with the tangent space of M at a point. The metric corresponds to an Ad_H invariant inner product on $\mathfrak{m}$. As we will see in § 2, one easily reduces to the case where G is compact and semisimple. Hence B, the negative of the Killing form of $\mathfrak{g}$, induces the standard normal homogeneous metric g_B on G/H and from now on we will assume that the metric is of this form. This also means that $\mathfrak{m}$ is chosen so that $\mathfrak{h}$ and $\mathfrak{m}$ are perpendicular with respect to B. In particular, every automorphism of G that leaves H invariant induces an isometry of G/H.

Let H_0 be the identity component of H and let $\mathfrak{m}=\mathfrak{m}_1\oplus\cdots\oplus\mathfrak{m}_k$ be a decomposition of $\mathfrak{m}$ into Ad_{H_0} irreducible summands. Since we assume that G/H is not strongly isotropy irreducible, we have $k>1$. For G/H to be isotropy irreducible H/H_0 must act via inner or outer automorphisms on G_0, leaving $H\cap G_0$ invariant, and permuting transitively the irreducible factors $\mathfrak{m}_i$, at least if all $\mathfrak{m}_i$ are inequivalent. See (2.3) for a precise statement. Hence the dominant weights of the irreducible factors $\mathfrak{m}_i$ must in particular be equivalent to each other up to some inner or outer automorphism of H.

We end this section with a number of illustrative examples.

Example 1. Let G be a compact connected simple Lie group with a biinvariant metric. As a symmetric space this is $[G\times G]/\triangle G$ and is strongly isotropy irreducible. But there are many other ways to write it as a homogeneous space, most of which are not isotropy irreducible. It will turn out that the only other way to write it as an isotropy irreducible homogeneous space is described as follows.

Let $\Gamma\subset G$ be a finite subgroup, acting on G via the adjoint representation. Then $G=[G\ltimes\Gamma]/\Gamma$ with isotropy representation $\mathrm{Ad}|_\Gamma$. Hence, if $\mathrm{Ad}|_\Gamma$ is irreducible, $[G\ltimes\Gamma]/\Gamma$ is isotropy irreducible, but not strongly isotropy irreducible. We can also use Γ to construct subcoverings of G since the isotropy representation of G/Γ is also $\mathrm{Ad}|_\Gamma$. Hence if $\mathrm{Ad}|_\Gamma$ is irreducible, G/Γ is isotropy irreducible, but not strongly isotropy irreducible.

More generally, we can take any subgroup Γ of $\mathrm{Aut}(G)$ such that the natural representation on $\mathfrak{g}$ is irreducible. Then $[G\ltimes\Gamma]/\Gamma$ becomes isotropy irreducible and if $\Gamma^*\subseteq\Gamma\cap\mathrm{Int}(G)$ is a normal subgroup of Γ, then G/Γ^* becomes isotropy irreducible if we adjoin to G the isometries induced by Γ but not in Γ^*.

This example, and the flat metric on $\mathbf{R}^n$, are the only examples of isotropy irreducible Riemannian manifolds which can be written in several ways as isotropy irreducible homogeneous spaces such that the id-components of the transitive groups are distinct.

At first sight the condition on Γ appears to be a very restrictive one. However one can find many examples satisfying this condition, including many of the sporadic finite simple groups. We first explain the condition more explicitly for the classical groups.

If $\pi\colon\Gamma\to\mathrm{SO}(n)$ is a faithful orthogonal representation of Γ, then since the adjoint representation of $\mathrm{SO}(n)$ is just the second exterior power of the standard n-dimensional representation, our condition says that $\wedge^2\pi$ is irreducible. Clearly we may assume that π is absolutely irreducible. For $\pi\colon\Gamma\to\mathrm{Sp}(n)$, since the adjoint representation of $\mathrm{Sp}(n)$ is the second symmetric power of the standard $2n$-dimensional representation, the condition becomes $S^2\pi$ is irreducible. Finally, if $\pi\colon\Gamma\to\mathrm{SU}(n)$, then the condition is that $\pi\otimes\pi^*-1$ is irreducible. Here we can assume that π is irreducible and $\pi\neq\pi^*$. Below we give some of the examples we found satisfying these conditions.

(1) The natural representation of the symmetric group S_n on $\mathbf{R}^n$ is orthogonal and splits into the trivial representation on the diagonal and an irreducible representation on the orthogonal complement. Let $\pi\colon S_n\to\mathrm{O}(n-1)$ denote this representation. Then, if $n\geq4$, $\wedge^2\pi$ is irreducible, so that $\mathrm{O}(n-1)/S_n$ is isotropy irreducible. The restriction to the alternating group A_n maps into $\mathrm{SO}(n-1)$ and yields the isotropy irreducible space $\mathrm{SO}(n-1)/A_n$ $(n\neq5)$, which as a manifold is of course the same as $\mathrm{O}(n-1)/S_n$. More

generally, if W is the Weyl group of a compact simple Lie group of rank n, it acts naturally on $\mathbf{R}^n$ and $O(n)/W$ is isotropy irreducible.

(2) A finite subgroup of SO(3) (respectively SU(2)) satisfies the isotropy irreducibility condition if it is conjugate to the tetrahedral, icosahedral, or octahedral (resp. binary tetrahedral, binary octahedral, or binary icosahedral) group. The subgroups of SO(3) of course follow from (1) since they contain the alternating group and if $\Gamma\subset\Phi\subset G$ with G/Γ isotropy irreducible, then G/Φ is also isotropy irreducible.

(3) All the Mathieu groups have irreducible orthogonal representations satisfying the isotropy irreducibility condition. For M_{11} the representations have dimension 10 and 11, for M_{12} there are two 11-dimensional representations and for M_{22}, M_{23}, M_{24} there is a representation of dimension 21, 22, 23 respectively. As for unitary representations such that $\pi\otimes\pi^*-1$ is irreducible, there are none for M_{11}, M_{12}, or M_{22}, but one each for M_{23}, M_{24}, both of dimension 45. All this is verified using character tables in [CCNPW].

(4) Among the other sporadic finite simple groups one obtains a unitary representation of dimension 1333 for the Janko group J_4, and orthogonal representations for the Baby Monster group B, the Fischer group Fi_{22}, the Harada–Norton group HN, the Thompson group Th, the McLaughlin group McL, and the Conway groups Co_2, Co_3 of dimension 4371, 78, 133, 248, 22, 23, and 23 respectively. Similarly one can easily obtain a number of orthogonal, unitary, and symplectic representations from [CCNPW] among the finite simple groups of Lie type.

Although a complete enumeration of all such groups is probably complicated, an answer to the following questions would be of interest:

(a) For each compact connected simple Lie group G, does there exist a finite subgroup $\Gamma\subset\mathrm{Aut}(G)$ such that Γ acts irreducibly on $\mathfrak{g}$?

(b) Is there a characterization of those finite groups Γ which admit a faithful representation $\pi:\Gamma\to\mathrm{Aut}(G)$ for some compact connected simple Lie group G such that Γ acts irreducibly on $\mathfrak{g}$?

Example 2. Let H be a compact, simply connected, simple Lie group with center Z and let $\triangle H$ be the diagonal subgroup of $G=H\times\cdots\times H$ (n times). The isotropy representation of $G/\triangle H$ is equal to $(n-1)\,\mathrm{Ad}_H$, where $\mathfrak{m}=\{(X_1,\ldots,X_n)|\ X_i\in\mathfrak{h}$ and $\Sigma X_i=0\}$. Notice though that the decomposition of $\mathfrak{m}$ into Ad_H irreducible summands is highly non-unique. The symmetric group S_n acts on $G=H\times\cdots\times H$ as outer automorphisms by permuting the factors. Since this action keeps $\triangle H$ invariant, it induces isometries of the standard normal homogeneous metric on $M=G/\triangle H$. If we add these isometries to G, then M becomes an isotropy irreducible homogeneous space. Notice that S_n does not interchange the $\mathrm{Ad}_{\triangle H}$ irreducible factors of any fixed decomposition of $\mathfrak{m}$, and that it acts trivially on the collection of dominant weights since they are all the same. M is simply connected, but not yet effective, $([H\times\cdots\times H]/\triangle Z)/(\triangle H/\triangle Z)$ being the effective version.

Isotropy irreducible subcoverings are obtained by dividing G by a central subgroup such that S_n still acts by isometries. E.g. $\{[(H/A)\times\cdots\times(H/A)]/\triangle Z\}/\{\triangle(H/A)/\triangle Z\}$ is isotropy irreducible for any $A\subset Z$.

M is a symmetric space if $n=2$ (and hence strongly isotropy irreducible), but is not strongly isotropy irreducible if $n>2$.

Another description of this example is obtained by observing that as a manifold $G/\triangle H$ is equivariantly diffeomorphic to $H\times\cdots\times H$ ($n-1$ times), an explicit map being given by

$$(g_1,\dots,g_n)\,\triangle H\in G/\triangle H\to(g_1g_n^{-1},\dots,g_{n-1}g_n^{-1})\in H\times\cdots\times H.$$

The isotropy irreducible metric on $G/\triangle H$ corresponds under this map to a left invariant metric on $H\times\cdots\times H$, which one easily checks is given at the identity by

$$|(X_1,\dots,X_{n-1})|^2=\sum|X_i|^2-\frac{1}{n}\left|\sum X_i\right|^2.$$

This metric can thus be viewed also as follows. Starting with the metric on $H\times\cdots\times H$ ($n-1$ times) which is the $(n-1)$-fold product of a biinvariant metric on H with itself, we can fibre $H\times\cdots\times H$ by left cosets of $\triangle H$ and multiply the metric in direction of these fibres by $(n-1)/n$, while leaving the metric unchanged in the direction perpendicular to the fibres.

If one starts with this left invariant metric on $H\times\cdots\times H$ ($n-1$ times) it is more difficult to see all the isometries that are necessary in order to make it isotropy irreducible. Besides the left translations on $H\times\cdots\times H$ one needs the isometries $(\mathrm{Ad}(g),\dots,\mathrm{Ad}(g))$, $g\in H$, the interchange of any two factors, and the special isometries

$$(g_1,\dots,g_{n-1})\to(g_1g_i^{-1},\dots,g_{i-1}g_i^{-1},g_i^{-1},g_{i+1}g_i^{-1},\dots,g_{n-1}g_i^{-1})$$

for each $i=1,\dots,n-1$.

Notice that on this manifold there exists another isotropy irreducible metric, namely the $(n-1)$-fold product of a biinvariant metric with itself. This metric is not isometric to the above example, since the metric on $G/\triangle H$ is de Rham irreducible, as follows from [KN, Theorem X.5.3] and the fact that the metric on $G/\triangle H$ is given by the standard normal homogeneous metric. Hence this manifold has two non-isometric isotropy irreducible metrics. This phenomenon does not occur among strongly isotropy irreducible manifolds.

Example 3. Let $H=G_2 \subset \mathrm{Spin}(7) \subset \mathrm{Spin}(8)=G$. The isotropy representation of G/H is $\phi_7 \oplus \phi_7$, where ϕ_7 is the seven dimensional representation of G_2. G_2 has no outer automorphisms and one easily sees that the normalizer of G_2 in Spin(8) is $Z(G)\,H$, and hence does not induce any new isometries on G/H. But there are outer automorphisms of Spin(8) which keep G_2 invariant and hence induce further isometries of G/H. For example, the triality automorphism of Spin(8) fixes G_2 and one easily checks that it does not keep the decomposition of $\mathfrak{m}$ into irreducible summands invariant. Hence, if we add this isometry to Spin(8), G/H becomes an isotropy irreducible homogeneous space. On the other hand, the order 2 automorphism keeps the decomposition $\phi_7 \oplus \phi_7$ invariant.

As a manifold G/H is diffeomorphic to $S^7 \times S^7$ (see e.g. [WZ 2, p. 575, Example 4]). But the Riemannian metric is de Rham irreducible, as follows from [KN, Corollary X.5.4]. The product metric of a round sphere metric with itself is of course also isotropy irreducible. Hence this manifold is again an example which carries two non-isometric isotropy irreducible homogeneous metrics. From our classification it seems to follow that Example 2 and 3 are the only manifolds which can carry more than one isotropy irreducible metric. Although this is most likely the case, a rigorous proof would involve showing that none of the examples in Table I and II are diffeomorphic to each other or to a strongly isotropy irreducible space or to products of such, which is not easy to do. As far as we know, even among the strongly isotropy irreducible homogeneous spaces, such a program has not been carried out.

There are several Riemannian homogeneous subcoverings, obtained by dividing Spin(8) by a subgroup of its center, but the only subcovering for which the triality automorphism descends to an isometry is that obtained by dividing by the full center, i.e. $(SO(8)/\mathbf{Z}_2)/G_2$, which is diffeomorphic to $\mathbf{R}P^7 \times \mathbf{R}P^7$.

This example also arises naturally in the following context. As a homogeneous space the Cayley plane is $F_4/\mathrm{Spin}(9)$. Then $\mathrm{Spin}(8) \subset \mathrm{Spin}(9)$ acts on the Cayley plane with principal orbits $\mathrm{Spin}(8)/G_2$. As we move through the principal orbits, the induced metric achieves, up to scaling, every Spin(8)-invariant metric on $\mathrm{Spin}(8)/G_2$. In particular, the standard normal homogeneous metric is achieved on a specific principal orbit.

Example 4. Let $H=SO(k)\times\cdots\times SO(k)$ (n times)$\subset SO(nk)=G$ be given by the usual diagonal embedding. One easily sees that $\Sigma_{i<j}[\mathrm{id}\,\hat{\otimes}\ldots\hat{\otimes}\varrho_k\hat{\otimes}\ldots\hat{\otimes}\varrho_k\hat{\otimes}\ldots\hat{\otimes}\,\mathrm{id}]$ is the

isotropy representation of H and that the outer automorphisms of H which interchange the $\mathrm{SO}(k)$ factors extend to inner automorphisms of G, which therefore lie in the normalizer of H. Thus conjugation by these elements induce isometries fixing (H) and if we add these to G, we obtain an isotropy irreducible homogeneous space.

There are further isometries which are not needed to make G/H isotropy irreducible, but which can be used to produce subcoverings that are also isotropy irreducible. They are all of the form $\mathrm{Ad}(\mathrm{diag}\{\pm 1, \ldots, \pm 1\})$.

The space G/H can also be described as the set of all oriented flags $V_1 \subset V_2 \subset \ldots \subset V_n = \mathbf{R}^{nk}$ with $\dim V_i = ik$. These flags can be represented by an element A of $\mathrm{SO}(nk)$ such that the first ik columns span V_i. If we denote the columns of A by $w_1, \ldots, w_{nk}$, then the isometry which interchanges the ith and jth factor in H, but viewed as a right translation on G/H, acts on the flags by interchanging the vectors $w_{(i-1)k+1}, \ldots, w_{ik}$ with the vectors $w_{(j-1)k+1}, \ldots, w_{jk}$. They form a group S_n of isometries which are fixed point free. Additional isometries are obtained by changing the orientation of one or several of the subspaces V_i, $i=1, \ldots, n-1$. These isometries can be viewed as right translations on G/H by $D = \mathrm{diag}\{\pm 1, \ldots, \pm 1\}$ where D contains an even number of -1 on the diagonal. This group $\mathbf{Z}_2^{n-1}$, which acts freely on G/H, gives rise to many isotropy irreducible subcoverings of G/H, including the space of unoriented flags in $\mathbf{R}^{nk}$. One can also divide out by S_n or subgroups of S_n to obtain isotropy irreducible subcoverings. This subgroup has to be chosen so that it acts transitively on the set of unordered pairs. If nk is even, there exists one additional isometry, given by reflecting a flag in a hyperplane. But the component of the full isometry group containing this isometry does not contain any fixed point free isometries. For more details see § 4 and § 5.

Similar examples arise by using $\mathrm{SU}(k)$ and $\mathrm{Sp}(k)$ instead of $\mathrm{SO}(k)$.

Example 5. If rank H=rank G, we will see that a necessary and sufficient condition for G/H to be isotropy irreducible as a Riemannian manifold is the existence of a group of outer automorphisms of $\mathfrak{h}$ which acts transitively on the set of dominant weights of the Ad_{H_0} irreducible factors of $\mathfrak{m}$. Hence for almost all the examples in Table I it is obvious that G/H is isotropy irreducible, without any computation.

One particularly interesting example is $[\mathrm{SU}(5) \times \mathrm{SU}(5)]/\triangle \mathbf{Z}_5 \subset \mathrm{E}_8$, which is the only maximal subalgebra of maximal rank such that the corresponding homogeneous space is not strongly isotropy irreducible. Hence all maximal subalgebras of maximal rank give rise to isotropy irreducible homogeneous spaces.

Another interesting example is G/T, where G is a compact, simple, centerless Lie group such that all roots of $\mathfrak{g}$ have the same length, and T is a maximal torus in G. Then the Weyl group acts transitively on all roots and hence transitively on the irreducible factors of the isotropy representation of G/T. If we add these isometries to $G, G/T$ becomes an isotropy irreducible space. Further isometries are obtained from the diagram automorphisms of $\mathfrak{g}$.

Example 6. Let $G_0=E_7$ or E_8 and $H_0=[SU(2)]^7/\mathbf{Z}_2^4$ or $[SU(2)]^8/\mathbf{Z}_2^4$. The isotropy representations are given in Table I. It turns out that in order to make G_0/H_0 isotropy irreducible, we have to add a group of automorphisms of G_0 which is a subgroup of the group of automorphisms of the projective plane (resp. affine 3-space) over the field $\mathbf{F}_2$ of 2 elements. To explain the connection, we consider first the case of E_7.

Each copy in the isotropy representation of G_0/H_0 is described by a 4-tuple of integers from 1 to 7, where e.g. (1234) corresponds to the representation

$$\overset{1}{\circ}\,\widehat{\otimes}\,\overset{1}{\circ}\,\widehat{\otimes}\,\overset{1}{\circ}\,\widehat{\otimes}\,\overset{1}{\circ}\,\widehat{\otimes}\,\circ\,\widehat{\otimes}\,\circ\,\widehat{\otimes}\,\circ$$

of $[SU(2)]^7$. We can instead describe this representation by the complementary 3-tuple of numbers (567) and thus obtain 7 such 3-tuples. These 3-tuples correspond to all possible lines on the projective plane over $\mathbf{F}_2$ as follows. Label the points in this projective plane by $1=(1,0,0)$, $2=(0,1,0)$, $3=(0,0,1)$, $4=(1,1,1)$, $5=(0,1,1)$, $6=(1,0,1)$, $7=(1,1,0)$. Then each line goes through precisely 3 points and we can label a line by these points. We again obtain 7 3-tuples, and the 3-tuples in both cases agree. An element of $\hat{H}/H_0$ corresponds to a permutation of the integers 1 to 7, which preserves the 4-tuples and hence the 3-tuples which describe the isotropy representation. Thus it is described by a transformation of the projective plane which preserves lines, and therefore by an element of $\mathrm{Aut}(P^2(\mathbf{F}_2))=PGL(3,2)=GL(3,2)$. Hence $\hat{H}/H_0=GL(3,2)$, a simple group of order $8\cdot 7\cdot 3$. The 7-Sylow subgroup $\mathbf{Z}_7$ is generated by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

or equivalently by the permutation (1254673). It permutes the lines transitively and hence, if $\Gamma\supset\mathbf{Z}_7$, then $[G_0\ltimes\Gamma]/[H_0\ltimes\Gamma]$ is isotropy irreducible.

For the case E_8 we have to relabel the circles according to the scheme $1\to 0$, $2\to 5$, $3\to 4$, $4\to 1$, $5\to 3$, $6\to 7$, $7\to 2$, and $8\to 6$. Then the isotropy representation becomes

$$(0145)\oplus(2367)\oplus(0347)\oplus(1256)\oplus(0246)\oplus(1357)\oplus(0136)\oplus$$

$$(2457)\oplus(0127)\oplus(3456)\oplus(0235)\oplus(1467)\oplus(0567)\oplus(1234)$$

We now put these 4-tuples in 1–1 correspondence with the set of 2-planes in 3-dimensional affine space over $\mathbf{F}_2$. We label the points of this space with the integers from 0 to 7 as before. Then each 2-plane contains exactly 4 points, and if we describe the 2-plane by this 4-tuple of numbers, we obtain the same set of 4-tuples as above. Hence an element of $\hat{H}/H_0$ corresponds to a transformation of affine 3-space which takes 2-planes to 2-planes, and therefore an element of $T(3,2)\ltimes GL(3,2)$, where $T(3,2)$ are the translations in $\mathbf{F}_2^3$. This group has order $2^3\cdot 168$. If $\mathbf{Z}_7$ denotes the 7-Sylow subgroup of $GL(3,2)$ as above, then $T(3,2)\ltimes\mathbf{Z}_7$ already acts transitively on the 2-planes, and one easily sees that there is no subgroup of smaller order which does. Hence, if $\Gamma\supset T(3,2)\ltimes\mathbf{Z}_7$, then $[G_0\ltimes\Gamma]/[H_0\ltimes\Gamma]$ is isotropy irreducible.

It would be interesting to obtain a better understanding of the relationship of these examples with the corresponding finite geometries.

§ 2. General theory

The first result is essentially already contained in [Be, 7.46], but we include a proof here for completeness.

(2.1) THEOREM. *Let $M=G/H$ be an isotropy irreducible Riemannian homogeneous space. Then*:

(a) *If M is non-compact, M is either flat or G/H is a symmetric space of non-compact type. In either case, M is simply connected.*

(b) *If $\mathfrak{g}$ is not semisimple, then M is flat.*

Proof. Let B be the Killing form of $\mathfrak{g}$. Since G/H is effective and H is compact, it easily follows that $B|_{\mathfrak{h}}$ is negative definite (see e.g. [Be, 7.35 and 7.36]). Hence we can choose an Ad_H invariant complement $\mathfrak{m}$ with $B(\mathfrak{h},\mathfrak{m})=0$. $B|_{\mathfrak{m}}$ is a symmetric Ad_H invariant bilinear form, and so is the metric g. Hence $B|_{\mathfrak{m}}=cg$ for some constant c. If $c>0$, then $(\mathfrak{g},\mathfrak{h})$ is a symmetric pair of non-compact type and hence so is G/H. If $c<0$, then $B<0$ and hence $\mathfrak{g}$ is semisimple and compact. If $c=0$, then $\mathfrak{m}$ is the nullspace of B and hence an ideal. In particular $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{m}$ and since $[\mathfrak{m},\mathfrak{m}]$ is Ad_H invariant, isotropy irreducibility implies $[\mathfrak{m},\mathfrak{m}]=\mathfrak{m}$ or $[\mathfrak{m},\mathfrak{m}]=0$. But if $[\mathfrak{m},\mathfrak{m}]=\mathfrak{m}$, then $\mathfrak{m}$ would be semisimple, which would contradict $B|_{\mathfrak{m}}=0$, since $\mathfrak{m}$ is an ideal. Hence $\mathfrak{m}$ is abelian, which implies that any G-invariant metric on G/H is flat. A flat homogeneous metric is isometric to the product of the flat metric on $\mathbf{R}^k$ and a flat metric on T^{n-k} (see e.g. [Wo 2, 2.7.1]). Since M is isotropy irreducible and non-compact, M must be isometric to $\mathbf{R}^n$. This proves (a) and (b). □

We already saw in § 1 that we can assume that M is simply connected and de Rham irreducible. We will do so now.

(2.2) THEOREM. *Let $M=G/H$ be a simply connected, compact, isotropy irreducible Riemannian homogeneous space which is also de Rham irreducible. If $\mathfrak{g}$ is semisimple but not simple, then G/H is the Example* 2 *of* § 1.

Proof. Let G_0 and H_0 be the id-component of G, H. Since M is compact and simply connected, G_0 acts transitively on M and the isotropy group, which must be connected, is equal to H_0. We can assume that the metric on M is the standard homogeneous metric and hence [KN, Corollary X.5.3] implies that M is de Rham reducible iff $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_k$ and $\mathfrak{h}=\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_k$ with $\mathfrak{h}_i\subset\mathfrak{g}_i$ and $k\geq 2$.

Let $\mathfrak{m}$ be an Ad_H invariant complement of $\mathfrak{h}$. If $\mathfrak{g}$ is semisimple but not simple, let $\mathfrak{g}_0$ be a non-trivial simple ideal of $\mathfrak{g}$ and let $\mathfrak{g}_1$ be the ideal $\mathrm{Ad}_H(\mathfrak{g}_0)$. If $\mathfrak{g}_1$ is not all of $\mathfrak{g}$, there exists a complementary Ad_H invariant ideal $\mathfrak{g}_2$. Decompose $\mathfrak{h}$ into ideals such that $\mathfrak{h}=\mathfrak{h}_1\oplus\mathfrak{h}_2\oplus\triangle\mathfrak{h}_3$, where $\mathfrak{h}_i\oplus\mathfrak{h}_3$ are subalgebras of $\mathfrak{g}_i$ and $\triangle\mathfrak{h}_3$ is embedded diagonally into $\mathfrak{h}_3\oplus\mathfrak{h}_3\subset\mathfrak{g}_1\oplus\mathfrak{g}_2$. Then $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\triangle\mathfrak{m}_3$, where $\mathfrak{m}_i$ are orthogonal complements of $\mathfrak{h}_i\oplus\mathfrak{h}_3$ in $\mathfrak{g}_i$ and $\triangle\mathfrak{m}_3=\{(X,-X)|X\in\mathfrak{h}_3\}\subset\mathfrak{h}_3\oplus\mathfrak{h}_3\subset\mathfrak{g}_1\oplus\mathfrak{g}_2$. $\mathfrak{h}_3\neq 0$ since otherwise M would be de Rham reducible. Hence $\triangle\mathfrak{m}_3\neq 0$ and since $\mathfrak{m}_i$ and $\triangle\mathfrak{m}_3$ are Ad_H invariant, we have $\mathfrak{m}_1=\mathfrak{m}_2=0$. Since G/H is effective, this implies $\mathfrak{h}_1=\mathfrak{h}_2=0$ and hence $\mathfrak{g}_1=\mathfrak{g}_2=\mathfrak{h}_3$ with $\mathfrak{h}$ embedded diagonally. $\mathfrak{h}$ must then be simple, since otherwise M would be de Rham reducible. Hence G/H is as in Example 2.

We can therefore assume that $\mathrm{Ad}_H(\mathfrak{g}_0)=\mathfrak{g}$. Hence $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_k$, $k\geq 2$, where $\mathfrak{g}_i$ are isomorphic simple ideals permuted transitively by Ad_H. In particular $\mathfrak{h}$ is invariant under the automorphisms which permute the $\mathfrak{g}_i$. Let π_i be the projection of $\mathfrak{g}$ onto $\mathfrak{g}_i$. Then $\mathfrak{h}^*=\pi_1(\mathfrak{h})\oplus\cdots\oplus\pi_k(\mathfrak{h})$ is Ad_H invariant and contains $\mathfrak{h}$. Hence either $\mathfrak{h}^*=\mathfrak{h}$ or $\mathfrak{h}^*=\mathfrak{g}$. If $\mathfrak{h}^*=\mathfrak{h}$ (which includes the possibility of $\mathfrak{h}=0$), then $\mathfrak{h}=\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_k$ with $\mathfrak{h}_i\subset\mathfrak{g}_i$, which implies that M is de Rham reducible. Hence $\mathfrak{h}^*=\mathfrak{g}$, in other words, $\pi_i(\mathfrak{h})=\mathfrak{g}_i$. For each simple ideal $\mathfrak{h}_1$ of $\mathfrak{h}$, $\pi_i|_{\mathfrak{h}_1}$ is either an isomorphism onto $\mathfrak{g}_i$ or trivial, and if $\mathfrak{h}_2$ is a second simple ideal of $\mathfrak{h}$, then $\pi_i|_{\mathfrak{h}_1}\neq 0$ implies $\pi_i|_{\mathfrak{h}_2}=0$ since $\mathfrak{h}_1$ and $\mathfrak{h}_2$ commute. Hence, if $\mathfrak{h}$ is not simple, M is again de Rham reducible. Thus $\mathfrak{h}$ is simple and $\pi_i|_{\mathfrak{h}}$ is an isomorphism onto $\mathfrak{g}_i$ for each i. We can therefore assume that $\mathfrak{g}=\mathfrak{h}\oplus\cdots\oplus\mathfrak{h}$ and $\mathfrak{h}$ is embedded diagonally, which is Example 2 in § 1. □

We can now assume that G is compact and simple and hence the metric is given by the standard normal homogeneous metric g_B, which also must be Einstein. In [WZ 1] we classified all simply connected effective homogeneous spaces such that G is simple, compact, and such that the standard normal homogeneous metric is Einstein. To select among these manifolds the ones that are isotropy irreducible, we prove:

(2.3) THEOREM. *Let G/H be an isotropy irreducible homogeneous space which is not strongly isotropy irreducible and assume that $\mathfrak{h}\neq 0$. Let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ be an Ad_H invariant decomposition, $\mathfrak{t}$ a maximal abelian subalgebra of $\mathfrak{h}$ and Φ a fundamental system of roots of $(\mathfrak{h},\mathfrak{t})$. Given any Ad_{H_0} irreducible summand $\mathfrak{m}_0$ of $\mathfrak{m}$, there exist elements $h_1,\dots,h_k$ of H such that $\mathfrak{m}=\mathrm{Ad}_{h_1}(\mathfrak{m}_0)\oplus\cdots\oplus\mathrm{Ad}_{h_k}(\mathfrak{m}_0)$ is a decomposition of $\mathfrak{m}$ into Ad_{H_0} irreducible summands. Furthermore $\Gamma=\{h\in H\mid \mathrm{Ad}_h(\mathfrak{t})\subset\mathfrak{t}$ and $\mathrm{Ad}_h^*\Phi\subset\Phi\}$ permutes transitively the dominant weights of the Ad_{H_0} irreducible summands of $\mathfrak{m}$.*

Proof. We choose a set of representatives $\{h_i\}$ of H/H_0 such that Ad_{h_i} leaves $\mathfrak{t}$ and Φ invariant. Then $\Sigma_i\mathrm{Ad}_{h_i}(\mathfrak{m}_0)\subset\mathfrak{m}$ is Ad_H invariant and hence $\mathfrak{m}=\Sigma_i\mathrm{Ad}_{h_i}(\mathfrak{m}_0)$. After reindexing, there is a subset $\{h_1,\dots,h_k\}$ such that $\mathfrak{m}=\oplus_i\mathrm{Ad}_{h_i}(\mathfrak{m}_0)$. If $h\in\Gamma$, $\mathfrak{m}=\mathrm{Ad}_h(\mathfrak{m})=\oplus_i\mathrm{Ad}_h(\mathrm{Ad}_{h_i}(\mathfrak{m}_0))$ is another decomposition of $\mathfrak{m}$ into Ad_{H_0} irreducible summands and hence Ad_h permutes the dominant weights. For G/H to be isotropy irreducible, Γ must permute these weights transitively. □

Remarks. As we saw in Example 2 and 3 in § 1, $\mathrm{Ad}_h, h\in\Gamma$, need not permute the irreducible summands $\mathfrak{m}_i=\mathrm{Ad}_{h_i}(\mathfrak{m}_0)$, although this will necessarily be the case if all $\mathfrak{m}_i$ are inequivalent as Ad_{H_0} representations. The same examples also show that there can be only a single dominant weight among the irreducible summands.

As a consequence of (2.3), all the Ad_{H_0} irreducible summands of $\mathfrak{m}$ must be equivalent to each other up to some inner or outer automorphism of $\mathfrak{h}$. Using this criterion, one can easily select from [WZ 1] all the candidates for isotropy irreducible spaces with G simple. The result is given in Table I and II, except that item 4 in Table II should be deleted. We will actually see that all these spaces are isotropy irreducible.

To proceed further, we have to determine the full isometry group of all the candidates. We first do this for the id-component, and then (in the next section) for the full isometry group.

(2.4) THEOREM. *Let $M=G/H$ be a simply connected isotropy irreducible homogeneous space which is not strongly isotropy irreducible. Then G_0 is the* id-*componnt of the isometry group unless*

(a) *$G/H=[\mathbf{R}^n\ltimes H]/H$ is flat with isometry group $\mathbf{R}^n\ltimes\mathrm{O}(n)$.*

(b) *$G/H=[K\ltimes\Gamma]/\Gamma$ with K connected, compact, and simply connected and*

$\Gamma \subset \mathrm{Aut}(K)$ *a finite subgroup acting irreducibly on* $\mathfrak{k}$. *Here the* id-*component of the isometry group is* $(K \times K)/\triangle Z(K)$ *acting via left and right translations on* K.

Proof. If M is flat, we are in case (a). If M is not flat, the de Rham reducible case immediately reduces to the de Rham irreducible case by the comments in § 1, and by (2.1) we can assume that G is semisimple. By [WZ 1, Theorem 5.1] and the note added in proof in [WZ 1], if G_0 is a simple Lie group, then G_0 is the id-component of the isometry group, unless we are in the case of a biinvariant metric on a compact simple Lie group, which is case (b).

If G_0 is semisimple but not simple, we are in the case of Example 2 of § 1 as follows from (2.2). For this example $G_0 = [H \times \cdots \times H]/\triangle Z(H)$ and we can argue as follows. If $\hat{G}_0$ is the id-component of the full isometry group of M, then $\hat{G}_0$ is semisimple by (2.1). If $\hat{G}_0$ is simple, it easily follows from [O, Table VII] that $\hat{G}_0 = G_0$. If $\hat{G}_0$ is not simple, then $\hat{G}_0/\hat{H}_0$ is again as in Example 2 of § 1 and hence M is diffeomorphic to $L \times \cdots \times L$ for some compact, simple, simply connected Lie group L with $\hat{G}_0 = [L \times \cdots \times L]/\triangle Z(L)$. But from $H \times \cdots \times H = L \times \cdots \times L$ it easily follows, using rational and $\mathbf{Z}_2$ cohomology rings, that both products contain the same number of factors and that $H = L$. Hence $\hat{G}_0 = G_0$. $\square$

Remark. If G/H is an irreducible symmetric space, it is well known that G_0 is the id-component of the isometry group. For strongly isotropy irreducible homogeneous spaces, Wolf showed [Wo, Theorem 7.1] that G_0 is the id-component of the isometry group, unless $G/H = \mathrm{Spin}(7)/G_2 = S^7$ with isometry group O(8) or $G/H = \mathrm{G}_2/\mathrm{SU}(3) = S^6$ with isometry group O(7).

§ 3. Extensions and restrictions of automorphisms

In this section we let $M = G_0/H_0$ be a simply connected, effective homogeneous space with G_0 connected, compact, semisimple, and H_0 compact. On M we consider the standard normal homogeneous metric g_B and we assume furthermore that G_0 is the id-component of the isometry group of g_B. Let $\hat{G}$ be the full isometry group of g_B with isotropy group $\hat{H}$ and hence $M = \hat{G}/\hat{H}$. The goal of this section is to determine $\hat{G}/G_0$ or equivalently $\hat{H}/H_0$. Every element of $\hat{H}$ gives rise to an automorphism of $\mathfrak{g}$ via the adjoint representation $\mathrm{Ad}_{\hat{G}}$ of $\hat{G}$, whose kernel is $C_{\hat{G}}(G_0)$ = centralizer of G_0 in $\hat{G}$.

Let

$$\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) = \{\text{automorphisms of } \mathfrak{g} \text{ leaving } \mathfrak{h} \text{ invariant}\}$$

and

$$\mathrm{Int}(\mathfrak{g}, \mathfrak{h}) = \{\text{inner automorphisms of } \mathfrak{g} \text{ leaving } \mathfrak{h} \text{ invariant}\}.$$

Then we have:

(3.1) THEOREM. *Under the above assumptions, the restriction of* $\mathrm{Ad}_{\hat{G}}$ *to* $\hat{H}$ *induces isomorphisms*

(a) $\hat{H}\approx\mathrm{Aut}(\mathfrak{g},\mathfrak{h})$ *which is also equal to the set of automorphisms of* G_0 *leaving* H_0 *invariant.*

(b) $\hat{H}/H_0\approx\mathrm{Aut}(\mathfrak{g},\mathfrak{h})/\mathrm{Ad}_{\hat{G}}(H_0)\approx\mathrm{Aut}(\mathfrak{g},\mathfrak{h})/H_0$

(c) $N_{G_0}(H_0)/Z(G_0)\approx\mathrm{Int}(\mathfrak{g},\mathfrak{h})$

(d) $N_{G_0}(H_0)/H_0Z(G_0)\approx\mathrm{Int}(\mathfrak{g},\mathfrak{h})/\mathrm{Ad}_{\hat{G}}(H_0)$.

Proof. $\mathrm{Ad}_{\hat{G}}$ clearly maps $\hat{H}$ into $\mathrm{Aut}(\mathfrak{g},\mathfrak{h})$. The kernel is $C_{\hat{G}}(G_0)\cap\hat{H}$. If $g\in C_{\hat{G}}(G_0)\cap\hat{H}$, then g is an isometry of $\hat{G}/\hat{H}$ fixing the coset $(\hat{H})$, whose differential at $(\hat{H})$ is the identity. Thus $g=1$, which shows that $\mathrm{Ad}_{\hat{G}}|_{\hat{H}}$ is injective. To show surjectivity, let $\varphi\in\mathrm{Aut}(\mathfrak{g},\mathfrak{h})$. Then φ induces an automorphism $\tilde{\varphi}$ of $\tilde{G}_0$, the universal cover of G_0, which leaves invariant $\tilde{H}_0$, the connected subgroup of $\tilde{G}_0$ with Lie algebra $\mathfrak{h}$. Hence $\tilde{\varphi}$ induces an isometry of $(\tilde{G}_0/\tilde{H}_0, g_B)$. Since G_0/H_0 is simply connected, $\tilde{G}_0/\tilde{H}_0=G_0/H_0$ and so $\tilde{\varphi}$ also induces an automorphism of G_0 leaving H_0 invariant and hence an isometry of $(G_0/H_0, g_B)$ fixing (H_0). So $\tilde{\varphi}\in\hat{H}$, which proves (a) and (b). Restricting $\mathrm{Ad}_{\hat{G}}$ to $N_{G_0}(H_0)$, the kernel is $C_{\hat{G}}(G_0)\cap N_{G_0}(H_0)=Z(G_0)$ and the image clearly lies in $\mathrm{Int}(\mathfrak{g},\mathfrak{h})$. Surjectivity is clear in this case, which proves (c) and (d). □

We will determine $\hat{H}/H_0$ and $N_{G_0}(H_0)/H_0Z(G_0)$ in all our examples, the second group being necessary to determine the Riemannian subcoverings of G_0/H_0 which are isotropy irreducible. We proceed as follows. Let φ be an automorphism of $\mathfrak{h}$. We must determine all extensions (if any) of φ to automorphisms of $\mathfrak{g}$. Clearly, for the existence of extensions we need only consider outer automorphisms of $\mathfrak{h}$. To determine the number of extensions modulo $\mathrm{Ad}_{\hat{G}}(H_0)$, we have to determine all automorphisms φ of $\mathfrak{g}$ which fix $\mathfrak{h}$. If φ is inner, then $\varphi=\mathrm{Ad}(g)$ with $g\in C_{G_0}(H_0)$. Hence we have to determine $C_{G_0}(H_0)/Z(G_0)Z(H_0)$ as well as those outer automorphisms of $\mathfrak{g}$ that fix $\mathfrak{h}$. For the latter, we only need to determine which elements of $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ have representatives that fix $\mathfrak{h}$.

If $\mathrm{rank}\,\mathfrak{h}=\mathrm{rank}\,\mathfrak{g}$, all these questions are answered by the following result:

(3.2) THEOREM. *Under the above assumptions and* $\mathrm{rank}\,\mathfrak{h}=\mathrm{rank}\,\mathfrak{g}$ *the following hold:*

(a) *Let* φ *be an automorphism of* $\mathfrak{h}$, *which, after composing with a suitable inner automorphism of* $\mathfrak{h}$, *satisfies* $\varphi(\mathfrak{t})\subset\mathfrak{t}$ *and* $\varphi^*(\Phi)\subset\Phi$, *where* $\mathfrak{t}$ *is a maximal abelian* *subalgebra of* $\mathfrak{h}$ *and* Φ *a fundamental system of roots of* $(\mathfrak{h}, \mathfrak{t})$. *Then* φ *extends to an automorphism of* $\mathfrak{g}$ *iff it permutes the dominant weights of the* $\mathrm{Ad}_{\hat{G}}|_{H_0}$ *irreducible summands of* $\mathfrak{m}$.

(b) $Z(G_0)=1$ *and* $C_{G_0}(H_0)=Z(H_0)$. *Furthermore any two extensions of an automorphism of* $\mathfrak{h}$ *are equal mod* $\mathrm{Ad}_{\hat{G}}(Z(H_0))$. *In particular, any two extensions are either both inner or both outer.*

(c) *If* $\mathfrak{h}=\mathfrak{h}_s\oplus\mathfrak{z}$, *where* $\mathfrak{h}_s$ *is semisimple and* $\mathfrak{z}$ *abelian, then* $\hat{H}/H_0$ *is isomorphic to the subgroup* Γ *of automorphisms of* $\mathfrak{h}$ *which are sums of automorphisms of* $\mathfrak{h}_s$ *and linear isomorphisms of* $\mathfrak{z}$ *and which permute the dominant weights of the* $\mathrm{Ad}_{\hat{G}}|_{H_0}$ *irreducible summands of* $\mathfrak{m}$. *Hence* G_0/H_0 *is isotropy irreducible as a Riemannian manifold if* Γ *in addition acts transitively on these weights.*

Proof. Let π_{λ_i} be the $\mathrm{Ad}_{\hat{G}}|_{H_0}$ irreducible summands of $\mathfrak{m}$ with dominant weight λ_i. Since $\operatorname{rank}\mathfrak{h}=\operatorname{rank}\mathfrak{g}$, $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$ and hence all the weights of π_{λ_i} are roots of $\mathfrak{g}$. This implies that all π_{λ_i} are inequivalent representations. Assume that φ permutes the λ_i, then it permutes the π_{λ_i} and their weights, and hence all the roots of $\mathfrak{g}$. So $\varphi|_{\mathfrak{t}}$ admits an extension to an automorphism σ of $\mathfrak{g}$ which leaves $\mathfrak{h}$ invariant. $\sigma|_{\mathfrak{h}}\circ\varphi^{-1}$ fixes $\mathfrak{t}\subset\mathfrak{h}$ and hence $\sigma|_{\mathfrak{h}}\circ\varphi^{-1}=\mathrm{Ad}_x$ for some $x\in T$. Therefore φ extends to $\mathrm{Ad}_x^{-1}\circ\sigma$. The converse clearly holds also and hence (a) is proved.

$Z(G_0)=1$ since G_0/H_0 is effective and $Z(G_0)$ is contained in every maximal torus of G_0 and hence in $T\subset H_0$. Similarly $C_{G_0}(H_0)\subset\bigcap_{T\subset H_0}T=Z(H_0)$ and hence $C_{G_0}(H_0)=Z(H_0)$. If φ and ψ are two automorphisms of $\mathfrak{g}$ with $\varphi|_{\mathfrak{h}}=\psi|_{\mathfrak{h}}$, then φ and ψ agree on every maximal abelian subalgebra of $\mathfrak{h}$ and hence $\varphi\circ\psi^{-1}=\mathrm{Ad}_x$ with $x\in Z(H_0)$. This proves (b), and (c) follows from (a) and (b). □

Therefore in the equal rank case, the fact that every entry in Table I is isotropy irreducible is almost obvious from the representation χ_0 of $\mathrm{Ad}_{\hat{G}}|_{H_0}$ on $\mathfrak{m}$. In the unequal rank case the following observations will help.

(3.3) THEOREM. *Under the above assumptions and* $\operatorname{rank}\mathfrak{h}<\operatorname{rank}\mathfrak{g}$ *the following hold:*

(a) *If* $\mathfrak{h}$ *is a maximal subalgebra in* $\mathfrak{g}$, *then* $C_{G_0}(H_0)=Z(G_0)$ *and* $Z(H_0)=1$.

(b) *Let* G_0 *be simple but not of type* D_4, *and assume that* H_0 *is semisimple and that the representation of* $\mathrm{Ad}_{\hat{G}}|_{H_0}$ *on* $\mathfrak{m}$ *contains no trivial representations. If* φ *is an outer automorphism of* $\mathfrak{g}$ *that fixes* $\mathfrak{h}$, *then there exists some integer* q *such that* φ^q *is an involutive automorphism with fixed point set* $\mathfrak{k}\supseteq\mathfrak{h}$. *In particular* $(\mathfrak{g}, \mathfrak{k})$ *is a symmetric pair.*

(c) *If in addition to the assumptions in* (b), $\mathfrak{h}$ *is a maximal subalgebra in* $\mathfrak{g}$ *and* $(\mathfrak{g},\mathfrak{h})$ *is not a symmetric pair, then* $\hat{H}/H_0$ *is isomorphic to the subgroup of diagram automorphisms of* $\mathfrak{h}$ *that extend to automorphisms of* $\mathfrak{g}$.

Proof. Let $g\in C_{G_0}(H_0)$ and let C_g be the centralizer of g in G_0. Then $H_0\subset C_g^0$ and since g is contained in a maximal torus T of G_0, we have $g\in T\subset C_g^0$, and rank $C_g^0=\operatorname{rank} G_0$. The maximality of $\mathfrak{h}$ in $\mathfrak{g}$ now implies $C_g^0=G_0$ since $C_g^0=H_0$ would contradict rank $\mathfrak{h}<\operatorname{rank}\mathfrak{g}$. Thus $g\in Z(G_0)$ and so $C_{G_0}(H_0)=Z(G_0)$. In particular $Z(H_0)=1$ since G_0/H_0 is effective. This proves (a).

To prove (b), we first observe that the assumptions imply that $C_{G_0}(H_0)$ is finite. Indeed the Lie algebra of $C_{G_0}(H_0)$ is equal to $\mathfrak{z}(\mathfrak{h})\oplus\mathfrak{m}_0$, where $\mathfrak{m}_0$ is the subalgebra of $\mathfrak{m}$ on which $\mathrm{Ad}_{\hat{G}}|_{H_0}$ acts trivially. Now φ induces a diagram automorphism of $\mathfrak{g}$ of order 2 since $\mathfrak{g}$ is simple $\neq D_4$. Then there exists an outer automorphism σ of $\mathfrak{g}$ with order 2 inducing the same diagram automorphism as φ. A theorem of de Siebenthal [Wo 2, 8.6.9], applied to the group $\mathrm{Aut}(\mathfrak{g})$, implies that we can find $x,g\in G_0$ such that $\varphi=\mathrm{Ad}_g\circ\sigma\circ\mathrm{Ad}_x\circ\mathrm{Ad}_g^{-1}$ and $[\sigma,\mathrm{Ad}_x]=0$. Then $\varphi^2=\mathrm{Ad}_g\circ\mathrm{Ad}_x^2\circ\mathrm{Ad}_g^{-1}=\mathrm{Ad}_{gx^2g^{-1}}$ is inner. Now $y=gx^2g^{-1}\in C_{G_0}(H_0)$ because φ fixes $\mathfrak{h}$ and hence $y^q=1$ for some integer q. But then $\varphi^{2q}=\mathrm{id}$, and φ^q is an involutive automorphism, which implies (b). (c) follows from (a) and (b). □

§ 4. The case of equal ranks

(4.1) THEOREM. *Let G/H be a simply connected, compact isotropy irreducible homogeneous space which is de Rham irreducible, but not strongly isotropy irreducible. If* rank $H=\operatorname{rank} G$, *then* (G_0,H_0) *can be found in Table* I *and conversely, every entry in Table* I *is an isotropy irreducible Riemannian manifold with full isotropy group* $\hat{H}$.

Proof. As explained earlier, the entries in Table I and χ_0 are collected from [WZ 1] and in most cases (3.2) easily implies that G_0/H_0 is isotropy irreducible and also explains the entries for $\hat{H}/H_0$ and $(\hat{H}/H_0)_{\min}$. The global form of G_0 and H_0 (which is not given in [WZ 1]), easily follows in most cases from the fact that $Z(G_0)=1$ and $H_0=\tilde{H}_0/\mathrm{Ker}(\chi_0)$, where $\tilde{H}_0$ is the universal cover of H_0. The only somewhat tricky case is no. 7, which we discuss separately. In this case the isotropy representation χ_0 was listed incorrectly in [WZ 1].

For the entry $N_{G_0}(H_0)/H_0Z(G_0)=N_{G_0}(H_0)/H_0$ we only have to determine which outer automorphisms of $\mathfrak{h}$ extend to inner automorphisms of $\mathfrak{g}$. Of course, in most of the cases $\mathfrak{g}$ has no outer automorphisms and hence $N_{G_0}(H_0)/H_0Z(G_0)=\hat{H}/H_0$.

We now discuss some of the cases separately and use the numbering in Table I.

Example no. 2. $S(\mathrm{U}(k)\times\cdots\times\mathrm{U}(k))/\mathbf{Z}_{nk}\subset\mathrm{SU}(nk)/\mathbf{Z}_{nk}$ (n factors).

It is clear that the outer automorphism on the semisimple part determines what linear isomorphism is needed on the center of $\mathfrak{h}$ in order to permute the Ad_{H_0} irreducible summands. The group S_n acts as outer automorphisms on $\mathfrak{h}$ by permuting the simple factors of $\mathfrak{h}$. The interchange of the ith and jth simple factor extends to an inner automorphism of $\mathfrak{g}$, i.e. conjugation by

$$\begin{pmatrix}
I &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& I &&&&&&&& \\
&&& 0 &&&& I &&& \\
&&&& I &&&&&& \\
&&&&& \ddots &&&&& \\
&&&&&& I &&&& \\
&&& -I &&&& 0 &&& \\
&&&&&&&& I && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& I
\end{pmatrix}$$

where I is the $k\times k$ identity matrix. The diagram automorphism of a single factor does not extend to $\mathfrak{g}$, since it does not preserve χ_0, but if we take the product of all diagram automorphisms of all simple factors, then it coincides with $X\to -X^t$, which is an outer automorphism of $\mathfrak{g}$. This outer automorphism clearly commutes with S_n and hence $\hat{H}/H_0=S_n\times\mathbf{Z}_2$ and $N_{G_0}(H_0)/H_0Z(G_0)=S_n$.

Example no. 3. $[\mathrm{SO}(k)\times\cdots\times\mathrm{SO}(k)]/\mathbf{Z}_2\subset\mathrm{SO}(nk)/\mathbf{Z}_2$ (n factors, k even).

This is similar to the previous example, except that the outer automorphism $\mathrm{Ad}(I_{1,k-1})$ of each simple factor (where $I_{1,k-1}$ is the matrix $\mathrm{diag}\{-1,1,\ldots,1\}$) does extend to the outer automorphism $\mathrm{Ad}(\mathrm{diag}\{I,\ldots,I_{1,k-1},\ldots,I\})$ of $\mathfrak{g}$. Thus the product of an even number of such outer automorphisms of $\mathfrak{h}$ extends to an inner automorphism of $\mathfrak{g}$. Hence $\hat{H}/H_0=(\mathbf{Z}_2)^n\ltimes S_n$ and $N_{G_0}(H_0)/H_0=(\mathbf{Z}_2)^{n-1}\ltimes S_n$, where S_n acts on $(\mathbf{Z}_2)^n$ by permuting the generators.

In both no. 2 and no. 3 (and similarly no. 4), it is difficult to determine subgroups of $\hat{H}/H_0$ of minimal order that still make G_0/H_0 isotropy irreducible. This is equivalent to finding subgroups of S_n of minimal order which act transitively on the set of unordered pairs and depends strongly on the value of n.

Example no. 6. $[\mathrm{Spin}(8)\times \mathrm{U}(1)\times \mathrm{U}(1)]/(\mathbf{Z}_4\times\mathbf{Z}_2)\subset \mathrm{E}_6/\mathbf{Z}_3$.

As we will see, the representation χ_0 of G_0/H_0 was given incorrectly in [WZ 1].

We first determine the global form of H_0. On the level of Lie algebras we have inclusions $\mathfrak{spin}(8)\oplus\mathfrak{u}(1)\oplus\mathfrak{u}(1)\subset\mathfrak{spin}(10)\oplus\mathfrak{u}(1)\subset\mathfrak{e}_6$. Since $Z(G_0)=1$, the global form of G_0 is $\mathrm{E}_6/\mathbf{Z}_3$ and since $Z(\mathrm{Spin}(10))=\mathbf{Z}_4$, $Z(\mathrm{E}_6)$ must be contained in the $\mathfrak{u}(1)$ factor of $\mathfrak{spin}(10)\oplus\mathfrak{u}(1)$. The isotropy representation of $\mathfrak{spin}(10)\oplus\mathfrak{u}(1)$ in $\mathfrak{e}_6$ is

$$\left[\circ-\circ-\circ<^{\circ\,1}_{\circ}\otimes\overset{1}{t}\right]_{\mathbf{R}}$$

which has kernel $\triangle\mathbf{Z}_4\subset\mathbf{Z}_4\times\mathrm{U}(1)=Z(\mathrm{Spin}(10))\times\mathrm{U}(1)$ and hence $[\mathrm{Spin}(10)\times\mathrm{U}(1)]/(\mathbf{Z}_3\times\triangle\mathbf{Z}_4)\subset\mathrm{E}_6/\mathbf{Z}_3$, which we can also write as $[\mathrm{Spin}(10)\times\mathrm{U}(1)]/\triangle\mathbf{Z}_4\subset\mathrm{E}_6/\mathbf{Z}_3$ since $\mathbf{Z}_3$ is contained in $\mathrm{U}(1)$.

Next we claim that the global form of $\mathfrak{so}(8)\oplus\mathfrak{u}(1)$ in $\mathfrak{spin}(10)$ is $[\mathrm{Spin}(8)\times\mathrm{U}(1)]/\triangle\mathbf{Z}_2\subset\mathrm{Spin}(10)$. Indeed, if we lift the inclusion $\mathrm{SO}(8)\times\mathrm{SO}(2)\hookrightarrow\mathrm{SO}(10)$ to a homomorphism $\mathrm{Spin}(8)\times\mathrm{U}(1)\to\mathrm{Spin}(10)$, it has kernel $(-1,-1)$. Furthermore, the generator of $Z(\mathrm{Spin}(10))=\mathbf{Z}_4$ is equal to (e,i), where e is the kernel of one of the spin representations of $\mathrm{Spin}(8)$. Hence

$$H_0=\{[\mathrm{Spin}(8)\times\mathrm{U}(1)]/\triangle\mathbf{Z}_2\times\mathrm{U}(1)\}/\triangle\mathbf{Z}_4=[\mathrm{Spin}(8)\times\mathrm{U}(1)\times\mathrm{U}(1)]/(\mathbf{Z}_2\times\mathbf{Z}_4)$$

where $\mathbf{Z}_2$ is generated by $(-1,-1,1)$ and $\mathbf{Z}_4$ is generated by (e,i,i).

To determine the isotropy representation χ_0, we first observe that the isotropy representation of $\mathrm{SO}(8)\times\mathrm{SO}(2)\subset\mathrm{SO}(10)$ is given by

$$\left[\overset{1}{\circ}-\circ<^{\circ}_{\circ}\otimes\overset{1}{t}\right]_{\mathbf{R}}$$

and since $\mathrm{U}(1)\to\mathrm{SO}(2)$ is the double cover, the isotropy representation of $[\mathrm{Spin}(8)\times\mathrm{U}(1)]/\triangle\mathbf{Z}_2\hookrightarrow\mathrm{Spin}(10)$ is equal to

$$\left[\overset{1}{\circ}-\circ<^{\circ}_{\circ}\otimes\overset{2}{t}\right]_{\mathbf{R}}.$$

Furthermore the representation $\circ-\circ-\circ<^{\circ\,1}_{\circ}$ of $\mathfrak{spin}(10)$ restricted to $\mathfrak{spin}(8)\oplus\mathfrak{u}(1)$ becomes

$$\left[\circ-\circ<^{\circ\,1}_{\circ}\otimes\overset{1}{t}\right]_{\mathbf{R}}\oplus\left[\circ-\circ<^{\circ}_{\circ\,1}\otimes\overset{-1}{t}\right]_{\mathbf{R}}.$$

Hence χ_0 is equal to

$$\left[\overset{1}{\circ}-\circ<^{\circ}_{\circ}\otimes\overset{2}{\mathfrak{t}}\otimes\mathfrak{t}\right]_{\mathbf{R}}\oplus\left[\circ-\circ<^{\circ\,1}_{\circ}\otimes\overset{1}{\mathfrak{t}}\otimes\overset{1}{\mathfrak{t}}\right]_{\mathbf{R}}\oplus\left[\circ-\circ<^{\circ}_{\circ\,1}\otimes\overset{-1}{\mathfrak{t}}\otimes\overset{1}{\mathfrak{t}}\right]_{\mathbf{R}}.$$

In [WZ 1] the 2 over $\mathfrak{t}$ was mistakenly replaced by 1.

We now check that G_0/H_0 is isotropy irreducible. If φ is an outer automorphism of $\mathfrak{h}$ such that $\varphi|_{\mathfrak{spin}(8)}$ is the triality automorphism, we need a linear isomorphism on $\mathfrak{t}\oplus\mathfrak{t}$ whose transpose takes $(2,0)$ to $\pm(1,1)$, $(1,1)$ to $\pm(-1,1)$, and $(-1,1)$ to $\pm(2,0)$. One easily checks that the only possibilities are

$$\pm\tfrac{1}{2}\begin{pmatrix}1 & -3\\ 1 & 1\end{pmatrix}$$

which takes $(2,0)$ to $\pm(1,1)$, $(1,1)$ to $\pm(-1,1)$, and $(-1,1)$ to $\mp(2,0)$. These $\varphi_\pm$ extend to automorphisms $\tilde{\varphi}_\pm$ of $\mathfrak{e}_6$. Hence G_0/H_0 is isotropy irreducible as a Riemannian manifold.

Next, if τ is an outer automorphism of $\mathfrak{h}$ such that $\tau|_{\mathfrak{spin}(8)}$ is the order 2 diagram automorphism of $\mathfrak{spin}(8)$, then $\tau|_{\mathfrak{t}\oplus\mathfrak{t}}$ must take $(2,0)$ to $\pm(2,0)$, $(-1,1)$ to $\pm(1,1)$ and $(1,1)$ to $\pm(-1,1)$. The only possible choices are

$$\pm\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

These $\tau_\pm$ again extend to automorphisms $\tilde{\tau}_\pm$ of $\mathfrak{e}_6$.

To determine the group $\hat{H}/H_0$, one easily checks the following identities mod Ad_{H_0}:

$$\tilde{\varphi}_+^6=1,\quad \tilde{\varphi}_-^3=1,\quad \tilde{\tau}_\pm^2=1,\quad \tilde{\varphi}_+^4=\tilde{\varphi}_-,\quad \tilde{\tau}_+\tilde{\varphi}_+^3=\tilde{\varphi}_+^3\tilde{\tau}_+=\tilde{\tau}_-,\quad \tilde{\tau}_+\tilde{\varphi}_+^2\tilde{\tau}_+^{-1}=\tilde{\varphi}_+^{-2}$$

by verifying them on $\mathfrak{spin}(8)$ and $\mathfrak{t}\oplus\mathfrak{t}$ separately. Hence $\hat{H}/H_0$ is isomorphic to $S_3\times\mathbf{Z}_2$, with S_3 generated by $\tilde{\varphi}_+^2$ and $\tilde{\tau}_+$, and $\mathbf{Z}_2$ generated by $\tilde{\varphi}_+^3$. Notice that $\tilde{\varphi}_+^3$ acts as id on $\mathfrak{spin}(8)$ and as $-\mathrm{id}$ on $\mathfrak{t}\oplus\mathfrak{t}$.

To determine $N_{G_0}(H_0)/H_0Z(G_0)$, we examine which one of these automorphisms extend to inner automorphisms of $\mathfrak{e}_6$. First observe that $\tilde{\varphi}_+^2$ must be inner since the group of diagram automorphisms of $\mathfrak{e}_6$ has order 2. Next observe that $\tilde{\varphi}_+^3$ and $\tilde{\tau}_+$ must be outer, since they act non-trivially on the center of $\mathfrak{e}_6$, which is generated by $(1,1,e^{2\pi i/3})$ in $[\mathrm{Spin}(8)\times\mathrm{U}(1)\times\mathrm{U}(1)]/(\mathbf{Z}_4\times\mathbf{Z}_2)$. Hence $N_{G_0}(H_0)/H_0Z(G_0)$ is isomorphic to S_3, sitting diagonally in $S_3\times\mathbf{Z}_2$, and is generated by $\tilde{\varphi}_+^2$ and $\tilde{\tau}_+\tilde{\varphi}_+^3$.

Examples 8 and 12 were discussed in § 1 and hence this finishes the proof of (4.1). □

Remark. To determine the group structure of $\hat{G}$, we observe that since $Z(G_0)=1$, every extension of G_0 is determined by the character $\hat{G}/G_0 \to \mathrm{Aut}(G_0)/\mathrm{Int}(G_0)$ and since there exists a lift $\mathrm{Aut}(G_0)/\mathrm{Int}(G_0) \to \mathrm{Aut}(G_0)$, $\hat{G}$ is the semidirect product of G_0 with $\hat{G}/G_0 = \hat{H}/H_0$. In most cases G_0 has no outer automorphisms and hence $\hat{G} = G_0 \times (\hat{H}/H_0)$. In the remaining 4 cases the discussion of the examples usually contains an explicit lift $\hat{H}/H_0 \to \mathrm{Aut}(G_0)$ which describes the semidirect product structure of $\hat{G}$.

§ 5. The case of unequal ranks

(5.1) THEOREM. *Let G/H be a simply connected, compact isotropy irreducible homogeneous space which is de Rham irreducible, but not strongly isotropy irreducible. If* $\operatorname{rank} H < \operatorname{rank} G$, *then (G_0, H_0) can be found in Table* II *and conversely every entry in Table* II *is an isotropy irreducible Riemannian manifold with full isotropy group $\hat{H}$, with the exception of no.* 5.

Case no. 5 is isotropy irreducible as a Riemannian manifold, but G_0 is not the id-component of the isometry group. See § 1, Example 1 for a discussion of this case.

Proof. As explained earlier, the entries in Table II are obtained from [WZ 1] and (2.2). We will now check for each entry that G_0/H_0 is isotropy irreducible as a Riemannian manifold. We also need to determine the global form of G_0 and H_0 as well as the groups $\hat{H}/H_0$ and $N_{G_0}(H_0)/H_0 Z(G_0)$. We will use the numbering in Table II.

Example no. 1. $\mathrm{SO}(k) \times \cdots \times \mathrm{SO}(k) \subset \mathrm{SO}(nk)$ (n factors, k odd).

The diagram automorphisms which interchange two simple factors extend to inner automorphisms of $\mathfrak{g}$ as in the corresponding equal rank case. But they do not extend uniquely. Indeed $C_{G_0}(H_0) = \mathbf{Z}_2^{n-1}$ since $C_{G_0}(H_0) \subset S(\mathrm{O}(k) \times \cdots \times \mathrm{O}(k))$ by Schur's lemma. If n is odd, then $Z(G_0)=1$ and G_0 has no outer automorphisms. Hence $\hat{H}/H_0 = N_{G_0}(H_0)/H_0 Z(G_0) = (\mathbf{Z}_2)^{n-1} \rtimes S_n$.

If n is even, then $Z(G_0) = \mathbf{Z}_2$ and hence $C_{G_0}(H_0)/Z(G_0) = \mathbf{Z}_2^{n-2}$, which implies that $N_{G_0}(H_0)/H_0 Z(G_0) = \mathbf{Z}_2^{n-2} \rtimes S_n$. But now G_0 has, modulo $\mathrm{Int}(\mathfrak{g})$, exactly one outer automorphism and we have to determine whether a representative of it fixes $\mathfrak{h}$. This is the case since $\mathrm{Ad}(\mathrm{diag}\{-I_k, I_k, \ldots, I_k\})$ is outer and fixes $\mathfrak{h}$. Hence we have again $\hat{H}/H_0 = (\mathbf{Z}_2)^{n-1} \rtimes S_n$. Notice that in both cases, n even and n odd, $(\mathbf{Z}_2)^{n-1}$ consists of $\mathrm{Ad}(g)$ with $g = (\pm I_k, \ldots, \pm I_k)$ and hence S_n acts on $(\mathbf{Z}_2)^{n-1} = (\mathbf{Z}_2)^n/\mathbf{Z}_2$ via permutations on $\mathbf{Z}_2^n$.

Example no. 2. $\mathfrak{so}(n) \oplus \mathfrak{so}(n) \subset \mathfrak{so}(n^2)$.

We first describe the global form of G_0 and H_0. The homomorphism $\mathrm{SO}(n)\times\mathrm{SO}(n)\to\mathrm{SO}(n^2)$ given by the tensor product representation is injective if n is odd and has kernel $\mathbf{Z}_2=\{\pm(1,1)\}$ if n is even. To determine if the quotient is simply connected, we observe that $\mathrm{SO}(n)\times\{1\}\subset\mathrm{SO}(n)\times\mathrm{SO}(n)\to\mathrm{SO}(n^2)$ is just the diagonal embedding of $\mathrm{SO}(n)$ into $\mathrm{SO}(n^2)$: $A\to(A,\dots,A)$. Since $A\to(A,1,\dots,1)$ is always injective on π_1, it follows that if n is odd, $\mathrm{SO}(n^2)/[\mathrm{SO}(n)\times\mathrm{SO}(n)]$ is simply connected and effective.

If n is even, say $n=2k$, then the fundamental group of each $\mathrm{SO}(n)$ factor goes to 0 in $\mathrm{SO}(n^2)$. But $[\mathrm{SO}(n)\times\mathrm{SO}(n)]/\mathbf{Z}_2$ has a third generator in π_1, represented by $\sigma\colon S^1\to[\mathrm{SO}(n)\times\mathrm{SO}(n)]/\mathbf{Z}_2$ where $\sigma(\theta)=(R(\theta),\dots,R(\theta))\times(R(\theta),\dots,R(\theta))$, $0\leqslant\theta\leqslant\pi$, and $R(\theta)$ is a 2×2 rotation of angle θ. Going into $\mathrm{SO}(n^2)$ this becomes $(R(2\theta),\dots,R(2\theta),1,\dots,1)$ with k^2 rotations $R(2\theta)$. This loop in $\mathrm{SO}(n^2)$ is therefore nontrivial iff k is odd. So if k is odd, i.e. $n\equiv 2 \pmod 4$, $\mathrm{SO}(n^2)/([\mathrm{SO}(n)\times\mathrm{SO}(n)]/\mathbf{Z}_2)$ is simply connected. To make it effective one divides out by $Z(\mathrm{SO}(n^2))=\mathbf{Z}_2$ to obtain $G_0=\mathrm{SO}(n^2)/\mathbf{Z}_2$ and $H_0=(\mathrm{SO}(n^2)/\mathbf{Z}_2)\times(\mathrm{SO}(n^2)/\mathbf{Z}_2)$.

If k is even, i.e. $n\equiv 0 \pmod 4$, this space has $\pi_1=\mathbf{Z}_2$. To obtain the simply connected space, consider the lift $i^*\colon[\mathrm{SO}(n)\times\mathrm{SO}(n)]/\mathbf{Z}_2\to\mathrm{Spin}(n^2)$ of the inclusion $i\colon[\mathrm{SO}(n)\times\mathrm{SO}(n)]/\mathbf{Z}_2\to\mathrm{SO}(n^2)$. The lift i^* is injective since i is. Furthermore $(-I,I)\in[\mathrm{SO}(n)\times\mathrm{SO}(n)]/\mathbf{Z}_2$ maps to the center of $\mathrm{Spin}(n^2)$ under i^* since $i((-I,I))=-I$. Therefore $\mathrm{Spin}(n^2)/i^*([\mathrm{SO}(n)\times\mathrm{SO}(n)]/\mathbf{Z}_2)$ is simply connected, but not effective. The effective version is $(\mathrm{Spin}(n^2)/\mathbf{Z}_2)/[(\mathrm{SO}(n)/\mathbf{Z}_2)\times(\mathrm{SO}(n)/\mathbf{Z}_2)]$. Note that $\mathrm{Spin}(n^2)/\mathbf{Z}_2$ is not equal to $\mathrm{SO}(n^2)$.

We proceed to determine $\hat{H}/H_0$. We first observe that any automorphism of $\mathfrak{h}$ which extends to $\mathfrak{g}$ does so uniquely by (3.3) (c). Next consider the outer automorphism φ of $\mathfrak{h}$ which interchanges the simple factors. Let $\sigma\colon\mathbf{R}^n\otimes\mathbf{R}^n\to\mathbf{R}^n\otimes\mathbf{R}^n$ be given by $\sigma(u\otimes v)=v\otimes u$. Then $\sigma\in\mathrm{O}(\mathbf{R}^{n^2})$ and one checks easily that $\mathrm{Ad}_\sigma|_{\mathfrak{h}}=\varphi$. Since $\mathbf{R}^n\otimes\mathbf{R}^n=\Lambda^2\mathbf{R}^n\oplus S^2\mathbf{R}^n$ and since $\sigma|_{\Lambda^2\mathbf{R}^n}=-\mathrm{id}$ and $\sigma|_{S^2\mathbf{R}^n}=\mathrm{id}$, it follows that the fixed point set of Ad_σ is $\mathrm{SO}(\Lambda^2\mathbf{R}^n)\times\mathrm{SO}(S^2\mathbf{R}^n)$. Since an involutive automorphism is inner iff the fixed point group has equal rank, Ad_σ is outer iff $n\equiv 2 \pmod 4$. If n is even, we must also consider the order 2 automorphism of each simple factor. One checks that the involution corresponding to the first simple factor is induced by Ad_τ, where $\tau\colon\mathbf{R}^n\otimes\mathbf{R}^n\to\mathbf{R}^n\otimes\mathbf{R}^n$ is given by $\tau(u\otimes v)=I_{1,n-1}(u)\otimes v$. Clearly $\tau=I_{n,n^2-n}$ and hence Ad_τ is inner. Furthermore $\mathrm{Ad}_{\sigma\tau\sigma}$ induces the order 2 automorphism of the second simple factor. Hence $\hat{H}/H_0=N_{G_0}(H_0)/H_0Z(G_0)=\mathbf{Z}_2$ if n is odd. If n is even $\hat{H}/H_0=D_8$, the dihedral group of order 8, and $N_{G_0}(H_0)/H_0Z(G_0)=D_8$ if $n\equiv 0 \pmod 4$ and $=\mathbf{Z}_2\times\mathbf{Z}_2$ if $n\equiv 2 \pmod 4$. In all cases the group $(\hat{H}/H_0)_{\min}$ is equal to $\mathbf{Z}_2$ and given by Ad_σ.

Example no. 3 is similar and is left to the reader.

Example no. 4. $\triangle(K/Z(K))\subset(K\times\cdots\times K)/\triangle Z(K)$

If $g=(g_1,\ldots,g_n)\in N_{G_0}(H_0)$, then $g_1^{-1}g_i\in Z(K)$ for all i and hence $g\in H_0Z(G_0)$, i.e. $N_{G_0}(H_0)/H_0Z(G_0)=1$.

Secondly, any outer automorphism A of $\mathfrak{h}$ extends to the outer automorphism $(A,\ldots,A)$ and one easily sees by (3.3) that, modulo Ad_{H_0}, any outer automorphism of $\mathfrak{g}$ that fixes $\mathfrak{h}$ is an outer automorphism that permutes the simple factors of $\mathfrak{g}$. Hence $\hat{H}/H_0=S_n\times D$, where D is the group of diagram automorphisms of $\mathfrak{k}$.

Example no. 5 was already discussed in § 1.

Example no. 6. $G_2\subset\mathrm{Spin}(8)$.

Since G_2 has no outer automorphisms, to determine $N_{G_0}(H_0)/H_0Z(G_0)$, we only need to compute $C_{G_0}(H_0)$. But if $g\in C_{G_0}(H_0)$ and $g\notin Z(G_0)$, then g and H_0 are contained in C_g^0, and hence in a connected maximal subgroup of maximal rank in Spin(8). But the only such groups have Lie algebras $\mathfrak{u}(4)$, $\mathfrak{so}(2)\oplus\mathfrak{so}(6)$, or $\mathfrak{so}(4)\oplus\mathfrak{so}(4)$, which do not contain $\mathfrak{g}_2$. Thus $C_{G_0}(H_0)=Z(G_0)$ and hence $N_{G_0}(H_0)/H_0Z(G_0)=1$.

The triality automorphism of Spin(8) has fixed point set G_2 and there exists an order 2 outer automorphism of Spin(8) which has fixed point set $\mathrm{Spin}(7)\supset G_2$. Hence $\hat{H}/H_0=S_3$ and $(\hat{H}/H_0)_{\min}=\mathbf{Z}_3$.

Example no. 7. $\mathfrak{so}(8)\subset\mathfrak{su}(8)\subset\mathfrak{e}_7$.

The understanding of this case rests upon a description of E_7 due to E. Cartan. Let $\langle\ ,\ \rangle$ denote the negative of the Killing form of $\mathfrak{h}=\mathfrak{so}(8)$ as well as its extension to a hermitian form on $\mathfrak{h}_C$. On the 56-dimensional space $\mathfrak{h}_C\oplus\mathfrak{h}_C$ there is a skew-symmetric form ω as well as the Hermitian form $\langle\ ,\ \rangle\perp\langle\ ,\ \rangle$ which we again denote by $\langle\ ,\ \rangle$. Then $\mathrm{Sp}(\mathfrak{h}_C\oplus\mathfrak{h}_C,\omega)\cap\mathrm{SU}((\mathfrak{h}_C\oplus\mathfrak{h}_C,\langle\ ,\ \rangle)$ is a compact connected Lie group isomorphic to Sp(28). In the following we will refer to this Lie group as Sp(28). As usual we describe the elements of Sp(28) as matrices of the form

$$\begin{pmatrix} A & B \\ -\bar{B} & \bar{A}\end{pmatrix}$$

with $AA^*+BB^*=I$ and $AB^t=BA^t$. In particular the subgroup $\mathrm{SU}(28)\subset\mathrm{Sp}(28)$ has the form

$$\begin{pmatrix} A & 0 \\ 0 & \bar{A}\end{pmatrix}.$$

On $\mathfrak{h}_{\mathbf{C}}\oplus\mathfrak{h}_{\mathbf{C}}$ we have the following homogeneous polynomial of degree 4:

$$J(X,Y)=\mathrm{Pf}(X)+\mathrm{Pf}(Y)-\frac{1}{4}\mathrm{tr}(XYXY)+\frac{1}{16}(\mathrm{tr}(XY))^2, \quad X, Y\in\mathfrak{h}_{\mathbf{C}}=\mathfrak{so}(8,\mathbf{C})$$

where $\mathrm{Pf}(X)$ denotes the Pfaffian. E. Cartan has shown that E_7 is the largest connected subgroup of $\mathrm{Sp}(28)$ that leaves J invariant (see [C 2, 143–144] and [Fr 1, 2]). We first observe that the invariance group of J is actually connected:

(5.2) LEMMA. *E_7 is the largest subgroup of* $\mathrm{Sp}(28)$ *leaving J invariant.*

Proof. Let L be the invariance group of J. Then it follows from the result of Cartan that $\mathrm{E}_7\subset L\subset\mathrm{Sp}(28)$. But $\mathrm{Sp}(28)/\mathrm{E}_7$ is strongly isotropy irreducible (see [Wo 1]) and hence $\mathfrak{e}_7$ is a maximal subalgebra in $\mathfrak{sp}(28)$. This implies $L\subset N_{\mathrm{Sp}(28)}(\mathrm{E}_7)$. Since E_7 has no outer automorphisms $N_{\mathrm{Sp}(28)}(\mathrm{E}_7)/\mathrm{E}_7=C_{\mathrm{Sp}(28)}(\mathrm{E}_7)/Z(\mathrm{E}_7)$ and the argument in (3.3) (a) implies $C_{\mathrm{Sp}(28)}(\mathrm{E}_7)=Z(\mathrm{Sp}(28))=Z(\mathrm{E}_7)$ (notice that $\mathrm{Sp}(28)/\mathrm{E}_7$ is not effective). Hence $L=\mathrm{E}_7$. □

With this description of E_7, $\mathrm{SO}(8)/\mathbf{Z}_2$ has a natural embedding into it given by the composite

$$\mathrm{SO}(8)/\mathbf{Z}_2\xrightarrow{\mathrm{Ad}}\mathrm{SO}(\mathfrak{h},\langle\ ,\ \rangle)\subset\mathrm{SU}(\mathfrak{h}_{\mathbf{C}},\langle\ ,\ \rangle)=\mathrm{SU}(28)\subset\mathrm{Sp}(28)$$

because $A\in\mathrm{SO}(8)$ is mapped to the transformation of $\mathfrak{h}_{\mathbf{C}}\oplus\mathfrak{h}_{\mathbf{C}}$ given by $(X,Y)\to(AXA^t, AYA^t)$, which clearly leaves J invariant. Thus $\mathrm{E}_7/(\mathrm{SO}(8)/\mathbf{Z}_2)$ is simply connected and effective.

We can also easily describe $\mathrm{SU}(8)/\mathbf{Z}_2\subset\mathrm{E}_7$ with $\mathrm{SO}(8)/\mathbf{Z}_2\subset\mathrm{SU}(8)/\mathbf{Z}_2$. Indeed $\mathrm{SU}(8)/\mathbf{Z}_2\subset\mathrm{SU}(\mathfrak{h}_{\mathbf{C}},\langle\ ,\ \rangle)$ is given by $A\to(X\to AXA^t)$ and hence $A\in\mathrm{SU}(8)/\mathbf{Z}_2$ acts on $\mathfrak{h}_{\mathbf{C}}\oplus\mathfrak{h}_{\mathbf{C}}$ via the embedding $\mathrm{SU}(\mathfrak{h}_{\mathbf{C}})\subset\mathrm{Sp}(28)$ as $(X,Y)\to(AXA^t,\bar{A}Y\bar{A}^t)$ and this clearly leaves J invariant and contains $\mathrm{SO}(8)/\mathbf{Z}_2$.

Since E_7 has no outer automorphisms, $\hat{H}/H_0=N_{G_0}(H_0)/H_0Z(G_0)$. An order 2 outer automorphism of $\mathrm{SO}(8)/\mathbf{Z}_2$ is given by $\mathrm{Ad}(I_{7,1})$,which extends to the inner automorphism $\mathrm{Ad}(\mathrm{diag}\{\xi,\dots,\xi,-\xi\})$, $\xi=e^{\pi i/8}$, of $\mathrm{SU}(8)/\mathbf{Z}_2$ and hence to an inner automorphism of E_7.

To see that the triality automorphism τ of $\mathfrak{h}$ extends to an inner automorphism of $\mathfrak{e}_7$ will be more complicated and is discussed in the remainder of this section.

First we observe that τ extends to an inner automorphism of $\mathfrak{so}(\mathfrak{h})$. In fact $\tau\in\mathrm{SO}(\mathfrak{h})$ since $\tau^3=1$ implies $\det\tau=1$. The embedding of $\mathfrak{h}$ into $\mathfrak{so}(\mathfrak{h})$ is given by $X\to\mathrm{ad}_X$

and since $\tau \circ \mathrm{ad}_X \circ \tau^{-1} = \mathrm{ad}_{\tau(X)}$ it follows that $\mathrm{Ad}(\tau)$, $\tau \in \mathrm{SO}(\mathfrak{h})$, induces τ on $\mathfrak{h}$. $\tau \in \mathrm{SO}(\mathfrak{h})$ goes into

$$T = \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} \in \mathrm{Sp}(28)$$

and hence the automorphism τ of $\mathfrak{h}$ extends to the inner automorphism Ad_T of $\mathfrak{sp}(28)$. We will see that $T \notin \mathrm{E}_7$, but that it will be in E_7 if we modify T with an element of $C_{\mathrm{Sp}(28)}(\mathrm{SO}(8)/\mathbf{Z}_2)$. In fact τ has an extension to an inner automorphism of $\mathfrak{e}_7$ iff $T\Lambda \in \mathrm{E}_7$ for some $\Lambda \in C_{\mathrm{Sp}(28)}(\mathrm{SO}(8)/\mathbf{Z}_2)$.

(5.3) LEMMA.

(a) $$C_{\mathrm{Sp}(28)}(\mathrm{SO}(8)/\mathbf{Z}_2) = \left\{ \begin{pmatrix} \lambda I & \mu I \\ -\bar{\mu} I & \bar{\lambda} I \end{pmatrix} \,\middle|\, |\lambda|^2 + |\mu|^2 = 1,\ \lambda, \mu \in \mathbf{C} \right\} \approx \mathrm{Sp}(1).$$

(b) *$C_{\mathrm{E}_7}(\mathrm{SO}(8)/\mathbf{Z}_2)$ is the quaternion subgroup of* $\mathrm{Sp}(1)$ *of order* 8. *The center of* E_7 *corresponds to the center of the quaternion subgroup and hence*

$$C_{\mathrm{E}_7}(\mathrm{SO}(8)/\mathbf{Z}_2)/Z(\mathrm{E}_7) = \mathbf{Z}_2 \times \mathbf{Z}_2.$$

Proof. If

$$\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$$

commutes with

$$\begin{pmatrix} \mathrm{Ad}(C) & 0 \\ 0 & \mathrm{Ad}(C) \end{pmatrix}$$

then A and B commute with $\mathrm{Ad}(C)$ and hence $A = \lambda I$ and $B = \mu I$, which implies (a).

To prove (b), let

$$g = \begin{pmatrix} \lambda I & \mu I \\ -\bar{\mu} I & \bar{\lambda} I \end{pmatrix} \in C_{\mathrm{E}_7}(\mathrm{SO}(8)/\mathbf{Z}_2).$$

Then g maps $\mathfrak{t}_{\mathbf{C}} \oplus \mathfrak{t}_{\mathbf{C}} \subset \mathfrak{h}_{\mathbf{C}} \oplus \mathfrak{h}_{\mathbf{C}}$ to itself and leaves J invariant. Let

$$X = \begin{pmatrix} 0 & x_1 & & & & \\ -x_1 & 0 & & & & \\ & & \ddots & & & \\ & & & & 0 & x_4 \\ & & & & -x_4 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & y_1 & & & & \\ -y_1 & 0 & & & & \\ & & \ddots & & & \\ & & & & 0 & y_4 \\ & & & & -y_4 & 0 \end{pmatrix}$$

and notice that

$$J(X, Y) = x_1x_2x_3x_4+y_1y_2y_3y_4-\frac{1}{2}(x_1^2y_1^2+x_2^2y_2^2+x_3^2y_3^2+x_4^2y_4^2)$$
$$+\frac{1}{4}(x_1y_1+x_2y_2+x_3y_3+x_4y_4)^2.$$

Comparing coefficients of the monomials $x_1x_2x_3x_4$ and $x_1^2y_1^2$ in $J(X, Y)=J(g(X, Y))=J(\lambda X+\mu Y, -\bar{\mu}X+\bar{\lambda}Y)$, we obtain $\lambda^4+\bar{\mu}^4=1$ and $|\lambda|^4+|\mu|^4-4|\lambda|^2|\mu|^2=1$. Since $|\lambda|^2+|\mu|^2=1$, the only solutions are $\lambda=0$ and $\mu^4=1$, or $\mu=0$ and $\lambda^4=1$. Now one easily checks that all possibilities actually leave J invariant and together they form the quaternion group of order 8. $\square$

To proceed further, we need the following identities involving the triality automorphism:

(5.4) LEMMA. *Let $X, Y\in \mathfrak{h}_{\mathbf{C}}$. Then we have:*

(a) $\mathrm{Pf}(\tau(X))=-\frac{1}{2}\mathrm{Pf}(X)-\frac{1}{16}\mathrm{tr}(X^4)+\frac{1}{64}(\mathrm{tr}(X^2))^2$

(b) $\mathrm{tr}(\tau(X)^4)=12\,\mathrm{Pf}(X)+\frac{3}{8}(\mathrm{tr}(X^2))^2-\frac{1}{2}\mathrm{tr}(X^4)$

(c) $\mathrm{tr}(\tau(X)\tau(Y)\tau(X)\tau(Y))-\mathrm{tr}(\tau(X)^2\tau(Y)^2)=\mathrm{tr}(XYXY)-\mathrm{tr}(X^2Y^2)$

(d) $\mathrm{tr}(X^4)+\mathrm{tr}(\tau(X)^4)+\mathrm{tr}([\tau^2(X)]^4)=\frac{3}{4}(\mathrm{tr}(X^2))^2$

(e) $\mathrm{tr}(\tau(X)^2\tau(Y)^2)+\mathrm{tr}([\tau^2(X)]^2[\tau^2(Y)]^2)=\frac{1}{4}\mathrm{tr}(X^2)\,\mathrm{tr}(Y^2)+\frac{1}{2}\mathrm{tr}((XY)^2)-\mathrm{tr}(XYXY)$.

Proof. Let $X, Y\in\mathfrak{h}=\mathfrak{so}(8)$. (a) and (b) are equations in a single variable X, and since each entry in the equations is $\mathrm{Ad}_{SO(8)}$ invariant, it is sufficient to check them on a maximal abelian subalgebra $\mathfrak{t}\subset\mathfrak{so}(8)$, which is easily done. Since both equations are polynomials in the entries of X and Y, they also hold for $X, Y\in\mathfrak{h}_{\mathbf{C}}$.

To prove (c) we use the fact that τ is an automorphism of $\mathfrak{h}_{\mathbf{C}}$ and that the Killing form is invariant under τ. Hence $\mathrm{tr}([\tau(X),\tau(Y)]^2)=\mathrm{tr}(\tau([X, Y])^2)=\mathrm{tr}([X, Y])^2)$. Writing this out using $[X, Y]=XY-YX$, we obtain (c).

To prove (d), we observe that (a) and (b) imply

$$\begin{aligned}\mathrm{tr}(\tau(X)^4)+\mathrm{tr}(\tau^2(X)^4) &= 12\,\mathrm{Pf}(X)+\frac{3}{8}(\mathrm{tr}(X^2))^2-\frac{1}{2}\mathrm{tr}(X^4)\\ &\quad+12\,\mathrm{Pf}(\tau(X))+\frac{3}{8}(\mathrm{tr}(\tau(X)^2))^2-\frac{1}{2}\mathrm{tr}(\tau(X)^4)\\ &= -\mathrm{tr}(X^4)+\frac{3}{4}(\mathrm{tr}(X^2))^2.\end{aligned}$$

Finally, (e) is obtained from (d) by polarization. First note that

$$\mathrm{tr}((X+Y)^4)+\mathrm{tr}((X-Y)^4) = 2\,\mathrm{tr}(X^4)+2\,\mathrm{tr}(Y^4)+4\,\mathrm{tr}(XYXY)+8\,\mathrm{tr}(X^2Y^2).$$

Using this equation to expand the left hand and right hand side of

$$\begin{aligned}\operatorname{tr}\{X+Y)^4+[\tau(X+Y)]^4+[\tau^2(X+Y)]^4\}+\operatorname{tr}\{(X-Y)^4+[\tau(X-Y)]^4+[\tau^2(X-Y)]^4\}\\ =\frac{3}{4}\{\operatorname{tr}((X+Y)^2)\}^2+\frac{3}{4}\{\operatorname{tr}((X-Y)^2)\}^2\end{aligned}$$

we obtain (e). □

We now prove that τ extends to an automorphism of $\mathfrak{e}_7$. Let

$$\Lambda=\begin{pmatrix}\lambda I & \mu I\\ -\bar{\mu}I & \bar{\lambda}I\end{pmatrix}\in C_{\mathrm{Sp}(28)}(\mathrm{SO}(8)/\mathbf{Z}_2).$$

We will try to find λ,μ such that $T\Lambda\in \mathrm{E}_7$. This will be satisfied if

$$J(X,Y)=J(\lambda\tau(X)+\mu\tau(Y),\,-\bar{\mu}\tau(X)+\bar{\lambda}\tau(Y)).$$

Since $T\Lambda$ maps $\mathfrak{t}_{\mathbf{C}}\oplus\mathfrak{t}_{\mathbf{C}}$ to itself, we may compare coefficients of monomials on the left and right hand side for $(X,Y)\in\mathfrak{t}_{\mathbf{C}}\oplus\mathfrak{t}_{\mathbf{C}}$. Doing this for $x_1^2y_1^2$, we get $\lambda^2\mu^2+\bar{\lambda}^2\bar{\mu}^2+2|\lambda|^2|\mu|^2=1$, while comparing the coefficients for $x_1x_2x_3x_4$ we obtain $\lambda^4+\bar{\mu}^4+6\lambda^2\bar{\mu}^2=-2$. Since $|\lambda|^2+|\mu|^2=1$ also, we easily see that $\lambda^4=-1/4$ and either $\lambda=\bar{\mu}$ or $\lambda=-\bar{\mu}$. So a necessary condition is

$$\Lambda=\begin{pmatrix}\lambda I & \bar{\lambda}I\\ -\lambda I & \bar{\lambda}I\end{pmatrix}\quad\text{or}\quad\Lambda=\begin{pmatrix}\lambda I & -\bar{\lambda}I\\ \lambda I & \bar{\lambda}I\end{pmatrix}\quad\text{with}\quad\lambda^4=-1/4.$$

We may focus on the first possibility as the two differ by

$$\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix},$$

which lies in $C_{\mathrm{E}_7}(\mathrm{SO}(8)/\mathbf{Z}_2)$. We will show that with this choice of Λ, $T\Lambda\in\mathrm{E}_7$.

Let $A=\bar{\lambda}Y$ and $B=\lambda X$. Then

$$\begin{aligned}&J(\lambda\tau(X)+\bar{\lambda}\tau(Y),\,-\lambda\tau(X)+\bar{\lambda}\tau(Y))\\ &\quad=J(\tau(A+B),\tau(A-B))\\ &\quad=\mathrm{Pf}(\tau(A+B))+\mathrm{Pf}(\tau(A-B))\\ &\qquad-\frac{1}{4}\operatorname{tr}(\tau(A+B)\,\tau(A-B)\,\tau(A+B)\,\tau(A-B))+\frac{1}{16}[\operatorname{tr}(\tau(A+B)\,\tau(A-B))]^2\\ &\quad=\frac{1}{12}\operatorname{tr}(\tau^2(A+B)^4)+\frac{1}{24}\operatorname{tr}(\tau(A+B)^4)-\frac{1}{32}(\operatorname{tr}(\tau(A+B)^2))^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{12}\operatorname{tr}(\tau^2(A-B)^4)+\frac{1}{24}\operatorname{tr}(\tau(A-B)^4)-\frac{1}{32}(\operatorname{tr}(\tau(A-B)^2))^2\\
&-\frac{1}{4}\operatorname{tr}((A+B)(A-B)(A+B)(A-B))+\frac{1}{4}\operatorname{tr}((A+B)^2(A-B)^2)\\
&-\frac{1}{4}\operatorname{tr}(\tau(A+B)^2\,\tau(A-B)^2)+\frac{1}{16}(\operatorname{tr}(A^2-B^2))^2\\
=&-\frac{1}{24}\operatorname{tr}(\tau(A+B)^4)-\frac{1}{24}\operatorname{tr}(\tau(A-B)^4)+\frac{1}{16}(\operatorname{tr}((A+B)^2))^2+\frac{1}{16}(\operatorname{tr}((A-B)^2))^2\\
&-\frac{1}{12}\operatorname{tr}((A+B)^4)-\frac{1}{12}\operatorname{tr}((A-B)^4)-\frac{1}{32}(\operatorname{tr}((A+B)^2))^2-\frac{1}{32}(\operatorname{tr}((A-B)^2))^2\\
&-\operatorname{tr}(ABAB)+\operatorname{tr}(A^2B^2)-\frac{1}{4}\operatorname{tr}(\tau(A+B)^2\,\tau(A-B)^2)+\frac{1}{16}(\operatorname{tr}(A^2-B^2))^2\\
=&-\frac{1}{12}[\operatorname{tr}(\tau(A)^4)+\operatorname{tr}(\tau(B)^4)+2\operatorname{tr}(\tau(A)\,\tau(B)\,\tau(A)\,\tau(B))+4\operatorname{tr}(\tau(A)^2\,\tau(B)^2)]\\
&-\frac{1}{6}[\operatorname{tr}(A^4)+\operatorname{tr}(B^4)+2\operatorname{tr}(ABAB)+4\operatorname{tr}(A^2B^2)]\\
&+\frac{1}{16}[(\operatorname{tr}(A^2))^2+(\operatorname{tr}(B^2))^2+2\operatorname{tr}(A^2)\operatorname{tr}(B^2)+4(\operatorname{tr}(AB))^2]\\
&-\operatorname{tr}(ABAB)+\operatorname{tr}(A^2B^2)-\frac{1}{4}\operatorname{tr}(\tau(A+B)^2\,\tau(A-B)^2)+\frac{1}{16}(\operatorname{tr}(A^2-B^2))^2\\
=&-\frac{1}{3}[\operatorname{tr}(\tau(A)^4)+\operatorname{tr}(\tau(B)^4)-\operatorname{tr}(\tau(A)\,\tau(B)\,\tau(A)\,\tau(B))+\operatorname{tr}(\tau(A)^2\,\tau(B)^2)]\\
&-\frac{1}{6}[\operatorname{tr}(A^4)+\operatorname{tr}(B^4)+2\operatorname{tr}(ABAB)]+\frac{1}{3}\operatorname{tr}(A^2B^2)\\
&+\frac{1}{8}[(\operatorname{tr}(A^2))^2+(\operatorname{tr}(B^2))^2]+\frac{1}{4}(\operatorname{tr}(AB))^2-\operatorname{tr}(ABAB)\\
=&-\frac{1}{6}[2\operatorname{tr}(\tau(A)^4)+\operatorname{tr}(A^4)+2\operatorname{tr}(\tau(B)^4)+\operatorname{tr}(B^4)]\\
&+\frac{1}{8}(\operatorname{tr}(A^2))^2+\frac{1}{8}(\operatorname{tr}(B^2))^2+\frac{1}{4}(\operatorname{tr}(AB))^2-\operatorname{tr}(ABAB)\\
=&-\frac{1}{6}\left[24\operatorname{Pf}(A)+\frac{3}{4}(\operatorname{tr}(A^2))^2+24\operatorname{Pf}(B)+\frac{3}{4}(\operatorname{tr}(B^2))^2\right]\\
&+\frac{1}{8}(\operatorname{tr}(A^2))^2+\frac{1}{8}(\operatorname{tr}(B^2))^2+\frac{1}{4}(\operatorname{tr}(AB))^2-\operatorname{tr}(ABAB)\\
=&\operatorname{Pf}(X)+\operatorname{Pf}(Y)+\frac{1}{16}(\operatorname{tr}(XY))^2-\frac{1}{4}\operatorname{tr}(XYXY)\\
=&J(X,Y).
\end{aligned}
$$

Hence $T\Lambda\in E_7$.

We have now showed that all outer automorphisms of $\mathfrak{h}$ extend to inner automorphisms of $\mathfrak{e}_7$. Each one has four extensions since we showed in (5.4)(b) that $C_{G_0}(H_0)/Z(G_0)=\mathbf{Z}_2\times\mathbf{Z}_2$. Hence $\hat{H}/H_0=(\mathbf{Z}_2\times\mathbf{Z}_2)\rtimes S_3$. To determine the group structure, choose the generators

$$\alpha=\pm\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix} \quad\text{and}\quad \beta=\pm\begin{pmatrix}0 & iI\\ -iI & 0\end{pmatrix}$$

of $C_{G_0}(H_0)$. If $\tau\in S_3$ represents the triality automorphism and $\sigma\in S_3$ represents the order 2 automorphism, then one easily verifies, by a computation with the corresponding matrices in Sp(28), that

$$\tau\alpha\tau^{-1}=\alpha\beta,\quad \tau\beta\tau^{-1}=\alpha,\quad \sigma\alpha\sigma^{-1}=\beta,\quad \sigma\beta\sigma^{-1}=\alpha.$$

$(\hat{H}/H_0)_{\min}$ is of course $\mathbf{Z}_3\subset S_3$.

This finishes this example and hence the proof of (5.1). □

Remark. To determine the group structure of $\hat{G}$, we observe that the discussion of each example contains an explicit map $\hat{H}/H_0\to\mathrm{Aut}(G_0)$ and hence $\hat{G}$ is the semidirect product of G_0 with $\hat{H}/H_0$.

§ 6. Isotropy irreducible subcoverings and applications

In this section we describe how the general isotropy irreducible homogeneous space $M^*=G^*/H^*$ can be constructed from Table I and II. As explained in § 1, the transitive action of G^* lifts to an effective transitive action of G on the universal cover M of M^* such that G is a covering of G^*, and M becomes isotropy irreducible under G. If M is compact and de Rham irreducible but not strongly isotropy irreducible, then (G_0, H_0) must appear in Table I or II.

If M is non-compact, it follows from (2.1) that M has no isotropy irreducible subcoverings.

If M^* is compact and flat, then $M^*=\mathbf{R}^n/\Gamma$ for some lattice Γ and $G_0^*=\mathbf{R}^n, H_0^*=1$. The full isometry group of $\mathbf{R}^n/\Gamma$ is $(\mathbf{R}^n/\Gamma)\rtimes N_{O(n)}(\Gamma)$ and hence M^* is isotropy irreducible iff $N_{O(n)}(\Gamma)$ acts irreducibly on $\mathbf{R}^n$. We do not try to enumerate such examples.

(6.1) Theorem. *Let $M^*=G^*/H^*$ be a compact, non-flat, isotropy irreducible homogeneous space.*

(a) *If the universal cover of M^* is isometric to a compact simply connected Lie group K with $G_0=K$ and $H_0=1$, then there exists a central subgroup N and a finite* *subgroup* Γ *of* K *containing* N *and such that* $M^*=K/\Gamma=(K/N)/(\Gamma/N)$. *Furthermore* $G^*=K\cdot\Phi$, $H^*=\Phi$ *for some finite subgroup* Φ *of* $\mathrm{Aut}(K)$ *such that* Φ *acts irreducibly on* $\mathfrak{k}$, $\mathrm{Ad}(\Gamma)\subseteq\Phi$, *and* Φ *leaves* Γ *invariant*.

(b) *If* G_0 *is the* id-*component of the full isometry group, then there exists a central subgroup* $N\subseteq Z(G_0)$, *compact subgroups* P *and* L *satisfying* $H_0\times N\subseteq P\subseteq N_{G_0}(H_0)$, $H_0\subseteq P/N\subseteq L\subseteq\hat{H}$ (*with* $\hat{H}$ *the full isotropy group*), *such that* Ad_L *acts irreducibly on* $\mathfrak{m}$ *and leaves* N *and* P *invariant, i.e.* P/N *is normal in* L. *Furthermore,* $G^*=(G_0/N)\ltimes(L/\{P/N\})$, $H^*=(P/N)\ltimes(L/\{P/N\})\simeq L$, $M^*=G^*/H^*=G_0/P=(G_0/N)/(P/N)$ *and* $\pi_1(M^*)\simeq P/H_0$.

Proof. Let M be the universal cover of M^* as above with deck group D and transitive group action G. Since by construction all isometries in G project to isometries in G^*, D must be normal in G and hence $D\cap G_0=N$ is a central subgroup of G_0 with $G_0^*=G_0/N$. Let P^* be such that $M^*=G_0^*/P^*$ and let $\pi\colon G_0\to G_0^*=G_0/N$ be the canonical projection. Then $M^*=G_0/P=(G_0/N)/(P/N)$ with $P=\pi^{-1}(P^*)$ and hence the id-component of P is H_0. Furthermore $H_0\cap N=1$ and therefore $H_0\times N\subset P\subset N_{G_0}(H_0)$ and $D\simeq\pi_1(M^*)\simeq P/H_0$. If P/N does not yet act irreducibly on $\mathfrak{m}$, then G^* and hence H^* must contain further isometries.

First, assume that we are in case (a). Then $H_0=1$ and hence $P=\Gamma$ is a finite group. The further isometries in H^* become, via the adjoint representation, automorphisms of $\mathfrak{k}$ and hence there exists a finite group Φ with $\mathrm{Ad}(\Gamma)\subset\Phi\subset\mathrm{Aut}(K)$ such that Φ acts irreducibly on $\mathfrak{k}$ and Φ leaves Γ invariant.

If we are in case (b), notice that $N_{G_0}(H_0)/H_0Z(G_0)\subset\hat{H}$ via right multiplication on G_0/H_0 and hence $P/N\subset\hat{H}$. To make G_0/P isotropy irreducible we need a compact group L with $H_0\subset P/N\subset L\subset\hat{H}$ such that Ad_L acts irreducibly on $\mathfrak{m}$ and keeps N and P invariant. Then L induces further isometries on G_0/P and we obtain an isotropy irreducible form of M^*. □

Notice that by (2.4) the two cases in (6.1) cover all possible isotropy irreducible homogeneous spaces which are not strongly isotropy irreducible.

We now discuss some examples of (6.1) (b). Notice that the possible manifolds that can arise as subcoverings are always of the form G_0/P with $H_0\subset P\subset N_{G_0}(H_0)$.

If $\mathrm{rank}\, G_0=\mathrm{rank}\, H_0$, then $Z(G_0)=1$ and hence $G_0^*=G_0$. By examining Table I, we can see that a minimal subgroup needed to make G_0/H_0 isotropy irreducible can always be chosen to be in $N_{G_0}(H_0)$ and hence we can choose $P\subseteq L\subseteq N_{G_0}(H_0)$ with P normal in L, and such that Ad_L acts irreducibly on $\mathfrak{m}$. One possible choice that always works is $P=L=N_{G_0}(H_0)$ and hence all equal rank examples have isotropy irreducible subcoverings. In several cases $N_{G_0}(H_0)/H_0Z(G_0)=S_3$ with $(\hat{H}/H_0)_{\min}=\mathbf{Z}_3\subset S_3$. Here the only possible choices are $P/H_0=\mathbf{Z}_3$ with $L/H_0=\mathbf{Z}_3$ or S_3 and $P/H_0=L/H_0=S_3$. But in many other examples the possible isotropy irreducible subcoverings become almost impossible to list, since there are so many of them. E.g. in Example 3 in Table I there are many subgroups of $(\mathbf{Z}_2)^{n-1}$ invariant under S_n giving rise to spaces of partially oriented flags in $\mathbf{R}^{nk}$ (see the discussion in § 1, Example 4). There are also many subgroups of S_n which act transitively on the set of unordered pairs and hence give rise to further isotropy irreducible subcoverings.

If $\operatorname{rank} H_0<\operatorname{rank} G_0$, $(\hat{H}/H_0)_{\min}$ sometimes consists of outer automorphisms of G_0. For case (1) in Table II there exists many isotropy irreducible subcoverings as in the equal rank case. For case 2(a) there clearly exists only one subcovering which becomes isotropy irreducible, whereas for case 3(a) there exists none. Case 2(c) is interesting since the autmorphism in $(\hat{H}/H_0)_{\min}$ is outer. There are two isotropy irreducible subcovers corresponding to the whole group $N_{G_0}(H_0)/H_0Z(G_0)=\mathbf{Z}_2\times\mathbf{Z}_2$ or to the subgroup $\triangle\mathbf{Z}_2\subset\mathbf{Z}_2\times\mathbf{Z}_2$. Case 2(b) has many isotropy irreducible subcoverings including e.g. the ones corresponding to $P=N_{G_0}(H_0)/H_0Z(G_0)=D_8$, two subgroups in D_8 isomorphic to $\mathbf{Z}_2$, the product of the previous 3 groups with $Z(G_0)=\mathbf{Z}_2$, and two groups $\mathbf{Z}_2$ embedded diagonally in $D_8\times\mathbf{Z}_2$. An interesting example is case (6). Here $N_{G_0}(H_0)/H_0Z(G_0)=1$ and $Z(G_0)=\mathbf{Z}_2\times\mathbf{Z}_2$, but the only subgroup of $Z(G_0)$ which is invariant under the triality automorphism, which generates $\mathbf{Z}_3=(\hat{H}/H_0)_{\min}$, is the whole center. Hence the only isotropy irreducible subcovering is $(\mathrm{SO}(8)/\mathbf{Z}_2)/\mathrm{G}_2=\mathbf{R}P^7\times\mathbf{R}P^7$. Finally we discuss case (4). Here we have again $N_{G_0}(H_0)/H_0Z(G_0)=1$ and hence we can only divide out by a central subgroup of G_0. An obvious choice is $N=(A\times\cdots\times A)/\triangle(A)$ for any subgroup $A\subset Z(K)$, in fact this may be the only possibility.

To obtain examples which are not locally de Rham irreducible, we can take for M a product of N k times with itself, where N is an entry in Table I or II. P is then a product of the groups discussed above, but for L we need to choose in addition to the product of the above groups a subgroup of the symmetric group S_k acting transitively on the de Rham factors of M.

It is natural to ask the question what the isotropy irreducible homogeneous spaces $M^*=G^*/H^*$ are with G^* connected. We can now easily answer this more restrictive question. Of course, if M^* is simply connected, this can only happen if G^*/H^* is strongly isotropy irreducible. In general we have:

(6.2) Theorem. *Let $M^*=G^*/H^*$ be a non-flat isotropy irreducible homogeneous space with G^* connected and with universal cover $M=G_0/H_0$. Then M is de Rham* *irreducible. If* $H_0=1$, *there exists a central subgroup* N *of* G_0 *and a finite group* $\Gamma\subset G_0$ *such that* $N\subset\Gamma$ *and* $M^*=G_0/\Gamma=(G_0/N)/(\Gamma/N)$ *with* $\mathrm{Ad}|_\Gamma$ *acting irreducibly on* $\mathfrak{g}$. *In all other cases there exists a central subgroup* N *of* G_0 *and a compact group* P *with* $H_0\times N\subseteq P\subseteq N_{G_0}(H_0)$ *such that* Ad_P *acts irreducibly on* $\mathfrak{m}$ *and* $M^*=G^*/H^*=(G_0/N)/(P/N)=G_0/P$. *For every entry in Table* I *and* II, *such a group* P *exists, with the exception of Examples* 2(c), 3(a), 4, *and* 6 *in Table* II.

Proof. Most of the statements in this theorem follow as in the proof of (6.1). Here we only need in addition $L=P/N$. That M must be de Rham irreducible follows from the fact that in the de Rham reducible case all isometries which interchange the de Rham factors of M are outer automorphisms of G_0. Similarly, the four cases in Table II have no isotropy irreducible subcovers of this form since the isometries needed to make G_0/H_0 isotropy irreducible come from outer automorphisms of G_0. □

We finally make some comments about the minimal isometric immersions mentioned in the introduction. In general, if $\phi: M\to S^N(r)$ is an isometric minimal immersion, then the coordinate functions $\phi_1,\ldots,\phi_{N+1}$ are eigenfunctions of the Laplace operator with eigenvalue $\lambda=\dim M/r^2$. Conversely, as was observed by Takahashi [T], if $M=G/H$ is an isotropy irreducible homogeneous space and if $\phi_1,\ldots,\phi_{N+1}$ is an orthonormal basis of the eigenspace E_λ of the Laplace operator with eigenvalue $\lambda>0$, then $\phi=(\phi_1,\ldots,\phi_{N+1})$ is an isometric minimal immersion into $S^N(r)$ with $r^2=\dim M/\lambda$. Indeed, the pull back of the metric on $S^N(r)$ under ϕ is clearly invariant under G and hence is either 0 or a G-invariant metric. But it cannot be 0 since $\lambda>0$. A different choice of orthonormal basis gives rise to a congruent immersion and we call this class of isometric immersions the standard eigenspace immersion. There are in general many other minimal isometric immersions into $S^N(r)$, but they are all given by choosing eigenfunctions in E_λ, $\lambda=\dim M/r^2$, as coordinates. If the coordinates form a basis of E_λ, then we call such a minimal isometric immersion a full eigenspace immersion, otherwise a partial eigenspace immersion. For example, if there is a subspace of E_λ invariant under G, then an orthonormal basis of this subspace gives rise to a partial eigenspace immersion. It was shown in [L] that the space of all isometric minimal immersions of G/H into $S^N(r), r^2=\dim M/\lambda$, forms a compact convex body in a finite dimensional vector space with interior points corresponding to the full eigenspace immersions and boundary points corresponding to partial eigenspace immersions.

We now discuss some of the other results in [L], since not all of them are correct. In [L, Lemma 3] it was shown that for a full eigenspace immersion $\phi: M=G/H\to S^N(r)$, $M^*=\phi(M)$ is an embedded submanifold and $\phi: M\to M^*$ is a covering map. This is then applied in [L, Theorem 4] to show that every eigenspace immersion is a covering onto its image, that the homogeneous structure on M^* is the one induced from M, and that the deck group of the covering is contained in the center of G. This is actually false, but partially true for the full eigenspace immersions. A good counterexample is given by K/Γ where K is a connected, compact, simple Lie group and Γ a finite subgroup such that $\mathrm{Ad}|_\Gamma$ acts irreducibly on $\mathfrak{k}$. Using a full eigenspace immersion for K/Γ, we obtain an isometric minimal immersion $K\to K/\Gamma\to S^N(r)$ whose image does not carry the homogeneous structure induced from K. Indeed the id-component of the isometry group is equal to $[K\times K]/\triangle K$ for K and equal to K for K/Γ. Also, even for a full eigenspace immersion, the deck group need not be contained in $Z(G)$. The correct statement is:

(6.3) THEOREM (P. Li). *Let $M=G/H$ be a compact isotropy irreducible homogeneous space and $\phi: M\to S^N(r)$ a full eigenspace immersion. Then $\phi(M)=M^*$ is an embedded submanifold, $\phi: M\to M^*$ a covering map, and the deck group D of the covering is normal in the full isometry group $\hat{G}$ of M. Hence D commutes with G_0 and $D\cap G_0\subset Z(G_0)$. The isometry group of M^* is $\hat{G}/D$, and hence M^* must be an isotropy irreducible homogeneous space.*

Proof. We can regard $\phi=(\phi_1, \ldots, \phi_{N+1})$ as an immersion $\phi: M\to \mathbf{R}^{N+1}$. The isometric action of $\hat{G}$ on M induces an orthogonal representation of $\hat{G}$ on E_λ. By using the basis $\phi_1, \ldots, \phi_{N+1}$ of E_λ to identify E_λ with $\mathbf{R}^{N+1}$, this induces a linear action of $\hat{G}$ on $\mathbf{R}^{N+1}$ and the immersion $\phi: M\to\mathbf{R}^{N+1}$ becomes equivariant with respect to this action. Notice though that $\phi: M\to S^N(r)$ is equivariant only for the standard eigenspace immersion. In any case, the equivariance implies that the immersion has no double points and since M is compact $\phi: M\to M^*$ is a covering map. The equivariance also implies that the isometries in $\hat{G}$ induce isometries on M^* and hence D is normal in $\hat{G}$, which implies that $D\cap G_0$ must centralize G_0. $\square$

The fact that D is normal in $\hat{G}$ is actually a strong property, which is not satisfied by many of our isotropy irreducible subcoverings. E.g. for the space of real flags in $\mathbf{R}^{nk}$ as in Example 4 in § 1, if we divide out by the symmetric group S_n, we obtain an isotropy irreducible subcovering which cannot be the image of any full eigenspace immersion of the space of oriented flags. This also implies that if we choose an eigenspace immersion for this quotient and lift it to the space of oriented flags, it must become a partial eigenspace immersion of the space of oriented flags. For the Example 3(a) in Table II, (6.3) implies that every full eigenspace immersion must be an embedding since there are no isotropy irreducible subcoverings.

Notice that [L, Theorem 5] is false, as the example S^3/Γ shows, where Γ is e.g. the binary dodecahedral group (see Example 1 in § 1). But it is true that a full eigenspace immersion of $S^n(1)$ is either an embedding of $S^n(1)$ or an embedding of $\mathbf{R}P^n(1)$. It is an interesting question whether the image of some partial eigenspace immersion of $S^n(1)$ can be a lens space or more generally if every manifold of constant positive curvature admits a minimal isometric immersion into S^N.

[L, Theorem 7] is again false, since among our examples there are coverings $M \to L \to N$ with M and N isotropy irreducible, but L not.

In [L, Theorem 8] it is shown that if $Z(G)$ is cyclic, then there are infinitely many eigenfunctions which are not invariant under any subgroup of $Z(G)$. Unfortunately, by the above remarks this does not imply [L, Corollary 9] which claims that infinitely many of the full eigenspace immersions are embeddings. It would be interesting to obtain a criterion which guarantees that some of the full eigenspace immersions are embeddings. We suspect that for each isotropy irreducible homogeneous space there exists at least one of these.

Note added in proof. After the preparation of this manuscript it was pointed out to us, that the mistake in Li's paper [L] was already observed by K. Mashimo in "Minimal immersions of 3-dimensional sphere into spheres", Osaka J. Math., 21 (1984), 721–732.

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Received January 9, 1990